MATH 323 - MID TERM
SAMPLE QUESTIONS: SOLUTIONS

1. (a) *Four letters are to be selected from the first eight letters of the alphabet, abcdefgh. How many distinct letter patterns can be formed if the selection is made* ***without replacement*** *?*

   By the multiplication principle, this is $8 \times 7 \times 6 \times 5 = 8!/4! = P_4^8 = 1680$. 2 Marks

   How many patterns are possible if the selection is made ***with replacement*** *?*

   By the multiplication principle, this is $8 \times 8 \times 8 \times 8 = 8^4 = 4096$. 2 Marks

   (b) *How many* ***distinct*** *letter patterns can be created by rearranging the letters abcccdde ?*

   This is
$$\frac{8!}{3! \times 2! \times 1! \times 1! \times 1!} = 3360.$$

   There are $8!$ permutations of the letters, but we do not want to double count the multiple permutations where different copies of the same letter. For example, labelling the three cs c_1, c_2, c_3 and the two ds d_1, d_2, we want to avoid counting the sequences

$$\begin{array}{cccccccc}
a & b & c_1 & c_2 & c_3 & d_1 & d_2 & e \\
a & b & c_2 & c_1 & c_3 & d_1 & d_2 & e \\
a & b & c_2 & c_3 & c_1 & d_1 & d_2 & e \\
 & & & & \vdots & & & \\
a & b & c_3 & c_2 & c_1 & d_2 & d_1 & e
\end{array}$$

   as being different. There are $3! \times 2!$ such sequences. 3 Marks

   (c) *In each month of a given year, a company selects one from six contractors to fulfil a contract in that month, with all selections being equally likely within a month, and with selections made in successive months being independent.*

   Compute the probability that a given contractor is selected ***fewer than three times*** *during the year.*

   In any month, the probability that the contractor is selected is $1/6$. To be selected fewer than three times, the contractor needs to be selected precisely zero, one or two times: these possibilities form a partition of the event of interest. There are C_y^{12} sequences of selections where the contractor is selected precisely y times during the twelve months in the year, and the probability of being selected precisely y times **in a specified collection of months** is

$$\left(\frac{1}{6}\right)^y \left(\frac{5}{6}\right)^{12-y}$$

   (i.e. the contractor is selected y times with probability $1/6$ and not selected $12 - y$ times with probability $1 - 1/6 = 5/6$.) Therefore, using the partition, the required probability is

$$\binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12-0} + \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{12-1} + \binom{12}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{12-2}$$

   or

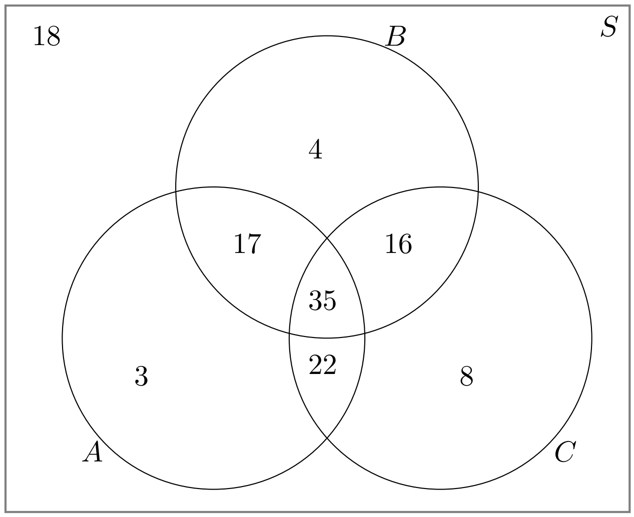
$$\frac{5^{12}}{6^{12}} + 12 \times \frac{5^{11}}{6^{12}} + 66 \times \frac{5^{10}}{6^{12}} = 0.6774262$$

   3 Marks

2. *A total of 123 people were interviewed as part of an urban mass transportation study.*

   - *Some people live more that 10km from city centre (event A),*
   - *some people regularly drive to work (event B),*
   - *some people would use a public mass transportation if it was available (event C).*

   The Venn diagram below represents the counts of people that fall into each partition element:



This information can be displayed in the following table for ease of computation:

	C: Yes		C: No	
	B: Yes	B: No	B: Yes	B: No
A: Yes	35	22	17	3
A: No	16	8	4	18

A further questionnaire is going to be given to one person who is selected from the 123 people in the study population. Compute the probabilities of the following events:

(a) *the selected person drives to work;*

This is $P(B) = (4 + 17 + 35 + 16)/123 = 72/123 = 0.5853.$ 2 Marks

(b) the selected person drives to work and would use a public mass transportation system if it were available;

This is $P(B \cap C) = (35 + 16)/123 = 51/123 = 0.4146.$ 2 Marks

(c) *the selected person lives more than 10km from the city centre;*

This is $P(A) = (3 + 17 + 35 + 22)/123 = 77/123 = 0.6260.$ 2 Marks

(d) *the selected person lives more than 10km from the city centre, drives to work, but would not use a mass transportation system.*

This is $P(A \cap B \cap C') = 17/123 = 0.1382.$ 2 Marks

Write down a symbolic expression of the event "exactly of two of A, B and C occur", and compute the probability of this event.

$$(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)$$

which happens with probability

$$\frac{17}{123} + \frac{22}{123} + \frac{16}{123} = \frac{55}{123} = 0.4472.$$

2 Marks

3. *In a certain population, a genetic marker at a given genomic locus takes one of three forms (or genotypes), AA, Aa, or aa, and the genotype for an individual is thought to determine their risk of a specific disease in later life.*

- *60% of genotype AA individuals develop the disease;*
- *35% of genotype Aa individuals develop the disease;*
- *5% of genotype aa individuals develop the disease.*

The proportions of type AA and aa individuals are 0.04 and 0.64 respectively.

Assuming that the disease mechanism described above is correct:

(a) *What proportion of the population develop the disease ?*

This is a Theorem of Total Probability question. Let D denote the event that a person from the population develops the disease, with AA, Aa and aa denoting the events that the person has the denoted genotype. Then we have that

$$P(AA) = 0.04 \qquad P(aa) = 0.64 \qquad P(Aa) = 1 - P(AA) - P(aa) = 0.32$$

and hence

$$\begin{aligned} P(D) &= P(D|AA)P(AA) + P(D|Aa)P(Aa) + P(D|aa)P(aa) \\ &= (0.6 \times 0.04) + (0.35 \times 0.32) + (0.05 \times 0.64) = 0.168. \end{aligned}$$

3 Marks

(b) *Given that a person develops the disease, what is the probability that their genotype is AA ?*

This is a Bayes Theorem (or conditional probability) question. We have

$$P(AA|D) = \frac{P(D|AA)P(AA)}{P(D)} = \frac{(0.6 \times 0.04)}{0.168} = 0.1429.$$

3 Marks

Suppose that new research determines that in fact, the disease can also develop due to non genetic reasons: each person may develop the disease with probability 0.1 ***irrespective*** *of their genotype, and only 70% of cases are determined by genetics.*

In light of the new research, what proportion of population develop the disease ?

This is also a Theorem of Total Probability question. Let G denote the event that the disease is determined by genetic causes. Then we have

$$P(D) = P(D|G)P(G) + P(D|G')P(G')$$

with $P(G) = 0.70$. Now $P(D|G)$ is precisely what was computed in (a): formally, we have

$$\begin{aligned} P(D|G) &= P(D|G \cap AA)P(AA|G) + P(D|G \cap Aa)P(Aa|G) + P(D|G \cap aa)P(aa|G) \\ &= P(D|G \cap AA)P(AA) + P(D|G \cap Aa)P(Aa) + P(D|G \cap aa)P(aa) \end{aligned}$$

as G relates to the disease-causing mechanism, not the genotype. Thus as $P(D|G') = 0.1$,

$$P(D) = (0.168 \times 0.70) + (0.1 \times 0.30) = 0.1476.$$

4 Marks

4. *A University committee is to comprise five members drawn from Faculties of Arts, Engineering and Science. Each Faculty nominates four professors for consideration for the committee, and the committee is selected with all selections of five members from the nominees being equally likely.*

We want to select five professors from twelve, with at most four from the Science Faculty.

(a) *Let discrete random variable Y denote the number of Science professors selected for the committee. Write down the set of possible values, denoted $\mathcal{Y}$, that Y can take and for which the probability is positive.*

We have that $\mathcal{Y} = \{0, 1, 2, 3, 4\}$: it is possible to fill the committee with non-Science professors, but also that all of the four science professors could be selected.

2 Marks

(b) *Write down the form of the probability mass function (pmf), $p(y)$, for Y.*

We have that $p(y) = P(Y = y)$. To select precisely y Science professors, that is y Science professors and $5 - y$ non-Science professors, we can achieve this in

$$\binom{4}{y} \times \binom{8}{5-y}$$

ways. The total number of ways of selecting the committee is $\binom{12}{5}$, so

$$p(y) = \frac{\binom{4}{y}\binom{8}{5-y}}{\binom{12}{5}} \qquad y = 0, 1, 2, 3, 4$$

and $p(y) = 0$ otherwise.

3 Marks

(c) *Suppose that discrete random variable Z records the number of different Faculties represented on the committee. Identify the set of possible values, denoted $\mathcal{Z}$, that Z can take and for which the probability is positive, and find $p(z)$ for $z \in \mathcal{Z}$.*

As there are at most four members from each Faculty, we need to have representation from at least two Faculties, and it is possible for all three to be represented, so we must have that $\mathcal{Z} = \{2, 3\}$. Focus on $P(Z = 3)$. The possible configurations of selections of (Arts, Engineering, Science) that can fill the committee are

$$(1, 1, 3) \quad (1, 2, 2) \quad (1, 3, 1) \quad (2, 1, 2) \quad (2, 2, 1) \quad (3, 1, 1)$$

and by the multinomial calculation, we have that the total number of committees for each configuration is

$$\binom{4}{1}\binom{4}{1}\binom{4}{3} + \binom{4}{1}\binom{4}{2}\binom{4}{2} + \binom{4}{1}\binom{4}{3}\binom{4}{1} + \binom{4}{2}\binom{4}{1}\binom{4}{2} + \binom{4}{2}\binom{4}{2}\binom{4}{1} + \binom{4}{3}\binom{4}{1}\binom{4}{1}$$

or

$$64 + 144 + 64 + 144 + 144 + 64 = 624.$$

The total number of committees available is $\binom{12}{5} = 792$, so

$$P(Z = 3) = \frac{624}{792} = 0.7879 \qquad P(Z = 2) = \frac{168}{792} = 0.2121.$$

Thus

$$p(z) = \begin{cases} 0.2121 & z = 2 \\ 0.7879 & z = 3 \end{cases}$$

and $p(z) = 0$ otherwise.

5 Marks