Assignment 3
MATH 323 - Probability
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1. Suppose Y is a continuous random variable with the following cdf:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/2 & 0 \leq y \leq 2 \\ 1 & y > 2 \end{cases}$$

Let $X = Y^2$. Find

(a) $P(1 \leq Y \leq 2)$;

$$P(1 \leq Y \leq 2) = F_Y(2) - F_Y(1) = \frac{2}{2} - \frac{1}{2} = \frac{1}{2}$$

(b) $P(X \leq Y)$;

$$P(X \leq Y) = P(Y^2 \leq Y) = P(0 \leq Y \leq 1) = F_Y(1) - F_Y(0) = \frac{1}{2}$$

(c) $P(Y \leq 2X)$;

$$P(Y \leq 2X) = P(Y \leq 2Y^2) = P\left(\left(Y - \frac{1}{4}\right)^2 \geq \frac{1}{16}\right) = P(Y \leq 0) + P\left(Y \geq \frac{1}{2}\right) = 0 + \left(1 - P\left(Y \leq \frac{1}{2}\right)\right) = 1 - F_Y\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

(d) $P(X + Y \leq 3/4)$;

$$P(X + Y \leq 3/4) = P(Y^2 + Y - 3/4 \leq 0) = P((Y + 1/2)^2 - 1 \leq 0) = P((Y - 1/2)(Y + 3/2) \leq 0) = P(-3/2 \leq Y \leq 1/2) = F_Y(1/2) = \frac{1}{4}$$

(e) the *covariance* between Y and X, $\mathrm{Cov}[Y, X]$, defined by

$$\mathrm{Cov}[Y, X] = \mathbb{E}[YX] - \mathbb{E}[Y]\mathbb{E}[X]$$

Note that here, as $X = Y^2$, we have that

$$\mathbb{E}[YX] = \mathbb{E}[Y^3].$$

$$\mathrm{Cov}[Y, X] = \mathbb{E}[Y^3] - \mathbb{E}[Y]\mathbb{E}[Y^2]$$

$$\mathbb{E}[Y^3] = \int_{-\infty}^{\infty} y^3 \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y}\mathrm{d}y = \frac{1}{2}\int_0^2 y^3 \mathrm{d}y = \frac{1}{2}\frac{2^4}{4} = 2$$

$$\mathbb{E}[Y] = \frac{1}{2}\int_0^2 y \mathrm{d}y = \frac{1}{2}\frac{2^2}{2} = 1$$

$$\mathbb{E}[Y^2] = \frac{1}{2}\int_0^2 y^2 \mathrm{d}y = \frac{1}{2}\frac{2^3}{3} = \frac{4}{3}$$

Then,

$$\mathrm{Cov}[Y, X] = 2 - (1)\left(\frac{4}{3}\right) = \frac{2}{3}$$

2. Suppose that Y_1 and Y_2 are continuous random variables with joint pdf given by

$$f_{Y_1,Y_2}(y_1, y_2) = c(3y_1y_2 + y_1^2 + y_2^2) \qquad 0 < y_1 < 1, 0 < y_2 < 1$$

and zero otherwise, for some constant $c > 0$.

(a) Find the value of c.

$$\begin{aligned} F_{Y_1,Y_2}(y_1, y_2) &= \int_0^{y_1}\int_0^{y_2} c(3t_1t_2 + t_1^2 + t_2^2)\mathrm{d}t_2\mathrm{d}t_1 \\ &= c\int_0^{y_1} \frac{3}{2}t_1y_2^2 + t_1^2y_2 + \frac{1}{3}y_2^3\mathrm{d}t_1 \\ &= c\left(\frac{3}{4}y_1^2y_2^2 + \frac{1}{3}y_1^3y_2 + \frac{1}{3}y_1y_2^3\right) \end{aligned}$$

$$F_{Y_1,Y_2}(1,1) = 1 \Rightarrow c\left(\frac{3}{4} + \frac{1}{3} + \frac{1}{3}\right) = 1 \Rightarrow c = \frac{12}{17}$$

(b) Find the joint cdf, $F_{Y_1,Y_2}(y_1, y_2)$, for all values $(y_1, y_2) \in \mathbb{R}^2$.

$$F_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{12}{17}\left(\frac{3}{4}y_1^2y_2^2 + \frac{1}{3}y_1^3y_2 + \frac{1}{3}y_1y_2^3\right) & 0 \le y_1, y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the marginal pdf of Y_1, $f_{Y_1}(y_1)$ (taking care to note the support of this pdf).

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^1 f_{Y_1,Y_2}(y_1, y_2)\mathrm{d}y_2 \\ &= \frac{12}{17}\int_0^1 3y_1y_2 + y_1^2 + y_2^2\mathrm{d}y_2 \\ &= \frac{12}{17}y_1^2 + \frac{18}{17}y_1 + \frac{4}{17} \end{aligned}$$

(d) Are Y_1 and Y_2 independent ? Justify your answer

$$f_{Y_2}(y_2) = \frac{12}{17}\left(\frac{3}{2}y_2 + \frac{1}{3} + y_2^2\right) = \frac{12}{17}y_2^2 + \frac{18}{17}y_2 + \frac{4}{17}$$

$$f_{Y_1}(y_1)f_{Y_2}(y_2) = \left(\frac{12}{17}\right)^2\left(\frac{9}{4}y_1y_2 + \frac{3}{2}(y_1y_2^2 + y_1^2y_2) + \frac{1}{2}(y_1 + y_2) + \frac{1}{3}(y_1^2 + y_2^2) + y_1^2y_2^2 + \frac{1}{9}\right)$$

$$f_{Y_1}(y_1)f_{Y_2}(y_2) \neq f_{Y_1,Y_2}(y_1, y_2)$$

$$\Rightarrow Y_1 \text{ and } Y_2 \text{ not independent.}$$

3. Suppose that Y_1 and Y_2 are continuous random variables with joint pdf given by

$$f_{Y_1,Y_2}(y_1,y_2) = cy_1^2\text{exp}\{-4(y_1+y_2)\} \qquad y_1>0, y_2>0$$

and zero otherwise, for some constant c > 0.

(a) Find the value of c.

$$c\int_0^\infty\int_0^\infty y_1^2\text{exp}\{-4(y_1+y_2)\}\text{d}y_2\text{d}y_1 = 1$$
$$\Rightarrow c\int_0^\infty \frac{y_1^2}{4}e^{-4y_1}\text{d}y_1 = 1$$
$$\Rightarrow \frac{c}{4}\left(y_1^2\frac{e^{-4y_1}}{-4}\bigg|_0^\infty + \frac{1}{2}\int_0^\infty e^{-4y_1}y_1\text{d}y_1\right) = 1$$
$$\Rightarrow \frac{c}{4}\left(\frac{1}{2}\left(y_1\frac{e^{-4y_1}}{-4}\bigg|_0^\infty + \frac{1}{4}\int_0^\infty e^{-4y_1}\text{d}y_1\right)\right) = 1$$
$$\Rightarrow \frac{c}{4}\left(\frac{1}{2}\left(\frac{1}{16}\right)\right) = 1$$
$$\Rightarrow c = 128$$

(b) Are Y_1 and Y_2 independent ? Justify your answer.

$$f_{Y_1}(y_1) = 32y_1^2e^{-4y_1}$$
$$\begin{aligned} f_{Y_2}(y_2) &= 128\int_0^\infty y_1^2\text{exp}\{-4(y_1+y_2)\}\text{d}y_1 \\ &= 4e^{-4y_2} \end{aligned}$$

$$f_{Y_1}(y_1)f_{Y_2}(y_2) = 128y_1^2\text{exp}\{-4(y_1+y_2)\} = f_{Y_1,Y_2}(y_1,y_2)$$

$$\Rightarrow Y_1 \text{ and } Y_2 \text{ independent.}$$

(c) Let $Y = Y_1 + Y_2$. Compute the probability $P(Y \le 3)$.

$$\begin{aligned} P(Y\le 3) &= \int_0^3\int_0^{3-y_1} f_{Y_1,Y_2}(y_1,y_2)\text{d}y_2\text{d}y_1 \\ &= 128\int_0^3\int_0^{3-y_1} y_1^2\text{exp}\{-4(y_1+y_2)\}\text{d}y_2\text{d}y_1 \\ &= -32\int_0^3 y_1^2(e^{-12}-e^{-4y_1})\text{d}y_1 \\ &= -32e^{-12}\int_0^3 y_1^2\text{d}y_1 + 32\int_0^3 y_1^2e^{-4y_1}\text{d}y_1 \end{aligned}$$

$$
\begin{aligned}
&= -288e^{-12} + 32\left(y_1^2\frac{e^{-4y_1}}{-4}\Big|_0^3 + \frac{1}{2}\int_0^3 e^{-4y_1}y_1\mathrm{d}y_1\right)\\
&-288e^{-12} + 32\left(-\frac{9}{4}e^{-12} + \frac{1}{2}\left(y_1\frac{e^{-4y_1}}{-4}\Big|_0^3 + \frac{1}{4}\int_0^3 e^{-4y_1}\mathrm{d}y_1\right)\right)\\
&= -288e^{-12} + \left(-72e^{-12} + 16\left(-\frac{3}{4}e^{-12} + \frac{1}{4}\left(\frac{e^{-4y_1}}{-4}\right)\Big|_0^3\right)\right)\\
&= -288e^{-12} + \left(-72e^{-12} + \left(-12e^{-12} + 4\left(\frac{e^{-12}}{-4} + \frac{1}{4}\right)\right)\right)\\
&= -288e^{-12} + (-72e^{-12} + (-12e^{-12} - e^{-12} + 1))\\
&= -373e^{-12} + 1\\
&\approx 0.99771
\end{aligned}
$$

(d) Let U and V be independent continuous random variables having the same (marginal) distribution as Y_2. Identify the distribution of random variable W defined by

$$W = U + V.$$

$$f_{Y_2}(y_2) = 4e^{-4y_2} = \frac{1}{1/4}e^{y_1/(1/4)}$$

$$\Rightarrow Y_2 \sim \text{Exponential with } \beta = 1/4 \text{ (or Gamma with } \alpha = 1, \beta = 1/4)$$

$$m_U(t) = m_V(t) = m_{Y_2}(t) = \frac{1}{1 - \frac{1}{4}t}$$

$$m_W(t) = \mathbb{E}[e^{t(U+V)}] = \mathbb{E}[e^{tU}]\mathbb{E}[e^{tV}] = m_U(t)m_V(t) = \left(\frac{1}{1 - \frac{1}{4}t}\right)^2$$

$$\Rightarrow W \sim \text{ Gamma with } \alpha = 2, \beta = 1/4$$