Statistical and Mathematical Methods DS-GA 1002
, 2019

Recitation Session 7 Solutions

Problem

1. *Hunter and Duck* Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size 10 flies overhead.

   SOLUTION

   Let X_i equal 1 if the ith duck escapes unhurt and 0 otherwise, for $i = 1, 2, \dots, 10$. The expected number of ducks to escape can be expressed as

   $$E[X_1 + \cdots + X_{10}] = E[x_1] + \cdots + E[x_{10}]$$

   To compute $E[X_i] = P\{X_i = 1\}$, we note that each of the hunters will, independently, hit the ith duck with probability $p/10$, so

   $$P\{X_i = 1\} = \left(1 - \frac{p}{10}\right)^{10}$$

   Hence,

   $$E[X] = 10\left(1 - \frac{p}{10}\right)^{10}$$

2. *Conditional Expectation.* S_n be the number of successes in n trials, what is $E(S_m|S_n = k)$ for $m \leq n$, what is $E(S_m|S_n)$? If X and Y are independent what is $E(X + Y|X = x)$.

   SOLUTION

   a. First argue intuitively to get $\frac{m}{n}S_n$. Let $S_n = \sum X_i$,

   $$E(X_i|S_n = k) = \mathrm{P}(i^{th}\text{trial is success given there are } k \text{ successes}) \tag{1}$$

   $$= \frac{\mathrm{P}(i^{th} \text{ is success and } k \text{ successes})}{\mathrm{P}(k \text{ successes})} \tag{2}$$

   $$= \frac{p\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{k}{n} \tag{3}$$

   $$E(S_m|S_n = k) = \sum_{i=1}^{m} E(X_i|S_n = k) = \frac{mk}{n} \tag{4}$$

   b. First term of the RHS is x because X is constant, and the second term is $E(Y)$ because of independence.

$$
\begin{aligned}
E(X+Y|X=x) &= E(X|X=x) + E(Y|X=x) && (5)\\
&= x + E(Y) && (6)
\end{aligned}
$$

3. *Correlation and Dependence.* Suppose that a random variable X takes on three values $\{-1, 0, 1\}$ with equal probability. Let $Y = X^2$.

   a. Are X and Y independent?

   b. Are X and Y correlated?

   SOLUTION

   a. No. X fully determines the value of Y.

   b. X and Y are uncorrelated. Note that X^3 is the same random variable as X. We have

   $$E(XY) = E(X^3) = E(X) = 0.$$

4. *Random walk.* A particle moves on a line at integer sites. At each step it can move one step to the left, one step to the right, or stay at the same place, all with equal probability. After 10,000 steps what is the probability that the particle is 100 steps away from its starting point.

   SOLUTION

   Let X_i be the ith step. $\mathrm{E}(X_i) = (-1)\frac{1}{3} + (0)\frac{1}{3} + (+1)\frac{1}{3} = 0$, and $\mathrm{Var}(X_i) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = (-1)^2\frac{1}{3} + (0)^2\frac{1}{3} + (+1)^2\frac{1}{3} = 2/3$ hence its standard deviation is 0.8165. The position at the 10,000th step is given by $S_{10,000} = \sum_{i=1}^{10,000} X_i$. By the normal approximation $S_{10,000}$ is approximately normal with mean zero and standard deviation 81.65. Then $\mathrm{P}(S_{10,000} > 100) = \mathrm{P}(S_{10,000}/81.65 > 100/81.65) = Q(100/81.65) \approx 11\%$.