Recitation Session 1

Problem

1. *Dependence.* From a well shuffled standard deck of 52 cards, pick two. What is the probability that the *second* card is black?

2. *Set.* Show that $\mathbb{P}(E^C \cap F^C) = 1 - \mathbb{P}(E) - \mathbb{P}(F) + \mathbb{P}(E \cap F)$.

3. *Conditional probability.* The probability that a child has blue eyes is 1/4. Assume independence between children. Consider a family with 3 children.

   a. If it is known that at least one child has blue eyes, what is the probability that at least two children have blue eyes?

   b. if it is known that the youngest child has blue eyes, what is the probability that at least two children have blue eyes?

4. *Conditional Independence.* Let's define X_1 and X_2 as conditionally independent given Y if $\mathbb{P}(X_1|X_2,\ Y) = \mathbb{P}(X_1|Y)$. Does independence imply conditional independence? Give an example.

5. *Bayes' Formula.* Given 1,000 coins, 1 coin has heads on both sides while the other are fair coins. One coin is randomly chosen and tossed 10 times. Each time, it turns up heads. What's the probability that the coin is the unfair one?

Recap

1. Independence: Two events A and B are independent if and only if $\mathbb{P}(A|B) = P(A)$.

2. Conditional Probability: If $\mathbb{P}(B) > 0$ then the conditional probability of A given B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

3. Bayes' Rule : Let $\{A_i\}_{i=1}^k$ be a partition of the sample space and B be any event. Then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^k \mathbb{P}(B|A_j)\mathbb{P}(A_j)},\ \forall i$$

where the numerator equals $\mathbb{P}(B \cap A_i)$ and the denominator is $\mathbb{P}(B)$.

4. The law of total probability: Let $\{A_i\}_{i=1}^k$ be a partition of the sample space and B be any event. Then

$$\mathbb{P}(B) = \sum_{j=1}^{k} \mathbb{P}(B \cap A_j) = \sum_{j=1}^{k} \mathbb{P}(B|A_j)\mathbb{P}(A_j)$$

where $\{B \cap A_j\}_{j=1}^k$ is a disjoint partition of B.