Recitation Session 1

Problem

1. *Dependence.* From a well shuffled standard deck of 52 cards, pick two. What is the probability that the *second* card is black?

   Solution

   a. One might wonder whether the outcome of the first card has any effect. One can compute considering all possible cases for the first card. Let $B_{1,2}$ and $R_{1,2}$ denote the events that the first and the second card turns black and red, respectively:

   $$\begin{aligned} \mathrm{P}(B_2) &= \mathrm{P}(B_2 \;\&\; B_1) + \mathrm{P}(B_2 \;\&\; R_1) && (1)\\ &= \mathrm{P}(B_2|B_1)\,\mathrm{P}(B_1) + \mathrm{P}(B_2|R_1)\,\mathrm{P}(R_1) && (2)\\ &= \frac{25}{51}\frac{26}{52} + \frac{26}{51}\frac{26}{52} && (3)\\ &= \frac{1}{2} && (4) \end{aligned}$$

   Discussion: The result is as if it didn't matter what card I picked for the first one. This makes sense because there is no information on the outcome of the first card. Therefore we could have argued that $P(\text{first card is black}) = P(\text{second card is black})$. In fact in the extreme case think of the following problem: given a deck, pick 52 cards, throw away the first 51 ones that are picked and, what is the probability that the last card is Black. This is the same as picking just one card from the deck!

2. *Set.* Show that $\mathbb{P}(E^C \cap F^C) = 1 - \mathbb{P}(E) - \mathbb{P}(F) + \mathbb{P}(E \cap F)$.

   Solution

   $$\begin{aligned} E^C \bigcap F^C &= \Omega - E \bigcup F \\ &= \Omega - (E + F - E \bigcap F) \\ \Rightarrow \Pr(E^C \bigcap F^C) &= 1 - \Pr(E) - \Pr(F) + \Pr(E \bigcap F) \end{aligned}$$

3. *Conditional probability.* The probability that a child has blue eyes is 1/4. Assume independence between children. Consider a family with 3 children.

   a. If it is known that at least one child has blue eyes, what is the probability that at least two children have blue eyes?

   Solution

   $$\begin{aligned} \frac{1 - \Pr(0 \text{ kid has blue eyes}) - \Pr(1 \text{ kid has blue eyes})}{1 - \Pr(0 \text{ kid has blue eyes})} &= \frac{1 - \left(\frac{3}{4}\right)^3 - \binom{3}{1}\frac{1}{4}\left(\frac{3}{4}\right)^2}{1 - \left(\frac{3}{4}\right)^3} \\ &\approx 0.27 \end{aligned}$$

a. if it is known that the youngest child has blue eyes, what is the probability that at least two children have blue eyes? **Solution**

$$\begin{aligned}
A_1 &= \{\text{first kid has blue eyes}\};\\
A_2 &= \{\text{second kid has blue eyes}\};\\
A_3 &= \{\text{third kid has blue eyes}\};
\end{aligned}$$

$$\begin{aligned}
1 - \Pr(A_1^C \bigcap A_2^C | A_3) &= 1 - \frac{\Pr(A_1^C \bigcap A_2^C \bigcap A_3)}{\Pr(A_3)}\\
&= 1 - \frac{\Pr(A_1^C)\Pr(A_2^C)\Pr(A_3)}{\Pr(A_3)} \text{ because } A_1 A_2 A_3\\
&= 1 - \Pr(A_1^C)\Pr(A_2^C)\\
&= 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}
\end{aligned}$$

4. *Conditional Independence.* Let's define X_1 and X_2 as conditionally independent given Y if $\mathbb{P}(X_1|X_2,\ Y) = \mathbb{P}(X_1|Y)$. Does independence imply conditional independence? Give an example.

   Solution Assume X_1, X_2 are two independent variables, taking on values $\{0, 1\}$. $X_1, X_2 \sim Bernoulli(0.5)$. They are not conditionally independent given their sum Y. e.g. $\Pr(X_1 = 1|X_2 = 1, Y = 1) = 0$.

5. *Bayes' Formula.* Given 1,000 coins, 1 coin has heads on both sides while the other are fair coins. One coin is randomly chosen and tossed 10 times. Each time, it turns up heads. What's the probability that the coin is the unfair one?

   Solution Assume

$$\begin{aligned}
\mathbb{P}(Bias|10H) &= \frac{\mathbb{P}(Bias \cap 10H)}{\mathbb{P}(10H)}\\
&= \frac{1 * \frac{1}{1000}}{1 * \frac{1}{1000} + 0.5^{10} * \frac{999}{1000}}\\
&= \frac{1}{1 + 0.5^{10} * 999}\\
&= \frac{1024}{1024 + 999}\\
&= 0.506
\end{aligned}$$

Recap

1. Independence: Two events A and B are independent if and only if $\mathbb{P}(A|B) = P(A)$.
2. Conditional Probability: If $\mathbb{P}(B) > 0$ then the conditional probability of A given B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

3. Bayes' Rule : Let $\{A_i\}_{i=1}^k$ be a partition of the sample space and B be any event. Then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^k \mathbb{P}(B|A_j)\mathbb{P}(A_j)}, \ \forall i$$

where the numerator equals $\mathbb{P}(B \cap A_i)$ and the denominator is $\mathbb{P}(B)$.

4. The law of total probability: Let $\{A_i\}_{i=1}^k$ be a partition of the sample space and B be any event. Then

$$\mathbb{P}(B) = \sum_{j=1}^k \mathbb{P}(B \cap A_j) = \sum_{j=1}^k \mathbb{P}(B|A_j)\mathbb{P}(A_j)$$

where $\{B \cap A_j\}_{j=1}^k$ is a disjoint partition of B.