MATH 223 Linear Algebra
Midterm - Answer Key

Question 1.

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Question 2.

See lecture notes.

Question 3.

First check that the Pauli matrices are hermitian:

$$P_1^H = \overline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P_1,$$

$$P_2^H = \overline{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = P_2,$$

$$P_3^H = \overline{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P_3,$$

Now let $A \in M_{2\times 2}(\mathbb{C})$ be any hermitian matrix. Write

$$A = \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix},$$

where a's and b's are real numbers. As $A = A^H$, and

$$A^H = \overline{\begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix}}^T = \begin{bmatrix} a_1 - b_1 i & a_2 - b_2 i \\ a_3 - b_3 i & a_4 - b_4 i \end{bmatrix}^T = \begin{bmatrix} a_1 - b_1 i & a_3 - b_3 i \\ a_2 - b_2 i & a_4 - b_4 i \end{bmatrix},$$

we get: $b_1 = b_4 = 0$, $a_2 = a_3$, $b_2 = -b_3$. Thus, A is of the form

$$A = \begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

We look for $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that

$$A = k_1 P_1 + k_2 P_2 + k_3 P_3 + k_4 I.$$

$$\begin{aligned}
\begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix} &= k_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & k_1 \\ k_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -k_2 i \\ k_2 i & 0 \end{bmatrix} + \begin{bmatrix} k_3 & 0 \\ 0 & -k_3 \end{bmatrix} + \begin{bmatrix} k_4 & 0 \\ 0 & k_4 \end{bmatrix} \\
&= \begin{bmatrix} k_3 + k_4 & k_1 - k_2 i \\ k_1 + k_2 i & k_4 - k_3 \end{bmatrix}.
\end{aligned}$$

Then,

$$\begin{cases} a = k_3 + k_4 \\ b = k_1 \\ c = -k_2 \\ d = k_4 - k_3 \end{cases} \qquad \text{so} \qquad \begin{cases} k_1 = b \\ k_2 = -c \\ k_3 = \frac{a-d}{2} \\ k_4 = \frac{a+d}{2} \end{cases}$$

Hence,

$$A = \begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix} = bP_1 - cP_2 + \frac{a-d}{2} P_3 + \frac{a+d}{2} I$$

as desired.

Question 4.

See solution to Assignment 2, Exercise 4.

Question 5.

See lecture notes.