Oseledets theorem

Random Dynamical system

→ deterministic $\varphi : M \to M$

$\varphi^t = \varphi \circ \varphi^{t-1}$ (discrete)

$$\frac{d\varphi^t(x)}{dt} = v(\varphi^t x) \quad \text{(cont)}$$

$$\to \frac{d\varphi^t(x)}{dt} = v(t, \varphi^t x) \quad \begin{matrix}\text{deterministic}\\ \text{(nonautonomous)}\end{matrix}$$

$$\to \varphi(\underset{\text{time}}{t}, \underset{\text{"seed"}}{\omega}, x) \rightarrow \text{state}$$

$$\varphi(t, \omega, \cdot) : M \to M$$

Cocycle property

$$\varphi(t+s, \omega, \cdot) = \varphi(s, \theta(t)\omega, \cdot) \circ \varphi(t, \omega, \cdot)$$

Cocycles: derivatives of deterministic or stochastic (random) dynamics (cont or discrete)

Lyapunov exponents

Matrix cocycle: (linear random dynamics)

$$\underset{\mathbb{R}^{d\times d}}{\underset{\cap}{A(t+s, x)}} = A(s, \varphi(x))\, A(t, x) \quad \text{(cocycle property)}$$

E.g. $A(t, x) = d\varphi^t(x)$

$$= d\varphi(\varphi^{t-1}(x))\, d\varphi(\varphi^{t-2}(x)) \dots d\varphi(x)$$

(chain rule)

$d\varphi^{-1}(x)$, $(d\varphi)^T(x)$ along orbits of x.

Let $v \in \mathbb{R}^d$ $T_x M$

$$\chi(x, v) = \limsup_{t\to\infty} \frac{1}{t} \log \| A(t, x)\, v \|$$

(Characteristic exponent associated with x and vector v)

$$a_t = \frac{1}{t} \log \| A(t, x)\, v \|$$

(Rudin: lim sup)

Oseledets theorem

Assumptions:

→ $A(t, x) = A(\varphi^t x)\, A(\varphi^{t-1}(x)) \dots A(x)$ (discrete time case)

→ $\frac{1}{t} \log \| A(\varphi^t x) \| \to 0$ as $t \to \infty$

→ $\max\{0, \log\|A(x)\|\}$ is integrable wrt μ.

$$\int \max\{0, \log\|A(x)\|\}\, d\mu(x) < \infty.$$

→ The characteristic exponents are limits, i.e. for any $v \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{1}{t} \log \| A(t,x)\, v \| = \lambda(x, v) < \infty$$

→ $\lambda(x,v)$ can take only finitely many values $\lambda_1(x) > \lambda_2(x) > \dots > \lambda_{p(x)}(x)$ at μ-a.e. x

$p(x)$: # of distinct values

$\lambda_i(x)$: Lyapunov exponents

→ Correspondingly, there are subspaces of $\mathbb{R}^d$

$$\mathbb{R}^d = V_1(x) \supseteq V_2(x) \supseteq V_3(x) \dots \supseteq V_p(x) \supseteq V_{p+1}(x) = \{0\}$$

V_i is the set of all vectors v such that $\lambda(x, v) \leq \lambda_i$

$$V_{p+1}(x) = \{ v \in \mathbb{R}^d : \lambda(x, v) < \lambda_p \}$$

→ $v \in V_i(x) \setminus V_{i+1}(x)$,

$$\lambda(x, v) = \lambda_i$$

Invariant along orbits

→ $\lambda(x, v) = \lambda_i(x)$

$$\lambda_i(x) = \lambda(x, v) = \lim_{t\to\infty} \frac{1}{t} \log \| A(t, x)\, v \|$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \| A(\varphi^t x)\, A(\varphi^{t-1} x) \dots A(\varphi x) \underbrace{A(x)\, v} \|$$

$$= \lambda(\varphi x, A(x) v)$$

→ $A(x)\, V_i(x) \subseteq V_i(\varphi x)$

$v \in V_i(x)$

$$\lambda(x, v) = \lim_{t\to\infty} \frac{1}{t} \log \| A(t, x)\, v \|$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \| A(t, \varphi x)\, \omega \| = \lambda(\varphi x, A(x) v)$$

$\omega = A(x)\, v$ $\leq \lambda_i(\varphi x)$

$V_i(\varphi x)$

Simplification in the case of ergodic systems

(φ, μ) μ is ergodic

$$\Rightarrow \frac{1}{T} \sum_{t=0}^{T-1} J(\varphi^t x) \xrightarrow{T\to\infty} \langle J, \mu \rangle \quad \mathbb{E}_{x\sim\mu} J(x)$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} J(\varphi^t x) = \bar{J}(x)$$

$$\bar{J}(\varphi x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} J(\varphi^{t+1} x)$$

$$= \lim_{T\to\infty} \frac{1}{T} \left(\sum_{t=0}^{T-1} J(\varphi^t x) - J(x) \right)$$

$$= \bar{J}(x)$$

$\bar{J}$ is invariant along orbits

Any invariant function is constant $\iff$ ergodic.

Specialization of OMET to ergodic systems

(φ, μ) : μ is ergodic for φ.

$\rightarrow$ There are finitely many L.E. for μ-a.e x, and any $v \in \mathbb{R}^d$,

$$\lambda(x, v) = \lim_{t \to \infty} \frac{1}{t} \log \| A(t,x) v \|$$
$$= \{\lambda_1, \lambda_2, \dots, \lambda_p\}$$

$\rightarrow$ $V_i(\varphi x) \supseteq A(x) V_i(x)$

$$V_i(x) = \{ v \in \mathbb{R}^d : \lambda(x, v) \leq \lambda_i \}$$

$$A(1, x) = A(\varphi x) A(x) \qquad A(0, x) = A(x)$$
$$A(-1, x) = (A(1, x))^{-1}$$
$$= (A(x))^{-1} (A(\varphi x))^{-1}$$

if $A(x)$ are invertible at each x, one can define OMET for negative cocycle

$t \geq 0$

$$B(t, x) = A(-t, x)$$

$$\bar{\lambda}(x, v) = \lim_{t \to \infty} \frac{1}{t} \log \| B(t, x) v \|$$

$$= \lim_{t \to \infty} \frac{1}{t} \log \| A(-t, x) v \|$$

$$= \lim_{t \to \infty} \frac{1}{t} \log \| (A(x))^{-1} \cdots (A(\varphi^t x))^{-1} v \|$$

$$= \lim_{t \to \infty} \frac{1}{t} \log \| (A(t, x))^{-1} v \|$$

λ_i are eigenvalues of

$$\log^{\mathrm{eig}} \left(\lim_{t \to \infty} \left(A(t,x)^T A(t,x) \right)^{\frac{1}{2t}} \right)$$

Backward cocycle : same LEs.