Last time: Linear stability analysis
: tangent equation

* Linear dynamics
  phase portraits

Today: First examples of nonlinear dynamics

→ linear stability analysis with an example

Tangent equation:

Flows

$$\frac{d\varphi^t}{dt}(x) = v(\varphi^t x)$$

$$\frac{du_t}{dt} = dv\, u_t$$

$$\left(\frac{du_t}{dt}(\varphi^t x) = dv(\varphi^t x)\, u_t(\varphi^t x) \right)$$

Maps

$$x_{t+1} = F(x_t)$$

$$u_{t+1}(x_{t+1}) = dF(x_t)\, u_t(x_t)$$

$$u_{t+1} \circ F = dF\, u_t$$

Flows

Fixed point

$v(x^*) = 0$

$\Lambda(x^*) = \mathrm{spec}(dv(x^*))$

$\lambda_1 > \lambda_2 \cdots$

$\Rightarrow \lambda_n > 0$

or

Maps

$F(x^*) = x^*$

$\Lambda(x^*) = \mathrm{Spec}(dF(x^*))$

$\lambda_n > 1$
(unstable)
(smallest eigenvalue)

Dynamics of linear perturbations / tangent equation

$$\frac{d\,u_t(x^*)}{dt} = \underbrace{dv(x^*)}\, u_t(x^*)$$

$$u_{t+1}(x^*) = dF(x^*)\, u_t(x^*)$$

in 2D

$\rightarrow \lambda_1 > 0,\ \lambda_n < 0$ (hyperbolic linear)
$\lambda_i \neq 0$ (almost surely unstable)

$\lambda_1 > 1,\ \lambda_n < 1$
$\lambda_i \neq 1$

Hyperbolic fixed points ≡ saddle points

$\rightarrow \lambda_i = 0$ (non hyperbolic fixed point)

$\lambda_i = 1$

Normal form for a saddle node bifurcation

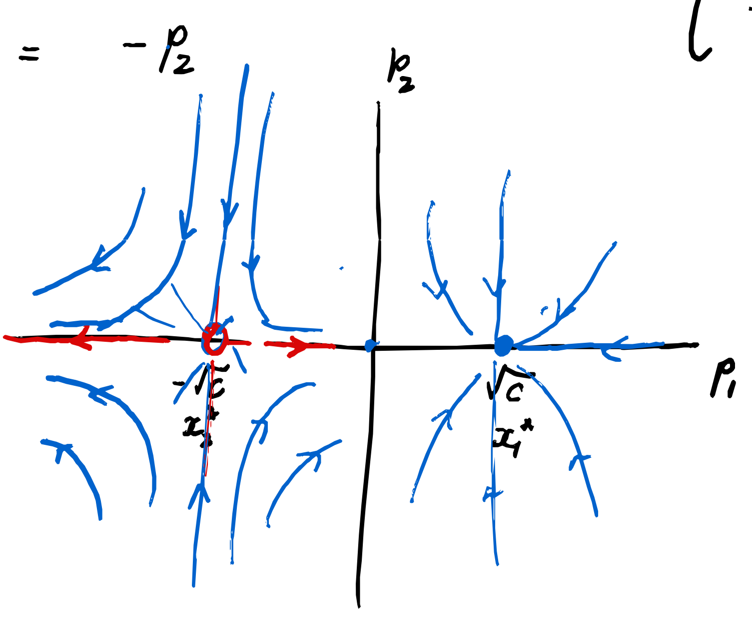
$$\frac{d}{dt}\varphi^t(x) = v(\varphi^t x) \qquad \varphi^t : M \circlearrowleft$$

$M \equiv \mathbb{R}^2$

$x \equiv (P_1, P_2) \qquad P_i : M \to \mathbb{R}$

$$\frac{dP_1}{dt} = c - P_1^2 \qquad v = \begin{bmatrix} c - P_1^2 \\ -P_2 \end{bmatrix}$$

$$\frac{dP_2}{dt} = -P_2$$



$$dv(x) = \begin{bmatrix} -2P_1(x) & 0 \\ 0 & -1 \end{bmatrix}$$

$$dv(x_1^*) = \begin{bmatrix} -2\sqrt{c} & 0 \\ 0 & -1 \end{bmatrix}$$

$$\frac{dP_2}{dt} = -P_2 \ :$$

$$\frac{dP_1}{dt} = -P_1^2$$

(Jacobian at fixed point has 1 zero eigenvalue)

$$\int -\frac{dP_1}{P_1^2} = \int dt$$

$$\frac{1}{P_1} = t + \text{Const}$$

In eigenspaces corresponding to eigenvalue of 0 (flows) or 1 (maps), linear perturbations grow or decay sub-exponentially

Bifurcations

Qualitative changes in response to parameter perturbations

$$\frac{dP_1}{dt} = c - P_1^2$$

$$\frac{dP_2}{dt} = -P_2$$

Saddle node bifurcation: along an axis, fixed points appear / disappear

Zero eigenvalues at fixed points appear in other types of bifurcations

→ Climate science / tipping points

→ Hopf bifurcation

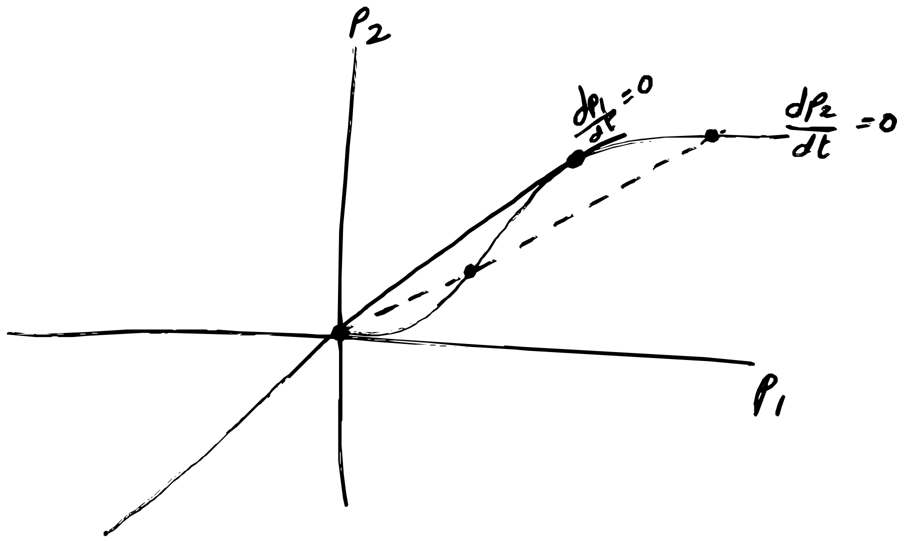
→ Aerospace

→ Rate-induced tipping
   Noise-induced tipping

Griffith 1971 Strogatz 8.1.1.

$$\frac{dp_1}{dt} = -ap_1 + p_2 \qquad b, a > 0$$

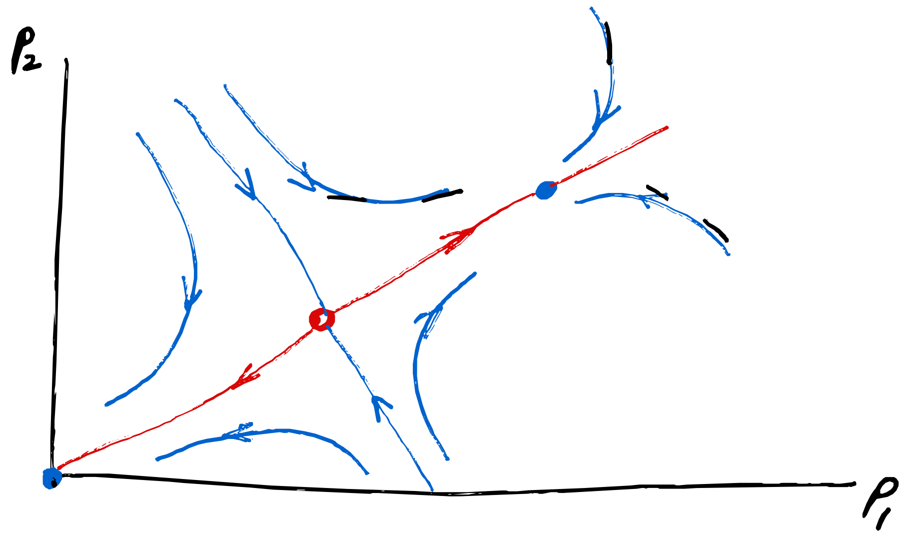
$$\frac{dp_2}{dt} = \frac{p_1^2}{1+p_1^2} - bp_2$$



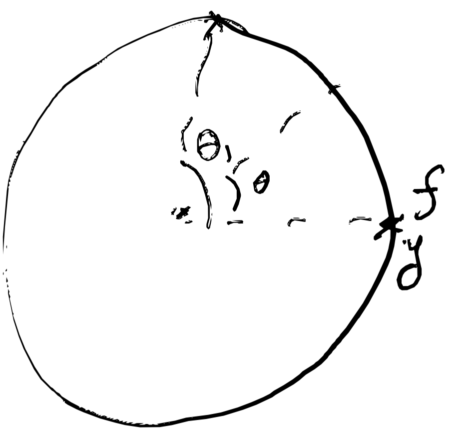
$$dv(x) = \begin{bmatrix} -a & 1 \\ \frac{2p_1}{1+p_1^2} - \frac{p_1^2(2p_1)}{(1+p_1^2)^2} & -b \end{bmatrix}$$

$$Tr(dv(x)) = -(a+b)$$

$$\det dv(x^*) = ab\left(\frac{p_1^{*2}-1}{1+p_1^{*2}}\right)$$



- Basins of attraction of the fixed points are separated by the stable manifold of the saddle point
- As $t \to \infty$, orbits converge to unstable manifold!

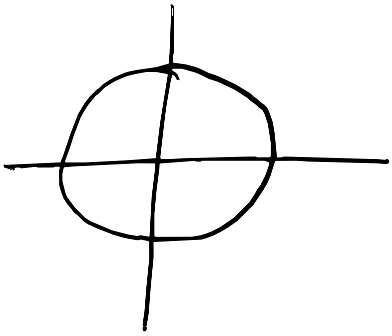


$$\frac{d\theta_y}{dt} = \omega_y \qquad \omega_y > \omega_f$$

$$\frac{d\theta_f}{dt} = \omega_f$$

$\theta_{t+1} = F(\theta_t)$ → if speeds are rational, F has a periodic orbit

→ S^1



S^n : surface of sphere in $\mathbb{R}^{n+1}$

Time-t map :

$$x_{t+1} = (x_t + \alpha) \bmod 1$$

($x_t > 1$ → $x_t \bmod 1 = x_t - 1$)

→ Maps on Torus (next time)

$$\log \frac{2}{1+s} \cdot \frac{1}{2} + \log \frac{2}{1-s} \cdot \frac{1}{2}$$

$$\varphi_s(x_t) = x_{t+1}$$

$$\frac{1}{T} \sum_{t=1}^{T} \log |\varphi_s'(x_t)| \to$$