Assignment 3 | **LE, Nhat Hung**
COMP 424 - Artificial Intelligence | McGill ID: 260793376
Prof. Jackie Chi Kit Cheung | Date: March 19, 2019
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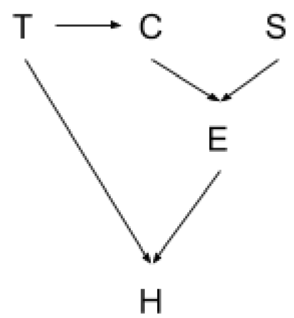
Question 1: Designing a Bayesian Network
Toby the cat is not having a good day. His sister Lucy ate all the food provided by their owner, Claire, so he has to find a way to feed himself. He can try to catch a squirrel outside, and if he succeeds, he is going to eat it. If Toby is tired, he is less likely to catch the squirrel. Toby can also try to steal Claire's sandwich, which takes less effort, so it does not depend on whether he is tired. However even if he succeeds, he might not get to eat it (for example Claire may be quick enough to snatch it back). Finally, if Toby manages to eat at least something, he might feel happy, despite all the events. Though if Toby is tired, he is less likely to feel happy in general.

Consider the Boolean variables: H (happy), E (eats at least one item), C (catches squirrel), S (steals sandwich) and T (tired).

a. Draw a Bayesian network for this domain. Only include the Boolean variables listed above, so your network should have 5 nodes.



b. Is your network a polytree? Why or why not? (A polytree is a graph that has no directed or undirected cycles.)

A polytree is a directed acyclic graph (DAG) whose underlying undirected graph is a tree. In other words, if we replace its directed edges with undirected edges, we obtain an undirected graph that is both connected and acyclic.

The underlying undirected graph has a cycle: TCEH. Therefore, the network is not a polytree.

c. Suppose the probability that Toby catches the squirrel is x when he is tired, and y when he is not tired. Give the conditional probability table associated with C.

$P(C\|T)$		
	$C = 1$	$C = 0$
$T = 1$	x	$1 - x$
$T = 0$	y	$1 - y$

d. Suppose that if Toby catches the squirrel, he will eat it with probability 1, and if he successfully steals a sandwich, he will eat it with probability 0.4. If he fails at both hunting and stealing, then he will not eat anything. Give the conditional probability table associated with E.

$P(E\|C, S)$		
	$E = 1$	$E = 0$
$C = 1, S = 1$	1	0
$C = 1, S = 0$	1	0
$C = 0, S = 1$	0.4	0.6
$C = 0, S = 0$	0	1

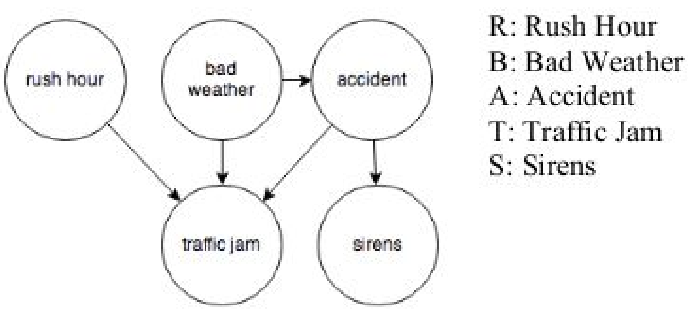
e. Suppose Toby is happy. Write down the expression for the probability that Toby is tired, in terms of the various conditional probabilities in the network.

$$\begin{aligned} P(T=1|H=1) &= \frac{\sum_{c,e,s} \mathrm{P}(T=1, H=1, C, E, S)}{\sum_{t,c,e,s} \mathrm{P}(T, H=1, C, E, S)} \\ &= \frac{\sum_{c,e,s} \mathrm{P}(T=1)\mathrm{P}(H=1|T=1, E)\mathrm{P}(C|T=1)\mathrm{P}(E|C,S)\mathrm{P}(S)}{\sum_{t,c,e,s} \mathrm{P}(T)\mathrm{P}(H=1|T,E)\mathrm{P}(C|T)\mathrm{P}(E|C,S)\mathrm{P}(S)} \end{aligned}$$

with E shorthand for $E = e$, etc..

Question 2: Inference in Bayesian Networks

Consider the following Bayesian Network



We will denote random variables with capital letters (e.g., R), and the binary outcomes with lowercase letters (e.g., r, and ¬r).

The network has the following parameters:

P(b)=0.3
P(r)=0.15
P(t|r, b,a) = 0.98
P(t|r, ¬b,a) = 0.9
P(t|r,b,¬a) = 0.88
P(t|r,¬b,¬a) = 0.85

P(t|¬r,b,a) = 0.5
P(t|¬r,b,¬a) = 0.4
P(t|¬r,¬b,a) = 0.6
P(t|¬r,¬b,¬a) = 0.05
P(s|a) = 0.9
P(s|¬a) = 0.2
P(a|b) = 0.6
P(a|¬b) = 0.3

Compute the following terms using basic axioms of probability and the conditional independence properties encoded in the above graph. You can use Bayes Ball properties to simplify the computation, if applicable.

a. P(a, r)

$$\begin{aligned}
\mathrm{P}(a,r) &= \sum_{B,T,S} \mathrm{P}(a,r,B,T,S) \\
&= \sum_{B,T,S} \mathrm{P}(a|B)\mathrm{P}(r)\mathrm{P}(B)\mathrm{P}(T|B,a,r)\mathrm{P}(S) \\
&= \mathrm{P}(r)\sum_{B,T} \mathrm{P}(a|B)\mathrm{P}(B)\mathrm{P}(T|B,a,r)\sum_{S}\mathrm{P}(S) \\
&= 0.15\sum_{B,T} \mathrm{P}(a|B)\mathrm{P}(B)\mathrm{P}(T|B,a,r)(1) \\
&= 0.15\sum_{B} \mathrm{P}(a|B)\mathrm{P}(B)\sum_{T}\mathrm{P}(T|B,a,r) \\
&= 0.15\sum_{B} \mathrm{P}(a|B)\mathrm{P}(B) \\
&= (0.15)(\mathrm{P}(a|b)\mathrm{P}(b)+\mathrm{P}(a|\neg b)\mathrm{P}(\neg b)) \\
&= (0.15)((0.6)(0.3)+(0.3)(0.7)) \\
&= 0.0585
\end{aligned}$$

b. P(b, ¬a)

$$\begin{aligned}
\mathrm{P}(b,\neg a) &= \sum_{R,T,S} \mathrm{P}(\neg a,b,R,T,S) \\
&= \sum_{R,T,S} \mathrm{P}(\neg a|b)\mathrm{P}(b)\mathrm{P}(R)\mathrm{P}(T|R,\neg a,b)\mathrm{P}(S|\neg a) \\
&= \mathrm{P}(\neg a|b)\mathrm{P}(b) \\
&= (0.4)(0.3) \\
&= 0.12
\end{aligned}$$

c. P(b|s)

$$\begin{aligned}
\mathrm{P}(b|s) &= \mathrm{P}(b,s)/\mathrm{P}(s) \\
&= \frac{\sum_{R,T,A}\mathrm{P}(s,b,R,T,A)}{\sum_{B,R,T,A}\mathrm{P}(s,B,R,T,A)} \\
&= \frac{\sum_{R,T,A}\mathrm{P}(s|A)\mathrm{P}(b)\mathrm{P}(R)\mathrm{P}(T|R,b,A)\mathrm{P}(A|b)}{\sum_{B,R,T,A}\mathrm{P}(s|A)\mathrm{P}(B)\mathrm{P}(R)\mathrm{P}(T|R,B,A)\mathrm{P}(A|B)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{P(b)\sum_A P(s|A)P(A|b)}{\sum_B P(B)\sum_A P(s|A)P(A|B)} \\
&= \frac{(0.3)((0.9)(0.6)+(0.2)(0.4))}{(0.3)((0.9)(0.6)+(0.2)(0.4))+(0.7)((0.9)(0.3)+(0.2)(0.7))} \\
&\approx 0.393
\end{aligned}
$$

Question 3: Variable Elimination

For the graph above, compute the MAP result of querying P(T|b) using variable elimination with the following order: S, A, R, T.

Clearly explain each step. For each of the intermediate factors created, explain what probabilistic function it represents.

Take the variable ordering S, A, R, B, T, because I don't know what to do with just S, A, R, T.

Active factor list:

$$P(S|A), P(A|B), P(R), P(T|R,B,A), P(B), \delta(B,1)$$

with

$$\delta(B,1)$$

the evidence potential of the evidence variable B, and

$$\delta(x,y) = 1 \text{ iff } x = y \text{ else } 0$$

Eliminate S:

$\cancel{P(S|A)}, P(A|B), P(R), P(T|R,B,A), P(B), \delta(B,1),$ $\boldsymbol{m_S(A)}$ $\quad m_S(A) = \Sigma_S P(S|A)$

Eliminate A:

$\cancel{P(A|B)}, P(R), \cancel{P(T|R,B,A)}, P(B), \delta(B,1), \cancel{m_S(A)},$ $\boldsymbol{m_A(T,R,B)}$ $\quad m_A(T,R,B) = \Sigma_A P(A|B)P(T|R,B,A)m_S(A)$

Eliminate R:

$\cancel{P(R)}, P(B), \delta(B,1), \cancel{m_A(T,R,B)},$ $\boldsymbol{m_R(T,B)}$ $\quad m_R(T,B) = \Sigma_R P(R)m_A(T,R,B)$

Eliminate B:

$\cancel{P(B), \delta(B,1), m_R(T,B)},$ $\boldsymbol{m_B(T)}$ $\quad m_B(T) = \Sigma_B P(B)\delta(B,1)m_R(T,B)$

Query T:

$$
\begin{aligned}
m_B(t) &= \sum_B P(B)\delta(B,1)m_R(t,B) \\
&= P(b)m_R(t,b) \\
&= P(b)\sum_R P(R)m_A(t,R,b) \\
&= P(b)\sum_R P(R)\sum_A P(A|b)P(t|R,b,A)m_S(A)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{P}(b)\sum_R \mathrm{P}(R)\sum_A \mathrm{P}(A|b)\mathrm{P}(t|R,b,A)(1) \\
&= \mathrm{P}(b)\sum_R \mathrm{P}(R)[\mathrm{P}(a|b)\mathrm{P}(t|R,b,a) + \mathrm{P}(\neg a|b)\mathrm{P}(t|R,b,\neg a)] \\
&= \mathrm{P}(b)(\mathrm{P}(r)[\mathrm{P}(a|b)\mathrm{P}(t|r,b,a) + \mathrm{P}(\neg a|b)\mathrm{P}(t|r,b,\neg a)] \\
&+\mathrm{P}(\neg r)[\mathrm{P}(a|b)\mathrm{P}(t|\neg r,b,a) + \mathrm{P}(\neg a|b)\mathrm{P}(t|\neg r,b,\neg a)]) \\
&= 0.3(0.15[(0.6)(0.98) + (0.4)(0.88)] + 0.85[(0.6)(0.5) + (0.4)(0.4)]) \\
&= 0.1596
\end{aligned}
$$

$$
\begin{aligned}
m_B(\neg t) &= \mathrm{P}(b)(\mathrm{P}(r)[\mathrm{P}(a|b)\mathrm{P}(\neg t|r,b,a) + \mathrm{P}(\neg a|b)\mathrm{P}(\neg t|r,b,\neg a)] \\
&+\mathrm{P}(\neg r)[\mathrm{P}(a|b)\mathrm{P}(\neg t|\neg r,b,a) + \mathrm{P}(\neg a|b)\mathrm{P}(\neg t|\neg r,b,\neg a)]) \\
&= 0.3(0.15[(0.6)(0.02) + (0.4)(0.12)] + 0.85[(0.6)(0.5) + (0.4)(0.6)]) \\
&= 0.1404
\end{aligned}
$$

$$m_B(t) > m_B(\neg t)$$

Therefore,

$$\mathrm{MAP}(T|b) = \mathrm{argmax}_T \mathrm{P}(T|b) = t$$