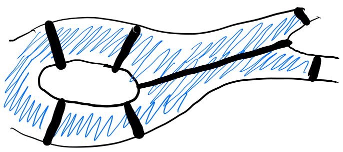
Graph Theory

Bridges of Königsberg



Euler: Can you start anywhere in the city, cross every bridge exactly once and finish where you started?

A graph G is a pair (V,E) where V is a set (the elements are the vertices of the graph) and E is a set of unordered pairs of elements of V (called the edges of G).

We often write $V(G)$ and $E(G)$ for the vertices/edges of $G=(V,E)$

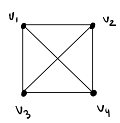
example $V = \{v_1, v_2, v_3, v_4\}$

$E = \{v_1v_2,\ v_1v_3,\ v_1v_4,\ v_2v_3,\ v_2v_4,\ v_3v_4\}$

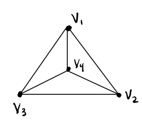
↳ unordered set $(v_1v_2 = v_2v_1)$

Note: for brevity, we write $uv \in E(G)$ for the edge $\{u,v\}$

Graphs are often represented pictorially where $V(G)$ is a set of points or dots and each edge is a segment or curve between points.



if we cared about edges crossing eachother is there another way we could draw this?



⇒ These are 2 different representations of the same graph

A drawing of a graph is not the graph itself.

Use of Graphs

① Routing Problems

V = locations

E = direct passage between locations

② Social networks

V = users

E = "Friends"

- "twitter" → we'd want ordered pairs (one can follow someone and not be followed by that person)

③ Scheduling Problems

E = events

V = pairs of events which cannot coincide

Notation

- Given $v \in V(G)$ and $e \in E(G)$ if $v \in e$ then we say e is incident to v.
- If $uv \in E(G)$ we say u and v are adjacent (or neighbours).
- The neighbourhood of $v \in V(G)$, denoted $N(v)$ is $N(v) = \{u \mid uv \in E(G)\}$
- The degree of $v \in V(G)$ in G is $d(v) = |N(v)|$ (or $\deg(v)$). We sometimes write $N_G(v)$, $d_G(v)$ if we wish to make it clear to which graph we are referring
- In general, $0 \leq \deg(v) \leq |V(G)| - 1$

Degree sequence of G is a list of degrees (in increasing order)

THM: If G is a graph, then $\exists\ u,v \in V(E)$ s.t. $\deg(u) = \deg(v)$

PROOF: Case 1: Suppose $\deg(u) \neq 0 \ \forall u \in V(G)$ Let $|V(G)| = n$

$\Rightarrow 1 \leq \deg(n) \leq n-1$

There are n vertices and $n-1$ possible degrees $\Rightarrow$ by PHP two vertices have the same degree

Case 2: $\exists u$ s.t. $\deg(v) = 0 \Rightarrow 0 \leq \deg(u) \leq n-2$, $n-1$ possible degrees, n vertices $\Rightarrow$ 2 have same degree by PHP.

Handshaking Lemma

In any graph G, there are an even number of vertices with odd degree

THM: If G is a graph then $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

PROOF: Count the number of pairs (v,e) such that e is incident to v.

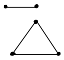
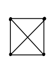
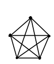
- each vertex v appears $\deg(v)$ times in the set of pairs
- each edge appears exactly twice

$\Rightarrow \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} 2 = 2|E(G)|$ ■

PROOF OF HL: $\sum_{\substack{v \in V(G) \\ \deg(v)\text{ odd}}} \deg(v) + \sum_{\substack{v \in V(G) \\ \deg(v)\text{ even}}} \deg(v) = 2|E(G)| \Rightarrow \sum_{\substack{v \in V(G) \\ \deg(v)\text{ odd}}} \deg(v)$ is even

$\Rightarrow$ the number of vertices in the sum is even ■

Special Graphs

- Empty graph: $E(G) = \emptyset$
- Complete graph: $E(G)$ = all possible pairs (K_n if $|N(G)| = n$)
  - K_1 = •
  - K_2 = 
  - K_3 = 
  - K_4 = 
  - K_5 = 

$|E(K_n)| = \binom{n}{2}$

In general, $0 \leq |E(G)| \leq \binom{n}{2}$

- Complete bipartite graph $K_{m,n}$

$V(K_{m,n}) = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$

$E(K_{m,n}) = \{u_i v_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$

$|V(K_{m,n})| = m+n$

$|E(K_{m,n})| = mn$