Statistical and Mathematical Methods DS-GA 1002
, 2019

Recitation Session 9 Solutions

Problem

1. *Sinusoidal Signal.* Find mean, autocorrelation and autocovariance of:
   a. Define $X(t) = A\cos(2\pi t)$ where $t \geq 0$ and A some R.V.
   b. Define $Y(t) = \cos(ct + \Theta)$ where $t \geq 0$, $\Theta \sim Unif[0, 2\pi]$ and c is constant.

   Solution

   a. $E(X(t)) = E[A]cos2\pi t$
   $R_X(t_1, t_2) = E[A\cos 2\pi t_1 A \cos 2\pi t_2] = E[A^2]\cos 2\pi t_1 A \cos 2\pi t_2$
   $C_X(t_1, t_2) = R_X(t_1, t_2) - E(X(t_1))E(X(t_2)) = \mathrm{Var}(A)\cos 2\pi t_1 A \cos 2\pi t_2$
   b. By symmetry, we have mean
   $$E(Y(t)) = E(\cos(\omega t + \Theta)) = 0$$
   and
   $C_Y(t_1, t_2) = R_Y(t_1, t_2) = E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)]$
   $= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[\cos[\omega(t_1 - t_2)] + \cos[\omega(t_1 + t_2) + 2\theta]]d\theta = \frac{1}{2}\cos(\omega(t_1 - t_2))$

2. *Random Walk with Two People.* Two people each move in a one-dimensional random walk. The properties of the random walks are identical and independent for the two, that is, each takes a forward step with probability p and a backward step with probability $q = 1 - p$. The steps are of equal size and are taken at the same times. Assume that the two people start in the same location.
   a. For each person, what is the distribution of being D steps away (to the right) from the origin?
   b. Given that the two start their random walk at the same location, what is the probability that they meet back at that location after they have each taken n steps? Assume each direction has an equal probability.

   Solution

   a. Let R represent the number of steps to the right. Then $D = R - (n - R) = 2R - n$. The probability of taking R steps to the right given n total steps is binomial with $n = n$:
   $$P_n(r) = \frac{n!}{r!(n-r)!}p^r q^{n-r}. \quad (1)$$

b. We know $R = \frac{D+n}{2}$ and $n - R = \frac{n-D}{2}$ so we have

$$P_n(r) = \frac{n!}{\left(\frac{D+n}{2}\right)!\left(\frac{n-D}{2}\right)!}p^{\frac{D+n}{2}}q^{\frac{n-D}{2}}. \quad (2)$$

There are $2n$ steps in total and for both people to meet at the origin, $D = 0$ so

$$P_{2n}(n/2) = \frac{(2n)!}{\left(\frac{2n}{2}\right)!\left(\frac{2n}{2}\right)!}p^n q^n \quad (3)$$

3. *Poisson Process* Suppose an office receives two different types of inquiry: persons who walk in off the street, and persons who call by telephone. Suppose the two types of arrivals are described by independent Poisson process, with rate λ_w for walk-ins, and rate λ_c for the callers. What is the distribution of the number of telephone calls received before the first walk-in customer?

Solution

Let T for the arrival time of the first walk-in, and let N be the number of calls in $[0, T)$. The time T has a continuous distribution, with the exponential density

$$f(t) = \lambda_w e^{-\lambda_w t}, \quad t > 0$$

We need to calculate $P\{N = i\}$ for $i = 0, 1, 2, \ldots$ Then

$$P\{N = i\} = \int_0^\infty P\{N = i|T = t\}f(t)dt$$

The conditional distribution of N is affected by the walk-in process only insofar as that process determines the length of the time interval over which N counts. Given $T = t$, the random variable N has a poisson($\lambda_c t$) conditional distribution. Thus,

$$\begin{aligned} P\{N = i\} &= \int_0^\infty \frac{e^{-\lambda_c t(\lambda_c t)^i}}{i!}\lambda_w e^{-\lambda_w t}dt \\ &= \lambda_w \frac{\lambda_c^i}{i!}\int_0^\infty \left(\frac{x}{\lambda_c + \lambda_w}\right)^i e^{-x}\frac{dx}{\lambda_c + \lambda_w} \text{ putting } x = (\lambda_c + \lambda_w)t \\ &= \frac{\lambda_w}{\lambda_w + \lambda_c}\left(\frac{\lambda_c}{\lambda_c + \lambda_w}\right)^i \frac{1}{i!}\int_0^\infty x^i e^{-x}dx \end{aligned}$$

The $1/i!$ and the last integral cancel. (Compare with $\Gamma(i + 1)$) Writing p for $\lambda_w/(\lambda_c + \lambda_w)$ we have

$$P\{N = i\} = p(1 - p)^i \text{ for } i = 0, 1, 2, ...$$

Compare with the geometric(p) distribution. The random variable N has the distribution of the number of tails tosses before the first head, for independent tosses of a coin that lands heads with proability p.

Process of all arrivals is a Poisson process with rate $\lambda_w + \lambda_c$.(Why?)

Now consider an interval of length t in which there are X walk-ins and Y calls. Given that $X + Y = n$, the conditional distribution of X is $B(n, p)$, where

$$p = \frac{\lambda_w t}{\lambda_w t + \lambda_c t} = \frac{\lambda_w}{\lambda_w + \lambda_c}$$

Thas is, X has the conditional distribution that would be generated by the following mechanism:
(1) Generate inquiries as a Poisson process with rate $\lambda_w + \lambda_c$
(2) For each inquiry, toss a coin that lands heads with probability $p = \lambda_w/\lambda_w + \lambda_c$. For a head, decalre the arrival to be a walk-in, for a tail decalre it to be a call.