Stellar Wind Theories

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Abstract. We describe the basic theory of stellar winds with momentum input due to a force or with energy input and we formulate the five laws of stellar winds.

Stellar winds can be driven by various mechanisms, specified by the main force that is responsible for the wind. These are:
(1) Coronal winds, driven by gas pressure at high temperature
(2) Line driven winds, due to radiation pressure in spectral lines
(3) Dust driven winds, due to radiation pressure on dust
(4) Pulsation driven winds, due to oscillating motions of the photosphere
(5) Sound wave driven winds, due to wave pressure of acoustic waves
(6) Alfvén wave driven winds, due to wave pressure by Alfvén waves
(7) Magnetic rotating winds, due to magnetic corotation
We briefly review these mechanisms.

1 Introduction

Most stars are loosing mass in the form of a stellar wind. The mass loss rate of the sun is about $10^{-14}\ M_\odot \mathrm{yr}^{-1}$. This is such a small amount that it does not affect the evolution. The mass loss by the solar wind is even smaller than the mass loss rate in the interior of the sun due to the nuclear fusion: $\dot{M}_{\mathrm{nucl}} = L/c^2$. However in a later evolutionary phase low mass stars suffer much higher mass loss rates up to about $10^{-5}\ M_\odot \mathrm{yr}^{-1}$ when they reach the Asymptotic Giant Branch. The late evolution of low mass stars is dominated by mass loss.

Massive stars with $M \gtrsim 30 M_\odot$ experience significant mass loss already on the main sequence. The typical mass loss rates of O stars during the main sequence is about 10^{-6} to $10^{-5}\ M_\odot \mathrm{yr}^{-1}$. For those stars the whole evolution is seriously affected by mass loss.

Stellar winds can have very different forms:
(1): cold ($T \approx T_*$), slow ($v << v_{\mathrm{esc}}$) and dense for late-type supergiant stars
(2): cold ($T \approx T_*$), fast ($v \gtrsim v_{\mathrm{esc}}$) and dense for luminous hot stars
(3): hot ($T >> T_*$), fast ($v \gtrsim v_{\mathrm{esc}}$) and teneous around cool dwarfs and giants.
This shows that there are different mechanisms which produce these winds. At least seven different mechanisms have been proposed to explain the observed properties of stellar winds in different types of stars. They are listed in the abstract.

The two most important theories for mass loss are the "dust driven wind theory" and the "line driven wind theory". In both theories the wind is driven by radiation pressure. These theories are important because they are successful in explaining the gross properties of the observed winds in the two evolutionary phases with the highest mass loss rates: the cool luminous stars and the hot luminous stars. These two theories will be discussed in some detail in two chapters in this volume.

The main emphasis of this paper is on explaining the basic physical principles of stellar wind theories. We first discuss the theory of isothermal winds without extra forces other than the one due to the gas pressure gradient. We show that for a wind with a given density at its lower boundary, the velocity law and the mass loss rate are fixed by the condition that the wind should start subsonic at the photosphere and reach supersonic velocities further outwards. Then we describe the effects of momentum input (due to an extra outward force) and energy input (due to heating) on the structure of a stellar wind and on its mass loss rate. These principles hold for (almost) all wind theories. We show the results of forces acting at different distances in the wind. Based on these results we formulate the five laws of stellar wind theories. In the last section we briefly review the different wind theories.

An extensive discussion of the theories and observations of stellar winds will be published in *Introduction to Stellar Winds* by H.J.G.L.M. Lamers and J.P. Cassinelli (Cambridge University Press), in preparation.

2 Basic Concepts of Wind Theories

2.1 The Mass Continuity Equation

For a time-independent stellar wind with a constant mass loss rate, the amount of gas passing through any sphere of radius r is constant. This is expressed in the *equation of mass conservation*

$$\dot{M} = 4\pi r^2 \rho(r) v(r) \ . \tag{1}$$

Differentiation of this equation gives

$$\frac{1}{v}\frac{dv}{dr} = -\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r} \ . \tag{2}$$

2.2 The Momentum Equation

The motion of the gas in a stellar wind is described by Newton's law $F = m.a$ or $F = \rho.dv/dt$ if F is the force per unit volume and $f = F/\rho$ is the force per unit mass. The velocity gradient in Newton's law is

$$\frac{dv(r,t)}{dt} = \frac{\delta v(r,t)}{\delta t} + \frac{\delta v(r,t)}{\delta r}\frac{dr(t)}{dt} = v(r)\frac{dv}{dr} \ . \tag{3}$$

In a stationary wind $\delta v(r,t)/\delta t = 0$. The *equation of motion* or the *momentum equation* in a wind is

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM_*}{r^2} + f(r) \ . \qquad (4)$$

The first term is the force due to the gradient of the gas pressure. This force is directed outwards (positive) because $dp/dr < 0$. The second term is the inward directed gravity and the third term describes any extra outward directed force.

2.3 The Energy Equation

The gas pressure gradient depends on the temperature structure of the wind, which depends on the heat input and cooling by expansion. The first law of thermodynamics states that

$$\frac{dQ}{dt} = \frac{du}{dt} + p\frac{d\rho^{-1}}{dt} \qquad (5)$$

where Q is the heat energy per gram, and $u = (3/2)(\mathcal{R}T/\mu)$ is the internal energy for an ideal gas with mean particle weight μm_H. The gas pressure for an ideal gas is $p = \rho\mathcal{R}T/\mu$. For a stationary wind the time derivative is $d/dt = v\ d/dr$. Define $q(r) = dQ/dr$ as the heat input (positive) or heat loss (negative) per gram per cm in the wind, then the energy equation becomes

$$q = \frac{3}{2}\frac{\mathcal{R}}{\mu}\frac{dT}{dr} + p\frac{d\rho^{-1}}{dr} \ . \qquad (6)$$

By substituting $pd(1/\rho)dr = d(p/\rho)/dr - (1/\rho)dp/dr$ with the ideal gas law we find

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{5}{2}\frac{\mathcal{R}}{\mu}\frac{dT}{dr} - q(r) \ . \qquad (7)$$

Combining (4) and (7) gives the *energy equation* of stellar winds

$$\frac{d}{dr}\left\{\frac{v^2}{2} + \frac{5}{2}\frac{\mathcal{R}T}{\mu} - \frac{GM_*}{r}\right\} = f(r) + q(r) \ . \qquad (8)$$

This equation states that the change in energy of the gas as it moves 1 cm outwards is equal to the momentum input by the force and the heat input. The first left hand term is the kinetic energy of the flow, the second term is the enthalpy of the gas (the internal kinetic energy plus the capacity to do work) and the third term is the potential energy.

This energy equation can also be written in the integral form

$$e(r) \equiv \frac{v^2}{2} + \frac{5}{2}\frac{\mathcal{R}T}{\mu} - \frac{GM_*}{r} = e(r_0) + \int_{r_0}^{r} f(r)dr + \int_{r_0}^{r} q(r)dr \qquad (9)$$

where r_0 is some arbitrarily chosen lower limit.

Compare the energy of the wind just above the photosphere and at ∞. In or just above the photosphere the potential energy is much larger than the enthalpy and the kinetic energy because for a normal star $v_{\rm esc} >> \mathcal{R}T_*/\mu$ and $v(R_*) << v_{\rm esc}$ so the total energy is $e(r_0) \approx -GM_*/r_0$, which is negative. At large distance $r \to \infty$ the potential energy and the enthalpy both go to 0, so the total energy is the kinetic energy. This means that

$$\frac{{v_\infty}^2}{2} \approx -\frac{GM_*}{R_*} + \int_{R_*}^{\infty} f(r)dr + \int_{R_*}^{\infty} q(r)dr \ . \tag{10}$$

We see that *a stellar wind can only escape if there is an outward force that provides sufficient momentum input or an energy source that provides sufficient heat input for the wind to escape the potential well.*

3 Isothermal Winds Driven by Gas Pressure

Let us first consider an isothermal stellar wind. Such a wind obviously requires energy input, otherwise the gas would cool as it expands adiabatically. If the wind is isothermal, we do not have to worry about the energy equation and we can concentrate on the momentum equation and the resulting flow velocity. If there is no extra force, the wind is *driven by gas pressure only.* This is the case for the solar wind from the hot corona. Winds driven by gas pressure produce only small mass loss rates and are not important for stellar evolution. However, their theory gives insight into some of the basic physical principles of stellar wind theories.

3.1 The Critical Point of the Momentum Equation

The momentum equation of an isothermal wind with gas pressure only is

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM_*}{r^2} \ . \tag{11}$$

In an isothermal wind consisting of an ideal gas of temperature T the force due to the pressure gradient can be written as

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{\mathcal{R}}{\mu}\frac{dT}{dr} + \frac{\mathcal{R}T}{\mu\rho}\frac{d\rho}{dr} = \left(\frac{\mathcal{R}T}{\mu}\right)\frac{1}{\rho}\frac{d\rho}{dr} \ . \tag{12}$$

The density gradient can be expressed in a velocity gradient by (2). Substituting (2) and (12) into (11) yields

$$\frac{1}{v}\frac{dv}{dr} = \left\{\frac{2a^2}{r} - \frac{GM_*}{r^2}\right\} / \left\{v^2 - a^2\right\} \tag{13}$$

with

$$a = (\mathcal{R}T/\mu)^{1/2} \tag{14}$$

is the isothermal speed of sound, which is constant in an isothermal wind. The lower boundary condition of (13) is the bottom of the isothermal region, located at r_0 where $v(r_0) = v_0$. In general r_0 is about the photospheric radius or a bit larger if the star is surrounded by an isothermal corona.

The momentum equation (13) has a singularity at the point where $v(r) = a$. We will show below that this singularity is extremely important, because it implies that the mass loss rate is fixed.

The numerator goes to 0 at a distance

$$r = r_c \equiv GM_*/2a^2 \ . \tag{15}$$

This is the *critical distance*, or the distance of the *critical point*. The velocity gradient at the critical distance will be zero, because the numerator equals zero, unless $v(r_c) = a$. Similarly, the velocity gradient at the distance where $v = a$ will be $\pm\infty$, because the denominator $= 0$, unless $r = r_c$ when $v = a$. So the only solution which can have a positive velocity gradient at all distances is the one that goes through the critical point. This is the *critical solution* for which

$$v(r_c) = a \qquad \text{at} \qquad r_c = \frac{GM_*}{2a^2} \ . \tag{16}$$

So we find that at the critical point

$$v(r_c) = a = \frac{v_{\rm esc}(r_c)}{2} \tag{17}$$

where $v_{\rm esc}(r_c) = \sqrt{2GM_*/r_c}$ is the escape velocity at the critical point. The point in the wind where $v(r) = a$ is called the *sonic point*. In an isothermal wind the critical point coincides with the sonic point, but this is not necessarily true for other wind models. The critical solution is transonic, because it starts subsonic at small distances and reaches a supersonic velocity at large distances.

The topology of the solutions of (13) is shown in Fig. 1 for various initial velocities $v(r_0)$. Notice that only one solution (thick line) starts subsonic and ends supersonic.

The slope of the velocity law through the critical point can be derived by applying de l'Hopital's rule which states that the right hand side of a critical equation like (13) is equal to the quotient of the derivatives of the numerator and the denominator. (This can be proven by expressing both factors as a series around their zero point: $f(r) = f(r_c) + (r - r_c).(df/dr)_c$). De l'Hopital's rule results in

$$\frac{1}{v}\left(\frac{dv}{dr}\right)_{r_c} = \left\{-\frac{2a^2}{r_c^2} + \frac{2GM_*}{r_c^3}\right\} / \left\{\frac{2v\ dv}{dr}\right\}_{r_c} = \frac{a^2}{r_c^2}\left(\frac{v\ dv}{dr}\right)^{-1}_{r_c} \ . \tag{18}$$

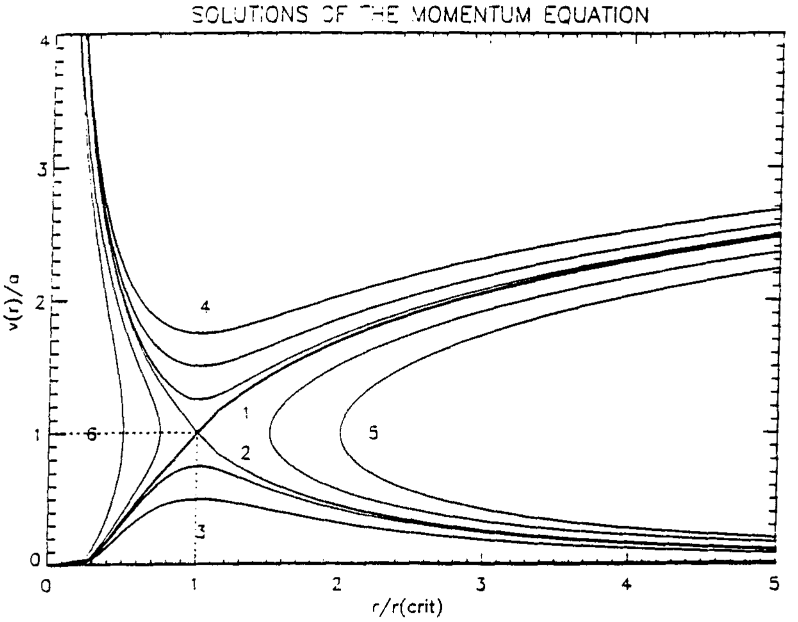

Fig. 1. Different types of solutions of the momentum equation for an isothermal wind with gas pressure only. For this particular case r_c is at 5 r_0. The thick line passes through the critical point. It is the only transonic solution

This gives

$$\left(\frac{dv}{dr}\right)_{r_c} = \frac{\pm 2a^3}{GM_*} \ . \tag{19}$$

The positive or negative sign is a result of the fact that de l'Hopital's rule gives an expression for $(dv/dr)^2$. For a wind with an outward increasing velocity we obviously chose the positive gradient.

This discussion has shown that there is only one solution which starts subsonic and ends supersonic. This critical solution occurs for only one particular value of the velocity at the lower boundary: v_0(crit). This implies that an isothermal envelope with given density ρ_0 at its bottom can only produce a transonic wind if

$$\dot{M} = 4\pi r_0^2 \rho_0 v_0(\mathrm{crit}) \ . \tag{20}$$

This is a very important result which shows that *an isothermal wind with a given lower boundary (ρ_0, T_0 and gravity) can reach supersonic velocities for only one specific value of the mass loss rate!*

3.2 The Velocity Law of Isothermal Winds Driven by Gas Pressure

The momentum equation (13) has an analytic solution

$$\frac{v^2}{2} - a^2 \ln(v) = 2a^2 \ln(r) + \frac{GM_*}{r} + \text{ constant} \cdot \tag{21}$$

The constant is fixed by the condition $v(r_c) = a$ at the critical point. This gives an expression for the velocity law of an isothermal wind driven by gas pressure only

$$v \exp\left(-\frac{v^2}{2a^2}\right) = a\left(\frac{r_c}{r}\right)^2 \exp\left\{-\frac{2r_c}{r} + \frac{3}{2}\right\} \tag{22}$$

with $r_c = GM_*/2a^2$. This velocity law is shown in Fig. 2.

The initial velocity at the lower boundary of the isothermal region can be derived by applying (22) at r_0. At the bottom of a gravitationally bound subsonic wind with $v_0 \ll a < v_{\rm esc}$ one finds

$$\begin{aligned} v_0 &\approx a\left(\frac{r_c}{r_0}\right)^2 \exp\left\{-\frac{2r_c}{r_0} + \frac{3}{2}\right\} \\ &= a\left(\frac{v_{\rm esc}(r_0)}{2a}\right)^2 \exp\left\{-\frac{v_{\rm esc}^2(r_0)}{2a^2} + \frac{3}{2}\right\} \quad . \end{aligned} \tag{23}$$

Equation (22) can also be written as

$$\frac{v}{v_0} \exp\left(-\frac{v^2}{2a^2}\right) = \left(\frac{r_0}{r}\right)^2 \exp\left\{\frac{GM_*}{a^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right\} \quad . \tag{24}$$

At large distances where $r \gg r_0$, the velocity law approaches

$$v(r \to \infty) \approx 2a\sqrt{\ln(r/r_0)} \ , \tag{25}$$

which increases infinitely. This is a consequence of the assumption that the wind is isothermal up to very large distances. It requires the continuous addition of energy and the resulting gas pressure then accelerates the wind indefinitely. Clearly this is an unrealistic situation. In reality the winds are approximately isothermal only up to a certain distance and the velocity does not increase beyond that distance.

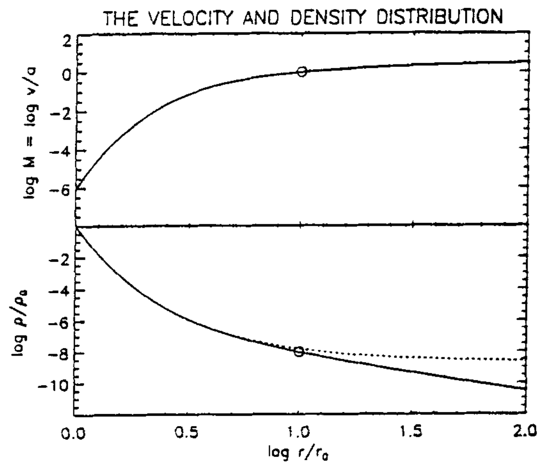

Fig. 2. The top part of the figure shows the velocity law, in terms of Mach-number ($M = v/a$) for an isothermal wind as a function of distance normalized to the lower boundary of the isothermal region r_0. The location of the critical point is indicated by a dot. The lower part shows the logarithmic density distribution of the wind normalized to the density at r_0 (full line). The dashed line shows the density distribution of a hydrostatic atmosphere with the same temperature. The two density distributions are very similar in the subsonic part of the wind. The difference at the critical point is only a factor $e^{-0.5}$

3.3 The Density Structure of Isothermal Winds Driven by Gas Pressure

The density structure is given by the mass continuity equation (1) and the velocity law (22) which yields

$$\frac{\rho}{\rho_0}\exp\left\{+\frac{1}{2}\left(\frac{v_0\rho_0 r_0^2}{a\rho r^2}\right)^2\right\} = \exp\left\{-\frac{GM_*}{a^2}\left(\frac{1}{r_0}-\frac{1}{r}\right)\right\} \quad . \tag{26}$$

This equation can be solved numerically to give $\rho(r)/\rho_0$. The result is shown in Fig. 2 for a wind with a temperature such that $a^2 = 0.05GM_*/r_0$, which implies a critical point at $r_c = 10r_0$. Let us now compare this with the density distribution of a hydrostatic atmosphere.

In a static atmosphere the density is given by the hydrostatic equation

$$\frac{1}{\rho}\frac{dp}{dr} + \frac{GM_*}{r^2} = 0 \tag{27}$$

which transforms, with (12), into

$$\frac{r^2}{\rho}\frac{d\rho}{dr} = -\frac{GM_*}{a^2} \tag{28}$$

if the atmosphere is isothermal. The solution is

$$\frac{\rho(r)}{\rho_0} = \exp\left\{-\frac{GM_*}{a^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right\} = \exp\left\{-\frac{(r-r_0)}{\mathcal{H}_0}\frac{r_0}{r}\right\} \tag{29}$$

where

$$\mathcal{H}_0 = \mathcal{R}T/\mu g_0 \quad \text{with} \quad g_0 = GM_*/r_0^2 \tag{30}$$

is the density scale height at the bottom of the isothermal region.

The density structure (29) of the hydrostatic region is very similar to that of an isothermal wind. In fact, (29) is equal to (26) at the bottom of a wind if v_0 is highly subsonic. The difference at the critical or sonic point is exactly a factor exp (-0.5) because $v_0\rho_0 r_0^2 = a\rho(r_c)r_c^2$.

The close agreement between the hydrostatic and wind density structure in the subsonic region where $v \lesssim 0.5a$ is due to the fact that the term vdv/dr in the momentum equation (4) is much smaller than the pressure gradient term. In other words: *the structure of the subsonic region is mainly determined by the hydrostatic density structure and not by the velocity law!*

3.4 The Mass Loss Rate of Isothermal Winds Driven by Gas Pressure

The mass loss rate of an isothermal wind driven by gas pressure follows from the equation of mass continuity at either the lower boundary or at the critical point.

$$\dot{M} = 4\pi r_0^2 \rho_0 v_0 = 4\pi r_c^2 \rho_c a \ . \tag{31}$$

This gives

$$\begin{aligned}\dot{M} &= 4\pi\rho_0\ a\ r_0^2 \left\{\frac{v_{\rm esc}(r_0)}{2a}\right\}^2 \exp\left\{-\frac{(r_c-r_0)}{\mathcal{H}_0}\frac{r_0}{r_c} - \frac{1}{2}\right\} \\ &= 4\pi\rho_0\ a\ r_0^2 \left\{\frac{v_{\rm esc}(r_0)}{2a}\right\}^2 \exp\left\{-\frac{v_{\rm esc}^2(r_0)}{2a^2} + \frac{3}{2}\right\} \ . \end{aligned} \tag{32}$$

Estimates of the mass loss for a few characteristic stars are given in Table 1. Notice the extreme sensitivity of the mass loss rate to the height of the critical point in terms of the pressure scale height (column 8) because the density at the critical point decreases exponentially as as $(r_c - r_0)/\mathcal{H}_0$. The predicted mass loss rate of a solar type star with a corona of 1×10^6 K and $\rho_0 = 10^{-14}$ g cm^{-3} is $1.6 \times 10^{-14}\ M_\odot$ yr^{-1}. This is in reasonable agreement with the observed rate of $2 \times 10^{-14} M_\odot \text{yr}^{-1}$.

Table 1. Characteristics of isothermal winds with a density at the lower boundary of $\rho_0 = 10^{-14}$ g/cm^3

M_* ($M_\odot$)	R_* ($R_\odot$)	v_{esc} (km/s)	T (K)	a (km/s)	$\mathcal{H}_0$ (R$_*$)	r_c (R$_*$)	$\frac{r_c - R_*}{\mathcal{H}_0}$	$\dot{M}$ ($M_\odot$/yr)
1	1	617.5	1.10^5	37.2	$7.3\ 10^{-3}$	68.7	$9.3\ 10^3$	$1.2\ 10^{-68}$
			3.10^5	64.5	$2.2\ 10^{-2}$	22.9	$1.0\ 10^3$	$1.5\ 10^{-28}$
			1.10^6	117.7	$7.3\ 10^{-2}$	6.9	$8.0\ 10^1$	$1.6\ 10^{-14}$
			3.10^6	203.9	$2.2\ 10^{-1}$	2.3	5.9	$8.2\ 10^{-11}$
			5.10^6	263.2	$3.6\ 10^{-1}$	1.4	1.1	$4.0\ 10^{-10}$
1	100	61.7	3.10^3	6.4	$2.2\ 10^{-2}$	22.9	$1.0\ 10^3$	$1.5\ 10^{-25}$
			1.10^4	11.8	$7.3\ 10^{-2}$	6.9	$8.1\ 10^1$	$1.6\ 10^{-11}$
			3.10^4	20.4	$2.2\ 10^{-1}$	2.3	5.9	$8.2\ 10^{-8}$
			5.10^4	26.3	$3.6\ 10^{-1}$	1.4	1.1	$4.0\ 10^{-7}$
10	10	617.5	1.10^5	37.2	$7.3\ 10^{-3}$	68.7	$9.3\ 10^3$	$1.2\ 10^{-66}$
			3.10^5	64.5	$2.2\ 10^{-2}$	22.9	$1.0\ 10^3$	$1.5\ 10^{-26}$
			1.10^6	117.7	$7.3\ 10^{-2}$	6.9	$8.0\ 10^1$	$1.6\ 10^{-12}$
			3.10^6	203.9	$2.2\ 10^{-1}$	2.3	5.9	$8.2\ 10^{-9}$
			5.10^6	263.2	$3.6\ 10^{-1}$	1.4	1.1	$4.0\ 10^{-8}$
10	1000	61.7	3.10^3	6.4	$2.2\ 10^{-2}$	22.9	$1.0\ 10^3$	$1.5\ 10^{-23}$
			1.10^4	11.8	$7.3\ 10^{-2}$	6.9	$8.1\ 10^1$	$1.6\ 10^{-9}$
			3.10^4	20.4	$2.2\ 10^{-1}$	2.3	5.9	$8.2\ 10^{-6}$
			5.10^4	26.3	$3.6\ 10^{-1}$	1.4	1.1	$4.0\ 10^{-5}$

1 M$_\odot$/yr = 6.303 10^{25} g/s, $\mu = 0.60$

4 Isothermal Winds with an Outward Force

The momentum equation of an isothermal wind with an extra outward force f per unit mass can easily be derived from (4), (12) and (2)

$$\frac{1}{v}\frac{dv}{dr} = \frac{\frac{2a^2}{r} - \frac{GM_*}{r^2} + f(r)}{v^2 - a^2} \quad . \tag{33}$$

We can immediately see from this equation that the effect of a force is very different for the subsonic and the supersonic regions.

(1). Supersonic: The denominator is positive so an extra force f results in an *increase* of the velocity gradient. The wind will reach a higher velocity.

(2). Subsonic: The denominator is negative so an extra force f results in a *decrease* of dv/dr. This may seem surprising, because it implies that if the wind is pushed outwards in the subsonic region it will accelerate slower, instead of faster. One can understand this by remembering the conclusion of the previous section, that the density structure of the subsonic region is mainly

determined by the hydrostatic equilibrium. So applying an outward force in the subsonic region has the same effect as reducing the gravity, which results in an increase in the pressure scaleheight. An increase in the pressure scaleheight gives a slower outward decrease of the density and, since the velocity has to follow the density structure (because of the mass continuity equation), a slower outward decrease in density means a slower outward increase of the velocity! (Note: this does not mean that the subsonic velocity is smaller if there is an outward force, but that the *subsonic velocity gradient* is smaller. We will argue below that the subsonic velocity is in fact larger).

What will be the result of the force on the mass loss rate? We have shown above that the mass loss rate is set by the condition that the solution of the momentum equation has to pass through the critical point. The conditions for the critical point are

$$r_c = \frac{GM_*}{2a^2} - \frac{f(r_c)r_c^2}{2a^2} \qquad \text{and} \qquad v_c = a \ . \tag{34}$$

We see that the critical velocity is again the isothermal sound speed, but the critical point is now closer to the star than for $f = 0$. Since the velocity at r_c is the same as for $f = 0$, but the velocity *gradient* in the subsonic region is smaller due to the force, the velocity below the critical point must be *higher* than without a force! This is also true at the lower boundary r_0 of the isothermal region. This means that the mass loss rate $\dot{M} = 4\pi r_0^2 v_0 \rho_0$ is higher!

This can also be understood in terms of the change in the density structure. Applying a force in the subsonic region results in a slower outward density decrease. Moreover, the critical point is closer to the star. Both effects imply that the density at the critical point, where $v = a$, will be higher than for $f = 0$. If the density at the critical point is higher and the velocity at the critical point is the same, the mass loss rate will be higher. We conclude that applying a force in the subsonic region results in a higher mass loss rate.

5 The Effect of Energy Input on a Stellar Wind

In Sect. 2.3 we have derived the energy equation (8) of a stellar wind with momentum and energy input. In this section we study the effect of energy input on the velocity law and the mass loss rate. The wind is no longer assumed to be isothermal.

The momentum equation (4) in a wind with energy and momentum deposition does not contain the thermal energy deposition $q(r)$. This does not mean, however, that the velocity in the wind is unaffected by heat deposition. Heat input changes the temperature structure of the wind and hence also the gas pressure. Since the momentum equation contains the force due to the gas pressure gradient, heat input affects the velocity law, density structure and the mass loss rate of a stellar wind. The effect of energy input on the velocity can be derived from the momentum equation by taking into account its effects on the pressure gradient. This will be studied here.

The pressure term $\rho^{-1}dp/dr$ in the momentum equation can be expressed in terms of p/ρ

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{d(p/\rho)}{dr} - p\frac{d\rho^{-1}}{dr} = \frac{d(p/\rho)}{dr} + \frac{p}{\rho}\frac{d\ \ln\rho}{dr} \ . \tag{35}$$

The density gradient can be eliminated by means of the mass continuity equation $d\ \ln\rho = -d\ \ln v - 2d\ \ln r$. Substitution of the isothermal sound speed $a(r)^2 = p/\rho = \mathcal{R}T/\mu$ for a perfect gas yields

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{da^2}{dr} - \frac{2a^2}{r} - a^2\frac{d\ \ln v}{dr} \ . \tag{36}$$

This results in the momentum equation in a general form that involves both the effects of a change in temperature and the external force

$$v\frac{dv}{dr} - \frac{a^2}{v}\frac{dv}{dr} + \frac{da^2}{dr} - \frac{2a^2}{r} + \frac{GM_*}{r^2} = f \tag{37}$$

or

$$\frac{1}{v}\frac{dv}{dr} = \left\{\frac{2a^2}{r} - \frac{da^2}{dr} - \frac{GM_*}{r^2} + f\right\} / \{v^2 - a^2\} \ . \tag{38}$$

This equation contains the term $da^2/dr = (\mathcal{R}/\mu)dT/dr$. The energy equation (8) shows that in a wind with momentum and energy deposition

$$\frac{da^2}{dr} = \frac{\mathcal{R}}{\mu}\frac{dT}{dr} = \frac{\gamma-1}{\gamma}\left\{f + q - \frac{GM_*}{r^2} - v\frac{dv}{dr}\right\} \ . \tag{39}$$

With $\gamma = 5/3$ for an ideal gas. Substituting this expression for da^2/dr in the momentum equation one finds after multiplication of the result by γ

$$\frac{1}{v}\frac{dv}{dr} = \left\{\frac{2c_s^2}{r} - \frac{GM_*}{r^2} + f - (\gamma-1)q\right\} / \{v^2 - c_s^2\} \ . \tag{40}$$

In this expression $c_s = \sqrt{\gamma a^2}$ is the adiabatic speed of sound.

This is the most general form of the momentum equation of a spherically symmetric stellar wind with energy input and momentum input. Notice that *the energy input, $q > 0$, produces an inward directed force $(\gamma - 1)q$ which counteracts the outward force f.* This is because the energy input heats the gas which reduces the negative temperature gradient and thus the outward force of the pressure gradient.

The two forms of the momentum equation (38) and (40) show a curious difference. The first one suggests that the critical point occurs where $v = a$ whereas the second one suggests that the critical point occurs where $v = c_s$. The difference is due to the fact that (38) still contains the temperature gradient in the numerator through the term da^2/dr. This temperature gradient depends on the velocity gradient as can easily be seen from (39). This extra vdv/dr term means that the critical point is not at $v = a$ but at $v = c_s$.

Equation (40) seems to suggest that in the most general case of a wind with energy and momentum deposition the critical point is always the sonic point. This, however, is only true if both f and q do not depend on the velocity gradient dv/dr. If the momentum or energy deposition depends on dv/dr, it will produce additional terms of type dv/dr in the right hand side of (40) and move the critical point to another location. This is the case for line driven winds (see second contribution Lamers, this Volume).

For a wind with a given force $f(r)$ and given momentum input $q(r)$, the momentum equation (40) and the energy equation (8) can be solved simultaneously by standard numerical methods. This gives the velocity law and the temperature structure. The mass loss rate is set by the condition that the velocity law has to pass smoothly through the critical point.

6 A Wind with an $f \sim r^{-2}$ Force

Let us now consider the effect of an extra force on the mass loss rate and the velocity law. We take a simple $f \sim r^{-2}$ force that starts at different distances from the star. Such a force can be produced by radiation pressure due to optically thin lines or due to dust. This is because the radiative flux F varies as r^{-2} and thus the radiative acceleration is $g_{\rm rad} = \kappa_F F(r)/c = \kappa_F F(R_*)(r/R_*)^{-2}/c \sim r^{-2}$ where κ_F is the flux-mean opacity. The last equality is only valid if κ_F is independent of distance.

The wind is again assumed to be isothermal. This simplifies the problems because the energy equation is reduced to $T(r) = T$ and it allows the isolation of the effects of the forces on the winds. The presence of a positive r^{-2} force which is smaller than the acceleration of gravity and which acts throughout the whole wind will obviously have the same effect as a reduction of the gravity or the mass of the star by a constant factor. In that case the mass loss rate and the velocity can simply be solutions of (24) and (32). However, if the force operates only in the lower part of the wind or only in the upper part, the solutions will be different.

The momentum equation of an isothermal wind with an additional positive force $f = Ar^{-2}$ is

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM_*}{r^2} + \frac{A(r)}{r^2} \tag{41}$$

with

$$\begin{aligned} A(r) &= 0 \qquad \text{for} \qquad r < r_d \\ A(r) &= A \qquad \text{for} \qquad r \geq r_d \ . \end{aligned} \tag{42}$$

In this equation A is a positive constant in the region $r > r_d$ where the force operates. The boundary is called r_d because of the similarity of this model to the dust driven wind, where the radiation pressure is switched on at the dust-formation radius. Expressing the pressure gradient in terms of the velocity gradient by means of the mass continuity equation (1), with (12) and (2) results in the momentum equation

$$\frac{r^2}{v}\frac{dv}{dr} = \frac{2a^2 r - GM_* + A(r)}{v^2 - a^2} = \frac{2a^2 r - GM_*(1 - \Gamma(r))}{v^2 - a^2} \tag{43}$$

with $\Gamma(r) = 0$ if $r < r_d$ and $\Gamma(r) = A/GM_*$ if $r > r_d$. The critical point occurs where

$$\frac{r_c}{1 - \Gamma(r_c)} = \frac{GM_*}{2a^2} \ . \tag{44}$$

The models for different values of r_d are shown in Fig. 3 for $\Gamma(r) = 0.5$ at $r > r_d$. The temperature of the wind model is chosen such that $\mathcal{R}T/\mu = a^2 = (GM_*/r_0)/2\sqrt{10}$. As the location of the critical point depends on Γ, we will indicate it as $r_c(\Gamma)$. The dependence of $\dot{M}$ in the five cases can be judged from the velocity laws by realizing that $\dot{M}$ is proportional to $v(r_0)$ because $\rho(r_0)$ is a fixed boundary condition. The top figure shows the location of $r_c(0)$ in case of $\Gamma = 0$. The bottom figure shows the location of $r_c(\Gamma)$ in case of $\Gamma = 0.5$ throughout the wind. These are the two extreme cases.

If $r_d > r_c(0)$ the structure of the subsonic region nor the location of the critical point is affected by Γ. So the mass loss rate will be the same as in the case of $\Gamma = 0$. This shows a very important characteristic of stellar winds: *an outward force applied to the wind above the critical point does not affect the mass loss rate.* The velocity in the supersonic regions at $r > r_d$ will be larger however than in the case of $\Gamma = 0$, because the numerator of (43) is larger when $\Gamma > 0$ and the denominator of (43) is positive. So increasing Γ above the critical point will result in a steeper velocity law and larger velocities at $r > r_d$. In the supersonic region the value of Γ may exceed 1.

If $r_d < r_c(\Gamma)$ the critical point occurs at $r_c(\Gamma)$. This is due to the fact that the location of r_c is given by the *local* condition that the numerator of (43) is zero. The mass loss depends on the value of Γ through the whole subcritical region $r_0 < r < r_c(\Gamma)$ because the velocity law in this region is affected by $\Gamma(r)$ according to (43). A positive value of $\Gamma > 0$ implies a smaller velocity gradient and a smaller density gradient at $r_d < r < r_c(0)$ than in the case of $\Gamma = 0$. This smaller density gradient results in a higher value of ρ at $r_c(\Gamma)$ and thus a higher mass loss rate.

If the value of Γ becomes positive somewhere between $r_c(\Gamma)$ and $r_c(0)$ then the location of the critical point depends sensitively on the shape of the $\Gamma(r)$ function. If Γ jumps from 0 to a value larger than 1 at r_d, the critical point will occur at r_d. This resembles models with radiation pressure due to dust.

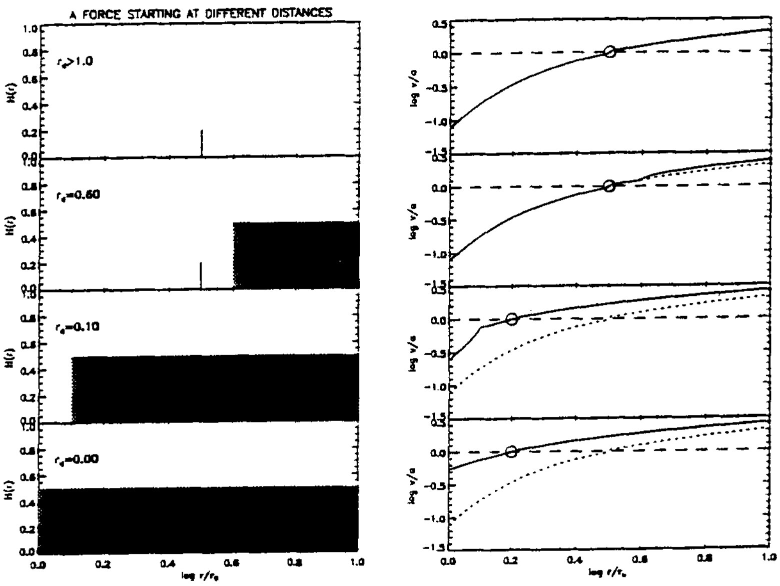

Fig. 3. The effect of an outward force $f(r) = \Gamma(r)GM_*/r^2$ on the velocity structure of an isothermal stellar wind. The left hand side shows the various distributions of $\Gamma(r)$ and the right hand side shows the resulting wind velocity. The wind velocity for $\Gamma(r) = 0$ is shown by a dotted line. The location of the critical point is indicated by a dot in the right hand figure and by a tickmark in the left figure. Notice the changes in the location of the critical point and in the mass loss rate $\dot{M} \sim v(r_0)$ if $\Gamma(r) > 0$ in the subsonic region

The mass loss rate can easily be predicted in the following way. Below the critical point, the density structure will be approximately as in hydrostatic equilibrium. This means that the density at r_d can simply be derived from the density at the lower boundary and the outward decrease with the pressure scale height. Since the sonic point is at r_d, the mass loss rate follows from the mass continuity equation.

7 The Five Laws of Stellar Wind Theory

Based on the arguments and the simple models described above, we can formulate five important laws for stellar winds.

1. *The first law of stellar winds*:
   The mass loss rate is set by the condition that the velocity law should pass smoothly through the critical point.

2. *The second law of stellar winds*:
   The mass loss rate is determined by the forces and the energy in the *subcritical* region of the wind.
3. *The third law of stellar winds*:
   The input of energy (heat) or momentum (by an outward directed force) in the *subcritical* region results in a smaller velocity gradient but a higher velocity and a higher mass loss rate.
4. *The fourth law of stellar winds*:
   The input of energy (heat) or momentum (by an outward directed force) in the *supercritical* region results in an increase in the terminal velocity of the wind, but does not affect the mass loss rate.
5. *The fifth law of stellar winds*:
   In the subsonic part of the wind the density structure is very similar to that of a hydrostatic wind. The velocity law then follows from the mass continuity equation. In the supersonic part of the wind the forces determine the velocity structure. The density law then follows from the mass continuity equation.

8 Mass Loss Mechanisms

Several mechanisms have been suggested to explain the mass loss and stellar winds of different types of stars. We briefly summarize these, together with some references to the basic theory.

8.1 Coronal Winds

Coronal winds are driven by gas pressure due to the high temperature of stellar coronae. Stars with a sub-photospheric convection zone can be surrounded by a hot corona of a few 10^6 K. The theory of coronal winds is very similar to the basic theory of hot isothermal winds that was discussed in Sect. 3, apart from the fact that coronal winds are not really isothermal but have a slowly outward decreasing temperature. The temperature gradient is small and the wind remains hot up to a large distance because of thermal conduction. The mass loss rates of coronal winds is rather small (see Table 1) except for very low gravity stars. However, for these stars other mechanisms are more important.

References: Parker (1958), Brandt (1970)

8.2 Dust Driven Winds

The driving force of dust driven winds is the radiation pressure on dust. Dust can form in the envelopes of cool stars when the temperature has dropped to below $\sim 10^3$ K and the density is still sufficiently large. Since dust is a very good continuum absorber, the dust grains will be radiatively accelerated outwards. Interactions with atoms provide the momentum-sharing needed to drag the gas component along. The wind theory is basically similar to that discussed in Sect. 5 for a wind with an $f \sim r^{-2}$ force from a certain distance onwards. The mass loss of the dust driven winds depends crucially on the location of the dust formation.

Reference: Sedlmayr (this Volume)

8.3 Line Driven Winds

The winds of hot stars are driven by radiation pressure on spectral lines due to ions of abundant elements that have very large numbers of absorption lines in the UV and the far-UV (below 912 Å). The radiation pressure depends strongly on the Doppler effect, because it allows the ions to intercept stellar radiation that was not absorbed in the lower layers of the wind. In this respect the line driven wind theory is very different from the dust driven wind theory and the basic theory described for gas-pressure driven winds because the force depends on the velocity *gradient*. The line driven wind theory is discussed in my other paper in this Volume.

References: Castor et al. (1975), Lamers (second contribution to this Volume).

8.4 Pulsation Driven Wind Theory

Cool supergiants such as the Mira stars and the AGB stars pulsate. During each expansion phase the atmosphere is tossed upward and it falls back during the contraction of the star. However, due to the low gravity, the layers in the outer atmosphere fall back so slowly that they are hit by the next expansion wave before they have reached their initial position. So the outer layers get a kick during every pulsation cycle. This results in a slow acceleration of outer layers of the atmosphere. This mass loss mechanism can be much more efficient if dust formation is taken into account because of the resulting radiation pressure on dust. The combination of pulsation and radiation pressure on dust provides a very efficient mass loss mechanism for luminous cool stars. Figure 4 shows the basic mechanism.

References: Bowen (1988).

8.5 Sound Wave Driven Winds

The convection zone below the photospheres of cool stars generates sound waves in the atmospheres. Sound waves generate a pressure very similar to the gas pressure by thermal motions. Outward travelling soundwaves produce a pressure gradient that results in an outward directed force. This force could in principle drive a stellar wind. The problem with the sound wave driven wind models is the same as for wind models driven by gas pressure (coronal

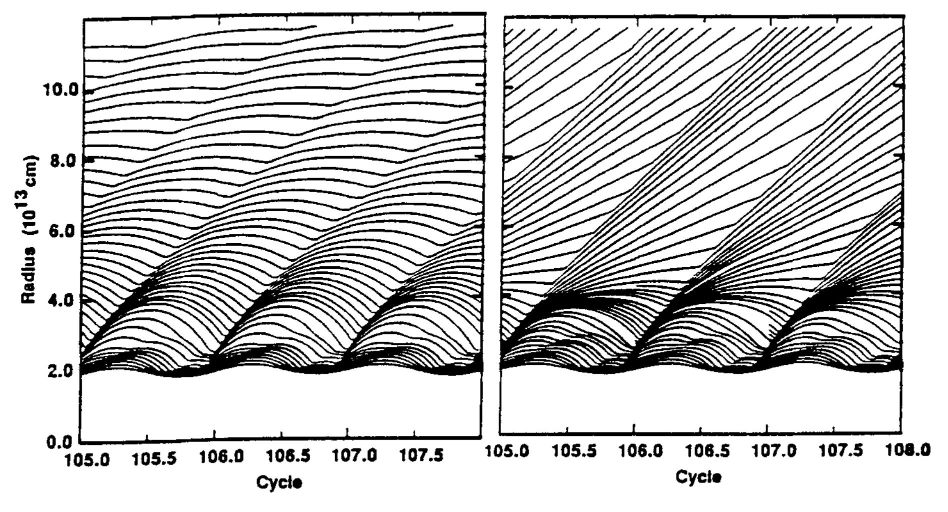

Fig. 4. The mass loss mechanism of pulsation. Left: pulsation only; right: pulsation and radiation pressure due to dust. The dust is formed at a distance of 4 10^{13} cm. The figure shows the motions of the different layers during a pulsation cycle (from Bowen 1988)

winds): the mechanism produces only small mass loss rates. This is because the amplitude of the waves has to be small, otherwise dissipation would quickly decrease the wave pressure.

References: Pijpers and Hearn (1989), Pijpers and Habing (1989)

8.6 Alfvén Wave Driven Wind Models

Stars with magnetic fields can have a mass loss due to the wave pressure by Alfvén waves. This requires open magnetic field lines with their footpoints in the photosphere. When these footpoints are oscillating, a magnetic wave travels outwards with the Alfvén speed, $v_A = B/\sqrt{4\pi\rho}$. This mechanism is quite similar to that of the sound wave driven winds but it is more efficient because the Alfvén speed of magnetic waves is much larger than the sound speed. It can result in high mass loss rates and high wind speeds of several times the Alfvén speed. This mechanism is important for stars that are not luminous enough to have a strong radiation pressure , i.e. $L_* < 10^3 L_\odot$. It is the dominant mechanism for the fast wind from the coronal holes of the sun.

References: Hartmann and MacGregor (1980, 1982).

8.7 Magnetic Rotating Winds

In these models the wind is basically driven by the rotation. Since the plasma can only move along the magnetic field lines, material in the equatorial plane of the star will be "flung out" by the magnetic field lines that corotate with the star. (This is similar to the motion of a marble in a hollow flexible plastic-tube. When the tube is swayed around, the marble will be flung out of the tube). This process converts rotational energy into expansion from the outer atmosphere. The mass loss rate is mainly determined by the rotation, and the terminal velocity of the wind is determined by the strength of the magnetic field. This mechanism explains the fast deceleration of rapidly rotating pre-main sequence stars.

References: Weber and Davis (1967), Brandt (1970)

8.8 Summary of Wind Theories

Table 2 gives a summary of the characteristics of the mechanisms for mass loss not due to rotation.

Table 2. Mass loss mechanisms

Mechanism and Stars	Types	Characteristics
CORONAL WINDS		
Solar type	G,K	low $\dot{M}$, high v_∞
Giants ?		
LINE DRIVEN WINDS		
Hot stars	O,B,A CPN, WD WR?	high $\dot{M}$, high v_∞
DUST DRIVEN WINDS		
Cool supergiants	M, AGB	high $\dot{M}$, low v_∞
PULSATION DRIVEN WINDS		
Cool pulsating stars	Mira's, AGB?	high $\dot{M}$, low v_∞
ALFVEN WAVE DRIVEN WINDS		
Cool stars with magnetic fields	F,G,K,M,	low $\dot{M}$?, high v_∞
MAGNETIC ROTATING WINDS		
Magnetic fast rotators	WR?	high $\dot{M}$, high v_∞
SOUND WAVE DRIVEN WINDS		
Stars with convective envelopes	??	low $\dot{M}$, low v_∞

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