Introduction to Numerical Techniques: Numerical Integration **Noah Huffman**

Ph 20.2

1 Simpson's Rule Error

First, we Taylor expand the Integral's true value using the Lagrange remainder.

$$I = f(a)H + f'(a)\frac{H^2}{2} + f''(\eta)\frac{H^3}{3!} + f'''(\eta)\frac{H^4}{4!} + f''''(\eta)\frac{H^5}{5!} \tag{1}$$

Where $H = b - a$. Then, we expand Simpson's Rule.

$$I_{Simpson} = H(\frac{f(a)}{6} + \frac{4f(a)}{6} + \frac{f(b)}{6}) \tag{2}$$

$$= H(\frac{f(a)}{6} + \frac{4}{6}(f'(a)(c-a) + \frac{f''(\eta)}{2!}(c-a)^2 + ...) + \frac{1}{6}(f'(a)(b-a) + \frac{f''(\eta)}{2!}(b-a)^2 + ...)$$

Noting that $b - a = H$ and $c - a = \frac{b-a}{2} = \frac{H}{2}$, and expanding to fifth order in H, we have:

$$= H(\frac{f(a)}{6} + \frac{4}{6}(f'(a)(\frac{H}{2}) + \frac{f''(\eta)}{2!}(\frac{H}{2})^2 + \frac{f''(\eta)}{3!}(\frac{H}{2})^3 + \frac{f''''(\eta)}{4!}(\frac{H}{2})^4)$$

$$+\frac{1}{6}(f'(a)(H) + \frac{f''(\eta)}{2!}(H)^2 + \frac{f'''(\eta)}{3!}(H)^3\frac{f''''(\eta)}{4!}(H)^4)$$

Simplifying yields

$$f(a)H + f'(a)\frac{H^2}{2} + f''(\eta)\frac{H^3}{6} + f'''(\eta)\frac{H^4}{24} + f''''(\eta)H^5\frac{5}{576}$$

Therefore, the error is

$$I_{Simpson} - I = f''''(\eta)H^5(\frac{5}{576} - \frac{1}{120}) = \frac{f''''(\eta)H^5}{2880} \tag{3}$$

2 Error Plot

For the extended Trapezoid and Simpson's rule estimates, the error of the estimate from the known value of an integral can be computed as a function of the number of sub-intervals, N, created between the bounds of integration, [a,b]. In the notes for this lab, we saw that the global error of the Trapezoid rule is given by

$$f''(\xi)\frac{h_N^3}{12} * N = (b-a)f''(\xi)\frac{h_N^2}{12} = (b-a)f''(\xi)\frac{H^2}{12N^2}, \text{ with } \xi \in [a,b] \tag{4}$$

Which implies the global Simpson's error is:

$$f''''(\xi)\frac{h_N^5}{2880} * N = (b-a)f''(\xi)\frac{h_N^4}{2880} = (b-a)f''(\xi)\frac{H^4}{2880N^4}, \text{ with } \xi \in [a,b] \tag{5}$$

Plotting on a log-log scale gives:

$$\log[\text{global error}] = \log[(b-a)f''(\xi)\frac{H^2}{12}] - 2\log[N] \;\; \textit{Trapezoid}$$

$$\log[\text{global error}] = \log[(b-a)f''(\xi)\frac{H^4}{2880}] - 4\log[N] \ \textit{Simpson}$$

For the Integral, $\int_0^1 e^x dx$ we choose $\xi = a = 0$ for simplicity in plotting our expected error intercepts, so we have:

$$\log[\text{expected error } (\int_0^1 e^x dx)] = \log[\frac{1}{12}] - 2\log[N] \ \textit{Trapezoid}$$

$$\log[\text{expected error } (\int_0^1 e^x dx)] = \log[\frac{1}{2880}] - 4\log[N] \ \textit{Simpson}$$

Below are plots of the numerical errors vs the expected ones for both methods plotted on a log-log scale. The relationship of the errors are clearly linear and match our expectations within a constant offset.

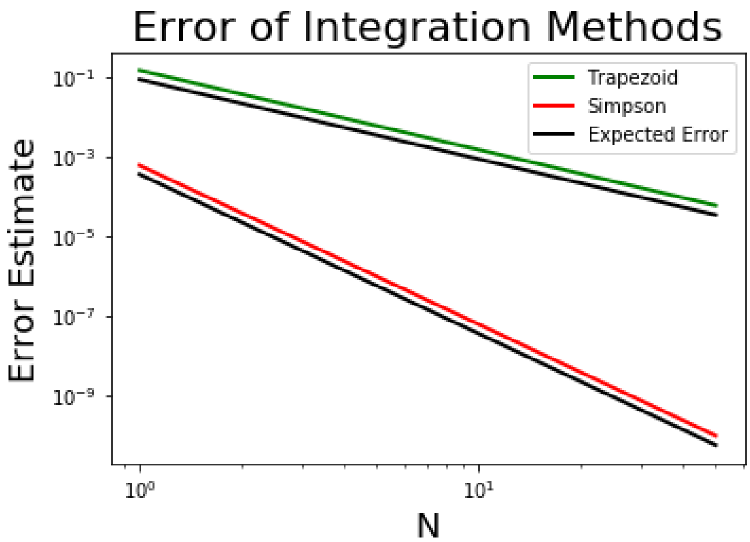

Figure 1: Errors of different integration methods evaluating $\int_0^1 e^x dx$ vs number of intervals plotted on a log-log scale.

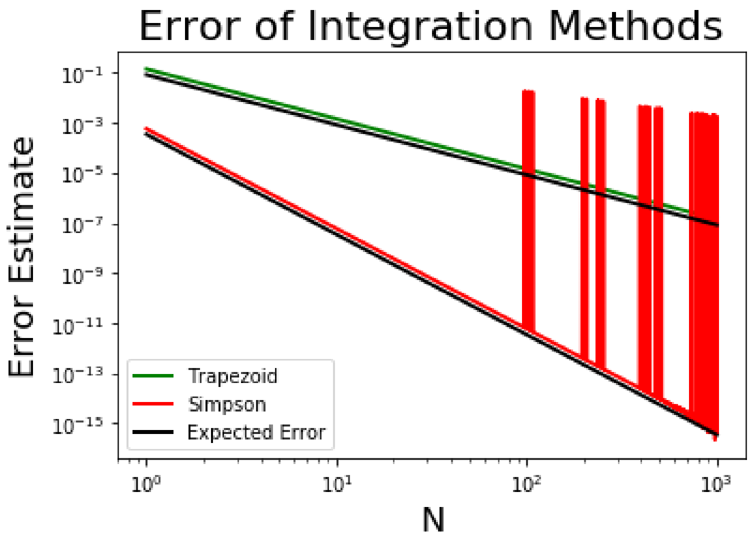

Figure 2: At higher values of N, the error saturates due to the numerical limits of the computer (i.e. roundoff error).