Oseledets spaces

Kuptsov and Parlitz 2012

Dynamical system

$$\frac{du}{dt} = g(u, \boxed{t})$$

$g \to$ Vector field

$u(t) \in \mathbb{R}^m$

$\downarrow$

phase space

$$\frac{d\varphi^t(x)}{dt} = v_t(\varphi^t(x))$$

$$u(t) \equiv \varphi^t(x)$$

$$u \equiv x \quad \text{(phase point)}$$

$$g \equiv v_t$$

Can be non-autonomous

Linearized dynamics

Infinitesimal linear perturbations
$\equiv$ vector fields

Evolution of vector fields : subject of Oseledets theorem

$$\frac{d\varphi^t(u)}{dt} = g(\varphi^t(u), t)$$

$$J(\varphi^t(u), t) \equiv dg(\varphi^t(u), t) \quad \left(\underset{\mathbb{R}^{m\times m}}{\text{Jacobian}}\right)$$

$$g(\varphi^t(u), t) = [\, g_1(\varphi^t(u), t),\ g_2(\varphi^t(u), t), \dots,\ g_m(\varphi^t(u), t)]^T$$

$$[J(\overset{u(t)}{\varphi^t(u)}, t)]_{ij} = \partial_j g_i(\varphi^t(u), t)$$

$[\partial_1, \dots, \partial_m]$: partial derivatives in Euclidean space $\mathbb{R}^m$

$$v(t) \in T_{\underset{u(t)}{\underset{\parallel}{\varphi^t_u}}} \mathbb{R}^m \quad (\text{isomorphic to } \mathbb{R}^m)$$

Tangent space evolution

$$\frac{dv(t)}{dt} = J(\varphi^t(u), t)\, v(t)$$

$$v(0) \in T_u \mathbb{R}^m$$

$$v(t) = \underbrace{e^{\int_0^t J(\varphi^{t'}(u), t')\, dt'}}_{M(t)} v(0)$$

$$\mathcal{F}(t_1, t_2)\, v(t_1) = v(t_2)$$

$$v(t_1) = M(t_1)\, v(0)$$

$$v(t_2) = M(t_2)\, v(0)$$

$$t_2 > t_1$$

$$\mathcal{F}(t_1, t_2)\, M(t_1)\, v(0) = M(t_2)\, v(0)$$

$$\mathcal{F}(t_1, t_2)\, M(t_1) = M(t_2)$$

$$\mathcal{F}(t_1, t_2) = M(t_2)(M(t_1))^{-1}$$

→ There is an underlying fixed orbit $\{(\varphi^t_u)\}$

→ $\mathcal{F}(t_1, t_2) \in \mathbb{R}^{m\times m}$ maps vectors in $T_{\varphi^{t_1}_u} \mathbb{R}^m$ to vectors in $T_{\varphi^{t_2}_u} \mathbb{R}^m$

→ $M(t)$ is invertible

$\mathcal{F}(t_1, t_2)$: tangent propagator

Adjoint space

→ $Ax \cdot y = x \cdot A^T y$

→ $v(u) \in \underbrace{T_u \mathbb{R}^m}_{\text{target space at } u}$ $\quad v \in \underbrace{T\mathbb{R}^m}_{\text{tangent bundle}}$

$f : \mathbb{R}^m \to \mathbb{R}$

$$\underbrace{v}(f)(u) = \lim_{\varepsilon \to 0} \frac{f(u + \varepsilon v(u)) - f(u)}{\varepsilon}$$

linear operator
on space of functions
that gives "directional derivative"

→ $v(u)$: vector that represents
an infinitesimal perturbation

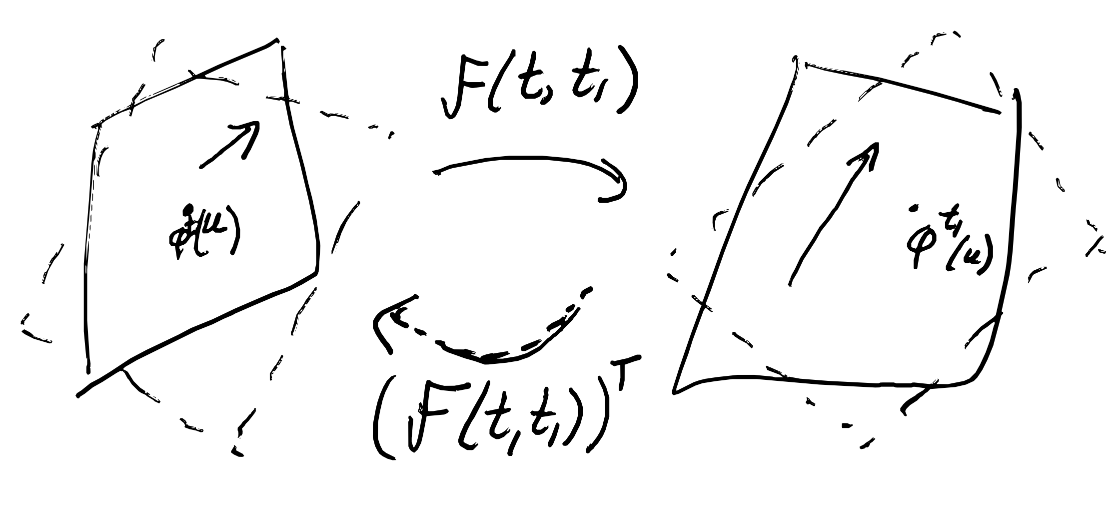
→ $$\mathcal{F}(t, t_1)\, \underset{\substack{\cap \\ T_{\varphi^t(u)} \mathbb{R}^m}}{v(t)} = \underset{\substack{\cap \\ T_{\varphi^{t_1}_u} \mathbb{R}^m}}{v(t_1)}$$

$$\underset{\substack{\cap \\ T_{\varphi^t(u)} \mathbb{R}^m}}{\underbrace{\mathcal{F}(t, t_1)\, v(t)}} \cdot \underset{\substack{\cap \\ T^*_{\varphi^t(u)} \mathbb{R}^m}}{w(t_1)} = v(t_1) \cdot w(t_1)$$

→ Dot product → Inner product
→ Works for Riemannian manifold

$$\mathcal{F}(t, t_1) v(t) \cdot w(t_1) = v(t) \cdot (\mathcal{F}(t, t_1))^T w(t_1)$$
$$= v(t_1) \cdot w(t_1)$$

→ $$\underset{\substack{\cap \\ T^*_{\varphi^t(u)} \mathbb{R}^m}}{w(t)} = (\mathcal{F}(t, t_1))^T w(t_1)$$
↳ adjoint propagator



→ $v(f)(u) =$ directional derivative of function f in the direction of v at u

f is smooth

$$= \underset{\substack{\cap \\ T^*_u \mathbb{R}^m}}{df(u)} \cdot v(u)$$

→ $$w(\varphi^{t_1}(u)) = (df \cdot (\varphi^{t_1}(u)))^T$$

$$\mathcal{F}(t, t_1)^T w(\varphi^{t_1}(u))$$
$$= \underbrace{w(\varphi^t(u))}_{?}$$

$\varphi(u)$ $\quad$ Discrete-time

→ $$(J(u))^T \underbrace{w(\varphi(u))}_{(df(\varphi(u)))^T}$$

$$= w(u) = (d(f \circ \varphi)(u))^T$$

$$d(f \circ \varphi)(u) = df(\varphi(u))\, d\varphi(u)$$
$$= df(\varphi(u))\, J(u)$$

row vector

$$(J(u))^T \underset{\substack{||| \\ w(\varphi(u))}}{(df(\varphi(u)))^T} = \underset{\substack{||| \\ w(u)}}{(d(f \circ \varphi)(u))^T}$$

→ Intuitive explanation for adjoint propagator:

if $w(\varphi^t u) = (df(\varphi^t u))^T$

$$(M(t))^T w(\varphi^t u) = \underset{\substack{\cap \\ T^*_u \mathbb{R}^m}}{w(u)}$$

$$= (d(\underbrace{f \circ \varphi^t})(u))^T$$

$f : \mathbb{R}^m \to \mathbb{R}$

$\underbrace{f \circ \varphi^t} : \mathbb{R}^m \to \mathbb{R}$

Matrices $\mathbb{R}^{m\times m}$

$\mathcal{F}(t_1, t_2)$: tangent propagator

$(\mathcal{F}(t_1, t_2))^T$: adjoint propagator

→ acts backward

→ functions change along orbits

$$\left((\mathcal{F}(t_1,t_2))^T\right)^{-1} = \left((\mathcal{F}(t_1,t_2))^{-1}\right)^T$$

$$= (\mathcal{F}(t_1,t_2))^{-T}.$$

$$\mathcal{G}(t_1,t_2) \equiv (\mathcal{F}(t_1,t_2))^{-T}$$

- adjoint tangent propagator
- forward adjoint propagator

Discrete-time

$\to u \to \varphi(u)$

$J(u) : \quad T_u\mathbb{R}^m \to T_{\varphi(u)}\mathbb{R}^m$

$(J(u))^T : \quad T^*_{\varphi(u)}\mathbb{R}^m \to T^*_u\mathbb{R}^m$

$(J(u))^{-1} : \quad T_{\varphi(u)}\mathbb{R}^m \to T_u\mathbb{R}^m$

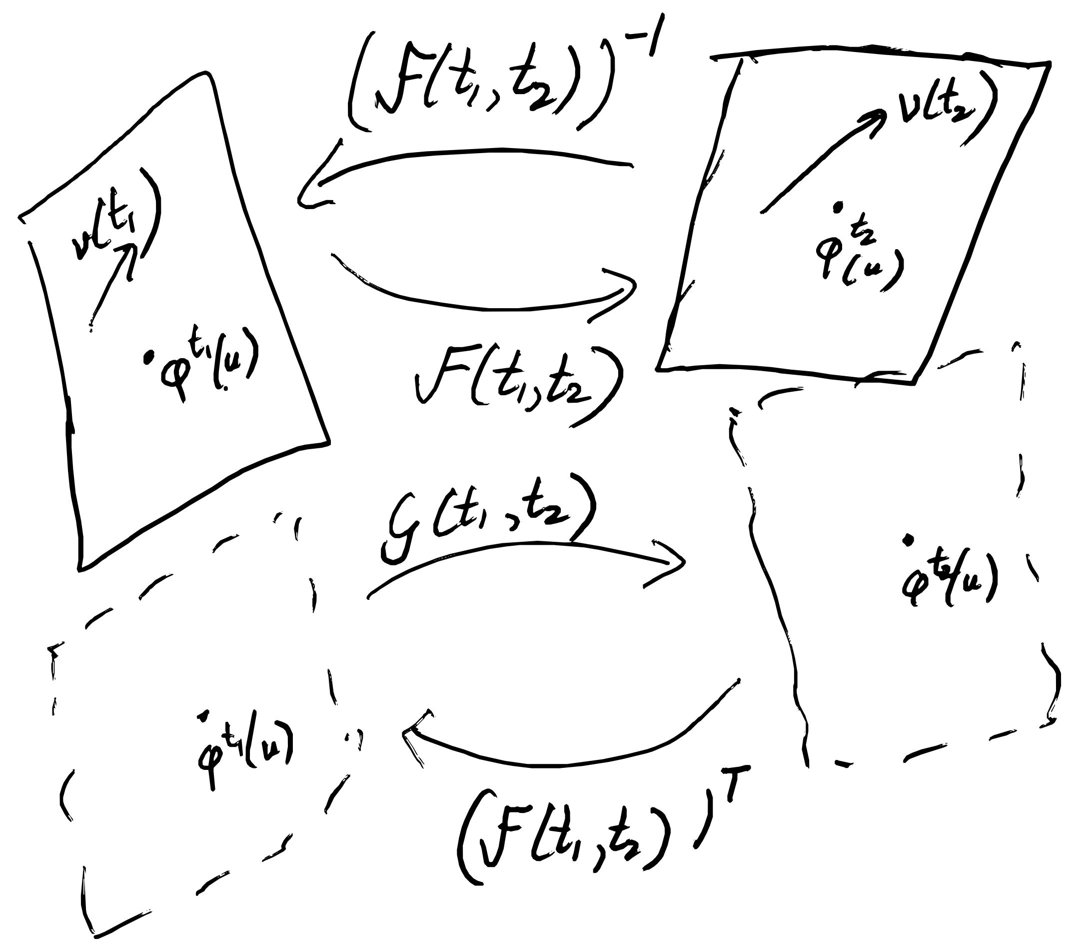
What does $(\mathcal{F}(t_1,t_2))^{-1}$ do?

$\vec{\mathcal{F}}(t_1,t_2)\, v(t_1) = v(t_2)$

$$\to \quad v(t_1) = \left((\mathcal{F}(t_1,t_2))\right)^{-1} \underline{v(t_2)}$$

$$= \left(M(t_2)(M(t_1))^{-1}\right)^{-1}$$

$$= M(t_1)(M(t_2))^{-1}$$



$$\mathcal{G}(t_1,t_2) = (\mathcal{F}(t_1,t_2))^{-T}.$$

SVD

$$\underset{\mathbb{R}^{m\times n}}{A} = \underset{\mathbb{R}^{m\times m}}{U}\,\Sigma\,V^T$$

$\Sigma \to \mathbb{R}^{m\times n}$

$V^T \to \mathbb{R}^{m\times n}$

σ_i : singular values >0

$$A v_i = \sigma_i u_i$$

Volumes

Subspaces : $v \in T_u M$

$$V_2 = \begin{bmatrix} v_1^T \\ \hline v_2^T \end{bmatrix}$$

$$\mathrm{Vol}(V_2) = \det(V_2)$$

$v_1, v_2, \ldots, v_n$

$$V_n = \begin{bmatrix} v_1^T \\ \hline \vdots \\ \hline v_n^T \end{bmatrix}$$

$\det(V_n)$ = vol of n dimensional subspace formed by $v_1, v_2 .., v_n$