Lab 12

1). $X_1, X_2, \dots, X_n \sim \text{Poisson}(\lambda)$; $\{X_i\}$ i.i.d.

a) Find the method of moment $\hat{\lambda}_{MM}$ of λ;

Let M_1 be the first ~~empirical moment~~ sample

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$E(X) = \lambda$. (Poisson ~~dist~~ mean).

Set $M_1 = E(X)$

$$\Rightarrow \frac{1}{n}\sum^{n} X_i = \lambda = \bar{X}_n$$

$$\Rightarrow \hat{\lambda}_{MM} = \bar{X}_n$$

Method of moment = a method to estimate ~~of~~ popular parameters. ~~Method~~ by expressing the popular moments as functions of the parameters of interest. Those expressions are then set equal to the sample moments.

Meth

$M_1 = E(W)$

$M_2 = E(W^2)$.

$M_k = E(W^k)$.

b) Find the maximum likelihood estimator $\hat{\lambda}_{ML}$ of λ

$$L(\lambda | X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} f(\lambda, x_i).$$

$$= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

(poisson).

Maximum likelihood estimator (MLE) = a method of estimating the parameters of a prob distribution by maximizing a likelihood func. So that under the assumed statistical model the observed data is most probable.

the log likelihood of n.

$$l(\lambda | X_1, X_2, \dots, X_n) = \log(L(\lambda | X_1, \dots X_n))$$

$$= \log\left(\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)$$

$$= \sum \log \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum \log \lambda^{x_i} + \log e^{-\lambda} - \log x_i!$$

$$= \sum x_i \log \lambda - \lambda - \log(x_i!)$$

Max $l(\lambda | X_1, .. X_n)$ → set derivative of $l(\lambda | X_1, .. X_n)$ to 0

$$\frac{\partial l(\lambda | X_1, .. X_n)}{\partial \lambda} = 0$$

$$\Rightarrow \sum \left(\frac{x_i}{\lambda} - 1 \right) = 0$$

$$\Rightarrow \sum \frac{x_i}{\lambda} = \sum 1 = n.$$

$$\Rightarrow \hat{\lambda}_{MLE} = \frac{\sum x_i}{n}$$

2) $X_1, \ldots X_n \sim \text{Uni}[0, \theta]$, $\theta > 0$; $\{X_i\}$ iid.

a) $\hat{\theta}_{MM} = ?$

Let M_1 be the first empirical moment $\ni$ $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n$

~~Let~~ $E(X_i) = \frac{\theta}{2}$ (Uniform mean).

Set $M_1 = E(X_i) \Rightarrow \overline{X}_n = \frac{\theta}{2} \Rightarrow \hat{\theta}_{MM} = 2\overline{X}_n$.

__Note__: Need to find 1 parameter, so only need 1 moment.

b) Find MSE, bias, var of $\hat{\theta}_{MM}$

Mean square err.

$\Rightarrow E(\hat{\theta}_{MM}) = E(2\overline{X}_n) = E(2.\frac{\sum X_i}{n}) = \frac{2}{n}.\sum E(X_i)$

$= \frac{2}{n}.n.E(X_i) = 2E(X_i) = 2.\frac{\theta}{2} = \theta.$

Since $E(\hat{\theta}_{MM}) = \theta$, $\hat{\theta}_{MM}$ is unbiased.

$\Rightarrow \text{biased}(\hat{\theta}_{MM}) = 0.$

$\Rightarrow Var(\hat{\theta}_{MM}) = Var(2\overline{X}_n) = 4Var(\overline{X}_n) = 4 Var(\frac{\sum X_i}{n})$

$= 4.\frac{1}{n^2}.\sum Var(X_i) = \frac{4}{n}\frac{1}{12}.\theta^2 = \frac{\theta^2}{3n}$

$\Rightarrow MSE(\hat{\theta}_{MM}) = Var(\hat{\theta}_{MM}) + bias^2(\hat{\theta}_{MM}) = \frac{\theta^2}{3n} + 0^2 = \frac{\theta^2}{3n}$

3) $\hat{\theta}_{ML} = \max_{1 \le i \le n} X_i$

Find MSE, bias, var of $\hat{\theta}_{ML}$

Known: $f_{\hat{\theta}_{ML}}(x) = \frac{n}{\theta^n} x^{n-1} \quad 0 < x < \theta.$

$$E(\hat{\theta}_{ML}) = \int_0^\theta x f(x)\,\partial x = \frac{n}{\theta^n}\int_0^\theta x^n \partial x = \frac{n}{\theta^n} \cdot \frac{x^{n+1}}{n+1}\Big|_0^\theta$$

$$= \frac{n}{\theta^n} \cdot \left(\frac{\theta^{n+1}}{n+1}\right) = \frac{n\theta}{n+1}.$$

$$E(\hat{\theta}^2_{ML}) = \int_0^\theta x^2 f(x)\,\partial x = \frac{n}{\theta^n}\int_0^\theta x^{n+1}\,\partial x = \frac{n}{\theta^n} \cdot \frac{x^{n+2}}{n+2}\Big|_0^\theta$$

$$= \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$$

$\rightarrow$ bias $(\hat{\theta}_{ML}) = E(\hat{\theta}_{ML}) - \theta = \frac{n\theta}{n+1} - \theta = \theta\left(\frac{-1}{n+1}\right).$

$\rightarrow$ var $(\hat{\theta}_{ML}) = E(\hat{\theta}^2_{ML}) - \left[E(\hat{\theta}_{ML})\right]^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2.$

$$= \theta^2\left(\frac{n}{(n+2)(n+1)^2}\right)$$

$\rightarrow MSE(\hat{\theta}_{ML}) = bias^2(\hat{\theta}_{ML}) + Var(\hat{\theta}_{ML})$

$$= \left(\frac{-\theta}{n+1}\right)^2 + \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$= \frac{(n+n+2)\theta^2}{(n+1)^2(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

d) $\lim MSE(\hat{\theta}_{MM}) = \lim \left(\frac{\theta^2}{3n}\right) = 0$ when $n \to \infty$

$\Rightarrow \hat{\theta}_{MM}$ converges to θ in MSE

$\lim MSE(\hat{\theta}_{MLE}) = \lim \left(\frac{2\theta^2}{(n+1)(n+2)}\right) = 0$, $n \to \infty$.

$\Rightarrow \hat{\theta}_{MLE}$ converges to θ in MSE.

$\Rightarrow \hat{\theta}_{MM}$ and $\hat{\theta}_{MLE}$ are consistent estimators of θ.

3) $X_1, \ldots X_n \sim Geo(\theta)$;

The prior of θ is $Beta(a,b)$ PDF: $\propto x^{a-1}(1-x)^{\beta-1)}$

a) Posterior distribution of $\pi(\theta|x)$?

$\underbrace{\pi(\theta|x)}_{\text{posterior distribution}} = \underbrace{\pi(\theta)}_{\text{prior distribution}} . \underbrace{f(x|\theta)}_{\text{likelihood function}}$

-) likelihood function:

$$f_X(x|\theta) = \prod (1-\theta)^{x_i-1}\theta$$

$$= (1-\theta)^{\Sigma(x_i)-n}\theta^n$$

$(geo(\theta))$

-) Prior distribution: $Beta(a,b)$: $\pi(\theta) = \theta^{a-1}(1-\theta)^{b-1}$

-) Posterior distribution: $\pi(\theta|x) = f(x|\theta).\pi(\theta)$

$$= (1-\theta)^{\Sigma(x_i)-n}\theta . \theta^{a-1}(1-\theta)^{b-1}$$

$$= (1-\theta)^{(\Sigma x_i)-n+b-1}.\theta^{a+n-1}$$

$$\propto Beta(a+n, \Sigma x_i - n + b).$$

b) Find Minimum mean square error (MMSE) $\hat{\theta}_{MMSE}$, $\lim \hat{\theta}_{MMSE}$?

The expectation of $Beta(a+n, \Sigma x_i - n + b)$ is:

$$\hat{\theta}_{MMSE} = \frac{a+n}{a+n+\Sigma x_i - n + b} = \frac{a+n}{a+b+\Sigma x_i}.$$

3b)
When $n \to \infty$, $\lim \hat{\theta}_{MMSE} = \lim \dfrac{a+n}{\underbrace{n\bar{X}_n}_{\Sigma X_i} + a + b} = \dfrac{1}{\lim_{n\to\infty} \bar{X}_n} = \theta.$

($\lim \bar{X}_n = \frac{1}{\theta}$ b/c of law of large number,

$\lim (\bar{X}_n) \to E(X) = 1/\theta$ due to geo).

c) Find MAP, $\hat{\theta}_{MAP}$.

To find the max of $\pi(\theta \mid x)$, it suffices to find the max of $\log \pi(\theta \mid x)$. $\hat{\theta}_{MAP} = \arg\max \pi(\theta \mid x)$.

$\cdot)\ \log(\pi(\theta \mid x)) = \log\left((1-\theta)^{\Sigma X_i - n + b - 1}\ \theta^{a+n-1}\right).$

$= \left(\Sigma X_i - n + b - 1\right) \log(1-\theta) + (a+n-1)\log\theta$

$$\frac{\partial \log(\pi(\theta \mid x))}{\partial \theta} = \frac{(a+n-1)}{\theta} - \frac{(\Sigma X_i) - n + b - 1}{1-\theta} = 0.$$

$$\Rightarrow \hat{\theta}_{MAP} = \frac{a+n-1}{\Sigma X_i + a + b - 2} = \frac{a+n-1}{n\bar{X}_n + a + b - 2}.$$

$n \to \infty$, $\lim_{n\to\infty} \hat{\theta}_{MAP} = \dfrac{1}{\lim \bar{X}_n} = \theta.$

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