MATH 323 - EXERCISES 5- SOLUTIONS

1 Need to show p non-negative, and sums to one over the range of Y. Sum of geometric progression gives result, that is,

$$\sum_{y=0}^{\infty} p(y) = \sum_{y=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^y = \frac{1}{1+\lambda}\left(1-\frac{\lambda}{1+\lambda}\right)^{-1} = 1$$

Distribution function F given, for $y = 0, 1, \ldots$ by

$$F(y) = \sum_{i=0}^{y} p(i) = \sum_{i=0}^{y} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^i = (1-p)\sum_{i=0}^{y} p^i = (1-p)\frac{1-p^{y+1}}{1-p} = 1 - p^{y+1}$$

where $p = \lambda/(1+\lambda)$. This is the *Geometric* distribution in a slightly different form.

2 $X = Y - n \implies \mathcal{X} = \{0, 1, 2, \ldots\}$ and

$$p(x) = P(X = x) = P(Y - n = x) = P(Y = x + n) = \binom{n+x-1}{n-1} p^n (1-p)^x$$

3 We can see that the approximation works for $Y \sim Hypergeometric(N, R, n)$ by manipulating the combinatorial terms. For $y \in \{\max(0, n-N+R), \ldots, \min(n, R)\}$

$$\begin{aligned} p(y) &= \frac{\binom{R}{y}\binom{N-R}{n-y}}{\binom{N}{n}} = \frac{R!}{y!(R-y)!}\frac{(N-R)!}{(n-y)!(N-R-n+y)!}\frac{n!(N-n)!}{N!} \\ &= \binom{n}{y}\frac{R!}{(R-y)!}\frac{(N-R)!}{(N-R-n+y)!}\frac{(N-n)!}{N!} \\ &\approx \binom{n}{y} R^y (N-R)^{n-y}\frac{1}{N^n} = \binom{n}{y} p^y (1-p)^{n-y} \end{aligned}$$

where $p = R/N$. Now for N and R large, with $p = R/N$ fixed, the hypergeometric distribution is tends to the binomial distribution, and thus sampling without replacement from a large population is approximately equivalent to sampling with replacement.

4 Let U ="Number of Heads"; U represents the number of 'successes' in a sequence of binary trials, so from lectures we know that

$$U \sim Binomial(n, 1/2),$$

and $Y = U - (n - U) = 2U - n$. Thus

$$\mathcal{Y} = \{-n, -n+2, -n+4, \ldots, n-4, n-2, n\}$$

and for $y \in \mathcal{Y}$,

$$p(y) = P(Y = y) = P(2U - n = y) = P(U = (y+n)/2) = \binom{n}{(y+n)/2}\left(\frac{1}{2}\right)^n$$

with $p(y) = 0$ otherwise.

5 If $Y \sim Geometric(p)$, then

$$p(y) = (1-p)^{y-1}p \qquad\qquad F(y) = 1-(1-p)^y$$

for $y \in \{1, 2, 3 \ldots\}$. Thus $P(Y > n) = (1-p)^n$, and hence

$$\begin{aligned} P(Y = n+k \mid Y > n) &= \frac{P(Y = n+k, Y > n)}{P(Y > n)} = \frac{P(Y = n+k)}{P(Y > n)} \\ &= \frac{(1-p)^{n+k-1}p}{(1-p)^n} = (1-p)^{k-1}p = P(Y = k\ ) \end{aligned}$$

6 Need $p(y)$ non-negative and convergent;

(i) always non-negative and convergent; need $k = 1$
(ii) always non-negative and convergent if $\alpha < -1$; no closed form for k.

7 (i) $Y = \min\{Y_1, \ldots, Y_n\}$, so $\mathcal{Y} = \{0, 1\}$.

$$\begin{aligned} P(Y = 1) &= P(\min\{Y_1, \ldots, Y_n\} = 1) = P(Y_1 = 1, Y_2 = 1, \ldots, Y_n = 1) = p^n \\ P(Y = 0) &= 1 - p^n \end{aligned}$$

Hence

$$p(y) = \begin{cases} 1 - p^n & y = 0 \\ p^n & y = 1 \end{cases}$$

(ii) $Z = \max\{Y_1, \ldots, Y_n\}$, so $\mathcal{Z} = \{0, 1\}$.

$$\begin{aligned} P(Z = 0) &= P(\max\{Y_1, \ldots, Y_n\} = 0) = P(Y_1 = 0, Y_2 = 0, \ldots, Y_n = 0) = (1-p)^n \\ P(Z = 1) &= 1 - (1-p)^n \end{aligned}$$

Hence

$$p(z) = \begin{cases} (1-p)^n & z = 0 \\ 1 - (1-p)^n & z = 1 \end{cases}$$

8 We have that the function I_A is a mapping acting on S, $I_A : S \longrightarrow \{0, 1\}$ and

$$P(I_A(s) = 1) \equiv P(s \in A) = P(A) = p$$

and hence I_A is a Bernoulli random variable with parameter p. If Y is discrete, can always write $\mathcal{Y} = \{y_1, y_2, \ldots\}$ as a list of those values for which $p(y) > 0$, and then express

$$Y = \sum_{i=1}^{\infty} I_{A_i} y_i$$

where $A_i = \{s : Y(s) = y_i\}$.