MATH 323: PROBABILITY
SOME USEFUL GENERAL RESULTS

- **SERIES SUMMATIONS**

  GEOMETRIC: For $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots = \sum_{k=0}^{\infty} z^k$$

  EXPONENTIAL:

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

  We also have by the definition of the exponential function that for real $x > 0$

$$\begin{aligned} e^x &= \lim_{n \longrightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n \longrightarrow \infty} \left(1 - \frac{x}{n}\right)^{-n} \\ e^{-x} &= \lim_{n \longrightarrow \infty} \left(1 - \frac{x}{n}\right)^n &= \lim_{n \longrightarrow \infty} \left(1 + \frac{x}{n}\right)^{-n} \end{aligned}$$

  BINOMIAL: For $n > 0$,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \ldots + nz^{n-1} + z^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

- **TAYLOR SERIES:** For real function f and real number x_0

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} f^k(x_0)$$

where

$$f^k(x_0) = \frac{d^k}{dx^k} \left\{ f(x) \right\}_{x=x_0}$$

assuming that the derivatives exist.

- **RESULTS RELATING TO THE GEOMETRIC SERIES:** We have from above that for $|t| < 1$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots = \sum_{k=0}^{\infty} t^k. \tag{1}$$

Differentiating both sides of equation (1), we have

$$\begin{array}{rcll}
\dfrac{d}{dt} & : & \dfrac{1}{(1-t)^2} & = 1 + 2t + 3t^2 + 4t^3 + \cdots = \sum\limits_{k=1}^{\infty} k t^{k-1} \\
\dfrac{d^2}{dt^2} & : & \dfrac{2}{(1-t)^3} & = 2 + 6t + 12t^2 + \cdots = \sum\limits_{k=2}^{\infty} k(k-1) t^{k-2} \\
\dfrac{d^3}{dt^3} & : & \dfrac{6}{(1-t)^4} & = 6 + 24t + \cdots = \sum\limits_{k=3}^{\infty} k(k-1)(k-2) t^{k-3}
\end{array}$$

etc. Extending this logic, we may deduce the following **NEGATIVE BINOMIAL** expansion: for $n > 0$ and $|t| < 1$

$$\frac{1}{(1-t)^{n+1}} = 1 + (n+1)\,t + \frac{(n+1)\,(n+2)}{2!} t^2 + \cdots = \sum_{k=0}^{\infty} \binom{n+k}{k} t^k$$

Furthermore, if we *integrate* rather than differentiate the geometric series in equation (1), we obtain the following:

$$\int_0^t \frac{1}{1-x}\,dx = \int_0^t \left\{ \sum_{k=0}^{\infty} x^k \right\} dx = \sum_{k=0}^{\infty} \left\{ \int_0^t x^k\,dx \right\}$$

that is

$$-\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \frac{t^5}{5} + \cdots = \sum_{k=1}^{\infty} \frac{t^k}{k}$$

which yields the **LOGARITHMIC** series expansions; for $|t| < 1$,

$$-\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots = \sum_{k=1}^{\infty} \frac{t^k}{k}$$

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ldots = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{t^k}{k}$$