Assignment 3 **LE, Nhat Hung**
MATH 240 February 10, 2018

1. (a)
$$\begin{aligned}(A\backslash B)\backslash C &= (A \cap \overline{B}) \cap \overline{C} \\ &= A \cap (\overline{B} \cap \overline{C}) \\ &= A \cap (\overline{B \cup C}) \\ &= A\backslash(B \cup C)\end{aligned}$$

   (b)
$$\begin{aligned}A \oplus B = A \oplus C &\Rightarrow A \oplus (A \oplus B) = A \oplus (A \oplus C) \\ &\Rightarrow (A \oplus A) \oplus B = (A \oplus A) \oplus C \\ &\Rightarrow \emptyset \oplus B = \emptyset \oplus C \\ &\Rightarrow B = C\end{aligned}$$

   (c) Let $A = D = \{1\}$, $B = C = \{2\}$. Counterexample:
$$\begin{aligned}(1, 1) &\in (A \cup B) \times (C \cup D) \\ &\notin (A \times C) \cup (B \times D)\end{aligned}$$

   Thus, $(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$.

2. (a) i. Let $X = \emptyset$.
   $X \in \wp(A)$, $X \cap X = \emptyset$.
   $(X, X) \notin \mathcal{R}$. $\mathcal{R}$ not reflexive.

$$\begin{aligned}(X, Y) \in \mathcal{R} &\Rightarrow X, Y \in \wp(A) \wedge (X \cap Y) \neq \emptyset \\ &\Rightarrow (Y, X) \in (\wp(A))^2 \wedge (Y \cap X) \neq \emptyset \\ &\Rightarrow (Y, X) \in \mathcal{R} \\ &\Rightarrow \mathcal{R} \text{ symmetric}\end{aligned}$$

   Let $(X, Y) \in \mathcal{R}$, $X \neq Y$.
   $\mathcal{R}$ symmetric, so $(Y, X) \in \mathcal{R}$. Still, $X \neq Y$.
   Thus, $\mathcal{R}$ not antisymmetric.

   Let $(X, Y), (Y, Z) \in \mathcal{R}$, $Z \subseteq Y$ and $X = Y\backslash Z$.
   $X \cap Z = \emptyset$. Thus $(X, Z) \notin \mathcal{R}$.
   $\mathcal{R}$ not transitive.

   $\mathcal{R}$ isn't reflexive, antisymmetric or transitive, thus isn't a partial order nor total order.

   ii. Let $a \in \mathbb{N}$.
   $a|a$. Thus $(a, a) \in \mathcal{R}$. $\mathcal{R}$ reflexive.

   $1|2$. So $(1, 2) \in \mathcal{R}$.
   2 doesn't divide 1.
   Thus $(2, 1) \notin \mathcal{R}$. $\mathcal{R}$ not symmetric.

$$\begin{aligned}(a, b), (b, a) \in \mathcal{R} &\Rightarrow a|b \wedge b|a \\ &\Rightarrow a \leq b \wedge a \geq b \\ &\Rightarrow a = b \\ &\Rightarrow \mathcal{R} \text{ antisymmetric}\end{aligned}$$

$$\begin{aligned}(a,b)(b,c)\in\mathcal{R} &\Rightarrow \exists k,l\in\mathbb{Z}, b=ka\wedge c=lb\\ &\Rightarrow c=(lk)a\\ &\Rightarrow (a,c)\in\mathcal{R}\\ &\Rightarrow \mathcal{R}\text{ transitive}\end{aligned}$$

$\mathcal{R}$, reflexive, antisymmetric and transitive, is thus a partial order.

Take 2 and 3.
$2,3\in\mathbb{N}$, but $(2,3),(2,3)\notin\mathcal{R}$.
Thus, $\mathcal{R}$ not a total order.

(b) $\mathcal{R}=\{(a,b)\in(\mathbb{R}\backslash\{0\})^2|\frac{a}{b}\in\mathbb{Q}\}$

Let $x\in\mathbb{R}\backslash\{0\}$. Then, $\frac{x}{x}\in\mathbb{Q}$.
Thus $(x,x)\in\mathcal{R}$. $\mathcal{R}$ reflexive.

$$\begin{aligned}(x,y)\in\mathcal{R} &\Rightarrow \tfrac{x}{y}\in\mathbb{Q}\\ &\Rightarrow \tfrac{y}{x}\in\mathbb{Q}\\ &\Rightarrow (y,x)\in\mathcal{R}\\ &\Rightarrow \mathcal{R}\text{ symmetric}\end{aligned}$$

$$\begin{aligned}(x,y),(y,z)\in\mathcal{R} &\Rightarrow \tfrac{x}{y},\tfrac{y}{z}\in\mathbb{Q}\\ &\Rightarrow \tfrac{xy}{yz}\in\mathbb{Q}\\ &\Rightarrow \tfrac{x}{z}\in\mathbb{Q}\\ &\Rightarrow (x,z)\in\mathcal{R}\\ &\Rightarrow \mathcal{R}\text{ transitive}\end{aligned}$$

$\mathcal{R}$, or $\sim$, reflexive, symmetric and transitive, is thus an equivalence relation.

$$\begin{aligned}\frac{\frac{9-\sqrt{5}}{1-\sqrt{5}}}{\frac{2}{3-6\sqrt{5}}} &= \tfrac{57-57\sqrt{5}}{2-2\sqrt{5}}\\ &= \tfrac{57}{2}\\ &\in\mathbb{Q}\\ &\Rightarrow [\tfrac{9-\sqrt{5}}{1-\sqrt{5}}]=[\tfrac{2}{3-6\sqrt{5}}]\end{aligned}$$

3. (a)

$$\begin{aligned}a\pm b\in\mathbb{Q} &\Rightarrow a\pm b=\tfrac{m}{n};\, m,n\in\mathbb{Z}\\ &\Rightarrow \pm b=\tfrac{m}{n}-a\\ &\Rightarrow \pm b=\tfrac{m-na}{n}\\ &\Rightarrow \pm b\in\mathbb{Q}\end{aligned}$$

which contradicts $b\notin\mathbb{Q}$.
Thus, $a\in\mathbb{Q}\wedge b\notin\mathbb{Q}\Rightarrow a\pm b\notin\mathbb{Q}$.

(b) Let $a,b,c,d\in\mathbb{Q}$

$$\begin{aligned}a\le 11, b\le 10, c\le 9, d\le 8 &\Rightarrow \tfrac{a+b+c+d}{4}\le 9.5\\ &\Rightarrow \tfrac{a+b+c+d}{4}\ne 10\end{aligned}$$

which contradicts "the average of 4 distinct integers is 10".
Thus, if the average of 4 distinct integers is 10, then at least one of the integers is greater than 11.