Lecture 1: Introduction to dynamical systems theory

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January 8, 2024

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- How to control a dynamical system?
- How to design/model a dynamical system?

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- How to predict the future of a dynamical system along with quantifying uncertainties in our prediction?
- How stable are the orbits?
- How to represent the system using a simpler/fewer equations?

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- Our brief study of stochastic systems uses stochastic analysis

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- Office hours: Friday 10-11 am, CODA S1323.

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- M is a d-dimensional smooth manifold. This means that M can be mapped locally to a Euclidean space with a smooth transformation.

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- Vector fields can also be interpreted as linear functionals on the space of smooth functions on the manifold.
- $v(f) = \langle v, df \rangle$, where df is the differential of $f \in \mathcal{C}^\infty(M)$, $\langle \cdot, \cdot \rangle$ is an inner product on TM.

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- Can be obtained from a flow by taking the time-δt map, e.g., from time-integration of the ODE $F(x) = \varphi^{\delta t}(x)$.

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- S is a stochastic process, dealt with later in the course.
- Delay differential equations, Partial differential equations, etc. also have infinite dimensional phase spaces.

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- Read Chapters 1-5, 7, 9 and 11 of Rudin's Principles of Mathematical Analysis.