COMP 546

Lecture 17

Linear Systems 2:

Fourier transform, convolution theorem, filtering

Thurs. March 21, 2019

Recall last lecture

- convolution
- impulse function $\delta(x - x_0)$
- special behavior of sines and cosines under convolution
- complex numbers and Euler's formula

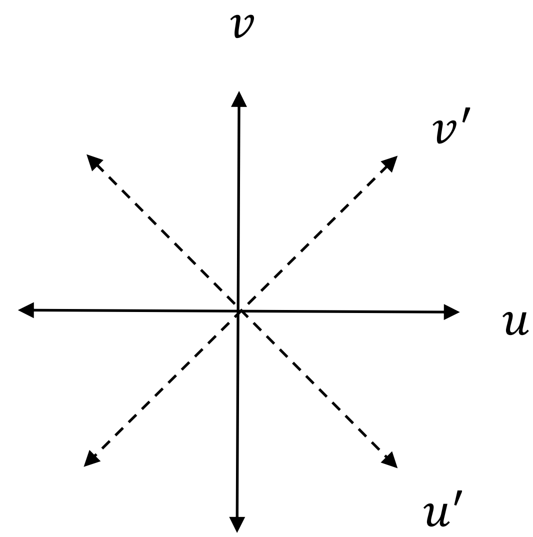
Today

- Fourier transform $\hat{I}(k) = \mathbf{F}\, I(x)$
- convolution theorem
- filtering

Recall from linear algebra: orthonormal basis vectors for a vector space

Example:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$



The inverse is just the transpose:

$$\begin{bmatrix} u \\ V \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$I(x) = \sum_{u=0}^{N-1} \delta(x-u)\, I(u)$$

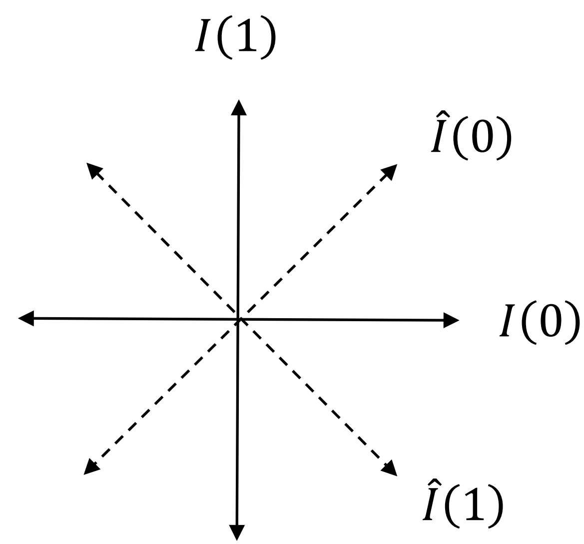
An image can be thought of as a sum of delta functions.

Think of an image with N pixels as a vector in an N-d vector space.

Fourier transform uses ortho*gonal* (but not ortho*normal*) basis vectors

Example $N = 2$:

$$\begin{bmatrix} \hat{I}(0) \\ \hat{I}(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} I(0) \\ I(1) \end{bmatrix}$$



The inverse is just the transpose (with a scale factor):

$$\begin{bmatrix} I(0) \\ I(1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{I}(0) \\ \hat{I}(1) \end{bmatrix}$$

We consider 2N basis vectors for the N-D vector space of images with N pixels.
These basis vectors are sines and cosines.

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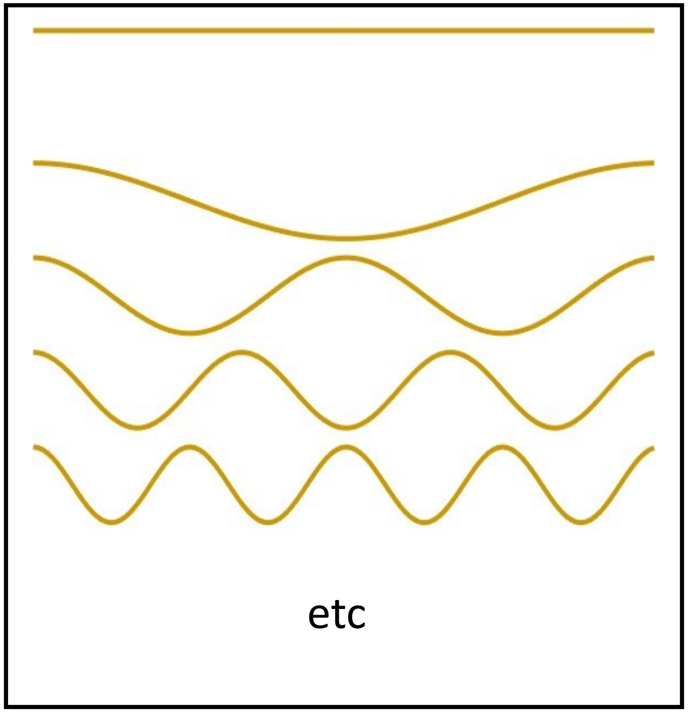

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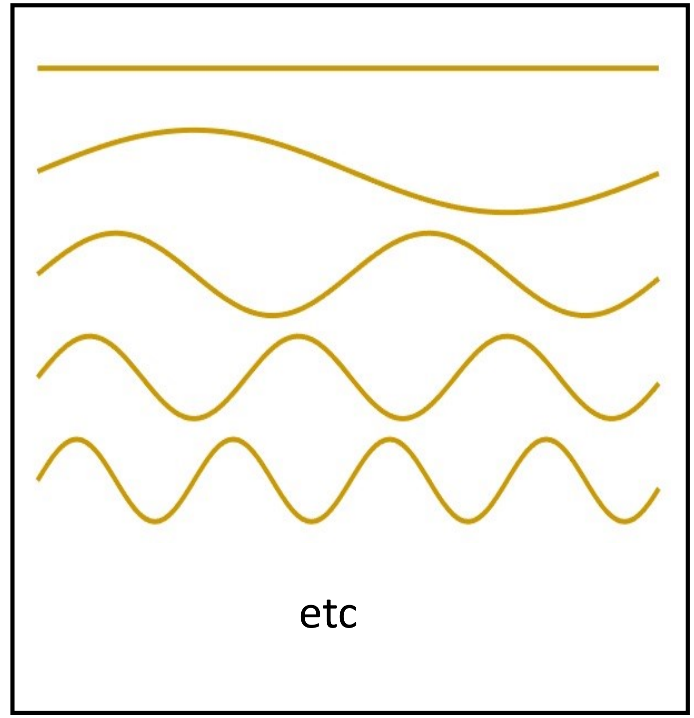


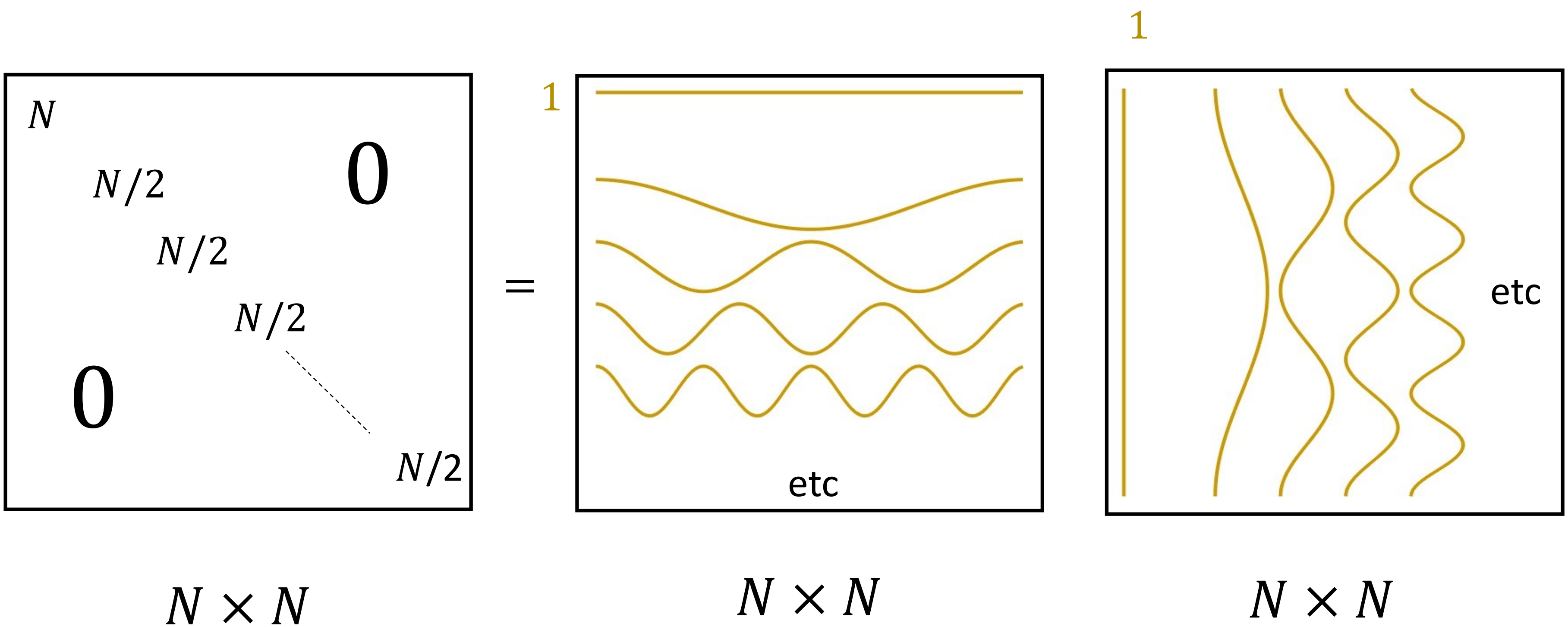
$$N \times N$$

$$N \times N$$

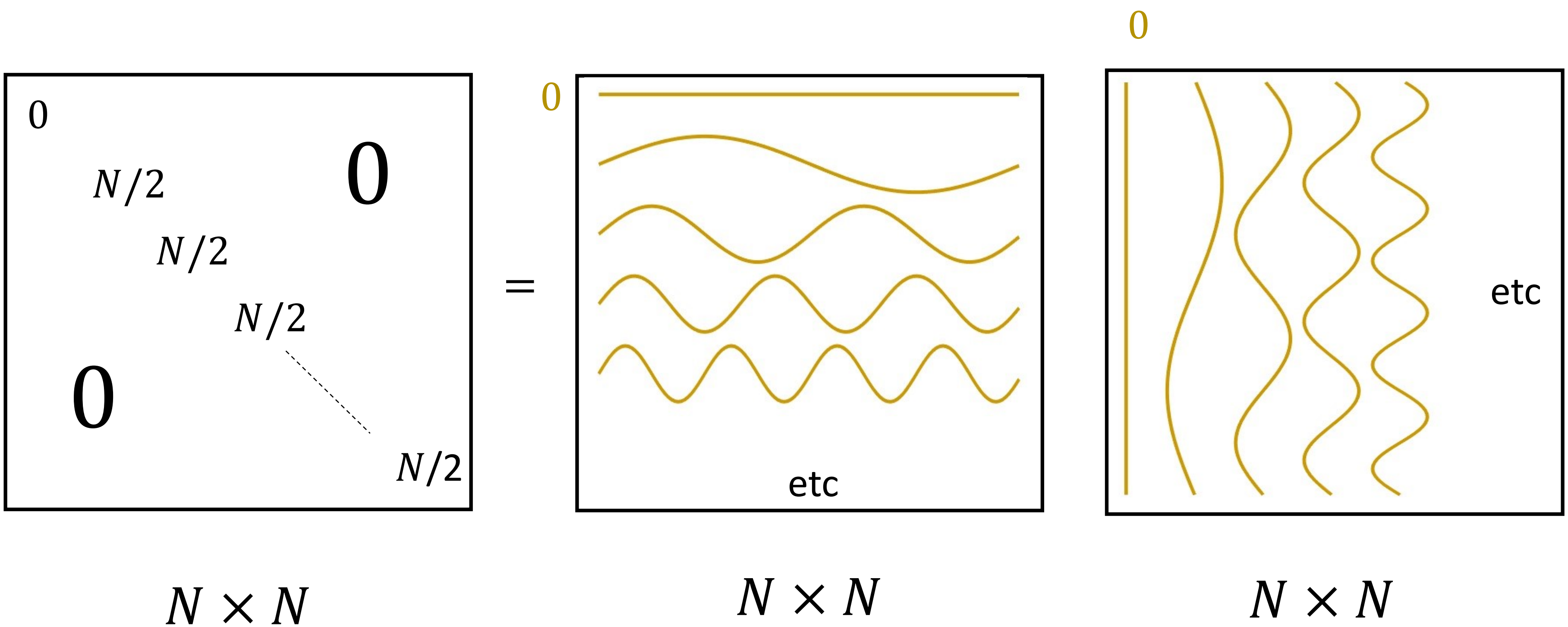
$$\cos\left(\frac{2\pi}{N} kx\right)$$

$$\sin\left(\frac{2\pi}{N} kx\right)$$

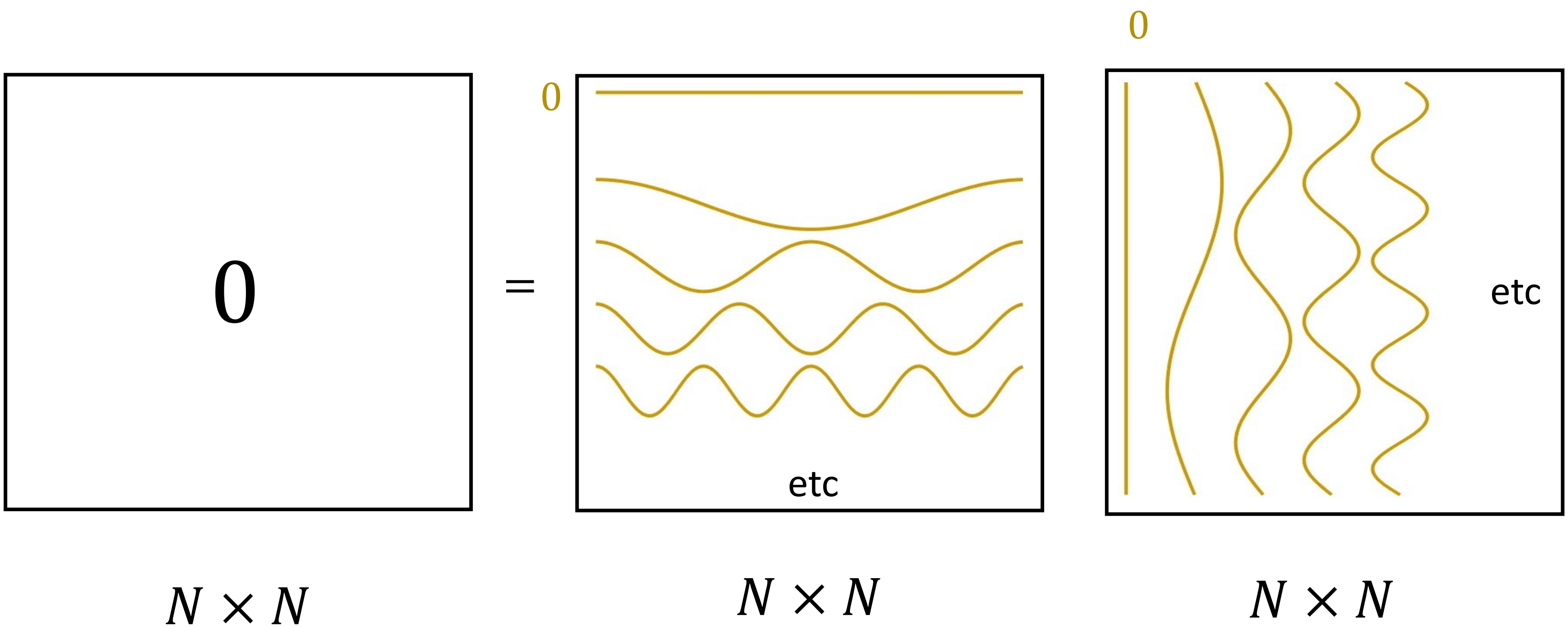
$$\sum_{x=0}^{N-1} \cos\left(\frac{2\pi}{N} k_1 x\right) \cos\left(\frac{2\pi}{N} k_2 x\right) = \ ?$$



$$\sum_{x=0}^{N-1} \sin\left(\frac{2\pi}{N} k_1 x\right) \ \sin\left(\frac{2\pi}{N} k_2 x\right) = \ ?$$



$$\sum_{x=0}^{N-1} \cos\left(\frac{2\pi}{N} k_1 x\right) \sin\left(\frac{2\pi}{N} k_2 x\right) = ?$$



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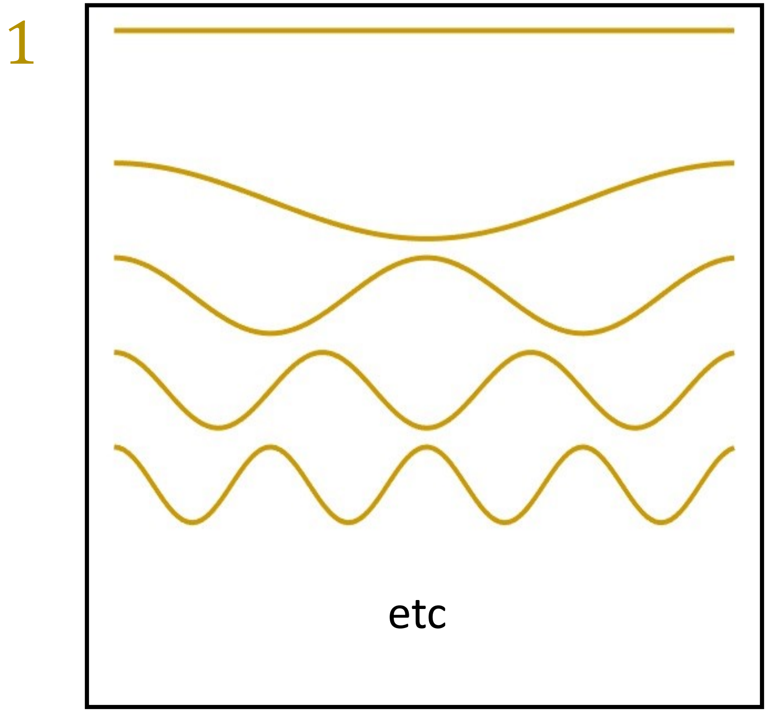
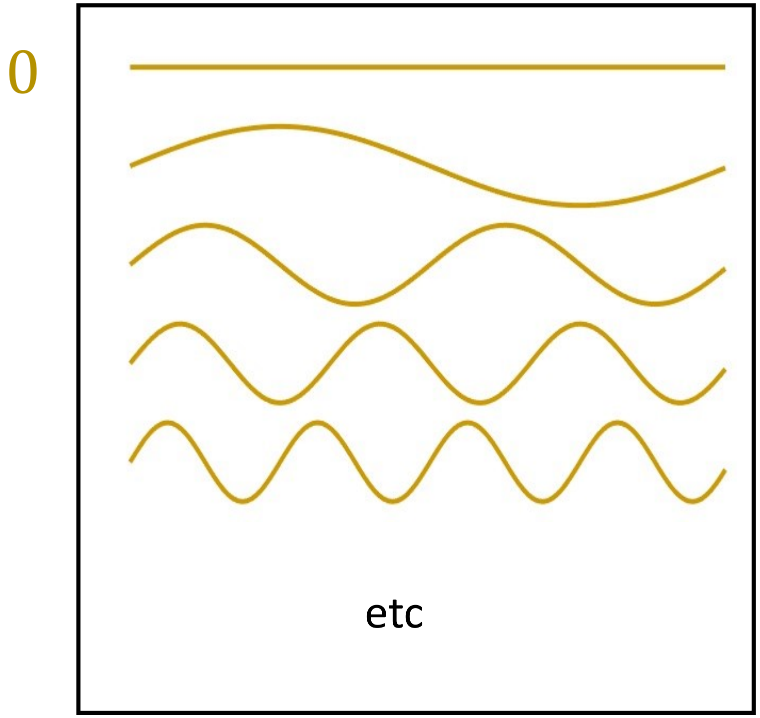




$$N \times N$$

$$N \times N$$

$$\cos\left(\frac{2\pi}{N}kx\right)$$

$$\sin\left(\frac{2\pi}{N}kx\right)$$

Fourier Transform

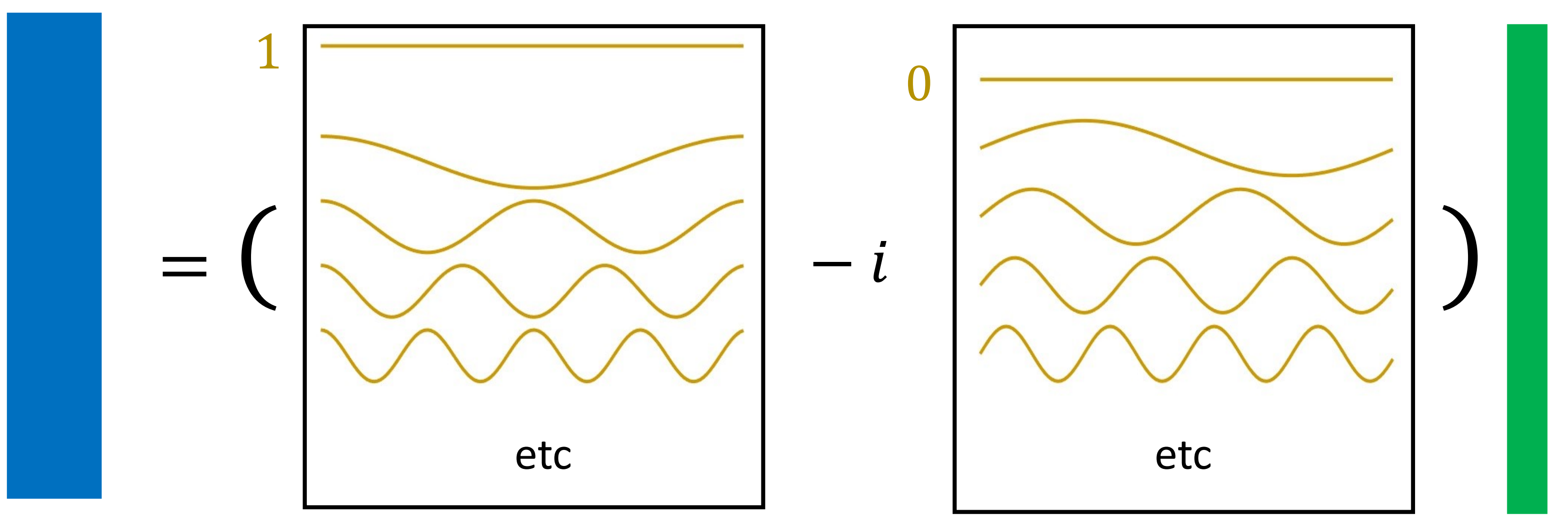
$$\hat{I}(k) = \sum_{x=0}^{N-1} \left(\cos\left(\frac{2\pi}{N} kx\right) - \; i \sin\left(\frac{2\pi}{N} kx\right) \right) I(x)$$

$$e^{-i \frac{2\,\pi}{N} k\,x}$$

$$= \mathbf{F}\, I(x)$$

Project the N-D image $I(x)$ onto 2N vectors of cosines and sines, and keep track of results using real and imaginary components of complex numbers.

$$\hat{I}(k) = \sum_{x=0}^{N-1} \left(\cos\left(\frac{2\pi}{N} kx\right) - i \sin\left(\frac{2\pi}{N} kx\right) \right) I(x)$$



$N \times 1$
(complex)

$N \times N$

$\cos\left(\frac{2\pi}{N} kx\right)$

$N \times N$

$\sin\left(\frac{2\pi}{N} kx\right)$

$N \times 1$

The Fourier transform is well defined for any frequency k.

Let’s look at some of the basic properties:

- Conjugacy property
- Periodicity property

Recall trig identities (see Exercises last lecture)…

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(A + B) = \sin(B)\cos(A) + \sin(A)\cos(B)$$

Recall trig identities   (see Exercises last lecture)…

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(A+B) = \sin(B)\cos(A) + \sin(A)\cos(B)$$

Thus (check for yourself)…

$$\cos\left(\frac{2\pi}{N}(N-k)\,x\right) = \cos\left(\frac{2\pi}{N}k\,x\right)$$

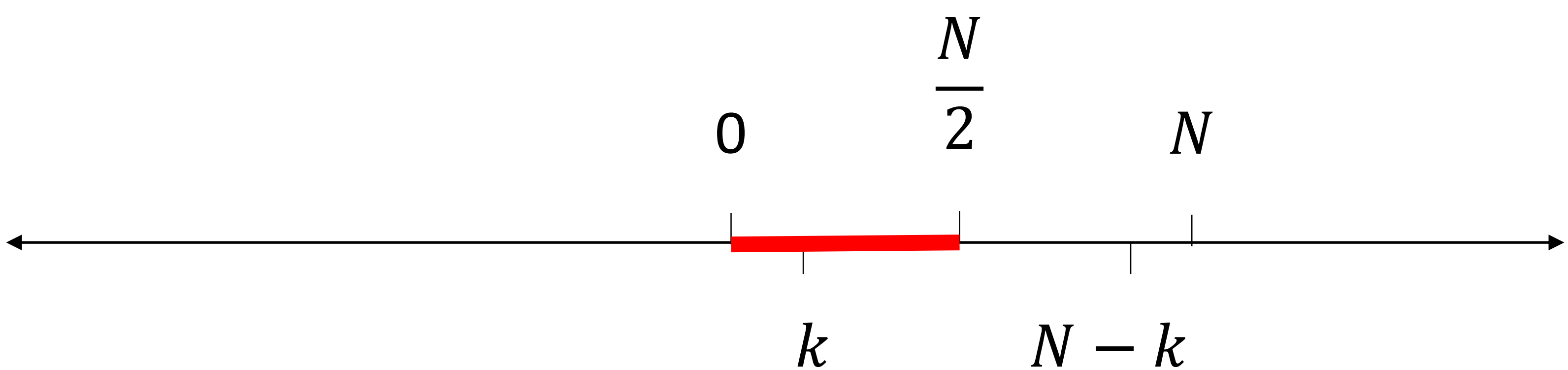
$$\sin\left(\frac{2\pi}{N}(N-k)\,x\right) = -\sin\left(\frac{2\pi}{N}k\,x\right)$$

Conjugacy Property of Fourier transform

Let $h(x)$ be a real valued function.

Then, for any integer k, $\hat{h}(k) = \overline{\hat{h}(N-k)}$.

Proof: see the lecture notes.



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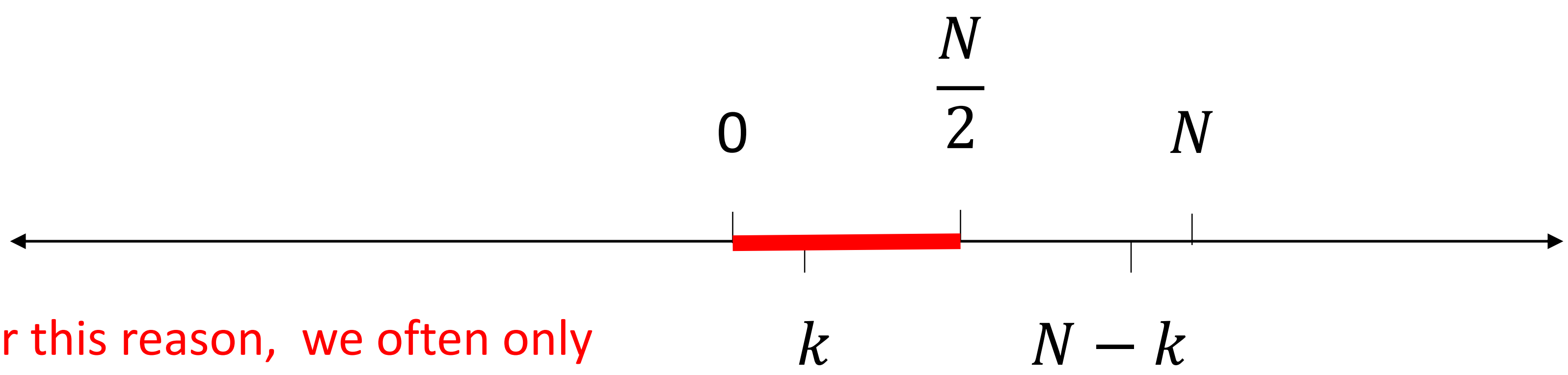
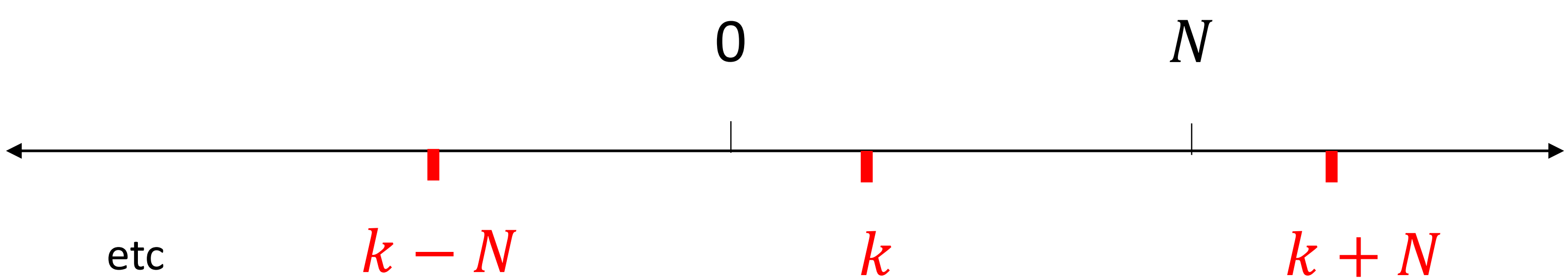


For this reason, we often only talk about the values of the Fourier transform on frequencies from 0 to $\frac{N}{2}$.

Periodicity Property of Fourier transform

For any positive or negative integer m,

$$\hat{h}(k) = \hat{h}(k + mN) .$$



Fourier transform values are the same for all three of these frequencies.

Periodicity Property of Fourier transform

For any positive or negative integer m,

$$\hat{h}(k) = \hat{h}(k + mN) .$$

Proof: Use this:

$$e^{-i\frac{2\pi}{N}(k+mN)x} = e^{-i\frac{2\pi}{N}kx} \underbrace{e^{-i\frac{2\pi}{N}mNx}}_{1}$$

Inverse Fourier transform

Fourier transform

map N-dimensional delta function basis
to an N-dimensional sinusoid function basis

$$\hat{I}(k) = \mathbf{F}\, I(x)$$

$$\begin{bmatrix} \hat{I}(0) \\ \vdots \\ \vdots \\ \hat{I}(N-1) \end{bmatrix} \quad \underset{\mathbf{F}^{-1}}{\overset{\mathbf{F}}{\leftrightarrows}} \quad \begin{bmatrix} I(0) \\ \vdots \\ \vdots \\ I(N-1) \end{bmatrix}$$

$$I(x) = \mathbf{F}^{-1} \hat{I}(k)$$

Inverse Fourier transform

The N x N Fourier transform can be represented as a matrix

$$\mathbf{F}_{k,x} \equiv e^{-i\frac{2\pi}{N}kx}$$

Claim: (see lecture notes for proof)

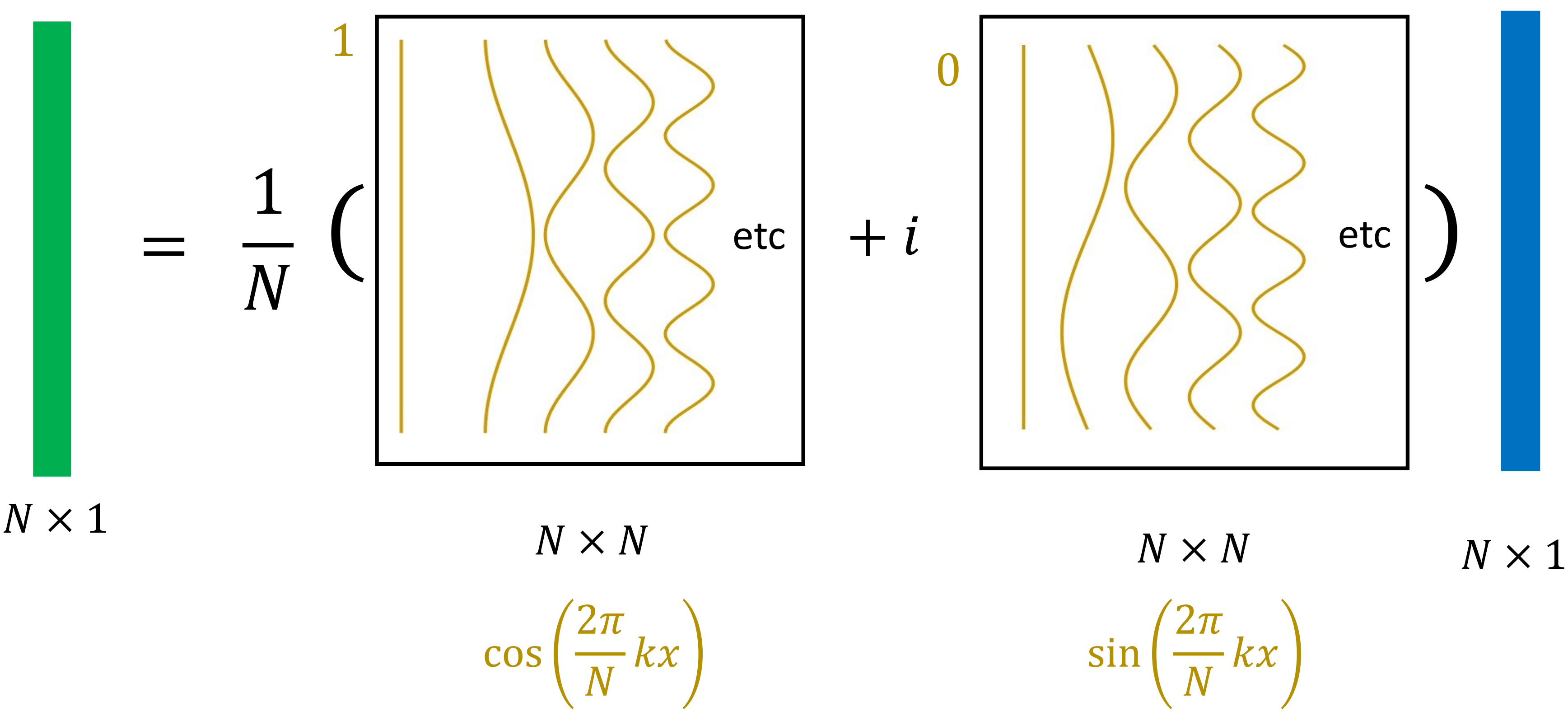
$$\mathbf{F}^{-1} = \frac{1}{N}\bar{\mathbf{F}}$$

where

$$\bar{\mathbf{F}}_{k,x} \equiv e^{i\frac{2\pi}{N}kx}$$

Visualizing this sinusoidal basis representation of $I(x)$ by arranging cosines and sines into columns of a matrix :

$$I(x) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\cos\left(\frac{2\pi}{N}kx\right) + i\sin\left(\frac{2\pi}{N}kx\right)\right)\hat{I}(k)$$



Recall last lecture

- convolution
- impulse function $\delta(x - x_0)$
- special behavior of sines and cosines under convolution
- complex numbers and Euler's formula

Today

- Fourier transform $\hat{I}(k) = \mathbf{F}\, I(x)$
- convolution theorem
- filtering

$$c = r(\cos\theta + i\,\sin\theta) = re^{i\theta}$$

$$\hat{I}(k) = \left|\hat{I}(k)\right|\, e^{\,i\,\phi(k)}$$

amplitude spectrum

phase spectrum

Convolution Theorem

Let $I(x)$ and $h(x)$ be defined on $x \in \{0, 1, \ldots, N-1\}$.

$$\boldsymbol{F} \{ \ I(x) * h(x) \ \} \ = \ \boldsymbol{F}\, I(x) \ \ \boldsymbol{F}\, h(x)$$

$$= \ \hat{I}(k) \quad \hat{h}(k)$$

See lecture notes for proof.

Convolution Theorem

Let $I(x)$ and $h(x)$ be defined on $x \in \{0, 1, \ldots, N-1\}$.

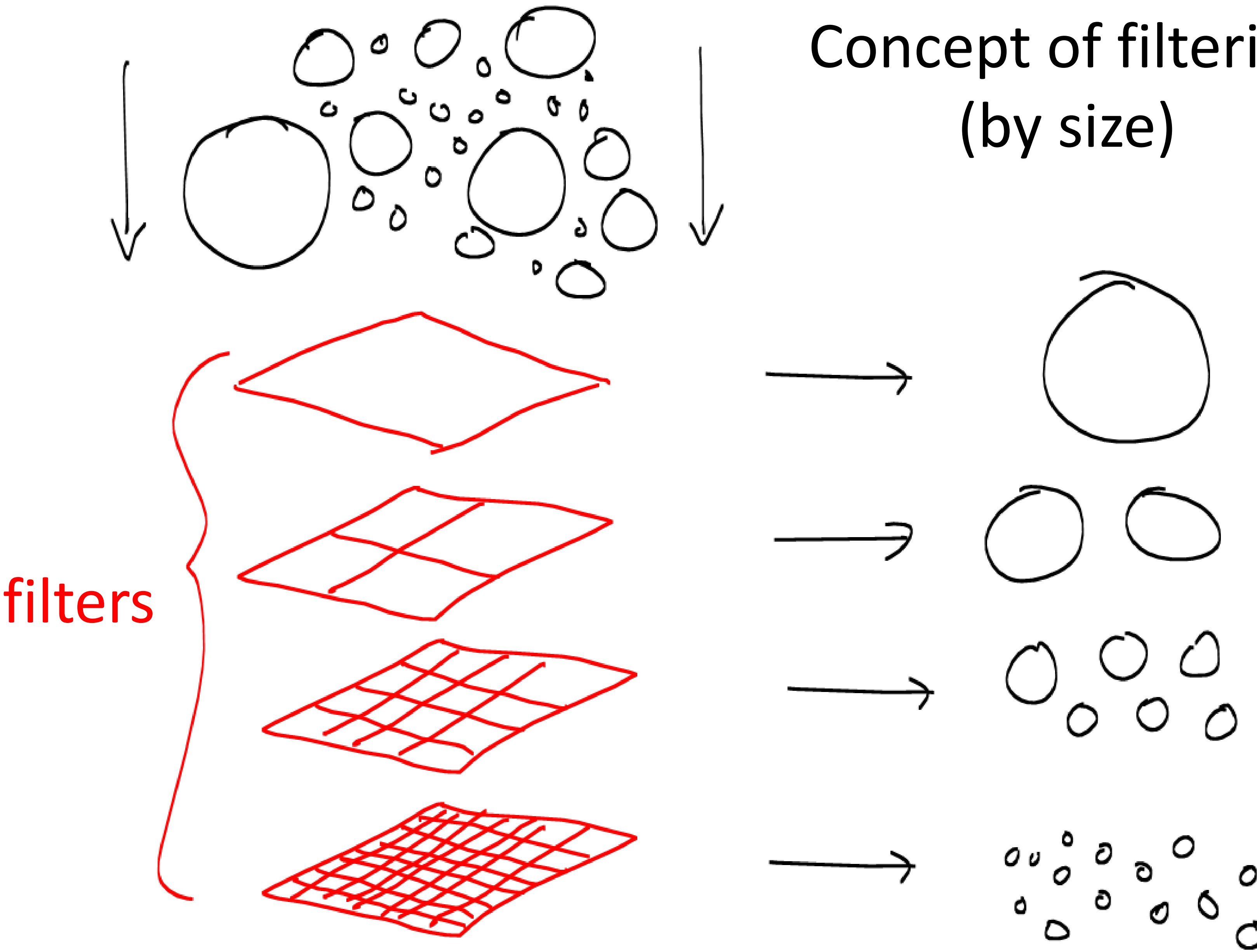
$$\boldsymbol{F}\{\ I(x) * h(x)\ \} = \boldsymbol{F}\, I(x)\ \boldsymbol{F}\, h(x)$$

$$= \hat{I}(k) \quad \hat{h}(k)$$

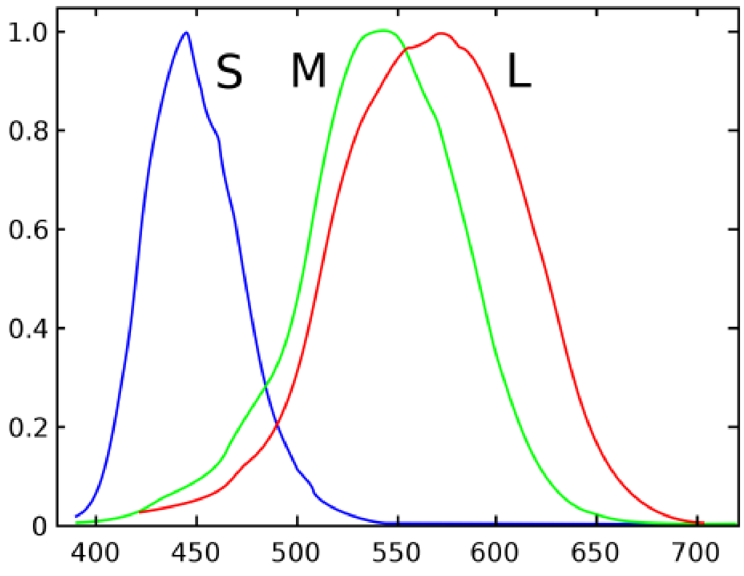
$$= \left|\hat{I}(k)\right| \quad \left|\hat{h}(k)\right| \quad e^{-i\,\phi_I(k)}\ e^{-i\,\phi_h(k)}$$

Convolving an image $I(x)$ with a filter $h(x)$ changes the amplitude and phase of each frequency component.

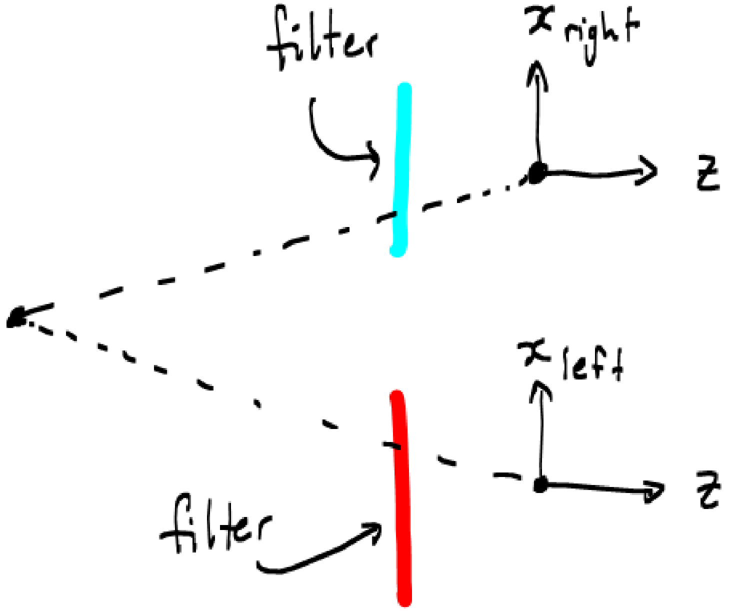
Concept of filtering (by size)



Color filtering
(by frequency or wavelength)



wavelength (1/frequency)



Linear Filtering (by frequency “band”)