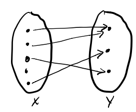
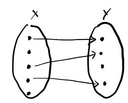
FUNCTIONS

A function f from X to Y written $f: X \to Y$ is a binary relation $f \subseteq X \times Y$ such that for every $x \in X$ there is at most one $y \in Y$ such that $(x,y) \in f$. Since y is unique for x we can write $y = f(x)$

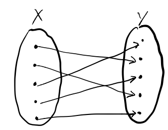
Surjection: $\forall y \in Y \ \exists x \in X$ s.t. $y = f(x)$
[f is "onto"]



Injection: $\forall y \in Y$ there is at most one $x \in X$ such that $y = f(x)$
[f is "one to one"]



Bijection: $\forall y \in Y$ there is exactly one $x \in X$ such that $y = f(x)$
[Both surjective and injective]



Relation to counting: Recall $|X|$ denotes the cardinality of X (the # of elements)

1. $\exists$ a surjection $f: X \to Y \Leftrightarrow |X| \geq |Y|$
2. $\exists$ an injection $f: X \to Y$ using every element of $X \Leftrightarrow |X| \leq |Y|$
3. $\exists$ a bijection $f: X \to Y \Leftrightarrow |X| = |Y|$

Principles of Counting

1. $|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i|$ if each pair A_i, A_j is disjoint $(A_i \cap A_j = 0)$
2. $|A_1 \times A_2 \times \ldots \times A_n| = \prod_{i=1}^{n} |A_i|$

Translated: Let A_i be ways in which a task can be done.

1. The number of ways in which a collection of tasks can be done if no two sets of ways of of completing tasks overlap is the sum of the # of ways each task can be done.
2. The # of ways a sequence of tasks can be done is the product of the number of ways each task can be done.

$\boxed{\text{examples}}$ ① How many 4 digit numbers have no repeated digits?

$\boxed{\text{Solution}}$ The tasks are to choose each digit → _ _ _ _

$9 \times 9 \times 8 \times 7 = 4536$

no "0" allowed because then it is no longer 4 digits

② How many 4 digit even numbers have no repeated digits?

$\boxed{\text{Solution}}$ We split into 2 cases: Working R to L

Ⓐ Last digit 0 → _ _ _ _

$$\underset{\uparrow}{9} \times \underset{\uparrow}{8} \times \underset{\uparrow}{7} \times \underset{\uparrow}{1} = 504$$

Ⓑ Last digit NOT 0 → $\underset{\uparrow}{8} \times \underset{\uparrow}{8} \times \underset{\uparrow}{7} \times \underset{\uparrow}{4} = 1792$

total # of ways is $504 + 1792 = 2296$

Counting Objects / structures

① How many functions are there from an n-set to a K-set? ["m-set" = set with m elements]

Sequence of events: · Let $|X| = n$ $|Y| = K$

· For each $x \in X$, choose $y \in Y$ such that $y = f(x)$

· # of functions from n-set to K-set? → $\underbrace{K \times K \times K \ldots K}_{n \text{ times}} = K^n$

② How many subsets of an n-set are there?

Sequence of events: · take each element and decide to put it in subset or not.

· n tasks having 2 possibilities

↳ # of subsets $= \underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n \text{ times}} = 2^n$ ⇝ Size of power set!!

③ How many bijections are there from an n-set to itself?

Sequence of events: · 1st element has n possible values

· 2nd element has n-1 possible values (b/c its a bijection)

↳ # of bijections $= (n)(n-1)(n-2) \ldots (2)(1) = n!$

A permutation of an n-set is an ordering of its elements. This is precisely the same as a bijection, so there are n! permutations of an n-set.

ex) $X = \{1, 2, 3, 4\}$

BIJ: $\{(1,4), (2,1), (3,2), (4,3)\}$

PERM: 4 1 2 3

④ A K-permutation of an n-set X is a choice of K elements of X in some order.

· # of K-permutations $= (n)(n-1)(n-2) \ldots (n-K+1)$ ⇝ leads to K terms

· This is usually denoted $P(n,K)$ → looks like n! with some terms dropped

# of elements ↲ ↳ elements to choose

· $P(n,K) = \dfrac{n!}{(n-K)!}$

ex) How many ways can you choose a president, vice-president, treasurer, secretary from a group of 7 people?

$P(7,4) = \dfrac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

⑤ How many subsets of size K does an n-set have?

· Count the # of K-permutations (from ④) again.

$= (\text{\# of K-subsets})(K!) = \dfrac{n!}{(n-K)!}$ ⇝ choose set first, then order

⇒ # of K subsets $= \dfrac{n!}{K!\,(n-K)!}$ Notation: $C(n,K)$ or $\binom{n}{K}$ ⇝ # of ways to take an n-set and choose K things from it.

PROPOSITION: $\binom{n}{K} = \binom{n}{n-K}$

PROOF: ① $\binom{n}{n-K} = \dfrac{n!}{(n-K)!\,(n-(n-K))!} = \dfrac{n!}{(n-K)!\,K!} = \binom{n}{K}$

② Choosing K elements to be <u>in</u> your set is equivalent to choosing the n-K elements in its complement.