Lecture 14: Convolutional Neural Networks and Intro to PCA

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- Visualization, interpretation
- Better generalization (avoid overfitting)

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- Both parameterized as Neural Networks.

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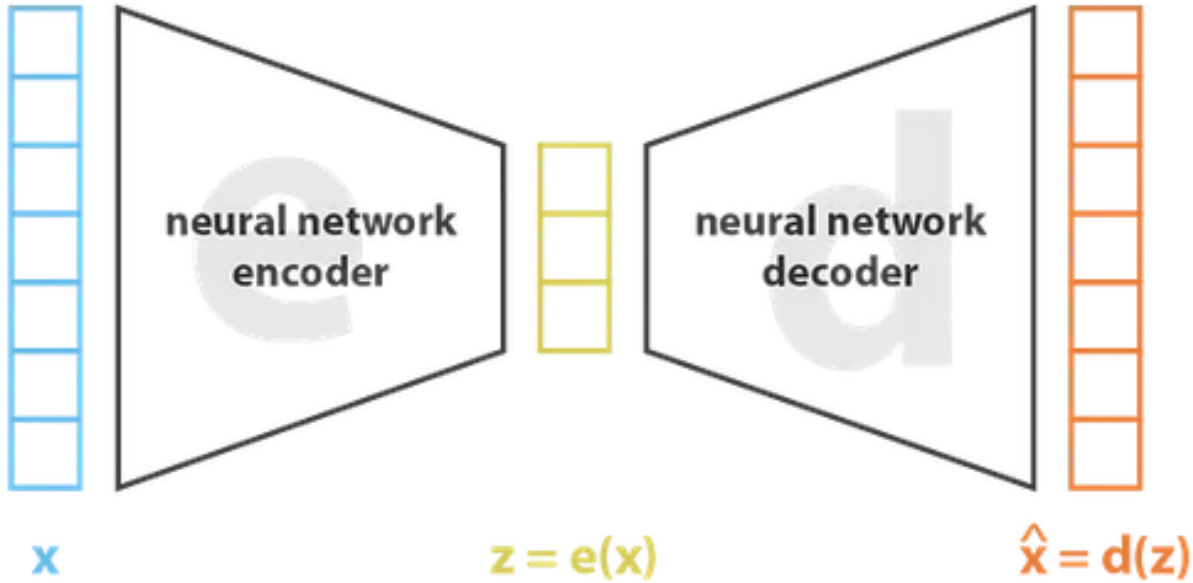
- Encoder: $q(z|x)$, Decoder: $p(x|z)$



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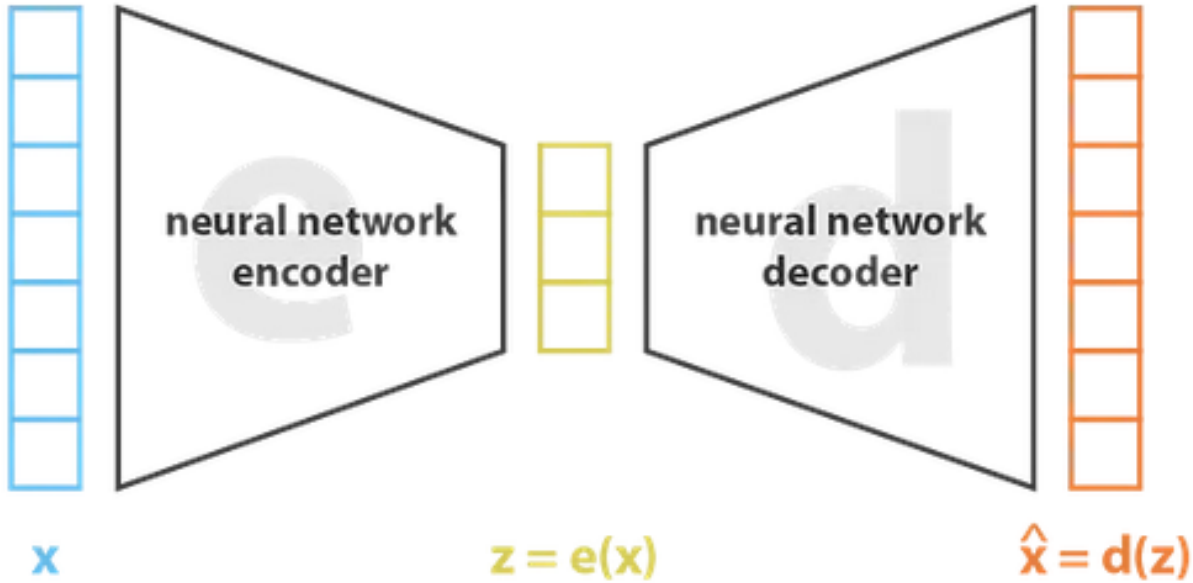

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- VAE: uses VI to regularize the latent space.

Variational Inference

- Minimize KL divergence between $q(z|x)$ and $p(z|x)$.
- Recall KL divergence: $D_{\mathrm{KL}}(p||q) = \int p(x) \log \frac{p(x)}{q(x)} dx$

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$$
\begin{aligned}
D_{\mathrm{KL}}(q_\theta(z|x)||p(z|x)) &= \int q_\theta(z|x) \log \frac{q_\theta(z|x)}{p(z|x)} dz) \\
&= E_{z\sim q_\theta(z|x)} [\log p(x|z)] \\
&- D_{\mathrm{KL}}(q_\theta(z|x)||p(z)) + c(x)
\end{aligned}
$$

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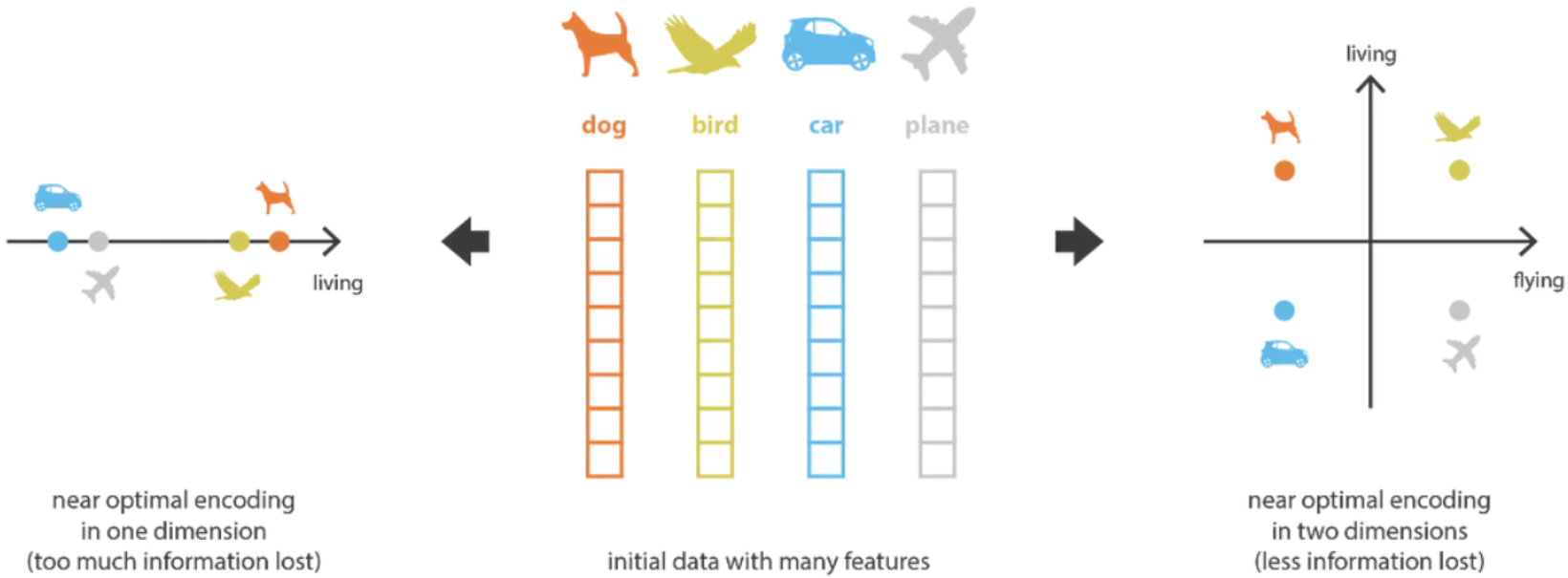
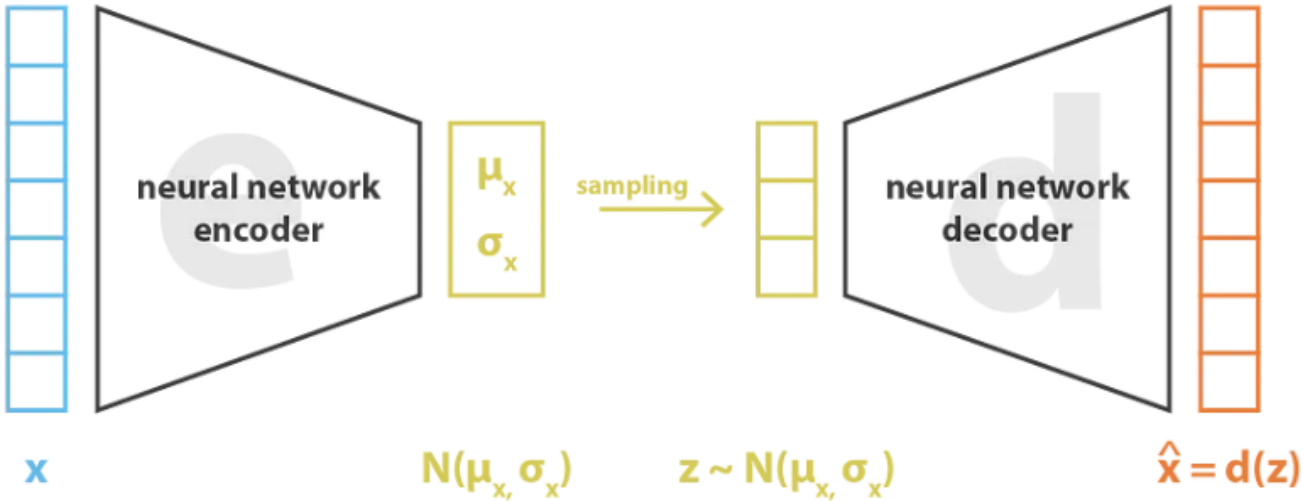
- Typically q_θ is parameterized with its mean and variance as Neural Networks.
- $$\mathrm{argmax}_{\theta,f} \sum_{i=1}^{m} \log p(x_i|z_i) - D_{\mathrm{KL}}(q_\theta(z_i|x_i)||p(z_i)) \qquad (2)$$



Courtesy: https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73



$$\text{loss} = \|x - \hat{x}\|^2 + KL[N(\mu_x, \sigma_x), N(0, I)] = \|x - d(z)\|^2 + KL[N(\mu_x, \sigma_x), N(0, I)]$$

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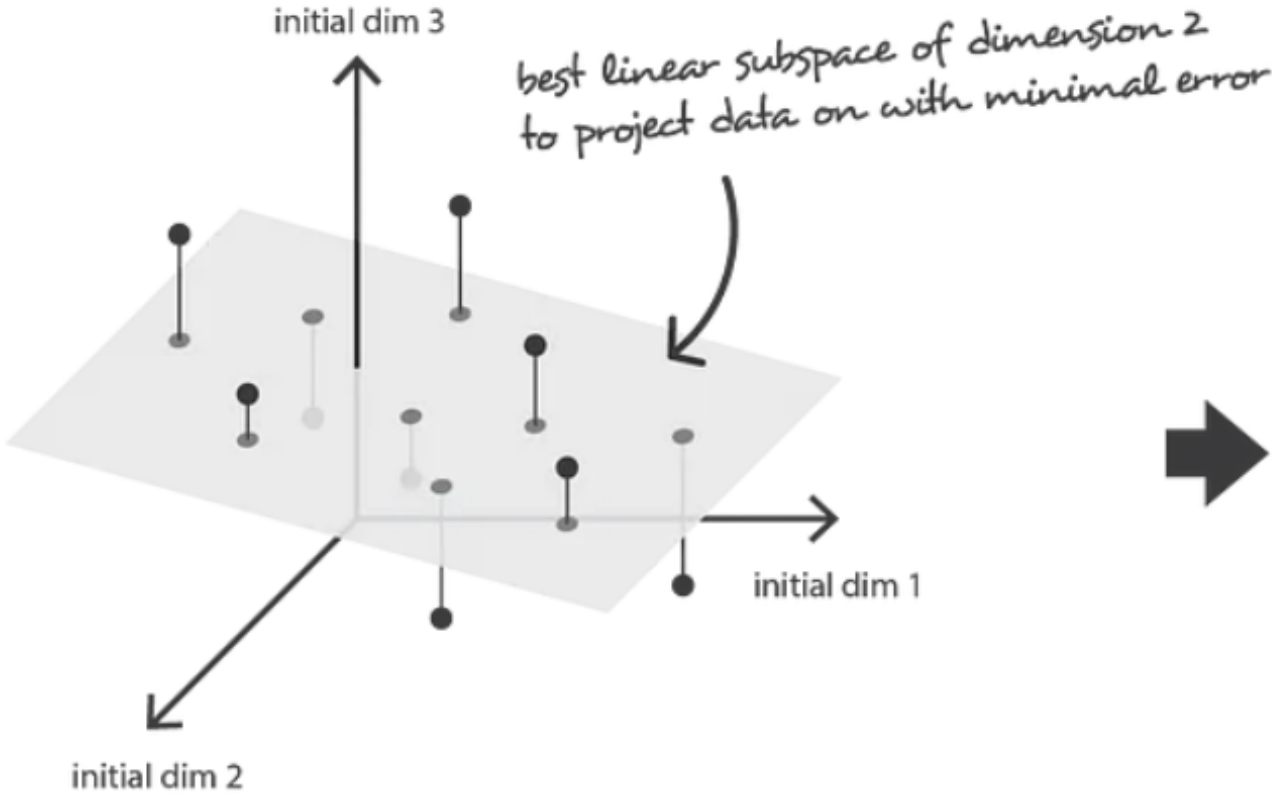
- Let $C = \sum_{i=1}^{m} x_i x_i^\top = X^\top X$ be the data correlation matrix.
- C is symmetric and positive semi-definite, $C = V \Lambda V^\top$.
- Theorem PCA: among linear hypothesis classes, $E^* = V^\top$, $D^* = V$, where V is the matrix of eigenvectors of $C = X^\top X$.

Best linear subspace



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- Correlation matrix $C = X^T X = V\Sigma^2 V^T$.
- principal components: $XV = U\Sigma$. Numerical stability

Convolutional Neural Networks (source: cs231n.stanford.edu)

- Suitable for image recognition. Won the 2012 ImageNet competition and subsequent ones.
- Three types of layers: convolutional, FC, pooling
- Convolutional layer: accepts a volume of size $W_1 \times H_1 \times D_1$ and outputs a volume of size $W_2 \times H_2 \times D_2$ where $W_2 = (W_1 - F + 2P)/S + 1$ and $H_2 = (H_1 - F + 2P)/S + 1$ and $D_2 = K$.
- K is number of filters, F is filter size, S is stride, P is padding.
- Pooling layer: downsamples along width and height, and optionally along depth.
- FC layer: computes class scores, resulting in volume of size $1 \times 1 \times K$.