Math 240: Discrete Structures I (W18) – Assignment 6

Solutions must typed or very neatly written and uploaded to MyCourses no later than **6 pm** on **Saturday, March 17, 2018**. Up to 4 bonus marks will be awarded for solutions typeset in LaTeX; both the .tex file and .pdf file must be uploaded.

You may use theorems proven or stated in class, but you must state the theorem you are using. All work must be shown for full marks.

1. (a) Let a be any integer. Prove that $a^n - an + n - 1$ is divisible by $(a-1)^2$ when $n \geq 2$.

   (b) We saw in class that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

   If we look at the following two sums, one might see a pattern emerging:

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$
$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

   Prove the following, which generalizes the three summations above, for $n \geq 1$ where $m \geq 0$ is some fixed integer:

$$\sum_{k=1}^{n} \frac{(k+m)!}{(k-1)!} = \frac{(n+m+1)!}{(n-1)!(m+2)}$$

   (c) You know about binary representations of integers, and I've asked you to prove things about integers in base 10. Now, show that every positive integer has a factorial representation. That is, prove that for every integer $n \geq 1$, we can write

$$n = \sum_{i=1}^{k} c_i i!$$

   where the integer coefficients c_i satisfy $0 \leq c_i \leq i$ for each i.

2. (a) Prove that $f_1 - f_2 + f_3 + \ldots + (-1)^n f_{n+1} = (-1)^n f_n + 1$ for all $n \geq 1$.

   (b) Prove that $f_1 f_2 + f_2 f_3 + f_3 f_4 + \ldots + f_{2n-1} f_{2n} = f_{2n}^2$ for all $n \geq 1$.

3. **Recurrence relations.** You've won a contest! You're going to win money! Your prize is determined as follows. You are given \$40, then asked to sit in a chair. At each minute mark of you being in the chair, your winnings are re-calculated as being 150% of the amount you held during the previous minute but deducted from that is 25% of the amount you held the minute before that (note that you held \$0 before the contest started). Whoever is holding the contest is no fool; it's not hard to see that there needs to be some cost to you sitting in the chair, or they'll go bankrupt! So, at each minute mark, you're going to lose \$6 for every minute you've been in the chair (after the first minute you'll lose \$6, after the second minute you'll lose an *another* \$12, after the third minute you'll lose *another* \$18, and so on). You can leave the chair any time you want, collect your winnings, and walk away.

   (c) Write a new recurrence relation that expresses the amount of money you win if you leave the chair after the m^{th} minute (but before minute $m + 1$).

   (d) Solve this recurrence relation to find an explicit function of m for your winnings after m minutes.

   (e) How long should you stay in the chair to maximize your winnings? If you make any claims about the behaviour of the function after a given point, make sure you justify your answer (this can be done using basic calculus or by other means).