Metric spaces

(M, d) is a metric space if the "distance" function d satisfies the properties of a metric:

- $d(x,y) = d(y,x)$ symmetricity
- $d(x,x) \geq 0$ and $d(x,x) = 0$ iff $x = 0$ positive definiteness
- $d(x,y) + d(y,z) \geq d(x,z)$ triangle inequality

Examples: Euclidean space with standard metric

$C(M)$: space of continuous functions on compact manifold M with

$$d(f, g) = \sup_{x \in M} |f(x) - g(x)|$$

Space of probability measures on M with Wasserstein metrics (W1 metric)

$$d_{W_1}(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \mathbb{E}_{x,y \sim \gamma} d(x,y)$$

$\Gamma(\mu,\nu)$: set of all joint probability measures with marginals μ & ν respectively

ie. $\gamma \in \Gamma(\mu,\nu)$ if

$$\int_M \gamma(x,y)\, dy = \mu(x)$$

$$\int_M \gamma(x,y)\, dx = \nu(y)$$

All normed spaces are metric spaces

$$d(x,y) = \|x - y\|$$

Norm $\|\cdot\|$ is a function on M that satisfies

- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0 \in M$ (positive definiteness)
- $\|ax\| = |a|\,\|x\|$ where $a \in \mathbb{C}$. (absolute homogeneity)
- $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

Examples above are all normed spaces.

Inner product spaces with inner products defined

Complete (see Rudin for definition) [cauchy sequences converge] inner product spaces are called Hilbert spaces

Complete normed spaces are called Banach spaces

Inner product:

$$\langle x, x \rangle = \|x\|^2$$

Variation of triangle inequality for norms:

Put $x = x - y$

$$\|x\| \leq \|x - y\| + \|y\|$$

$$\|x\| - \|y\| \leq \|x - y\|$$

Put $y = x - y$

$$\|y\| - \|x\| \leq \|x - y\|$$

$$\Rightarrow \quad \big|\|x\| - \|y\|\big| < \|x - y\|$$

Norm is uniformly continuous on M.

Recall continuity definition:

- A function $f: M \to N$ is continuous at a point $x \in M$ if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$d_N(f(x), f(y)) \leq \varepsilon$$

whenever

$$d_M(x,y) \leq \delta$$

((M, d_M) & (N, d_N) are metric spaces)

- function f is continuous on M if it is continuous at every point on M
- Uniformly continuous if ε & δ donot depend on x.
- Continuous but not uniformly continuous: $\frac{1}{x}$ on $(0,1)$: $\delta \to 0$ for fixed $\varepsilon > 0$ as $x \to 0$; x^2 on unbounded domain
- Lipschitz continuity: $\forall\, x, y \in M$, $d_N(f(x), f(y)) < L\, d_M(x,y)$
- Lipschitz continuity $\Rightarrow$ uniform continuity

Compactness

- For some $r > 0$, a set $B_r(x)$, consisting of all points $y \in M$ s.t. $d(x,y) < r$, is called a neighborhood of x.

- An open set is a set that contains a neighborhood of each of its points

- A point $x \in M$ is a limit point if every neighborhood of x contains a point $y \in M$, $y \neq x$.

- A closed set is a set that contains all its limit points.

Examples: $(0,2)$ is open, $[0,2]$ is closed
every point in $[0,2]$ is a limit point.

Sets of rational numbers, irrational numbers in $[0,1]$ are closed.

- A subset $E \subset M$ is dense if every point of M is either in E or is a limit point of E (i.e. is arbitrarily close to an element of M).

e.g.
- Rational numbers, irrational numbers are dense on reals.

- Set of polynomials on compact interval dense on continuous functions of the interval.

- Compact sets are "small". Two definitions, which are equivalent on metric spaces:
  - every open cover has a finite sub cover
  - every sequence has a converging subsequence.

e.g. closed & bdd subsets of $\mathbb{R}^d$.

- Cover: $\bigcup_\alpha G_\alpha$ is an open cover of X if $X \subset \bigcup_\alpha G_\alpha$ & G_α are open subsets of M (metric space)

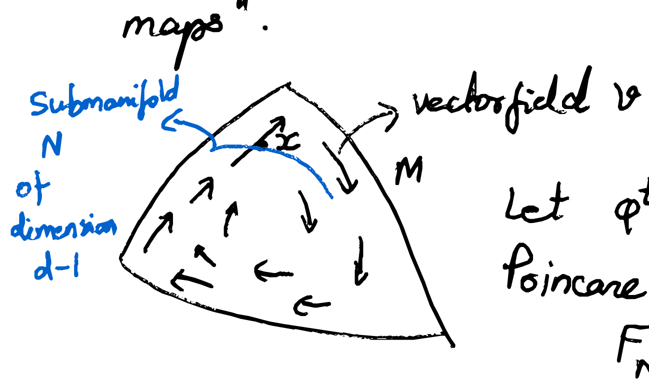
- 2.36 Corollary: If K_n is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, then $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

- In a "phase space", each point can identify a state of the dynamical system.

- Phase space will be denoted by M, which will be a compact Riemannian manifold or compact (closed and bounded) subset of Euclidean space. Dimension of M is d.

- First we will look at discrete-time dynamics that can come from time-integration of ODEs or time-integration of spatially discretized PDEs.

- Another way of generating a discrete-time system or map is to use "Poincare maps".



$v(x)$ is not tangent to N at any x on N.

Let $\varphi^t(x) = x$ (x is a periodic point)

Poincare map:

$F_N : U \to N$ (neighborhood of x in N)

s.t. $F_N(x) = x$

* $F_N(y)$ gives the point of (intersection or return with N) of the orbit starting at y.

* We know there is a ngbd U that returns because the orbit starting at x returns to x after time t.

Another example (from Strogatz pg 271)

- M is 2 dimensional. $x \equiv [\theta, y]$

  θ : phase difference across a Josephson junction

  $y := \frac{d\theta}{dt}$

  $I :=$ applied current

  $\alpha :=$ dimensionless damping

- $\theta(x)$: θ coordinate at $x \in M$
  $y(x)$: y coordinate at $x \in M$

Flow:

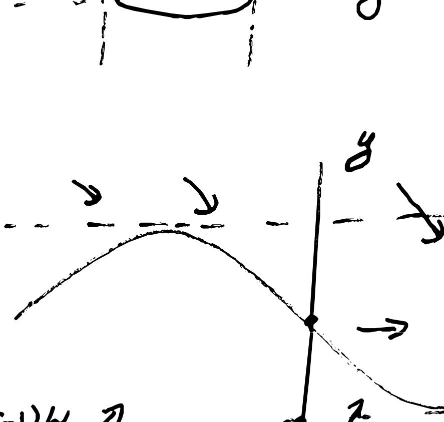
- $$\frac{d\,\varphi^t(x)}{dt} = \frac{d}{dt}\begin{bmatrix}\theta(\varphi^t x)\\ y(\varphi^t x)\end{bmatrix} = v(\varphi^t x)$$

$$= \begin{bmatrix} y(\varphi^t x) \\ I - \sin\theta(\varphi^t x) - \alpha\, y(\varphi^t x)\end{bmatrix}$$

ODE notation:

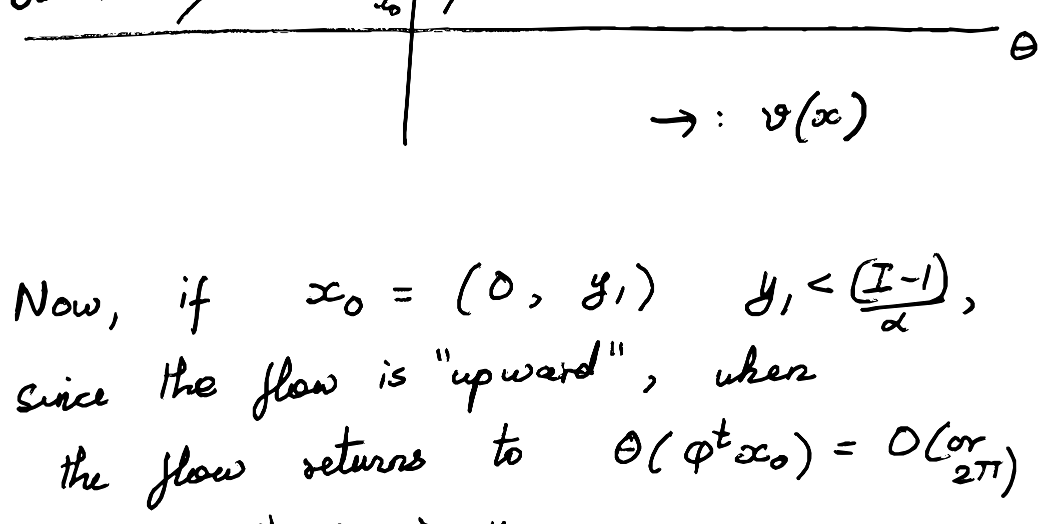
$$\left(\frac{dx}{dt} = \frac{d}{dt}\begin{bmatrix}\theta\\ y\end{bmatrix} = \begin{bmatrix} y \\ I - \sin\theta - \alpha y\end{bmatrix}\right)$$

- M can be thought of as a surface of a cylinder



$0 < \theta \leq 2\pi$

$y \in \mathbb{R}$

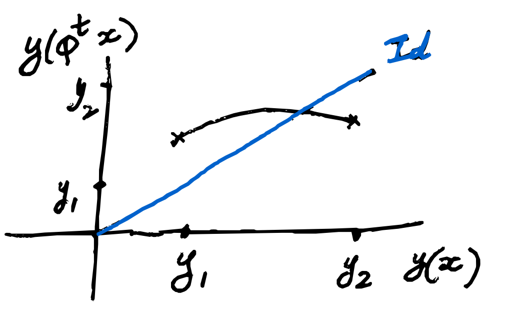
- 

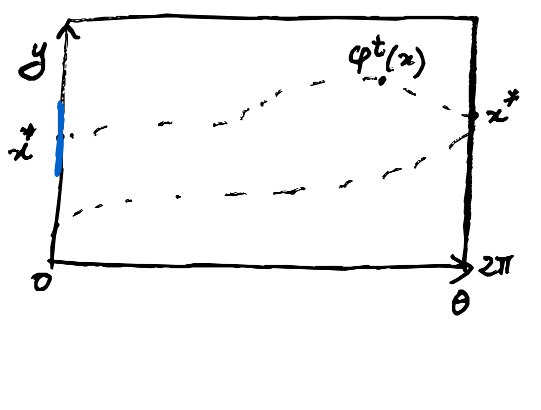
- Now, if $x_0 = (0, y_1)$ $y_1 < \frac{(I-1)}{\alpha}$, since the flow is "upward", when the flow returns to $\theta(\varphi^t x_0) = 0$ (or 2π) $y(\varphi^t x_0) > y_1$.

- Similarly, with $x_0 = (0, y_2)$, $\theta(\varphi^t x_0) = 0 \Rightarrow y(\varphi^t x_0) < y_2$. where $y_2 > \frac{(I+1)}{\alpha}$.

- Thus, there exists an $x^* = (0, y^*)$ s.t. $y(\varphi^t x^*) = y^*$ and $\theta(\varphi^t x^*) = 0$

  x^* is a fixed point of φ^t.



- 

Poincare "first return" map

| : U

| : N

$F_N(x) = \varphi^t(x)$

s.t. $\theta(\varphi^t x) = 0$

Contraction map

Recall : we are interested in behavior of all orbits

$F: M \to M$ is a contraction if $\exists\ \lambda < 1$ s.t. for $x, y \in M$,
$$d(F(x), F(y)) \leq \lambda\, d(x, y)$$

e.g. take any Lipschitz function with Lip constant < 1.

Theorem: F has a fixed point on M

$m \geq n$
$$d(F^m(x), F^n(x)) \leq \sum_{k=n}^{m-1} d(F^{k+1}(x), F^k(x))$$
$$\leq \sum_{k=n}^{m-1} d(F(x), x) \times \lambda^k \leq \lambda^n \sum_{k=0}^{m+n-1} \lambda^k \times d(F(x), x)$$
$$\leq \frac{\lambda^n}{1-\lambda} d(F(x), x)$$

Thus $\{F^n(x)\}$ is Cauchy & converges at every x. Since $d(F^m(x), F^m(y)) \xrightarrow{m \to \infty} 0$, limit is the same for all $x \in M$.

Thus, $\lim_{n \to \infty} F^n(x) = x^* \in M$ exists & is independent of x.

For any x, by triangle inequality,
$$d(F(x^*), x^*) \leq d(x^*, F^n(x)) + d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F(x^*))$$
$$\leq (1+\lambda)\, d(x^*, F^n(x)) + \lambda^n d(F(x), x)$$

as $n \to \infty$, $d(x^*, F^n(x)) = 0$
and $\lambda^n d(F(x), x) = 0$

$\Rightarrow$ $d(F(x^*), x^*) = 0$
x^* is a fixed point of F.

- Stronger existence results for fixed points on arbitrary metric spaces (including spaces of functions) exist.

- Browder fixed point theorem:
Let K be a nonempty closed, bounded, convex subset of a Banach space. If $F: K \to K$ is continuous with compact image, then F has a fixed point.