Recitation Session 11

Problem

1. *Convergence* A fair dice is casted independently 600 times. Give an approximation (by CLT) for the probability that the number of 6s falls between 95 and 110.

2. *Empirical covariance* Suppose $\{(X_i, Y_i)\}_{i=1}^n$ are n independent samples from a two-variable pdf $f_{X,Y}(x, y)$ where
$$\mu_X = \mathbb{E}(X), \quad \mu_Y = \mathbb{E}(Y)$$
and
$$\sigma_X^2 = \mathrm{Var}(X), \quad \sigma_Y^2 = \mathrm{Var}(Y), \quad \sigma_{XY} = \mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))].$$
Consider the empirical covariance between $\{X_i\}$ and $\{Y_i\}$:
$$\widehat{\sigma}_{XY} = \frac{1}{n-1} \sum_{i=1}^n (X_i - \overline{X}_n)(Y_i - \overline{Y}_n).$$
Prove that
$$\widehat{\sigma}_{XY} = \frac{1}{n-1} \left(\sum_{i=1}^n X_i Y_i - n\overline{X}_n \overline{Y}_n \right)$$

3. *PCA* Let $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$ be a (sample) covariance matrix with an eigenvalue/eigenvector decomposition as
$$\boldsymbol{\Sigma} = [\boldsymbol{u}_1, \cdots, \boldsymbol{u}_d] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix} [\boldsymbol{u}_1, \cdots, \boldsymbol{u}_d]^\top$$
where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, $\boldsymbol{u}_i \in \mathbb{R}^{d\times 1}$, and $\boldsymbol{u}_i \perp \boldsymbol{u}_j$, i.e., $\boldsymbol{u}_i^\top \boldsymbol{u}_j = 0$ for $i \neq j$.

   a. Prove
   $$\lambda_1 = \max_{\|\boldsymbol{v}\|=1} \boldsymbol{v}^\top \boldsymbol{\Sigma} \boldsymbol{v}, \quad \boldsymbol{u}_1 = \mathrm{argmax}_{\|\boldsymbol{v}\|=1} \boldsymbol{v}^\top \boldsymbol{\Sigma} \boldsymbol{v}$$
   (Hint: Rewrite $\boldsymbol{\Sigma}$ into $\boldsymbol{\Sigma} = \sum_{i=1}^d \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^\top$ and use the fact that $\sum_{i=1}^d \boldsymbol{u}_i \boldsymbol{u}_i^\top = \boldsymbol{I}_d$.)

   b. Prove that
   $$\lambda_2 = \max_{\|\boldsymbol{v}\|=1, \boldsymbol{v}\perp \boldsymbol{u}_1} \boldsymbol{v}^\top \boldsymbol{\Sigma} \boldsymbol{v}, \quad \boldsymbol{u}_2 = \mathrm{argmax}_{\|\boldsymbol{v}\|=1, \boldsymbol{v}\perp \boldsymbol{u}_1} \boldsymbol{v}^\top \boldsymbol{\Sigma} \boldsymbol{v}$$
   where $\|\boldsymbol{v}\|^2 = \boldsymbol{v}^\top \boldsymbol{v} = \sum_{i=1}^d v_i^2 = 1$.