

Assignment 4

MATH 323 - Probability
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1. Suppose Y is a continuous random variable with the following cdf:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/2 & 0 \leq y \leq 2 \\ 1 & y > 2 \end{cases}$$

Let $X = Y^2$. Find

(a) $P(1 \leq Y \leq 2)$;

$$P(1 \leq Y \leq 2) = F_Y(2) - F_Y(1) = \frac{2}{2} - \frac{1}{2} = \frac{1}{2}$$

(b) $P(X \leq Y)$;

$$P(X \leq Y) = P(Y^2 \leq Y) = P(0 \leq Y \leq 1) = F_Y(1) - F_Y(0) = \frac{1}{2}$$

(c) $P(Y \leq 2X)$;

$$\begin{aligned} P(Y \leq 2X) &= P(Y \leq 2Y^2) = P\left(\left(Y - \frac{1}{4}\right)^2 \geq \frac{1}{16}\right) = P(Y \leq 0) + \\ &P\left(Y \geq \frac{1}{2}\right) = 0 + \left(1 - P\left(Y \leq \frac{1}{2}\right)\right) = 1 - F_Y\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

(d) $P(X + Y \leq 3/4)$;

$$\begin{aligned} P(X + Y \leq 3/4) &= P(Y^2 + Y - 3/4 \leq 0) = P((Y + 1/2)^2 - 1 \leq 0) = \\ &P((Y - 1/2)(Y + 3/2) \leq 0) = P(-3/2 \leq Y \leq 1/2) = F_Y(1/2) = \frac{1}{4} \end{aligned}$$

(e) the covariance between Y and X , $\text{Cov}[Y, X]$, defined by

$$\text{Cov}[Y, X] = \mathbb{E}[YX] - \mathbb{E}[Y]\mathbb{E}[X]$$

Note that here, as $X = Y^2$, we have that

$$\mathbb{E}[YX] = \mathbb{E}[Y^3].$$

$$\text{Cov}[Y, X] = \mathbb{E}[Y^3] - \mathbb{E}[Y]\mathbb{E}[Y^2]$$

$$\mathbb{E}[Y^3] = \int_{-\infty}^{\infty} y^3 \frac{dF_Y(y)}{dy} dy = \frac{1}{2} \int_0^2 y^3 dy = \frac{1}{2} \frac{2^4}{4} = 2$$

$$\mathbb{E}[Y] = \frac{1}{2} \int_0^2 y dy = \frac{1}{2} \frac{2^2}{2} = 1$$

$$\mathbb{E}[Y^2] = \frac{1}{2} \int_0^2 y^2 dy = \frac{1}{2} \frac{2^3}{3} = \frac{4}{3}$$

Then,

$$\text{Cov}[Y, X] = 2 - (1) \left(\frac{4}{3} \right) = \frac{2}{3}$$

2. Suppose that Y_1 and Y_2 are continuous random variables with joint pdf given by

$$f_{Y_1, Y_2}(y_1, y_2) = c(3y_1y_2 + y_1^2 + y_2^2) \quad 0 < y_1 < 1, 0 < y_2 < 1$$

and zero otherwise, for some constant $c > 0$.

(a) Find the value of c .

$$\begin{aligned} F_{Y_1, Y_2}(y_1, y_2) &= \int_0^{y_1} \int_0^{y_2} c(3t_1t_2 + t_1^2 + t_2^2) dt_2 dt_1 \\ &= c \int_0^{y_1} \left(\frac{3}{2} t_1 y_2^2 + t_1^2 y_2 + \frac{1}{3} y_2^3 \right) dt_1 \\ &= c \left(\frac{3}{4} y_1^2 y_2^2 + \frac{1}{3} y_1^3 y_2 + \frac{1}{3} y_1 y_2^3 \right) \end{aligned}$$

$$F_{Y_1, Y_2}(1, 1) = 1 \Rightarrow c \left(\frac{3}{4} + \frac{1}{3} + \frac{1}{3} \right) = 1 \Rightarrow c = \frac{12}{17}$$

(b) Find the joint cdf, $F_{Y_1, Y_2}(y_1, y_2)$, for all values $(y_1, y_2) \in \mathbb{R}^2$.

$$F_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{12}{17} \left(\frac{3}{4} y_1^2 y_2^2 + \frac{1}{3} y_1^3 y_2 + \frac{1}{3} y_1 y_2^3 \right) & 0 \leq y_1, y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the marginal pdf of Y_1 , $f_{Y_1}(y_1)$ (taking care to note the support of this pdf).

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^1 f_{Y_1, Y_2}(y_1, y_2) dy_2 \\ &= \frac{12}{17} \int_0^1 (3y_1y_2 + y_1^2 + y_2^2) dy_2 \\ &= \frac{12}{17} y_1^2 + \frac{18}{17} y_1 + \frac{4}{17} \end{aligned}$$

(d) Are Y_1 and Y_2 independent? Justify your answer

$$\begin{aligned} f_{Y_2}(y_2) &= \frac{12}{17} \left(\frac{3}{2} y_2 + \frac{1}{3} + y_2^2 \right) = \frac{12}{17} y_2^2 + \frac{18}{17} y_2 + \frac{4}{17} \\ f_{Y_1}(y_1) f_{Y_2}(y_2) &= \left(\frac{12}{17} \right)^2 \left(\frac{9}{4} y_1 y_2 + \frac{3}{2} (y_1 y_2^2 + y_1^2 y_2) + \frac{1}{2} (y_1 + y_2) + \frac{1}{3} (y_1^2 + y_2^2) + y_1^2 y_2^2 + \frac{1}{9} \right) \\ f_{Y_1}(y_1) f_{Y_2}(y_2) &\neq f_{Y_1, Y_2}(y_1, y_2) \end{aligned}$$

$\Rightarrow Y_1$ and Y_2 not independent.

3. Suppose that Y_1 and Y_2 are continuous random variables with joint pdf given by

$$f_{Y_1, Y_2}(y_1, y_2) = cy_1^2 \exp\{-4(y_1 + y_2)\} \quad y_1 > 0, y_2 > 0$$

and zero otherwise, for some constant $c > 0$.

(a) Find the value of c .

$$\begin{aligned} c \int_0^\infty \int_0^\infty y_1^2 \exp\{-4(y_1 + y_2)\} dy_2 dy_1 &= 1 \\ \Rightarrow c \int_0^\infty \frac{y_1^2}{4} e^{-4y_1} dy_1 &= 1 \\ \Rightarrow \frac{c}{4} \left(y_1^2 \frac{e^{-4y_1}}{-4} \Big|_0^\infty + \frac{1}{2} \int_0^\infty e^{-4y_1} y_1 dy_1 \right) &= 1 \\ \Rightarrow \frac{c}{4} \left(\frac{1}{2} \left(y_1 \frac{e^{-4y_1}}{-4} \Big|_0^\infty + \frac{1}{4} \int_0^\infty e^{-4y_1} dy_1 \right) \right) &= 1 \\ \Rightarrow \frac{c}{4} \left(\frac{1}{2} \left(\frac{1}{16} \right) \right) &= 1 \\ \Rightarrow c &= 128 \end{aligned}$$

(b) Are Y_1 and Y_2 independent? Justify your answer.

$$\begin{aligned} f_{Y_1}(y_1) &= 32y_1^2 e^{-4y_1} \\ f_{Y_2}(y_2) &= 128 \int_0^\infty y_1^2 \exp\{-4(y_1 + y_2)\} dy_1 \\ &= 4e^{-4y_2} \\ f_{Y_1}(y_1)f_{Y_2}(y_2) &= 128y_1^2 \exp\{-4(y_1 + y_2)\} = f_{Y_1, Y_2}(y_1, y_2) \\ \Rightarrow Y_1 \text{ and } Y_2 &\text{ independent.} \end{aligned}$$

(c) Let $Y = Y_1 + Y_2$. Compute the probability $P(Y \leq 3)$.

$$\begin{aligned} P(Y \leq 3) &= \int_0^3 \int_0^{3-y_1} f_{Y_1, Y_2}(y_1, y_2) dy_2 dy_1 \\ &= 128 \int_0^3 \int_0^{3-y_1} y_1^2 \exp\{-4(y_1 + y_2)\} dy_2 dy_1 \\ &= -32 \int_0^3 y_1^2 (e^{-12} - e^{-4y_1}) dy_1 \\ &= -32e^{-12} \int_0^3 y_1^2 dy_1 + 32 \int_0^3 y_1^2 e^{-4y_1} dy_1 \end{aligned}$$

$$\begin{aligned}
&= -288e^{-12} + 32 \left(y_1^2 \frac{e^{-4y_1}}{-4} \Big|_0^3 + \frac{1}{2} \int_0^3 e^{-4y_1} y_1 dy_1 \right) \\
&-288e^{-12} + 32 \left(-\frac{9}{4}e^{-12} + \frac{1}{2} \left(y_1 \frac{e^{-4y_1}}{-4} \Big|_0^3 + \frac{1}{4} \int_0^3 e^{-4y_1} dy_1 \right) \right) \\
&= -288e^{-12} + \left(-72e^{-12} + 16 \left(-\frac{3}{4}e^{-12} + \frac{1}{4} \left(\frac{e^{-4y_1}}{-4} \right) \Big|_0^3 \right) \right) \\
&= -288e^{-12} + \left(-72e^{-12} + \left(-12e^{-12} + 4 \left(\frac{e^{-12}}{-4} + \frac{1}{4} \right) \right) \right) \\
&= -288e^{-12} + (-72e^{-12} + (-12e^{-12} - e^{-12} + 1)) \\
&= -373e^{-12} + 1 \\
&\approx 0.99771
\end{aligned}$$

(d) Let U and V be independent continuous random variables having the same (marginal) distribution as Y_2 . Identify the distribution of random variable W defined by

$$W = U + V.$$

$$f_{Y_2}(y_2) = 4e^{-4y_2} = \frac{1}{1/4} e^{y_2/(1/4)}$$

$\Rightarrow Y_2 \sim$ Exponential with $\beta = 1/4$ (or Gamma with $\alpha = 1, \beta = 1/4$)

$$m_U(t) = m_V(t) = m_{Y_2}(t) = \frac{1}{1 - \frac{1}{4}t}$$

$$m_W(t) = \mathbb{E}[e^{t(U+V)}] = \mathbb{E}[e^{tU}] \mathbb{E}[e^{tV}] = m_U(t) m_V(t) = \left(\frac{1}{1 - \frac{1}{4}t} \right)^2$$

$\Rightarrow W \sim$ Gamma with $\alpha = 2, \beta = 1/4$