Lecture 1: Introduction to dynamical systems theory

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January 8, 2024

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- How to control a dynamical system?
- How to design/model a dynamical system?

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- How to predict the future of a dynamical system?
- How to understand the behavior of a dynamical system?
- How to understand the behavior of a dynamical system in the presence of noise?

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- “Take the dynamical systems approach” to a computational math problem.
- Understand problems at the interface of dynamical systems and machine learning.

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- Office hours: Friday 10-11 am, CODA S1323.

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- M is a d-dimensional smooth manifold.

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- Vector fields can also be interpreted as linear functionals on the space of smooth functions on the manifold.
- $X(f) = \langle v, df \rangle$, where df is the differential of $f \in \mathcal{C}^\infty(M)$.

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- Can be obtained from a flow by taking the time-1 map, e.g., from time-integration of the ODE $F(x) = \varphi^1(x)$.

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- S is the price of the stock, μ is the drift, σ is the volatility, W is the Wiener process.
- S is a stochastic process, dealt with later in the course.
- Delay differential equations, Partial differential equations, etc. also have infinite dimensional phase spaces.

Analysis and basic topology