Last time :

Contraction map on a complete metric space converges exponentially to its unique fixed point

Takeaway:

Hyperbolic linear maps with a nontrivial unstable subspace asymptotically exponentially diverge to infinity [almost surely]

Linear stability analysis (introduced in the last lecture)

* $F : M \circlearrowleft$ map
* $dF : TM \circlearrowleft$ (Derivatives map tangent spaces of the domain to that of the range)

* At time 0, we introduce an infinitesimal linear perturbation along vector field εv_0, $\|v_0(x)\| = 1$ (at all x)

* $x \longrightarrow x + \varepsilon v_0(x)$

$F(x) \longrightarrow F(x + \varepsilon v_0(x))$

At time 1,

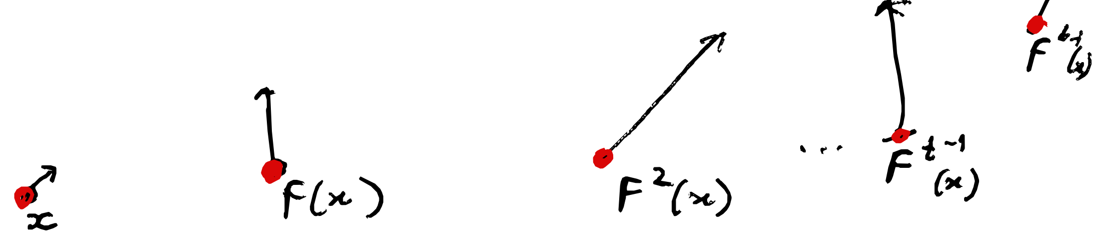
* $$v_1(F(x)) := \lim_{\varepsilon \to 0} \frac{F(x + \varepsilon v_0(x)) - F(x)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{F(x) + \varepsilon\, dF(x)\, v_0(x) + O(\varepsilon^2) - F(x)}{\varepsilon}$$

$$= dF(x)\, v_0(x)$$

* $\| v_1(F(x)) \| = \| dF(x)\, v_0(x) \|$

* $v_t(F^t(x)) = dF(F^{t-1}(x))\, v_{t-1}(F^{t-1} x)$



- $$v_t(F^t(x)) = \lim_{\varepsilon \to 0} \frac{F^t(x + \varepsilon v_0(x)) - F^t(x)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{F(F^{t-1}(x) + \varepsilon v_{t-1}(F^{t-1}(x))) - F(F^{t-1}(x))}{\varepsilon}$$

$$= dF(F^{t-1}(x))\, v_{t-1}(F^{t-1}(x))$$

- Engg notation: (Fixing an orbit)

$$v_t = (dF)_t\, v_{t-1}$$

(Tangent equation)

- $v_0(x) \quad v_1(F(x)) \quad v_2(F^2(x)) \quad \cdots$

$\quad \cap \qquad\quad \cap \qquad\quad \cap$

$T_x M \quad T_{F(x)} M \quad T_{F^2(x)} M$

- $v_2(x) = dF(F^{-2}(x))\, v_0(F^{-2}(x))$

- M is a smooth Riemannian manifold

- $(dF)_\#$

- Adjont equation (Dual of tangent equations)

- Backpropagation algorithm

- $dF(x) = A$

Want: all possible orbits asymptotic behavior

Linear Maps in $\mathbb{R}^d$

<u>Maps</u> $t \in \mathbb{Z}$

- $F(x) = Ax$

<u>Flows</u> $t \in \mathbb{R}$

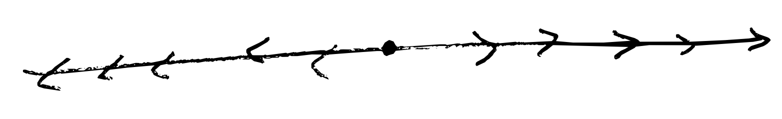
- $\frac{d}{dt}\varphi^t(x) = v(\varphi^t(x))$
$= \log A \ \varphi^t(x)$

- $$\frac{d\varphi^t(x)}{dt} = \log A \ \varphi^t(x)$$
$$\int \frac{d\varphi^t(x)}{\varphi^t(x)} = \int \log A \ dt$$
$$\log \varphi^t(x) = \log A \ t + C$$
$$\log \varphi^t(x) = t \log A + \log x$$
$$\varphi^t(x) = x A^t$$

- $|A| < 1$

Map: time-1 map of φ^t
$$F(x) = \varphi^1(x) = Ax$$

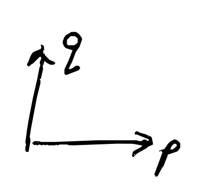
- $|A| > 1$



Linear maps in $\mathbb{R}^2$

* $\frac{d\varphi^t(x)}{dt} = \log A \ \varphi^t(x)$ $\qquad F(x) = Ax$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

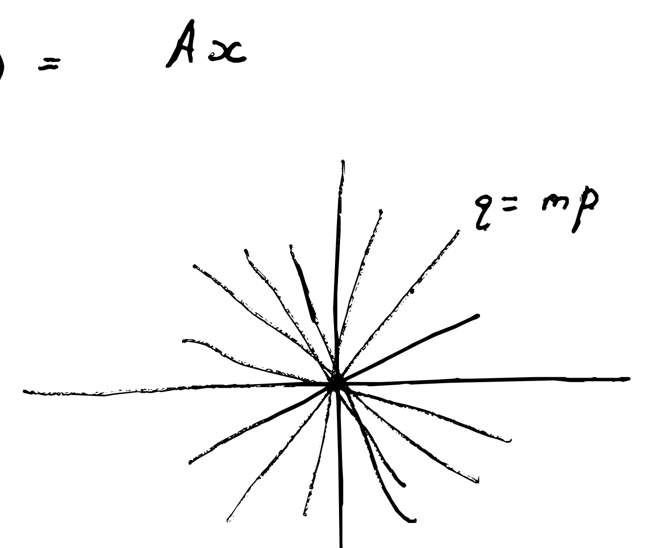


$$x = \begin{bmatrix} p \\ q \end{bmatrix}$$

- $$\frac{d\varphi^t(x)}{dt} = \frac{d}{dt}\begin{bmatrix} p(\varphi^t(x)) \\ q(\varphi^t(x)) \end{bmatrix} = \begin{bmatrix} \log\lambda \ p(\varphi^t(x)) \\ \log\lambda \ q(\varphi^t(x)) \end{bmatrix}$$

- $\varphi^t(x) = A^t x$
- $F(x) = Ax$

- $|\lambda| < 1$



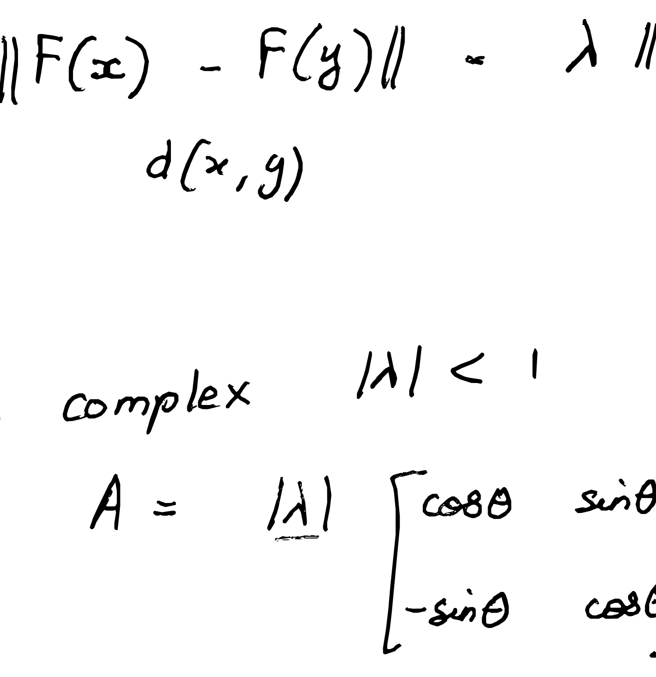
"straight lines in $\mathbb{R}^2$ that pass through the origin are invariant under the flow"

- Fixed point : $(0,0)$

- $\|F(x) - F(y)\| = \lambda \|x - y\|$
$d(x,y)$

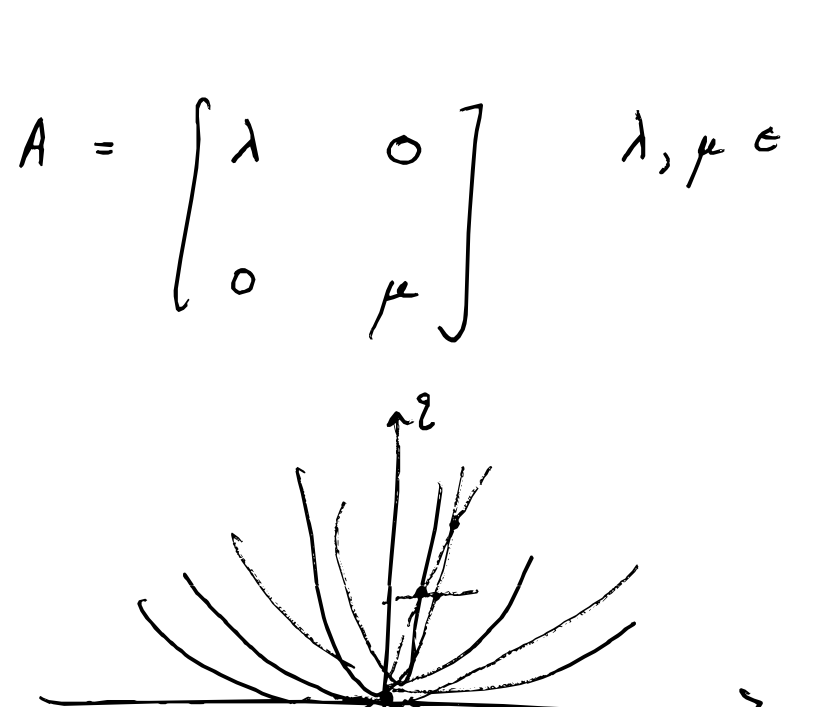
Chapter 1
(K H)
Strogatz

- λ complex $\quad |\lambda| < 1$
$$A = \underline{|\lambda|}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$eig(A) = |A| e^{\pm i\theta}$$



- $|\lambda| = 1$: Rotations of the circle

- $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ $\qquad \lambda, \mu \in (0,1)$



Ex: Contraction
$$p^{\log\mu} = c \, q^{\log\lambda}$$
(check: $(\lambda p)^{\log\mu} = \lambda^{\log\mu} p^{\log\mu}$
$= \lambda^{\log\mu} c \, q^{\log\lambda}$
$= c \, \mu^{\log\lambda} q^{\log\lambda}$
$= c(\mu q)^{\log\lambda}$)

* $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

- A map $F(x) = Ax$ is linear hyperbolic if the eigenvalues of A are not norm 1.

- Unstable subspace:
$$E^u = \{ v \in \mathbb{R}^d : (A - \lambda I)^k v = 0 \text{ for some } k \text{ and some } \lambda \text{ with } |\lambda| > 1 \}$$

- Stable subspace:
$$E^s = \{ v \in \mathbb{R}^d : (A - \lambda I)^k v = 0 \text{ for some } k, \lambda \text{ with } |\lambda| < 1 \}$$

- Center / Neutral subspace
$$E^c = \{ v \in \mathbb{R}^d : (A - \lambda I)^k v = 0 \ \ |\lambda| = 1 \}$$

$$\mathbb{R}^d = E^u \oplus E^s \oplus E^c$$ (Ex: direct sum)

Stable subspace:

$\rightarrow$ $F(x) = Ax$

$x \in E^s$

- $A|_{E^s}$ is a contraction map.

- 0 is the unique fixed point
- all orbits on E^s converge exponentially
- If A is invertible, all orbits of A^{-1} on E^u converge exponentially to 0

- All orbits of A on E^u go to infinity exponentially

- Define E^s as the space of initial points that converge exponentially to 0.

- $x \in \mathbb{R}^d$ $\qquad$ A is hyperbolic

$x = \underset{E^u}{x^u} + \underset{E^s}{x^s}$ $\qquad$ $(\mathbb{R}^d = E^u \oplus E^s)$

$$\begin{aligned} \|F^t(x)\| &= \|A^t x\| \\ &= \|A^t(x^u + x^s)\| \\ &= \|A^t x^u + A^t x^s\| \quad \text{(reverse triangle ineq)} \\ &\geq \|A^t x^u\| - \|A^t x^s\| \end{aligned}$$

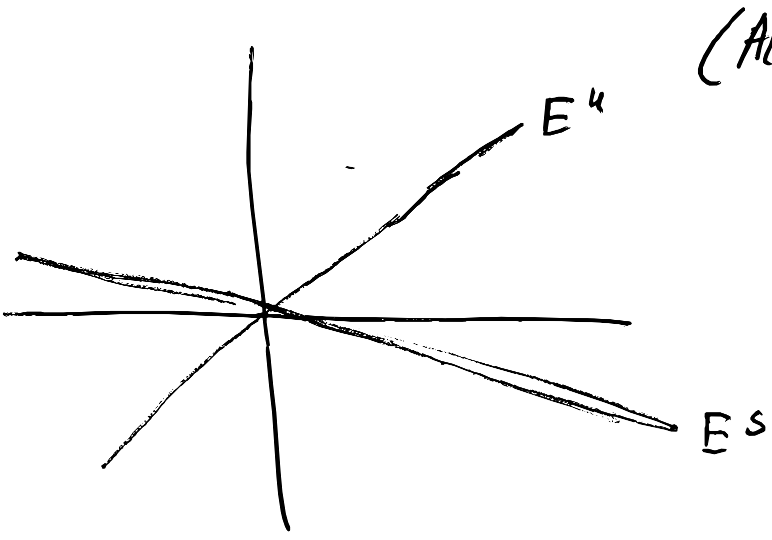
$(\|x+y\| < \|x\| + \|y\|)$ (Ex.)

$\|x+y\| - \|y\| < \|x\|$

$$\geq \lambda^t \|x^u\| - \lambda^{-t} \|x^s\|$$

$(\lambda > 1)$

$-\|A^t x^s\| \geq -\lambda^{-t} \|x^s\|$

- 

(Almost sure convergence)

a.s.

a.e.