Last time: Existence of Lyapunov functions $\Rightarrow$ asymptotic stability of fixed points.

Linear stability: long-term behavior of linear perturbations.

$$\left.\begin{aligned} F(x) &= Ax \\ \varphi^t(x) &= e^{tA}x \end{aligned}\right] \quad \text{eigenvalues of } A$$

if any eigenvalue of A has a real part > 0 (> 1 for maps), then unstable.

if there exists a function $\mathcal{L}$ in ngbd of Origin such that $\frac{d}{dt}\mathcal{L}\circ\varphi^t(x) := \nabla\mathcal{L}(\varphi^t x)\cdot\frac{d\varphi^t(x)}{dt}$

≥ 0 on orbits $\Rightarrow$ lack of uniform stability

$\rightarrow$ Finding Lyapunov functions is not systematic
$\rightarrow$ Global stability extensions
$\rightarrow$ Random dynamical systems
$\rightarrow$ stability behaviors more general than convergence to fixed points

Jordan Smith Ch 10

Weakly nonlinear systems

$$\frac{d\varphi^t(x)}{dt} = Ax + h(x)$$

$\rightarrow$ regular system (existence & uniqueness thm for IVP in $\mathcal{N}\times[0,T]$).

$\rightarrow$ $h(0) = 0$ and

$$\lim_{\|x\|\to 0} \frac{\|h(x)\|}{\|x\|} = 0.$$

Origin is asymptotically stable provided all eigenvalues of A are < 0.

Proof: Application of Lyapunov function analysis

$$\mathcal{L}(x) = x^T L x$$

$$\frac{d\,\mathcal{L}\circ\varphi^t(x)}{dt} = \frac{d}{dt}(\varphi^t x)^T L (\varphi^t x)$$

$$\frac{d\,\mathcal{L}\circ\varphi^t(x)}{dt} = \nabla\mathcal{L}(\varphi^t x)\cdot \frac{d\varphi^t(x)}{dt}$$

$$= \nabla\mathcal{L}(\varphi^t x)\cdot\left(A\varphi^t(x) + h(\varphi^t(x))\right)$$

$$= (\varphi^t(x))^T(L+L^T)\left(A\varphi^t(x) + h(\varphi^t(x))\right)$$

$$V(x) = \underbrace{x^T(LA+L^TA)x} + x^T(L+L^T)h(x)$$

$$L = \int_0^\infty e^{tA^T}e^{tA}\,dt$$

$$e^{tA} = Id + tA + \frac{t^2A^2}{2!} + \cdots$$

$-I =$

$$\left[e^{tA^T}e^{tA}\right]_0^\infty = \int_0^\infty \underbrace{\frac{d}{dt}e^{tA^T}e^{tA}}\,dt =$$

$$\int_0^\infty \left(A^T e^{tA^T}e^{tA} + e^{tA^T}e^{tA}A\right)dt$$

$$= A^T\int_0^\infty e^{tA^T}e^{tA}\,dt + \left(\int_0^\infty e^{tA^T}e^{tA}\,dt\right)A$$

$$= A^TL + L^TA$$

$$A^TL + L^TA = -I$$

$$V(x) = \underbrace{x^T(LA^T+L^TA)x} + x^T(L+L^T)h(x)$$

$$= -x^Tx + 2x^TLh(x)$$

$$= -\|x\|^2 + 2x^TLh(x)$$

$$|2x^TLh(x)| \leq 2\|x\|\,\|L\|\,\|h(x)\|$$

[For every $\varepsilon > 0$, $\exists\ \delta > 0$
s.t. $\|h(x)\| < \varepsilon\|x\|$
when $\|x\| < \delta$.]

$$\leq 2\|x\|^2\varepsilon\|L\|$$

$$2\|x\|^2\varepsilon\|L\| < \|x\|^2$$

$$\varepsilon < \frac{1}{2\|L\|}$$

QED.

$$x_1^2 + x_2^2 + \cdots + x_d^2$$

$$\int_0 e^{tA^T}e^{tA}\,dt$$

Oseledets Multiplicative ergodic theorem

There exist a finite set of "Lyapunov exponents" that characterize asymptotic exponential growth/decay of infinitesimal linear perturbations along orbits of random dynamical systems

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$$x_{t+1} = A x_t$$

$$x_{t+1} = A_t x_t \qquad\qquad x_{t+1} = F(x_t)$$

$$dx_t = f(x_t)\,dt + g(x_t) \circ dW_t$$

(Stratonovich)

<u>RDS</u>

$$(x_{t+1}, \omega_{t+1}) = \varphi(t, \omega_t)\, x_t$$

$$(x_t, \omega_t) = \varphi(t, \omega_0)\, x_0$$

$$d\underbrace{\varphi(t,\omega)}(x) : T_x M \longrightarrow T_{\varphi(t,\omega)x} M$$

phase space differential

point of evaluation

$$d\varphi(t,\omega)(x)\, v$$

$$v \in T_x M$$

Floquet theory