→ NITMB Workshop (4^{th} - 8^{th})

→ HW3 (11/11, Sunday)

→ 1-page proposal (11/14) Week of 19^{th}
- Email to set up a 10-minute meeting
- Dynamical system + computation
  - → perturbations
  - → stability / control theory
  - → $\boxed{\text{Dimension reduction}}$ / feature extraction
  - → optimization (parameter or optimization dynamics)
  - → data assimilation / inverse problems
  - → Reinforcement learning ✗
- Introduction
  solution strategy
  learning outcome

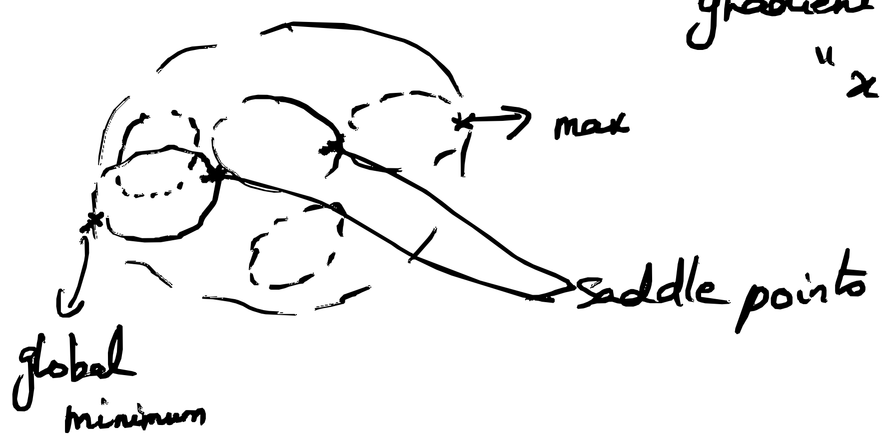
Plan: linear stability (tangent equations)
→ Lyapunov function method
- uniform
- asymptotic

Oseledets multiplicative ergodic theorem

(infinite-dim) → spectral methods for operators associated with dynamical systems

(finite-dim) → Most general linear perturbation analysis

Homework 3



gradient flow of "x" coordinate function

Recap

Theorem: $\dfrac{dF^t x}{dt} = v(F^t x)$

$v(0) = 0$

Define $x \to V(x)$ that is +ve definite

$V(0) = 0$

$\dfrac{d}{dt} V \circ F^t(x) \leq 0$ uniform stability

$= 0$ at $x = 0$

$\forall \; x \in N$.

All orbits asymptotically go to 0.
(asymp stability)

Domain of attraction : "Largest" subset of $\mathbb{R}^d$ that goes to 0 asymptotically

$\left(\{ x \in \mathbb{R}^d : \lim_{t \to \infty} F^t x = 0 \} \right)$.

How to obtain Lyapunov functions?

$$\frac{dF^t_x}{dt} = AF^t_x$$

(ODE form)

$$\frac{dx(t)}{dt} = Ax(t)$$

Ansatz : $V(x) = x^T K x$

$K \in \mathbb{R}^{d \times d}$

- $V(x) \geq 0$ iff K is SPD.

(SPD is a cone)

- $\frac{d\, V \circ F^t_x}{dt} < 0$

$\Rightarrow \nabla V(F^t_x) A(F^t_x) < 0$

$V(x) = x^T K x$

$x^T (AK + KA^T) x < 0$

Want K s.t. $AK + KA^T$

$\nabla V(x) = (K^T x + Kx)^T$

$(K^T x + Kx)^T A x$

$AK + KA^T = -C$

C is +ve definite

- "mildly nonlinear"

$$\frac{d}{dt} F^t_x = A(F^t_x) + h(F^t_x)$$

where $h(0) = 0$ and $\nabla h(0) = 0$
is stable whenever $\frac{dF^t_x}{dt} = A(F^t_x)$

<u>Proof</u> : $V(x) = x^T K x$
is a Lyapunov function

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Solving for <u>Lyapunov</u> functions

Lyapunov equation
solve K SPD

$$AK + KA^T = -C$$

$C > 0$

$K_{ij} \quad i, j = 1, 2, \dots, d$
Linear system of equations for d^2 variables
$O(d^6)$
when no solution:
- A has +ve eigenvalue
- $A = A^T$
  if $Tr(AK + KA) > 0$

Thm: if A is negative definite,
$AK + KA^T = -C$ has
a soln for any $C > 0$.

<u>Proof</u> : $K = -\int_0^\infty e^{tA^T} C e^{tA} dt$

$\left(e^{tA} = I + tA + \frac{t^2 A^2}{2} + \cdots \right.$

$\left. e^{tA^T} = (e^{tA})^T \right)$

$$\frac{d}{dt} e^{tA} = t e^{tA}$$

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classical approach (matrix computation)
- K is symmetric so $O(\frac{d^2}{2})$ variables

$$AK + KA^T = -C$$

- Sylvester equation:
  Solve for K in $AK + KB = C$

- $\underbrace{Q^T A Q = U}$ Schur decomposition
  $U^T = Q^T A^T Q$ U is upper triangular/ hessenberg
  $(Q^{-1} = Q^T)$ Q is unitary.
- let $K = QXQ^T$ for some X

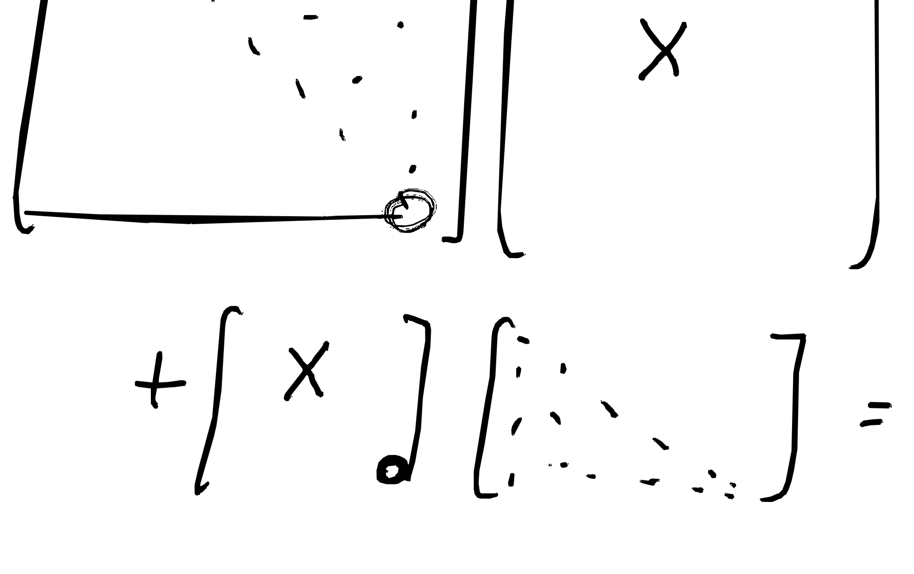
$AK + KA^T = -C$

$QU\underbrace{Q^TQ}XQ^T + QX\underbrace{Q^TQ}U^TQ^T$

$= QUXQ^T + QXU^TQ^T = -C$

$\Rightarrow UX + XU^T = -Q^TCQ$

Solve for X.



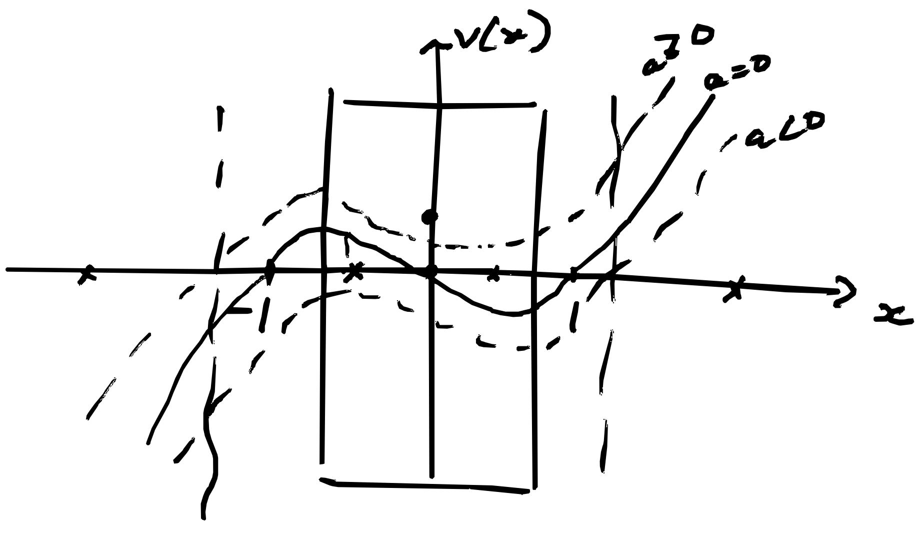
$O((d^2)^2)$ algorithm.

Example

(Source: Underactuated robotics Tedrake)

$-c \leq a \leq c$

$$\frac{d\,x(t)}{dt} = -x(t) + (x(t))^3 + a$$



- $V(x) = \frac{x^2}{2} \geq 0$

- $\frac{d}{dt} V(x(t)) = x(-x + x^3 + a)$

$$= -x^2 + x^4 + ax$$

$$< 0.$$

- quadratic : good ansatz

- Pessimistic estimate / "inner" estimate of region of attraction can be found from defn of Lyapunov function

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Barrier function

$$\frac{d\, B \circ F^t(x)}{dt} < 0$$

$B(\boxed{\text{bad } x}) = 1$ e.g. robot falls down, self-driving car brakes suddenly

$B(0) = 0$

Then, if x is such that $B(x) < 0$

$F^t x \neq$ bad for any t.

$\{x : B(x) = 0\}$: barrier

Remark: Proof of Lyap function method, sub-levels sets of V are invariant.

$\{x : V(x) \leq \varepsilon\}$ is invariant

Applications of finding Lyapunov function

- is 0 asymptotically stable?
- "inner" region of attraction
- Barrier functions to avoid bad regions.

Sum-of-squares optimization

Polynomial ansatz:

$$V(x) = a_0 + a_1 x^{(1)} + a_2 x^{(2)} + \cdots + a_d x^{(1)^2} + \cdots$$

For poly. of degree n, solve for coefficients

$$c = (a_0, a_1, \ldots\ldots)$$

$V(x)$ is a sum of squares

$$\min_{c \in \mathbb{R}^D} \quad V(x) \text{ is SOS}$$

$$\text{and } -\frac{d}{dt} V \circ F^t(x) \text{ is SOS}$$

Convex optimization : semi-definite programming