Last time

→ Computation of top backward Lyapunov vector

Review: power iteration

Review:

→ QR iteration

Today

→ Computation of LEs

→ Computation of backward LVs

→ Covariant Lyapunov vectors

Future

→ Stochastic analysis

→ BYO dynamical system

→ Long-term / ensemble

→ stability under perturbations

Review

QR iteration (Trefethen - Bau)

A : symmetric $\qquad A \in \mathbb{R}^{d \times d}$

$Q_1 = [v_1 | v_2 | \cdots v_d]$

for $i = 1, 2, \ldots$

$$Q_i = A Q_{i-1}$$

$$Q_i R_i = Q_i \quad \text{(do QR factorization)}$$

$$B_i = Q_i^T A Q_i$$

T.B: $Q_i^T A Q_i \to$ diagonal matrix

triangular matrix

Schur form

$\boxed{QR}$: $Q R = A_0$

$A_1 = R Q$

$A_0 = A$

$Q_1 Q_2 \ldots Q_k = Q$ at k^{th} iteration of $\boxed{QR}$

$B_i = Q_i^T A Q_i$ similar to A.

→ double exponential precision

→ columns of Q_i converge to eigenvectors of A.

→ $\bar{R}_k = (R_k R_{k-1} \cdots R_1)^{1/k}$

Upper triangular matrix

$\frac{1}{k} \sum_{i=1}^{k} \log \operatorname{diag}(R_i) \to$ log of eigenvalues of A.

$F(0,t)$, $(F(0,t))^T$, $(F(0,t))^{-1}$,

$(F(0,t))^{-T}$: Cocycles.

Fix some orbit $x, \varphi x, \ldots, \varphi^t x, \ldots$

Far-future matrix / {Oseledets matrix}

$$O^+(x) = \lim_{t \to \infty} \left((F(0,t))^T (F(0,t)) \right)^{\frac{1}{2t}}$$

Far past matrix

$$O^-(x) = \lim_{t \to -\infty} \left((F(t,0))^{-T} (F(t,0))^{-1} \right)^{\frac{1}{2|t|}}$$

- Using adjoint propagators, we still get O^+ or O^-.

Log.
- Eigenvalues of $O^+(x)$:
  LEs: $\lambda_1 > \lambda_2 > \cdots > \lambda_p$ $p \le d$
  p : # of distinct LEs.

- log eigenvalues of $O^-(x)$:
  LEs : $-\lambda_p > -\lambda_{p-1} > \cdots > -\lambda_1$

→ If system is ergodic LEs are const fns of x.

→ Forward Lyapunov vectors (x)
$\equiv$ Eigenvectors of $O^+(x)$
$\equiv$ Right singular vectors of $F(0,t)$ as $t \to \infty$
$\equiv f_i^+(x)$ $i = 1, 2, \ldots, d$

→ Backward Lyapunov vectors (x)
$\equiv$ Eigenvectors of $O^-(x)$
$\equiv$ Left singular vectors of $F(0,t)$ as $t \to \infty$
$\equiv f_i^-(x)$ $i = 1, 2, \ldots d$

$$F(0,t) = \overset{\text{left}}{U(0,t)} \Sigma(0,t) \overset{\text{right}}{V(0,t)^T}$$

(dim(span of eigenvectors corresp. to λ_i) = multiplicity of λ_i
$= k_i$

$$\sum_{i=1}^{p} k_i = d$$)

$$(F(0,t))^{-1} = (V(0,t))^{-T} \Sigma^{-1}(0,t) (U(0,t))^{-1}$$

$$= V(0,t) \Sigma^{-1}(0,t) U(0,t)^T$$

$(U^T U = U U^T = I)$

Right singular vectors of $(F(0,t))^{-1}$ as $t \to \infty$?
$\equiv$ Backward Lyapunov vectors

$$(F(0,t))^T = V(0,t) \Sigma(0,t) U(0,t)^T$$

Right singular vectors of adjoint propagator as $t \to \infty$
→ backward LVs.

Basic idea of iterative algorithms

$v_0, v_1, \ldots, v_k$

$\text{span}\{A^j \{v_0, \ldots, v_k\}\}$

$= \text{span}\{q_1, \ldots, q_k\}$

$q_i = i^{th}$ top eigenvector of A.

descending order of magnitude.

$F(0,t)\ \{v_0, \ldots, v_k\}$

$\subseteq T_x M$

$\longrightarrow T_{\varphi^t_x} M$

$F(0,t)\, f_i^+(0) = q_i(0,t)\, f_i^-(t)$

also read as $\left(f_i^+(x)\right)$

i^{th} backward LV at φ^t_x.

$\underline{F(0,t)} = A(t-1)\, A(t-2) \cdots A(0)$

$A(i) = d\varphi^i(x)$

$Q_0 \in T_x M \qquad \mathbb{R}^{d \times k}$

for $i = 1, 2, \ldots$

$Q_i = A(i-1)\, Q_{i-1}$

$Q_i R_i = Q_i$

- Q_i for large $i \longrightarrow$ backward LV at φ^i_x.
- $\frac{1}{i} \sum_{j=1}^{i} \log \text{diag}(R_j) \longrightarrow$ LEs

- $(F(0,t))^{-1} =$

$\left(A(t-1)\, A(t-2) \cdots A_0\right)^{-1}$

$= A_0^{-1} A_1^{-1} \cdots (A(t-1))^{-1}$

$Q_0 \in T_{\varphi^t_x} M \qquad Q_t$: Forward LV at x.

- $(F(0,t))^T$:

adjoint LV.

$Q_0 \in T^*_{\varphi^t_x} M \qquad Q_t$: adjoint LVs at x.

- $(F(0,t))^{-T}$:

forward adjoint LVs.

$Q_0 \in T^*_x M \qquad Q_t$: forward adjoint LVs at φ^t_x.

Characteristic exponents

$$\chi(x, \underset{T_xM}{v}) = \lim_{t\to\infty} \frac{1}{t} \log \| F(0,t) v \|$$

(rate of asymptotic exponential growth/decay)

$E_i(x)$

$= \{ v : \chi(x, v) \leq \lambda_i \}$

$= \text{span} \{ f_j^+(x) : j = i, i+1, \ldots p \}$

$\lambda_1 > \lambda_2 > \ldots \lambda_p$

$E_1 = \mathbb{R}^d$

$E_{p+1} =$ empty