Lecture 20: Brownian motion

Nisha Chandramoorthy

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Limit theorems review

- $X_1, X_2, \cdots$,, where X_i is independent and identically distributed (iid) with mean μ and std $\sigma < \infty$. Let $S_n = \sum_{i=1}^n X_i$.
- Weak Law of Large Numbers: $\lim_{n\to\infty} \mathbb{P}(|S_n/n - \mu| > \epsilon) = 0$ for every $\epsilon > 0$. That is, $S_n/n \xrightarrow{P} \mu$ as $n \to \infty$. Convergence in probability. (Proof: Chebyshev's inequality when $\sigma < \infty$, but LLNs do not require finite variance in general)
- Strong LLN: $\mathbb{P}(\lim_{n\to\infty} S_n/n = \mu) = 1$. Also written as $S_n/n \xrightarrow{\text{a.s.}} \mu$. Almost sure convergence. (requires boundedness of expected value)
- Central limit theorem: $\lim_{n\to\infty} \mathrm{CDF}((S_n - n\mu)/(\sqrt{n}\sigma))(t) = \mathrm{CDF}(Z)(t)$, where Z is a standard Gaussian. That is, $((S_n - n\mu)/(\sqrt{n}\sigma)) \xrightarrow{\text{d}} \mathcal{N}(0, 1)$. Convergence in distribution. (Requires finite variance)

- We will see limit theorems when $X_i = f \circ \varphi^i(x_0)$, where φ is a random or deterministic DS. x_0 is a random variable sampled from any distribution.
- In deterministic dynamics, randomness comes from initial condition
- $\varphi(x) = Ax$, linear hyperbolic
- $X_0 \sim \rho_0$, then, $X_t \sim \rho_t$.
- $X_t \sim \rho$ and $X_{t+1} \sim \rho$, then, ρ is an invariant density.
- For Cat map, ρ is uniform on unit square
- $\rho_{t+1} = \varphi_\sharp \rho_t$
- Markov process: $\mathbb{P}(X_t|X_1, \cdots, X_{t-1}) = \mathbb{P}(X_t|X_{t-1})$
- Transition kernel of a Markov process: $\mathcal{K}(x, A) = \mathbb{P}(X_{t+1} \in A|X_t = x)$
- $\rho_{t+1}(A) = \int_X \mathcal{K}(x, A)\rho_t(dx)$.

- $\rho_{t+1}(A) = \int_x \mathcal{K}(x, A)\rho_t(dx)$.
- Transition operator: $\mathcal{T}\rho(A) = \int \mathcal{K}(x, A)\rho(dx)$ (Function on the space of measures)
- $\mathcal{T}(\rho_1 + \rho_2) = \mathcal{T}\rho_1 + \mathcal{T}\rho_2$ (linear operator)
- Alternatively: ρ is an invariant measure if it is an eigenfunction of $\mathcal{T}$ with eigenvalue 1.
- Limit theorems also valid for some "weakly" dependent RVs
- If $X_1, \cdots, X_n \cdots$, is generated by a hyperbolic dynamical system, CLT is valid.
- $|(1/T)\sum_{t \leqslant T} f(X_t) - E_{x\sim\rho} f(x)|$ behaves like a normal RV for an idealized class of chaotic systems

Random walk

- Start on a 1D lattice at 0. With probability $1/2$, go left or right.
- $\mathbb{P}(X_t = k) = \binom{t}{(t+k)/2}\frac{1}{2^t}$. Let a and b be the number of times you go right and left respectively. Clearly, $a + b = t$. Also, $a - b = k$.
- Stirling's approximations of these probabilities for large t. $\log n! \approx n \log n - n$
- $\mathcal{K}(x, \{x+1\}) = 1/2$
- Diffusion processes.

Monte Carlo integration

- Monte Carlo integration: a way to estimate integrals that uses the SLLN.
- Want: $E_{x\sim\mu} f(x) = \int f \, d\mu$ (statistical physics, quantum physics, finance...)
- $\int f \, d\mu \approx (1/N) \sum_{i=1}^{N} f(X_i)$, $X_i \sim \mu$
- MC Estimator: $I_N = (1/N) \sum_{i=1}^{N} f(X_i)$
- Law of the iterated logarithm: $|I_N - E_{x\sim\mu} f(x)| \sim \mathcal{O}(\frac{\sqrt{\log\log N}}{\sqrt{N}})$ as $N \to \infty$

Random walks in 1D

- X_i can take values ϵ or $-\epsilon$ with equal probability.
- Connection with Brownian motion: particles "diffuse" in a fluid bath.
- Random walker takes step every δt units
- $\mathrm{Var}(X_i) = \epsilon^2$
- $S_n = (X_1 + \cdots + X_n)$. CLT says that $S_n/(\sqrt{n}\epsilon) \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$.
- If we $\epsilon^2/\delta t$ to be constant, as $n = \lfloor t/\delta t \rfloor \to \infty$, we can define $B_n(1) = \sqrt{\delta t} S_n/(\sqrt{t}\epsilon)$
- $\rho(x,t) = \dfrac{1}{\sqrt{2\pi\sigma}} e^{-x^2/(2\sigma^2)}$; For a variance $\sigma^2 = \epsilon^2 t/\delta t$, distribution of a random walker

Wiener process or Brownian dynamics

- Take $\epsilon^2 = \delta t = 1$.
- Wiener process: $B_n(\xi) = S_{\lfloor \xi n \rfloor}/(\sqrt{t}), \xi \in [0, 1]$
- W_t continuous limit of random walk as $n \to \infty$. $W_0 = 0$ almost surely.
- $W_{t+s} - W_t$ is an independent Gaussian for all s with variance s.
- W_t are almost surely continuous but nowhere differentiable

Diffusion equation

- $dX_t = dW_t$ (SDE)
- $dX_t = v(X_t, t)dt + \sigma(X_t, t)dW_t$
- $dx/dt = v(x, t)$
- dW_t : differential of a Wiener process; Ito and Stratanovich calculus
- Long-term behavior of the solutions?
- Fokker-Planck equation: describes the evolution of probability densities of the states
- $\dfrac{d\rho_t}{dt} = d^2\rho_t/dx^2$. Verify FPE for the Wiener process.
- Diffusion models in ML.