Assignment 2
MATH 324 - Statistics
Prof. Masoud Asgharian
Winter 2019

LE, Nhat Hung
McGill ID: 260793376
Date: February 9, 2019
Due date: February 16, 2019

9.3 p. 447

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from the uniform distribution on the interval $(\theta, \theta + 1)$. Let

$$\hat{\theta}_1 = \overline{Y} - \frac{1}{2} \text{ and } \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}$$

a. Show that both $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are unbiased estimators of θ.

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_1] &= \mathbb{E}\left[\overline{Y} - \frac{1}{2}\right] \\
&= \mu - \frac{1}{2} \\
&= \frac{\theta + (\theta+1)}{2} - \frac{1}{2} \\
&= \theta
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_2] &= \mathbb{E}\left[Y_{(n)} - \frac{n}{n+1}\right] \\
&= \mathbb{E}[Y_{(n)}] - \frac{n}{n+1}
\end{aligned}$$

Let $f_{Y_{(n)}}(y)$ the density function of $Y_{(n)}$. Then

$$\mathbb{E}[Y_{(n)}] = \int_{\theta}^{\theta+1} y f_{Y_{(n)}}(y) \mathrm{d}y$$

$$f_{Y_{(n)}} = nF(y)^{n-1} f(y) = n\left(\frac{y-\theta}{(\theta+1)-\theta}\right)^{n-1} \frac{1}{(\theta+1)-\theta} = n(y-\theta)^{n-1}$$

$$\begin{aligned}
E[Y_{(n)}] &= \int_{\theta}^{\theta+1} yn(y-\theta)^{n-1} \mathrm{d}y \\
&= y(n-\theta)^n|_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1} (y-\theta)^n \mathrm{d}y \\
&= \theta + 1 - \frac{1}{n+1} \\
&= \theta + \frac{n}{n+1}
\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\hat{\theta}_2] &= \mathbb{E}[Y_{(n)}] - \frac{n}{n+1} \\ &= \theta + \frac{n}{n+1} - \frac{n}{n+1} \\ &= \theta \quad \square\end{aligned}$$

b. Find the efficiency of $\widehat{\theta}_1$ relative to $\widehat{\theta}_2$.

$$\mathbb{V}[\hat{\theta}_1] = \mathbb{V}\left[\overline{Y} - \frac{1}{2}\right] = \mathbb{V}(\overline{Y}) = \frac{\sigma^2}{n} = \frac{\frac{((\theta+1)-\theta)^2}{12}}{n} = \frac{1}{12n}$$

$$\mathbb{V}[\hat{\theta}_2] = \mathbb{V}\left[\overline{Y} - \frac{n}{n+1}\right] = \mathbb{V}[Y_{(n)}]$$

$$\mathbb{V}[Y_{(n)}] = \mathbb{E}[Y_{(n)}^2] - \mathbb{E}[Y_{(n)}]^2 = \mathbb{E}[Y_{(n)}^2] - \left(\theta + \frac{n}{n+1}\right)^2$$

$$\mathbb{E}[Y_{(n)}^2] = \int_\theta^{\theta+1} y^2 n(y-\theta)^2 \mathrm{d}y = (\theta+1)^2 - \frac{2(\theta+1)(n+2)-2}{(n+1)(n+2)}$$

$$\begin{aligned}\mathbb{V}[Y_{(n)}] &= \mathbb{E}[Y_{(n)}^2] - \left(\theta + \frac{n}{n+1}\right)^2 \\ &= (\theta+1)^2 - \frac{2(\theta+1)(n+2)-2}{(n+1)(n+2)} - \left(\theta + \frac{n}{n+1}\right)^2 \\ &= \frac{n}{(n+2)(n+1)^2}\end{aligned}$$

Therefore,

$$\mathbb{V}[\hat{\theta}_2] = \mathbb{V}[Y_{(n)}] = \frac{n}{(n+2)(n+1)^2}$$

and

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathbb{V}[\hat{\theta}_2]}{\mathbb{V}[\hat{\theta}_1]} = \frac{12n^2}{(n+2)(n+1)^2}$$

9.7 p. 447

Suppose that $Y_1, Y_2, ..., Y_n$ denote a random sample of size *n* from an exponential distribution with density function given by

$$f(y) = \begin{cases}(1/\theta)e^{-y/\theta}, & 0 < y, \\ 0, & \text{elsewhere.}\end{cases}$$

In Exercise 8.19, we determined that $\widehat{\theta}_1 = nY_{(1)}$ is an unbiased estimator of θ with MSE $(\widehat{\theta}_1) = \theta^2$. Consider the estimator $\widehat{\theta}_2 = \overline{Y}$ and find the efficiency of $\widehat{\theta}_1$ relative to $\widehat{\theta}_2$.

$$Y \sim \text{Exponential}(\theta)$$

Because $\widehat{\theta}_1$ is an unbiased estimator of θ,

$$\mathbb{V}[\hat{\theta}_1] = \text{MSE}(\hat{\theta}_1) = \theta^2$$

$$\mathbb{V}[\hat{\theta}_2] = \mathbb{V}[\overline{Y}] = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$$

$$\mathrm{eff}(\hat\theta_1,\hat\theta_2)=\frac{\mathbb{V}[\hat\theta_2]}{\mathbb{V}[\hat\theta_1]}=\frac{1}{n}$$

9.17 p. 456

Suppose that $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that $\overline{X}-\overline{Y}$ is a consistent estimator of $\mu_1-\mu_2$.

$$\lim_{n\to\infty}\mathbb{V}[\overline{X}]=\lim_{n\to\infty}\mathbb{V}[\overline{Y}]=\lim_{n\to\infty}\frac{\sigma_1^2}{n}=\lim_{n\to\infty}\frac{\sigma_2^2}{n}=0$$

Therefore, $\overline{X}$ and $\overline{Y}$ are consistent estimators of μ_1 and μ_2.

Because $\overline{X}$ and $\overline{Y}$ converge in probability to μ_1 and μ_2, $\overline{X}-\overline{Y}$ converges in probability to $\mu_1-\mu_2$ (theorem 9.2 p. 451).

Therefore, $\overline{X}-\overline{Y}$ is a consistent estimator of $\mu_1-\mu_2$. □

9.18 p. 456

In Exercise 9.17, suppose that the populations are normally distributed with $\sigma_1^2=\sigma_2^2=\sigma^2$. Show that

$$\frac{\sum_{i=1}^n (X_i-\overline{X})^2+\sum_{i=1}^n (Y_i-\overline{Y})^2}{2n-2}$$

is a consistent estimator of σ^2.

$$\frac{\sum_{i=1}^n (X_i-\overline{X})^2+\sum_{i=1}^n (Y_i-\overline{Y})^2}{2n-2}=\left(\frac{\sum_{i=1}^n (X_i-\overline{X})^2}{n-1}+\frac{\sum_{i=1}^n (Y_i-\overline{Y})^2}{n-1}\right)/2$$

Let $S_1^2=\Sigma(X_i-\overline{X})^2/(n-1)$ and $S_2^2=\Sigma(Y_i-\overline{Y})^2/(n-1)$.
S_1^2 and S_2^2 are sample variances and estimators of σ_1^2 and σ_2^2 and are both estimators of σ^2.

$$\begin{aligned}
&\mathbb{E}[S_1^2]\\
&=\mathbb{E}\left[\frac{\sum_{i=1}^n (X_i-\overline{X})^2}{n-1}\right]\\
&=\frac{1}{n-1}\mathbb{E}\left[\sum((X_i-\mu)-(\overline{X}-\mu))^2\right]\\
&=\frac{1}{n-1}\mathbb{E}\left[\sum((X_i-\mu)^2-2(\overline{X}-\mu)(X_i-\mu)+(\overline{X}-\mu)^2)\right]\\
&=\frac{1}{n-1}\mathbb{E}\left[\sum(X_i-\mu)^2-\sum 2(\overline{X}-\mu)(X_i-\mu)+\sum(\overline{X}-\mu)^2\right]\\
&=\frac{1}{n-1}\left(\mathbb{E}\left[\sum(X_i-\mu)^2\right]+\mathbb{E}\left[-2(\overline{X}-\mu)\sum(X_i-\mu)+n(\overline{X}-\mu)^2\right]\right)\\
&=\frac{1}{n-1}\left(\sum\mathbb{E}\left[(X_i-\mu)^2\right]+\mathbb{E}\left[-2(\overline{X}-\mu)(\sum X_i-\sum\mu)+n(\overline{X}-\mu)^2\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-1}\left(n\sigma^2 + \mathbb{E}\left[-2(\overline{X}-\mu)(n\overline{X}-n\mu) + n(\overline{X}-\mu)^2\right]\right)\\
&= \frac{1}{n-1}\left(n\sigma^2 + \mathbb{E}\left[-2n(\overline{X}-\mu)^2 + n(\overline{X}-\mu)^2\right]\right)\\
&= \frac{1}{n-1}\left(n\sigma^2 - n\mathbb{E}\left[(\overline{X}-\mu)^2\right]\right)\\
&= \frac{1}{n-1}(n\sigma^2 - n(\mathbb{E}[\overline{X}^2] - 2\mu^2 + \mu^2))\\
&= \frac{1}{n-1}(n\sigma^2 - n(\mathbb{V}[\overline{X}] + \mathbb{E}[\overline{X}]^2 - \mu^2))\\
&= \frac{1}{n-1}\left(n\sigma^2 - n\left(\frac{\sigma^2}{n} + \mu^2 - \mu^2\right)\right)\\
&= \sigma^2
\end{aligned}
$$

Similarly with $S_2^{\;2}$, both $S_1^{\;2}$ and $S_2^{\;2}$ are therefore unbiased estimators of σ^2.

Without loss of generality, consider $S_1^{\;2}$.

$$X_i \sim^{\text{iid}} \text{Normal}(\mu, \sigma^2) \Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Therefore,

$$\mathbb{V}\left[\frac{(n-1)S^2}{\sigma_1^2}\right] = 2(n-1) \Rightarrow \mathbb{V}[S_1^2] = \frac{2\sigma_1^4}{n-1}$$

$$\lim_{n\to\infty} \mathbb{V}[S_1^2] = 0$$

Thus, $S_1^{\;2}$ and $S_2^{\;2}$ are consistent estimators of σ^2.

Therefore, $(S_1^{\;2} + S_2^{\;2})/2$ is a consistent estimator of $(\sigma^2 + \sigma^2)/2 = \sigma^2$. □

9.33 p. 458

An experimenter wishes to compare the numbers of bacteria of types A and B in samples of water. A total of *n* independent water samples are taken, and counts are made for each sample. Let X_i denote the number of type A bacteria and Y_i denote the number of type B bacteria for sample i. Assume that the two bacteria types are sparsely distributed within a water sample so that $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ can be considered independent random samples from Poisson distributions with means $\lambda_!$ and λ_2, respectively. Suggest an estimator of $\lambda_1/(\lambda_1 + \lambda_2)$. What properties does your estimator have?

Take $\overline{X}$ and $\overline{Y}$ sample means of type A bacteria and type B bacteria, which are unbiased and consistent estimators of $\lambda_!$ and λ_2. Then

$$\overline{X}/(\overline{X}+\overline{Y})$$

is a consistent estimator of $\lambda_1/(\lambda_1 + \lambda_2)$.

9.36 p. 458

Suppose that Y has a binomial distribution based on n trials and success probability p. Then $\widehat{p}_n = Y/n$ is an unbiased estimator of p. Use Theorem 9.3 to prove that the distribution of $(\widehat{p}_n - p)/\sqrt{\widehat{p}_n\widehat{q}_n/n}$ converges to a standard normal distribution. [Hint: Write Y as we did in Section 7.5.]

Write Y as the sum of a sample consisting of 1s and 0s

$$Y = \sum_{i=1}^{n} X_i,$$

where $X_i = 1$ if the ith trial is a success and 0 otherwise.

Then

$$X_i \sim^{\text{idd}} \text{Bernouilli}(p)$$
$$\mathbb{E}[X_i] = p$$
$$\mathbb{V}[X_i] = pq$$

Define U_n such that

$$U_n = \frac{\sum X_i - np}{\sqrt{pqn}} = \frac{Y - np}{\sqrt{pqn}} = \frac{\hat{p}_n - p}{\sqrt{pq/n}}$$

Then, from the Central Limit Theorem, the distribution of U_n converges to the standard normal distribution.

We want W_n such that

$$\frac{U_n}{W_n} = \frac{\hat{p}_n - p}{\sqrt{\hat{p}_n\hat{q}_n/n}} \Rightarrow W_n = \sqrt{\frac{\hat{p}_n\hat{q}_n}{pq}}$$

Want to prove $\widehat{p}_n\widehat{q}_n$ is a consistent estimator of pq.

$$\mathbb{E}[\hat{p}_n] = \frac{1}{n}\sum \mathbb{E}[X_i] = p$$
$$\lim_{n\to\infty} \mathbb{V}[\hat{p}_n] = \lim_{n\to\infty} \frac{pq}{n} = 0$$

Therefore, $\widehat{p}_n$ is a consistent estimator of p, and $\widehat{q}_n = 1 - \widehat{p}_n$ is a consistent estimator of $1 - p = q$.

Thus, $\widehat{p}_n\widehat{q}_n$ is a consistent estimator of pq, and W_n converges in probability to 1.

Now, the distribution of U_n converges to the standard normal distribution and W_n converges in probability to 1. From theorem 9.3 p. 453,

$$\frac{U_n}{W_n} = \frac{\hat{p}_n - p}{\sqrt{\hat{p}_n\hat{q}_n/n}}$$

converges to a standard normal distribution. □

9.39 p. 463

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a Poisson distribution with parameter λ. Show by conditioning that

$$\sum_{i=1}^{n} Y_i$$

is sufficient for λ.

Let $U = \Sigma Y_i$. Then, from independence of Y_i

$$U \sim \text{Poisson}(n\lambda)$$

$$\begin{aligned} P(Y_1 = y_1, ..., Y_n = y_n | U = u) &= \frac{P(Y_1 = y_1, ..., Y_n = y_n)}{P(U = u)} \\ &= \frac{\prod \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}}{\frac{e^{-n\lambda}(n\lambda)^u}{u!}} \\ &= \frac{\frac{e^{-n\lambda}\lambda^{\sum y_i}}{\prod y_i!}}{\frac{e^{-n\lambda}(n\lambda)^u}{u!}} \\ &= \frac{\frac{e^{-n\lambda}\lambda^{u}}{\prod y_i!}}{\frac{e^{-n\lambda}(n\lambda)^u}{u!}} \\ &= \frac{u!}{n^u \prod y_i!} \\ &= \begin{cases} \frac{u!}{n^u \prod y_i!}, & \text{if } \sum y_i = u, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$P(Y_1 = y_1, ..., Y_n = y_n | U = u)$ does not depend on λ.
Therefore, U is sufficient for λ. □

9.41 p. 463

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a Weibull distribution with known m and unknown α. (Refer to Exercise 6.26.) Show that

$$\sum_{i=1}^{n} Y_i^m$$

is sufficient for α.

The Weibull density function is

$$f(y) = \begin{cases} \frac{1}{\alpha} m y^{m-1} e^{-y^m/\alpha}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $U = \Sigma Y_i^m$.
The likelihood is

$$L(\alpha) = \prod f(y_i|\alpha) = \left(\frac{m}{\alpha}\right)^n \prod y_i^{m-1} e^{-y_i^m/\alpha} = \alpha^{-n} e^{-u/\alpha} \times m^n \left(\prod y_i\right)^{m-1}$$

Let

$$g(u, \alpha) = \alpha^{-n} e^{-u/\alpha},$$
$$h(y_1, ..., y_n) = m^n \left(\prod y_i\right)^{m-1}$$

Both $g(u, \alpha)$ and $h(y_1, ..., y_n)$ are non negative.
Therefore, from the factorization criterion, U is sufficient for α. □

9.65 p. 471

In this exercise, we illustrate the direct use of the Rao–Blackwell theorem. Let $Y_1, Y_2, ..., Y_n$ be independent Bernoulli random variables with

$$p(y_i|p) = p^{y_i}(1-p)^{1-y_i}, \quad y_i = 0, 1.$$

That is, $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$. Find the MVUE of $p(1-p)$, which is a term in the variance of Y_i or $W = \Sigma Y_i$, by the following steps.

a. Let

$$T = \begin{cases} 1, & \text{if } Y_1 = 1 \text{ and } Y_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $E(T) = p(1-p)$.

$$\mathbb{E}[T] = P(T=1) = P(Y_1 = 1)P(Y_2 = 0) = p(1-p) \quad \square$$

b. Show that

$$P(T = 1|W = w) = \frac{w(n-w)}{n(n-1)}.$$

$$\begin{aligned} P(T = 1|W = w) &= \frac{P(T=1, W=w)}{P(W=w)} \\ &= \frac{P(Y_1 = 1, Y_2 = 0, \sum_{i=3}^n Y_i = w-1)}{\binom{n}{w} p^w (1-p)^{n-w}} \\ &= \frac{P(Y_1 = 1)P(Y_2 = 0)P\left(\sum_{i=3}^n Y_i = w-1\right)}{\binom{n}{w} p^w (1-p)^{n-w}} \\ &= \frac{p(1-p)\binom{n-2}{w-1} p^{w-1}(1-p)^{(n-2)-(w-1)}}{\binom{n}{w} p^w (1-p)^{n-w}} \\ &= \frac{\binom{n-2}{w-1}}{\binom{n}{w}} \\ &= \frac{\frac{(n-2)!}{(w-1)!(n-w-1)!}}{\frac{n!}{w!(n-w)!}} \\ &= \frac{w(n-w)}{n(n-1)} \quad \square \end{aligned}$$

c. Show that

$$E(T|W) = \frac{n}{n-1}\left[\frac{W}{n}\left(1-\frac{W}{n}\right)\right] = \frac{n}{n-1}\overline{Y}(1-\overline{Y})$$

and hence that $n\overline{Y}(1-\overline{Y})/(n-1)$ is the MVUE of $p(1-p)$.

$$\mathbb{E}[T|W] = P(T=1|W) = \frac{W(n-W)}{n(n-1)} = \frac{n}{n-1}\left[\frac{W(n-W)}{n^2}\right] = \frac{n}{n-1}\left[\frac{W}{n}\left(1-\frac{W}{n}\right)\right]$$

Want to show W is sufficient for $p(1-p)$.

$$\begin{aligned} P(Y_1 = y_1, ..., Y_n = y_n | W = w) &= \frac{P(Y_1 = y_1, ..., Y_n = y_n, W = w)}{P(W = w)} \\ &= \frac{p^w(1-p)^{n-w}}{\binom{n}{w}p^w(1-p)^{n-w}} \\ & \begin{cases} \frac{1}{\binom{n}{w}}, & \text{if } \sum y_i = w, \\ 0, & \text{otherwise.} \end{cases}. \end{aligned}$$

The condition distribution of $Y_1, Y_2, ..., Y_n$ given W does not depend on $p(1-p)$.
Therefore, W is sufficient for $p(1-p)$.

T is unbiased (from part a.).

Therefore, $n\overline{Y}(1-\overline{Y})/(n-1)$ is the MVUE of $p(1-p)$. □

9.86 p. 482

Suppose that $X_1, X_2, ..., X_m$, representing yields per acre for corn variety A, constitute a random sample from a normal distribution with mean μ_1 and variance σ^2. Also, $Y_1, Y_2, ..., Y_n$, representing yields for corn variety B, constitute a random sample from a normal distribution with mean μ_2 and variance σ^2. If the X's and Y's are independent, find the MLE for the common variance σ^2. Assume that μ_1 and μ_2 are unknown.

The likelihood for corn variety A is

$$\begin{aligned} L(\mu_1, \sigma^2) &= f(X_1 = x_1, ..., X_m = x_m | \mu_1 \sigma^2) \\ &= \prod f(X_i = x_i | \mu_1 \sigma^2) \\ &= (\sqrt{2\pi\sigma^2})^{-m} \prod e^{-(x_i-\mu_1)^2/(2\sigma^2)} \\ &= (2\pi\sigma^2)^{-m/2} e^{\sum[(x_i-\mu_1)^2]/(-2\sigma^2)} \end{aligned}$$

$$\ln L(\mu_1, \sigma^2) = -\frac{m}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i - \mu_1)^2$$

$$\frac{\partial \ln L(\mu_1, \sigma^2)}{\partial \mu_1} = \frac{1}{\sigma^2}\sum(x_i - \mu_1)$$

$$\frac{\partial \ln L(\mu_1, \sigma^2)}{\partial \mu_1} = 0 \Rightarrow \frac{1}{\sigma^2}\sum(x_i - \mu_1) = 0 \Rightarrow \mu_1 = \overline{X}$$

The MLE of μ_1 is therefore $\overline{X}$. Similarly, the MLE of μ_2 is $\overline{Y}$.

$$\begin{aligned} L(\sigma^2) &= f(X_1 = x_1, ..., X_m = x_m, Y_1 = y_1, ..., Y_n = y_n | \mu_1, \mu_2, \sigma^2) \\ &= \prod^m f(X_i = x_i) \prod^n f(Y_i = y_i) \\ &= (2\pi\sigma^2)^{-(n+m)/2} e^{-\sum^m (x_i - \mu_1)^2/(2\sigma^2)} e^{-\sum^n (y_i - \mu_2)^2/(2\sigma^2)} \\ &= (2\pi\sigma^2)^{-(n+m)/2} \exp\left\{ -\frac{1}{2\sigma^2}\left(\sum^m (x_i - \mu_1)^2 + \sum^n (y_i - \mu_2)^2\right)\right\} \end{aligned}$$

$$\ln L(\sigma^2) = -\frac{n+m}{2}\ln(2\pi) - \frac{n+m}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\left(\sum^m (x_i - \mu_1)^2 + \sum^n (y_i - \mu_2)^2\right)$$

$$\frac{\mathrm{d}\ln L(\sigma^2)}{\mathrm{d}\sigma^2} = \frac{-n-m}{2\sigma^2} + \frac{1}{2\sigma^4}\left(\sum^m (x_i - \mu_1)^2 + \sum^n (y_i - \mu_2)^2\right)$$

$$\begin{aligned} \frac{\mathrm{d}\ln L(\sigma^2)}{\mathrm{d}\sigma^2} &\Rightarrow \frac{-n-m}{2\sigma^2} + \frac{1}{\sigma^4}\left(\sum^m (x_i - \mu_1)^2 + \sum^n (y_i - \mu_2)^2\right) = 0 \\ &\Rightarrow -n - m + \frac{1}{2\sigma^2}\left(\sum^m (x_i - \mu_1)^2 + \sum^n (y_i - \mu_2)^2\right) = 0 \\ &\Rightarrow \sigma^2 = \frac{\sum^m (x_i - \mu_1)^2 + \sum^n (y_i - \mu_2)^2}{n+m} \end{aligned}$$

Plugging in the MLEs of μ_1 and μ_2, $\overline{X}$ and $\overline{Y}$, the MLE of σ^2 is therefore

$$\hat{\sigma}^2 = \frac{\sum^m (x_i - \overline{X})^2 + \sum^n (y_i - \overline{Y})^2}{n+m}$$

9.90 p. 482

A random sample of 100 men produced a total of 25 who favored a controversial local issue. An independent random sample of 100 women produced a total of 30 who favored the issue. Assume that p_M is the true underlying proportion of men who favor the issue and that p_W is the true underlying proportion of women who favor of the issue. If it actually is true that $p_W = p_M = p$, find the MLE of the common proportion p.

Let Y the number of people in the sample favoring the issue.
Then, the likelihood is

$$L(p) = p^Y (1-p)^{n-Y}$$

The log-likelihood is

$$\ln L(p) = Y \ln p + (n - Y)\ln(1-p)$$

$$\frac{\mathrm{d}\ln L(p)}{\mathrm{d}p} = \frac{Y}{p} - \frac{n-Y}{1-p}$$

$$\frac{\mathrm{d}\ln L(p)}{\mathrm{d}p} = 0 \Rightarrow \frac{Y}{p} - \frac{n-Y}{1-p} = 0 \Rightarrow p = \frac{Y}{n}$$

Therefore, the MLE of p is Y/n, and the sample MLE is $(25+30)/(100+100) = 11/20$.

9.112 p. 487

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a Poisson distribution with mean λ and define

$$W_n = \frac{\overline{Y} - \lambda}{\sqrt{\overline{Y}/n}}$$

a. Show that the distribution of W_n converges to a standard normal distribution.

From the Central Limit Theorem,

$$\frac{\overline{Y} - \mathbb{E}[Y_i]}{\sqrt{\mathbb{V}[Y_i]/n}} = \frac{\overline{Y} - \lambda}{\sqrt{\lambda/n}}$$

converges to a standard normal distribution.

Want U such that

$$\frac{\frac{\overline{Y}-\lambda}{\sqrt{\lambda/n}}}{U} = W_n \Rightarrow U = \sqrt{\overline{Y}/\lambda}$$

From the law of Large Numbers, U converges to 1 in probability.

Therefore, W_n converges to a standard normal distribution. □

b. Use W_n and the result in part (a) to derive the formula for an approximate 95% confidence interval for λ.

The standard error is

$$\sqrt{\overline{Y}} n$$

Therefore, the 95% confidence interval for λ is

$$\overline{Y} \pm \mathrm{z}_{0.025}\sqrt{\overline{Y}/n} = \overline{Y} \pm 1.96\sqrt{\overline{Y}/n}$$