Statistical and Mathematical Methods DS-GA 1002
, 2019

Recitation Session 8 Solutions

Problem

1. *Sinusoidal Signal with Random Phase.* Define $X(t) = \alpha \cos(\omega t + \Theta)$ where $t \geq 0$, $\Theta \sim Unif[0, 2\pi]$ and α, ω are constant.

   a. Find CDF and PDF of $X(t)$

   b. Find mean of $X(t)$

   SOLUTION

   a. Note that, since uniform is invariant to shift, $\omega t + \Theta$ has the same distribution as Θ. Thus we have

   $$F_{X(t)}(x) = 1 - \frac{1}{\pi} \cos^{-1} \frac{x}{\alpha} \text{ and } f_{X(t)}(x) = \frac{1}{\alpha\pi\sqrt{1 - (\frac{x}{\alpha})^2}} \text{ where } -\alpha < x < \alpha$$

   b. By symmetry, we have mean

   $$E(\alpha \cos(\omega t + \Theta)) = 0$$

2. *Dice Rolling.* You can roll a die 3 times. You win X where X is the last roll you get. After each roll, you decide whether you should continue rolling or stop.

   a. What is the expected value of each roll?

   b. How should you decide if you want to re-roll the die to maximize your winnings?

   c. What is the expected value of your winnings?

   SOLUTION

   a. Let Y_i be the outcome of the ith roll.

   $$E[Y_i] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

   b. We should re-roll if the current roll is less than the expected value of the next roll.

   c.

   $$E[X] = E[X|R_1 = 1]P(R_1 = 1) + E[X|R_2 = 1]P(R_2 = 1) + E[X|R_3 = 1]P(R_3 = 1) \quad (1)$$

   First, we note that $E[X|R_3 = 1] = 3.5$. Knowing this, we would only roll the die a third time if we roll a 3 or smaller in the second roll. That means $E[X|R_2 = 1] = 5$ because we would only stop rolling after the second roll if we get a 4 or larger. Knowing this, we only stop rolling the die in the first throw if we get a 5 or 6. Therefore,

$$= E[X|R_1 = 1]P(X_1 \geq 5) + E[X|R_2 = 1]P(X_1 \leq 4, X_2 \geq 4) + E[X|R_3 = 1]P(X_1 \leq 4, X_2 \leq 3) \tag{2}$$

$$= 5.5 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} + 3.5 \cdot \frac{1}{3} = 4.667. \tag{3}$$

Note: As a few of you pointed out during lab, the fair value we should compare with for the first roll should be

$$4.25 = 0.5 * 5 + 0.5 * 3.5.$$

I convinced myself by assuming you are able to get fraction for the first roll. Suppose you get 4.5, then you should not roll the dice anymore because your expected payoff for continuing rolling is 4.25, which takes the payoff from both rolls into account. The answer just happened to be the same, but my reasoning is incorrect. My apologies for the confusion.

3. *Waiting Time.* Suppose $H_i \sim Unif[a, b]$ i.i.d. with $a < b$ and $i = 0, 1, 2...$ is the height of person you observed, sequentially. Let H_0 be your initial observation, and denote T the number of observations (in addition to H_0) it takes to find someone taller. What is $E(T)$? (Hint: (a) The distribution of H_i does not matter (b) Ordered Statistics)

   Solution

   When conditioned on the number of observations, which are observed, the event of interest is simply some particular permutations of the observed values. Suppose you observations are some ordered values $h_0 < h_1 < \cdots < h_k$, then T corresponds to $H_0 = h_{k-1}$, $H_k = h_k$ and H_i being any of the rest for $i \neq 0, k$. And hence we have

$$E(T) = E(E(T \mid \# \text{ of observations})) = \sum_L L \cdot \frac{1}{L+1} \cdot \frac{1}{L} \cdot 1 = +\infty$$

4. *Spaghetti in a Bowl.* Suppose you have a plate of spaghetti, no sauce, where you randomly choose two ends and tie them together until there's no end. Find the expected number of loops given n noddles.

   Solution

   Note $E(1) = 1$ and, in general, condition on whether you pick the same noddle, we would have

$$E(n) = E(n-1 \mid same)p(same) + E(n-1 \mid not\ same)p(not\ same) \tag{4}$$

$$= (E(n-1)+1)\frac{1}{2n-1} + E(n-1)(1 - \frac{1}{2n-1}) \tag{5}$$

$$\text{rearrange gives } E(n) = E(n-1) + \frac{1}{2n-1}, \tag{6}$$

   and we can solve $E(n)$ recursively.