COMP 350 Numerical Computing

Assignment #3: Solving linear systems.
Date given: Thursday, September 27. Date due: Thursday, October 11, 2018, 11:59pm
Responsible TAs: Mr. Zhilong Chen, Mr. Sitao Luan (zhilong.chen, sitao.luan@mail.mcgill.ca)
TA office hours: Thursday 4:00pm–5:30pm, Trottier 3110.

1. (a) (3 points) Solve the following system using GEPP (Gaussian elimination with partial pivoting):

$$A = \begin{bmatrix} 3 & 1 & -3 & 15 \\ -1 & -3 & -4 & 4 \\ 4 & 4 & -4 & 8 \\ 2 & -2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \\ 8 \\ 16 \end{bmatrix}$$

   Show intermediate matrices, vectors and multipliers at each step.

   (b) (2 points) Compute the LU factorization of the matrix in the previous question with partial pivoting: $PA = LU$. You have to show intermediate results at each step. This question and the previous one can be answered together.

   Note: Do the computations by your hands and don't consider any rounding errors.

2. (3 points) Suppose we have a complex linear system $Ax = b$, where $A \in \mathbb{C}^{n\times n}$ is nonsingular and $b \in \mathbb{C}^n$. Can you solve it in real arithmetic operations? What is the cost?

   Hint: Rewrite $Ax = b$ as an equivalent $2n \times 2n$ real linear system.

3. Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular.

   (a) (4 points) Given $B \in \mathbb{R}^{n\times p}$, show how to use the LU factorization with partial pivoting to solve $AX = B$. What is the cost of your method?

   Hint: $AX = B$ is equivalent to $AX(:, j) = B(:, j)$ for $j = 1 : p$.

   (b) (5 points) Use `lupp.m` given in the lecture notes to solve $AX = B$, where

$$10 \times 10 \;\; \text{Hilbert matrix} \;\; A = (a_{ij}),\; a_{ij} = 1/(i + j - 1);$$

$$10 \times 5 \;\; B = A * \texttt{randn}(10, 5).$$

   Here `randn` is a MATLAB built-in function to generate a random matrix. Denote this $\texttt{randn}(10, 5)$ by X_t and your computed solution by X_c.

   - Compute $\|X_c - X_t\|_F/\|X_t\|_F$ and $\epsilon\|A\|_F\|A^{-1}\|_F$, where ϵ is the machine epsilon. Check MTALAB built-in functions or constants `norm`, `cond` and `eps`, to see how to compute or get related quantities.
   - Compute the relative residual $\|B - AX_{comp}\|_F/(\|A\|_F\|X_{comp}\|_F)$.

   i. Run your code 10 times (you may use a loop). Notice each time you have different B, since X_{true} is random. Answer the following questions:
   ii. Do you see any rough relation between $\|X_c - X_t\|_F/\|X_t\|_F$ and $\epsilon\|A\|_F\|A^{-1}\|_F$?
   iii. Do you see any rough relation between $\|B - AX_c\|_F/(\|A\|_F\|X_c\|_F)$ and ϵ ?

   Print out your MATLAB code and the results.

(c) Computing A^{-1}

i. (3 points) Show how to use the LU factorization with partial pivoting to compute the inverse of a general $n \times n$ nonsingular matrix A. What is the cost of your method? (Hint: Think about what matrix equation $AX = B$ you should solve to get A^{-1}).
Compute the inverse of a 5×5 Hilbert matrix defined in 3(b). Print out your MATLAB code and the result.

ii. (3 bonus points) For the sake of simplicity, suppose we use the LU factorization with no pivoting to compute A^{-1}. Briefly state an algorithm which costs $2n^3$ flops. You need to explain why its cost is $2n^3$ flops.