Assignment 1 - Time Value of Money, IRR, Effective Annual Cost, Bond Returns, Bond Pricing and Bond Arbitrage, Forward Rates and Arbitrages

MGCR 341 - Introduction to Financial Accounting

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Time Value of Money

1)

For each of the below, select a, b, or c. Justify your answer briefly. Rates are annual with annual compounding.

Solution:

A. Interest rate is positive and the inflation rate is positive. Do you prefer receiving

a. $1 at t = 0

b. 1 real dollar at t = 10

c. Not enough information to determine whether a or b is preferred

Interest rate and inflation rate unknown ⇒ real interest rate unknown ⇒ present value of b is unknown

B. Interest rate is 5% and the inflation rate is 5%. Do you prefer receiving:

a. 10 real dollars at t = 10

b. 11 real dollars at t = 12

c. Not enough information to determine whether a or b is preferred

$$1 + r = (1 + r_r)(1 + i) \Rightarrow r_r \approx r - i$$
$$r \text{ nominal interest rate}$$
$$r_r \text{ real interest rate}$$
$$i \text{ inflation rate}$$

From above, the real interest rate is 0.

Therefore, b discounted to t = 10 is

$$\frac{11}{(1 + r_r)^2} = \frac{11}{(1 + 0)^2} = 11 > 10$$

C. Inflation rate is positive and the real risk free rate is positive. Do you prefer receiving:

a. $10 at t = 10

b. $11 at t = 11

c. Not enough information to determine whether a or b is preferred

Don't know real risk free rate ⇒ can't find value of b @ t = 10 from above formula.

D. Real interest rate is negative and inflation is positive. Do you prefer:

a. 1 real dollar at t = 10

b. $1 at t = 0

c. Not enough information to determine whether a or b is preferred

PV of a is

$$\frac{1}{(1+r_r)^{10}} > 1$$

because real interest rate < 0

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2)

The risk free rate is 8%. You will deposit $600 in the bank each year from t = 13 to t = 28. After that you will let the money sit in the bank until t = 38. At t = 38 you will withdraw $2,000. You will then let the money sit in the bank until t = 48. What is your account balance at t = 48?

Solve this in 3 ways.

First, by using the PV of an annuity formula to discount the 16 deposits back to t = 12 and then future valuing that amount 26 periods forward. Then subtract off $2,000 from this value, and then future value this result forward another 10 years.

Second, by using the PV of an annuity formula to discount the 16 deposits back to t = 0 and then future valuing that amount 38 periods forward. Then subtract off $2,000 from this value, and then future value this result forward another 10 years.

Third, by using the conservation of money approach: PV(deposits) = PV(withdrawals) + PV(final balance)

Solution:

PV of annuity formula:

$$\sum_{t=1}^{n} \frac{C}{(1+r)^t} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Method 1:

$$\left[\frac{600}{0.08}\left(1 - \frac{1}{1.08^{16}}\right)(1.08^{26}) - 2000\right](1.08^{10}) \approx 80486.26$$

Method 2:

$$\left[\frac{600}{0.08}\left(1 - \frac{1}{1.08^{16}}\right)\frac{1}{1.08^{12}}(1.08^{38}) - 2000\right](1.08^{10}) \approx 80486.26$$

Method 3:

$$PV(deps) = PV(withs) + PV(final\ balance)$$

$$\Rightarrow \frac{600}{0.08}\left(1 - \frac{1}{1.08^{16}}\right)\frac{1}{1.08^{12}} = \frac{2000}{1.08^{38}} + \frac{C_{48}}{1.08^{48}}$$

$$\Rightarrow C_{48} \approx 80486.26$$

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3)

The inflation rate from t = 0 to t = 1 is 3%. The inflation rate from t = 1 to t = 2 is X. The risk free rate is constant at 5%. At t = 0, 1 hamburger costs $1. If you had $100 at t = 0, and invested this money in the bank for 2 years, you could take your money out of the bank at t = 2 and with that money you could buy 103 hamburgers. What does this imply about X?

Solution:

$$1(1.03)(1+X)(103) = 100(1.05^2) \Rightarrow X = 3.92\%$$

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4)

a) At t = 0 you deposit $100 in the bank, where the APR is Z. Two years later at t = 2, you have $121. What is the EAR?

b) The APR is X with semi-annual compounding. The EAR is 12%. What is X?

c) The monthly interest rate is 1%. What is the EAR?

d) A bank tells you that for the first 6 months, your money will get multiplied once by a factor of 1.02, and that for the second 6 months your money will get multiplied once by a factor of 1.03. Another bank tells you that their APR is Z with monthly compounding. If you have to leave your money in either bank for a full year, for what value of Z are you indifferent between the two banks?

Solution:

EAR formula:

$$1 + EAR = \left(1 + \frac{APR}{k}\right)^k$$

a)

$$100(1+EAR)^2 = 121 \Rightarrow EAR = 10\%$$

b)

$$1.12 = \left(1 + \frac{X}{2}\right)^2 \Rightarrow X = 11.66\%$$

c)

$$1 + EAR = 1.01^{12} \Rightarrow EAR = 12.12\%$$

d)

$$1.02(1.03) = (1 + Z/12)^{12} \Rightarrow Z = 4.95\%$$

5)

For each of the below, indicate True or False. No explanation is required.

a) For a project with risk free cash flows, the appropriate discount rate is the project's IRR.

b) For a risky project, the IRR rule states that you should accept the project if the cost of capital is greater than the required rate of return.

c) For a risky project, the IRR is always greater than the risk free rate, even if the NPV is negative.

d) If project A has a higher IRR than B, then it will also have a higher NPV.

e) Given: i) project A is riskier than project B and thus has a higher required rate of return. ii) project A has a higher IRR than B. Statement: It is possible that the IRR rule accepts project B but not A.

Solution:

Net Present Value (NPV) formula:

$$NPV = -PV(cost) + PV(revenue) = C_0 + \sum_t \frac{C_t}{(1+r)^t}$$

Internal Rate of Return (IRR): hypothetical discount rate such that NPV = 0

IRR > discount rate ⇒ accept project

IRR < discount rate ⇒ reject project

IRR = discount rate ⇒ indifferent

a) False. The appropriate discount rate would be the risk free rate

b) False

c) False

d) False

e) True

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6)

Today is t= 0. The interest rate is 8%. The inflation rate is 1%. You will make your first deposit of $14,000 at t = 10. Each year thereafter you will deposit 5% more money into the account, until t = 16. Each year after that you will deposit 7% more money into the account, until your last deposit at t = 20. At t = 26 you will make withdrawals that will allow you to consume the same amount of goods that $4,000 can buy today. For each year thereafter up to t = 45 you will withdraw enough money to allow you to consume 3% more goods than the previous year. You will then let the money sit in the bank until t = 50.

a) How much money will be left in the bank account at t = 50? Use the nominal approach.

b) How much money will be left in the bank account at t = 50? Use the real approach.

c) Download "Assignment 1 Winter2018Table.xlsx" from the assignments folder online. Fill in the table as explained on the spreadsheet. Make sure that you don't type 0.08in any of your formulas. Instead, when you need to use the risk free rate, reference the cell G1. This is necessary for part d. Also, you will find it easier to compute relevant nominal growth rates in the indicated cells, and reference those cells in your table. Compute the nominal growth rates using the appropriate formulas directly within those cells so that you avoid any rounding errors.

PLEASE MAKE SURE TO PRINT YOUR TABLE SO THAT IT FITS ON 1 PAGE. You only need to hand in this one page for part cof this question. (The goal of this question is twofold: first, to allow you to visualize what is happening in part a; and second to familiarize yourself with Excel. Excel is widely used in finance and in business in general. You would be wise to become very good at Excel. Note, however, that you will not be tested on Excel in your midterm or final exam.)

d) Now we want to make use of the Excel Solver function to answer the following question, which cannot be solved analytically. Suppose therisk free rate were not 8%. For what value of the risk free rate would the above deposits and withdrawals lead to a final balance of exactly $4M at t = 50?

Solution:

a) Nominal approach:

PV of growing annuity with growth rate formula:

$$\sum_{t=1}^{n} C[(1+r_f)(1+g)]^t = \frac{C}{r_f - g}\left(1 - \left(\frac{1+g}{1+r_f}\right)^n\right)$$
$$g \text{ growth rate}$$
$$r_f \text{ risk free rate}$$

Recall:

$$PV(deposits) = PV(withdrawals) + PV(final\ balance)$$

For *PV(deposits)*, g = 5% for t from 11 to 16, and = 7% for t from 17 to 20.

$$PV(deposits) = \frac{14000}{0.08-0.05}\left(1-\left(\frac{1.05}{1.08}\right)^{16-10+1}\right)\frac{1}{1.08^{10-1}} + \frac{14000(1.05^6)(1.07)}{0.08-0.07}\left(1-\left(\frac{1.07}{1.08}\right)^{20-17+1}\right)\frac{1}{1.08^{17-1}}$$

For *PV(withdrawals)*:

$$g = (1+i)(1+g_r) - 1 = (1.01)(1.03) - 1 = 4.03\%$$

From above,

$$PV(withdrawals) = \frac{4000(1.01^{26})}{0.08-0.0403}\left(1-\left(\frac{1.0403}{1.08}\right)^{45-26+1}\right)\frac{1}{1.08^{26-1}}$$

For *PV(final balance)*:

$$PV(final\ balance) = \frac{C_{50}}{1.08^{50}}$$

Therefore:

$$C_{50} = (PV(deposits) - PV(withdrawals))(1.08^{50})$$

$$\Rightarrow C_{50} \approx 2492224.28$$

b) Real approach:

Real deposit at t = 10:

$$C_{10,r} = 14000/(1+i)^{10} = 14000/1.01^{10}$$

Real interest rate:

$$r_r = (1+r_f)/(1+i) - 1 = (1.08)/(1.01) - 1 = 6.93\%$$

Real growth rates:

$$g_{1,r} = (1+g_1)/(1+i) - 1 = 1.05/1.01 - 1 = 3.96\%$$
$$g_{2,r} = (1+g_2)/(1+i) - 1 = 1.07/1.01 - 1 = 5.94\%$$

PVs:

$$PV(deposits) = \frac{C_{10,r}}{r_r - g_1}\left(1 - \left(\frac{1+g_1}{1+r_r}\right)^{16-10+1}\right)\frac{1}{(1+r_r)^{10-1}}$$
$$+\frac{C_{10,r}(1+g_1)^6(1+g_2)}{r_r - g_2}\left(1 - \left(\frac{1+g_2}{1+r_r}\right)^{20-17+1}\right)\frac{1}{(1+r_r)^{17-1}}$$

$$PV(withdrawals) = \frac{4000}{r_r - 0.03}\left(1 - \left(\frac{1.03}{1+r_r}\right)^{45-26+1}\right)\frac{1}{(1+r_r)^{26-1}}$$

$$PV(final\ balance) = \frac{C_{50}/1.01^{50}}{(1+r_r)^{50}}$$

$$\Rightarrow C_{50} \approx 2492224.28$$

c) d) See appendixes.

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7)

(Note: This is the exact same question as from Q2 in Assignment 1 Summer 2012. Thus, as long as you attempt all parts of the question, you will receive full credit. Remember that the main purpose of this assignment is to better learn the material and prepare you for the midterm/final. Thus, I would strongly advise you to try to solve this question without looking at the solutions.)

Today is month t = 0. A bank is offering you the following financial product:

You make monthly payments to the bank of $500 per month from month t = 25 to month t = 264. In exchange, the bank will give you monthly payments of X from month t = 265 to month t = 504.

The annual interest rate (APR) is 12%, with monthly compounding.

The inflation rate is 0.2% per month.

a) Assuming the contract is priced in such a way that the bank makes a $1,000 profit in PV terms, what would X be?

Suppose that at t = 264, the bank makes you the following offer: Instead of giving you monthly payments of X from t= 265 to t = 504, the bank will pay you a lump sum of $280,000 at t = 264, and another lump sum of $250,000 at t = 364.

b) Should you accept the above lump sum offer?

Suppose your goal in retirement is to consume the same quantity of hamburgers every month from t = 265 to t = 504.

c) If a hamburger costs $1 at t = 0, what constant amount of hamburgers would you be able to purchase and consume each month from t = 265 to t = 504, based on the constant annuity with level X from part a).

d) If a hamburger costs $1 at t = 0, what constant amount of hamburgers would you be able to purchase and consume each month from t = 265 to t = 504, based on the alternative lump sum offer that the bank proposed?

e) Based on your answer to part d), how much money are you spending at t = 504 to purchase the hamburgers you consume at t = 504

Solution:

a)

Monthly interest rate = APR / 12 = 1%

$$\frac{500}{0.01}\left(1-\frac{1}{1.01^{264-25+1}}\right)\frac{1}{1.01^{25-1}}-\frac{X}{0.01}\left(1-\frac{1}{1.01^{504-265+1}}\right)\frac{1}{1.01^{265-1}}=1000$$

$$\Rightarrow X = 5293.99$$

b)

PV of constant annuity paid by bank:

$$\frac{5293.99}{0.01}\left(1-\frac{1}{1.01^{504-265+1}}\right)\frac{1}{1.01^{265-1}}=34763.15$$

PV of lump sum paid by bank:

$$280000/1.01^{264}+250000/1.01^{364}=26927.21<\text{PV of constant annuity}$$

Therefore, reject the lump sum offer.

c)

To compensate for inflation, the required monthly amount increases.

Growth rate is:

$$g=(1+g_r)(1+i)-1=(1+0)(1+0.002)-1=0.2\%$$

From above,

$$\frac{C}{0.01-0.002}\left(1-\left(\frac{1.002}{1.01}\right)^{504-265+1}\right)\frac{1}{1.01^{265-1}}=34763.15$$

$$\Rightarrow C = 4516.08$$

$$\Rightarrow C/1.002^{265} = 2659.60$$

One burger cost $1 at t=0 ⇒ the constant monthly amount of hamburgers is 2659.

d)

$$\frac{C}{0.01-0.002}\left(1-\left(\frac{1.002}{1.01}\right)^{504-265+1}\right)\frac{1}{1.01^{265-1}}=26927.21$$

$$\Rightarrow C = 3498.18$$

$$\Rightarrow C/1.002^{265} = 2060.14$$

e)

$$2060.14(1.002^{504}) = 5639.33$$

IRR

8)

(Note: This is the exact same question as from Q5 Winter 2015. Thus, as long as you attempt all parts of the question, you will receive full credit.)

Your company is considering two mutually exclusive projects A and B.

Project A requires an initial investment of $1M at t = 0, and starting at t = 1 makes constant perpetual annual cash flows equal to Q. Your boss tells you that Project A has an IRR of 15% and an NPV of $500,000.

Project B has the same level of risk as project A, and thus has the same cost of capital. Project B requires an initial investment of Z at t = 0, and starting at t = 3 makes constant perpetual annual cash flows equal to $250,000.

What is the minimum IRR on Project B that would make you want to choose Project B over Project A?

Solution

$$0 = -1000000 + Q/0.15 \Rightarrow Q = 150000$$

Find discount rate r of project A, and therefore also that of project B's:

$$500000 = -1000000 + 150000/r \Rightarrow r = 10\%$$

Find Z such that NPV of project B = NPV of project A:

$$500000 = -Z + \frac{250000}{0.10}\frac{1}{1.10^2} \Rightarrow Z = 1566115.7$$

From above, the minimum IRR on project B which would make it favorable is:

$$0 = -1566115.7 + \frac{250000}{IRR}\frac{1}{(1+IRR)^2} \Rightarrow IRR = 12.59\%$$

Effective Annual Cost

9)

Machine A costs $10,000 at t = 0 and needs to be replaced every 10 years. The price of Machine A increases with inflation, which is 3% per year. Machine A has annual maintenance costs where the cost is $2,000(1.03) at t = 1, and maintenance costs increase with inflation each year. Machine A also has an additional "refurbishment cost" of 4,000 real dollars that has to be paid halfway through its lifecycle, each cycle (t = 5, 15, 25, etc...).

If you pay $Z at t = 0 you can use Machine B. For machine B, there is no replacement cost. You do, however, have to pay a $600 annual maintenance fee, which we assume is paid at the end of each year. Machine B produces $340 more revenue per year than machine A. Assume all cash flows are risk free and that the real risk free rate is 10%. For what value of Z are you indifferent between both machines?

(Hint: use the real approach to calculate an EAC in real terms for Machine A. Then compute the PV of all costs to infinity for A using the PV of constant perpetuity formula where the level of the perpetuity is the real EAC. Next, set this equal to the PV of all costs for Machine B, where you can think about the extra revenue generated by B as going towards lowering the annual maintenance fee.

Solution:

Have:

$$r_r = 10\%$$
$$r_f = (1+r_r)(1+i) - 1 = (1.10)(1.03) - 1 = 13.3\%$$
$$r_r \text{ real risk free rate}$$
$$r_f \text{ nominal risk free rate}$$

Then:

$$NPV_{cycle,A,real} = 10000 + \frac{2000}{r_r}\left(1 - \frac{1}{(1+r_r)^{10}}\right) + \frac{4000}{(1+r_r)^5} = 24772.84$$

$$\frac{EAC_{A,r}}{0.10}\left(1 - \frac{1}{1.10^{10}}\right) = 24772.84 \Rightarrow EAC_{A,r} = 4031.68$$

$$PV(\text{all costs of A to infinity}) = EAC_{A,r}/r_r = 4031.68/0.10 = 40316.80$$

$$PV(\text{all costs of B to infinity}) = Z + \frac{600 - 340}{r_f} = Z + 260/0.133$$

$$PV(\text{all costs of B to infinity}) = PV(\text{all costs of A to infinity})$$
$$\Rightarrow Z + 260/0.133 = 40316.80$$
$$\Rightarrow Z = 38361.92$$

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10)

The following problem is adapted from a case study that a Masters student of mine was asked to prepare for a job interview at a company I will call BioFinancial.

Today is t = 0. BiotechX, a biotech company, developed a drug (DrugX) that helps fight a rare form of cancer.

PharmaX, a large pharmaceutical company, has purchased the rights to produce, market, and sell DrugX for the next 20 years. According to the terms of the deal, PharmaX has to pay BiotechX each year from t = 1 to t = 20, 3% of the annual sales revenue from the sales of DrugX.

BioFinancial wants to make an offer to BiotechX. BioFinancial will pay BiotechX a lump sum payment of Z at t = 0 in exchange for receiving all future cash flows that PharmaX will pay to BiotechX. Assume that the appropriate nominal discount rate for all cash flows is 5%.

Your job is to determine the maximum amount that BioFinancial should be willing to pay at t = 0 for the deal. Since the future payments BioFinancial will be entitled to depend on DrugX sales, your answer depends in part on the estimated future sales of DrugX.

Based on a combination of population growth estimates for various demographics and incidence rates in each demographic, you estimate that DrugX sales will be $100M at t = 1 and the real growth rate of sales will be 5% per year until t = 15. For t = 15 to 20 you estimate the real growth rate of sales will be -15% as the drug patent expires and competition enters the market. The -15% growth rate estimate is based on historical averages of similar drugs after patent expiration.

Assume that entering this deal would imply additional administrative costs for BioFinancial of $100,000 at t = 1, and where the administrative costs would grow at the inflation rate until t = 20, where the inflation rate is 2% per year.

a) Based on the above information, what is the maximum value of Z that BioFinancial should offer BiotechX at t = 0 for the deal?

b) For what value of Z would the deal imply a(nominal) IRR of 8% for BioFinancial? You will have to use Excel Solver to answer this question.

c) Think about the following: if the proposal has a positive NPV for BioFinancial, then why would BiotechX accept the offer since it would imply a negative NPV for BiotechX? You won't be graded on this part, but it's important to consider. Look for explanations when I post the solutions.

Solution:

a)

Nominal growth rates:

$$g_1 = (1+g_r)(1+i) - 1 = (1.05)(1.02) - 1 = 7.1\%$$
$$g_2 = (1-0.15)(1.02) - 1 = -13.3\%$$

From above:

$$PV(\text{DrugX sales}) = \frac{100\text{M}}{0.05-0.071}\left(1-\left(\frac{1.071}{1.05}^{15}\right)\right)$$
$$+\frac{100\text{M}(1.071^{14})(1-0.133)}{0.05+0.133}\left(1-\left(\frac{1-0.133}{1.05}\right)^{20-16+1}\right)\frac{1}{1.05^{16-1}}$$

For BioFinancial:

$$PV(\text{admin costs}) = \frac{100000}{0.05-0.02}\left(1-\left(\frac{1.02}{1.05}\right)^{20}\right)$$

$$NPV = 0 = -Z + PV(\text{DrugX sales})(3\%) - PV(\text{admin costs})$$
$$\Rightarrow Z = 58948389.53$$

b)

Substituting the discount rate for the IRR of 8%, and re-solving the above equation yields:

$$Z = 44558920.18$$

c)

- BiotechX could benefit from the deal if they really needed a large lump sum at t = 0 to invest in other projects deemed profitable.
- If BioFinancial is more optimistic of DrugX sales, then both BioFinancial and BiotechX would view the deal as having a positive NPV.

Bond Returns

11)

For each of the below, indicate True or False. No explanation is required.

a) Given: At t = 0 you bought a 3-year, 9% coupon bond with a 7% YTM. You held the position until t = 3. Each coupon that was received prior to t = 3 was reinvested and rolled over at a 7% interest rate. Statement: The realized compound yield on the investment was 7%.

b) Given: At t = 0 you bought a 4-year zero coupon bond with a 9% YTM. Two years later you sold the bond when it was trading at a 12% YTM. Statement: The realized compound yield on your two-year investment was somewhere between 9% and 12%.

c) Given: At t = 0 a 10-year zero coupon bond had an 8% YTM. You bought the bond at t = 2 when it had 8 years left to maturity and was trading at a 7% YTM. You sold the bond 3 years later when its YTM was greater than 7%. Statement: Your realized compound yield from t = 2 to t = 5 was less than 7%.

d) Given: From t = 3 to t = 4 the price of a risk free bond increased, and its YTM also increased. Statement: The coupon rate on this bond must be less than the YTM. (Hint: think of the pull-to-par graph and think of whether the statement would be possible for a bond with a coupon rate greater or equal to its YTM.)

e) As a general rule, if one expects interest rates are going to surprise to the upside, it would be wiser to invest in longer term bonds.

Solution:

Yield to Maturity (YTM): discount rate where PV of cash flows = price of bond

Realized compound yield:

$$y_{realized} = (FW/P_{bond})^{1/T} - 1$$
$$FW \text{ final wealth}$$
$$P_{bond} \text{ bond price at t=0}$$

FW = future value of coupons + present value of remaining cash flows (if bond sold before maturity)

Bond price = present value of all cash flows

$$P_{bond} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{F}{(1+r)^T}$$
$$C \text{ coupon value}$$
$$F \text{ face value}$$

a) True

b) False

c) True

d) True

e) False

Bond Pricing and Bond Arbitrage

12)

You are given the following information:

- Bond A is a 1-year, 6% coupon bond with face value $8,000 and YTM = 4%
- Bond B is a 2-year, zero coupon bond with face value $30,000 and YTM = 6%
- Bond C is a 3-year, 7% coupon bond with face value $20,000 and YTM = 7%
- Bond D is a 4-year, zero coupon bond with face value $8,000 and YTM = 9%
- Bond E is a 5-year, zero coupon bond with face value $2,000 and YTM = 11%
- A financial institution is offering the following product:

- the client pays the financial institution $200,000 at t = 0 and another $100,000 at t = 1
- the financial institution pays the client $40,000 at t = 2, X at t = 3, and $60,000 at t = 4

a) What would X have to be in order for the product to be fairly priced? (Hint: bootstrap the yield curve and then set PV(client cash flow to bank) = PV(bank cash flow to client).

b) Suppose a financial institution is offering the above product, but the payment at t = 3 is $100 greater than your answer to part a. You are a Hedge Fund manager. Describe how you would create an arbitrage by buying 1 unit of the product from the financial institution and trading the various bonds listed above at t = 0 (you may not have to trade all of them). Specify how many units of each bond you will buy or sell. Construct the arbitrage so that your profit is realized at t = 0.

c) Based on your strategy in part b, what is the magnitude of the arbitrage profit at t = 0?

d) Repeat part b, but this timeconstruct the arbitrage in such a way that the profit is realized at t = 3. Hint: in this case your inflow = outflow conditions should hold for t = 0, 1, 2 and 4.

e) Based on your strategy in part d, what is the magnitude of the arbitrage profit at t = 3?

f) What is the relationship between your answers to parts c and e?

Solution:

Recall:

- One year bond ⇒ risk free rate = YTM
- Multi-year zero coupon bond ⇒ risk free rate = YTM
- YTM = coupon rate ⇒ bond price = face value

a)

$$r_{0,1} = YTM_A = 4\%$$
$$r_{0,2} = YTM_B = 6\%$$

$$P_C = 20000 = 1400/1.04 + 1400/1.06^2 + 21400/(1 + r_{0,3})^3 \Rightarrow r_{0,3} = 7.12\%$$

$$r_{0,4} = 9\%$$
$$r_{0,5} = 11\%$$

From above:

$$20000 + 100000/1.04 = 40000/1.06^2 + X/1.0712^3 + 60000/1.09^4$$
$$\Rightarrow X = 268053.61$$

b)

Let A, B, C, and D the number of units we short of bonds A, B, C and D, respectively.

Want to realize profit at t = 0 ⇒ apply net income = 0 condition to all other t's.

Want the net income = 0 at each t = 1, 2, 3 and 4 i.e. cash outflows = inflows.

t = 1 condition: 100000 + C(1400) + A(8480) = 0

t = 2 condition: B(30000) + C(1400) = 40000

t = 3 condition: C(21400) = 268153.61

t = 4 condition: D(8000) = 60000

⇒ D = 7.5, C = 12.530, B = 0.7486, A = -13.86

⇒ Buy the product, buy 13.86 units of bond A, and short bonds B, C and D in the above quantities.

c)

$$profit = -200000 - 13.86P_A + 0.749P_B + 12.53P_C + 7.5P_D$$

$$\Rightarrow profit = -20000 - 13.86\frac{8480}{1.04} + 0.749\frac{30000}{1.06^2} + 12.53(20000) + 7.5\frac{8000}{1.09^4}$$

$$\Rightarrow profit = 81.8$$

d)

Realize profit at t = 3 ⇒ apply condition to t = 0, 1, 2 and 4.

t = 0 condition:

$$-200000 + A\frac{8480}{1.04} + B\frac{30000}{1.06^2} + C(20000) + D\frac{8000}{1.09^4} = 0$$

t = 1 condition: 100000 + C(1400) + A(8480) = 0

t = 2 condition: B(30000) + C(1400)= 40000

t = 4 condition: D(8000) = 60000

⇒ D = 7.5, C = 12.526, B = 0.7488, A = -13.86

e)

$$profit = 268153.61 - 12.526(214000) = 100$$

f)

Profit in part e due to bank paying $100 more than X at t = 3.

Profit in part c due to profit discounted at the 3 year spot rate (i.e. 7.12%).

Forward Rates and Arbitrage

13)

You are given the following information: at t = 0, the price of a 10 - year zero coupon bond with FV = $5,000 is $3,500; the price of a 3 - year zero coupon bond with FV = $10,000 is X; f3,12= 6%. A bank is offering the following product: for every $0.60 that you give the bank at t = 10, the bank will give you back $1 at t = 15, or for every $0.60 that you borrow from the bank at t = 10, you will have to pay back $1 at t = 15.

a) What should X be if there is no arbitrage?

b) Suppose X were $5 less than your answer to part a. Describe how you would construct an arbitrage strategy. Assume that part of your strategy involves buying or selling $1 worth of the 10 - year zero coupon bond, and also either buying or selling $1 worth of the 3 - year zero coupon bond. Construct the arbitrage in such a way that your profit is realized at t = 15, and the net cash flows at all other points in time are 0. You must specify for f3,12= 6% whether you are borrowing or lending at that rate, and how much. You must also specify how much you are borrowing or lending using the bank's product. Make sure to also specify for each bond if you are buying or shorting $1 worth of the bond. Finally, make sure to specify for each bond, how many units you are buying or selling.

c) What is the magnitude of your arbitrage profit at t = 15?

d) Repeat question b, but this time for 2 units of the 3 - year bond bought or sold at t = 0.

e) What is the magnitude of the arbitrage profit at t = 15 from part d?

Solution:

a)

$$\frac{5000}{3500}\frac{1}{0.6} = \frac{10000}{X}(f_{3,12})^{12} \Rightarrow X = 8451.23$$

b)

- Long $1 worth of the 3-year zero coupon bond (1/8446.23 units)
- Lock in lending rate of $f_{3,12}$= 6% for a notional value of 10000/8,44623 from t = 3 to t = 15.
- Short $1 worth of the 10-year zero coupon bond (1/3500 of a unit)
- Use the bank contract to borrow 5000/3500 from t = 10 to t = 15

c)

$$\frac{10000}{8446.23}(1.06^{15-3}) - \frac{5000}{3000}\frac{1}{0.6} = 0.001409$$

d)

Multiply all positions in b by factor of 2(8446.23).

- Long 2 units of 3-year zero coupon bond
- Lock in lending rate of $f_{3,12}$= 6% for a notional value of 20000 from t = 3 to t = 15.
- Short 2(8446.23) worth of the 10-year zero coupon bond (2(8446.23)/3500 = 4.83 units)
- Use the bank contract to borrow 2(8446.23)(5000/3500) = 24132.07 from t = 10 to t = 15

e)

$$20000(1.06^{12}) - 20(8446.23)\frac{5000}{3500}\frac{1}{0.6} = 23.81$$

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14)

(This question is from the 2016 Winter Final Exam. As long as you attempt all parts you will receive full credit.)

Today is t = 0.

- The price of a Dollar denominated 10-year zero coupon bond with Face Value = $10,000 is $8,000
- $r_{0,5}$= 2% (Dollar interest rate); $f_{20,5}$= 3% (Dollar forward interest rate)
- $fE_{20,5}$= 4% (Euro forward interest rate)
- Contract X: for every $1 you give the bank at t = 10 they give you back 2 Euros at t = 20
  - (or for every $1 you borrow at t = 10 you have to pay back 2 Euros at t = 20)
- Contract Y: for every $1 you give the bank at t = 0 they give you back K Euros at t = 25
  - (or for every $1 you borrow at t = 0 you have to pay back K Euros at t = 25)

Note: for a given currency, a forward interest rate in that currency allows you to lock in a lending or borrowing rate for that currency for a certain period of time.

a) Assuming there is no arbitrage, write down one equation where the only unknown is K? You do not need to solve or isolate for K.

b) Suppose K were 2 less than implied by your answer to part a. Describe how you would construct an arbitrage where you realize the profit (in Euros) at t = 25. Assume that your arbitrage involves buying or shorting 3 units of the 10-year zero coupon bond. Make sure that for each rate you are using you specify if you are borrowing or lending, in what currency, and how much. Make sure that for each contract X, Y, that you use that you specify whether you are using it for lending or borrowing and for how much, and in what currency. You do not need to report the magnitude of the profit.

Solution:

a)

$$\frac{10000}{8000}(2)(1.04^5) = K \Rightarrow K = 3.04$$

b)

- Long 3 units of bond
- Borrow 3(8000) = $24000
- Lend 3(10000) = $30000 using contract X
- Lend 30000(2) = 60000 Euros using Euro forward interest rate

Fill in the red cells in the table below based on the information in Assignment 1.
All values should be in *nominal* terms.

r =	8.00%
i =	1.00%
real growth rate of withdrawals =	3%
nominal growth rate of withdrawals =	4.03% <-- enter formula g = (1+g_r)(1+i) - 1

Period t =	Deposits	PV(Deposit)	Withdrawals	PV(Withdrawal)	Account balance before deposit/withdrawal	Account balance after deposit/withdrawal
0	0.00	0.00	0.00	0.00	0.00	0.00
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00
10	14,000.00	6,484.71	0.00	0.00	0.00	14,000.00
11	14,700.00	6,304.58	0.00	0.00	15,120.00	29,820.00
12	15,435.00	6,129.45	0.00	0.00	32,205.60	47,640.60
13	16,206.75	5,959.19	0.00	0.00	51,451.85	67,658.60
14	17,017.09	5,793.66	0.00	0.00	73,071.29	90,088.37
15	17,867.94	5,632.72	0.00	0.00	97,295.44	115,163.39
16	18,761.34	5,476.26	0.00	0.00	124,376.46	143,137.79
17	20,074.63	5,425.55	0.00	0.00	154,588.82	174,663.45
18	21,479.86	5,375.31	0.00	0.00	188,636.53	210,116.38
19	22,983.45	5,325.54	0.00	0.00	226,925.69	249,909.14
20	24,592.29	5,276.23	0.00	0.00	269,901.87	294,494.16
21	0.00	0.00	0.00	0.00	318,053.69	318,053.69
22	0.00	0.00	0.00	0.00	343,497.99	343,497.99
23	0.00	0.00	0.00	0.00	370,977.83	370,977.83
24	0.00	0.00	0.00	0.00	400,656.06	400,656.06
25	0.00	0.00	0.00	0.00	432,708.54	432,708.54
26	0.00	0.00	5,181.03	700.48	467,325.22	462,144.20
27	0.00	0.00	5,389.82	674.73	499,115.73	493,725.91
28	0.00	0.00	5,607.03	649.93	533,223.99	527,616.96
29	0.00	0.00	5,832.99	626.04	569,826.31	563,993.32
30	0.00	0.00	6,068.06	603.03	609,112.78	603,044.72
31	0.00	0.00	6,312.61	580.86	651,288.30	644,975.69
32	0.00	0.00	6,567.00	559.51	696,573.75	690,006.74
33	0.00	0.00	6,831.65	538.94	745,207.28	738,375.63
34	0.00	0.00	7,106.97	519.13	797,445.68	790,338.71
35	0.00	0.00	7,393.38	500.05	853,565.80	846,172.42
36	0.00	0.00	7,691.33	481.67	913,866.22	906,174.88
37	0.00	0.00	8,001.30	463.96	978,668.87	970,667.58
38	0.00	0.00	8,323.75	446.91	1,048,320.99	1,039,997.24
39	0.00	0.00	8,659.19	430.48	1,123,197.02	1,114,537.82
40	0.00	0.00	9,008.16	414.65	1,203,700.85	1,194,692.69
41	0.00	0.00	9,371.19	399.41	1,290,268.10	1,280,896.91
42	0.00	0.00	9,748.85	384.73	1,383,368.67	1,373,619.82
43	0.00	0.00	10,141.73	370.59	1,483,509.41	1,473,367.68
44	0.00	0.00	10,550.44	356.96	1,591,237.09	1,580,686.66
45	0.00	0.00	10,975.62	343.84	1,707,141.59	1,696,165.97
46	0.00	0.00	0.00	0.00	1,831,859.25	1,831,859.25
47	0.00	0.00	0.00	0.00	1,978,407.98	1,978,407.98
48	0.00	0.00	0.00	0.00	2,136,680.62	2,136,680.62
49	0.00	0.00	0.00	0.00	2,307,615.07	2,307,615.07
50	0.00	0.00	0.00	0.00	2,492,224.28	2,492,224.28 <--- this final balance should be your answer to a and b

PV(all deposits) --> (sum of column C)	63,183.19	PV(all withdrawals) -->	10,045.91 <-- (sum of column E)
		PV(final balance) -->	53,137.28
		sum	63,183.19

Note that PV(all deposits) should equal PV(all withdrawals) + PV(final balance)

Fill in the red cells in the table below based on the information in Assignment 1.
All values should be in *nominal* terms.

r =	9.34%	
i =	1.00%	
real growth rate of withdrawals =	3%	
nominal growth rate of withdrawals =	4.03%	<-- enter formula g = (1+g_r)(1+i) - 1

Period t =	Deposits	PV(Deposit)	Withdrawals	PV(Withdrawal)	Account balance before deposit/withdrawal	Account balance after deposit/withdrawal
0	0.00	0.00	0.00	0.00	0.00	0.00
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00
10	14,000.00	5,734.26	0.00	0.00	0.00	14,000.00
11	14,700.00	5,506.83	0.00	0.00	15,307.11	30,007.11
12	15,435.00	5,288.42	0.00	0.00	32,808.71	48,243.71
13	16,206.75	5,078.67	0.00	0.00	52,747.97	68,954.72
14	17,017.09	4,877.24	0.00	0.00	75,392.66	92,409.74
15	17,867.94	4,683.81	0.00	0.00	101,037.55	118,905.49
16	18,761.34	4,498.04	0.00	0.00	130,007.07	148,768.41
17	20,074.63	4,401.92	0.00	0.00	162,658.13	182,732.76
18	21,479.86	4,307.85	0.00	0.00	199,793.55	221,273.41
19	22,983.45	4,215.79	0.00	0.00	241,932.53	264,915.98
20	24,592.29	4,125.70	0.00	0.00	289,649.78	314,242.07
21	0.00	0.00	0.00	0.00	343,581.18	343,581.18
22	0.00	0.00	0.00	0.00	375,659.54	375,659.54
23	0.00	0.00	0.00	0.00	410,732.88	410,732.88
24	0.00	0.00	0.00	0.00	449,080.83	449,080.83
25	0.00	0.00	0.00	0.00	491,009.12	491,009.12
26	0.00	0.00	5,181.03	508.77	536,852.04	531,671.02
27	0.00	0.00	5,389.82	484.08	581,310.32	575,920.50
28	0.00	0.00	5,607.03	460.59	629,691.15	624,084.12
29	0.00	0.00	5,832.99	438.23	682,351.54	676,518.55
30	0.00	0.00	6,068.06	416.96	739,681.50	733,613.44
31	0.00	0.00	6,312.61	396.73	802,107.04	795,794.43
32	0.00	0.00	6,567.00	377.47	870,093.54	863,526.53
33	0.00	0.00	6,831.65	359.15	944,149.43	937,317.78
34	0.00	0.00	7,106.97	341.72	1,024,830.17	1,017,723.20
35	0.00	0.00	7,393.38	325.14	1,112,742.62	1,105,349.24
36	0.00	0.00	7,691.33	309.36	1,208,549.84	1,200,858.50
37	0.00	0.00	8,001.30	294.34	1,312,976.30	1,304,975.00
38	0.00	0.00	8,323.75	280.06	1,426,813.60	1,418,489.85
39	0.00	0.00	8,659.19	266.47	1,550,926.73	1,542,267.54
40	0.00	0.00	9,008.16	253.53	1,686,260.89	1,677,252.73
41	0.00	0.00	9,371.19	241.23	1,833,848.92	1,824,477.73
42	0.00	0.00	9,748.85	229.52	1,994,819.55	1,985,070.70
43	0.00	0.00	10,141.73	218.38	2,170,406.23	2,160,264.51
44	0.00	0.00	10,550.44	207.78	2,361,956.95	2,351,406.52
45	0.00	0.00	10,975.62	197.70	2,570,944.88	2,559,969.26
46	0.00	0.00	0.00	0.00	2,798,980.02	2,798,980.02
47	0.00	0.00	0.00	0.00	3,060,305.95	3,060,305.95
48	0.00	0.00	0.00	0.00	3,346,030.50	3,346,030.50
49	0.00	0.00	0.00	0.00	3,658,431.63	3,658,431.63
50	0.00	0.00	0.00	0.00	4,000,000.00	4,000,000.00

<--- this final balance should be your answer to a and b

PV(all deposits) --> (sum of column C)	52,718.54	PV(all withdrawals) -->	6,607.22	<-- (sum of column E)
		PV(final balance) -->	46,111.32	
		sum	52,718.54	

Note that PV(all deposits) should equal PV(all withdrawals) + PV(final balance)