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MATH 446
Project 2

The equation: $\mathbf{3x + 1 = 5x^2 + 9x^3 + \sin e^x}$

<u>Question 1:</u> Use Fixed Point Iteration to calculate the three roots, round each root to 6 correct decimal places.

The given equation can be rewritten as:
$f(x) = 9x^3 + 5x^2 - 3x - 1 + \sin e^2 = 0$

The below are the report in the format of:
$g_i(x)$: g(x) used to find roots
r_i: root found, rounded to 6 decimal places
x_0: initial guess I used
n = number of steps required

$g_1(x) = \frac{\sin(e^x)-1}{3-5x-9x^2}$
r1 = -0.058419
$x_0 = 1.5$
n = 21

Code:

```
g = @(x) (sin(exp(x)) - 1) / (3-5*x-9*x^2);
num_steps=21;
x(1)=1.5;
for i=1:num_steps
    x(i+1)=g(x(i));
end
r=x(num_steps+1);
x'
```

Output:

```
>> fpi1

ans =

                   1.5
    0.0797376401120131
   -0.0458438457339206
   -0.0571833295338694
   -0.0582966809049428
   -0.0584069031478175
   -0.0584178241818272
   -0.0584189063473963
   -0.0584190135800802
   -0.0584190242058644
    -0.058419025258783
   -0.0584190253631177
   -0.0584190253734563
   -0.0584190253744808
   -0.0584190253745823
   -0.0584190253745924
   -0.0584190253745934
   -0.0584190253745935
   -0.0584190253745935
   -0.0584190253745935
   -0.0584190253745935
   -0.0584190253745935
```

$g_2(x) = 3\sqrt{\frac{1}{9}(3x + 1 - 5x^2 + sine^x)}$

$r2 = -0.855869$

$x_0 = 1.5$

$n = 32$

Code:

```
g = @(x) nthroot(((-5/9)*x^2 - (1/9)*sin(exp(x)) + (1/3)*x + 1/9),3);
num_steps=32;
x(1)=1.5;
for i=1:num_steps
    x(i+1)=g(x(i));
end
r=x(num_steps+1);
x'
```

Output:

```
ans =

                       1.5
         -0.809634285797879
         -0.829506966951408
         -0.840899273427117
         -0.847388023362513
         -0.851070448900595
         -0.853155991621719
         -0.854335778474715
         -0.855002747217674
         -0.855379666109187
         -0.855592627197997
         -0.855712937202728
         -0.855780900509112
         -0.855819291650712
          -0.85584097759752
          -0.85585322715987
         -0.855860146422779
         -0.855864054808706
         -0.855866262478502
         -0.855867509489443
         -0.855868213867981
         -0.855868611738533
         -0.855868836476985
         -0.855868963421201
         -0.855869035126033
         -0.855869075628726
         -0.855869098506796
         -0.855869111429543
         -0.855869118728997
         -0.855869122852115
         -0.855869125181071
         -0.855869126496588
         -0.855869127239663
```

$g_3(x) = 3\sqrt{\frac{1}{9}(3x + 1 - 5x^2 - sine^x)}$

r3 = 0.364934

x_0 = 0.5

n = 28

Code:

```
g = @(x) nthroot(((-5/9)*x^2 - (1/9)*sin(exp(x)) + (1/3)*x + 1/9),3);
num_steps=28;
x(1)=0.5;
for i=1:num_steps
    x(i+1)=g(x(i));
end
r=x(num_steps+1);
x'
```

Output:

```
ans =

                     0.5
       0.304073897561328
       0.374584055230418
        0.36256796891474
       0.365476853321337
       0.364806824328463
       0.364963072627986
       0.364926738807312
       0.364935193408277
       0.364933226389018
       0.364933684045477
       0.364933577565738
       0.364933602339685
       0.364933596575695
       0.364933597916764
       0.364933597604746
       0.364933597677342
       0.364933597660451
       0.364933597664381
       0.364933597663467
       0.364933597663679
        0.36493359766363
       0.364933597663642
       0.364933597663639
       0.364933597663639
       0.364933597663639
       0.364933597663639
       0.364933597663639
       0.364933597663639
```

Question 2: Determine convergence rate S = |g'(r)|

$g_1(x) = \frac{\sin(e^x)-1}{3-5x-9x^2}$

$g_1'(x) = \frac{e^x\cos(e^x)}{-9x^2-5x+3} - \frac{(-18x-5)(\sin(e^x)-1)}{(-9x^2-5x+3)^2}$

$|\, g_1'(r_1)| = |\, g_1'(-0.058419025374593)| = 0.099090917650315$

$g_2(x) = 3\sqrt{\frac{1}{9}(3x+1-5x^2-sine^x)}$

$g_2'(x) = \frac{-e^x\cos(e^x)-10x+3}{3\sqrt[3]{9}(-\sin(e^x)-5x^2+3x+1)^{2/3}}$

$|\, g_2'(r_2)| = |\, g_2'(-0.855868213867981)| = 0.564853236573544$

$g_2(x) = 3\sqrt{\frac{1}{9}(3x+1-5x^2-sine^x)}$

$g_3'(x) = \frac{-e^x\cos(e^x)-10x+3}{3\sqrt[3]{9}(-\sin(e^x)-5x^2+3x+1)^{2/3}}$

$|\, g_3'(r_3)| = |\, g_3'(0.364933597663639)| = 0.232663394953845$

Question 3: For each fixed point r, use Matlab calculations to approximate the convergence rate:

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i}$$

With r_1 = -0.058419025374593, $\lim_{i\to\infty} \frac{e_{i+1}}{e_i}$ **= 0.099090917650315**

Looking at the output, I can see the consistent appearance of the first few digits from this ratio in the third column, please look at the highlighted part below.

Code:

```
g = @(x) (sin(exp(x)) - 1) / (3-5*x-9*x^2);
num_steps=20;
x(1)=0.5;
for i=1:num_steps
    x(i+1)=g(x(i));
end
r=x(num_steps+1);
e=x-r;
for i=1:num_steps
    ratio(i)=e(i+1)/e(i);
end
ratio(num_steps+1)=0;
[x' e' ratio']
```

Output:

0.500000000000000	0.558419025374593	0.107720344153447
0.001734064220590	0.060153089595183	0.095347363217642
-0.052683586892298	0.005735438482295	0.098710844289464
-0.057852875399636	0.000566149974958	0.099053153628938
-0.058362946434147	0.000056078940446	0.099087174566595
-0.058413468670832	0.000005556703762	0.099090546733926
-0.058418474757780	0.000000550616814	0.099090880871899
-0.058418970813488	0.000000054561105	0.099090914026863
-0.058419019968084	0.000000005406510	0.099090915674921
-0.058419024838857	0.000000000535736	0.099090874296921
-0.058419025321507	0.000000000053087	0.099090160246716
-0.058419025369333	0.000000000005260	0.099087583778857
-0.058419025374072	0.000000000000521	0.099070795282090
-0.058419025374542	0.000000000000052	0.098495028218221
-0.058419025374588	0.000000000000005	0.095497953615280
-0.058419025374593	0.000000000000000	0.057142857142857
-0.058419025374593	0.000000000000000	-0.250000000000000
-0.058419025374593	-0.000000000000000	0
-0.058419025374593	0	NaN
-0.058419025374593	0	NaN
-0.058419025374593	0	0

With r_2 = -0.855868213867981, $\lim_{i \to \infty} \frac{e_{i+1}}{e_i}$ **= 0.564853236573544**

Looking at the output, I can see the consistent appearance of the first few digits from this ratio in the third column, please look at the highlighted part below.

Code:

```
g = @(x) nthroot(((-5/9)*x^2 - (1/9)*sin(exp(x)) + (1/3)*x + 1/9),3);
num_steps=32;
x(1)=1.5;
for i=1:num_steps
    x(i+1)=g(x(i));
end
r=x(num_steps+1);
e=x-r;
for i=1:num_steps
    ratio(i)=e(i+1)/e(i);
end
ratio(num_steps+1)=0;
[x' e' ratio']
```

Output:

1.500000000000000	2.355869127239663	0.019625386192805
-0.809634285797879	0.046234841441784	0.570179532711241
-0.829506966951408	0.026362160288255	0.567853834771487
-0.840899273427117	0.014969853812546	0.566545537675337
-0.847388023362513	0.008481103877150	0.565808225978213
-0.851070448900595	0.004798678339067	0.565392265586011

-0.853155991621719	0.002713135617944	0.565157434374663
-0.854335778474715	0.001533348764948	0.565024762659603
-0.855002747217674	0.000866380021989	0.564949696499513
-0.855379666109187	0.000489461130476	0.564907046646657
-0.855592627197997	0.000276500041665	0.564882507769857
-0.855712937202728	0.000156190036934	0.564867851256673
-0.855780900509112	0.000088226730551	0.564858163047668
-0.855819291650712	0.000049835588951	0.564850195113367
-0.855840977597520	0.000028149642142	0.564841276195016
-0.855853227159870	0.000015900079792	0.564828416035689
-0.855860146422779	0.000008980816884	0.564807302348581
-0.855864054808706	0.000005072430957	0.564770853462157
-0.855866262478502	0.000002864761161	0.564706838881212
-0.855867509489443	0.000001617750219	0.564593761610667
-0.855868213867981	0.000000913371682	0.564393597603073
-0.855868611738533	0.000000515501129	0.564038876709275
-0.855868836476985	0.000000290762678	0.563409523775312
-0.855868963421201	0.000000163818462	0.562290898014229
-0.855869035126033	0.000000092113630	0.560296411959891
-0.855869075628726	0.000000051610936	0.556720503970241
-0.855869098506796	0.000000028732867	0.550245102973326
-0.855869111429543	0.000000015810119	0.538304993494851
-0.855869118728997	0.000000008510666	0.515535181172978
-0.855869122852115	0.000000004387548	0.469189641158823
-0.855869125181071	0.000000002058592	0.360962332950853
-0.855869126496588	0.000000000743074	0
-0.855869127239663	0	0

With r_3 = 0.364933597663639, $\lim_{i\to\infty} \frac{e_{i+1}}{e_i}$ **= 0.232663394953845**

Looking at the output, I can see the consistent appearance of the first few digits from this ratio in the third column, please look at the highlighted part below.

Code:

```
g = @(x) nthroot(((-5/9)*x^2 - (1/9)*sin(exp(x)) + (1/3)*x + 1/9),3);
num_steps=28;
x(1)=0.5;
for i=1:num_steps
    x(i+1)=g(x(i));
end
r=x(num_steps+1);
e=x-r;
for i=1:num_steps
    ratio(i)=e(i+1)/e(i);
end
ratio(num_steps+1)=0;
[x' e' ratio']
```

Output:

0.500000000000000	0.135066402336361	-0.450590961553491
0.304073897561328	-0.060859700102311	-0.158568930680812

0.374584055230418	0.009650457566778	-0.245131252329738
0.362567968914740	-0.002365628748899	-0.229645356631132
0.365476853321337	0.000543255657698	-0.233358518002414
0.364806824328463	-0.000126773335177	-0.232501293004758
0.364963072627986	0.000029474964347	-0.232701089867946
0.364926738807312	-0.000006858856327	-0.232654623620452
0.364935193408277	0.000001595744637	-0.232665435710877
0.364933226389018	-0.000000371274621	-0.232662919991936
0.364933684045477	0.000000086381838	-0.232663505419479
0.364933577565738	-0.000000020097901	-0.232663368978345
0.364933602339685	0.000000004676045	-0.232663401203842
0.364933596575695	-0.000000001087945	-0.232663362632603
0.364933597916764	0.000000000253125	-0.232663624489352
0.364933597604746	-0.000000000058893	-0.232662940972985
0.364933597677342	0.000000000013702	-0.232667711890843
0.364933597660451	-0.000000000003188	-0.232661802859083
0.364933597664381	0.000000000000742	-0.232749588384972
0.364933597663467	-0.000000000000173	-0.232797427652733
0.364933597663679	0.000000000000040	-0.233425414364641
0.364933597663630	-0.000000000000009	-0.236686390532544
0.364933597663642	0.000000000000002	-0.250000000000000
0.364933597663639	-0.000000000000001	-0.200000000000000
0.364933597663639	0.000000000000000	-1.000000000000000
0.364933597663639	-0.000000000000000	0.500000000000000
0.364933597663639	-0.000000000000000	0
0.364933597663639	0	NaN
0.364933597663639	0	0