Power iteration

$v_0 \not\perp q$

$A \in \mathbb{R}^{m \times m}$
invertible
top eigenvector $q_1$,
$|\lambda_1| > |\lambda_2| \geq \dots |\lambda_n|$

Algorithm

$$v_{t+1} = A v_t$$

$$v_{t+1} = \frac{v_{t+1}}{\| v_{t+1} \|}$$

Dynamics

$$F : \mathbb{R}^m$$

$$F(v) = \frac{Av}{\|Av\|}$$

Trefethen Bau: $\| v_t - \pm q_1 \| \sim O\left( \left| \frac{\lambda_2}{\lambda_1} \right|^t \right)$

Is this $F$ a contraction on $\mathbb{R}^m = E^1 \oplus E^2 \oplus \cdots \oplus E^n$

→ No!

$E^i$: eigenspace corresponding to $\lambda_i$.

→ Not a contraction, but converges to a fixed vector space.

→ $d(q_1, -q_1) = 0$

$$P_{E_1} = Id - q_1 q_1^T$$

$$d_q(x, y) = d(P_{E_1} x, P_{E_1} y)$$

$$P_{E_1} q_1 = q_1 - (q_1^T q_1) q_1 = 0$$

$$P_{E_1}(-q_1) = -q_1 + q_1 \langle q_1^T, q_1 \rangle = 0$$

→ In general, distances to $E_1^\perp$ are also not uniformly contracted for a general $A$.

→ $\| dF(x) \| < 1 \quad \forall x \in S^{d-1}$

differential $\qquad x \in S^{d-1}$

under condition on $A$,

then $F$ is a contraction on $S^{d-1}$

$$dF(x) = f\left( \sigma_{max}(A), \sigma_{max}(A x x^T A^T) \right)$$

Check if an intuitive condition on $A$ can be deduced from $f$.

# Pre-Lyapunov stability

Lyapunov : Asymptotic stability of vector fields under $d\varphi^t$ (or $dF^t$) (Jacobian) along orbits

Lyapunov functions ( control : nonautonomous)

Takeaway: if a "Lyapunov function" exists in a neighborhood, the neighborhood is a "basin of attraction".
Orbits entering the basin of attraction are uniformly & asymptotically stable.

No systematic way to construct Lyapunov functions

Existence & uniqueness for IVP

( Do Carmo )

$$\frac{d\varphi^t(x)}{dt} = v(\varphi^t(x), t)$$

→ if $v$ is continuous & partial derivatives are continuous

$$\frac{\partial}{\partial x_1} v(x,t) \text{ exists \& is continuous}$$

for all $x \in D$, $t \in I$.

→ then, starting from any $x \in D$, the solutions $\varphi^t(x)$ exist and are unique on $D \times I$.

→ $\varphi^t(x)$ either reach the boundary of $D \times I$ or are unbounded as $t \to \infty$.

Examples ( Jordan-Smith )

→ $$\frac{d\varphi^t(x)}{dt} = (\varphi^t(x))^2 t$$

$$\int \frac{d\varphi^t}{(\varphi^t)^2} = \int t \, dt$$

$$\frac{-1}{\varphi^t(x)} = \frac{t^2}{2} + c$$

$$-\frac{1}{x} = c$$

$$\frac{-1}{\varphi^t(x)} = \frac{t^2}{2} - \frac{1}{x}$$

$$\varphi^t(x) = \left(\frac{1}{x} - \frac{t^2}{2}\right)^{-1}$$

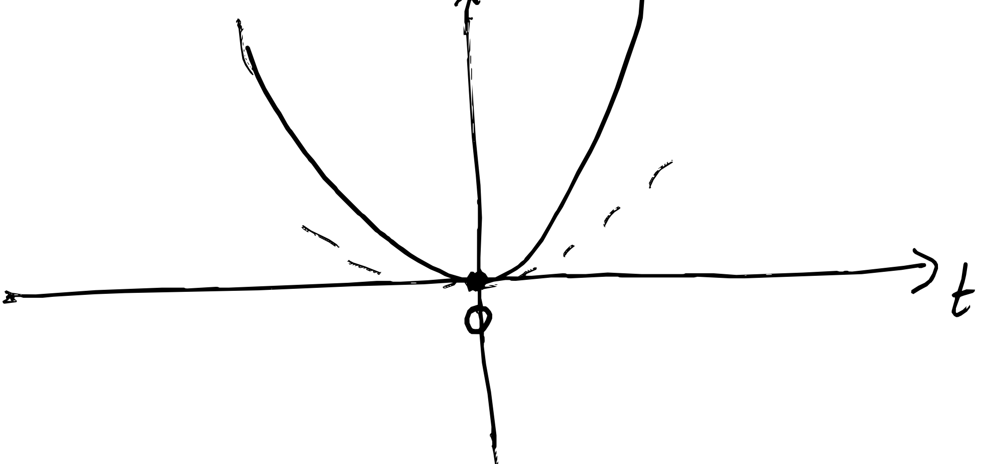
$D \equiv \mathbb{R}$ $\quad I \equiv \mathbb{R}$

$$\varphi^t(2) = \left(\frac{1}{2} - \frac{t^2}{2}\right)^{-1}$$

→ $$\frac{d}{dt}\varphi^t(x) = 2\frac{\varphi^t(x)}{t} \Big\}$$

$$\int \frac{d\varphi^t(x)}{\varphi^t(x)} = 2\int \frac{dt}{t}$$

$$\varphi^t(x) = ct^2$$




$D \equiv \mathbb{R}$ $\quad I : (-\infty, 0)$

$(0, \infty)$

# Lyapunov function - based stability

$\rightarrow \quad \dfrac{d\varphi^t(x)}{dt} = v(\varphi^t(x))$

"Regular" (Jordan-Smith)

$\rightarrow$ Lyapunov function, Continuous $\mathcal{L} : N(x^*) \to \mathbb{R}^+$
that is positive definite
$\mathcal{L}(x) > 0$ and $\mathcal{L}(x^*) = 0$
and decreases (strictly) along orbits

$$\frac{d}{dt} \mathcal{L} \circ \varphi^t(x) < 0 \quad \forall x \in N(x^*).$$

$\rightarrow$ If $\mathcal{L}$ satisfying above conditions exists, then, $\varphi^t$ is uniformly stable in a nghbd of $x^*$ & asymptotically converges to $x^*$.

$\rightarrow \quad N(x^*)$: basin of attraction in $x^*$.

$\rightarrow \quad N_\delta(x^*) = \{x : d(x, x^*) < \delta\}$

Uniformly:

For every $\varepsilon > 0$, $\exists\ \delta > 0$
s.t. whenever $x \in N_\delta(x^*)$
$d(\varphi^t(x), x^*) < \varepsilon$ for all $t$.


$v(x^*) = 0$

Asymptotic converge to $x^*$

For every $\varepsilon > 0$, $\exists\ \delta > 0$, s.t.
whenever $x \in N_\delta(x^*)$,

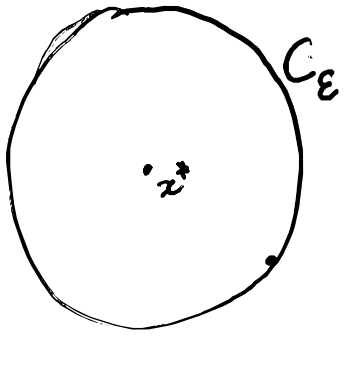
$$\lim_{t \to \infty} d(\varphi^t(x), x^*) = 0.$$

# Proof of uniform stability

→ Existence of uniqueness
→ $\mathcal{L}$ is cont, $\mathcal{L} \geq 0$ & $\frac{d\mathcal{L}\circ\varphi^t}{dt} < 0$.

$$C_\varepsilon = \{x : d(x, x^*) = \varepsilon\}$$

$C_\varepsilon$ is compact




(Rudin. A continuous fn on a compact set is bounded and its sup/inf are attained).

$\exists$ some point $y^* \in C_\varepsilon$ s.t.

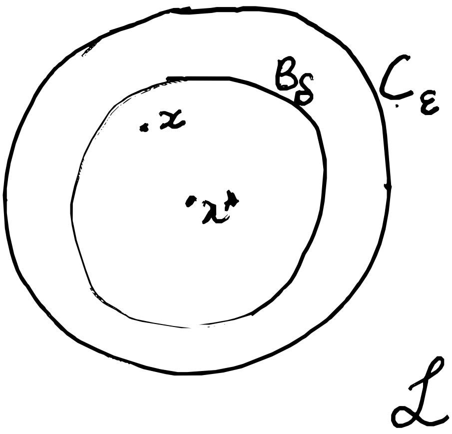
$$\inf_{y \in C_\varepsilon} \mathcal{L}(y) = \mathcal{L}(y^*) = \alpha > 0$$

$$\mathcal{L}(x^*) = 0$$

Since $\mathcal{L}$ is continuous, $\exists\ \delta > 0$ s.t. whenever $x \in B_\delta(x^*)$

$$d(\mathcal{L}(x), \mathcal{L}(x^*)) < \alpha$$

$$\mathcal{L}(x) < \alpha.$$




$$\frac{d\,\mathcal{L}\circ\varphi^t(x)}{dt} \leq 0$$

$$\mathcal{L}(x) < \alpha$$

$\varphi^t(x)$ does not attain $C_\varepsilon$ because $\mathcal{L}\circ\varphi^t(x) < \alpha$ for all $t$.

$\exists\ \delta$ s.t.
For all $t$,

$$|\varphi^t(x)| \in B_\delta$$

and $d(\varphi^t(x), x^*) < \varepsilon$.