MATH 323: PROBABILITY
CONTINUOUS DISTRIBUTIONS AND THE NORMAL DISTRIBUTION

For a continuous random variable Y, consider the cdf $F(y) = P(Y \leq y)$, for $y \in \mathbb{R}$ where $F(y)$ is a continuous function. We assume that $F(y)$ can be expressed

$$F(y) = \int_{-\infty}^{y} f(t)\ dt$$

for a function $f(y)$ called the pdf (probability density function). The fundamental law of calculus tells us that wherever $F(y)$ is differentiable

$$f(y) = \left.\frac{dF(t)}{dt}\right|_{t=y} \qquad \text{or, less formally,} \qquad f(y) = \frac{dF(y)}{dy}.$$

- The pdf $f(y)$ describes how the probability is distributed across the real line; we have the fundamental properties that for all y

  $$f(y) \geq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(y)\ dy = 1.$$

  but we could have $f(y) = 0$ for some y.
- Notice also that $f(y)$ is **NOT** constrained to be less than or equal to 1.
- The pdf $f(y)$ **does not specify probabilities directly**; that is, $f(y)$ does not represent $P(Y = y)$, instead it captures the *slope* of $F(.)$ at y.

Suppose now that we have a continuous random variable Y with pdf f_Y, and we consider a linear transformation from Y to X, where

$$X = aY + b.$$

where $a > 0$ and b are constants. We aim to compute the pdf f_X of X. To do this, notice that for any number $x \in \mathbb{R}$ we have

$$F_X(x) = P(X \leq x) = P(aY + b \leq x) = P\left(Y \leq \frac{x-b}{a}\right) = F_Y((x-b)/a)$$

so we have cdf of X in terms of the cdf of Y. Finally we can compute $f_X(x)$

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx}\left\{F_Y\left(\frac{x-b}{a}\right)\right\} = \frac{1}{a} f_Y\left(\frac{x-b}{a}\right)$$

by the chain rule for differentiation.

Notice that the identity $F_X(x) = F_Y((x-b)/a)$ can be written in integral form

$$\int_{-\infty}^{x} f_X(s)\ ds = \int_{-\infty}^{(x-b)/a} f_Y(t)\ dt$$

but changing variables $t \longrightarrow s = at + b$ in the second integral reveals that

$$\int_{-\infty}^{x} f_X(s)\ ds = \int_{-\infty}^{x} f_Y((s-b)/a)\frac{dt}{ds}\ ds = \int_{-\infty}^{x} f_Y((s-b)/a)\frac{1}{a}\ ds$$

This confirms our earlier result: by equating the integrands, we have

$$f_X(s) = f_Y((s-b)/a)\frac{1}{a}.$$

for $s \in \mathbb{R}$.

The Normal case: For the Normal (or Gaussian) distribution, we have the pdf

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$$

specified using parameters $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. Notice that this function achieves its maximum value at $y = \mu$, and

$$f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} = \frac{0.3989423}{\sigma}$$

Hence it is clear that $f(y)$ can exceed 1 if σ is chosen small enough. For the expectation, we have

$$\begin{aligned}
\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f(y)\, dy &= \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\} dy \\
&= \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\} dy \\
&= \int_{-\infty}^{\infty} (\mu+t) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(t+\mu-\mu)^2\right\} dt \quad (\text{setting } t = y-\mu) \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2\sigma^2}\right\} dt + \int_{-\infty}^{\infty} t \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2\sigma^2}\right\} dt.
\end{aligned}$$

However, for the first integral

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2\sigma^2}\right\} dt = 1$$

as the integrand is merely the Normal pdf with $\mu = 0$. Similarly, for the second integral, integrating directly

$$\int_{-\infty}^{\infty} t \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2\sigma^2}\right\} dt = \frac{1}{\sigma\sqrt{2\pi}} \left[\sigma^2 \exp\left\{-\frac{t^2}{2\sigma^2}\right\}\right]_{-\infty}^{\infty} = 0,$$

a result we can spot by noting that the integrand is an **odd function** around zero (i.e. $g(y)$ is an odd function around zero if $g(-y) = -g(y)$). Therefore $\mathbb{E}[Y] = \mu$.

Now suppose that $Z = (Y-\mu)/\sigma$ is a linear transformed version of Y

$$Z = \frac{Y-\mu}{\sigma} = \frac{1}{\sigma}Y - \frac{\mu}{\sigma}$$

so that $Y = \sigma Z + \mu$. Following the general method calculation from page 1, we have that the cdf of Z is given for each fixed $z \in \mathbb{R}$ by

$$F_Z(z) = P(Z \le z) = P\left(\frac{Y-\mu}{\sigma} \le z\right) = P(Y \le \sigma z + \mu) = F_Y(\sigma z + \mu).$$

Differentiating both sides with respect to z, we have that

$$f_Z(z) = \sigma f_Y(\sigma z + \mu) = \sigma \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\sigma z + \mu - \mu)^2\right\} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}$$

but this is **identical** to the Normal pdf with parameters $\mu = 0$ and $\sigma = 1$. Notice that the identity $F_Z(z) = F_Y(\sigma z + \mu)$ can be written in integral form as

$$\int_{-\infty}^{z} f_Z(s)\, ds = \int_{-\infty}^{\sigma z + \mu} f_Y(t)\, dt = \int_{-\infty}^{z} f_Y(\sigma s + \mu) \frac{dt}{ds}\, ds = \int_{-\infty}^{z} \sigma f_Y(\sigma s + \mu)\, ds$$

by changing variables $t = \sigma s + \mu$ in the second integral. We equate the integrands to conclude

$$f_Z(s) = \sigma f_Y(\sigma s + \mu)$$

for $s \in \mathbb{R}$ as before.