

CSE 8803 CDS: Homework 1

Due Jan 19, '24 (11:59 pm ET) on Gradescope

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Problem 1

This problem is about interchanging limits and the problems this can lead to. Source: Rudin Chapter 7.

Part I: Uniform convergence

1. Prove Theorem 7.13 from Rudin: Suppose $f_n, n \in \mathbb{N}$ is a sequence of continuous functions on a compact set E . Assume that i) f_n converge pointwise to a continuous function f and ii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in E$. Then, $f_n \rightarrow f$ uniformly on E . (5 points)
2. State in mathematical terms what uniform convergence of a sequence of continuous functions with respect to the supremum norm is. (1 point)
3. Consider $f_n = \sin nx$. Does this sequence converge uniformly on a compact subset of \mathbb{R} ? (1 point) Does it satisfy the assumptions i) and ii) of the Theorem in Part I.1? (1 point)
4. Consider $f_n = \sum_{k=0}^n x^2/(1+x^2)^k$. Does this sequence converge uniformly on $[0, 1]$? (1 point) Does it satisfy the assumptions i) and ii) of the Theorem in Part I.1? (1 point)

Part II: Limits and integration

Consider the sequence of functions $f_n(x) = nx(1-x^2)^n$, which is defined on $[0, 1]$ for $n \in \mathbb{N}$.

1. Is f_n uniformly continuous for all n ? (2 points)
2. Does f_n converge uniformly to $f(x) = 0$ on $(0, 1]$? (1 point)
3. Let $g_n = \int_0^1 f_n(x) dx$. Does the sequence g_n converge? (1 point)
4. Define h to be the integral of the limit f , $h := \int_0^1 f(x) dx$. Does $\lim_{n \rightarrow \infty} g_n = h$? (1 point)

5. Consider the following theorem (7.16 of Rudin). Let h_n be a sequence of Riemann integrable functions on $[a, b]$ for all n and let $h_n \rightarrow h$ uniformly on $[a, b]$ ¹. Then, h is Riemann integrable on $[a, b]$ and the sequence of limits of integrals converges to the integral of the limit of the sequence, that is,

$$\int_a^b h(x) dx = \lim_{n \rightarrow \infty} \int_a^b h_n(x) dx.$$

Does your answer to the previous part satisfy the conclusion of this theorem? Why or why not? (2 points)

Problem 2

Strogatz 2.3.6 (Language death). Consider the model $\dot{x} = s(1 - x)x^a - (1 - s)x(1 - x)^a$.

- (a) Show that this equation for \dot{x} has three fixed points. (2 points)
- (b) Show that for all $a > 1$, the fixed points at $x = 0$ and $x = 1$ are both stable. (2 points)
- (c) Show that the third fixed point, $0 < x^* < 1$, is unstable. (2 points)

¹Even pointwise convergence is sufficient for the same conclusion