MATH323 - Assignment 2

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Problem 1. *For the following discrete random variable Y and pmfs p(y) with support sets γ (containing the values y for which p(y) ¿ 0), compute the following quantities:*

(a) γ = {1, 2, 3, ... , 10}, p(y) = cy.
Compute c, E[Y] and Var[Y].

Solution.

$\sum$ p(y) = $\sum$ cy = 1, so $\sum$ y = $\frac{10\times 11}{2} = 55 = \frac{1}{c}$ and c = $\frac{1}{55}$

E[Y] = $\sum yp(y) = \sum cy^2 = \frac{1}{55} \times \frac{10\times 11\times 21}{6} = 7$

var[Y] = $E[(Y - E[Y])^2] = \sum(y - E[Y])^2 p(y) = \frac{1}{55}\sum(y-7)^2 y = \frac{1}{55}\sum(y^3 - 14y^2 + 49y)$
$= \frac{1}{55}[\frac{100\times 121}{4} - 14\frac{10\times 11\times 21}{6} + 49\frac{10\times 11}{2}] = 6$

(b) Y = {1, 2, 3, ... }, $p(y) = \frac{c}{y}(\frac{1}{2})^y$
Compute c and E[Y].

Solution.

$\sum p(y) = \sum \frac{c}{y}(\frac{1}{2})^y = 1, so \sum \frac{(\frac{1}{2})^y}{y} = -ln(1-\frac{1}{2}) = ln(2) = \frac{1}{c}$ and $c = \frac{1}{ln(2)}$

$E[Y] = \sum yp(y) = \sum y\frac{c}{y}(\frac{1}{2})^y = \sum c(\frac{1}{2})^y = \sum \frac{1}{ln(2)}(\frac{1}{2})^y = \frac{1}{2ln2}\sum(\frac{1}{2})^{y-1} = \frac{1}{ln2}$
Using the geometric series formula

(c) Y ∼ Binomial(20, 3/4): compute $E[Y^2]$.

Solution.

$E[Y^2] = E[Y(Y-1)] + E[Y] = n(n-1)p^2 + np$ by the proof seen in class
$= 20(19)(\frac{3}{4})^2 + 20(\frac{3}{4}) = \frac{915}{4}$

(d) Y ∼ Geometric(1/2): compute $E[Y^3]$.

Solution.

We know from the property of an infinite geometric series that
$\sum_{y=1} q^y = \frac{1}{1-q}$
$\frac{d}{dq}(\sum_{y=1} q^y) = \frac{1}{(1-q)^2}$
$\frac{d^2}{dq^2}(\sum_{y=1} q^y) = \frac{2}{(1-q)^3}$

$\frac{d^3}{dq^3}(\sum\limits_{y=1} q^y) = \frac{6}{(1-q)^4}$

Also, E[Y^3] $= E[Y(Y-1)(Y-2) + 3Y^2 - 2Y]$ (2)
$E[Y^2] = E[Y(Y-1)] + E[Y]$ (3)

So by (1) and (2)
$E[Y^3] = E[Y(Y-1)(Y-2)] + 3(E[Y(Y-1)] + E[Y]) - 2E[Y]$
$= E[Y(Y-1)(Y-2)] + 3E[Y(Y-1)] + E[Y]$ (3)

E[Y(Y-1)(Y-2)]
$= \sum y(y-1)(y-2)p(y)$
$= \sum y(y-1)(y-2)pq^{y-1}$
$= pq^2 \sum y(y-1)(y-2)q^{y-3}$
$= pq^2 \frac{d^3}{dq^3}(\sum q^y)$
$= pq^2 \frac{6}{(1-q)^4}$
$= pq^2 \frac{6}{p^4}$
$= q^2 \frac{6}{p^3}$

$3E[Y(Y1)]$
$= 3\sum y(y-1)pq^{y-1}$
$= 3pq \sum y(y-1)q^{y-2}$
$= 3pq \frac{d^2}{dq^2}(\sum q^y)$
$= 3pq \frac{2}{(1-q)^3}$
$= q\frac{6}{p^2}$

$E[Y] =$
$= \sum ypq^{y-1}$
$= p\sum yq^{y-1}$
$= p\frac{d}{dq}(\sum q^y)$
$= p\frac{1}{(1-q)^2}$
$= \frac{1}{p}$

So E[Y(Y-1)(Y-2)]+3E[Y(Y-1)] + E[Y]
$= q^2\frac{6}{p^3} + q\frac{6}{p^2} + \frac{1}{p}$
$= \frac{1}{4}6 \times 8 + \frac{1}{2}6 \times 4 + 2$
$= 12 + 12 + 2$
$= 26$

Problem 2. *Identify the pmfs for the stated random variables in the given experimental conditions.*

(a) A pair of dice are rolled repeatedly until the total score on a given roll is equal to 10. Random variable Y records the number of rolls until the experiment terminates.

Solution.

Geometric distribution
There is 3 possibilities to get 10 : 6-4, 4-6 or 5-5
p = $\frac{3}{6\times 6} = \frac{1}{12}$
q = 1 − p = $\frac{11}{12}$
p(y) = pq^{y-1} = $\frac{1}{12}(\frac{11}{12})^{y-1}$

(b) Two hundred students take a test. The probability that an individual student passes the test is 0.75, with the results of the tests for the two hundred students being considered mutually independent events. Random variable Y records the total number of students who pass the test.

Solution.

Binomial distribution
p(y) $= \binom{n}{y} p^y p^{n-y} = \binom{200}{y} (\frac{3}{4})^y (\frac{1}{4})^{200-y}$

(c) Two hundred students in class comprise 80 Arts students and 120 Science students. Each lecture, the professor selects 10 students at random, with all selections of 10 students being equally likely. Random variable Y records the number of lectures that go by until the selected students have equal representation from Arts and Science (that is, there are five from Arts and five from Science).

Solution.

Geometric distribution
p $= \frac{\binom{120}{5}\binom{80}{5}}{\binom{200}{10}}$
q $= 1 - \frac{\binom{120}{5}\binom{80}{5}}{\binom{200}{10}}$
p(y) $= pq^{y-1} = \frac{\binom{120}{5}\binom{80}{5}}{\binom{200}{10}} (1 - \frac{\binom{120}{5}\binom{80}{5}}{\binom{200}{10}})^{y-1}$

Problem 3. *A new driver can take their driving test at either Test Centre A, where their probability of passing a test is p A , or at Test Centre B, where their probability of passing the test is p B . The driver tosses a fair coin, and if the coin lands Heads, they decide to take the test at Test Centre A; if Tails, they take the test at Test Centre B. The driver continues to take a test using the coin-tossing procedure until they pass the test. Let random variable Y record the number of times the driver takes the test until they pass. Find the pmf of Y .*

Solution.

Geometric distribution
p $= \frac{1}{2}(p_{\mathrm{A}} + p_{\mathrm{B}})$
q $= 1 - \frac{1}{2}(p_{\mathrm{A}} + p_{\mathrm{B}})$
p(y) $= pq^{y-1} = \frac{1}{2}(p_{\mathrm{A}} + p_{\mathrm{B}})(1 - \frac{1}{2}(p_{\mathrm{A}} + p_{\mathrm{B}}))^{y-1}$