COMP 546

Lecture 16

Linear Systems 1:

convolution, impulse response functions, complex numbers, Euler's equation

Thurs. March 14, 2019

Recall definitions from lecture 5:

Cross correlation

$$f(x) \otimes I(x) \quad \equiv \sum_{u} f(u-x)\, I(u)$$

Convolution

$$f(x) * I\,(x\,) \quad \equiv \sum_{u} f(x-u)\, I(u)$$

Any cross-correlation can be written as a convolution (and vice-versa) just by flipping the function $f(\,)$.

Convolution (sum up shifted versions of $f(x)$)

$$f(x) * I(x) = \sum_{u} f(x-u)\, I(u)$$

$$= f(x)\, I(0) + \ ...$$

$$+ \ f(x+1)\, I(-1) + \ f(x+2)\, I(-2) + \ ...$$

$$+ \ f(x-1)\, I(1) \ + \ f(x-2)\, I(2) \ + \ ...$$

Some algebraic properties of convolution

For any $f_1(x),\ f_2(x)\ f_3(x)$:

$$f_1 * f_2 = f_2 * f_1$$

Cross correlation does not have this property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$$

$$(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$$

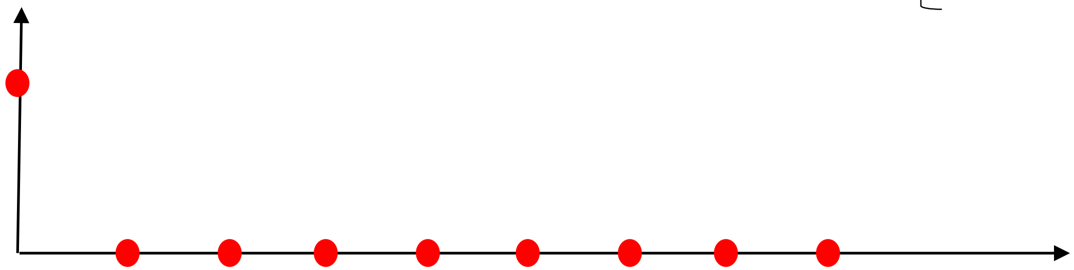
Boundary conditions

$$f(x) * I(x) = \sum_{u=0}^{N-1} f(x-u)\, I(u)$$

Assume $I(x) = 0$ outside range 0 to $N-1$.

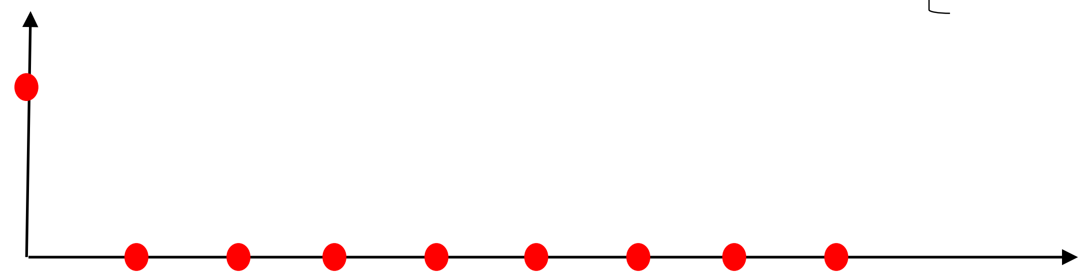
Impulse function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

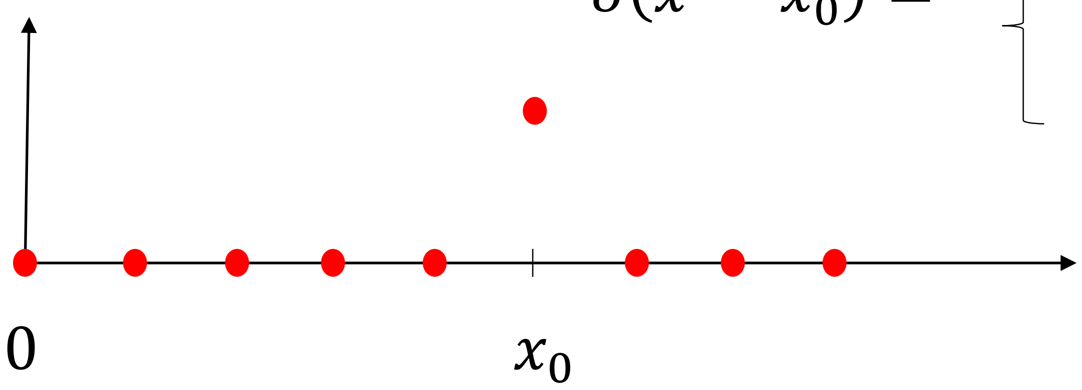


Impulse function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$



$$\delta(x - x_0) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$



$$\delta(x) * I\ (x\ )\ = \sum_{u} \delta(x-u)\ I(u)$$

$$= \ ?$$

$$\delta(x) * I(x) = \sum_u \delta(x-u)\, I(u)$$

$$= I(x)$$

An image can be thought of as a sum of delta functions.

Impulse Response function

If we think of $I(x) * f(x)$ as a mapping from a input function $I(x)$ to an output function, then we often call $f(x)$ an "impulse response" function since:

$$\delta(x) * f(x) = f(x)$$

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_u f(u-x)\, I(u)$$

Sliding a template across an image, and taking inner product.

Convolution

$$f(x) * I(x) \equiv \sum_u f(x-u)\, I(u)$$

Summing the impulse responses from all the pixels.

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_u f(u - x)\, I(u)$$

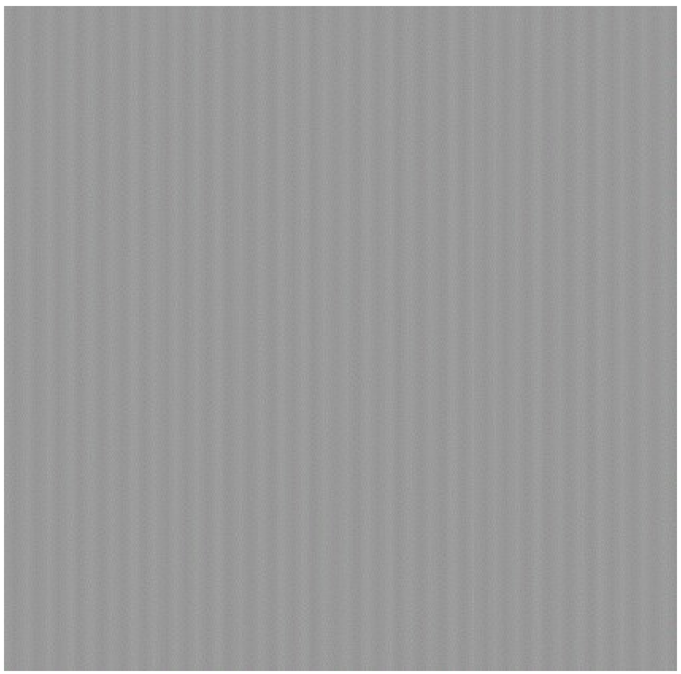
How well does the filter match the image, when filter is placed at position x ?

Convolution

$$f(x) * I(x) \equiv \sum_u f(x - u)\, I(u)$$

How much does image intensity at position x contribute to the filtered output?

Convolution with a sinusoid



In Assignments 1 and 2, you considered what happens when you cross-correlate 1D or 2D sinusoid images with some filter (DOG or Gabor). How did result depend on frequency?

This type of analysis will be especially important when we discuss sound and hearing.

In this lecture and the next, we prepare the theoretical groundwork.

Towards Fourier Analysis

Q: What happens when you convolve a sinusoid function with some other function ?

$$cos\left(\frac{2\pi k}{N}x\right) * \; h(x) = \; ?$$

$$sin\left(\frac{2\pi k}{N}x\right) * \; h(x) = \; ?$$

Claim 1: (See lecture notes for proof.)

Q: What happens when you convolve a sinusoid function with some other function ?

A: We get a sum of scaled sine and cosine functions of the same frequency.

$$cos\left(\frac{2\pi k}{N}x\right) * \ h(x) = \ a\ cos\left(\frac{2\pi k}{N}x\right) + b\ sin\left(\frac{2\pi k}{N}x\right)$$

where $a,\ b$ depend on $h(x)$ and frequency k.

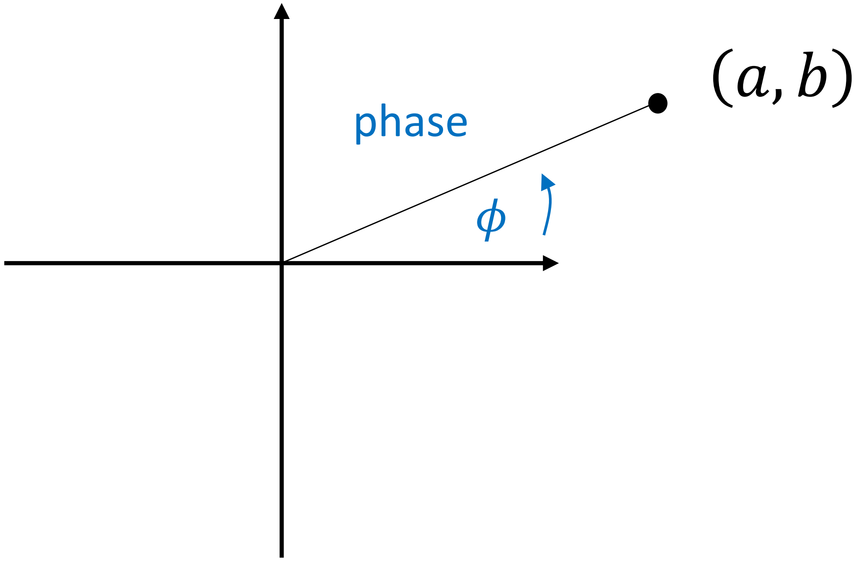
Claim 2:

When we add together a sine and cosine function of some frequency, the result is a shifted and scaled cosine function of the same frequency.

$$a \ cos\left(\frac{2\pi k}{N}x\right) + b \ sin\left(\frac{2\pi k}{N}x\right) \quad = \quad \underbrace{\sqrt{a^2 + b^2}}_{\text{“amplitude”}} \ \cos\left(\frac{2\pi k}{N}x - \underbrace{\phi}_{\text{“phase”}}\right)$$

where $a, \ b$ depend on $h(x)$ and frequency k.

To prove Claim 2, first we write (a, b) in polar coordinates



amplitude

$$(a, b) = \sqrt{a^2 + b^2} \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$$

$$= \sqrt{a^2 + b^2} \quad (\cos \phi \ , \sin \phi)$$

$$cos\left(\frac{2\pi k}{N}x\right) \ast \ h(x)$$

Claim 1 (see lecture notes)

$$= \ a \ cos\left(\frac{2\pi k}{N}x\right) + b \ sin\left(\frac{2\pi k}{N}x\right)$$

$$= \sqrt{a^2 + b^2} \ \left(cos\,\phi \ cos\left(\frac{2\pi k}{N}x\right) + sin\,\phi \ sin\left(\frac{2\pi k}{N}x\right)\right)$$

Claim 2: Using identity from Calculus 1 that $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.

$$= \sqrt{a^2 + b^2} \ \ cos\left(\frac{2\pi k}{N}x \ - \phi\right)$$

Exercise: Show

$$sin\left(\frac{2\pi k}{N}x\right) * \; h(x) \quad = \quad \sqrt{a^2+b^2} \;\; sin\left(\frac{2\pi k}{N}x \; - \; \phi\right)$$

same amplitude and phase
as for the cosine

Using identity from Calculus 1 that $\sin(A-B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Claim 3: (proof omitted)

We can write any function $I(x)$ defined on $x = 0$ to $N - 1$ as:

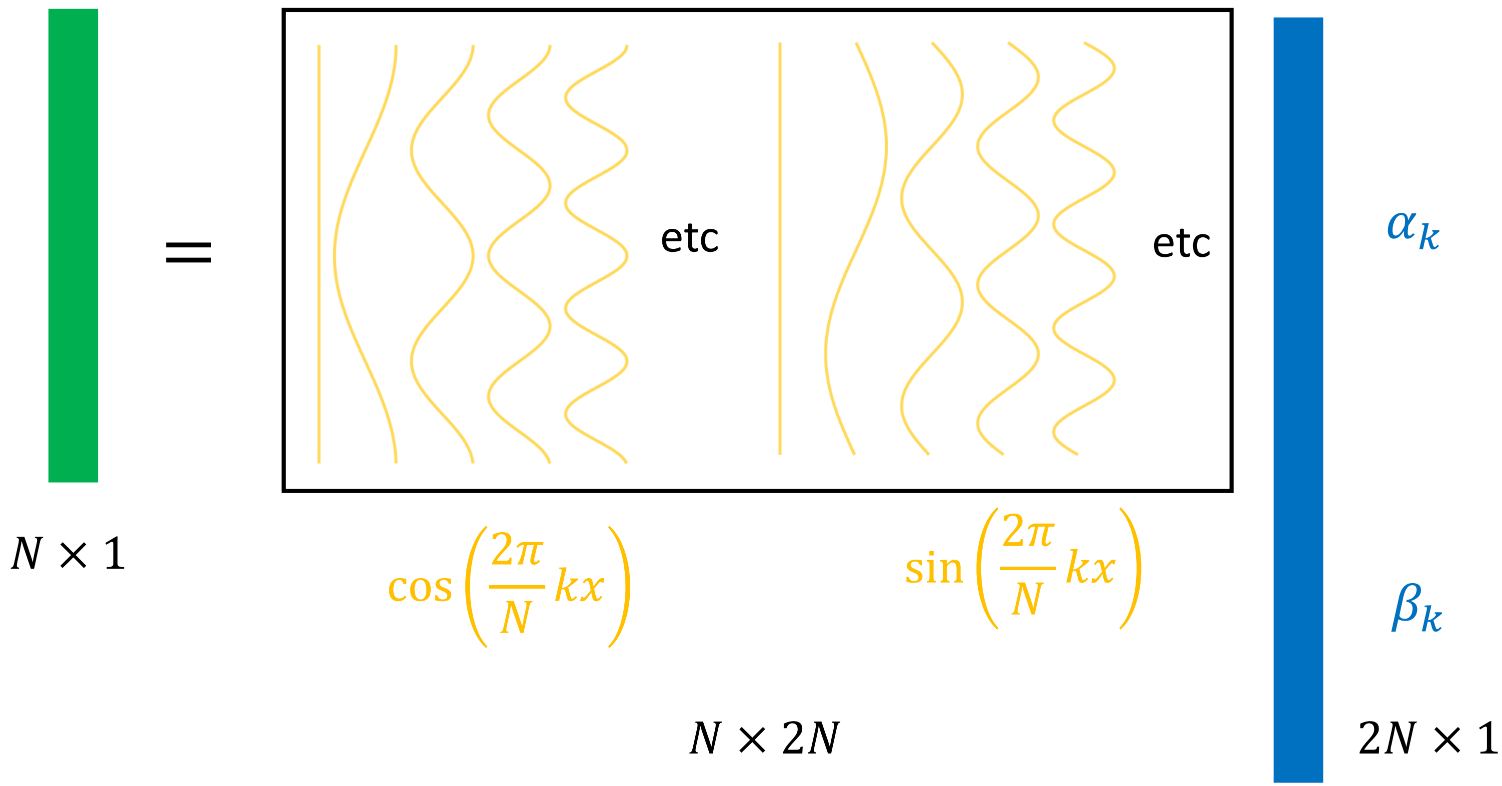
$$I(x) = \sum_{k=0}^{N-1} \alpha_k \cos\left(\frac{2\pi}{N} kx\right) + \sum_{k=0}^{N-1} \beta_k \sin\left(\frac{2\pi}{N} kx\right)$$

Thus, sines and cosines define an basis for functions $I(x)$.

Note there are $2N$ functions for N points. This leads to constraints on the α_k and β_k. More on this point next lecture…

Visualizing this sinusoidal basis representation of $I(x)$ by arranging cosines and sines into columns of a matrix :

$$I(x) = \sum_{k=0}^{N-1} \alpha_k \cos\left(\frac{2\pi}{N} kx\right) + \sum_{k=0}^{N-1} \beta_k \sin\left(\frac{2\pi}{N} kx\right)$$



Brief Summary

Make sure you understand what the following statements are saying:

$$cos\left(\frac{2\pi k}{N}x\right) * \ h(x) \ = \ \sqrt{a^2+b^2} \ \ cos\left(\frac{2\pi k}{N}x \ - \phi\right)$$

$$sin\left(\frac{2\pi k}{N}x\right) * \ h(x) \ = \ \sqrt{a^2+b^2} \ \ sin\left(\frac{2\pi k}{N}x \ - \phi\right)$$

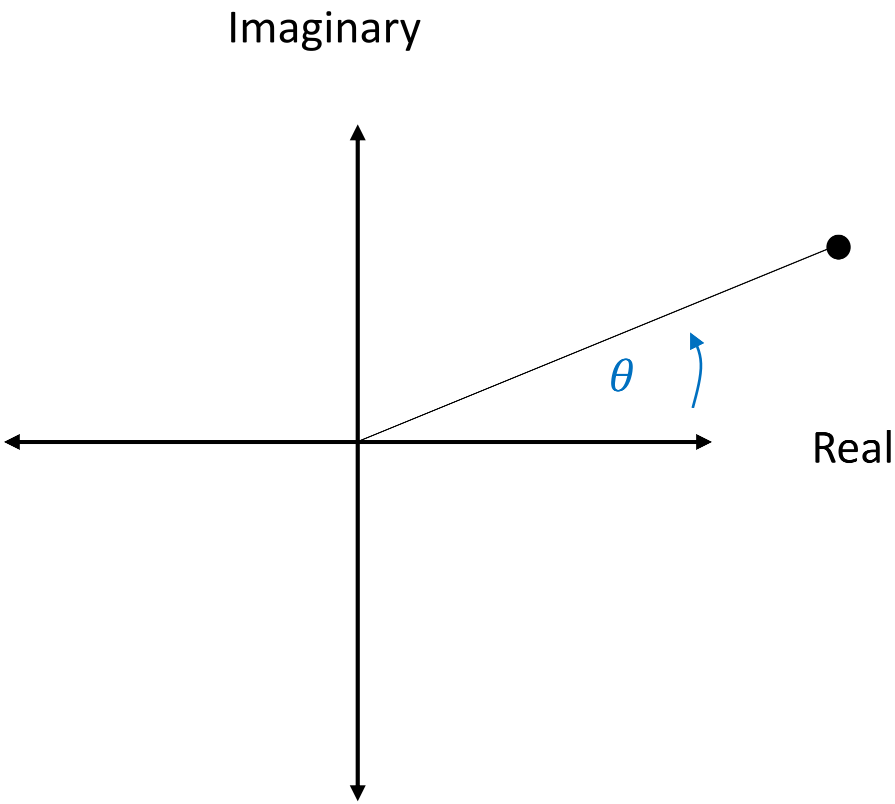
$$I(x) \ = \ \sum_{k=0}^{N-1} \alpha_k \cos\left(\frac{2\pi}{N}kx\right) + \sum_{k=0}^{N-1} \beta_k \sin\left(\frac{2\pi}{N}kx\right)$$

Fourier transform (next lecture)

- an alternative way of writing the above equations

- based on complex numbers, which I review next.

What is a complex number ?

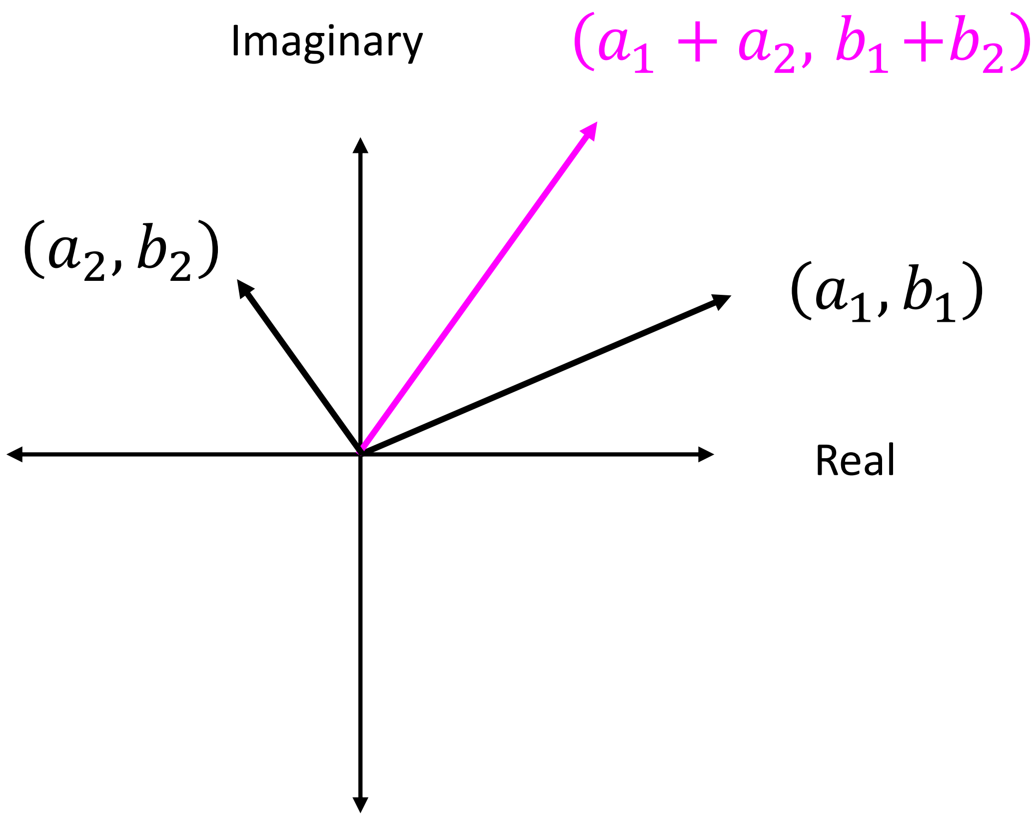


$$c = a + b\,i$$

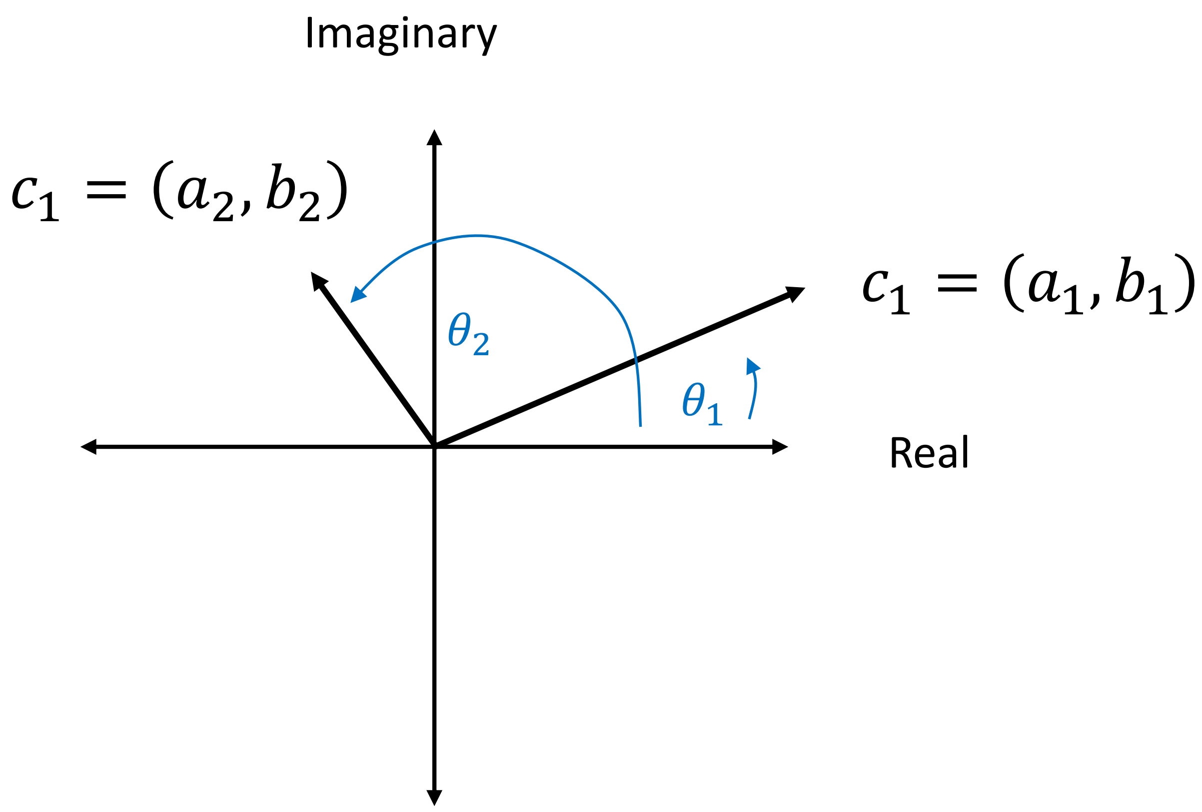
$$= \; r\cos\theta + i\,r\sin\theta$$

$$|\,c\,| = r = \sqrt{a^2 + b^2}$$

Addition of complex numbers

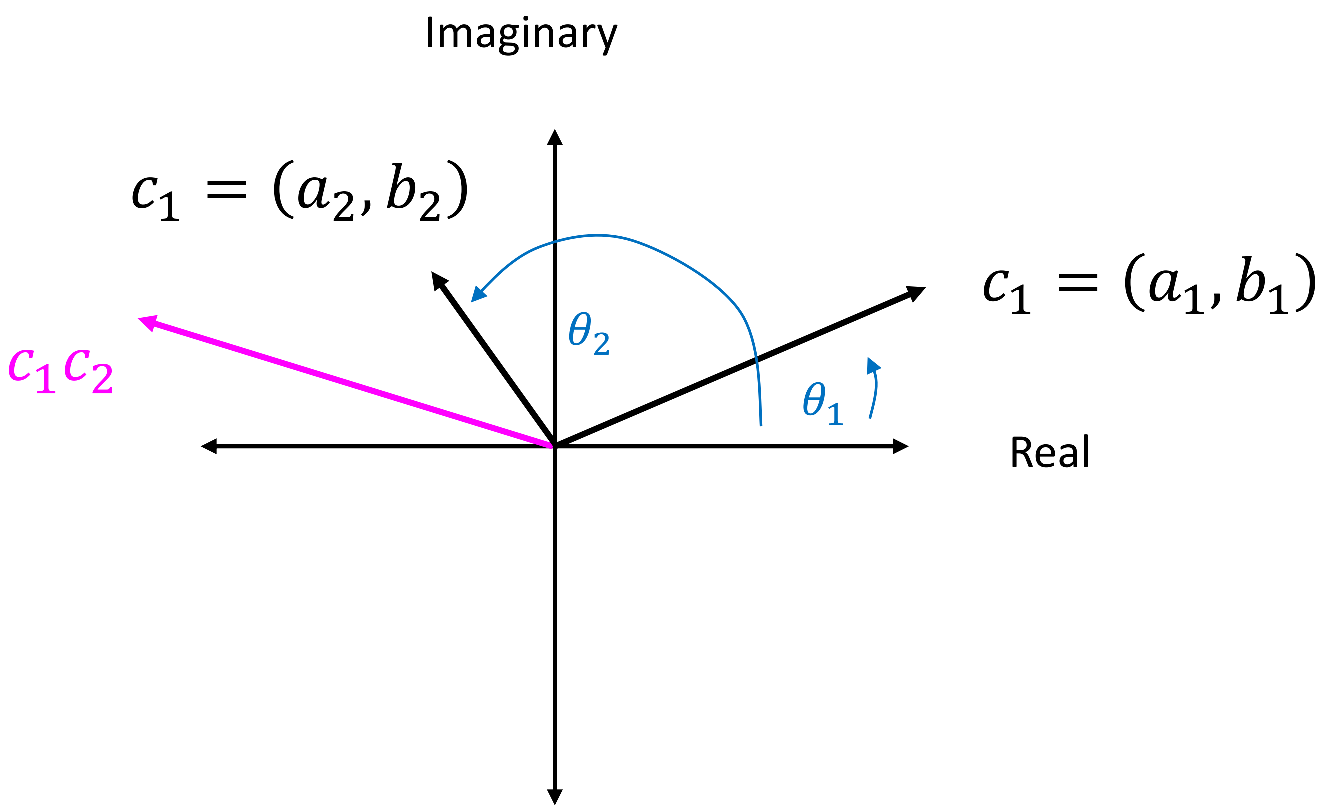


Multiplication of complex numbers



$$c_1 c_2 = \quad ?$$

Multiplication of complex numbers

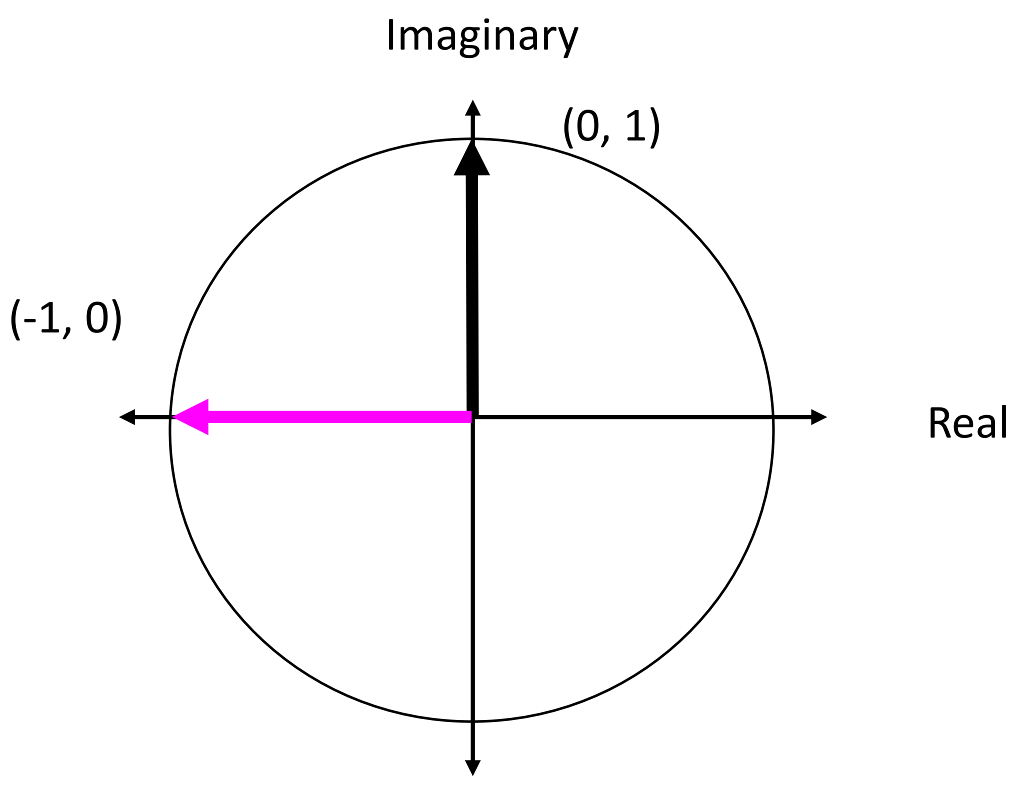


$$c_1c_2 = |c_1||c_2|\left(cos(\theta_1 + \theta_2),\ sin(\theta_1 + \theta_2)\right)$$

Multiply lengths

Add angles

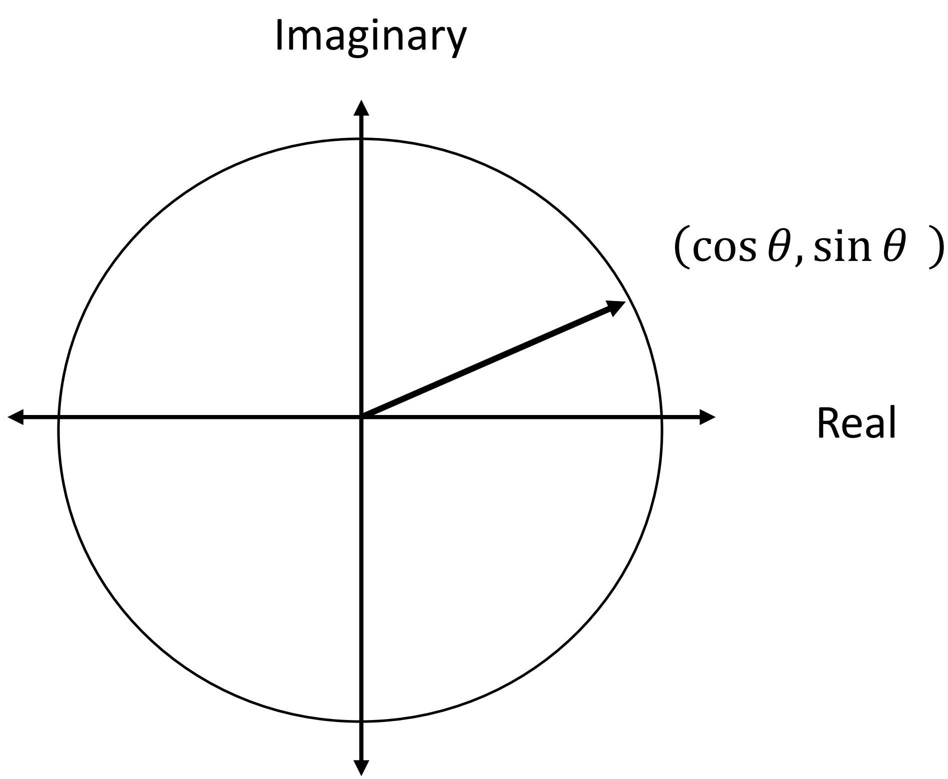
Example $i * i = -1$



Euler's equation

(Why? See Appendix to lecture notes)

$$e^{i\theta} = \cos\theta \; + i \; \sin\theta$$



Examples of Euler's equation

$$e^{i\cdot 0} = \cos(0) + i\sin(0) = 1$$

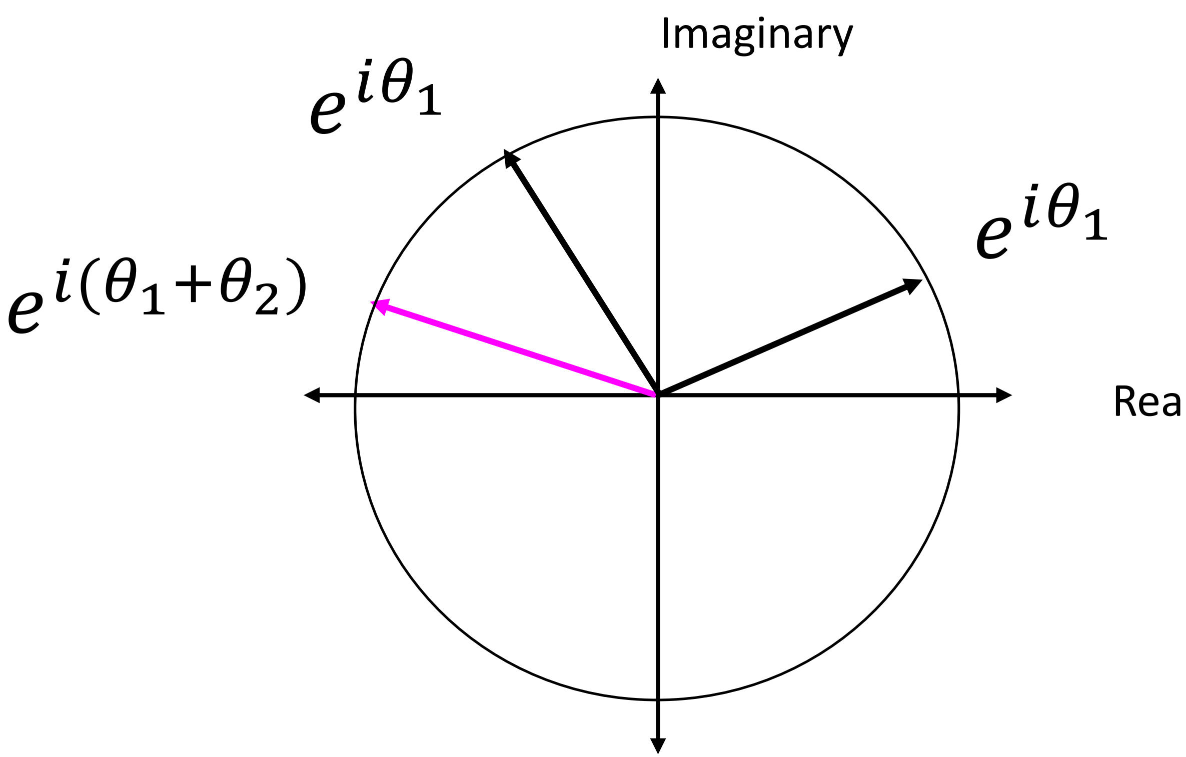
$$e^{i\cdot\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$$e^{i\cdot\pi} = \cos(\pi) + i\sin(\pi) = -1$$

$$e^{i\cdot\frac{3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$$

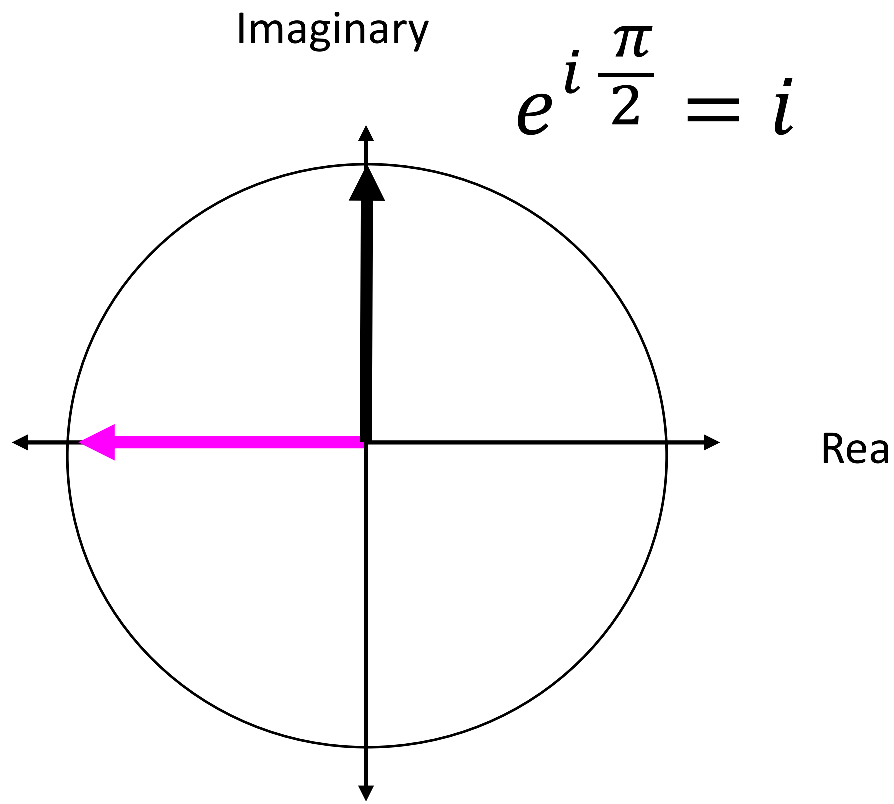
$$e^{i\cdot 2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$$

Multiplication of complex numbers using Euler's equation



$$e^{i\theta_1} \; e^{i\theta_1} = e^{i(\theta_1+\theta_2)}$$

Example $\quad i * i = -1$

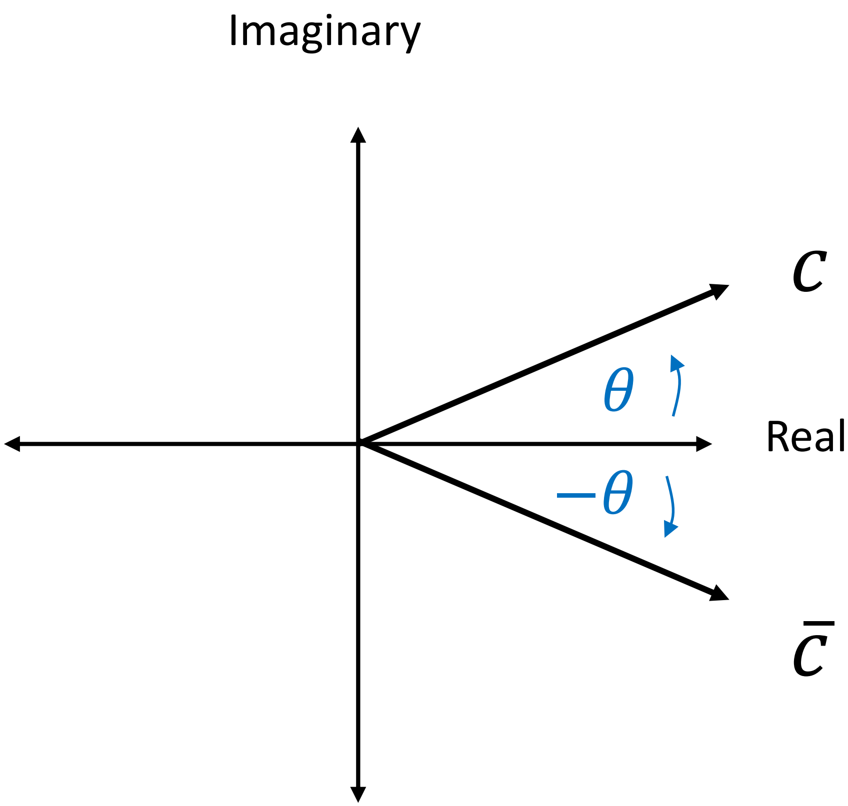


$$e^{i\,\frac{\pi}{2}} = i$$

$$e^{i\,\pi} = -1$$

$$e^{i\,\frac{\pi}{2}} \; e^{i\,\frac{\pi}{2}} \; = \; e^{i\,\pi}$$

Complex conjugate

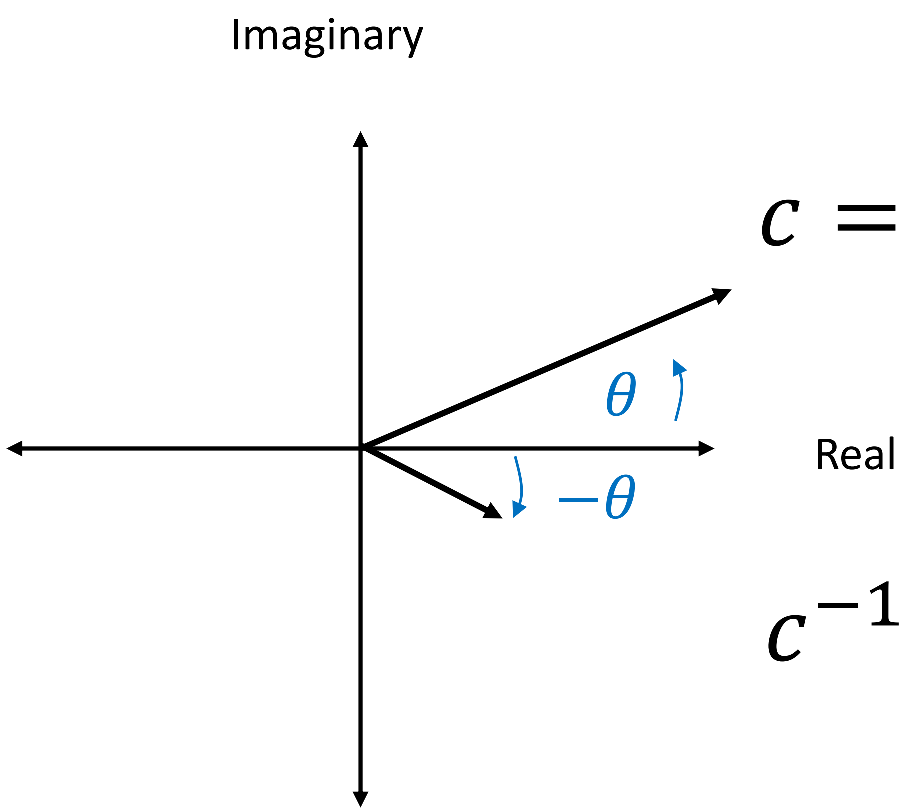


$$c = a + b\,i \;=\; r\,e^{i\theta}$$

$$\bar{c} = a - b\,i \;=\; r\,e^{-i\theta}$$

$$\bar{c}\,c \;= r^{2}$$

Multiplicative Inverse



$$c = r\, e^{i\theta}$$

$$c^{-1} \equiv \frac{1}{r} e^{-i\theta}$$

$$c\ c^{-1} = 1$$

Next Tuesday : posters

- Please read the PDF that I posted and let me know if you have any questions.
- I will bring tape
- Singles – you can merge (let me know soon)
- I assume you are judging on the same day. Let me know otherwise. (Google form to be announced.)
- Please show up on the other day too (but don't need to judge).