Assignment 2 | **LE, Nhat Hung**
MATH 323 - Probability | McGill ID: 260793376
Prof. David Stephens | Date: October 25, 2018
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1. For the following discrete random variables, Y, and pmfs $p(y)$ with support sets $\mathscr{Y}$ (containing the values y for which $p(y) > 0$), compute the following quantities:

(a) $\mathscr{Y} = \{1, 2, 3, \ldots, 10\}$,

$$p(y) = cy.$$

Compute c, $\mathbb{E}[Y]$ and $Var[Y]$.

$$\begin{aligned}\sum_{y=1}^{10} p(y) = 1 \quad &\Rightarrow \sum_{y=1}^{10} cy = 1\\ &\Rightarrow \sum_{y=1}^{10} y = \frac{1}{c}\\ &\Rightarrow 55 = \frac{1}{c}\\ &\Rightarrow c = \frac{1}{55}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y] \quad &= \sum_{y=1}^{10} yp(y)\\ &= \sum_{y=1}^{10} \frac{y^2}{55}\\ &= \frac{1+4+9+16+25+36+49+64+81+100}{55}\\ &= 7\end{aligned}$$

$$\begin{aligned}Var[Y] \quad &= \mathbb{E}[(Y-\mathbb{E}[Y])^2]\\ &= \mathbb{E}[(Y-7)^2]\\ &= \sum_{y=1}^{10} (y-7)^2 p(y)\\ &= \sum_{y=1}^{10} \frac{(y-7)^2 y}{55}\\ &= \frac{36+50+48+36+20+6+0+8+36+90}{55}\\ &= 6\end{aligned}$$

(b) $\mathscr{Y}= \{1,\ 2,\ 3,\ \ldots\}$,

$$p(y) = \frac{c}{y}\left(\frac{1}{2}\right)^y$$

Compute c, and $\mathbb{E}[Y]$.

$$\begin{aligned}
\sum_{y=1}^{\infty}\frac{(\frac{1}{2})^y}{y} = \frac{1}{c} \quad &\Rightarrow -ln(\frac{1}{2}) = \frac{1}{c}\\
&\Rightarrow ln(2) = \frac{1}{c}\\
&\Rightarrow c = \frac{1}{ln(2)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Y] \quad &= \sum_{y=1}^{\infty}\frac{y}{yln(2)}\left(\frac{1}{2}\right)^y\\
&= \frac{1}{2ln(2)}\sum_{y=1}^{\infty}\left(\frac{1}{2}\right)^{y-1}\\
&= \frac{1}{ln(2)}
\end{aligned}$$

(c) $Y \sim Binomial(20,\ 3/4)$: compute $\mathbb{E}[Y^2]$.

$$\begin{aligned}
\mathbb{E}[Y^2] \quad &= n(n-1)p^2 + np\\
&= (20)(19)\left(\frac{3}{4}\right)^2 + (20)\left(\frac{3}{4}\right)\\
&= \frac{915}{4}
\end{aligned}$$

(d) $Y \sim Geometric(1/2)$: compute $\mathbb{E}[Y^3]$.

$$\begin{aligned}
\mathbb{E}[Y^3] \quad &= \frac{d^3}{dt^3}m(t),\ with\, t = 0\\
&= \frac{d^3}{dt^3}\left(\sum_{y=1}^{\infty}e^{ty}p(y)\right)\\
&= \frac{d^3}{dt^3}\left(\sum_{y=1}^{\infty}e^{ty}q^{y-1}p\right)\\
&= \frac{d^3}{dt^3}\left(pe^t\sum_{y=1}^{\infty}(qe^t)^{y-1}\right)\\
&= \frac{d^3}{dt^3}\left(pe^t\sum_{j=0}^{\infty}(qe^t)^j\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d^3}{dt^3}\left(\frac{pe^t}{1-qe^t}\right) \\
&= \frac{d^3}{dt^3}\left(\frac{\frac{e^t}{2}}{\frac{2-e^t}{2}}\right) \\
&= \frac{d^3}{dt^3}\left(\frac{e^t}{2-e^t}\right) \\
&= \frac{d^2}{dt^2}\left(\frac{2e^t}{(2-e^t)^2}\right) \\
&= \frac{d}{dt}\left(\frac{4e^t+2e^{2t}}{(2-e^t)^3}\right) \\
&= \frac{16e^t+12e^{2t}-6e^{3t}+4e^{4t}}{(2-e^t)^5} \\
&= 26
\end{aligned}
$$

2. Identify the pmfs for the stated random variables in the given experimental conditions.

(a) A pair of dice are rolled repeatedly until the total score on a given roll is equal to 10. Random variable Y records the number of rolls until the experiment terminates.

$$
\begin{aligned}
Y &\sim Geometric(p), \\
with\, p &= \frac{3}{(6)(6)} = \frac{1}{12}
\end{aligned}
$$

$$
p(y) = \left(\frac{11}{12}\right)^{y-1}\left(\frac{1}{12}\right)
$$

(b) Two hundred students take a test. The probability that an individual student passes the test is 0.75, with the results of the tests for the two hundred students being considered mutually independent events. Random variable Y records the total number of students who pass the test.

$$
\begin{aligned}
Y &\sim Binomial(200, 0.75) \\
p(y) &= \binom{200}{y}(0.75^y)(0.25^{200-y})
\end{aligned}
$$

(c) Two hundred students in class comprise 80 Arts students and 120 Science students. Each lecture, the professor selects 10 students at random, with all selections of 10 students being equally likely. Random variable Y records the number of lectures that go by until the selected students have equal representation from Arts and Science (that is, there are five from Arts and five from Science).

$$
\begin{aligned}
Y &\sim Geometric(p), \\
with\, p &= \frac{\binom{80}{5}\binom{120}{5}}{\binom{200}{10}} \approx 0.20
\end{aligned}
$$

$$p(y) = \left(1 - \frac{\binom{80}{5}\binom{120}{5}}{\binom{200}{10}}\right)^{y-1} \frac{\binom{80}{5}\binom{120}{5}}{\binom{200}{10}}$$

3. A new driver can take their driving test at either Test Centre A, where their probability of passing a test is p_A, or at Test Centre B, where their probability of passing the test is p_B. The driver tosses a fair coin, and if the coin lands Heads, they decide to take the test at Test Centre A; if Tails, they take the test at Test Centre B. The driver continues to take a test using the coin-tossing procedure until they pass the test. Let random variable Y record the number of times the driver takes the test until they pass. Find the pmf of Y.

$$Y \sim Geometric(p),$$
$$with\, p = \frac{1}{2}p_A + \frac{1}{2}p_B = \frac{p_A + p_B}{2}$$

$$p(y) = \left(1 - \frac{p_A + p_B}{2}\right)^{y-1} \frac{p_A + p_B}{2}$$