Last lecture I introduced the idea that any function defined on $x \in 0, \ldots, N-1$ could be written a sum of sines and cosines. There are two different reasons why this is useful. The first is a general one, that sines and cosines behave nicely under convolution. If you convolve a sine function with some function $h(x)$, then you get a possibly shifted sine of the same frequency and some different amplitude. The second is more specific, that sines and cosines are a natural set of functions for describing sounds.

Today I will show you the most common way of writing a function in terms of sines and cosines, which is called the Fourier transform.

Sines, cosines, and linear algebra

Whenever we have an N dimensinal vector space, we can talk about a set of basis vectors for that space, e.g. any vector in the space can be written as a weighted sum of basis vectors. We say that the basis vectors form an *orthogonal* set if the inner product of any two different basis vectors is 0. We say that the basis vectors form an *orthonormal* set if the basis is orthogonal and if the length of each basis vector is 1.

Our discussion will focus on 1D signals, so our images will be denoted $I(x)$ which are defined on $x \in 0, \ldots, N-1$. So each image is a vector in an N-D space. The canonical basis for the space is the set of delta functions $\delta(x - x_0)$, where $x_0 \in 0, \ldots, N-1$. These basis vectors are orthonormal.

We are writing images in terms of sines and cosines that have k cycles per N pixels

$$\cos(\frac{2\pi}{N}kx), \sin(\frac{2\pi}{N}kx).$$

If we let $k \in 0, \ldots, N-1$, then we have $2N$ vectors. These vectors for a basis but obviously they cannot be an orthogonal basis since you can only have N orthogonal vecrors for an N dimensional space. So which of these $2N$ vectors are not orthogonal to each other?

Using triginometry identities for $\sin(A+B)$ and $\cos(A+B)$ (see slides), you can check that, for any k,

$$\cos(\frac{2\pi}{N}kx) = \cos(\frac{2\pi}{N}(N-k)x)$$

$$\sin(\frac{2\pi}{N}kx) = -\sin(\frac{2\pi}{N}(N-k)x).$$

Assuming N is even, we have $\frac{N}{2}+1$ cosine functions, namely for $k = 0, 1, \ldots, \frac{N}{2}+1$ and we have $\frac{N}{2}-1$ sine functions, namely for $k = 1, \ldots, \frac{N}{2}-1$, since when $k = 0, \frac{N}{2}$, the sine functions are identically 0. One can show that these $N = (\frac{N}{2}+1) + (\frac{N}{2}-1)$ functions define an orthogonal basis for the N-dimensional space of functions $I(x)$ on $x \in 0, \ldots, N-1$.

One can also show that for any $k \in 1, \ldots, \frac{N}{2}-1$,

$$\sum_{x=0}^{N-1} \cos(\frac{2\pi}{N}kx) \ \cos(\frac{2\pi}{N}kx) = \frac{N}{2}$$

$$\sum_{x=0}^{N-1} \sin(\frac{2\pi}{N}kx) \ \sin(\frac{2\pi}{N}kx) = \frac{N}{2}$$

and when $k = 0$,

$$\sum_{x=0}^{N-1} \cos(\frac{2\pi}{N}kx) \ \cos(\frac{2\pi}{N}kx) = N.$$

I mention the above because you should be familiar with these ideas already. In your basic Calculus courses, you had to comput integrals of sines and cosines and products of these functions. Although integrals are computed over $x \in \Re$ rather than discrete sums, you should be familiar with the ideas of sines and cosines being orthogonal in the sense that the sum/integral of their products can sometimes be 0.

Discrete Fourier Transform

With the review from last class and the review above, we are now ready to define the Fourier transform. The Fourier transform uses complex valued functions to define a basis for images. One of the challenges of understanding the Fourier transform is why we need complex numbers. This can only be explained once you have seen some examples of how the Fourier transform gets used, and this will take us the rest of the course. So you'll just have to bear with me for me. Here we go...

Consider 1D signals $I(x)$ which are defined on $x \in \{0, 1, ..., N-1\}$. Define the $N \times N$ *Fourier transform* matrix $\mathbf{F}$ whose k^{th} row and x^{th} column is:

$$\begin{aligned} \mathbf{F}_{k,x} &= \cos(\frac{2\pi}{N}kx) \ - \ i \ \sin(\frac{2\pi}{N}kx) \\ &\equiv e^{-i\frac{2\pi}{N}kx} \end{aligned}$$

This matrix is symmetric since $e^{-i\frac{2\pi}{N}kx} = e^{-i\frac{2\pi}{N}xk}$. Each row and column of the matrix $\mathbf{F}$ has a real part and an imaginary part. The real part is a sampled cosine function. The imaginary part is a sampled sine function. The leftmost and rightmost column of the matrix ($x = 0$ and $x = N-1$) are not identical. You would need to go to $x = N$ to reach the same value as at $x = 0$, but $x = N$ is not represented, as there are only N columns. Similarly, the first and last row ($k = 0$ and $k = N-1$) are not identical.

Right multiplying the matrix $\mathbf{F}$ by the $N \times 1$ vector $I(x)$ gives a vector $\hat{I}(k)$

$$\hat{I}(k) \ \equiv \ \mathbf{F} \ I(x) = \sum_{x=0}^{N-1} I(x) e^{-i\frac{2\pi}{N}kx} \tag{1}$$

which is called the *discrete Fourier transform* of $I(x)$. In general, $\hat{I}(k)$ is a complex number for each k. We can write it using Euler's equation:

$$\hat{I}(k) \ = A(k) e^{i\phi(k)}$$

$|\hat{I}(k)| = A(k)$ is called the *amplitude spectrum* and $\phi(k)$ is called the *phase spectrum*.

Conjugacy property of the Fourier transform

One of the main motivations of the Fourier transform is to have a set of basis vectors for functions $I(x)$. The Fourier transform seems to define $2N$ basis vectors, namely N sines and cosines. As I discussed at the beginning today, there are really only N such sine and cosine functions. What does this observation have to do with the Fourier transform?

The answer is that there is a redundancy in $\hat{I}(k)$ values. For example, since $\cos(\frac{2\pi}{N}kx) = \cos(\frac{2\pi}{N}(N-k)x)$, if we take the inner product of these two functions with $I(x)$ then will give the same value for frequency k as $N-k$. Similarly, $\sin(\frac{2\pi}{N}kx) = -\sin(\frac{2\pi}{N}(N-k)x)$ and so taking the inner product of $I(x)$ with these two functions will give the same value but with opposite sign. This property can be expressed as follows:

Conjugacy property: If $I(x)$ is a real valued function, then

$$\overline{\hat{I}(k)} = \hat{I}(N-k).$$

For the proof of the Conjugacy Property, see Appendix A. (I will not test you on this proof on the final exam. I include it here for completeness.)

Periodicity property of the Fourier transform

The Fourier transform definition assumed that the function $I(x)$ is defined on $x \in 0, \ldots, N-1$, and for frequencies k in $0, \ldots, N-1$. However, sometimes we will want to be more flexible with our range of x and k. For example, we may want to consider functions $h(x)$ that are defined on negative values of x such as a Gaussian which has mean 0, Gabor functions, etc. The point of the Fourier transform is to be able to write a function as a sum of sinuosoids. Since sine and cosine functions are defined over *all* integers, there is no reason why the Fourier transform needs to be defined only on functions that are defined on x in 0 to $N-1$.

We can define the Fourier transform of any function that is defined on a range of N consecutive values of x. For example, if we have a function defined on $-\frac{N}{2}, \cdots, -1, 0, 1, \frac{N}{2}-1$, then we can just write the Fourier transform as

$$\hat{I}(k) \ \equiv \ \mathbf{F} \ I(x) = \sum_{x=-\frac{N}{2}}^{\frac{N}{2}-1} h(x) e^{-i\frac{2\pi}{N}kx}$$

Essentially what we are doing here is treating this function $h(x)$ as periodic with period N, just like sine and cosine are, and defining the Fourier transform over a convenient sequence of N sample points.

The similar idea of periodicity holds for frequencies k. The definition of the Fourier transform above works for any integer k, not just the range that we have been working with so far, namely $0, \ldots, N-1$. It is strange to think of having a negative number of cycles per N pixels, but we can plug negative values into k nonetheless and the math still works fine.

Periodicity property: $\hat{I}(k)$ may be considered periodic in k with period N,

$$\hat{I}(k) \ = \ \hat{I}(k+mN) \ .$$

The idea is that, for any integer m,

$$e^{i2\pi m} = \cos(2\pi m) + i\sin(2\pi m) = 1$$

and so, since x is an integer, we have:

$$e^{i\frac{2\pi}{N}(k+mN)x} \;=\; e^{i\frac{2\pi}{N}kx}\, e^{i\frac{2\pi}{N}mNx} = \; e^{i\frac{2\pi}{N}kx}\, e^{i2\pi mx} \;=\; e^{i\frac{2\pi}{N}kx}$$

Thus, if we use frequency $k + mN$ instead of k in the definition of the Fourier transform, we get the same value, $\hat{I}(k) \;=\; \hat{I}(k + mN)$.

Inverse Fourier transform

Recall the discussion from the beginning today about how sine and cosines are orthogonal, and recall that we can represent the Fourier transform as a multipication by a matrix $\mathbf{F}$. This maps a function $I(x)$ – which is expressed as a $N \times 1$ column vector – to $\hat{I}(k)$ which is is also expressed as an $N \times 1$ column vector, i.e.

$$\hat{I} = \mathbf{F}I.$$

One can show (see Appendix B) that the inverse of this matrix $\mathbf{F}$ is

$$\mathbf{F}^{-1} = \frac{1}{N}\overline{\mathbf{F}}$$

where $\bar{\mathbf{F}}$ is the matrix of complex conjugates of $\mathbf{F}$.

$$\overline{\mathbf{F}}_{k,x} \;\equiv\; e^{i\frac{2\pi}{N}kx}.$$

That is,

$$\mathbf{I} = \frac{1}{N}\mathbf{F}\overline{\mathbf{F}}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix. We will not be using the inverse Fourier transform much, but should should be aware that it exists.

Convolution Theorem

One property of the Fourier transform that we will be using extensively in the rest of the course as as follows:

Convolution Theorem: for any two functions $I(x)$ and $h(x)$ that are defined on 0 to $N - 1$,

$$\mathbf{F}(I(x) * h(x))) = \mathbf{F}I(x)\ \mathbf{F}h(x) \;=\; \hat{I}(k)\ \hat{h}(k).$$

For the proof see Appendix C.

We will discuss the interpretation of this next lecture, but for now, notice how remarkably simple this property is! If you were to try to express something similar by using cosine and sine basis explicitly and without complex numbers, then you would not be able to. The expression would be much more complicated.

[The Appendices below are provided for completeness. I will not examine you on them. I do not even expect most of you to read through the carefully. But perhaps you could spend an hour or so, so that you can at least see what the flavour of the arguments are.]

Appendix A: Conjugacy property of the Fourier transform

Claim: If $I(x)$ is a real valued function, then

$$\overline{\hat{I}(k)} = \hat{I}(N-k).$$

Proof:

$$\begin{aligned}
\hat{I}(N-k) &= \sum_{x=0}^{N-1} I(x) e^{-i\frac{2\pi}{N}\,(N-k)x} \\
&= \sum_{x=0}^{N-1} I(x)\; e^{-i\;2\pi x}\; e^{i\;\frac{2\pi}{N}\;kx} \\
&= \sum_{x=0}^{N-1} I(x)\; e^{i\;\frac{2\pi}{N}\;kx}, \quad \text{since } e^{i2\pi x} = 1 \quad \text{for any integer } x \\
&= \sum_{x=0}^{N-1} I(x)\,\overline{e^{-i\frac{2\pi}{N}\;kx}} \\
&= \sum_{x=0}^{N-1} \overline{I(x)}\;\overline{e^{-i\frac{2\pi}{N}\;kx}} \quad \text{, if } I(x) \text{ is real} \\
&= \sum_{x=0}^{N-1} \overline{I(x)\; e^{-i\frac{2\pi}{N}\;kx}} \\
&= \overline{\hat{I}(k)}
\end{aligned}$$

Appendix B (inverse Fourier transform)

We want to show that $\mathbf{F}^{-1} = \frac{1}{N}\overline{\mathbf{F}}$, or equivalently that the matrix $\frac{1}{N}\mathbf{F}\bar{\mathbf{F}}$ is the $N \times N$ identity matrix. This means that we need to show that row k_0 and column k of the $N \times N$ matrix $\frac{1}{N}\mathbf{F}\ \overline{\mathbf{F}}$ has the value 1 when $k_0 = k$, i.e. diagonal, and it has the value 0 when $k_0 \neq k$, i.e. off-diagonal.

Claim: For any frequency k_0,

$$\mathbf{F}\ e^{\ i\frac{2\pi}{N}k_0 x} \ = \ N\delta(k - k_0).$$

That is,

$$\sum_{x=0}^{N-1} e^{\ i\ \frac{2\pi}{N}k_0 x} \ \ e^{-i\frac{2\pi}{N}kx} = \begin{cases} N, & k = k_0 \\ 0, & k \neq k_0 \end{cases}$$

Note that this claim essentially is essentially equivalent to saying that two cosine (or sine) functions of different frequencies are orthogonal; their inner product is 0.

Proof: Rewrite the left side of the above summation as

$$\sum_{x=0}^{N-1} e^{\ i\ \frac{2\pi}{N}(k_0-k)x} \quad . \tag{2}$$

If $k = k_0$, then the exponent is 0 and so we are just summing $e^0 = 1$ and the result is N.

That doesn't yet give us the result of the claim, because we still need to show that the summation is 0 when $k \neq k_0$. So, for the case $k \neq k_0$, observe that the summation is a finite geometric series and thus we can use the following identity which you know from Calculus:[1]: let γ be any number (real or complex) then

$$\sum_{x=0}^{N-1} \gamma^x = \frac{1-\gamma^N}{1-\gamma}.$$

Applying this identity for our case, namely $\gamma = e^{\ i\ \frac{2\pi}{N}(k-k_0)}$, lets us write (2) as

$$\sum_{x=0}^{N-1} e^{\ i\ \frac{2\pi}{N}(k-k_0)x} = \frac{1 - e^{\ i\ 2\pi(k-k_0)}}{1 - e^{\ i\ \frac{2\pi}{N}(k-k_0)}} \quad . \tag{3}$$

The numerator on the right hand side vanishes because $k - k_0$ is an integer and so

$$e^{\ i2\pi(k-k_0)} \ = 1 \ .$$

What about the denominator? Since k and k_0 are both in $0, \ldots, N-1$ and since we are considering the case that $k \neq k_0$, we know that $|k - k_0| < N$ and so $e^{\ i\ \frac{2\pi}{N}(k-k_0)} \neq 1$. Hence the denominator does not vanish. Since the numerator is 0 but the denominator is not 0, we can conclude that the right side of Eq. (3) is 0. Thus, the summation of (2) is 0, and so $\mathbf{F}\ e^{\ i\frac{2\pi}{N}k_0 x} = 0$ when $k \neq k_0$. This completes the derivation for the case $k \neq k_0$.

[1] If you are unsure where this comes from, see equations (1)-(6) of `http://mathworld.wolfram.com/GeometricSeries.html`.

Appendix C: Convolution Theorem

Claim: For any two functions $I(x)$ and $h(x)$ that are defined on N consecutive samples e.g. 0 to $N-1$,

$$\mathbf{F}(I(x) * h(x))) = \mathbf{F}I(x)\ \mathbf{F}h(x) \ = \ \hat{I}(k)\ \hat{h}(k).$$

Proof:

$$\begin{aligned}
\mathbf{F}\ \ I * h(x) \ &= \ \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}kx} \ \sum_{x'=0}^{N-1} I(x-x')h(x'), \ \text{ by definition} \\
&= \ \sum_{x'=0}^{N-1} h(x') \ \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}kx} \ \ I(x-x'), \ \text{ by switching order of sums} \\
&= \ \sum_{x'=0}^{N-1} h(x') \ \sum_{u=0}^{N-1} e^{-i\frac{2\pi}{N}k(u+x')} I(u) \ , \ \text{ where } u = x - x' \\
&= \ \sum_{x'=0}^{N-1} h(x') \ e^{-i\frac{2\pi}{N} \ kx'} \ \sum_{u=0}^{N-1} e^{-i\frac{2\pi}{N}ku} \ I(u) \\
&= \ \hat{h}(k)\ \hat{I}(k)
\end{aligned}$$

