Recitation Session 4

Problem

1. *Grizzlies in Yellowstone (Example 3.3.3 continued)* The scientist observes a bear with her binoculars. From their size she estimates that its weight is 180 kg. What is the probability that the bear is male?

2. *Bayesian coin flip (Example 3.3.6 continued)* Your uncle bets you ten dollars that a coin flip will turn out heads. You suspect that the coin is biased, but you are not sure to what extent. To model this uncertainty you represent the bias as a continuous random variable B with the following pdf:
$$f_B(b) = 2b \text{ for } b \in [0, 1].$$
The coin lands on tails. Compute the distribution of the bias conditioned on this information.

3. *Coin Toss* In a large collection of coins, the probability X that a head will be obtained when a coin is tossed varies from one coin to another, and the distribution of X in the collection is specified by the following pdf:
$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$
Suppose that a coin is selected at random from the collection and tossed once. Let Y be the event that a head is obtained.

   a. What is the conditional pmf of $Y|X = x$?

   b. What is the joint distribution of X and Y?

   c. Suppose the outcome of the coin flip is heads. What is the conditional distribution of X given that the outcome is heads?

   Solution

   a. The conditional pmf of $Y|X = x$ is
$$p(Y|X = x) = \begin{cases} x & \text{for } y = 1 \\ 1-x & \text{for } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

   b.
$$f_{X,Y}(x, y) = \begin{cases} 6x^2(1-x) & \text{for } 0 < x < 1 \text{ and } y = 1 \\ 6x(1-x)^2 & \text{for } 0 < x < 1 \text{ and } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

c. To find the conditional distribution of $X|Y = 1$, we first need to find the marginal distribution of Y.

$$\begin{aligned} P(Y = 1) &= \int_0^1 6x^2(1 - x)dx \\ &= \frac{1}{2}. \end{aligned}$$

Then we have

$$\begin{aligned} f_{X|Y}(X|Y = 1) &= \frac{f_{X,Y}(x, 1)}{P(Y = 1)} \\ &= 12x^2(1 - x). \end{aligned}$$

Extra Problem

1. *Normal Distribution* Suppose $X \sim \mathcal{N}(0, 1)$.

a. Find the cdf and pdf of $|X|$, i.e., the absolute value of a standard normal random variable. The distribution of $|X|$ is called the folded normal.

Solution $X \sim \mathcal{N}(0, 1)$, then cdf of X is $\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{\frac{-x^2}{2}} x$.

$$\begin{aligned} F_{|X|}(x) &= \Pr(|X| \leq x) \\ &= \Pr(-x \leq X \leq x) \\ &= \Pr(X \leq x) - \Pr(X \leq -x) \\ &= \Pr(X \leq x) - (\Pr(X \geq x)) \\ &= \Pr(X \leq x) - (1 - \Pr(X \leq x)) \\ &= \phi(x) - (1 - \phi(x)) \\ &= 2 \cdot \phi(x) - 1 \end{aligned}$$

$$\begin{aligned} f_{|X|}(x) &= \frac{F_X(x)}{x} \\ &= \frac{(-1 + 2 \cdot \phi(x))}{x} \\ &= 2 \cdot \frac{\phi(x)}{x} \\ &= 2 \cdot f_X(x) \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}} \\ &= \frac{2}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}} \end{aligned}$$

b. Find the cdf of $X^+ = \max\{X, 0\}$. In particular, find $\Pr(X^+ \leq 0)$. This distribution is a mixture of discrete and continuous random variables.

Solution

$$\begin{aligned}
F_X^+(x) &= \Pr(X^+ \leq x) \\
&= \Pr(max\{X, 0\} \leq x) \\
&= \Pr((X \leq x) \cap (0 \leq x)) \\
&= \Pr(X \leq x) \Pr(0 \leq x) \\
&= \begin{cases} 0 \text{ if } x < 0 \text{ because } \Pr(x < 0) = 0 \\ \phi(x) \text{ if } x \geq 0 \text{ because } \Pr(x \geq 0) = 1 \end{cases}
\end{aligned}$$

$$\Pr(X^+ \leq 0) = \phi(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\frac{-x^2}{2}x} = \frac{1}{2}$$