Binomial Theorem

For a positive integer n, what is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$?
How many ways can we choose k x's from the n factors and $n-k$ y's, choosing one variable from each factor?

We only need to count the # of ways to choose x's (get y's for free) $\Rightarrow$ coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$

Binomial Theorem: If $n \geq 1$ is an integer,

$$(x+y)^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\text{binomial coefficients}} x^k y^{n-k}$$

Exercise Use the binomial theorem to prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for any integer $n \geq 1$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Let $x = y = 1 \Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}(1)(1)$

- Combinatorially: $\underbrace{2^n}_{\#\text{ of subsets of an } n\text{-set}} = \underbrace{\binom{n}{0}}_{\#\text{ of subsets of cardinality } 0} + \underbrace{\binom{n}{1}}_{\#\text{ of subsets of cardinality } 1} + \underbrace{\binom{n}{2}}_{\text{cardinality } 2\ \ldots} + \cdots + \binom{n}{n-1} + \binom{n}{n}$

Exercise Prove that an n-set, $n \geq 1$ has the same number of even subsets as odd subsets.

- Combinatorially: Fix some element x.

$$\begin{array}{lcl} & \#\text{ odd subsets with } x & = & \#\text{ even without } x \\ + & \#\text{ odd subsets without } x & = & \#\text{ even with } x \\ \hline & \#\text{ odd} & = & \#\text{ even} \end{array}$$

- Binomial thm: $x = -1$ $y = 1$ $\Rightarrow (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cancel{(1)^{n-k}}$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \binom{n}{0} - \binom{n}{2} - \binom{n}{4} - \cdots$$

Exercise Prove $\underbrace{\binom{n}{k}}_{\#\text{ of subsets}} = \underbrace{\binom{n-1}{k}}_{x \text{ not in subset}} + \underbrace{\binom{n-1}{k-1}}_{x \text{ in the subset}}$

Exercise Prove $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{2}$$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \boxed{\binom{n}{k}\binom{n}{n-k}} + \cdots$$

$$\{1, \ldots, 2n\} = \{1, \ldots, n\} \cup \{n+1, \ldots, 2n\}$$

$\binom{2n}{n} = \#$ of n-subsets $\qquad \binom{n}{k}\binom{n}{n-k} = \#$ of ways to choose k elements from $\{1, \ldots, n\}$ and rest from $\{n+1, \ldots, 2n\}$

REPITITIONS

When order matters.

- How many anagrams of "EASY" are there? $4!$
- What about "CHEESE"? NOT $6!$ → C, H, E_1, E_2, S, E_3 → $6!$ ways to arrange
  $6! = (\#\text{ anagrams})(3!)$ ↳ # of ways to permute "E" ANS: $6!/3!$
- What about "MISSISSIPPI"? → $\dfrac{11!}{\underset{M}{(1!)}\underset{I}{(4!)}\underset{S}{(4!)}\underset{P}{(2!)}}$
- How many ways to arrange

When order doesn't matter

"Balls & Boxes" - How many ways can you distribute K identical balls between n boxes?

ex) $K=8$, $n=4$ corresponds to a binary string of length $n+K-1$ with $n-1$ 1s and K 0s.

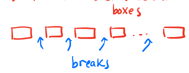
ANS: $\binom{n+K-1}{n-1} = \binom{n+K-1}{K}$ ↳ "How many binary strings of 11 can you construct with 3 1s?" → $\binom{11}{3}$

Exercise Jim Morton's offers 30 kinds of Jimbits. How many ways can you choose a dozen?

Boxes: 30 varieties (varieties)
Balls: 12 choices (choices)

ANS: $\binom{30+12-1}{29}$ or $\binom{30+12-1}{12} = 7{,}898{,}654{,}920$

"30-1" → # of breaks between boxes



Exercise A sandwich shop offers: Must make 20 sandwiches
- 3 kinds of bread
- 4 kinds of meat
- 10 kinds of toppings

(a) How many possibilities are there if each sandwich has 1 bread, 1 meat and 3 toppings?

$$\#\text{ possible sandwiches: } \binom{3}{1}\cdot\binom{4}{1}\cdot\binom{3}{10} = 1440$$

How many platters? ⇝ 1440 boxes, 20 balls

$$\binom{1440+20-1}{20} = 6.89\times 10^{44}$$

(b) What if you may now use at most 2 meats and any # of toppings?

$$\#\text{ possible sandwiches: } \binom{3}{1}\times\left[\binom{4}{0}+\binom{4}{1}+\binom{4}{2}\right]\times 2^{10} = 33792$$

# of platters: $\binom{33792+20-1}{20} = 1.55 \times 10^{72}$

$\boxed{\text{Exercise}}$ How many non-negative integer solutions are there to $x_1+x_2+x_3=48$?

48 balls
3 boxes $\binom{48+3-1}{2} = 1225$

What if $x_1 \geq 2$ and $x_3 \geq 5$?

2 balls already in x_1

5 balls already in x_3

Let $y_1 = x_1-2$ $\quad y_2 = x_2$ $\quad y_3 = x_3 - 5$

$(y_1+2) + y_2 + (y_3+5) = 48$

$y_1+y_2+y_3 = 41$

How many solutions to $x_1 x_2 x_3 = 48$?

$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \rightarrow$ Distribute four 2's to three boxes.
Distribute one 3 to three boxes.

$\binom{4+3-1}{2} \cdot 3 = 45$