Last time

→ Lyapunov vectors

singular vector of tangent / adjoint propagators

eigenvectors of far past / far future matrices

Future:

→ Machine learning + dynamical system (2)

→ Computational dynamics project (3)

→ Random dynamical systems (1)

Today

→ Review of Lyapunov vectors

→ Review of QR iteration (linear algebra)

→ Computation of LVs.

Kuptsov Parlitz 2012 / Trefethen Bau

Fix orbit
$x, \varphi x, \ldots \quad \varphi^k x, \ldots$

Far future operator

$$W^+(x) = \lim_{t\to\infty} \left((F(0,t))^T F(0,t) \right)^{\frac{1}{2t}}$$

tangent propagator

$$T_x M \to T_{\varphi^t x} M$$

$$(F(0,t))^T : T_{\varphi^t x} M \to T_x M$$

$$F(0,t) = U(0,t)\,\Sigma(0,t)\,V(0,t)^T$$

$$(F(0,t))^T F(0,t)$$
$$= V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$$

$$W^+(x) = \lim_{t\to\infty} \left(\underbrace{(F(0,t))^T (F(0,t))} \right)^{\frac{1}{2t}}$$

state at time 0

$$= \lim_{t\to\infty} V(0,t)\,\Sigma^{1/t}(0,t)\,V(0,t)^T$$

(Recall $A^T A$, $A A^T$ are SPSD matrices. eigenvalues ≥ 0, eigenvectors are orthogonal)

$$V(0,t) := \left[f_1^+(0,t) \middle| f_2^+(0,t) \middle| \cdots \middle| f_d^+(0,t) \right]$$

Existence of $\lim_{t\to\infty} V(0,t)$: Oseledets theorem

Logarithms of eigenvalues of $W^+(x)$:= Lyapunov exponents

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p$$

p : number of distinct LEs

$p \leq d$.

For degenerate LEs, $f_i^+(x)$ are $\boxed{\text{not}}$ unique

In Ergodic systems, LEs are constant functions of x.

$$\lim_{t\to\infty} f_i^+(0,t) = f_i^+(0)$$

at point x

Homework

→ Lorenz '63 system : first example of chaos in climate system

→ 

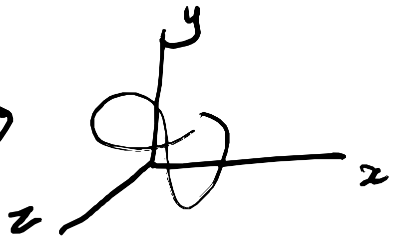
Navier-Stokes
Fourier

→ $F(0,t)$: $\underset{T_x M}{\underset{\cap}{v(0)}} \longrightarrow \underset{T_{\varphi^t x} M}{\underset{\cap}{v(t)}}$

→ $\varphi^t x$: solution of Lorenz system at time t

Runge Kutta
odeint (scipy)

$$\frac{d\varphi^t(x)}{dt} = g(\varphi^t x)$$

→ 

→ $v(0) \in T_x M$

$\rightsquigarrow$

$v(t) \in T_{\varphi^t x} M$

$$\frac{dv(t)}{dt} = dg(\varphi^t x)\, v(t)$$ ← tangent equation

$$F(0,t)\, v(0) = v(t)$$

↓

time integrator for tangent equations

Power iteration

Input: A

Output: q_0, λ_0 : top eigenvector & value

v_0 not orthogonal to its top eigenvector

for $i = 1, 2, 3, \dots$

$$v_{i+1} = A v_i$$
$$\alpha_i = \| A v_i \|$$
$$v_{i+1} = v_{i+1} / \alpha_i$$

$$v_i \to q_0 \qquad \alpha_i \to \lambda_0.$$

Input: $A_0, A_1, A_2 \dots$

Output: $q_0(\varphi^k x), q_0(\varphi^{k+1} x)$ ......

λ_0 : top LE

$$A_0 = d\varphi(x)$$
$$A_1 = d\varphi(\varphi x) \dots$$

$v_0 \in T_x M \equiv \mathrm{rand}(d)$

for $i = 1, 2, 3 \dots$

$$v_{i+1} = A_i v_i$$

$T_{\varphi^i x} M$

$$\alpha_i = \| A_i v_i \|$$
$$v_{i+1} = v_{i+1} / \| A_i v_i \|$$

$$v_i \to q_0(\varphi^i x)$$
$$\frac{1}{i} \sum_{j=1}^{i} \log \alpha_j \to \lambda_0$$

Inputs: $A_0, A_1, \dots A_k \dots$

$$A_0 = (d\varphi(\varphi^k x))^T$$
$$A_1 = (d\varphi(\varphi^{k-1} x))^T$$
$$\vdots$$
$$A_k = (d\varphi(x))^T$$

$$v_k \in T_x^* M$$
$$\approx p_0(x)$$

QR iteration

GS orthogonalization

Gram-Schmidt

$$\{v_1, \dots, v_n\} \longrightarrow \{q_1, \dots, q_n\}$$

$\mathbb{R}^d$

$q_i \perp q_j \quad i \neq j$

$$q_1 \leftarrow v_1/\|v_1\|$$

$$q_2 = v_2 - (v_2 \cdot q_1)\, q_1$$

$$q_2 = q_2/\|q_2\|$$

$$\left(q_2 \cdot q_1 = v_2 \cdot q_1 - (v_2 \cdot q_1) = 0 \right)$$

$$\vdots$$

$$q_3 = v_3 - (v_3 \cdot q_1)\, q_1 - (v_3 \cdot q_2)\, q_2$$

$\Updownarrow$ modified GS

$$\tilde{q}_3 = v_3 - (v_3 \cdot q_1)\, q_1$$

$$q_3 = \tilde{q}_3 - (\tilde{q}_3 \cdot q_2)\, q_2$$

$$A = Q R$$

A: $m \times n$

Q: $m \times n$ — orthonormal matrix

R: $n \times n$ — Upper triangular matrix

Q : form an orthonormal basis for the columns of A

R : off-diagonal elements : orthogonal projections

R : diagonal elements norms computed by modified GS.

Eigenvalue - Eigenvector of matrix A

$$\underset{\mathbb{R}^{d \times d}}{A^k} \underset{\substack{m\text{-dimensional} \\ \text{subspace} \\ \text{of } \mathbb{R}^d}}{E} \cong \text{span}\{q_0, q_1, \dots q_{k-1}\}$$

Combine power iteration + GS

Input: $A \in \mathbb{R}^{d \times d}$

Q_0 : random orthogonal matrix $\in \mathbb{R}^{d \times d}$

R_0 : Id

for $i = 1, 2, \dots$

$\hat{Q}_{i+1} = A Q_i$

$Q_{i+1} R_{i+1} = \hat{Q}_{i+1}$

$Q_i \longrightarrow$ eigenvectors of A

$$\prod_{j=1}^{i} R_i = R_i R_{i-1} \cdots R_0$$

(triangular matrix: eigenvalues are on the diagonal)

$$A^k = Q_1 Q_2 \cdots Q_k R_k \cdots R_1$$

$$= \tilde{Q}_k \tilde{R}_k$$

Schur form of A $= \tilde{Q}_k^* A \tilde{Q}_k$ (triangular, diagonal if A is symmetric)

Rayleigh quotient: v

$$r(v) = v^T A v$$