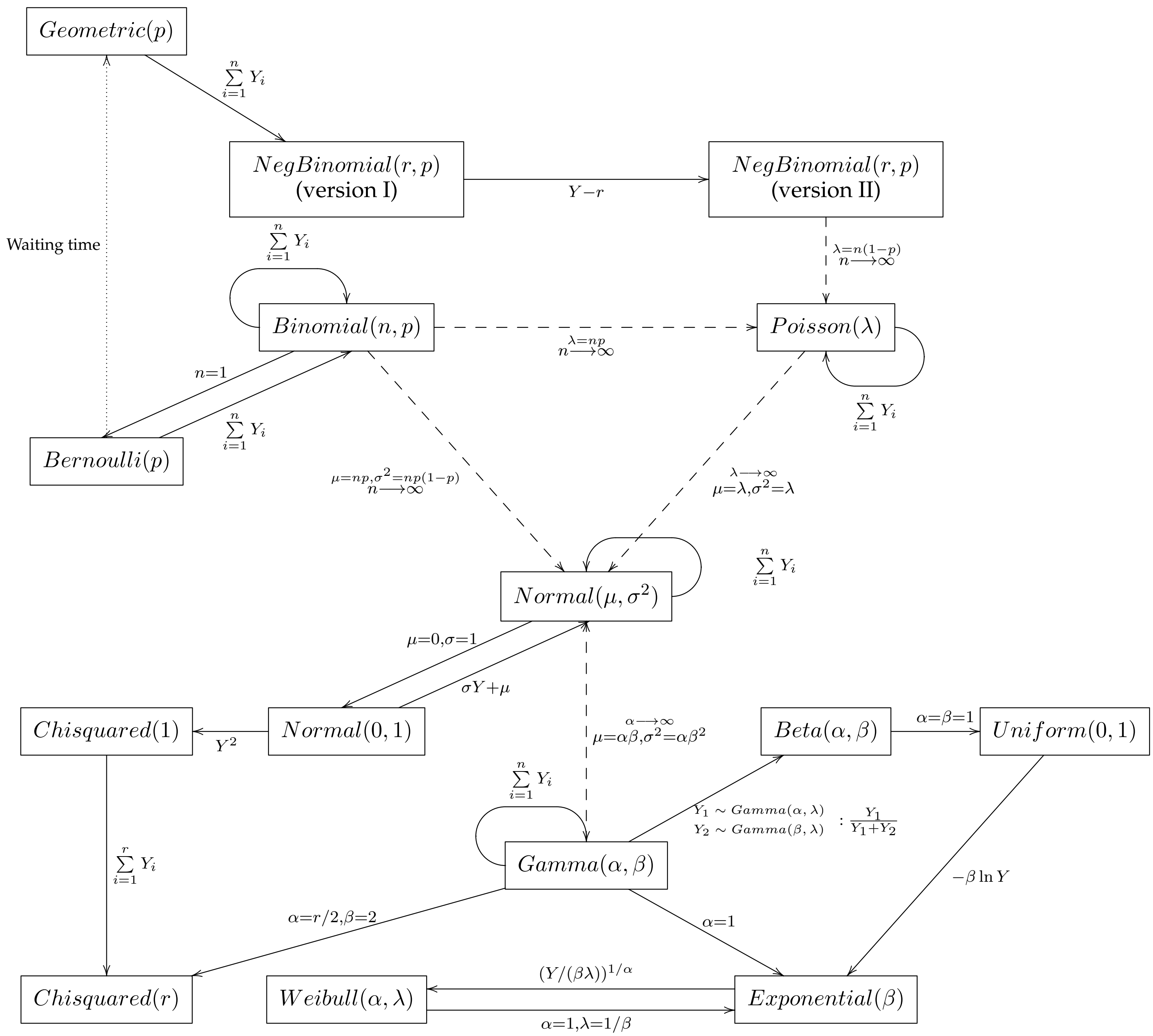
MATH 323: Probability
A Map of the Distributions



Direct link	$\longrightarrow$	Special case, transformation or summation
Indirect link	$- - - \rightarrow$	Limit case

After diagram in *Statistical Inference*, by G Casella and RL Berger.

DISCRETE DISTRIBUTIONS

Models based on an independent sequence of identical binary trials with success probability p.

- **BERNOULLI:** Y is the total number of successes in **one** trial.
- **BINOMIAL:** Y is the total number of successes in n trials.
- **GEOMETRIC:** Y is the total number of trials required to obtain **one** success.
- **NEGATIVE BINOMIAL:**
  - **Version I:** Y is the total number of trials required to obtain r successes.
  - **Version II:** Consider $X = Y - r$, to give a distribution on $\{0, 1, 2, \ldots\}$.
- **POISSON:** The Poisson distribution is obtained as the limiting form of the $Binomial(n, p)$ distribution, with $n \longrightarrow \infty$ but with $\lambda = np$ held fixed. Y is the count of the number of events in a given (continuous) time interval.

Connections:

- Bernoulli/Binomial

$$Y_1, \ldots, Y_n \sim Bernoulli(p) \qquad \Longrightarrow \qquad Y = \sum_{i=1}^{n} Y_i \sim Binomial(n, p)$$

- Geometric/Negative Binomial

$$Y_1, \ldots, Y_n \sim Geometric(p) \qquad \Longrightarrow \qquad Y = \sum_{i=1}^{n} Y_i \sim NegBinomial(n, p)$$

- Binomial/Poisson

$$Y_n \sim Binomial(n, p) \longrightarrow Y \sim Poisson(\lambda)$$

  where $\lambda = np$ is held fixed and $n \longrightarrow \infty$.
- Negative Binomial/Poisson

$$Y_n \sim NegBinomial(n, p) \qquad X_n = Y_n - n \longrightarrow X \sim Poisson(\lambda)$$

  where $\lambda = n(1 - p)$ is held fixed and $n \longrightarrow \infty$.

Sums of Independent Random Variables: Proved using mgfs.

- Binomial

$$\left.\begin{array}{r} Y_1 \sim Binomial(m, p) \\ Y_2 \sim Binomial(n, p) \end{array}\right\} \qquad \Longrightarrow \qquad Y = Y_1 + Y_2 \sim Binomial(m + n, p)$$

- Negative Binomial

$$\left.\begin{array}{r} Y_1 \sim NegBinomial(m, p) \\ Y_2 \sim NegBinomial(n, p) \end{array}\right\} \qquad \Longrightarrow \qquad Y = Y_1 + Y_2 \sim NegBinomial(m + n, p)$$

- Poisson

$$\left.\begin{array}{r} Y_1 \sim Poisson(\lambda_1) \\ Y_2 \sim Poisson(\lambda_2) \end{array}\right\} \qquad \Longrightarrow \qquad Y = Y_1 + Y_2 \sim Poisson(\lambda_1 + \lambda_2)$$

Continuous Distributions

- Distributions on $\mathbb{R}^+$: Begin with $Y \sim Uniform(0,1)$:
  - $U = -\beta \log Y \sim Exponential(\beta)$, for $\beta > 0$.
  - $X = (U/(\beta\lambda))^{1/\alpha} \sim Weibull(\alpha, \lambda)$, for $\alpha, \lambda > 0$.
  - If $X_1, \ldots, X_n \sim Exponential(\beta)$, independent, then $Z = \sum_{i=1}^{n} X_i \sim Gamma(n, \beta)$.
  - If $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent, then
    $$S = Y_1 + Y_2 \sim Gamma(\alpha_1 + \alpha_2, \beta)$$
- Distributions on $\mathbb{R}$: The Normal distribution and connections
  - Suppose $Y \sim N(0,1)$. Then $X = \mu + \sigma Y \sim N(\mu, \sigma^2)$.
  - Suppose $Y \sim N(0,1)$. Then $U = Y^2 \sim Gamma(1/2, 2) \equiv Chisquared(1)$.
  - If $Y_i \sim Gamma(\alpha_i/2, 2) \equiv Chisquared(\alpha_i)$ for $i = 1, \ldots, n$ are independent, then
    $$V = \sum_{i=1}^{n} Y_i \sim Gamma\,(\nu/2, 2) \equiv Chisquared(\nu)$$
    where
    $$\nu = \sum_{i=1}^{n} \alpha_i.$$
  - If $Y_1 \sim N(\mu_1, \sigma_1^2)$ and $Y_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then
    $$Y = Y_1 + Y_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$
- Distribution on $(0,1)$: The Beta distribution
  - If $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent, then
    $$Y = \frac{Y_1}{Y_1 + Y_2} \sim Beta(\alpha_1, \alpha_2)$$
    This result follows by multivariate transformations.

All the summation results proved using mgfs.