Assignment 3 **LE, Nhat Hung**
MATH 323 - Probability McGill ID: 260793376
Prof. David Stephens Date: November 9, 2018
Fall 2018 Due date: November 9, 2018

1. Suppose Y is a continuous random variable that has an Exponential distribution with parameter β :

$$f(y) = \frac{1}{\beta}e^{-y/\beta} \qquad y > 0$$

and $f(y) = 0$ otherwise.

(a) Compute the following probabilities:

(i) $P(Y \leq 2)$ if $\beta = 1$.

In general

$$P(Y \leq y) = \int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{y} \frac{1}{\beta}e^{-t/\beta}dt = 1 - e^{-y/\beta}$$

Then, with $\beta = 1$,

$$P(Y \leq 2) = 1 - e^{-2}$$

(ii) $P(Y > 4)$ if $\beta = 2$.

$$P(Y > y) = 1 - P(Y < y) = 1 - (1 - e^{-y/\beta}) = e^{-y/\beta}$$

Then, with $\beta = 2$,

$$P(Y > 4) = e^{-4/2} = e^{-1/2}$$

(iii) $P(Y > 4 \mid Y > 2)$ if $\beta = 1$.

$$\begin{aligned} P(Y > 4|Y > 2) &= \frac{P(Y > 4 \cap Y > 2)}{P(Y > 2)} \\ &= \frac{P(Y > 4)}{P(Y > 2)} \\ &= \frac{P(Y > 4)}{1 - P(Y \leq 2)} \\ &= \frac{e^{-4}}{e^{-2}} \\ &= e^{-2} \end{aligned}$$

(iv) $P(Y = 3 \mid Y > 2)$ if $\beta = 4$.

$$P(Y = 3|Y > 2) = \frac{P(Y = 3 \cap Y > 2)}{P(Y > 2)} = \frac{P(Y = 3)}{P(Y > 2)} = 0$$

(b) Now suppose that we define the new random variable $X = 3Y$. Find the pdf of X.

$$f_X(x) = \frac{d}{dx} F_Y\left(\frac{x}{3}\right) = \frac{d}{dx}\left(1 - e^{-x/(3\beta)}\right) = \frac{1}{3\beta} e^{-x/(3\beta)}$$

2. Suppose Y is a continuous random variable that has a Normal distribution with parameters $\mu \in R$ and $\sigma \in R^+$:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$$

for $y \in R$.

(a) Using the Normal probability tables (attached, also see page 2.), compute the following probabilities:

(i) $P(Y \leq 2)$ if $\mu = 0,\ \sigma = 1$.

$\mu = 0$ and $\sigma = 1$ give the standard Normal pmf. Therefore,

$$P(Y \leq 2) = 0.9772$$

(ii) $P(Y > 1)$ if $\mu = 1,\ \sigma = 1$.

$$Z = \frac{Y-\mu}{\sigma} = \frac{Y-1}{1} = Y - 1$$

$$P(Y > 1) = P(Y - 1 > 0) = P(Z > 0) = 0.5$$

(iii) $P(Y > 0 \mid Y > 2)$ if $\mu = 0,\ \sigma = 4$.

$$P(Y > 0 | Y > 2) = \frac{P(Y > 0 \cap Y > 2)}{P(Y > 2)} = \frac{P(Y > 2)}{P(Y > 2)} = 1$$

(iv) $P(Y > 2\ or\ Y < -1)$ if $\mu = 0,\ \sigma = 2$.

$$Z = \frac{Y}{2}$$

$$\begin{aligned} P(Z > 1 \cup Z < -0.5) &= P(Z > 1) + P(Z < -0.5) \\ &= P(Z > 1) + P(Z > 0.5) \\ &= 0.1587 + 0.3085 \\ &= 0.4672 \end{aligned}$$

(b) Now suppose that we define the new random variable $X = 2Y - 1$. Find the pdf of X.

$$F_X(x) = F_Y\left(\frac{X+1}{2}\right)$$

$$f_X(x) = \frac{d}{dx} F_Y\left(\frac{x+1}{2}\right) = \frac{1}{2} f_Y\left(\frac{x+1}{2}\right) = \frac{1}{2\sigma\sqrt{2\pi}} exp\left\{-\frac{1}{2\sigma^2}\left(\frac{x+1}{2} - \mu\right)^2\right\}$$

3. Suppose Y is a continuous random variable that has a pdf given by

$$f(y) = c\, exp\{-y - e^{-y}\}$$

for $y \in R$ for some constant c, and cdf

$$F(y) = \int_{-\infty}^{y} c\, exp\{-t - e^{-t}\}dt = c\, exp\{-e^{-y}\}$$

for $y \in R$.

(a) Write down the value of c.

$$\lim_{y\to\infty} F(y) = 1 \Rightarrow \lim_{y\to\infty} c\, exp\{-e^{-y}\} = 1 \Rightarrow c = 1$$

(b) Compute $P(Y > 1)$.

$$P(Y > 1) = 1 - F(1) = 1 - exp\{-e^{-1}\} = 1 - e^{-1/e}$$

(c) Find the pdf of random variable $X = e^{-Y}$, and hence identify the distribution of X.

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(e^{-Y} \leq x) \\ &= P(Y \geq -\ln x) \\ &= 1 - F_Y(-\ln x) \\ &= 1 - exp\{-e^{\ln x}\} \\ &= 1 - e^{-x} \end{aligned}$$

$$\begin{aligned} f_X(x) &= \frac{d}{dx}(1 - e^{-x}) \\ &= e^{-x} \end{aligned}$$

Therefore, X has an Exponential distribution with parameter $\beta = 1$.