→ Lyapunov vectors
→ Plan:

- 2 hws
- Final project

Meeting: by next Monday (04/01)

10 minutes

Project scope

Theory nonlinear dynamical systems

→ perturbation theory (linear stability Lyapunov Oseledets)
→ Ergodic theory (invariant measures, ergodic)
→ bifurcations → perturbations to dynamical model

→ Stochastic systems
→ Representations (Neural network e.g. FNO, PINNs, Neural ODEs, Neural SDEs)

→ Model order reduction (Nonlinear)

- Machine learning
- Climate dynamics

} BYO dyn. sys.

Example

Optimization algorithms + dimension reduction

- $\frac{1}{N}\sum_{i=1}^{N} f_i(x)$: objective function

i : i^{th} example / sample

$f_i(x)$: loss incurred on the i^{th} example by model with parameters x

$x \in \mathbb{R}^{(\text{no of weights + biases})}$

- $x_{t+1} = x_t - \eta_t \underbrace{\nabla\left(\frac{1}{N}\sum_{i=1}^{N} f_i(x)\right)}$

(Gradient descent)

learning rate / step size

- Training algorithm $\Leftrightarrow$

$x_{t+1} = \varphi(x_t)$

$\varphi(x) = x - \eta \nabla(x)$

- Efficiency
- Understanding of generalization / feature learning

Osdedets theory

- Fix orbit $x \in \mathbb{R}^d$
  $x, \varphi x, \varphi^2 x, \ldots \quad \varphi : \mathbb{R}^d \circlearrowleft$
- $d\varphi(x), d\varphi(\varphi x), \ldots$
- $F(0,t) = d\varphi(\varphi^{t-1}x) \cdots d\varphi(x)$ tangent
  $= d\varphi^t(x)$
- $(F(0,t))^T : T_{\varphi^t_x} M \to T_x M$ adjoint
- $(F(0,t))^{-T}$: forward adjoint propagator
- $(F(0,t))^{-1}$ backward/inverse tangent

Far future matrix
- $W(x) = \lim_{t \to \infty} \left((F(0,t))^T F(0,t) \right)^{\frac{1}{2t}}$
  (x: state at time 0)

$F(0,t) = U(0,t) \quad \Sigma(0,t) V(0,t)$

ergodic systems: Σ is independent of x

$\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ ↳ distinct LEs

$p \leq d.$

$\lambda_1 > \lambda_2 > \cdots > \lambda_d$

$$U(0,t) = \left[f_1^-(0,t) \Big| \cdots \Big| f_d^-(0,t) \right]$$

$$f_i^-(x) = \lim_{t \to \infty} f_i^-(0,t)$$

backward Lyapunov vectors

$$V(0,t) = \left[f_1^+(0,t) \Big| \cdots \Big| f_d^+(0,t) \right]$$

$$f_i^+(x) = \lim_{t \to \infty} f_i^+(0,t)$$

forward LVs.

- $E_1^+(x) = \text{span}\{ f_1^+(x), \ldots, f_d^+(x) \}$
  $= \mathbb{R}^d$

$E_2^+(x) = \text{span}\{ f_2^+(x), \ldots, f_d^+(x) \}$

for a large t, $\| F(0,t) E_2^+(x) \| \sim$
$O(e^{\lambda_2 t})$

$\eta(E) \; \lim_{t \to \infty} \frac{1}{t} \log \det (E \to F(0,t) E)$

$E_i^+(x) = \text{span}\{ f_i^+(x), \ldots, f_d^+(x) \}$

$\underline{\eta(E_i^+(x)) \leq \lambda_i}$

$E_d^+(x) = \text{span}\{ f_d^+(x) \}$

$E_{d-1}^+(x) = \text{span}\{ f_{d-1}^+(x), f_d^+(x) \}$

$a_1 f_{d-1}^+(x) + a_2 f_d^+(x)$

- $\eta(E_i^-) \leq -\lambda_i$
  $(F(0,t))^{-1} = (d\varphi(x))^{-1} \cdots (d\varphi(\varphi^{t-1}x))^{-1}$
  $\lambda_1 > \lambda_2 > \cdots > \lambda_d$
  $-\lambda_d > -\lambda_{d-1} > \cdots > -\lambda_1$

$E_i^-(x) = \text{Span}\{ f_1^-(x), \ldots, f_i^-(x) \}$

- $F(0,t) f_i^+(x) \; \cancel{\approx} \; e^{\lambda_i t} f_i^+(\varphi^t x)$
  ~~Covariant~~
  $\approx e^{\lambda_i t} f_i^-(\varphi^t x)$

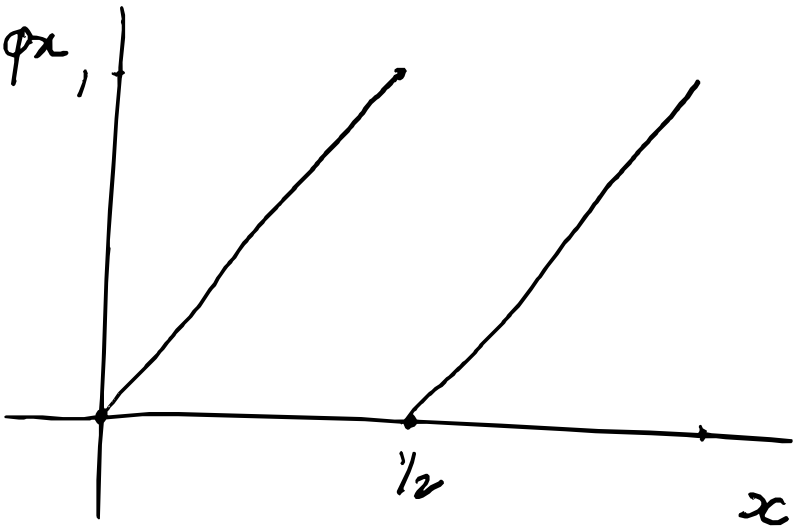
$F(0,t) E_i^+(x) \xrightarrow{t \to \infty} E_i^-(\varphi^t x)$

- $(F(0,t))^T$
  $\lambda_1 > \cdots > \lambda_d$
  $(F(0,t))^T f_i^-(\varphi^t x) \approx e^{\lambda_i t} f_i^+(x)$

- Chaos:
  - if almost every x, $\lambda_1(x) > 0$
  - $\varphi : M \circlearrowleft$ M compact
  - topological transitivity

- Ergodic chaotic:
  $\lambda_1 > 0$

Sawtooth map



$$\varphi(x) = 2x \,\%\, 1$$

$$d\varphi(x) = 2 \qquad \varphi : M \circlearrowleft$$

$$d\varphi^t(x) = d\varphi(\varphi^{t-1}x)\cdots d\varphi(x) = 2^t$$

Leb measure : Unif$([0,1])$

$$M = [0,1]$$

$$TM = [0,1] \times [0,1]$$

$$\chi(v) = \lim_{t\to\infty} \frac{1}{t} \log \frac{\|d\varphi^t(x)\, v(x)\|}{\|v(x)\|}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log 2^t$$

$$= \lg 2$$

"with prob. 1" $\equiv$ "Leb a.e."

Chaotic orbit