CSE 8803 CDS, Spr 2024, Georgia Tech

CSE 8803 CDS: Homework 1

Due Feb 4,'24 (11:59 pm ET) on Gradescope

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Problem 1

This problem asks you to think about an iterative numerical method as a discrete-time dynamical system (map). Consider the power iteration method for a square, non-singular, diagonalizable matrix $A \in \mathbb{R}^{d\times d}$. For $t \in \mathbb{N}$,

- $v_t \to Av_{t-1}$

(*) $v_{t+1} \to v_{t+1}/\|v_{t+1}\|$.

1. Write down a map $F(x_t) = x_{t+1}$ to describe the above algorithm, where F is defined on a set $M \subseteq \mathbb{R}^d$. (1 point)
2. Is M compact? (1 point)
3. Is F a contraction on M? (1 point)
4. How many fixed points does F have? (1 point) What are they? (1 point)
5. State the assumptions on A so that almost every initial condition converges to a fixed point. (1 point)
6. Under the assumptions in the part above, prove the convergence of almost every iterate to a fixed point of F. (3 points)
7. From here on, consider the power iteration without the normalization step (*). Write the corresponding new map, F, on $\mathbb{R}^d$ (1 point).
8. Give conditions on A for F to be a contraction map (1 point).
9. Give conditions on A for F to be a linear hyperbolic map (1 point).
10. Without the additional conditions in the above two parts (i.e., without hyperbolicity assumptions), describe the asymptotic behavior of all orbits of F. That is, give, with justification, a stable-unstable-center decomposition of $\mathbb{R}^d$ by F. (3 points).

Problem 2 (KH Proposition 1.1.5)

This question is a first excursion into perturbation theory of dynamical systems. Let $F : M \to M$ be a contraction map with fixed point x_F^* and contraction coefficient $\lambda_F \in (0,1)$: for all $x, y \in M$, $\|F(x)-F(y)\| \leq \lambda_F \|x-y\|$. Show that for every $\epsilon > 0$, there exist $\delta > 0$ and $\lambda_G \in (0,1)$ such that any map $G : M \to M$ with a contraction coefficient λ_G that is δ-close to F in the C^0-norm, i.e.,$\|F - G\| := \sup_{x \in M} \|F(x) - G(x)\| \leq \delta$ satisfies $\|x_F^* - x_G^*\| \leq \epsilon$. (5 points)

Problem 3

Strogatz Ex. 5.3 (Love affairs)

- 5.3.2 b (1 point)
- 5.3.3 "What happens?" -> Describe the asymptotic behavior of all orbits (1point)
- 5.3.4 (2 points)
- 5.3.5 (2 point)
- 5.3.6 (1 point)