Assignment 6 **LE, Nhat Hung**
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1. **(a)** $n = 2:$

$$\begin{aligned} a^2 - 2a + 2 - 1 &= a^2 - 2a + 1 \\ &= (a-1)^2 \end{aligned}$$

Assume $\forall n \geq 2$, $(a-1)^2 \mid a^n - an + n - 1$.

$$\begin{aligned} a^n - an + n - 1 = k(a-1)^2 \quad &\Rightarrow a^{n+1} - a^2n + an - a = ka(a-1)^2,\ k \in \mathbb{Z} \\ &\Rightarrow a^{n+1} - an - a + n = ka(a-1)^2 + a^2n - 2an + n \\ &\Rightarrow a^{n+1} - a(n+1) + n = ka(a-1)^2 + n(a-1)^2 \\ &\Rightarrow a^{n+1} - a(n+1) + n = (ka+n)(a-1)^2 \\ &\Rightarrow (a-1)^2 \mid a^{n+1} - a(n+1) + n \quad \square \end{aligned}$$

(b) $n = 1:$

$$\sum_{k=1}^{1} \frac{(k+m)!}{(k-1)!} = (1+m)!$$

$$\frac{(1+m+1)!}{(1-1)!(m+2)} = (1+m)!$$

$$\sum_{k=1}^{1} \frac{(k+m)!}{(k-1)!} = \frac{(1+m+1)!}{(1-1)!(m+2)}$$

Assume $\forall n \geq 1$,

$$\sum_{k=1}^{n} \frac{(k+m)!}{(k-1)!} = \frac{(n+m+1)!}{(n-1)!(m+2)}$$

Then,

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{(k+m)!}{(k-1)!} &= \sum_{k=1}^{n} \frac{(k+m)!}{(k-1)!} + \frac{(n+1+m)!}{n!} \\ &= \frac{(n+m+1)!}{(n-1)!(m+2)} + \frac{(n+1+m)!}{n!} \\ &= \frac{(n+m+1)!n! + (n+m+1)!(n-1)!(m+2)}{(n-1)!(m+2)n!} \\ &= \frac{(n+m+1)!(n! + (n-1)!(m+2))}{(n-1)!(m+2)n!} \\ &= \frac{(n+m+1)!(n-1)!(n+m+2)}{(n-1)!(m+2)n!} \\ &= \frac{(n+m+2)!}{n!(m+2)} \quad \square \end{aligned}$$

(c) $n = 1:$

$$1 = \sum_{i=1}^{1} 1 \cdot i!$$

Let $m \geq 1$ be given. Assume all $n \leq m$ has such a representation.
Now proving $\mathbf{n = m + 1}$ also has such a representation.

Let k largest integer such that

$$(1) \quad k! \leq m + 1 \leq (k+1)!$$

If $k! = m + 1$, then $m + 1$ has the desired representation.
If $k! < m + 1$, let m' such that $k! + m' = m + 1$

$$k! + m' = m + 1 \quad \Rightarrow m' = m + 1 - k!$$

$$\begin{aligned} 1 \leq k! < m + 1 \quad &\Rightarrow -m - 1 < -k! \leq -1 \\ &\Rightarrow 0 < m + 1 - k! \leq m \\ &\Rightarrow 0 \leq m' \leq m \end{aligned}$$

Thus m' has the desired representation.
Then, $k! + m' = m + 1$ is a sum of factorials.

Now checking if $k! + m'$ satisfies $0 \leq c_i < i$.

If $k!$ is absent from the representation of m', the condition is satisfied.

If $k!$ is present in the representation of m', we have 2 cases: $c_k \leq k - 1$ and $c_k = k$.

$\mathbf{c_k \leq k - 1}$:
$k! + m' = ... + (c_k + 1)k!$
with $c_k + 1 \leq k$, satisfying the condition.

$\mathbf{c_k = k}$:

$$\begin{aligned} m + 1 \quad &= m' + k! \\ &\geq d_k k! + k! \\ &\geq kk! + k! \\ &\geq (k+1)! \end{aligned}$$

which contradicts (1).
This case is thus impossible.

Therefore, n = m + 1 has the desired representation.

In conclusion, $\forall n \geq 1$,

$$n = \sum_{i=1}^{k} c_i i! \quad \square$$

2. (a) $n = 1$:

$$\begin{aligned} (-1)^1 f_1 + 1 \quad &= -1 + 1 \\ &= 0 \\ &= 1 - 1 \\ &= f_1 - f_2 \end{aligned}$$

Assume $\forall n \geq 1,\ f_1 - f_2 + f_3 + ... + (-1)^n f_{n+1} = (-1)^n f_n + 1$.

$$\begin{aligned} f_1 - f_2 + f_3 + ... + (-1)^{n+1} f_{n+2} &= (-1)^n f_n + 1 + (-1)^{n+1} f_{n+2} \\ &= (-1)^n f_n + 1 + (-1)^{n+1} f_n + (-1)^{n+1} f_{n+1} \\ &= (-1)^n f_n - (-1)^n f_n + (-1)^{n+1} f_{n+1} + 1 \\ &= (-1)^{n+1} f_{n+1} + 1 \quad \square \end{aligned}$$

(b) $n = 1:$

$$\begin{aligned} f_1 f_2 &= 1 \\ &= 1^2 \\ &= f_2^2 \end{aligned}$$

Assume $\forall n \geq 1,\ f_1 f_2 + f_2 f_3 + f_3 f_4 + ... + f_{2n-1} f_{2n} = f_{2n}^2$.

$$\begin{aligned} f_1 f_2 + f_2 f_3 + f_3 f_4 + ... + f_{2(n+1)-1} f_{2(n+1)} &= f_{2n}^2 + f_{2n} f_{2n+1} + f_{2(n+1)-1} f_{2(n+1)} \\ &= f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} \\ &= f_{2n}^2 + 2 f_{2n} f_{2n+1} + f_{2n+1}^2 \\ &= (f_{2n} + f2n + 1)^2 \\ &= f_{2n+2}^2 \\ &= f_{2(n+1)}^2 \quad \square \end{aligned}$$

3. (a) $a_m = \frac{3}{2} a_{m-1} - \frac{1}{4} a_{m-2} - 6m$

(b) Let $p_m = am + b$

$$\begin{aligned} am + b &= \tfrac{3}{2}\Big(a(m-1) + b\Big) - \tfrac{1}{4}\Big(a(m-2) + b\Big) - 6m \\ &= \tfrac{3}{2} am - \tfrac{3}{2} a + \tfrac{3}{2} b - \tfrac{1}{4} am + \tfrac{1}{2} a - \tfrac{1}{4} b - 6m \\ 0 &= \tfrac{1}{4} am - 6m - a + \tfrac{1}{4} b \\ &= -a + \tfrac{1}{4} b + \Big(\tfrac{1}{4} a - 6\Big) m \end{aligned}$$

$$\begin{cases} -a + \frac{1}{4} b = 0 \\ \frac{1}{4} a - 6 = 0 \end{cases} \Rightarrow \begin{cases} b = 96 \\ a = 24 \end{cases}$$
$$\Rightarrow p_m = 24m + 96$$

Now solving $a_m = \frac{3}{2} a_{m-1} - \frac{1}{4} a_{m-2}$

$$\begin{aligned} x^2 - \tfrac{3}{2} x + \tfrac{1}{4} &= \Big(x - \tfrac{3}{4}\Big)^2 - \tfrac{5}{16} \\ &= \Big(x - \tfrac{3+\sqrt{5}}{4}\Big)\Big(x - \tfrac{3-\sqrt{5}}{4}\Big) \end{aligned}$$

Let $q_m = c_1 \Big(\frac{3+\sqrt{5}}{4}\Big)^m + c_2 \Big(\frac{3-\sqrt{5}}{4}\Big)^m$

Then,

$$\begin{aligned} a_m &= p_n + q_n \\ &= 24m + 96 + c_1 \Big(\frac{3+\sqrt{5}}{4}\Big)^m + c_2 \Big(\frac{3-\sqrt{5}}{4}\Big)^m \end{aligned}$$

$$\begin{cases} a_0 = 40 \\ a_1 = 54 \end{cases} \Rightarrow \begin{cases} 40 = 96 + c_1 + c_2 \\ 54 = 24 + 96 + c_1\left(\frac{3+\sqrt{5}}{4}\right) + c_2\left(\frac{3-\sqrt{5}}{4}\right) \end{cases}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = -56 \\ c_1\left(\frac{3+\sqrt{5}}{4}\right) + c_2\left(\frac{3-\sqrt{5}}{4}\right) = -66 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = -c_2 - 56 \\ -c_2\left(\frac{3+\sqrt{5}}{4}\right) - 56\left(\frac{3+\sqrt{5}}{4}\right) + c_2\left(\frac{3-\sqrt{5}}{4}\right) = -66 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = -c_2 - 56 \\ -\frac{\sqrt{5}}{2}c_2 - 42 - 14\sqrt{5} = -66 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{-48\sqrt{5}-140}{5} \\ c_2 = \frac{48\sqrt{5}-140}{5} \end{cases}$$

Thus,

$$\begin{aligned} a_m &= 24m + 96 + c_1\Big(\frac{3+\sqrt{5}}{4}\Big)^m + c_2\Big(\frac{3-\sqrt{5}}{4}\Big)^m \\ &= 24m + 96 - \frac{140+48\sqrt{5}}{5}\Big(\frac{3+\sqrt{5}}{4}\Big)^m - \frac{140-48\sqrt{5}}{5}\Big(\frac{3-\sqrt{5}}{4}\Big)^m \end{aligned}$$

(c) Now proving $\frac{da_m}{dm} < 0$ for $m \geq 3$

$$\frac{da_m}{dm} = 24 - \frac{140+48\sqrt{5}}{5} ln\Big(\frac{3+\sqrt{5}}{4}\Big)\Big(\frac{3+\sqrt{5}}{4}\Big)^m - \frac{140-48\sqrt{5}}{5} ln\Big(\frac{3-\sqrt{5}}{4}\Big)\Big(\frac{3-\sqrt{5}}{4}\Big)^m$$

$$\lim_{m\to\infty} \frac{140+48\sqrt{5}}{5} ln\Big(\frac{3+\sqrt{5}}{4}\Big)\Big(\frac{3+\sqrt{5}}{4}\Big)^m = \infty$$

$$\lim_{m\to\infty} \frac{140-48\sqrt{5}}{5} ln\Big(\frac{3-\sqrt{5}}{4}\Big)\Big(\frac{3-\sqrt{5}}{4}\Big)^m = 0$$

Thus,

$$\lim_{m\to\infty} \frac{140+48\sqrt{5}}{5} ln\Big(\frac{3+\sqrt{5}}{4}\Big)\Big(\frac{3+\sqrt{5}}{4}\Big)^m + \frac{140-48\sqrt{5}}{5} ln\Big(\frac{3-\sqrt{5}}{4}\Big)\Big(\frac{3-\sqrt{5}}{4}\Big)^m = \infty$$

$$\lim_{m\to\infty} 24 - \frac{140+48\sqrt{5}}{5} ln\Big(\frac{3+\sqrt{5}}{4}\Big)\Big(\frac{3+\sqrt{5}}{4}\Big)^m - \frac{140-48\sqrt{5}}{5} ln\Big(\frac{3-\sqrt{5}}{4}\Big)\Big(\frac{3-\sqrt{5}}{4}\Big)^m = -\infty$$

$$\lim_{m\to\infty} \frac{da_m}{dm} = -\infty$$

Let $f'(m) = \frac{da_m}{dm}$

$$\begin{aligned} f'(3) &\approx -5.8 \\ &< 0 \end{aligned}$$

Thus, $\forall m \geq 3, \frac{da_m}{dm} < 0 \quad \square$

This means after 3 minutes, the prize money a_m only decreases.
Checking a_0, a_1, a_2 and a_3:
$a_0 = 40$
$a_1 = 54$
$a_2 = 59$
$a_3 = 57$

Thus, leaving the chair after 2 minutes maximizes the winnings.