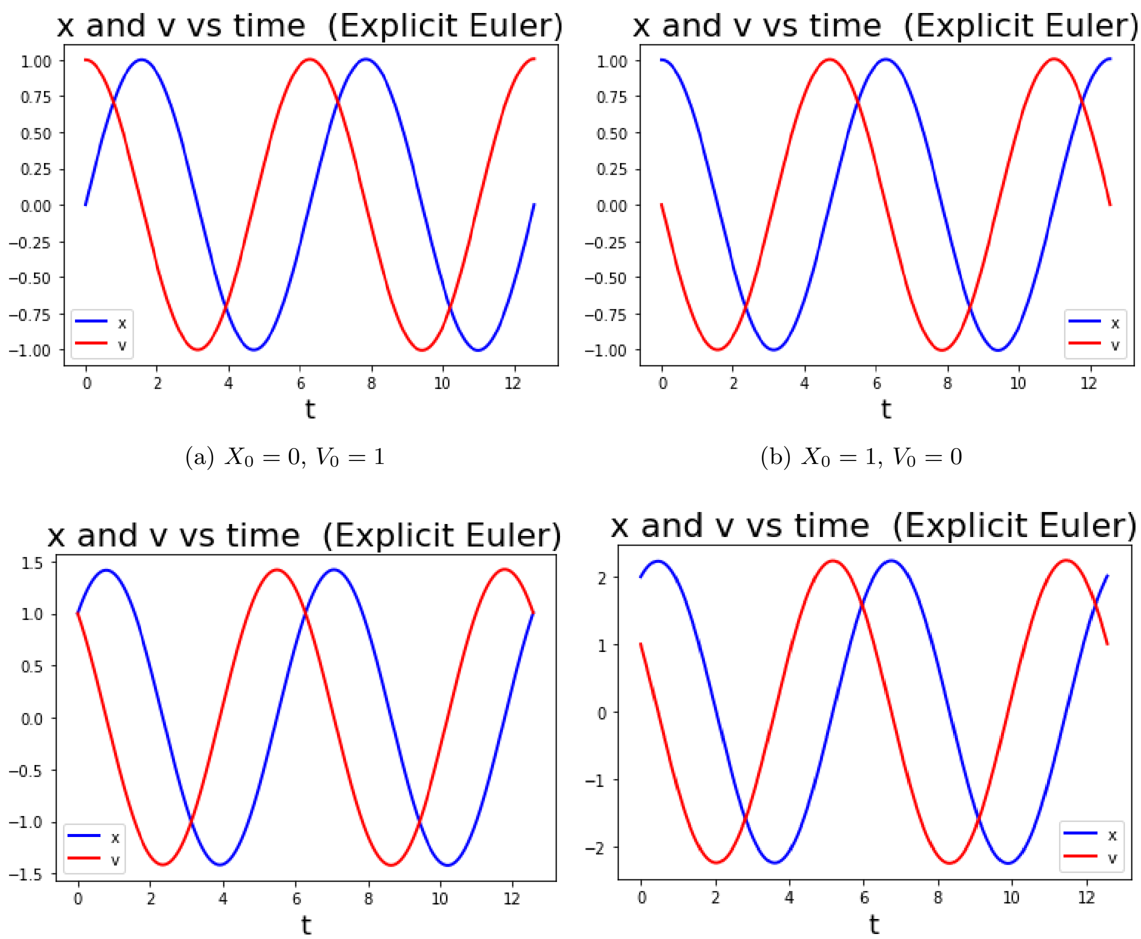
Numerical Solution of Ordinary Differential Equations **Noah Huffman**
Ph 20.3

1 Motion of a mass on a spring

First, Here are some plots of my numerical time evolutions of the system. Note that $X_0 = 0$, $V_0 = 0$ is an uninteresting initial condition because the system has not initial energy to make it dynamic.



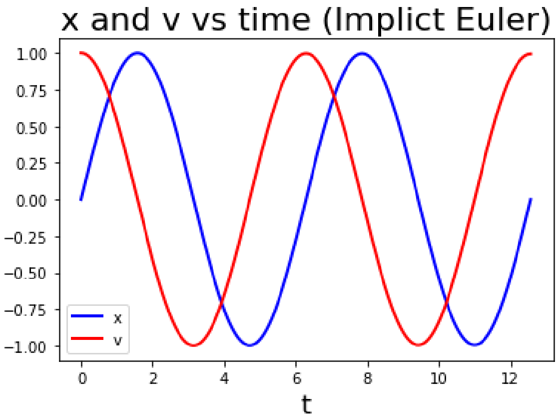
(a) $X_0 = 0$, $V_0 = 1$

(b) $X_0 = 1$, $V_0 = 0$

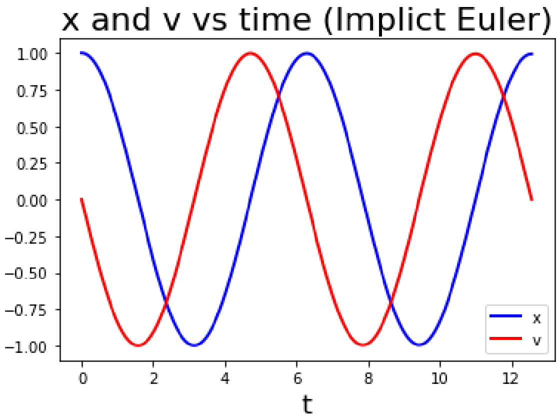
(c) $X_0 = 1$, $V_0 = 1$

(d) $X_0 = 2$, $V_0 = 1$

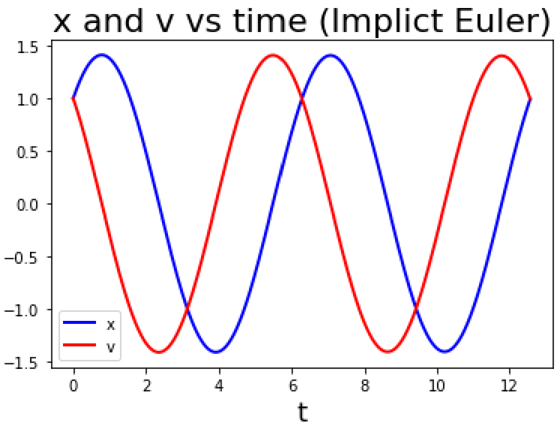
Figure 1: Explicit Euler numerical evolution of system over time for several initial conditions.



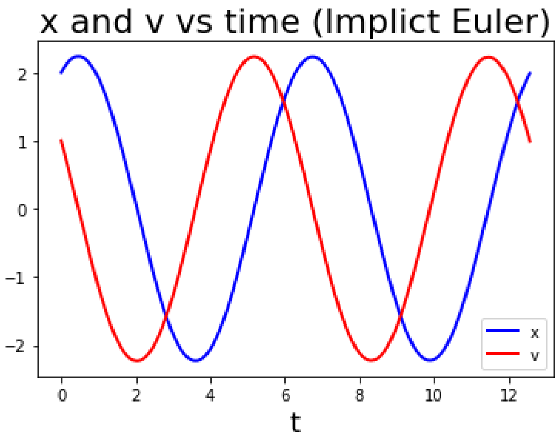
(a) $X_0 = 0$, $V_0 = 1$



(b) $X_0 = 1$, $V_0 = 0$



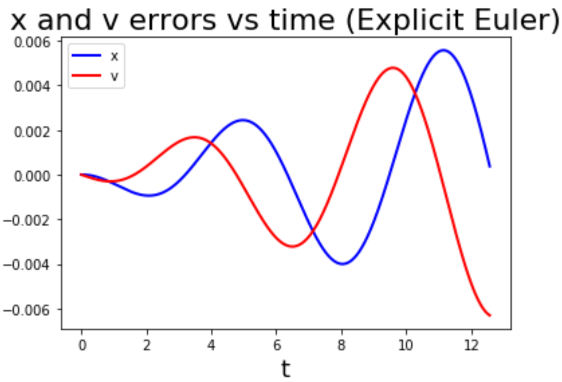
(c) $X_0 = 1$, $V_0 = 1$



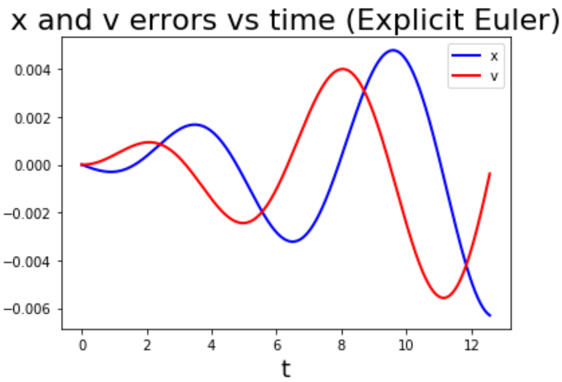
(d) $X_0 = 2$, $V_0 = 1$

Figure 2: Implicit Euler numerical evolution of system over time for same initial conditions.

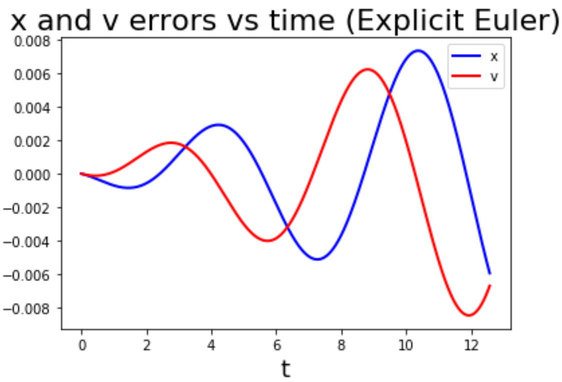
2 Global errors



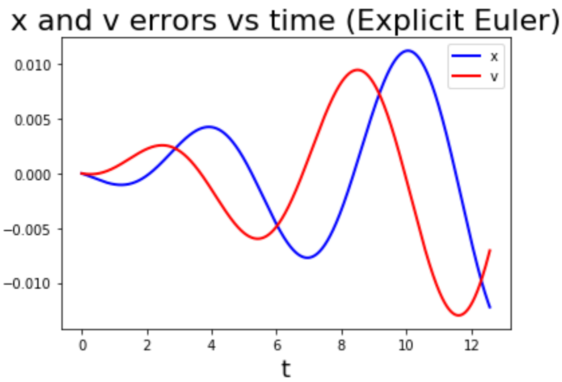
(a) $X_0 = 0$, $V_0 = 1$



(b) $X_0 = 1$, $V_0 = 0$

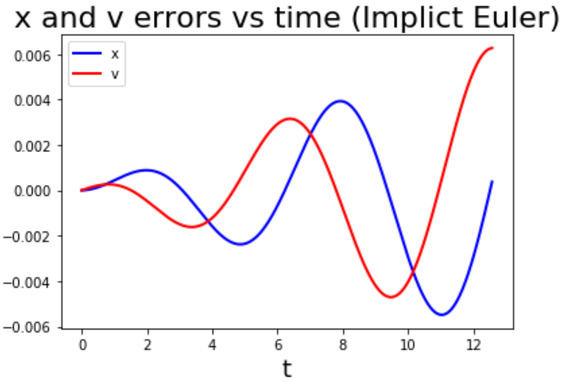


(c) $X_0 = 1$, $V_0 = 1$

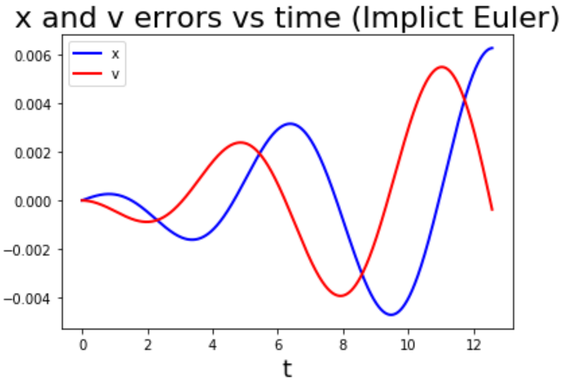


(d) $X_0 = 2$, $V_0 = 1$

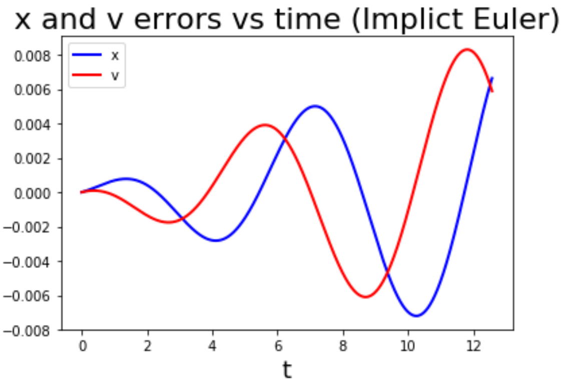
Figure 3: Explicit Euler global errors. Regardless of the choice of initial conditions, the errors grow over time.



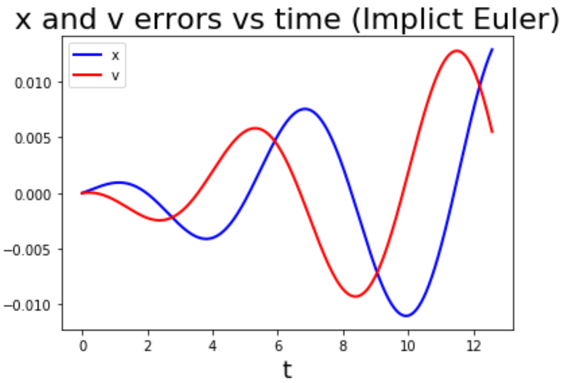
(a) $X_0 = 0$, $V_0 = 1$



(b) $X_0 = 1$, $V_0 = 0$



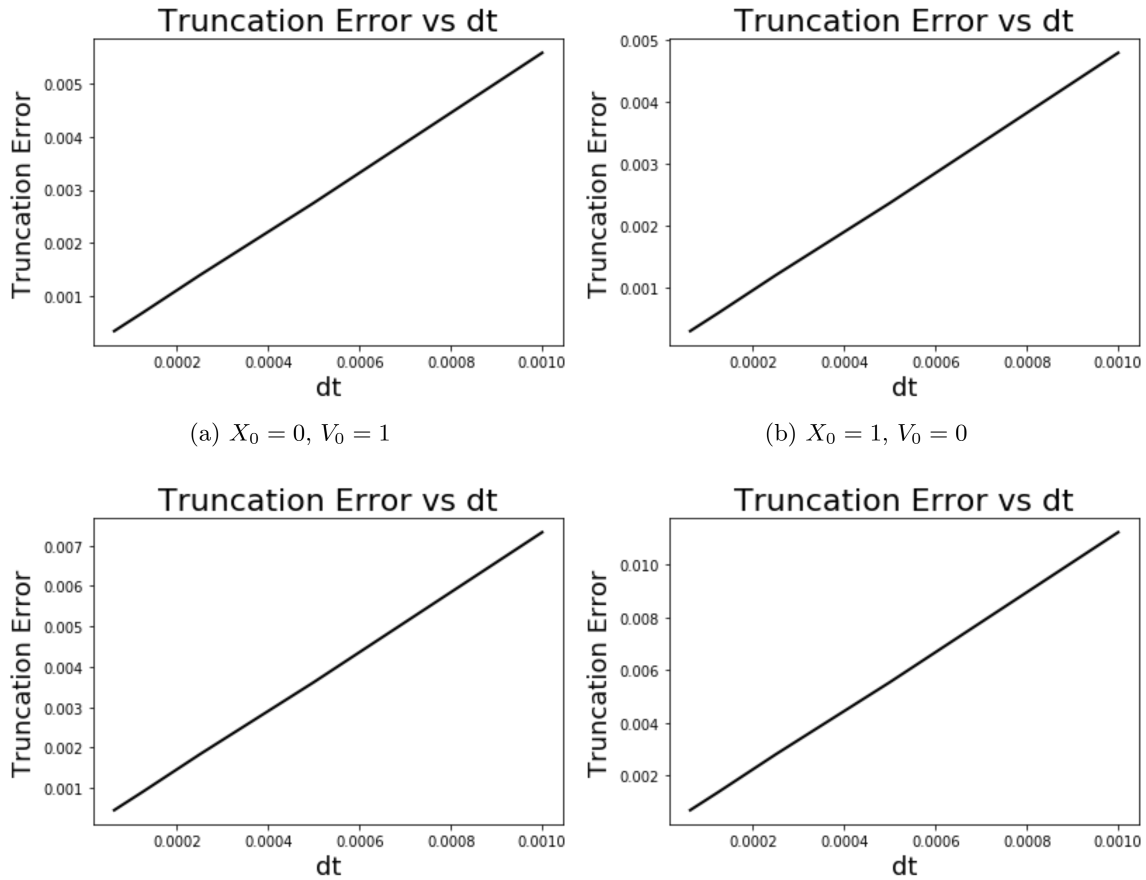
(c) $X_0 = 1$, $V_0 = 1$



(d) $X_0 = 2$, $V_0 = 1$

Figure 4: Implicit Euler global errors. Notice how they are just the negative of the Explicit ones. This makes sense since we are essentially going "backwards" with this method compared to the explicit method (i.e. using using new values to compute old ones rather than vice versa).

3 Truncation Error



(a) $X_0 = 0$, $V_0 = 1$

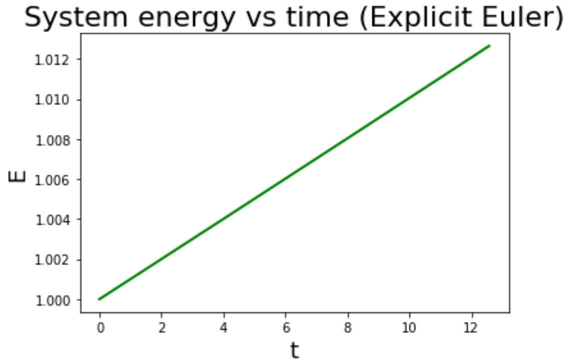
(b) $X_0 = 1$, $V_0 = 0$

(c) $X_0 = 1$, $V_0 = 1$

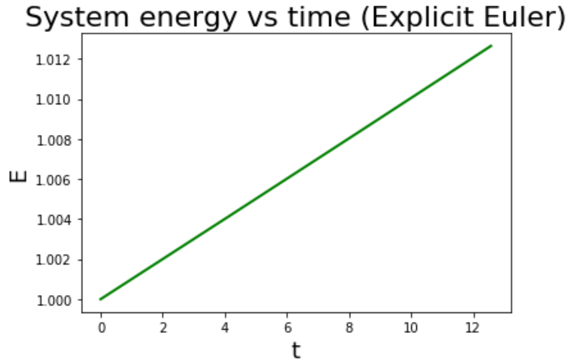
(d) $X_0 = 2$, $V_0 = 1$

Figure 5: Truncation error versus step size. For relatively small step sizes, the truncation error is proportional to the step size regardless of initial conditions. It is also independent of the Euler method we choose. This linear relationship between truncation error and step size is also positive as expected, implying that larger steps lead to more error.

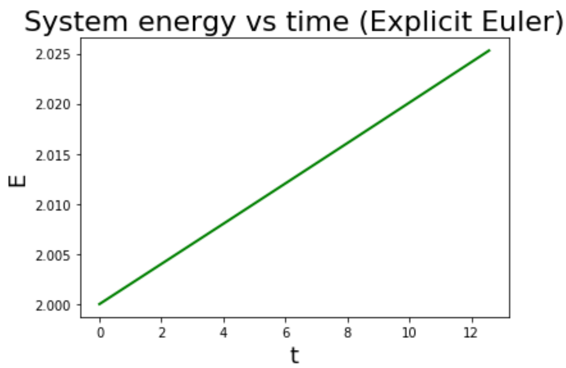
4 Total Energy



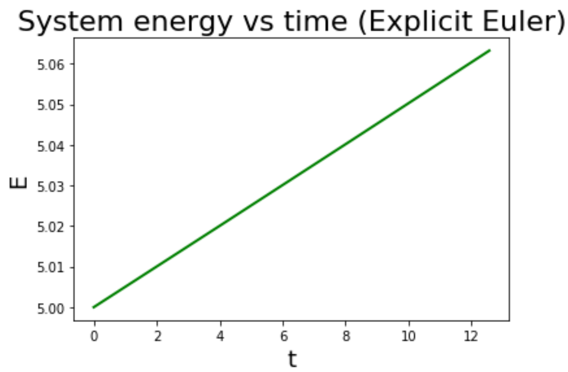
(a) $X_0 = 0$, $V_0 = 1$



(b) $X_0 = 1$, $V_0 = 0$

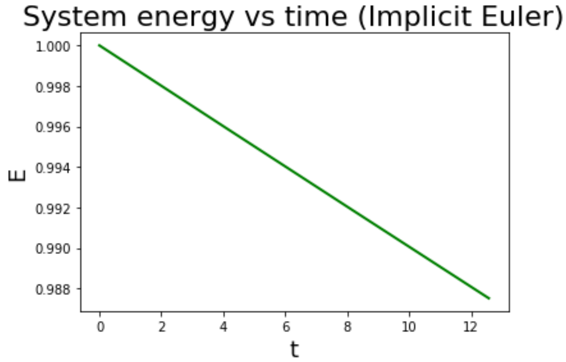


(c) $X_0 = 1$, $V_0 = 1$

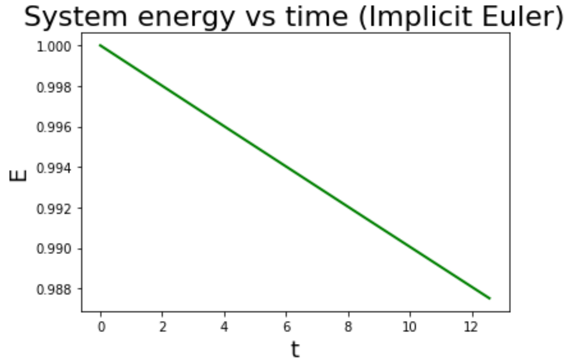


(d) $X_0 = 2$, $V_0 = 1$

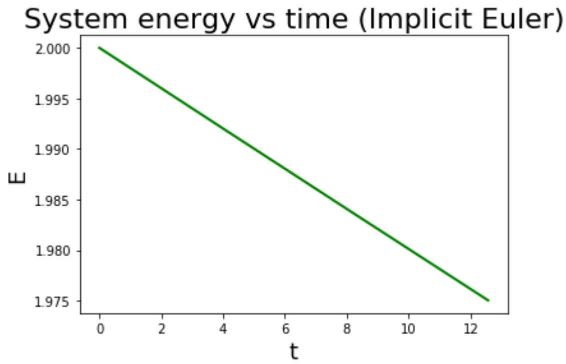
Figure 6: Total system energy given by the numerical explicit Euler method.



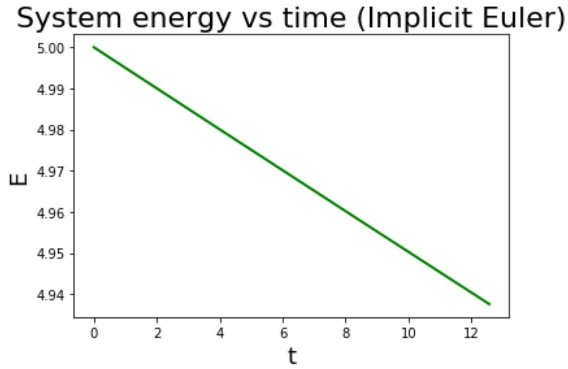
(a) $X_0 = 0$, $V_0 = 1$



(b) $X_0 = 1$, $V_0 = 0$



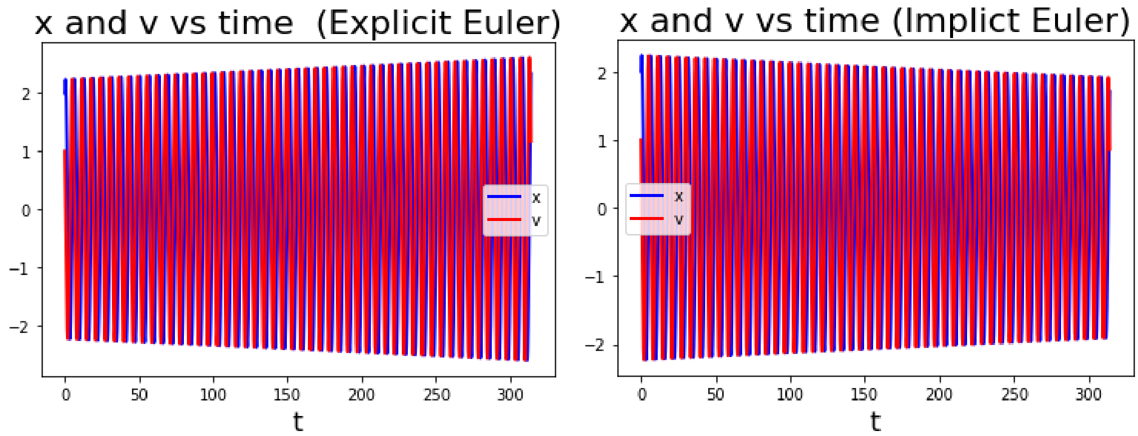
(c) $X_0 = 1$, $V_0 = 1$



(d) $X_0 = 2$, $V_0 = 1$

Figure 7: Total system energy given by the numerical implicit Euler method.

Notice how in both cases, the system energy increases over time when it should remain constant. This is because we are adding the squares of position and velocity, which is comparable to adding their amplitudes within a scale factor. However, as the global error plots show, the amplitude of our numerical solutions is increasing/decreasing (Explicit/Implicit) relative to that of the analytic answers, whose amplitudes remain constant. These changing amplitude of our estimates is why our energy grows or shrinks over time.



(a) $X_0 = 2$, $V_0 = 1$

(b) $X_0 = 2$, $V_0 = 1$

Figure 8: After many cycles, the explicit solution grows while the implicit one shrinks.