HW2 Due 4^{th} (Feb)

Last time

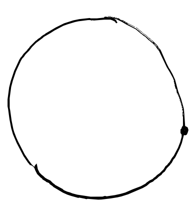
- Linear stability analysis (examples)
- Rotations on circle
- Bifurcations - saddle-node (creation / destruction of fixed points)
  - Hopf bifurcation

Rotations

- Nonlinear dynamics
- Periodic points
- quasiperiodicity (almost periodic)

Takeaway: Irrational rotations on
a) a circle are minimal
b) a torus are topologically transitive

Informal treatment



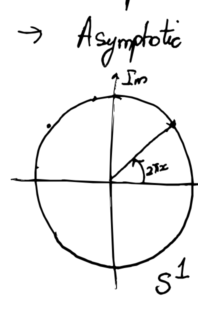
$S^1 \equiv \mathbb{R}/\mathbb{Z}$
(circle of radius 1 in $\mathbb{C}$) (quotient space)

$$\frac{d\varphi^t(x)}{dt} = v$$

(Flow with constant speed on S^1) $\varphi^t(x) = vt + x$

Time-1 map: $\varphi^1(x) = F(x) = (x + v) \bmod 1$

→ All possible orbits?
→ Asymptotic behavior?



ϕ (isomorphism)

$\phi(x) = e^{i2\pi x}$

$\mathbb{R}/\mathbb{Z}$
$[0,1)$
$(0,1]$

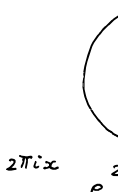
$x + k \equiv x \qquad x \in (0,1)$

$\mathbb{R}/\mathbb{Z}$: quotient space by equivalence relation of reals by integers.

$z = \phi(x) = e^{2\pi i x} \qquad \alpha = e^{2\pi i v}$

(on $\mathbb{R}/\mathbb{Z}$) $F(x) = x + v \bmod 1$ (additive)

(on S^1) $F(z) = z\alpha$ (multiplicative)



$$F(z) = e^{2\pi i x} \cdot e^{2\pi i v} = e^{2\pi i (x+v)}$$

$$F^{t+s} = F^t \circ F^s$$

- v is rational.

$v = \frac{p}{q} \qquad q \neq 0$, p, q coprime

$F^t(x) = (x + tv) \underbrace{\bmod 1}_{\text{nonlinearity}}$, $x \in \mathbb{R}/\mathbb{Z}$

$$F^q(x) = \left(x + q \times \frac{p}{q}\right) \bmod 1 = x$$

Every x is a periodic point

Topological transitivity: A map $F : M \circlearrowleft$ is topologically transitive if $\exists$ some $x \in M$ whose orbit $\{F^t(x)\}_{t \in \mathbb{Z}^+}$ is dense in M.

Minimality: If every orbit is dense in M, F is minimal

Density: A subset $A \subset M$ is dense in M if every point of M is either in A or is a limit point of A. (rational numbers are a dense set of $\mathbb{R}$)

- If v is irrational, F on S^1 (or $\mathbb{R}/\mathbb{Z}$) is minimal

Proof: (Proof by contradiction)

Assume that there is some orbit $\{F^t(x)\}_{t \in \mathbb{Z}^+}$ that is not dense.
v is irrational

- Let $A = \{F^t(x)\}_{t\in\mathbb{Z}^+}$, $\bar{A}$ be its closure.
- $M \setminus \bar{A}$ ($\mathbb{R}/\mathbb{Z}$) is some open set, which we can partition into a finite number of disjoint intervals, $I_1, \dots, I_k$
- Let I_1 be the interval with the largest length

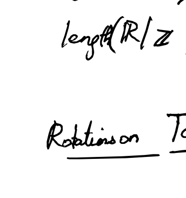
length$(F^t(I_1))$ = length(I_1) (F doesn't expand or contract)

- $F^t(I_i) \cap F^t(I_j) = \emptyset$

and

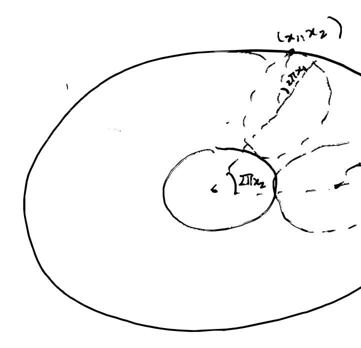
- $F^t(x_{endpoint}) \neq x_{endpoint}$ (v is irrational)
- $\left.\begin{array}{l}\text{length}(S^1) = 2\pi \\ \text{length}(\mathbb{R}/\mathbb{Z}) = 1\end{array}\right\}$ is finite.

Rotations on Torus

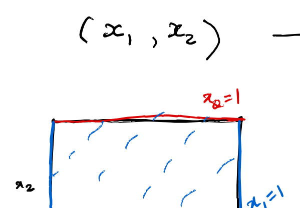


$$\mathbb{T}^d = \underbrace{S^1 \times S^1 \times \dots \times S^1}_{d \text{ times}} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \dots \mathbb{R}/\mathbb{Z}$$

Isomorphism

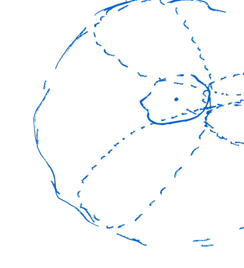


$$(x_1, x_2) \longrightarrow (e^{2\pi i x_1}, e^{2\pi i x_2})$$



$\mathbb{R}^2/\mathbb{Z}^2$

→ Knots
→ trefoil knot 2/3



→ $F(x_1, \dots, x_d) = (x_1 + v_1, \dots, x_d + v_d) \bmod 1$

→ Irrational rotation:
For every set of integers $k_1, \dots, k_d$ s.t. $\sum_{i=1}^{d} k_i v_i$ is not an integer unless $k_1 = k_2 = \dots = k_d = 0$

→ Irrational rotations on the torus are topologically transitive (KH)

Let M be a compact metric space.
→ A map $F : M \circlearrowleft$ is topologically transitive iff (if and only if) for any pair of open subsets $U, V \subset M$, there is an $N \in \mathbb{Z}^+$ ($N \equiv N(U,V)$) s.t. $F^N(U) \cap V$ is non-empty.

S1 F is topo. trans. $\Rightarrow$ "for any pair of open . . . . non-empty"

S2 "for any pair . . . . non-empty" $\Rightarrow$ F is topo. trans.

Proof:

S1 $\exists$ $A = \{F^t(x)\}_{t\in\mathbb{Z}^+}$ which is dense in M. At some t, $F^t(x) \in U$ and at another time τ, $F^\tau(x) \in V$. WLOG, $\tau > t$. $F^{\tau - t}(U) \cap V$ is non-empty.