Lecture 19: Introduction to RDS

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- Limit theorems review
- Goal: basic idea behind diffusion models
- Begin with Random walks
- Obtaining diffusion in the limit of random walks
- Ergodic theory basics
- Not much stochastic analysis
- References: [**Arnold Random Dynamical Systems**], [**Song et al 2020**], [**Kidger 2022 PhD thesis**]

So far...

- Linear hyperbolic systems: where the matrix has no eigenvalues equal to 1 in magnitude.
- More general hyperbolic systems: no zero Lyapunov exponent
- Statistical properties of dynamical systems: ergodic theory
- Oseledets theorem: applies to Random Dynamical System (RDS), the cocycle considered is a linear RDS
- $x_{t+1} = \varphi(x_t)$, $\varphi : M \to M$
- OMET deals with $d\varphi(x_t)$ as a dynamics on tangent vectors
- Linear RDS: $d\varphi(v_1 + v_2) = d\varphi v_1 + d\varphi v_2;\ v_1, v_2 \in TM$

Limit theorems review

- $X_1, X_2, \cdots$,, where X_i is independent and identically distributed (iid) with mean μ and std $\sigma < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$.
- Weak Law of Large Numbers: $\lim_{n\to\infty} \mathbb{P}(|S_n/n - \mu| > \epsilon) = 0$ for every $\epsilon > 0$. That is, $S_n/n \xrightarrow{P} \mu$ as $n \to \infty$. Convergence in probability. (Proof: Chebyshev's inequality when $\sigma < \infty$, but LLNs do not require finite variance in general)
- Strong LLN: $\mathbb{P}(\lim_{n\to\infty} S_n/n = \mu) = 1$. Also written as $S_n/n \xrightarrow{\text{a.s.}} \mu$. Almost sure convergence. (requires boundedness of expected value)
- Central limit theorem: $\lim_{n\to\infty} \mathrm{CDF}((S_n - n\mu)/(\sqrt{n}\sigma))(t) = \mathrm{CDF}(Z)(t)$, where Z is a standard Gaussian. That is, $((S_n - n\mu)/(\sqrt{n}\sigma)) \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1)$. Convergence in distribution. (Requires finite variance)

- We will see limit theorems when $X_i = f \circ \varphi^i(x_0)$, where φ is a random or deterministic DS. x_0 is a random variable sampled from any distribution.
- In deterministic dynamics, randomness comes from initial condition
- $\varphi(x) = Ax$, linear hyperbolic
- $X_0 \sim \rho_0$, then, $X_t \sim \rho_t$.
- $X_t \sim \rho$ and $X_{t+1} \sim \rho$, then, ρ is an invariant density.
- For Cat map, ρ is uniform on unit square
- $\rho_{t+1} = \varphi_\sharp \rho_t$
- Markov process: $\mathbb{P}(X_t | X_1, \cdots, X_{t-1}) = \mathbb{P}(X_t | X_{t-1})$
- Transition kernel of a Markov process: $\mathcal{K}(x, A) = \mathbb{P}(X_{t+1} \in A | X_t = x)$
- $\rho_{t+1}(A) = \int_x \mathcal{K}(x, A) \rho_t(dx)$.

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- Transition operator: $\mathcal{T}\rho(A) = \int \mathcal{K}(x, A)\rho(dx)$ (Function on the space of measures)
- $\mathcal{T}(\rho_1 + \rho_2) = \mathcal{T}\rho_1 + \mathcal{T}\rho_2$ (linear operator)
- Alternatively: ρ is an invariant measure if it is an eigenfunction of $\mathcal{T}$ with eigenvalue 1.
- Limit theorems also valid for some "weakly" dependent RVs
- If $X_1, \cdots, X_n \cdots$, is generated by a hyperbolic dynamical system, CLT is valid.
- $|(1/T)\sum_{t \leqslant T} f(X_t) - E_{x \sim \rho} f(x)|$ behaves like a normal RV for an idealized class of chaotic systems

Random walk

- Start on a 1D lattice at 0. With probability $1/2$, go left or right.
- $\mathbb{P}(X_t = k) = \binom{t}{(t+k)/2}\frac{1}{2^t}$. Let a and b be the number of times you go right and left respectively. Clearly, $a + b = t$. Also, $a - b = k$.
- Stirling's approximations of these probabilities for large t. $\log n! \approx n \log n - n$
- $\mathcal{K}(x, \{x + 1\}) = 1/2$
- Diffusion processes.