Applied Physics

&

PHY 495

Research Report - Spring 2023

Removing Unphysical Singularities in Photon Emission From Quark-Gluon Plasma

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Preface

To detect the spikes and modify them using modified Bessel functions of the first kind and second kind given the width of quasi-particles.

Abstract

Context: *The study on the quark-gluon plasma is essential to understand the fundamental physics. This paper is meant as the continuum of the effort in using mathematical models within the paper [1] and [2] to simulate the emission of electromagnetic waves within the plasma from particle, antiparticle splitting as well as particle-particle annihilation process in hope to handle the issues with infinite emission rate due to clean limit approximation in the derivation.*
Aims: *The goal is to detect where the infinite emission rates are and to modify the rates of emission using modified Bessel functions. Since it is shown that the emission rate of infinity within the plasma field is nonphysical, it is essential to find a clever way of approaching the limit approximation.*
Methods: *The local infinite emission rate can the be detected computationally when comparing to the surrounding emission rate due to its sudden raise and drop. One can also find these locations using an analytical approach. With the knowledge of where the singularities are, it is possible to show that these rates of emissions could be recalculated with the new approach of handling the limit approximation such that the total emission stays the same.*
Results: *It is possible to come up with a solution to work with a single singularity and transform to a smooth function. However, we find a challenging problem when handling the continuity between the singularities.*

Keywords

Applied Physics; Quark-Gluon plasma

Contents

1 Introduction

Two heavy ions approaching each other creates a magnetic field, but what important for our interest is magnetic field being partially trapped within the plasma. The collision of heavy ions and their destruction are the birth of a quark-gluon plasma which might also include a very strong magnetic field from our reasoning above. As the result of the proposed background magnetic field, the charged quarks will radiate emissions if they move in this hypothetical magnetic field. Why do we believe in the existence of this magnetic field? From our classical study on Electromagnetic, any moving charge will create currents which in turn creates a magnetic field. By studying how the rate of emission differs due to the magnetic field, we can learn a lot of information from the early universe around 20 microseconds when strong magnetic fields were present and how they would affect the evolution of our universe or the current physical phenomenons such that of thermal emission of Magnetars where Lorentz symmetry is believed to be broken at a relativistic motion. We can as well study the bulk properties of quark gluon plasma which is at a great importance.

In [1], the main approach was to study the properties of relativistic plasma in the regime of a strongly magnetized region. Through the study of professor Shovkovy on the field, he proposed the method of using Schwinger's result for the fermion Green's function in a background magnetic field to study the photon polarization properties. Accounting for the cyclotron motions for the charged particles within the background magnetic field, the interactions of particle, antiparticle splitting as well as the annihilation process of two charged particles will result in emissions of the electromagnetic waves. Note that the approach is to use QED formulations to approach QCD problems where the absorptive parts were treated as synchrotron emissions. The main difference of these two field formulations comes from the magnitude of the coupling constant. The approach in the paper assumes only the one loop Feynman diagram for a great simplification. In reality, there are infinite many ways how the emissions of photons can happen compared to the three main processes that the paper proposed. For example, processes like breaking radiations or Compton scatterings can also occur. Moreover, a photon can split into a pair of particles and then annihilate each other. Such that, they can do so infinitely and there appears to no restriction why the process should not happen. One can express these processes in Feynman diagrams and show that there are a lot more ways that the process could happen. Though, it is a good approximation since QFT shows that these processes happen at a very small percentage. At the end, by utilizing the conservation of energy, we can ultimately compute the locations of emissions in a polar coordinate. After working out the locations computationally and analytically, these emissions later could be shown as infinite rates of emissions which propose a problem in the approach of the limit.

The existence of spikes in numerical data usually means some unusual, interesting phenomenons or simply the ensuing results of certain assumptions. The project addresses the problem with the infinite rate of emission that professor Shovkovy was addressing in his paper. My hope is to show a clear physical and mathematical background of the approach alongside with the attempt that smooths out the spiky singularities.

By approaching the problem accounting for the interaction of the particles themselves, the problem could be accounted for after the new introduced value, the width of quasi-particle Γ. The approach is to introduce a Gaussian profile that replaces the delta function in the formulation to preserve the conservation of energy. The new approach leads to an approximation of utilizing the modified Bessel functions where these functions are well known for tackling singularities. The approximation is able to overcome the non physical singularities with a smooth distribution of rate of emission. The emission of the photons, the area under the singularities, is also shown to be conserved.

Here is how the paper will be organized. In chapter 2, we will overview the derivation as well as the physical process behind the formulation. Especially, we will go through how we replace the delta function with a Gaussian profile. We will go through several attempts in the approaches different from the approach proposed by the paper. At the end of chapter 3, we will summarize our research project and its implication.

2 Overview

2.1 The Derivation of Rate of Emission

Since all the derivations could be shown from paper [1], we will only try to show the important parts in the formulations where we want to look at to have a better sense of the physical process. Note that the whole derivation is based on working with a QED plasma where the fine constant is approximately 1/134, but the values of coupling constants or fine constants are much more different in a QCD plasma.

Let us assume the magnetic field to be pointing in the z direction. The choice of direction offers a great symmetry and easy to work with $\vec{A} = < -By, 0, 0 >$ where B is the magnetic field strength. In terms of field, the field strength tensor is $F^{\mu\nu} = -\epsilon^{0\mu\nu3}$. Now, we could take a look a the fermion green's function which gives the solution to inhomogeneous wave equations in QED.

$$G(t-t';\vec{r},\vec{r}') = e^{i\phi(r_\perp,r'_\perp)}\bar{G}(t-t';\vec{r}-\vec{r}')$$

where $\vec{r}$ denotes the position vector and $r_\perp$ is the position projection on the xy plane. Note that G is no longer translation invariant and could be broken under Lorentz transformation, but $\bar{G}$ remains translation invariant. Applying the Schwinger's result which only remains true for the leading one-loop order.

$$\phi(\vec{r}_\perp,\vec{r}'_\perp) \quad = -qB(x-x')(y+y')/2$$

We addressed the problem with one loop Feynman diagram that it is not the only situation where calculations should be accounted for. We also address that the work of QCD plasma at a finite temperature is not well developed because it violates Lorentz symmetry. The higher the temperature, the more broken the symmetry got. Nevertheless, we can use Matsubara formalism at a finite temperature for an one-loop order:

$$\Pi^{\mu\nu}(i\Omega_m;\vec{k}) = 4\pi N_f \alpha T \sum_{-\infty}^{\infty} \int \frac{dp_z}{2\pi} \int d^2\mathbf{r}_\perp e^{-i\mathbf{r}_\perp\cdot\mathbf{k}_\perp} tr[\gamma^\mu \bar{G}(i\omega_k,p_z;\mathbf{r}_p erp)]\gamma^\nu \bar{G}(i\omega_k - i\Omega_m, p_z-k_z;-\mathbf{r}_\perp)$$

Substituting Green's function, we can get the new one loop equation:

$$\Pi^{\mu\nu}(i\Omega_m;\vec{k}) = \frac{-\alpha N_f}{\pi l^4} \sum_{n,n'=0}^{\infty} \int \frac{dp_z}{2\pi} \sum_{\lambda=\pm1} \frac{(E_{n,p_z}-\lambda E_{n',p_z-k_z})(n_F(E_{n,p_z})-n_F(\lambda E_{n',p_z-k_z}))}{2E_{n,p_z}E_{n',p_z-k_z}[(E_{n,p_z}-\lambda E_{n',p_z-k_z})^2+\Omega_m^2]} \sum_{i=1}^{4} I_i^{\mu\nu}$$

where $E_n = \sqrt{m^2+p_z^2+2n|qB|}$ is the fermion energy at the nth Landau level. Since solving the real part alongside with the imaginary part of the equation could be very complicated, the paper only worked through the imaginary part such that $i\Omega_m \to \Omega + i\epsilon$. Thus, one can show the absorptive part of the equation to be:

$$\mathbf{Im}[\Pi^{\mu\nu}(i\Omega_m;\vec{k})] = \frac{\alpha N_f}{2l^4} \sum_{n,n'=0}^{\infty} \int \frac{dp_z}{2\pi} \sum_{\lambda=\pm1} \frac{n_F(E_{n,p_z})-n_F(\lambda E_{n',p_z-k_z})}{2\lambda\eta E_{n,p_z}E_{n',p_z-k_z}} \sum_{i=1}^{4} I_i^{\mu\nu}\delta(E_{n,p_z}-\lambda E_{n',p_z-k_z}+\eta\Omega)$$

The most important piece of this formulation is the argument of delta function when the absorptive part disappears. That is:

$$E_{n,p_z} - \lambda E_{n',p_z-k_z} - \eta\Omega = 0$$

One can analytically find what p_z is and perform calculations to find the energy of landau level at p_z. Note that we have both parameters λ and η where sign(λ) and sign(η) will decide the type of emission. The splitting process happens when $\lambda = +1$ where particle splitting is denoted as $\eta = -1$ while the antiparticle splitting is denoted as $\eta = +1$. The last process will be annihilation where $\eta = -1$ and $\lambda = -1$. Notice that when $\eta = +1$, the equation does not result in any physical situation. Therefore, we have three main processes in the calculation.

Using quantum field theory, we can find photon production rate from a thermally equilibrium charged plasma for our calculated polarization tensor:

$$\Omega\frac{d^3R}{d^3\mathbf{k}} = -\frac{1}{(2\pi)^3}\frac{\mathbf{Im}\Pi^{\mu\nu}(\Omega_m;\vec{k})}{exp(\frac{\Omega}{T})-1}$$

Where the photon energy and photon momentum are Ω nad $\mathbf{k}$ respectively. This rate of production is across all space, but we can use the symmetry of the problem to reduce it to a polar space. As a result, the polarization tensor reduces to $d^3\mathbf{k} = k^2dk\sin(\theta)d\theta d\phi$ for the photon momentum. This is the result of only accounting for θ such that ϕ can be neglected due to the symmetry of the problem. The rate of emission will reduce to a two polar dimensional rate of production.

$$\frac{d^2R}{kdkd(\cos\theta)} = -\frac{1}{(2\pi)^2}\frac{\mathbf{Im}\Pi^{\mu\nu}(k;\vec{k})}{exp(\frac{\Omega}{T})-1}$$

where $k = \Omega$. Since we want to find the rate of production in the reference frame of the magnetic field, we want to perform Lorentz contraction on the absorptive part from the frame of moving charged particles back to the origin of the background magnetic field.

The absorptive part becomes:

$$\mathbf{Im}\Pi^{\mu\nu}(\Omega;\vec{k}) = \frac{\alpha N_f}{4\pi l^4}\sum_{n>n'}^{\infty}\theta(k_-^2+k_z^2-\Omega^2)\sum_{s=\pm1}\frac{n_F(E_{n',p_z-k_z})-n_F(E_{n,p_z})}{\sqrt{(k_-^2+k_z^2-\Omega^2)(k_+^2+k_z^2-\Omega^2)}}\sum_{i=1}^{4}\mathcal{F}_i|_{p_z=p_z^{(s)},\lambda=1,=-1}$$
$$+\frac{\alpha N_f}{4\pi l^4}\sum_{n<n'}^{\infty}\theta(k_-^2+k_z^2-\Omega^2)\sum_{s=\pm1}\frac{n_F(E_{n,p_z})-n_F(E_{n',p_z-k_z})}{\sqrt{(k_-^2+k_z^2-\Omega^2)(k_+^2+k_z^2-\Omega^2)}}\sum_{i=1}^{4}\mathcal{F}_i|_{p_z=p_z^{(s)},\lambda=1,=1}$$
$$+\frac{\alpha N_f}{4\pi l^4}\sum_{n,n'=0}^{\infty}\theta(\Omega^2-k_-^2-k_z^2)\sum_{s=\pm1}\frac{n_F(E_{n',p_z-k_z})+n_F(E_{n,p_z})-1}{\sqrt{(\Omega^2-k_-^2-k_z^2)(\Omega^2-k_+^2-k_z^2)}}\sum_{i=1}^{4}\mathcal{F}_i|_{p_z=p_z^{(s)},\lambda=1,=-1}$$

After substituting the new absorptive part back into the rate of production formula, one will get:

$$\frac{d^2R}{kdkd(\cos\theta)} = \frac{\alpha N_f}{(2\pi)^3 l^4[exp(k/T)-1]}\sum_{n>n'}^{\infty}\frac{g(n,n')[\theta(\Omega^2-k_z^2-k_+^2)-\theta(k_z^2+k_+^2-\Omega^2)]}{\sqrt{(k_-^2+k_z^2-\Omega^2)(k_+^2+k_z^2-\Omega^2)}}(\mathcal{F}_1+\mathcal{F}_4)$$
$$+\frac{\alpha N_f}{(2\pi)^3 l^4[exp(k/T)-1]}\sum_{n=0}^{\infty}\frac{g_0(n)\theta(\Omega^2-k_z^2-k_+^2)}{\sqrt{\Omega^2-k_z^2)(\Omega^2-k_+^2-k_z^2)}}(\mathcal{F}_1+\mathcal{F}_4)$$

2.2 Replacing Delta Function by Gaussian Profile

As mentioned for the approach to the limit approximation, the results show many spiky shapes within the rate of production. To resolve this problem, one can introduce a new parameter, the width of a quasi-particle Γ. We have only accounted for the interactions of charged particles with the background magnetic field. In real interaction, one should also account for the interaction of charged particles with one another. As a result, charged particles should be introduced with a quasi-particle width. Therefore, it is reasonable to replace a delta function which is the main reason for spiky peaks by a Gaussian profile such that:

$$\begin{aligned} &\delta(E_{n,p_z}-\lambda E_{n',p_z-k_z}-\eta\Omega) \\ &\rightarrow \frac{1}{\sqrt{2\pi}\Gamma}\int_{-\infty}^{\infty} d\epsilon e^{\frac{-\epsilon^2}{2\Gamma^2}}\delta(E_{n,p_z}-\lambda E_{n',p_z-k_z}-\eta\Omega+\epsilon) \\ &= \frac{1}{\sqrt{2\pi}\Gamma}\int_{-\infty}^{\infty} d\Omega' e^{\frac{-(\Omega'-\Omega)^2}{2\Gamma^2}}\delta(E_{n,p_z}-\lambda E_{n',p_z-k_z}-\eta\Omega') \end{aligned}$$

where $\Omega' = \Omega+\epsilon$. The replacement of the Gaussian function will result in the change of the shape such that the spiky shapes will be replaced by Gaussian shapes determined by the width of quasi-particle. Thus, our absorptive part becomes:

$$\mathbf{Im}\Pi^{\mu\nu}(\Omega;\vec{k}) \to \frac{1}{\sqrt{2\pi}\Gamma}\int_{-\infty}^{\infty} d\Omega' e^{\frac{-(\Omega'-\Omega)^2}{2\Gamma^2}}\mathbf{Im}\Pi^{\mu\nu}(\Omega';\vec{k})$$

If we treat Ω_{thr} to be the positions where spiky results are present, the replacement with Bessel functions could be introduced to approximate the transformation of a smooth function to a peak:

$$\begin{aligned}
\frac{\theta(\Omega-\Omega_{thr})}{\sqrt{\Omega-\Omega_{thr}}}f(\Omega) &\to \frac{1}{\sqrt{2\pi}\Gamma}\int_{-\infty}^{\infty} d\Omega^{'} e^{-\dfrac{(\Omega^{'}-\Omega_{thr})^2}{2\Gamma^2}}\frac{\theta(\Omega^{'}-\Omega_{thr})}{\sqrt{\Omega^{'}-\Omega_{thr}}}f(\Omega^{'}) \\
&\approx \sqrt{\frac{\pi|\omega|}{8\Gamma}}e^{\frac{-\omega^2}{4}}\left\{\theta(\omega)\left[I_{-1/4}\left(\frac{\omega^2}{4}\right)+I_{1/4}\left(\frac{\omega^2}{4}\right)\right]+\frac{\sqrt{2}}{\pi}\theta(-\omega)K_{1/4}\left(\frac{\omega^2}{4}\right)\right\}f(\Omega_{thr}) \\
&\doteq f^{'}(\Omega_{thr})
\end{aligned}$$

where $\omega = (\Omega - \Omega_{thr})/\Gamma$ is a dimensionless parameter and $f(\Omega)$ is the smooth function. One could think about the formulation above as an operator acting on the smooth function $f(\Omega_{thr})$ to transform it into a peak of $f^{'}(\Omega_{thr})$. The meaning of an operator might be confusing since it has a lot of meanings, but it simply means a function that changes from a shape to another shape in our case. However, we still need to do some work before implementing them.

2.3 From Photon Energy Dependence to Angular Dependence

In the previous subsection, we have arrived the formulation that we will mainly work with in the project. However, it is still not the main formulation yet since we have some change of variable to work with. Looking at the figure $1(c)$, one could notice that we want to reproduce the polar rate of production that is angular dependence instead of energy dependence. Therefore, one must do a change of variable from Ω to θ before we do any simulation.

For some interesting takes from the graph, even after we try to replace the spiky emissions with Gaussian distribution, we still should preserve the properties of the physical phenomenons:

- The shape gets from a prolate shape to an oblate shape as the photon energy increases.

- The rates of emissions should be populated at $\pi/2$ which is perpendicular to the magnetic field as energy increases.

We want to handle the rate of emission as a function of angular dependence. The work was also done in the paper so we will present the equation after the change of variable from $\Omega \to \theta$

as follow:

$$\frac{\theta(\Omega\sin\theta - k_+)}{\sqrt{\Omega\sin\theta - k_+}}f(\theta) + g(\theta) \to \frac{1}{\sqrt{2\pi}\Gamma}\int_{-\infty}^{\infty} d\Omega' e^{-\dfrac{(\Omega' - \Omega)^2}{2\Gamma^2}} \frac{\theta(\Omega' - k_+)}{\sqrt{\Omega' - k_+}} f(\theta)$$

$$\approx \sqrt{\frac{\pi|\omega|}{8\Gamma\sin\theta}} e^{\frac{-\omega^2}{4}} \left\{\theta(\omega)\left[I_{-1/4}\left(\frac{\omega^2}{4}\right) + I_{1/4}\left(\frac{\omega^2}{4}\right)\right] + \frac{\sqrt{2}}{\pi}\theta(-\omega)K_{1/4}\left(\frac{\omega^2}{4}\right)\right\} f(\theta) + g(\theta)$$

$$\to f'(\theta) + g(\theta) \quad (1)$$

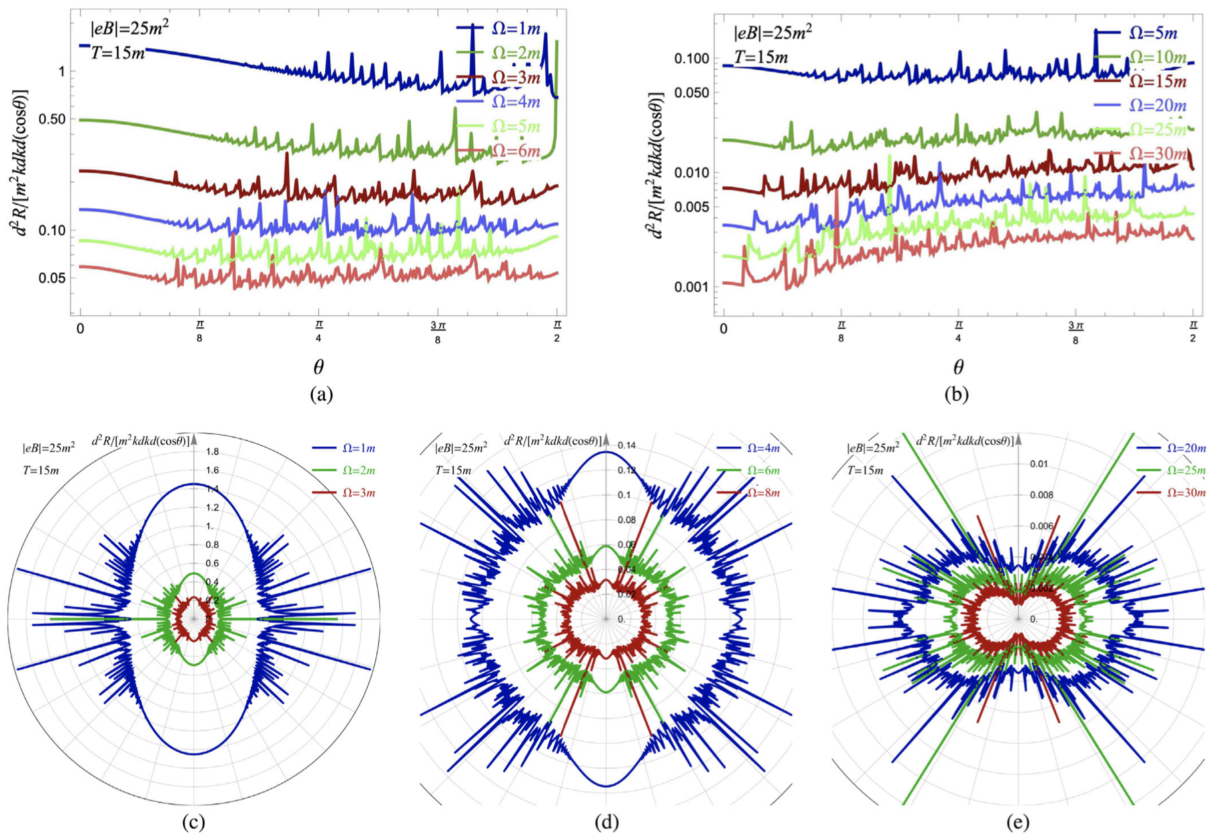

Figure 1: The differential photon production rate as a function of the angle θ for $|qB| = 25m^2$ and $T = 5m$ [1]

where $g(\theta)$ is a smooth function that $f'(\theta)$ will be added onto and $\omega = (\Omega\sin\theta - k_+)/(\Gamma\sin\theta)$. However, we want to rewrite ω such that the difference of step can be easy to regconize. We could see that ω becomes 0 when $\Omega\sin\theta_{thr} = k_+$ and solutions exist only when $k_+ \leq \Omega$ for a valid sinusoidal function. Thus, rewriting the term in terms of steps from singularities becomes:

$$\omega \approx \frac{\Omega\sqrt{\Omega^2 - k_+^2}}{k_+\Gamma}(\theta - \theta_{thr})$$

Note that the formula above is only valid for the annihilation process. If one wish to represent the process for splitting, the same derivation could be shown.

$$\frac{\theta(k_- - \Omega\sin\theta)}{\sqrt{k_- - \Omega\sin\theta}} f(\theta) + g(\theta) \rightarrow \frac{1}{\sqrt{2\pi}\Gamma}\int_{-\infty}^{\infty} d\Omega' e^{-\frac{(\Omega' - \Omega)^2}{2\Gamma^2}} \frac{\theta(k_- - \Omega')}{k_- - \sqrt{\Omega'}} f(\theta)$$
$$\approx \sqrt{\frac{\pi|\omega|}{8\Gamma\sin\theta}} e^{\frac{-\omega^2}{4}} \left\{\theta(\omega)\left[I_{-1/4}\left(\frac{\omega^2}{4}\right) + I_{1/4}\left(\frac{\omega^2}{4}\right)\right] + \frac{\sqrt{2}}{\pi}\theta(-\omega)K_{1/4}\left(\frac{\omega^2}{4}\right)\right\} f(\theta) + g(\theta)$$
$$\rightarrow f'(\theta) + g(\theta) \quad (2)$$

where $\omega = (k_- - \Omega\sin\theta)/(\Gamma\sin\theta)$ or in terms of steps as follow:

$$\omega \approx \frac{\Omega\sqrt{\Omega^2 - k_-^2}}{k_-\Gamma}(\theta_{thr} - \theta)$$

After transforming the function, there are a few things we want to double check. Firstly, it is the width of the peak:

$$\Delta\theta \approx \frac{k_-\Gamma}{\Omega\sqrt{\Omega^2 - k_-^2}}$$

One can notice that the change of θ gets smaller as energy increases. In another word, the quasi-particle width will not be great at smoothing a system with high energy. Seemingly, it suggests that particles' interaction become less important at higher energy.

When computing for data within a supercomputer, the analytical positions for peaks will differ to the ones generated due to resolution of data. Thus, it is essential to also find out the analytical values. We previously noticed that $\Omega\sin\theta = k_-$ and $\Omega\sin\theta = k_+$. Knowing the values:

$$\begin{aligned} k_- &= |\sqrt{m^2 + 2n|qB|} - \sqrt{m^2 + 2n'|qB|}| \\ k_+ &= |\sqrt{m^2 + 2n|qB|} + \sqrt{m^2 + 2n'|qB|}| \end{aligned}$$

Plugging k_- back in will give us the position of θ_{thr} for splitting process.

$$\begin{aligned} \frac{\Omega}{m}\sin\theta &= |\sqrt{1 + 2n\frac{|qB|}{m^2}} - \sqrt{1 + 2n'\frac{|qB|}{m^2}}| \\ \theta &= \arcsin\left(\frac{m}{\Omega}|\sqrt{1 + 2n\frac{|qB|}{m^2}} - \sqrt{1 + 2n'\frac{|qB|}{m^2}}|\right) \end{aligned}$$

Similarly for annihilation process, plugging k_+ will give us the location of annihilation process.

$$\begin{aligned} \frac{\Omega}{m}\sin\theta &= |\sqrt{1+2n\frac{|qB|}{m^2}}+\sqrt{1+2n'\frac{|qB|}{m^2}}| \\ \theta &= \arcsin\left(\frac{m}{\Omega}|\sqrt{1+2n\frac{|qB|}{m^2}}+\sqrt{1+2n'\frac{|qB|}{m^2}}|\right) \end{aligned}$$

where n' is the initial landau level and n is the final landau level. Note that, the whole argument within the sinusoidal function can not be bigger than one, and θ will be only from 0 to $\pi/2$ as a result. The emission process could be calculated only within the first quadrant of the polar coordinate. However, the symmetry of the problem shows that all other three quadrants will be the replicate of the first quadrant.

2.4 Miscellaneous

Before going into the next section, I want to note that it is not possible to transform the singularities into Gaussian profiles while maintaining the continuity. The properties of modified Bessel functions show that the end points can be only be the same as we go infinitely away from the singularity. Therefore, maintaining the continuity has to include some manipulations of data. However, I used the combination between a Bessel function and a Gaussian function to keep the continuity at one end if the peaks are not close together. Similarly, if the peaks are very packed together, I recommend using two Gaussian functions to smooth the peaks given valid ranges.

2.5 Simulation and Analysis

Firstly, we have defined the main frame work in the previous section. From (2), I want to define a peak operator W that is simply the expression before f as:

$$\sqrt{\frac{\pi|\omega|}{8\Gamma\sin\theta}}e^{\frac{-\omega^2}{4}}\left\{\theta(\omega)\left[I_{-1/4}\left(\frac{\omega^2}{4}\right)+I_{1/4}\left(\frac{\omega^2}{4}\right)\right]+\frac{\sqrt{2}}{\pi}\theta(-\omega)K_{1/4}\left(\frac{\omega^2}{4}\right)\right\} \tag{3}$$

We then define $\theta(\omega)$ as a heavy-side function such that

$$\theta(\omega)=\begin{cases}1 & \text{if } \omega>0\\ 0 & \text{if } x\leq 0\end{cases}$$

```python
def theta_heavyside(omega):
    """This function defines a heavy step function of omega"""
    return 1 if omega > 0 else 0
```

Given a smooth data or function, it is possible to transform it to a peak by applying W on f such that $W * f = f'$ becomes the new values of the peak. It is tempting to think that we can also go back from a peak to a smooth function with $W^{-1} * f' = f$ where W^{-1} is defined as the smooth operator. Because of the nature of the peak, either the left or the right side of the peaks will become 0 losing the necessary information to transform it back. However, we have the knowledge of the smooth function to be a Gaussian profile and the area is conserved. Let:

A_{peak} be the emission of the peak.

A_{rever} be the emission of the smooth function that is reversible by using modified Bessel functions.

A_{irrev} be the emission of the smooth function that is irreversible and defined as a Gaussian profile.

Knowing the peak is shifted by $\Omega^*_{thr} \approx \Omega_{thr} + .765\Gamma$ for annihilation process, we could make a change of variable with $\Omega' \sin\theta = k_+$. The shifted peak is located at:

$$\theta \approx \arcsin \frac{k_+}{\Omega' + .765\tau}$$

Similarly, one could derive the shifted peak for the splitting process:

$$\theta \approx \arcsin \frac{k_-}{\Omega' - .765\tau}$$

Where $\Omega^*_{thr} \approx \Omega_{thr} - .765\tau$. The location of the new peak indicates the new maximum value and a cutoff value. We have noticed that only nonzero values could be transformed with the smooth operator due to the nature of the peak while defining the zero values to be an evidence of where a Gaussian profile being acted by a peak operator. We can, therefore, find all the values of A_{peak}, A_{rever} and A_{irrev}.

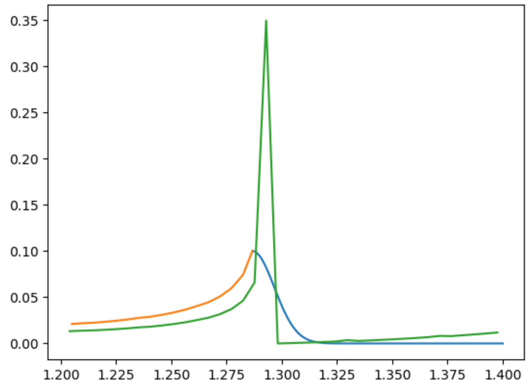

Figure 2: The smooth function in comparison with the peak function.

Knowing the irreversible half is a Gaussian profile:

$$f(x) = a \exp^{-\frac{(x-b)^2}{2\tau^2}}$$

where a is the maximum value at the cutoff value b with a width of τ. The values of a and b are known to the Gaussian profile. Knowing that:

$$\int_{-\infty}^{\infty} a \exp^{-\frac{(x)^2}{2\tau^2}} = a\sqrt{\pi 2\tau^2} = k$$

Solving for τ gives us:

$$\tau = \sqrt{(k/a)^2/(2\pi)}$$

Thus, we can recover the last half of the smooth function with the three parameters a, b and τ.

2.5.1 Background Function

The generations of data come from two main processes that we introduced that are splitting and annihilation. For this section, we introduce a simple way to define the smooth background function $g(\theta)$. Figure 1 shows the splitting process where the peak suddenly jumps off from the left side. Meanwhile, the annihilation process has one point jumps up to a peak from the left side. We took the advantage of the symmetry such that we will collect the data points on the right side of splitting process and the left side of annihilation process so that we can train the data points to arrive with the background linear line $g(\theta)$. However, this is only one of the many ways to generate the background smooth function.

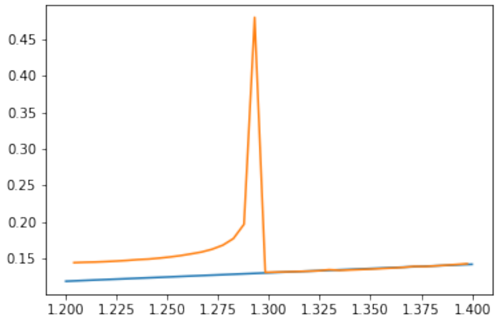

Figure 3: Background function as a linear line.

How exactly do we train the data points and get a linear line? It turns out that there is one method well explored and used widely in machine learning called linear regression. Before going to this, there is a difference between interpolation and linear regression technique, but we will not go into the topic. And, the discussion on the training has little to no interests to our study, but only for the sake of a discussion. We will start with a prediction or an hypothesis $h_\theta(x) = \theta_0 + \theta_1 x_1$. In fact, we can expand this if we have many training features $h(x) = \sum_{i=0}^{n} \theta_i x_i = \theta^T \vec{x}$ where $x_0 = 1$. But for our problem, there is only one feature that is x. Let's introduce the cost function which is $J(\theta) = 1/2 \sum_{i=1}^{n} (h(x^{(i)} - y^{(i)}))^2$ where $x^{(i)}$ and $y^{(i)}$ are the data points to be trained. The cost function comes from assuming a Gaussian distribution of our data such that the data is not correlated. By minimizing this cost function, we can find the parameters θ to fit the data points to a linear line.

$$\theta_j := \theta_j + \alpha(y^{(i)} - h(x^{(i)}))x_j^{(i)} \tag{4}$$

where α is the learning rate. We will repeat this process to convergence for θ_0 and θ_1.

This technique however limits us to the data being trained such that we only take a portion of the data to find the linear function. It becomes quite a task if we start dissecting the data from the main data to train each single peak. For example, smaller peaks will be hard to find the good data to train the system. And, we are also limited to a linear background function. One alternative way is to change our hypothesis function to higher order function for a polynomial fit, but this method will encounter large problems with outliners and usually overfit the data. To approach smaller peaks, it might be tempting to use a method such as weighted linear regression.

The weighted linear regression technique takes more weight on the local area of a certain position to come up with a linear line. Thus, peaks that are not close to the data point will not be accounted for much. However, our data acts as a non derivative function such that the prediction will change unexpectedly. Therefore, I will stick to linear regression in the approach.

2.5.2 Choices of Peak

We have mentioned the limitation of linear regression. There will be peaks that we do not have enough good data to handle. Besides that, there appear many infinite rates of emissions in the analytical approach that were not shown while found computationally. This shows another problem such that we will not able to smooth out every peak, but we can choose another feature to choose out the peaks we want to work with. Thus, we need to introduce a parameter Δh such that the peak's differnce with the smooth function will determine whether that peak should be chosen or not. Let us define:

$$\Delta h = f'(\theta_{thr}) - \mathbf{min}(g(\theta_k)) \tag{5}$$

where θ_k is any point near the threshold and $g(\theta_k)$ is our background function found by linear regression technique. The cutoff choice of Δh might be depending our interest. The smaller Δh is, the more peaks we will have to work with.

2.5.3 Translation

In an effort to transform the peak, one could naively let $g(\theta)$ become an x-axis such that we will do a horizontal, vertical, and rotational transition on $f(\theta)$. However, we are working with angular dependence such that a rotational translation would result in $f(\theta)$ being not a function as shown in Figure 4. This means that our function will not be able to become one to one mapping in practice. We also do not need to do a horizontal transition since the Bessel functions allow us to work with even not at the origin because our data does not have a negative θ. Therefore, we will only translate f vertically such that the peak function's minimum value lies on the x-axis. But before doing that, we must account for one problem that we previous mentioned.

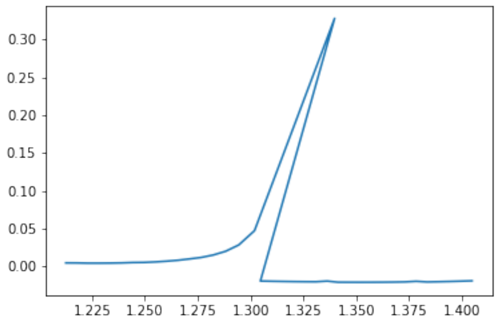

Figure 4: Peak function after the three main transitions

2.5.4 Locations of Peaks

We showed that it is possible to find the locations of thresholds analytically, but it is also possible to find the locations of the peaks through the data. Which one should we use? The green points are the positions of the peaks where we should get analytically while the orange points show the peaks through the data found computationally ($n_{max} = 29$). Notice however, if we increase the landau level's maximum, then the distribution of green points might be even more packed. The peaks from orange points and green points however differ in x-values when they practically should be equal because the resolutions in computing the data created a certain error range in step function.

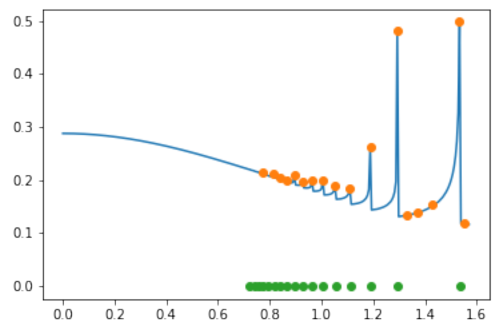

Figure 5: Peaks found analytically (green) and computationally (orange)

2.6 The Smooth Operator and The Python Simulation

There are a lot more work that could be done. However, I only partially worked out the problem, the code could be viewable through:

https://github.com/nhula01/polymath/tree/main/physics/projects/Smooth_operator

3 Summary and Conclusion

We were able to come up with a solution to recover the smooth function using the modified Bessel functions as well as the Gaussian profile. We found it to be challenging to deal with the continuity of peaks without manipulating the data. However, our algorithms can easily transform any function given from a peak to a smooth function or even a smooth function to a peak. Even though we were not able to work with the continuity of the peaks, we have made a great initial step to working with a singularity as well as finding a physical meaning for the alternative. To approach the continuity, we will have to make certain assumptions and it might be another great step to work with in our future study.

References

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