→ Signup sheet   Flipped class

→ HW3   Sat ___ (Sun allowed)

Last time :

Gradient flows

→ $$\frac{d\varphi^t(x)}{dt} = \nabla f(\varphi^t x)$$

$$f : M \to \mathbb{R}$$

$$\nabla f(x) = [\partial_1 f, \ldots, \partial_d f](x)$$

→ linear functional on tangent spaces

→ function from {functions} to scalar

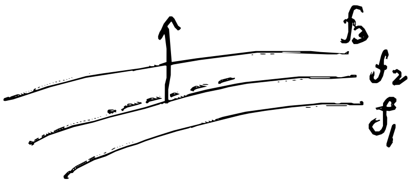
$v \in TM$, $\quad v(f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon v(x)) - f(x)}{\varepsilon}$

$$v(f)(x) = \nabla f(x) \cdot v(x)$$

"T^*M" Cotangent space

$$Ax \cdot y = x \cdot A^T y$$

$$= \langle \omega_f, v \rangle_M (x)$$



Flows in direction of Maximum ascent / increase in values of f.

→ Optimization → finite dimension
→ infinite dimensions

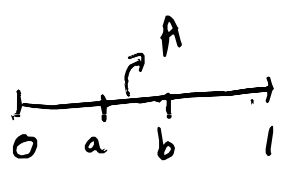
OMET

A probability distribution μ is ergodic for a dynamical system φ if for any φ-invariant set A,
$\mu(A) = 0$ or $\mu(A) = 1$

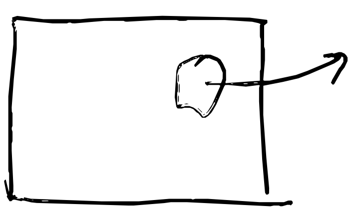
(Prob. of choosing a pt in A)
$\rightarrow$ $\mu(A) = \int_A \rho(x)\,dx$

ρ: density associated with μ
$\rho = \frac{d\mu}{d\lambda}$ ↳ Lebesgue

$\rightarrow$ Lebesgue measure "dx"



$|b-a| = \lambda(A)$



$\lambda(A) = \int_A 1\,dx$

$\rightarrow$ $\mu(A) = \int_A \rho(x)\,dx$

$\rightarrow$ Coin toss Bernoulli
$M = \{0, 1\}$
$\mu(0) = \frac{1}{2}$
$\mu(1) = \frac{1}{2}$

$\rightarrow$ Non-measurable

$\rightarrow$ $\mu(A) = \int_A \rho(x)\,dx$ $\left(\sum_{x \in A} \mu(x)\right)$

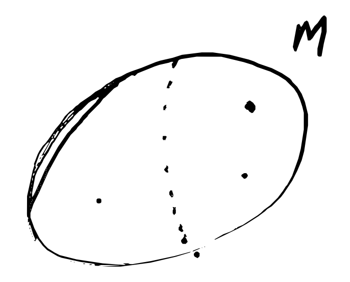
$\rightarrow$ $\mathbb{E}_{x\sim\mu} f(x) = \int_M f(x)\rho(x)\,dx = \int f(x)\,d\mu(x)$

$\rightarrow$ $\mathbb{E}_{x\sim\mu} 1_A(x) = \mu(A)$

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{o.w.} \end{cases}$$

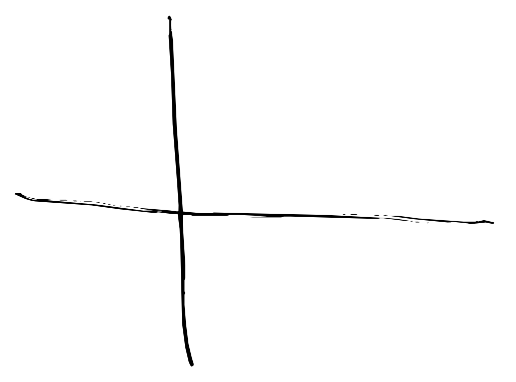
$\rightarrow$ A is φ-invariant if $\varphi^{-1}(A) = A$

$\varphi : M \circlearrowleft$



$\mu(A) = \frac{1}{2}$

$\rightarrow$ $\mu(A) = 1$
$x \in M$ with prob. 1, $x \in A$.
$\mu(A) = 0$



$\frac{1}{T}\sum_{t \leq T} 1_{x,t} \xrightarrow{T\to\infty} 0$

if μ is ergodic for φ,
$\rightarrow$ $f \in C^0(M)$,

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(\varphi^t x) \quad \text{const indep of } x$$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f \circ \varphi^t(x) = \mathbb{E}_{x\sim\mu} f$$

Birkhoff's ergodic theorem

Ergodicity $f(x), f(\varphi x), f(\varphi^2 x), \ldots$

$\rightarrow$ $X_1, \ldots, X_N$ iid according to μ
$\frac{1}{N}\sum_{i=1}^N X_i \xrightarrow{N\to\infty} EX_1$ with prob. 1

$\rightarrow$ $EX^2 < \infty$, $EX < \infty$
$\frac{1}{N}\sum_{i=1}^N X_i \xrightarrow{d} \mathcal{N}\left(EX, \frac{C}{N}\right)$

$\rightarrow$ $\mathbb{E}_{X\sim\mu} X = \int x\,d\mu \approx \frac{1}{N}\sum_{i=1}^N X_i$

OMET

→ (φ, μ) → μ is invariant for φ.

→ μ is preserved by φ.

Rmk: ergodic μ $\Rightarrow$ invariant μ

→ if for any set A, $\mu(A) = \mu(\varphi^{-1}(A))$

→ Lyapunov exponent

→ Cocycle

→ Random dynamical system

$\varphi : M \circlearrowleft \quad \mu$

→ $A(t, x) = A(\varphi^{t-1}(x)) \cdots\cdots A(x)$

$x \to A(x)$ (matrix in $\mathbb{R}^{d \times d}$)

d: $\dim(M)$

→ Cocycle property

$$A(t+s, x) = A(s, \phi^t x) A(t, x)$$

$$A(t+s, x) = \underbrace{A(\varphi^{t+s-1}(x)) \cdots}_{A(s, \varphi^t(x))} \underbrace{A(\varphi^{t-1}(x)) \cdots A(x)}_{A(t,x)}$$

$$\varphi^{t+s-1}(x) = \varphi^{s-1}(\varphi^{t}(x))$$

→ $A(x) = \begin{cases} d\varphi(x) \\ (d\varphi)^T(x) \\ (d\varphi)^{-1}(x) \end{cases}$

Lyapunov exponent

$$\eta(t, v) = \frac{1}{t} \log \| \underbrace{A(t, x) v}_{\in T_{\phi^t x} M} \|$$

$v \in \mathbb{R}^d$

$$\lambda(x, v) := \limsup_{t \to \infty} \frac{1}{t} \log \| A(t, x) v \|$$

$$\lim_{t \to \infty} \eta(t, v)$$

Assumptions: μ is preserved by φ

$\frac{1}{t} \log \| A(\varphi^t x) \| \to 0$ as $t \to \infty$.

$$\mathop{\mathbb{E}}_{x \sim \mu} \max \{ \log \| A(x) \|, 0 \} < \infty$$