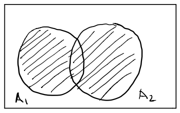
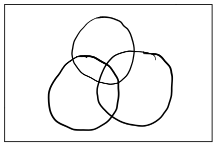
How to count $|A_1 \cup A_2 \cup \dots \cup A_n|$ if the sets are <u>not</u> disjoint?

n=2



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

n=3



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

example

40 students
- 10 in ANTH, 22 in BIOL, 16 in COMP
- 6 in A and B, 8 in B and C, 4 in A and C, 2 in all

How many students are not enrolled in <u>any</u> of the 3 courses?
How many students are in at least one?

$$|A \cup B \cup C| = |A| + |B| + |C| - \left(|A \cap B| + |A \cap C| + |B \cap C|\right) + |A \cap B \cap C|$$

$= (10+22+16+2) - (6+8+4) = 32$ $\therefore$ There are $40-32=8$ students in none of the courses

Principle of Inclusion-Exclusion

(PIE) If $A_1, A_2, \dots, A_n$ are finite sets, then

$$|A_1 \cup \dots \cup A_n| = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1} \left|\bigcap_{i \in S} A_i\right|$$

$$= (|A_1| + |A_2| \dots) - (\underbrace{|A_1 \cap A_2| + |A_1 \cap A_3| + \dots}_{\text{All pairs}}) + (\underbrace{|A_1 \cap A_2 \cap A_3| + \dots}_{\text{All triples}})$$

PROOF: Must show every $x \in A_1 \cup \dots \cup A_n$ is counted exactly once on the RHS. Say x appears in k of the sets $A_1, \dots, A_n$. How many times is x counted in · collections of one set: k

· intersections of 2 sets: $\binom{k}{2}$

· intersections of 3 sets: $\binom{k}{3}$

So x is counted $\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \dots + (-1)^{k+1}\binom{k}{k}$

Derangements

n people put their phones in a box. If every person takes a phone out, how likely is it that no one gets their phone back?

The number of permutations of $[n]$ (or bijections from $[n] \to [n]$) where no element is mapped to itself is denoted D_n; they are all called derangements.

Let A_i = set of permutations which map i to itself

$$D_n = |\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n|$$
$$= |\overline{A_1 \cap A_2 \cap \dots \cap A_n}|$$
$$= n! - |A_1 \cup A_2 \cup \dots \cup A_n|$$

$$|S| = (|A_1| + |A_2| + \dots) - (|A_1 \cap A_2| + |A_1 \cap A_2| + \dots) + (|A_1 \cap A_2 \cap A_3|) = n(n-1)! - \binom{n}{2}(n-2)! + \dots + (-1)^{n-1}\binom{n}{n}(n-n)!$$

$$= n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + (-1)^{n-1}\frac{1}{n!}\right)$$

$$D_n = n! - n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n-1}}{n!}\right) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

So how likely is it?

P(a permutation is a derangement) $= \frac{D_n}{\#\text{perms.}} = \frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \Rightarrow e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$

Recall: Taylor polynomial $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \ \forall x \in \mathbb{R}$

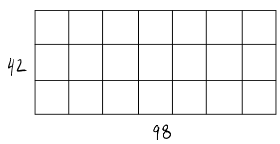
So for large n, $P \approx e^{-1} \approx 0.36788$

$$D_n \approx \frac{n!}{e} \text{ as } n \to \infty$$

Pidgeon Hole Principle (PHP)

If you distribute $>n$ objects to n boxes, some box has more than 1 object.

example 22 soccer players on a field measuring $42\text{m} \times 98\text{m}$. Show there are 2 players no more than 20 m apart.



21 boxes, 22 players

$\Rightarrow$ 2 in same square by PHP

$\Rightarrow d \le \sqrt{14^2+14^2} = \sqrt{196+196} < \sqrt{400} = 20$ ■

example Let S be a set of 9 points in $\mathbb{R}^3$ each with integer coordinates. Show $\exists\, p_i, p_j \in S$ such that the midpoint of $\overline{p_i p_j}$ has integer coordinates.

Let $p_i = (x_i, y_i, z_i)$ Want $\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2}, \frac{z_i+z_j}{2} \in \mathbb{Z} \iff x_i+x_j, y_i+y_j, z_i+z_j$ even

$p_j = (x_j, y_j, z_j)$

Possible parity combinations:

x	y	z
E	E	E
E	E	O
E	O	E
O	E	E
E	O	O
O	E	O
O	O	E
O	O	O

8 parity combos,

9 points $\Rightarrow$ 2 have the same parity

ie) $x_i \equiv x_j \bmod 2$

$y_i \equiv y_j \bmod 2$

$z_i \equiv z_j \bmod 2$

$\Rightarrow x_i + x_j,\ y_i+y_j,\ z_i+z_j$ all even ■

example Show that if $n+1$ numbers are chosen from $[2n]$ then there are

(a) 2 which differ by 1

(b) 2 which sum to $2n+1$

(a) Boxes: $\{1,2\}, \{3,4\}, \dots, \{2n-1, 2n\}$ n boxes, $n+1$ choices $\Rightarrow$ 2 integers in same box by PHP $\Rightarrow$ differ by 1.

(b) Boxes: $\{1, 2n\}, \{2, 2n-1\}, \{3, 2n-2\}, \dots$ Each pair sums to $2n+1$ $\Rightarrow$ 2 of the $n+1$ choices lie in one box by PHP.

Strong version: $m > n$, m objects, n boxes $\Rightarrow$ some box recieves $\ge \lceil \frac{m}{n} \rceil$ objects

example 16 students in 18 chairs. Show there are 6 consecutive occuppied seats

Split into 3 blocks of 6 chairs: 1-6 7-12 13-18

Distribute 16 people across 3 boxes $\Rightarrow$ some box gets $\ge \lceil \frac{16}{3} \rceil = 6$ people example

A student has 37 days to prepare for an exam. If she studies no more than 60 hours but at least 1 hour each day. Show there is some set of consecutive days over which she studied exactly 13 hours (whole hours each day)

Let s_i = # of hours studied on day i.

Let $a_i = s_1 + s_2 + \dots + s_i$ (# of hours studied from day 1 to day i)

$1 \le a_1 < a_2 < a_3 < \dots < a_{37} \le 60$ and $14 \le a_1+13 < a_2+13 < \dots < a_{37}+13 \le 73$

Consider the list $\boxed{\boxed{a_1, a_2, \dots, a_{37}}, a_1+13, a_2+13, \dots, a_{37}+13}$ There are 74 integers between 1 and 73 $\Rightarrow$ 2 are the same

$\Rightarrow \exists\, i\, j$ st $a_i = a_j + 13$

$\Rightarrow a_i - a_j = 13$

$\Rightarrow s_{j+1} + s_{j+2} + \dots + s_i = 13$ ▨