Oseledets spaces

Kuptsov and Parlitz 2012

Dynamical system

$$\frac{du}{dt} = g(u, \boxed{t})$$

$g \rightarrow$ Vector field

$u(t) \in R^m$ $\rightarrow$ phase space

$$\frac{d\varphi^t(x)}{dt} = v_t(\varphi^t(x))$$

$$u(t) \equiv \varphi^t(x)$$

$$u \equiv x \quad \text{(phase point)}$$

$$g \equiv v_t$$

Can be non-autonomous

Linearized dynamics

Infinitesimal linear perturbations
$\equiv$ vector fields

Evolution of vector fields : subject of Oseledets theorem

$$\frac{d\varphi^t(u)}{dt} = g(\varphi^t(u), t)$$

$J(\varphi^t(u), t) \equiv dg(\varphi^t(u), t)$ (Jacobian) $\mathbb{R}^{m\times m}$

$$g(\varphi^t(u), t) = [\, g_1(\varphi^t(u), t),\ g_2(\varphi^t(u), t), \dots,\ g_m(\varphi^t(u), t)]^T$$

$$[\, J(\overset{u(t)}{\nearrow}\!\!\!\varphi^t(u), t)]_{ij} = \partial_j g_i(\varphi^t(u), t)$$

$[\partial_1, \dots, \partial_m]$: partial derivatives in Euclidean space $\mathbb{R}^m$

$$v(t) \in T_{\underset{u(t)}{\varphi^t_u}} \mathbb{R}^m \quad \text{(isomorphic to } \mathbb{R}^m)$$

Tangent space evolution

$$\frac{dv(t)}{dt} = J(\varphi^t(u), t)\, v(t)$$

$$v(0) \in T_u \mathbb{R}^m$$

$$v(t) = \underbrace{e^{\int_0^t J(\varphi^{t'}(u), t')\,dt'}}_{M(t)}\, v(0)$$

$$\mathcal{F}(t_1, t_2)\, v(t_1) = v(t_2)$$

$$v(t_1) = M(t_1)\, v(0)$$

$$v(t_2) = M(t_2)\, v(0)$$

$$t_2 > t_1$$

$$\mathcal{F}(t_1, t_2)\, M(t_1)\, v(0) = M(t_2)\, v(0)$$

$$\mathcal{F}(t_1, t_2)\, M(t_1) = M(t_2)$$

$$\mathcal{F}(t_1, t_2) = M(t_2)(M(t_1))^{-1}$$

→ There is an underlying fixed orbit $\{(\varphi^t_u)\}$

→ $\mathcal{F}(t_1, t_2) \in \mathbb{R}^{m\times m}$ maps vectors in $T_{\varphi^{t_1}_u}\mathbb{R}^m$ to vectors in $T_{\varphi^{t_2}_u}\mathbb{R}^m$

→ $M(t)$ is invertible

$\mathcal{F}(t_1, t_2)$: tangent propagator

Adjoint space

$\rightarrow \quad Ax \cdot y = x \cdot A^T y$

$\rightarrow \quad v(u) \in \underbrace{T_u \mathbb{R}^m}_{\text{target space at } u} \qquad v \in \underbrace{T\mathbb{R}^m}_{\text{tangent bundle}}$

$f : \mathbb{R}^m \to \mathbb{R}$

$$\underbrace{v(f)}\,(u) = \lim_{\varepsilon \to 0} \frac{f(u + \varepsilon v(u)) - f(u)}{\varepsilon}$$

linear operator
on space of functions
that gives "directional derivative"

$\rightarrow \quad v(u)$: vector that represents an infinitesimal perturbation

$\rightarrow \quad F(t, t_1)\, v(t) = v(t_1)$

$v(t) \in T_{\varphi^t(u)} \mathbb{R}^m \qquad v(t_1) \in T_{\varphi^{t_1}_u} \mathbb{R}^m$

$$\underbrace{F(t, t_1)\, v(t)} \cdot w(t_1) = v(t_1) \cdot w(t_1)$$

$F(t,t_1)\,v(t) \in T_{\varphi^t(u)} \mathbb{R}^m \qquad w(t_1) \in T^*_{\varphi^{t_1}(u)} \mathbb{R}^m$

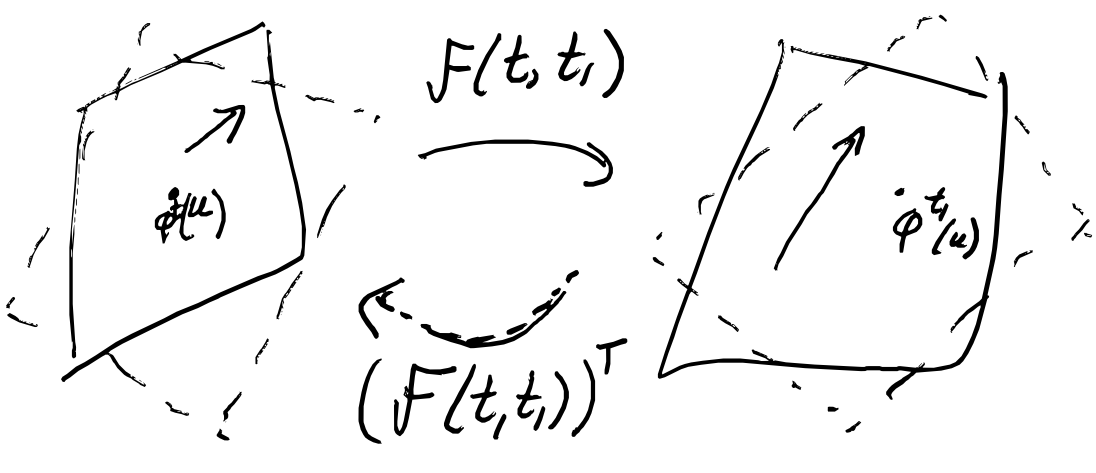
$\rightarrow$ Dot product $\rightarrow$ Inner product
$\rightarrow$ Works for Riemannian manifold

$$F(t,t_1)v(t) \cdot w(t_1) = v(t) \cdot (F(t,t_1))^T w(t_1)$$
$$= v(t_1) \cdot w(t_1)$$

$\rightarrow \quad w(t) = (F(t, t_1))^T w(t_1)$

$w(t) \in T^*_{\varphi^t(u)} \mathbb{R}^m$

$\hookrightarrow$ adjoint propagator



$\rightarrow \quad v(f)(u) =$ directional derivative of function f in the direction of v at u

f is smooth

$= df(u) \cdot v(u)$

$df(u) \in T^*_u \mathbb{R}^m$

$\rightarrow \quad w(\varphi^{t_1}(u)) = (df(\varphi^{t_1}(u)))^T$

$F(t, t_1)^T w(\varphi^{t_1}(u))$

$= \underbrace{w(\varphi^t(u))}_{?}$

$\varphi(u)$ — Discrete-time

$$\rightarrow \quad (J(u))^T \underbrace{w(\varphi(u))}_{(df(\varphi(u)))^T}$$

$$= w(u) = (d(f \circ \varphi)(u))^T$$

$d(f \circ \varphi)(u) = df(\varphi(u))\, d\varphi(u)$

row vector $\quad = df(\varphi(u))\, J(u)$

$$(J(u))^T (df(\varphi(u)))^T = (d(f \circ \varphi)(u))^T$$

$(df(\varphi(u)))^T \equiv w(\varphi(u)) \qquad (d(f\circ\varphi)(u))^T \equiv w(u)$

$\rightarrow$ Intuitive explanation for adjoint propagator:

if $\quad w(\varphi^t u) = (df(\varphi^t u))^T$

$(M(t))^T w(\varphi^t u) = w(u)$

$w(u) \in T^*_u \mathbb{R}^m$

$= (d(f \circ \underbrace{\varphi^t})(u))^T$

$f : \mathbb{R}^m \to \mathbb{R}$

$\underbrace{f \circ \varphi^t} : \mathbb{R}^m \to \mathbb{R}$

Matrices $\mathbb{R}^{m\times m}$

$\mathcal{F}(t_1, t_2)$: tangent propagator

$(\mathcal{F}(t_1, t_2))^T$: adjoint propagator

→ acts backward

→ functions change along orbits

$$((\mathcal{F}(t_1,t_2))^T)^{-1} = ((\mathcal{F}(t_1,t_2))^{-1})^T$$

$$= (\mathcal{F}(t_1,t_2))^{-T}.$$

$$\mathcal{G}(t_1,t_2) \equiv (\mathcal{F}(t_1,t_2))^{-T}$$

- adjoint tangent propagator
- forward adjoint propagator

Discrete-time

$\rightarrow u \rightarrow \varphi(u)$

$J(u) : T_u\mathbb{R}^m \rightarrow T_{\varphi(u)}\mathbb{R}^m$

$(J(u))^T : T^*_{\varphi(u)}\mathbb{R}^m \rightarrow T^*_u\mathbb{R}^m$

$(J(u))^{-1} : T_{\varphi(u)}\mathbb{R}^m \rightarrow T_u\mathbb{R}^m$

What does $(\mathcal{F}(t_1,t_2))^{-1}$ do?

$$\overrightarrow{\mathcal{F}(t_1,t_2)}\, v(t_1) = v(t_2)$$

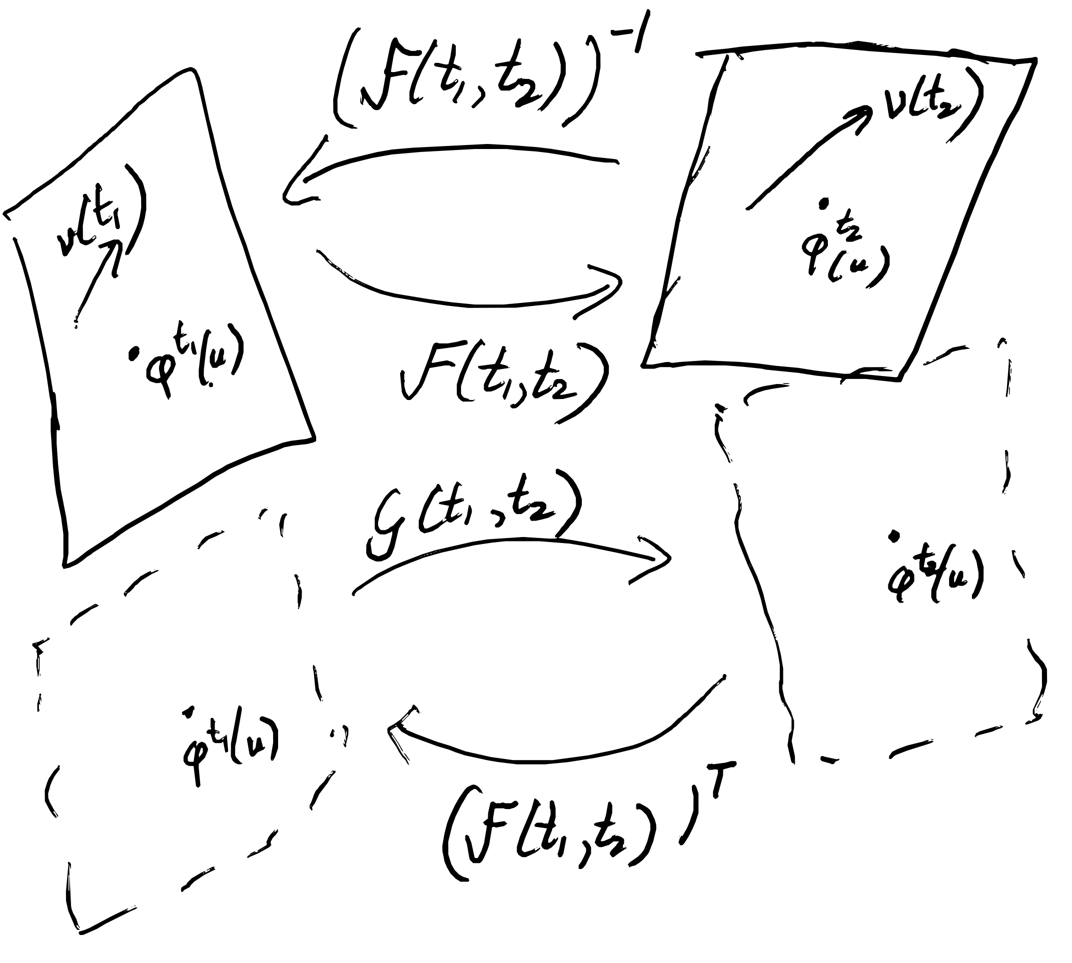
$$\rightarrow \quad v(t_1) = ((\mathcal{F}(t_1,t_2)))^{-1}\, \underline{v(t_2)}$$

$$\shortparallel$$

$$(M(t_2)(M(t_1))^{-1})^{-1}$$

$$\shortparallel$$

$$M(t_1)(M(t_2))^{-1}$$



$$\mathcal{G}(t_1,t_2) = (\mathcal{F}(t_1,t_2))^{-T}.$$

SVD

$$A = U \Sigma V^T$$

$A \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, $V^T \in \mathbb{R}^{n\times n}$

σ_i : singular values > 0

$$A v_i = \sigma_i u_i$$

Volumes

Subspaces : $v \in T_u M$

$$V_2 = \begin{bmatrix} v_1^T \\ \hline v_2^T \end{bmatrix}$$

$$\text{Vol}(V_2) = \det(V_2)$$

$v_1, v_2 \dots, v_n$

$$V_n = \begin{bmatrix} v_1^T \\ \hline \vdots \\ \hline v_n^T \end{bmatrix}$$

$\det(V_n)$
= vol of n dimensional subspace formed by $v_1, v_2 \dots, v_n$