Chapter 3
Image Derivatives

Image Processing and Computer Vision

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Overview

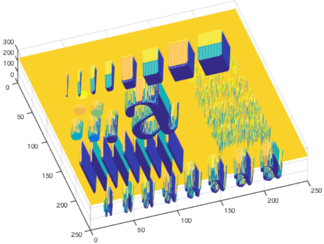
1. Image model
2. First-order derivative
3. Gradient
4. Second-order derivative

Image Model

Image model

- Image is a function of two variables x and y: $f(x, y)$
- It can seen as a surface on 2D-space.

An gray image



Mesh model of the image

Derivative of one variable

Taylor expansion for $f(x + \Delta x)$:

$$f(x + \Delta x) = f(x) + \Delta x \times f^{'}(x) + \frac{\Delta x^2}{2!} \times f^{''}(x) + O(\Delta x^3) \tag{1}$$

First-order derivative from Eq. (1)

$$f^{'}(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- This approximation has error $O(\Delta x)$

Derivative of one variable

Taylor expansion for $f(x - \Delta x)$:

$$f(x - \Delta x) = f(x) - \Delta x \times f^{'}(x) + \frac{\Delta x^2}{2!} \times f^{''}(x) - O(\Delta x^3) \tag{2}$$

First-order derivative from Eq. (2)

$$f^{'}(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

- This approximation has error $O(\Delta x)$

First-Order derivative of image

Partial Derivatives on x

Derivatives	Kernel of filters
$\frac{\partial f(x,y)}{\partial x} = \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$	$H_{conv} = \begin{bmatrix} 1 & -1 \end{bmatrix}$
$\frac{\partial f(x,y)}{\partial x} = \frac{f(x,y)-f(x-\Delta x,y)}{\Delta x}$	$H_{conv} = \begin{bmatrix} 1 & -1 \end{bmatrix}$
$\frac{\partial f(x,y)}{\partial x} = \frac{f(x+\Delta x,y)-f(x-\Delta x,y)}{2\Delta x}$	$H_{conv} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

First-Order derivative of image

Partial Derivatives on y

Derivatives	Kernel of filters
$\frac{\partial f(x,y)}{\partial y} = \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$	$H_{conv} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\frac{\partial f(x,y)}{\partial y} = \frac{f(x,y)-f(x,y-\Delta y)}{\Delta y}$	$H_{conv} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\frac{\partial f(x,y)}{\partial y} = \frac{f(x,y+\Delta y)-f(x,y-\Delta y)}{2\Delta y}$	$H_{conv} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

First-Order derivative of image

Other kernels for taking derivatives

Name	Derivative on x	Derivative on y
Prewitt	$H_x = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$	$H_y = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
Sobel	$H_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$	$H_y = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$
Robert	$H_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$H_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Gradient

Definition

Gradient at a pixel in a image $f(x,y)$ is a vector ∇f. It is defined as

$$\nabla f = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{bmatrix}$$

Shorted form

Let F_x and F_y be $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ respectively.

$$\nabla f = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

Gradient

Magnitude and angle of gradient vectors

1. **Magnitude** of gradient is computed by:
   $|\nabla f| = \sqrt{F_x^2 + F_y^2}$
   - Magnitude of a gradient at pixel (u, v) tells us the rate of change of intensities at (u, v)
   - In other words, it tells us the edge passing (u, v) is strong or not.
2. **Angle** of gradient is computed by:
   $\theta(\nabla f) = tan^{-1}(\frac{F_y}{F_x})$
   - Angle of a gradient at pixel (u, v) tells us the orientation of edge passing (u, v)

Second-order Derivative of one variable

Second-order differential can be approximated by

$$f^{''}(x) \cong f^{'}(x) - f^{'}(x+1)$$

First-order derivatives can be approximated as

$$f^{'}(x) \cong f(x) - f(x-1)$$
$$f^{'}(x+1) \cong f(x+1) - f(x)$$

Second-order derivative

$$f^{''}(x) \cong -f^{'}(x-1) + 2f^{'}(x) - f(x+1)$$

Second-Order derivative of image

1. Second-order derivative on x
   - Math:

$$\frac{\partial^2 f(x,y)}{\partial x^2} = -f(x-1,y) + 2f(x,y) - f(x+1,y)$$

   - Kernel: $H_{conv} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$

2. Second-order derivative on y
   - Math:

$$\frac{\partial^2 f(x,y)}{\partial y^2} = -f(x,y-1) + 2f(x,y) - f(x,y+1)$$

   - Kernel: $H_{conv} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

Second-Order derivative of image

Second-order derivative on x and y

Laplace operator:

$$\nabla^2 f = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

- Math:

$$\begin{aligned}\nabla^2 f = \quad & -f(x-1,y) + 2f(x,y) - f(x+1,y) \\ & -f(x,y-1) + 2f(x,y) - f(x,y+1)\end{aligned}$$

- Kernel: $H_{conv} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

Second-Order derivative of image

Second-order derivative on x, y, and diagonals:

Extended Laplace operator:

- Math:

$$\begin{aligned}\nabla^2 f = \quad & -f(x-1,y) + 2f(x,y) - f(x+1,y) \\ & -f(x,y-1) + 2f(x,y) - f(x,y+1)\end{aligned}$$

- Kernel: $H_{conv} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$