CSE 8803 CDS, Spr 2024, Georgia Tech

# CSE 8803 CDS: Homework 1

Due Jan 19, ’24 (11:59 pm ET) on Gradescope

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## Problem 1

This problem is about interchanging limits and the problems this can lead to. Source: Rudin Chapter 7.

### Part I: Uniform convergence

1. Prove Theorem 7.13 from Rudin: Suppose $f_n, n \in \mathbb{N}$ is a sequence of continuous functions on a compact set $E$. Assume that i) $f_n$ converge pointwise to a continuous function $f$ and ii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in E$. Then, $f_n \to f$ uniformly on $E$. (5 points)

2. State in mathematical terms what uniform convergence of a sequence of continuous functions with respect to the supremum norm is. (1 point)

3. Consider $f_n = \sin nx$. Does this sequence converge uniformly on a compact subset of $\mathbb{R}$? (1 point) Does it satisfy the assumptions i) and ii) of the Theorem in Part I.1? (1 point)

4. Consider $f_n = \sum_{k=0}^{n} x^2/(1+x^2)^k$. Does this sequence converge uniformly on $[0, 1]$? (1 point) Does it satisfy the assumptions i) and ii) of the Theorem in Part I.1? (1 point)

### Part II: Limits and integration

Consider the sequence of functions $f_n(x) = nx(1-x^2)^n$, which is defined on $[0, 1]$ for $n \in \mathbb{N}$.

1. Is $f_n$ uniformly continuous for all $n$? (2 points)

2. Does $f_n$ converge uniformly to $f(x) = 0$ on $(0, 1]$? (1 point)

3. Let $g_n = \int_0^1 f_n(x)\, dx$. Does the sequence $g_n$ converge? (1 point)

4. Define $h$ to be the integral of the limit $f$, $h := \int_0^1 f(x)\, dx$. Does $\lim_{n\to\infty} g_n = h$? (1 point)

5. Consider the following theorem (7.16 of Rudin). Let $h_n$ be a sequence of Riemann integrable functions on $[a, b]$ for all $n$ and let $h_n \to h$ uniformly on $[a, b]$[1]. Then, $h$ is Riemann integrable on $[a, b]$ and the sequence of limits of integrals converges to the integral of the limit of the sequence, that is,

$$\int_a^b h(x)\, dx = \lim_{n\to\infty} h_n(x)\, dx.$$

Does your answer to the previous part satisfy the conclusion of this theorem? Why or why not? ( 2 points)

## Problem 2

Strogatz 2.3.6 (Language death). Consider the model $\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a$.

(a) Show that this equation for $\dot{x}$ has three fixed points. (2 points)

(b) Show that for all $a > 1$, the fixed points at $x = 0$ and $x = 1$ are both stable. (2 points)

(c) Show that the third fixed point, $0 < x^* < 1$, is unstable. (2 points)

[1] Even pointwise convergence is sufficient for the same conclusion