CSE 8803 CDS, Spr 2024, Georgia Tech

CSE 8803 CDS: Homework 5

Due April 19,'24 (11:59 pm ET) on Gradescope

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1 Problem 1 (10 points)

Here we would like to proof a foundational result in ergodic theory: Birkhoff ergodic theorem. Consider a map $\varphi : M \to M$ with an ergodic measure, μ. Prove that for any function $f \in L^1(\mu)$,[1], its time average along an orbit, $(1/T)\sum_{t\leq T} f(\varphi^t(x))$ converges to a constant, independent of x for μ almost every x.

This result can be interpreted as a strong law of large numbers for the sequence $f \circ \varphi^t(x)$.

Hint: use Theorem 4.1.2 from KH. Then, any function that is constant along an orbit is constant almost everywhere for an ergodic system.

2 Problem 2 (20 points)

In this problem, we will implement an approximate score generative model (Song and Ermon 2019).

Part I Brownian motion or Wiener process is a popular choice for the forward process of an SGM. Simulate a Wiener process $W_t, t \leq 1$. Explain your code. (4 points)

Part II Let ρ_t be the solution of the Fokker Planck equation of $dX_t = dW_t$. Set ρ_0 to be a bimodal Gaussian of your choice. What is ρ_t? (3 points)

Part III The reverse SDE is defined as follows:

$$dY_t = \nabla \log \rho_{1-t}(Y_t)\, dt + dW_t, \tag{1}$$

with $Y_0 \sim \rho_1$. We will approximate the expression $\nabla \log \rho_{1-t}(Y_t)$ with data obtained from the forward process. Simulate the reverse process until $t = 1$. Plot the distribution of Y_1. (5 points)

Part IV Do we expect Y_1 to be distributed according to the target distribution (the bimodal Gaussian you set ρ_0 to)? Why or why not? (8 points)

[1] $L^1(\mu)$ is the space of functions with $\int |f|\, d\mu \leq \infty$.