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MATH 446
Project 3

The equation given:

$$f(x) = 98e^{5x^2} - (1225x^4 + 280x^2 + 66)e^{\frac{3}{7}} = 0$$

**Question 1:**

***Matlab Code:***

```
% Newton's Method
% By Ngoc Huynh
f=@(x) 98*exp(5*x^2) - (1225*x^4+280*x^2+66)*exp(3/7);
fp=@(x) 980*x*exp(5*x^2) - exp(3/7)*(4900*x^3+560*x);
num_steps=20;
x(1)=0.9;
for i=1:num_steps
    x(i+1) = x(i) - f(x(i)) / fp(x(i));
end
x'
```

***Output:***

r =
0.900000000000000
0.811157472559801
0.725013226764242
0.644638216032397
0.572743916164997
0.511402317867241
0.462292301561077
0.427311246863312
0.408300811664235
0.402888492031269
0.402498550900794
0.402496639950111
0.402496639904432
0.402496639904432
0.402496639904432
0.402496639904432
0.402496639904432
0.402496639904432
0.402496639904432
0.402496639904432
0.402496639904432

$$f(r) = f(0.402496639904432) = 0$$

The root I found is **r = 0.402496639904432** and this root has been the same since n=13 to n=20 so I am confident that all digits presented in **r = 0.402496639904432** are all correct.

## Question 2:

To show that this Newton's Method converges to the root quadratically or linearly, I will find $f'(x)$ and plug in the root found in part 1. If $f'(x) \neq 0$ then it is locally and quadratically convergent to r.

$$f'(x) = 980xe^{5x^2} - e^{\frac{3}{7}}(4900x^3 + 560x)$$
$$r = 0.402496639904431$$
$$f'(r) = 50.229128772220974 > 0$$

**So, Newton's Method converges to the root quadratically.**

***Matlab Code:***

```
% Newton's Method
% By Ngoc Huynh
f=@(x) 98*exp(5*x^2) - (1225*x^4+280*x^2+66)*exp(3/7);
fp=@(x) 980*x*exp(5*x^2) - exp(3/7)*(4900*x^3+560*x);
fpp=@(x) 9800*x^2*exp(5*x^2) + 980*exp(5*x^2) - exp(3/7)*(14700*x^2+560);
x(1) = 0.9;
num_steps=20;
for i=1:num_steps
    x(i+1) = x(i) - f(x(i)) / fp(x(i));
end

r=x(num_steps+1);
e=x-r;
for i=1:num_steps
    ratio(i)=e(i+1)/e(i)^2;

end
ratio(num_steps+1)=0;
[x' e' ratio']
```

Ratio $\lim_{i \to \infty} \frac{e_i+1}{{e_i}^2}$ from the Newton iteration:

| | | |
|---|---|---|
| 0.900000000000000 | 0.497503360095568 | 1.651090881756276 |
| 0.811157472559801 | 0.408660832655370 | 1.931194535961806 |
| 0.725013226764242 | 0.322516586859810 | 2.327905008963889 |
| 0.644638216032397 | 0.242141576127966 | 2.903631121384803 |
| 0.572743916164997 | 0.170247276260565 | 3.757423765940562 |
| 0.511402317867241 | 0.108905677962809 | 5.041602833918593 |
| 0.462292301561077 | 0.059795661656646 | 6.940137086971325 |
| 0.427311246863312 | 0.024814606958881 | 9.425957001431209 |
| 0.408300811664235 | 0.005804171759803 | 11.631659937726793 |
| 0.402888492031269 | 0.000391852126838 | 12.445588680602047 |
| 0.402498550900794 | 0.000001910996362 | 12.508324271206380 |
| 0.402496639950111 | 0.000000000045679 | 0 |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | NaN |
| 0.402496639904432 | 0 | 0 |

Theoretical value from calculus:

$$f'(x) = 980xe^{5x^2} - e^{\frac{3}{7}}(4900x^3 + 560x)$$
$$f''(x) = 9800x^2e^{5x^2} + 980e^{5x^2} - e^{\frac{3}{7}}(14700x^2 + 560)$$

$$r = 0.402496639904432$$

$$M = \frac{f''(r)}{2f'(r)} = 12.508624497410192$$

Comparing those two highlighted numbers above, we can see that they agree to each other at least up to 3 decimal places, which is **12.508.**

# Question 3:

## Question 3.1:

The equation given:

$$f(x) = 98e^{5x^2} - (1225x^4 + 280x^2 + 65)e^{\frac{3}{7}} = 0$$

***Matlab Code:***

```
% Newton's Method
% By Ngoc Huynh
f=@(x) 98*exp(5*x^2) - (1225*x^4+280*x^2+65)*exp(3/7);
fp=@(x) 980*x*exp(5*x^2) - exp(3/7)*(4900*x^3+560*x);
x(1) = 1;
num_steps=40;
for i=1:num_steps
    x(i+1) = x(i) - f(x(i)) / fp(x(i));
end
x'
```

***Output:***

r =
1.000000000000000
0.911468443992925
0.822466039088284
0.735721161005871
0.654234805004990
0.580606570253967
0.516549933553073
0.462765808605831
0.419083407294110
0.384702388814242
0.358434129736345
0.338910218579747
0.324752828610006
0.314700811935855
0.307684604497265
0.302851726218874
0.299555378302353
0.297322958951925
0.295818620608676
0.294808413628383
0.294131636766609
0.293678966496167
0.293376521025615
0.293174593510871
0.293039842622390
0.292949949431210
0.292889994444169
0.292850013379229
0.292823351705274
0.292805575582767

0.292793726599829
0.292785823085503
0.292780516723284
0.292776962585009
0.292774462195886
0.292772171192656
0.292772171192656
0.292772171192656
0.292772171192656
0.292772171192656
0.292772171192656

The fact that the root is known to be exactly $\sqrt{3/35}$.

```
>> sqrt(3/35)

ans =

   0.292770021884560
```

Comparing the root found from the iteration and the accurate root from $\sqrt{3/35}$, they agree with each other up to 5 decimal points, which is 0.29277.

**Question 3.2:**

To show that this Newton's Method converges to the root quadratically or linearly, I will find $f'(x)$ and plug in the root found in part 1. If $f'(x) \neq 0$ then it is locally and quadratically convergent to r.

$$f'(x) = 980xe^{5x^2} - e^{\frac{3}{7}}(4900x^3 + 560x)$$
$$r = 0.292772171192656$$
$$f'(r) = 8.7198941400856 71e-09 \text{ which is very small and close to 0}$$
Or plug in $\sqrt{3/35}$, I will have:
$$f'(\sqrt{3/35}) = 0$$

**So, Newton's Method converges to the root linearly.**

Find the multiplicity of the root r = $\sqrt{3/35}$ of $f(x) = 98e^{5x^2} - (1225x^4 + 280x^2 + 65)e^{\frac{3}{7}} = 0$. And estimate the number of steps of Newton's Method required to converge.
It is easy to check that:

$$f(x) = 98e^{5x^2} - (1225x^4 + 280x^2 + 65)e^{\frac{3}{7}}$$
$$f'(x) = 980xe^{5x^2} - e^{\frac{3}{7}}(4900x^3 + 560x)$$
$$f''(x) = 9800x^2e^{5x^2} + 980e^{5x^2} - e^{\frac{3}{7}}(14700x^2 + 560)$$

And that each evaluates to 0 at r = $\sqrt{3/35}$. The third derivative,

$$f''(x) = (98000x^3 + 29400x)e^{5x^2} - 29400xe^{\frac{3}{7}}$$

satisfies $f''\left(\sqrt{\frac{3}{35}}\right) = 3.775131618836147e + 03$, so the root r = $\sqrt{3/35}$ is a triple root, meaning that the multiplicity is m = 3. By Theorem 1.12, Newton should converge linearly with $e_{i+1} = 2e_i/3$.

***Matlab Code:***

```
% Newton's Method
% By Ngoc Huynh
f=@(x) 98*exp(5*x^2) - (1225*x^4+280*x^2+65)*exp(3/7);
fp=@(x) 980*x*exp(5*x^2) - exp(3/7)*(4900*x^3+560*x);
fpp=@(x) 9800*x^2*exp(5*x^2) + 980*exp(5*x^2) - exp(3/7)*(14700*x^2+560);
fppp=@(x) (98000*x^3 +29400*x)*exp(5*x^2) - 29400*exp(3/7)*x;
x(1) = 1;
num_steps=33;
for i=1:num_steps
    x(i+1) = x(i) - f(x(i)) / fp(x(i));
end
%r=x(num_steps+1);
r = sqrt(3/35);
e=x-r;
for i=1:num_steps
    ratio(i)=e(i+1)/e(i);

end
ratio(num_steps+1)=0;
[x' e' ratio']
```

***Output:***

| | | |
|---|---|---|
| 1.000000000000000 | 0.707229978115440 | 0.874819282628565 |
| 0.911468443992925 | 0.618698422108365 | 0.856145738013451 |
| 0.822466039088284 | 0.529696017203724 | 0.836236491751739 |
| 0.735721161005871 | 0.442951139121311 | 0.816037596917514 |
| 0.654234805004990 | 0.361464783120430 | 0.796305924700576 |
| 0.580606570253967 | 0.287836548369407 | 0.777454819188963 |
| 0.516549933553073 | 0.223779911668513 | 0.759656152573194 |
| 0.462765808605831 | 0.169995786721271 | 0.743038329630231 |
| 0.419083407294110 | 0.126313385409550 | 0.727811756700259 |
| 0.384702388814242 | 0.091932366929682 | 0.714265389272646 |
| 0.358434129736345 | 0.065664107851785 | 0.702669970013659 |
| 0.338910218579747 | 0.046140196695187 | 0.693165808042215 |
| 0.324752828610006 | 0.031982806725446 | 0.685705611754404 |
| 0.314700811935855 | 0.021930790051295 | 0.680075025925656 |
| 0.307684604497265 | 0.014914582612705 | 0.675962887873667 |
| 0.302851726218874 | 0.010081704334314 | 0.673036640709456 |
| 0.299555378302353 | 0.006785356417793 | 0.670994533968166 |
| 0.297322958951925 | 0.004552937067365 | 0.669589471369724 |
| 0.295818620608676 | 0.003048598724116 | 0.668632354825269 |
| 0.294808413628383 | 0.002038391743823 | 0.667984888663017 |
| 0.294131636766609 | 0.001361614882049 | 0.667548969675868 |
| 0.293678966496167 | 0.000908944611607 | 0.667256434892041 |
| 0.293376521025615 | 0.000606499141055 | 0.667060509941360 |
| 0.293174593510871 | 0.000404571626311 | 0.666929463862906 |
| 0.293039842622390 | 0.000269820737830 | 0.666841059351869 |
| 0.292949949431210 | 0.000179927546650 | 0.666782612464649 |

| | | |
|---|---|---|
| 0.292889994444169 | 0.000119972559609 | 0.666748254175257 |
| 0.292850013379229 | 0.000079991494669 | 0.666693639556151 |
| 0.292823351705274 | 0.000053329820714 | 0.666675749725370 |
| 0.292805575582767 | 0.000035553698207 | 0.666729945531494 |
| 0.292793726599829 | 0.000023704715269 | 0.666584718018009 |
| 0.292785823085503 | 0.000015801200943 | 0.664179815272219 |
| 0.292780516723284 | 0.000010494838724 | 0.661344174202699 |
| 0.292776962585009 | 0.000006940700449 | 0 |

Note the convergence of the error ratio in the right column to the predicted 2/3. Conclusion, the multiplicity of the root **m = 3**.