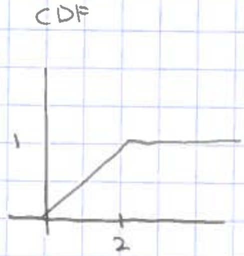
Math 323 ASSIGNMENT #4
Hyungjun Cho
260582309

#1. $F_Y(y) = \begin{cases} 0 & y<0 \\ y/2 & 0\le y\le 2 \\ 1 & y>2 \end{cases}$

Then we have pdf

$f_Y(y) = \int \frac{y}{2}\,dy = \frac{y^2}{4}$

$= \begin{cases} \frac{y^2}{4} & 0\le y\le 2 \\ 0 & \text{otherwise} \end{cases}$



a) $P(1\le Y\le 2)$

$= \int_1^2 \frac{y^2}{4}\,dy = \frac{y^3}{12}\Big|_1^2 = \frac{8}{12} - \frac{1}{12} = \frac{7}{12}$

b) $P(X\le Y)$ when $X=Y^2$

Then $P(Y^2\le Y) = P(Y^2-Y\le 0) = P(Y(Y-1)\le 0) = P(0\le Y\le 1)$
$= F_Y(1) - F_Y(0) = \frac{1}{2}$

c) $P(Y\le 2X)$

$= P(Y\le 2Y^2) = P((Y-\frac{1}{4})\ge\frac{1}{16}) = P(Y\le 0) + P(Y\ge\frac{1}{2})$
$= 0 + (1 - P(Y\le\frac{1}{2}))$
$= 1 - F_Y(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

d) $P(X+Y\le\frac{3}{4})$

$= P(Y^2+Y\le\frac{3}{4}) = P(Y^2+Y-\frac{3}{4}\le 0)$
$= P((Y-\frac{1}{2})(Y+\frac{3}{2})\le 0) = P(-\frac{3}{2}\le Y\le\frac{1}{2}) = F_Y(\frac{1}{2}) = \frac{1}{4}$

e) We want $\mathrm{cov}[Y, Y^2]$
$= E[Y^3] - E[Y]E[Y^2]$ where $E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y\cdot\frac{1}{2}\,dy$

$= \int_0^2 y^3\frac{1}{2}\,dy - \int_0^2 y\cdot\frac{1}{2}\,dy \cdot \int_0^2 y^2\frac{1}{2}\,dy$

$= \frac{y^4}{8}\Big|_0^2 - \frac{y^2}{4}\Big|_0^2 \cdot \frac{y^3}{6}\Big|_0^2$

$= \frac{2}{3}$

#2 $f_{Y_1,Y_2}(y_1,y_2) = c\,(3y_1y_2 + y_1^2 + y_2^2)$, $0 < y_1 < 1$, $0 < y_2 < 1$

for join pdf $\int_0^1\int_0^1 f_{Y_1,Y_2}(y_1,y_2)\,dy_2\,dy_1$ should sum to 1

Then $\int_0^1 \underbrace{\int_0^1 c\,(3y_1y_2 + y_1^2 + y_2^2)\,dy_2}\,dy_1 = 1$

Inner part: $\int_0^1 3y_1y_2 + y_1^2 + y_2^2\,dy_2 = y_1^2 + \frac{1}{3} + \frac{3}{2}y_1 - 0 = y_1^2 + \frac{3}{2}y_1 + \frac{1}{3}$

Then $c\cdot\int_0^1 \left(y_1^2 + \frac{3}{2}y_1 + \frac{1}{3}\right) dy_1 = c\cdot\left[\left(\frac{y_1^3}{3} + \frac{3}{4}y_1^2 + \frac{1}{3}y_1\right)\Big|_0^1\right]$

$= c\cdot\left(\frac{17}{12} - 0\right) = \frac{17}{12}\cdot c$

Finally $c\cdot\frac{17}{12} = 1 \rightarrow c = \frac{12}{17}$

b) The joint CDF $F_{Y_1,Y_2}(y_1,y_2)$ gives the probability $P(Y_1 \le y_1 \cap Y_2 \le y_2)$

Then we have $F_{Y_1,Y_2}(y_1,y_2) = \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} f_{Y_1,Y_2}(u,v)\,du\,dv$

$= \int_0^{y_2}\int_0^{y_1} \frac{12}{17}(3uv + u^2 + v^2)\,du\,dv$

Inner part: $\int_0^{y_1} \frac{12}{17}(3uv + u^2 + v^2)\,du = \frac{12}{17}\left(v^2y_1 + \frac{y_1^3}{3} + \frac{3vy_1^2}{2}\right) - 0$

Then: $\int_0^{y_2} \frac{12}{17}\left(v^2y_1 + \frac{y_1^3}{3} + \frac{3vy_1^2}{2}\right) dv$

$= \int_0^{y_2}\left(\frac{12y_1v^2 + 18y_1^2v + 4y_1^3}{17}\right) dv = \frac{1}{17}(4y_1v^3 + 9y_1^2v^2 + 4y_1^3v)\Big|_0^{y_2}$

$= \frac{1}{17}(4y_1y_2^3 + 9y_1^2y_2^2 + 4y_1^3y_2)$

* Finally for range
 - y_1 or $y_2 < 0 \rightarrow F_{Y_1Y_2}(y_1,y_2) = 0$
 - $0 < y_1 < 1,\ 0 < y_2 < 1 \rightarrow \frac{1}{17}(4y_1y_2^3 + 9y_1^2y_2^2 + 4y_1^3y)$
 - $0 < y_1 < 1,\ y_2 > 1 \rightarrow \frac{1}{17}(4y_1 + 9y_1^2 + 4y_1^3)$
 - $y_1 > 1,\ 0 < y_2 < 1 \rightarrow \frac{1}{17}(4y_2^3 + 9y_2^2 + 4y_2)$
 - $y_1 > 1,\ y_2 > 1 \rightarrow \frac{1}{17}(4 + 9 + 4) = 1$

c) $f_{Y_1}(y_1) = \int_0^1 \frac{12}{17}(3y_1y_2 + y_1^2 + y_2^2)\,dy_2 = \frac{12}{17}\left(y_1^2y_2 + \frac{y_2^3}{3} + \frac{3y_1y_2^2}{2}\right)\Big|_0^1$

$= \frac{12}{17}y_1^2 + \frac{18}{17}y_1 + \frac{4}{17}$

d) $f_{Y_1,Y_2}(y_1,y_2) = \frac{12}{17}(3y_1y_2 + y_1^2 + y_2^2)$, $f_{Y_1}(y_1) = \frac{12}{17}y_1^2 + \frac{18}{17}y_1 + \frac{4}{17}$, $f_{Y_2}(y_2) = \frac{12}{17}y_2^2 + \frac{18}{17}y_2 + \frac{4}{17}$

$f_{Y_1Y_2}(y_1,y_2) \ne f_{Y_1}\cdot f_{Y_2}$

$\therefore$ Not independent.

In order for ~~it~~ to be independent, $f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}\cdot f_{Y_2}$

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a) $f_{Y_1,Y_2}(y_1,y_2) = c y_1^2 \exp(-4(y_1+y_2))$, $y_1 \ge 0$, $y_2 > 0$

for joint pdf, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{y_1 y_2}(y_1,y_2)\,dy_1\,dy_2 = \int_0^\infty\int_0^\infty c y_1^2 \exp(-4(y_1+y_2))\,dy_1\,dy_2$

Inner part : $\int_0^\infty x_1^2 e^{-4(y_1+y_2)}\,dy_1 = \left[-\frac{1}{32}e^{-4(y_1+y_2)}(8y_1^2+4y_1+1)\right]\Big|_0^\infty$

$= 0 - (-\frac{1}{32}e^{-4y_2})$

Then : $\int_0^\infty \frac{1}{32}e^{-4y_1}\,dy_2 = \left[-\frac{1}{128}e^{-4y_1}\right]_0^\infty = 0-(-\frac{1}{128}) = \frac{1}{128}$

Thus $c\cdot\frac{1}{128} = 1 \to c = 128$

b) $f_{Y_1,Y_2}(y_1,y_2) = 128\, y_1^2 e^{-4(y_1+y_2)}$

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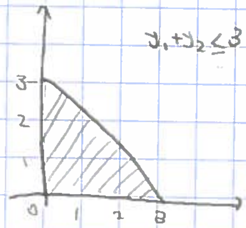
$f_{Y_1}(y_1) \times f_{Y_2}(y_2) = 32e^{-4y_1}y_1^2 \times 4e^{-4y_2} = 128\, y_1^2 e^{-4(y_1+y_2)}$

they are same. ∴ Y_1 and Y_2 are independent.

c) $P(Y\le 3) \to F_Y(3) \to P(Y_1+Y_2 \le 3)$



Then: $P(Y_1+Y_2 \le y) = \iint_{A_y} f_{y_1 y_2}(y_1,y_2)\,dy_1\,dy_2$

so, $F_Y(3) = \int_0^3\int_0^{3-y_1} f_{Y_1Y_2}(y_1,y_2)\,dy_2\,dy_1$

$= \int_0^3\int_0^{3-y_1} 128\, y_1^2 e^{-4(y_1+y_2)}\,dy_2\,dy_1$

$= \int_0^3 (-32y_1^2e^{-12} + 32y_1^2e^{-4y_1})\,dy_1$

$= \left[-32\frac{y_1^3}{3}e^{-12} - e^{-4y_1}(8y_1^2+4y_1+1)\right]_0^3$

$= -373e^{-12}+1 = 0.99771\cdots$

d) First get marginal $f_{Y_1}(y_1)$, $f_{Y_2}(y_2)$

$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_0^\infty 128\, y_1^2 e^{-4(y_1+y_2)}\,dy_2 = -32y_1^2 e^{-4(y_1+y_2)}\Big|_0^\infty$

$= 0 - (-32y_1^2e^{-4(y_1+y_2)}) = 32y_1^2e^{-4(y_1+y_2)}$

$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1,y_2)\,dy_1 = \int_0^\infty 128\, y_1^2 e^{-4(y_1+y_2)} = 4e^{-4y_2}$

U and V are independent random variables having same distribution as $f_{Y_2}(y_2)$ → exponential

The moment generating function : $M_Y(t) = \mathbb{E}[e^{tY}]$

Then $M_W(t) = M_{U+V}(t) = \mathbb{E}[e^{t(U+V)}]$

$= \mathbb{E}[e^{tU} \times e^{tV}]$

Then, using independence, $= \mathbb{E}[e^{tU}]\cdot\mathbb{E}[e^{tV}]$

$= M_U(t)\cdot M_V(t)$

$4e^{-4y_2} \to Y_2 \sim$ exponential(4)

Then both V and U have MGF of $\frac{\lambda}{\lambda - t}$

We see then $M_W(t) = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] = \prod_{i=1}^{n} M_{X_i}(t)$

Thus $U, V \sim Exp(\lambda)$ → W is gamma distribution with shape 2 and rate λ