Homework 0 : Solutions

1. $A = U \Sigma V^*$ $\quad A \in \mathbb{C}^{m \times n}$

Let $u_1, u_2, \cdots u_m \in \mathbb{C}^m$

$v_1, v_2 \cdots, v_m \in \mathbb{C}^m$ be columns of V. Let $\sigma_1, \sigma_2, \cdots, \sigma_m$ be diagonal elements of Σ. Then, $A v_i = \sigma_i u_i$ ($\because U, V$ are unitary). Also, $A^* = V \Sigma^* U^* = V \Sigma U^*$ and similarly, $A^* u_i = \sigma_i v_i$

Thus, $B \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \begin{bmatrix} A^* u_i \\ A v_i \end{bmatrix} = \sigma_i \begin{bmatrix} v_i \\ u_i \end{bmatrix}$

Hence, each σ_i is an eigenvalue with eigenvector $[v_i, u_i]^T$. Singular values are positive, and clearly $-\sigma_i$ is also an eigenvalue with eigenvector $-[v_i, u_i]^T$

2. Let X be a random variable representing the total score. $\mathbb{E} X = \sum_{i=1}^{n} p_i = \frac{1}{2} + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n} = \frac{n}{2}$.

$Var(X) = \sum_{i=1}^{n} p_i(1-p_i)$, assuming each Bernoulli RV denoting score to questions is independent of another. Notice that $Var(X) = s$. Hence, applying Chebyshev inequality, we obtain

$$\mathbb{E}\, \mathbb{1}_{\{|X - \frac{n}{2}| > 2\sqrt{s}\}} \leq 1/4.$$