CSE 6740 A/ISyE 6740: Computational Data Analysis: Introductory lecture

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- Overfitting, inductive bias, Intro to PAC learning

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- Generalization error or risk:

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- After that: Linear models.

Linear models

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- Equivalently, where X is $m \times (d+1)$ matrix with rows $X_i = (\Phi(x_i)^\top, 1)$, $W = [w_1, \cdots, w_d, 1]^\top$, $Y = [y_1, \cdots, y_m]^\top$,

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  $$\mathrm{argmin}_W \frac{1}{m} \|XW - Y\|^2$$
- Features may be defined by kernels

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- $X^T XW = X^T Y$. Can also get this by differentiating before writing in matrix form
- When is $X^T X = \sum_{i=1}^{m} \Phi(x_i) \Phi(x_i)^T$ invertible? When the training features span $\mathbb{R}^d$.

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- Can solve normal equations above directly, or use iterative methods for linear systems. Cost $\mathcal{O}(d^3)$

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- Since $DD^\top$ is positive semi-definite, qed.

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- Equivalent formulation: $\min_w \sum_{i=1}^m (w^\top \Phi(x_i) - y_i)^2$ subject to $\|w\|^2 \leqslant \Lambda^2$

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