Lecture 18: VAEs, Intro to PCA

Nisha Chandramoorthy

November 2, 2023

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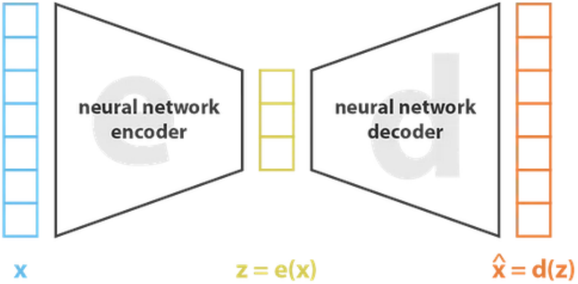

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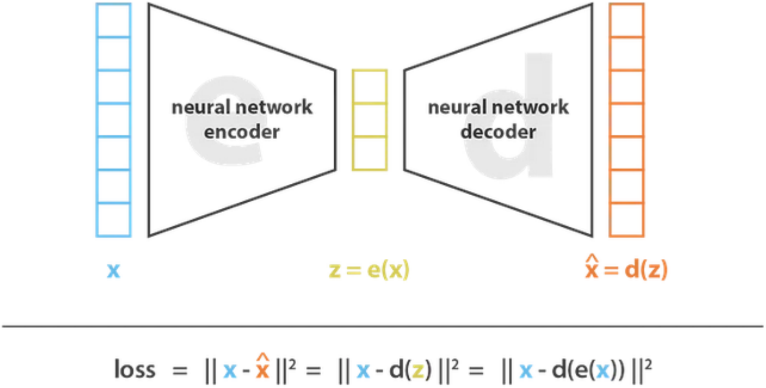

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- $$\begin{aligned} D_{\mathrm{KL}}(q_\theta(z|x)||p(z|x)) &= \int q_\theta(z|x) \log \frac{q_\theta(z|x)}{p(z|x)} dz) \\ &= E_{z\sim q_\theta(z|x)}[\log p(x|z)] \\ &- D_{\mathrm{KL}}(q_\theta(z|x)||p(z)) + c(x) \end{aligned}$$

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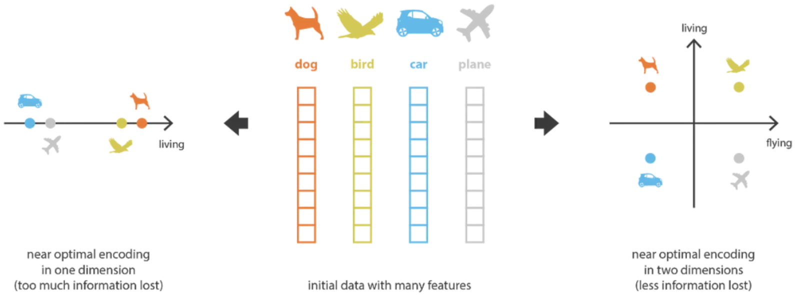
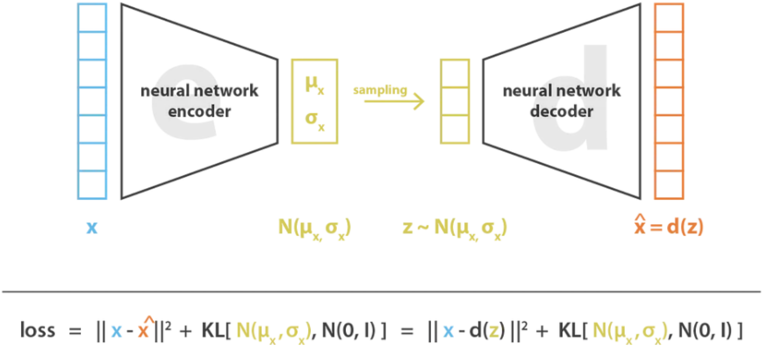


Courtesy: https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73



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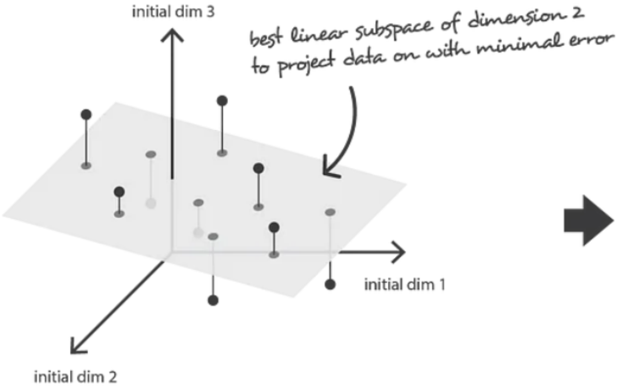

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Lecture 14: Convolutional Neural Networks and Intro to PCA

Nisha Chandramoorthy

October 26, 2023

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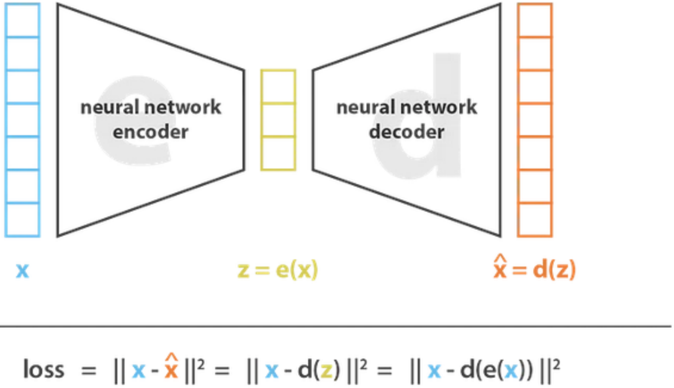

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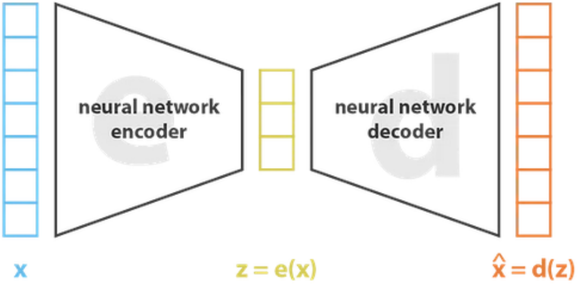

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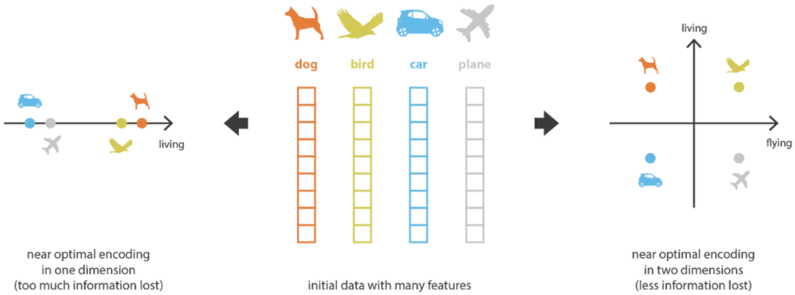

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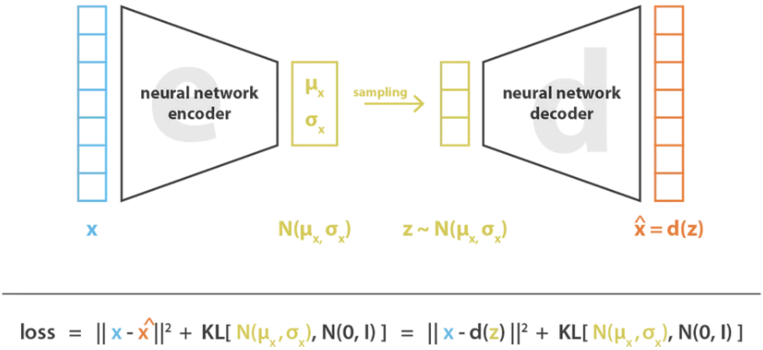

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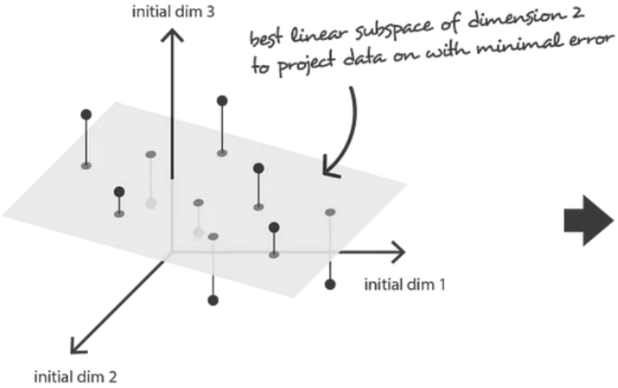

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Convolutional Neural Networks (source: cs231n.stanford.edu)

- Suitable for image recognition. Won the 2012 ImageNet competition and subsequent ones.
- Three types of layers: convolutional, FC, pooling
- Convolutional layer: accepts a volume of size $W_1 \times H_1 \times D_1$ and outputs a volume of size $W_2 \times H_2 \times D_2$ where $W_2 = (W_1 - F + 2P)/S + 1$ and $H_2 = (H_1 - F + 2P)/S + 1$ and $D_2 = K$.
- K is number of filters, F is filter size, S is stride, P is padding.
- Pooling layer: downsamples along width and height, and optionally along depth.
- FC layer: computes class scores, resulting in volume of size $1 \times 1 \times K$.