PLONK

Permutations over Lagrange-bases for Oecumenical Noninteractive arguments of Knowledge

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Reintroducing SNARKs

Scenario: A Prover wants to convince a Verifier that a certain computation was performed correctly. However:

- The Prover may want to keep parts of the computation secret
- The Verifier may want to verify the result without redoing the entire computation

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SNARKs (Succinct Non-interactive Arguments of Knowledge) allow the Prover to generate a *short proof* that:

- Convincingly proves correctness
- Hides specific parts of the input (if Prover want)
- Is quick for the Verifier to verify

Reintroducing SNARKs

SNARK : Succinct Non-interactive ARgument of Knowledge

- *Succinct* : The proof length is short and Verifier time is fast
- *Non-interactive* : The protocol does not require back-and-forth communication
- *ARgument* : Basically a proof
  (More precise: A computationally sound proof, which is secure against adversaries with bounded computational resources.)
- *of Knowledge* : The proof does not just show the system of equations has a solution; it also show that the Prover knows the solution

Zero-Knowledge SNARK (zk-SNARK): A SNARK where the Verifier learns nothing about the Prover's private input, except what the Prover intentionally reveals.

PLONK: A Modern zkSNARK

PLONK stands for:

Permutations over Lagrange-bases for Oecumenical Noninteractive arguments of Knowledge

It is a powerful type of **zkSNARK** that offers two major advantages:

- **Universal Trusted Setup:** A single setup can be reused for any circuit or program, regardless of the specific computation.
- **Updatable Setup:** Anyone can contribute randomness to the setup phase. As long as one participant is honest, the resulting setup remains secure - and can be updated over time to improve trust.

Programmable Cryptography

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The process of translating a computational problem into an algebraic form that a SNARK system can interpret and use to generate a proof called **Arithmetization**.

Circuit

A **circuit** (or Arithmetic Circuit) is an algebraic representation of a computational problem.

Given a problem, we can translate it into a circuit form that expresses its logic using arithmetic constraints.

Conversely, a circuit can be interpreted back into the original problem it was designed to represent.

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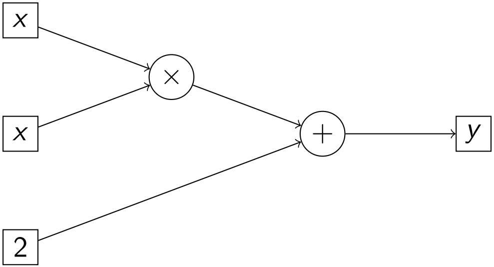
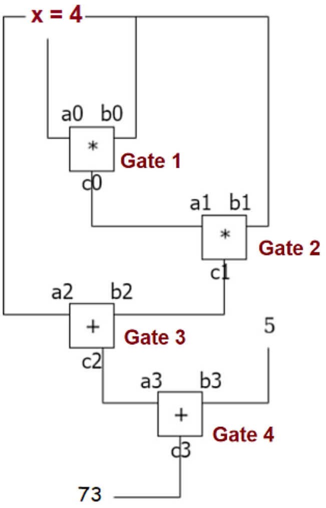
Example: Arithmetic Circuit for $y = x^2 + 2$



Arithmetic Circuit for PLONK



Vectors	a	b	c
Gate 1	4	4	16
Gate 2	16	4	64
Gate 3	4	64	68
Gate 4	68	5	73

Source: PlonK — Risen Crypto

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- **Copy Constraints:** Ensure that values on connected wires are consistent across the circuit.
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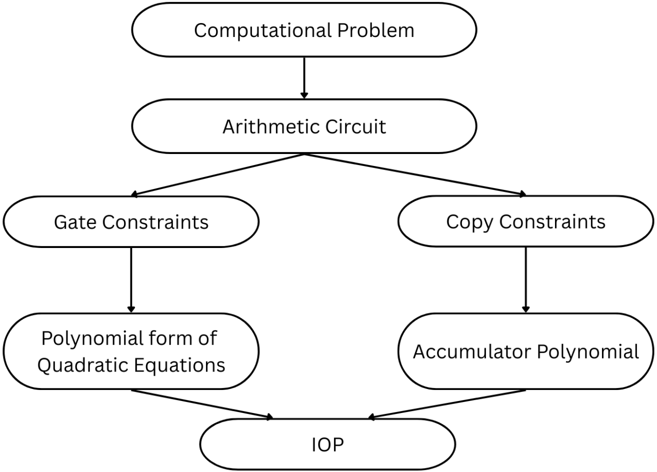
  $e.g.,\ c_0 = a_1, \quad c_1 = b_2, \quad \ldots$

To prove a valid PLONK instance, the Prover must demonstrate that both the **Gate Constraints** and **Copy Constraints** are satisfied.

PLONK Overview



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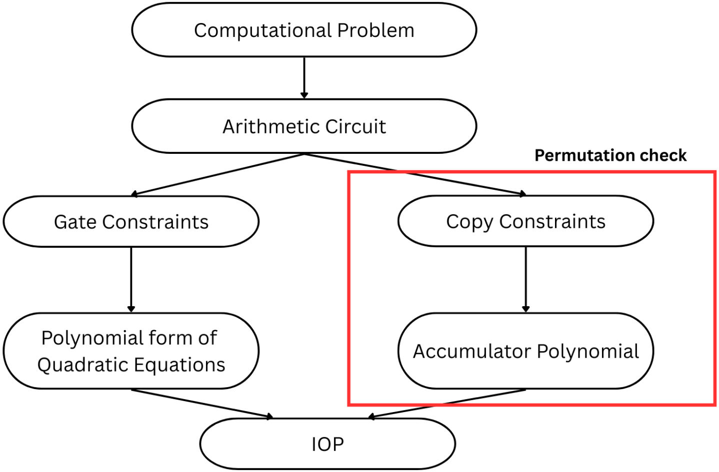


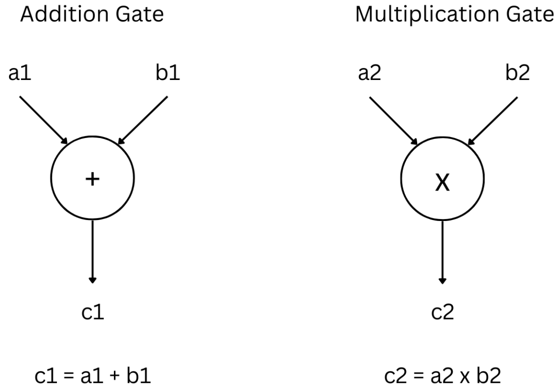
PLONK : All you need is a permutation check

Proving:

- **Gate Constraints**: $a_0 \cdot b_0 = c_0, a_2 + b_2 = c_2, \ldots$ is **easy**
- **Copy Constraints**: $c_0 = a_1, c_1 = b_2, \ldots$ is **hard**
- **Public input**: $a_2 = 5, c_3 = 9, \ldots$ is **easy**

So all you need is a permutation check - a technique to prove **Copy Constraints**.

PLONK Gate Constraint



PLONK Gate Constraint

The quadratic equation of Gate Constraint in PLONK has the form:

PLONK Gate Equation

$$q_{L,i} \cdot a_i + q_{R,i} \cdot b_i + q_{O,i} \cdot c_i + q_{M,i} \cdot a_i \cdot b_i + q_{C,i} = 0$$

For each equation i in the system:

- The $q_{*,i}$ are coefficients in $\mathbb{F}_p$, which are publicly known and often referred to as **selectors**.
- The subscripts denote the role of each term:
  - **L**: Left input
  - **R**: Right input
  - **O**: Output
  - **M**: Multiplication term
  - **C**: Constant term

PLONK Gate Constraint

Addition Gate : $a + b = c$

$$(q_{L_i}, q_{R_i}, q_{O,i}, q_{M,i}, q_{C,i}) = (1, 1, -1, 0, 0)$$

Multiplication Gate : $a \cdot b = c$

$$(q_{L_i}, q_{R_i}, q_{O,i}, q_{M,i}, q_{C,i}) = (0, 0, -1, 1, 0)$$

Assign Gate : $a = \tau$

$$(q_{L_i}, q_{R_i}, q_{O,i}, q_{M,i}, q_{C,i}) = (1, 0, 0, 0, -\tau)$$

PLONK Copy Constraints

We will explore the details of copy constraints later.

For now, it is enough to represent them as simple equalities between wires:

$$a_0 = b_1, \quad c_2 = a_3, \quad \ldots$$

PLONK Arithmetization Example

Goal: Encode the statement $y = x^2 + 2$ using PLONK constraints.

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Step 1: Break down the computation into basic operations

- **Multiplication:** $x \cdot x = x^2$
- **Constant assignment:** $t = 2$
- **Addition:** $x^2 + t = y$

PLONK Arithmetization Example

Step 2: Encode the operations using PLONK constraints

Gate Constraints:

- **Multiplication gate:** $(a_1, b_1, c_1) = (x, x, x^2)$

$$0 \cdot a_1 + 0 \cdot b_1 + (-1) \cdot c_1 + 1 \cdot a_1 \cdot b_1 + 0 = 0$$

- **Constant gate:** $(a_2, b_2, c_2) = (2, 0, 0)$

$$1 \cdot a_2 + 0 \cdot b_2 + 0 \cdot c_2 + 0 \cdot a_2 \cdot b_2 + (-2) = 0$$

- **Addition gate:** $(a_3, b_3, c_3) = (x^2, 2, y)$

$$1 \cdot a_3 + 1 \cdot b_3 - 1 \cdot c_3 + 0 \cdot a_3 \cdot b_3 + 0 = 0$$

Copy Constraints:

- $a_1 = b_1$ (for $x^2 = x \cdot x$)
- $c_1 = a_3$ (connect output of multiplication to input of addition)
- $a_2 = b_3$ (connect constant 2 to input of addition)

PLONK Flow: Setup

The Prover and Verifier are given a PLONK instance.

The Prover holds a **witness** — an assignment of values to each a_i, b_i, and c_i such that:

- All **gate constraints** are satisfied
- All **copy constraints** are satisfied

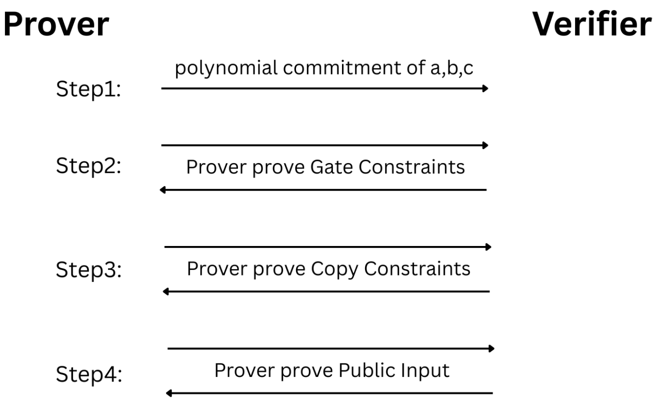
The goal: The Prover wants to convince the Verifier that a valid solution exists — **succinctly and without revealing the solution itself**.

PLONK Flow: Protocol Steps

The PLONK protocol proceeds as follows:

1. The Prover constructs and sends polynomial commitments corresponding to the wire assignments (a_i, b_i, c_i).
2. The Prover proves that these polynomials satisfy the **gate constraints** of the system.
3. The Prover proves that the same assignments also satisfy the **copy constraints**.
4. The Prover proves that the **Public Input** is match.

PLONK Flow


Prover
Verifier
Step1:
polynomial commitment of a,b,c
Step2:
Prover prove Gate Constraints
Step3:
Prover prove Copy Constraints
Step4:
Prover prove Public Input

Step 1: Commitment

We have n quadratic equations:

$$q_{L,1} \cdot a_1 + q_{R,1} \cdot b_1 + q_{O,1} \cdot c_1 + q_{M,1} \cdot a_1 \cdot b_1 + q_{C,1} = 0$$

$$q_{L,2} \cdot a_2 + q_{R,2} \cdot b_2 + q_{O,2} \cdot c_2 + q_{M,2} \cdot a_2 \cdot b_2 + q_{C,2} = 0$$

$$\vdots$$

$$q_{L,n} \cdot a_n + q_{R,n} \cdot b_n + q_{O,n} \cdot c_n + q_{M,n} \cdot a_n \cdot b_n + q_{C,n} = 0$$

Step 1: Commitments

To encode the witness, we move from field elements to polynomials.

Assume $n \mid (p-1)$ so that there exists a primitive n-th root of unity $\omega \in \mathbb{F}_p$:

$$\omega^n = 1 \quad \text{and} \quad \omega^i \neq 1 \text{ for } 1 \leq i < n$$

The Prover interpolates three polynomials:

$$\left.\begin{aligned} A(\omega^i) &= a_i \\ B(\omega^i) &= b_i \\ C(\omega^i) &= c_i \end{aligned}\right. \quad \text{for all } i = 1, \dots, n$$

These encode the witness values across all gates.

Step 1: Commitments

Commitment Protocol:

1. The Prover interpolates three polynomials: $A(X)$, $B(X)$, and $C(X)$ from the witness values.
2. The Prover sends $\texttt{Com}(A)$, $\texttt{Com}(B)$, $\texttt{Com}(C)$to the Verifier

These commitments hide the witness but bind the Prover to a specific assignment.

Step 2: Gate Constraint Check

Since the PLONK instance is public, both the Prover and Verifier can interpolate selector polynomials:

$$\begin{aligned} Q_L(\omega^i) &= q_{L,i} \\ Q_R(\omega^i) &= q_{R,i} \\ Q_O(\omega^i) &= q_{O,i} \quad \text{for } i = 1, \ldots, n \\ Q_M(\omega^i) &= q_{M,i} \\ Q_C(\omega^i) &= q_{C,i} \end{aligned}$$

These polynomials encode the gate constraints across the entire circuit.

Step 2: Gate Constraints Check

$$
\begin{aligned}
q_{L,1} \cdot a_1 + q_{R,1} \cdot b_1 + q_{O,1} \cdot c_1 + q_{M,1} \cdot a_1 \cdot b_1 + q_{C,1} &= 0 \\
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\end{aligned}
$$

Q_L Q_R Q_O Q_M Q_C

Step 2: Gate Constraints Check

The Prover must convince the Verifier that the following polynomial identity holds:

$$Q_L(x)A(x) + Q_R(x)B(x) + Q_O(x)C(x) + Q_M(x)A(x)B(x) + Q_C(x) = 0$$

This equation must be satisfied at all n evaluation points:

$$x = \omega^1, \omega^2, \ldots, \omega^n$$

This ensures that all n gate constraints are satisfied simultaneously.

Vanishing Polynomial Proof Protocol

Goal: Prove that a polynomial $F(X)$ vanishes on a finite set $S \subseteq \mathbb{F}_p$:

$$F(x) = 0 \quad \forall x \in S$$

Vanishing Polynomial Proof Protocol

1. The Prover sends a commitment $\texttt{Com}(F)$ to the Verifier.
2. Both Prover and Verifier compute the vanishing polynomial over S:
$$Z(X) := \prod_{z \in S}(X - z)$$
3. The Prover computes:
$$H(X) := \frac{F(X)}{Z(X)}$$
and sends the commitment $\texttt{Com}(H)$ to the Verifier.
4. The Verifier chooses a random challenge $\lambda \in \mathbb{F}_p$ and requests an opening of both commitments at λ.
5. The Prover responds with $F(\lambda)$ and $H(\lambda)$, along with proofs that these values match their respective commitments.
6. The Verifier verify the proofs and checks that:
$$F(\lambda) = Z(\lambda) \cdot H(\lambda)$$

KZG Polynomial Commitment Scheme

The *Kate–Zaverucha–Goldberg (KZG)* Polynomial Commitment Scheme allows a Prover to convince a Verifier that a committed polynomial $P(X)$ evaluates to y at a point z, i.e., $P(z) = y$.

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- The Verifier then queries the Prover for $P(z)$.
- The Prover responds with the claimed value $y = P(z)$, along with a short "proof" that convinces the Verifier that y is indeed the correct evaluation — without revealing $P(X)$.

Step 3: Proving Copy Constraints

Copy constraints ensure that values assigned to the same wire across different gates remain consistent.

To illustrate this, consider a circuit with 4 gates and the following copy constraints:

$$a_1 = a_4 = c_4 \quad \text{and} \quad b_2 = c_1$$

To prove that these constraints are satisfied, we use the technique named **permutation check**.

Permutation Check : Setup

Problem: Suppose we have polynomials P, Q with encode into two vectors of values

$$\vec{p} = \langle P(\omega^1), P(\omega^2), ..., P(\omega^n) \rangle$$

$$\vec{q} = \langle Q(\omega^1), Q(\omega^2), ..., Q(\omega^n) \rangle$$

Verify $\vec{p}$ is permutation of $\vec{q}$.

Permutation Check : Core Idea

Suppose $\vec{p}$ is a permutation of $\vec{q}$. Then the following identity holds:

$$(X + P(\omega^1))(X + P(\omega^2))...(X + P(\omega^n))$$
$$= (X + Q(\omega^1))(X + Q(\omega^2))...(X + Q(\omega^n))$$

$$\Leftrightarrow \prod_{i=1}^{n}(X + P(\omega^i)) = \prod_{i=1}^{n}(X + Q(\omega^i)) \qquad (*)$$

This is because both sides are degree-n polynomials in a single variable X, and they share the same set of roots (up to reordering).

Permutation Check : Randomness Argument

Lemma (Schwartz–Zippel Lemma)

If two degree-d polynomials over $\mathbb{F}_p$ are not equal, they can agree on at most d points in $\mathbb{F}_p$.

Therefore, if Equation $(*)$ from the previous slide does **not** hold, the probability that it still evaluates equally at a randomly chosen point $\lambda \in \mathbb{F}_p$ is:

$$\Pr[F(\lambda) = G(\lambda)] \leq \frac{n}{|\mathbb{F}_p|} \ll 1 \quad \text{(for large } p\text{, e.g., 256-bit field)}$$

This lets the verifier check polynomial equality using just one random evaluation.

Permutation Check : Accumulator Polynomial

We can use **Acummulator Polynomial** to let verifier check this equation at a random evaluation.

Create Accumulator Polynomials:

- Verifier picks a random challenge $\lambda \in \mathbb{F}_p$ and sends to Prover
- Prover then interpolate the **acummulator polynomial** F_P such that:

$$\begin{aligned}
F_P(\omega^1) &= \lambda + P(\omega^1) \\
F_P(\omega^2) &= (\lambda + P(\omega^1))(\lambda + P(\omega^2)) \\
&\dots \\
F_P(\omega^n) &= (\lambda + P(\omega^1))(\lambda + P(\omega^2))\dots(\lambda + P(\omega^n))
\end{aligned}$$

- Prover does similar for F_Q

Permutation Check — Protocol (Setup)

Suppose the Prover has committed to two polynomials via:

$$\texttt{Com}(P), \quad \texttt{Com}(Q)$$

Protocol Steps:

1. The Verifier sends a random challenge $\lambda \in \mathbb{F}_p$ to the Prover.
2. The Prover constructs two new polynomials:

$$F_P(X) := \prod_{i=1}^{n}(\lambda + P(\omega^i)), \quad F_Q(X) := \prod_{i=1}^{n}(\lambda + Q(\omega^i))$$

3. The Prover sends commitments $\texttt{Com}(F_P)$ and $\texttt{Com}(F_Q)$ to the Verifier.

Permutation Check : Verifier Checks

Using the **Vanishing Polynomial Proof Protocol** (described earlier), the Prover and Verifier now check the following statements:

1. $F_P(X) - (\lambda + P(X))$ vanishes at $X = \omega$
2. $F_P(\omega X) - (\lambda + P(X))F_P(X)$ vanishes at $X \in \{\omega, ..., \omega^{n-1}\}$
3. $F_Q(X) - (\lambda + Q(X))$ vanishes at $X = \omega$
4. $F_Q(\omega X) - (\lambda + Q(X))F_Q(X)$ vanishes at $X \in \{\omega, ..., \omega^{n-1}\}$
5. $F_P(X) - F_Q(X)$ vanishes at $X = 1$

Copy Constraint Check : Init Problem

Consider a circuit with 4 gates and the following copy constraints:

$$a_1 = a_4 = c_4 \quad \text{and} \quad b_2 = c_1$$

This means the assignments before and after applying the permutation (enforced by the copy constraints) must match:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} = \begin{bmatrix} {\color{red}a_4} & b_1 & {\color{blue}b_2} \\ a_2 & {\color{blue}c_1} & c_2 \\ a_3 & b_3 & c_3 \\ {\color{red}c_4} & b_4 & {\color{red}a_1} \end{bmatrix} \tag{1}$$

Copy Constraint Check : Tag

To transform the equality in (1) into a permutation check, we **tag** each entry in the matrix with a unique offset using powers of η, ω, and a random scalar μ. Specifically:

Each element at row k and column j is transformed as:

$$\text{entry} \quad \longrightarrow \quad \text{entry} + \eta^j \omega^k \mu$$

for $j = 0, 1, 2$ and $k = 1, ..., n$, where:

- $\omega \in \mathbb{F}_p$ is an n-th root of unity
- $\eta \in \mathbb{F}_p$ is a random value such that η^2 is not a power of ω
- $\mu \in \mathbb{F}_p$ is a random challenge

Copy Constraint Check : Tag

This transforms both matrices in equation (1) into:

$$\begin{bmatrix} a_1 + \omega^1\mu & b_1 + \eta\omega^1\mu & c_1 + \eta^2\omega^1\mu \\ a_2 + \omega^2\mu & b_2 + \eta\omega^2\mu & c_2 + \eta^2\omega^2\mu \\ a_3 + \omega^3\mu & b_3 + \eta\omega^3\mu & c_3 + \eta^2\omega^3\mu \\ a_4 + \omega^4\mu & b_4 + \eta\omega^4\mu & c_4 + \eta^2\omega^4\mu \end{bmatrix} = \begin{bmatrix} {\color{red}a_4} + \omega^1\mu & b_1 + \eta\omega^1\mu & {\color{blue}b_2} + \eta^2\omega^1\mu \\ a_2 + \omega^2\mu & {\color{blue}c_1} + \eta\omega^2\mu & c_2 + \eta^2\omega^2\mu \\ a_3 + \omega^3\mu & b_3 + \eta\omega^3\mu & c_3 + \eta^2\omega^3\mu \\ {\color{red}c_4} + \omega^4\mu & b_4 + \eta\omega^4\mu & {\color{red}a_1} + \eta^2\omega^4\mu \end{bmatrix} \tag{2}$$

This transformation ensures that only if the copy constraints hold, both tagged matrices are a permutation of each other.

Copy Constraint Check : Change to Permutation Check

From Equation (2) we have:

$$\begin{bmatrix} a_1 + \omega^1\mu & b_1 + \eta\omega^1\mu & c_1 + \eta^2\omega^1\mu \\ a_2 + \omega^2\mu & b_2 + \eta\omega^2\mu & c_2 + \eta^2\omega^2\mu \\ a_3 + \omega^3\mu & b_3 + \eta\omega^3\mu & c_3 + \eta^2\omega^3\mu \\ a_4 + \omega^4\mu & b_4 + \eta\omega^4\mu & c_4 + \eta^2\omega^4\mu \end{bmatrix}$$

is a permutation of

$$\begin{bmatrix} a_1 + {\color{red}\eta^2\omega^4\mu} & b_1 + \eta\omega^1\mu & c_1 + {\color{blue}\eta\omega^2\mu} \\ a_2 + \omega^2\mu & b_2 + {\color{blue}\eta^2\omega^1\mu} & c_2 + \eta^2\omega^2\mu \\ a_3 + \omega^3\mu & b_3 + \eta\omega^3\mu & c_3 + \eta^2\omega^3\mu \\ a_4 + {\color{red}\omega^1\mu} & b_4 + \eta\omega^4\mu & c_4 + {\color{red}\omega^4\mu} \end{bmatrix} \tag{3}$$

Copy Constraint Check : Permutation Check

Because permutation is publicly known, then both parties can interpolate the 3 polynomial $\sigma_a, \sigma_b, \sigma_c$:

- $\sigma_a(\omega^1) = \eta^2\omega^4, \quad \sigma_b(\omega^1) = \eta\omega^1, \quad \sigma_c(\omega^1) = \eta\omega^2$
- $\sigma_a(\omega^2) = \omega^2, \quad \sigma_b(\omega^2) = \eta^2\omega^1, \quad \sigma_c(\omega^2) = \eta^2\omega^2$
- $\sigma_a(\omega^3) = \omega^3, \quad \sigma_b(\omega^3) = \eta\omega^3, \quad \sigma_c(\omega^3) = \eta^2\omega^3$
- $\sigma_a(\omega^4) = \omega^1, \quad \sigma_b(\omega^4) = \eta\omega^4, \quad \sigma_c(\omega^4) = \omega^4$

Copy Constraint Check : Permutation Check

Prover defining accumulator polynomials after receive the random challenge λ from Verifier.

- $F_a(\omega^k) = \prod_{i \leq k}(a_i + \omega^i \mu + \lambda)$
- $F_b(\omega^k) = \prod_{i \leq k}(b_i + \eta\omega^i \mu + \lambda)$
- $F_c(\omega^k) = \prod_{i \leq k}(c_i + \eta^2\omega^i \mu + \lambda)$
- $F'_a(\omega^k) = \prod_{i \leq k}(a_i + \sigma_a(\omega^i)\mu + \lambda)$
- $F'_b(\omega^k) = \prod_{i \leq k}(b_i + \sigma_b(\omega^i)\mu + \lambda)$
- $F'_c(\omega^k) = \prod_{i \leq k}(c_i + \sigma_c(\omega^i)\mu + \lambda)$

And similar to the **Permutation Check** we have:

6 initialization conditions :

- $F_a(\omega^1) = A(\omega^1) + \omega^1\mu + \lambda$
- $F_b(\omega^1) = B(\omega^1) + \eta\omega^1\mu + \lambda$
- $F_c(\omega^1) = C(\omega^1) + \eta^2\omega^1\mu + \lambda$
- $F'_a(\omega^1) = A(\omega^1) + \sigma_a(\omega^1)\mu + \lambda$
- $F'_b(\omega^1) = B(\omega^1) + \sigma_b(\omega^1)\mu + \lambda$
- $F'_c(\omega^1) = C(\omega^1) + \sigma_c(\omega^1)\mu + \lambda$

Copy Constraint Check : Permutation Check

6 accumulation conditions :

- $F_a(\omega X) = F_a(X)(A(\omega X) + X\mu + \lambda)$
- $F_b(\omega X) = F_b(X)(B(\omega X) + \eta X\mu + \lambda)$
- $F_c(\omega X) = F_c(X)(C(\omega X) + \eta^2 X\mu + \lambda)$
- $F'_a(\omega X) = F'_a(X)(A(\omega X) + \sigma_a(X)\mu + \lambda)$
- $F'_b(\omega X) = F'_b(X)(B(\omega X) + \sigma_b(X)\mu + \lambda)$
- $F'_c(\omega X) = F'_c(X)(C(\omega X) + \sigma_c(X)\mu + \lambda)$

Copy Constraint Check : Permutation Check

And one product condition:

$$F_a(1)F_b(1)F_c(1) = F'_a(1)F'_b(1)F'_c(1)$$

We use **Vanishing Polynomial Proof Protocol** described above to verify these conditions.

Step 4: Handling Public and Private Inputs

Let $S \subseteq \{1, 2, \ldots, n\}$ be the set of indices corresponding to **public** inputs (i.e., specific a_i values that the Prover wants to reveal).

The Prover constructs a new polynomial $A^{\text{public}}(X)$ such that:

$$A^{\text{public}}(\omega^i) = \begin{cases} a_i & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Then, the Prover uses the **Vanishing Polynomial Proof Protocol** to prove:

$$A^{\text{public}}(X) - A(X) \text{ vanishes on } S$$

This ensures that the committed polynomial $A(X)$ agrees with the claimed public values at exactly the specified positions, without revealing the private ones.

Fiat-Shamir Transform

Fiat-Shamir is a cryptographic technique that transforms an interactive protocol into a non-interactive one.

Originally used for identification schemes, it plays a central role in SNARKs.

Core Idea:

- Replace the Verifier's random challenge with a deterministic value.
- The challenge is computed by hashing the transcript (e.g., using SHA256).
- This makes the protocol non-interactive while maintaining soundness under the *Random Oracle Model (ROM)*.

From Interactive to Non-Interactive

Interactive Protocol:

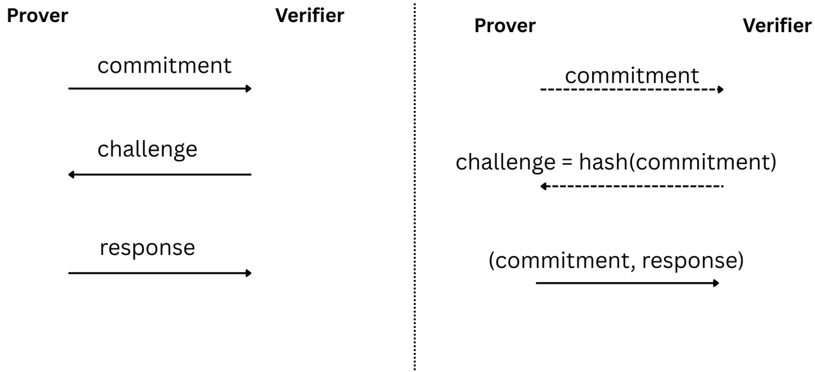
1. Prover sends a commitment to Verifier.
2. Verifier responds with a random challenge.
3. Prover replies with a response based on the challenge.

Fiat-Shamir:

1. Prover computes challenge $=$ Hash(commitment)
2. Uses this challenge to compute response.
3. Sends only (commitment, response) to Verifier.

No back-and-forth needed!

Fiat-Shamir Transform



Why Fiat-Shamir in SNARKs?

In SNARKs (e.g., PLONK, Groth16), many interactive steps involve the Verifier choosing random challenges.

To make the SNARK **succinct and non-interactive**, Fiat-Shamir is used to:

- Derive challenges deterministically from prior messages.
- Ensure the proof can be verified by anyone without interaction.
- Remove the need for trusted communication rounds.

Fiat-Shamir is what makes most zk-SNARKs practical!

KZG Polynomial Commitment Scheme

Structured Reference String (SRS)

$$g^1, g^\tau, g^{\tau^2}, \ldots, g^{\tau^d}$$

where $\tau \in \mathbb{F}_p$ is a secret scalar and g is a generator of an elliptic curve group.

Polynomial

$$p(x) = \sum_{i=0}^{d} a_i x^i$$

Commitment

$$\mathrm{Com}(p) = \prod_{i=0}^{d} (g^{\tau^i})^{a_i} = g^{\sum_{i=0}^{d} a_i \tau^i} = g^{p(\tau)}$$

KZG: Opening and Verification

Goal: Prove that a committed polynomial $p(x)$ evaluates to $p(z)$ at some point $z \in \mathbb{F}_p$.
Define:

$$h(x) := \frac{p(x) - p(z)}{x - z} \quad \Rightarrow \quad p(x) = h(x)(x - z) + p(z)$$

Opening Proof

$$\pi := g^{h(\tau)}$$

Verification Equation

$$e(\mathrm{Com}(p) - g^{p(z)}, g) \stackrel{?}{=} e(\pi, g^{\tau - z})$$

Why this works:

$$\begin{aligned} e(g^{p(\tau)-p(z)}, g) &= e(g^{h(\tau)(\tau-z)}, g) \\ &= e(g^{h(\tau)}, g^{\tau-z}) = e(\pi, g^{\tau-z}) \end{aligned}$$

Elliptic Curves

An elliptic curve over $\mathbb{F}_p$ is defined as:

$$E : y^2 = x^3 + ax + b$$

Properties:

- Points form a group with addition.
- Scalar multiplication: $g, 2g, 3g, \ldots$
- Secure due to hard discrete log problem.

Used in KZG: Commit to polynomials as group elements like $g^{p(\tau)}$.

Pairing Operation

A pairing is a map:

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

Key properties:

- Bilinear: $e(g^a, g^b) = e(g, g)^{ab}$
- Non-degenerate and efficient

Used in KZG to verify:

$$e(g^{p(\tau)-p(z)}, g) = e(g^{h(\tau)}, g^{\tau-z})$$