Lecture 21: Johnson-Lindenstrauss lemma, random projections, CNN

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- Informally, PCA rotates the data so that the variance is maximized along the first axis, then the second, and so on.
- Separates dissimilar points

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- For some PD kernel, if $K_{ij} = k(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle = (XX^\top)_{ij}$, can compute K only using kernel evaluations.

Graphical reprensentation of X

- Choose weighting, such as, $w_{ij} = \exp(-\|x_i - x_j\|^2/2\sigma^2)$. As $\sigma \to 0$, $w_{ij} \to \mathbb{1}_{i=j}$. The $m \times m$ matrix W is the adjacency matrix of a graph.

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- Graph laplacian: $L = D - W$.
- Detects local structure / clusters in data.

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- For any vector v, $v^\top L v = (1/2) \sum_{i,j=1}^{m} w_{ij}(v_i - v_j)^2$.
- L is positive semi-definite.

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- Another interpretation: top n eigenvectors of $L^\dagger$. $L^\dagger_{ij}$ represents expected time for random walk $i \to j \to i$.
- Kernel PCA with $K = L^\dagger$ is equivalent to Laplacian eigenmaps.

Stochastic neighbor embedding [Hinton and Roweis 2002]

- Stochastic neighbor embedding(SNE): conditional probability that x_i would pick x_j as its neighbor, given by

$$p(x_j|x_i) = \frac{\exp(-\|x_i - x_j\|^2/2\sigma_i^2)}{\sum_{k\neq i}\exp(-\|x_i - x_k\|^2/2\sigma_i^2)}.$$

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- SNE minimizes $\sum_{i=1}^{m} D_{\mathrm{KL}}(p_i\|q_i)$, where p_i and q_i are the conditional probabilities of x_i and y_i respectively.
- Penalizes large distances between x_i and x_j but also preserves local structure.

tSNE [Van der Maaten and Hinton 2008]

- tSNE cost function is $D_{\mathrm{KL}}(p\|q) = \sum_{i=1}^{m} \sum_{j \neq i} p_{ij} \log \frac{p_{ij}}{q_{ij}}$, where $p_i j$ and q_{ij} are the joint probabilities of (x_i, x_j) and (y_i, y_j) respectively.
- Changes joint distribution to a heavy-tailed distribution, $q(y_j, y_i) = \frac{(1+\|y_i - y_j\|^2)^{-1}}{\sum_{k \neq i}(1+\|y_i - y_k\|^2)^{-1}}$.

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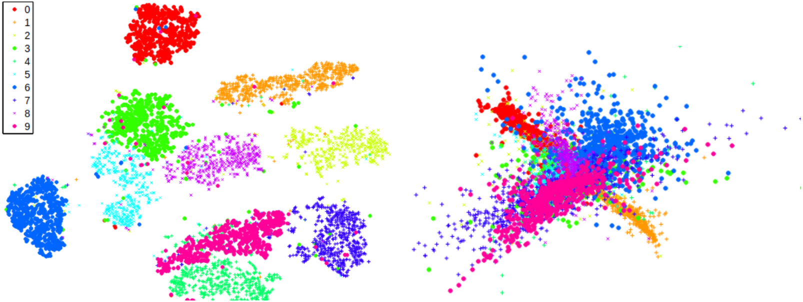
- approaches inverse square law on embedded space.

tSNE visualization



From Van der Maaten and Hinton 2008. tSNE (left) and LLE (right) on MNIST dataset.

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- Informal: any set of points in high-dimensional space can be mapped to a lower-dimensional space while approximately preserving the distances between the points.

Proof

- Distortion by Gaussian random matrices: for any $x \in \mathbb{R}^d$, when the entries A_{ij} are iid standard Gaussian,

$$\mathbb{P}(n(1-\epsilon)\|x\|^2 \leqslant \|Ax\|^2 \leqslant n(1+\epsilon)\|x\|^2) \\ \geqslant 1 - 2\exp(-(\epsilon^2 - \epsilon^3)n/4).$$

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- Then, deterministic statement of J-L lemma follows from union bound over all m^2 pairs of points.

To show: distortion by Gaussian random matrices

- Let A be a $n \times d$ matrix with iid standard Gaussian entries. Then, $E[(Ax)_j] = 0$ and $\mathrm{Var}((Ax)_j) = \|x\|^2$, for all $j \leqslant n$.

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- Models sum of squares of n independent standard normal random variables.

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- Use Markov inequality and moment generating function to prove.
- Use Lemma 15.2 to prove distortion by Gaussian random matrices.

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Tao 2005 If x is s-sparse, then,

$$x = \mathrm{argmin}_{z\in\mathbb{R}^d}\|z\|_1 \quad \text{s.t.} \quad Ax = Az. \tag{3}$$

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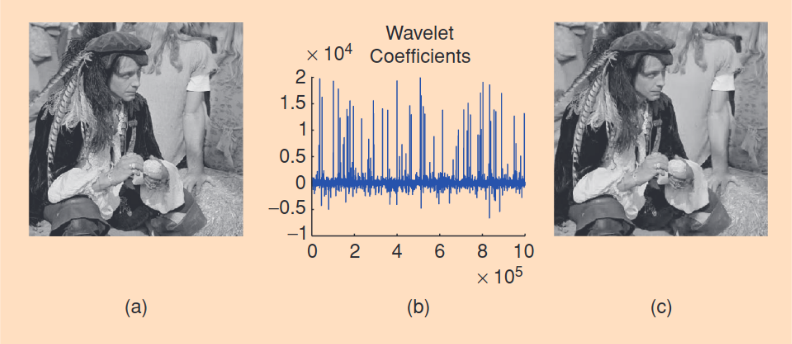
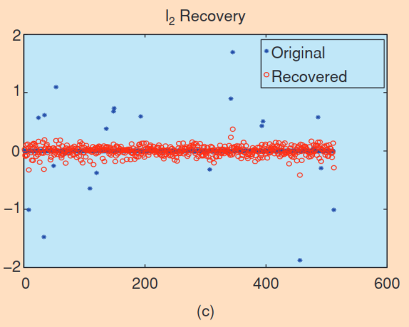
- Reconstruction obtained by solving a convex program.

Candes 2008

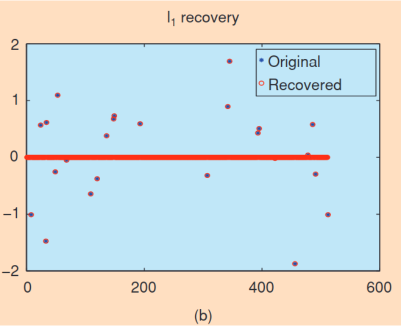


[FIG1] (a) Original megapixel image with pixel values in the range [0,255] and (b) its wavelet transform coefficients (arranged in random order for enhanced visibility). Relatively few wavelet coefficients capture most of the signal energy; many such images are highly compressible. (c) The reconstruction obtained by zeroing out all the coefficients in the wavelet expansion but the 25,000 largest (pixel values are thresholded to the range [0,255]). The difference with the original picture is hardly noticeable. As we describe in "Undersampling and Sparse Signal Recovery," this image can be perfectly recovered from just 96,000 incoherent measurements.

Candes 2008



l_2 Recovery
Original
Recovered
2
1
0
−1
−2
0
200
400
600
(c)



l_1 recovery
Original
Recovered
2
1
0
−1
−2
0
200
400
600
(b)

Convolutional Neural Networks (source: cs231n.stanford.edu)

- Suitable for image recognition. Won the 2012 ImageNet competition and subsequent ones.
- Three types of layers: convolutional, FC, pooling
- Convolutional layer: accepts a volume of size $W_1 \times H_1 \times D_1$ and outputs a volume of size $W_2 \times H_2 \times D_2$ where $W_2 = (W_1 - F + 2P)/S + 1$ and $H_2 = (H_1 - F + 2P)/S + 1$ and $D_2 = K$.
- K is number of filters, F is filter size, S is stride, P is padding.
- Pooling layer: downsamples along width and height, and optionally along depth.
- FC layer: computes class scores, resulting in volume of size $1 \times 1 \times K$.