Recap

HW1: • $X_1, X_2, \ldots$

$X_1 = f_{1,2}(Y_i)$ ← sentences

↑ embeddings

• SVD, choose top k left/right singular vectors

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Arzela-Ascoli: Infinite sequences of Equicontinuous families (subsets of continuous functions) have convergent subsequences (in supremum norm)

• Hölder, differentiable : σ-compact on Contin. fns.

• RVs: functions from $\Omega \Rightarrow$ metric space

• pushforward distributions

• Contraction maps

Contraction maps

Theorem (Banach fixed point theorem) A contraction map on a complete metric space has a unique fixed point, which is the limit of any arbitrary orbit of the map.

(M, d) is a complete metric space.
$F : M \circlearrowleft$ is a contraction map if $\exists\ \lambda \in (0,1)$ s.t. $\forall\ x, y \in M$,

$$d(F(x), F(y)) \leq \lambda\, d(x, y)$$

- "Contractive"

$$d(F(x), F(y)) < d(x, y)$$

Proof:

$m > n$

$$d(F^m(x), F^n(x))$$
$$\leq \sum_{k=0}^{m-n-1} d(F^{n+k+1}(x), F^{n+k}(x))$$

$$\leq d(F^m(x), F^{n+1}(x)) + d(F^{n+1}(x), F^n(x))$$
$$\leq d(F^m(x), F^{n+2}(x)) + d(F^{n+2}(x), F^{n+1}(x)) + d(F^{n+1}(x), F^n(x))$$

$$d(F^m(x), F^n(x)) \leq \sum_{k=0}^{m-n-1} \lambda^{n+k} d(F(x), x)$$
$$\leq \lambda^n / {1-\lambda}\ d(F(x), x) < \varepsilon$$

$$\Big(d(F(F(x)), F(x)) = d(F^2(x), F(x)) \leq \lambda\, d(F(x), x) \Big)$$

$$d(F^{n+1}(x), F^n(x))$$
$$\leq d(F(F^n(x)), F(F^{n-1}(x)))$$
$$\leq \lambda\, d(F^n(x), F^{n-1}(x))$$
$$\leq \dots \leq \lambda^n d(F(x), x)$$

For every $\varepsilon > 0$, $\exists$ N
s.t. $\forall$ $m, n \geq N$,

$$d(F^m(x), F^n(x)) < \varepsilon.$$

$\{F^n(x)\}$ is Cauchy $\rightarrow$ converges.
($\because$ M is complete)

$x \rightarrow x^*$
$y \rightarrow y^*$
$d(x^*, y^*)$

$$\rightarrow d(F^m(x), F^n(x)) \leq c\lambda^n d(F(x), x)$$
$$d(x^*, F^n(x)) \leq c\lambda^n d(F(x), x)$$

$x, F(x), F^2(x), \ldots$
$x_0 \quad x_1, \ldots$

$$\underbrace{F(x^*) = \lim_{n\to\infty} F^n(x)}_{\text{limit exists because } f \text{ is continuous}} = \lim_{n\to\infty} x_n = x^*$$

$$\rightarrow d(F(x^*), x^*)$$
$$\leq d(F(x^*), F(F^{n-1}x)) + d(F^n(x), x^*)$$
$$< \lambda d(x^*, F^{n-1}(x)) + c\lambda^{n-1} d(F(x), x)$$
$\rightarrow 0$ as $n \rightarrow \infty$.

x^*, y^*

$$d(F(x^*), F(y^*)) \leq \lambda d(x^*, y^*)$$
$$d(x^*, y^*) \leq \lambda d(x^*, y^*)$$

$$x^* = y^*$$

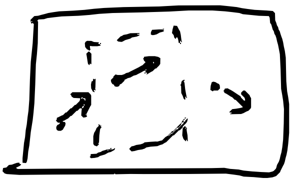
! fixed point

→ Inverse function theorem (Application of Contraction mapping principle)

* $F : X \to Y$

$dF : TX \to TY$ (linear map)

TX: Tangent bundle of X



(A) $v : X \to TX$

vector field

$v(x) : C^\infty(X) \to \mathbb{R}$

$$v(f)(x) = \lim_{\varepsilon \to 0} \frac{f(\underline{x} + \underbrace{\varepsilon v(x)}) - f(x)}{\varepsilon} \qquad \exp_v(x)$$

v: vector field (A)

directional derivative

dF represented in coordinates

$$\underset{m \times n}{(dF_x)_{ij}} = \langle \underbrace{dF_x b_i^n}_{dF(x)}, \underset{T_{F(x)}Y}{b_j^m} \rangle$$

$$F : \underset{n}{X} \to \underset{m}{Y}$$

$$\operatorname{span}\{b_j^m\} = T_{F(x)}Y$$

$$\operatorname{span}\{b_i^n\} = T_x X$$

where

$$dF_x b_i^n := b_i^n(F)(x)$$

— Inverse function theorem

"A continuously differentiable map can be approximated locally using its differential"

$F: M \to M \subseteq \mathbb{R}^d$ F is continuously differentiable at $a \in M$ (i.e. $dF(x)$ exists and $x \to dF(x)$ is continuous at a) and $dF(a)$ is invertible, then, $\exists$ G s.t.

$G(F(x)) = x$ for $x \in U \subset M$ containing a

[G is continuously differentiable in a neighborhood of $F(a)$]

↳ (needs $F \in C^1(U)$, $U \subseteq M$ containing a)

Proof: WLOG; assume $F(a) = 0 \in \mathbb{R}^d$.
(for any function H, take $F(x) = H(x) - H(a)$)

Want to prove invertibility of F in a ball around origin in $\mathbb{R}^d$. Take $y \in B(0, \delta) \supset F(U)$

$$\phi_y(x) = x + (dF(a))^{-1}(y - F(x))$$

Prove ϕ_y is a contraction on M.

(To be continued next time)

$\phi(x) = x + h(x)$
with $\|h'\| < \delta_1$

$\phi^{-1}(x) = x + g(x)$
$\|g'\| < \delta_2$

$h(x) = d\phi(y)(y - x)$

→ Conjugacy: Find h s.t.
$g \circ h = h \circ F$

$F: X \to X$ $\quad g: Y \to Y$
$h: X \to Y$

→ Compactness allows contractive maps to have unique fixed pts
$d(F(x), F(y)) < d(x, y)$