**Recitation Session 10**

# Problem

1. *Sample Average.* Suppose $\{X_1, \cdots, X_n\}$ are $n$ independent identically distributed (they are of the same distribution) random variables with finite expectation $\mu$, variance $\sigma^2$, and cdf $F_X(x)$. Denote the sample average as

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

   a. Compute the expectation of $\overline{X}_n$.

   b. Compute the variance of $\overline{X}_n$.

   c. Use (a) and (b) to compute the mean and variance of $F_n(x)$ for any fixed $x$. The answer should be written in terms of $F_X(x)$.

2. *Convergence.* Given the empirical cdf

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1\{X_i \leq x\}$$

   where $\{X_i\}_{i=1}^n$ are $n$ i.i.d. random variables with cdf $F_X(x)$.

   a. Show that for any fixed $x$, $F_n(x)$ converges to $F_X(x)$ in mean squares error as $n \to \infty$.

   b. Use (a) to show that $F_n(x)$ converges to $F_X(x)$ in probability as $n \to \infty$.

3. *Sample Variance.* Consider the sample variance

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

   where $\{X_i\}_{i=1}^n$ are $n$ i.i.d. random variables. Denote $\mu = \mathrm{E}(X_i)$ and $\sigma^2 = \mathrm{Var}(X_i)$.

   a. Show that

$$\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 = \sum_{i=1}^{n} X_i^2 - n\overline{X}_n^2.$$

   b. Compute $\mathrm{E}(S_n^2)$.

   c. Suppose $\mathrm{E}\,X_i^4 < \infty$. Justify why $S_n^2$ converges to $\sigma^2$ in probability.