* Presentations 10 minutes
  Sign-up sheet

* HW4 or HW5

Ergodic theory

$F^t : M \to M$

transformations on probability spaces

$(M, \Sigma, \mathbb{P}, F)$

→ Krylov-Bogolubov theorem:
If F is continuous and M is compact, there exists at least 1 invariant measure

Recall: μ is an invariant measure for F if for any Borel set A, $\mu \circ F^{-1}(A) = \mu(A)$.

Proof:

$\{ f_1, f_2, \dots, f_n \}$ sequence in $C(M)$.

$$\frac{1}{n} \sum_{k=0}^{n-1} f_i \circ F^k(x) =: a_n^{(i)}$$

We can find $\{ a_{n_k}^{(i)} \}$ that converges for every i.

$$\begin{array}{lll} a_1^{(1)} & a_2^{(1)} & \dots \\ a_1^{(2)} & a_2^{(2)} & \dots \end{array}$$

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} f_i \circ F^j(x)$$

for any $f \in C(M)$, s.t. $\| f - f_n \| < \frac{\varepsilon}{2}$

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} f \circ F^j(x) = a_{n_k}^{(i)} + \left(\frac{1}{n_k} \sum_{j=0}^{n_k-1} f \circ F^j(x) - \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_i \circ F^j(x) \right)$$

So, $\lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f \circ F^j(x)$ exists

$$\mathcal{L}(f) = \lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n-1} f \circ F^j(x)$$

Riesz Representation:

$\exists\ \mu_x,$

$$\mathcal{L}(f) = \int f \, d\mu_x = \langle f, \mu_x \rangle$$

$$\mathcal{L}(f \circ F) = \mathcal{L}(f)$$

$$\mu_x = \mu_{F(x)}$$

Thm: There exists at least one invariant, ergodic measure (when M is compact & F is continuous on M).

Recall: μ is an ergodic measure for F if any F-invariant set A has $\mu(A) = 0$ or 1.

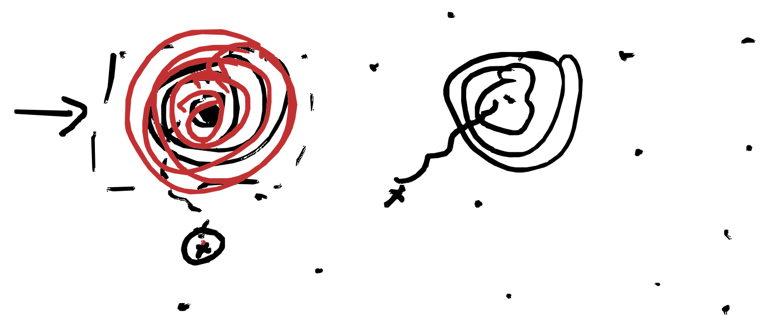
The set of invariant ergodic measures is convex and ergodic measures are extremal points

Birkhoff's ergodic theorem:
For any $f \in L^1(\mu)$,
$$\frac{1}{n}\sum_{i=0}^{n-1} f \circ F^i(x) \xrightarrow{n\to\infty} \bar{f}(x) \in L^1(\mu)$$
for μ - almost every x.

Remark:

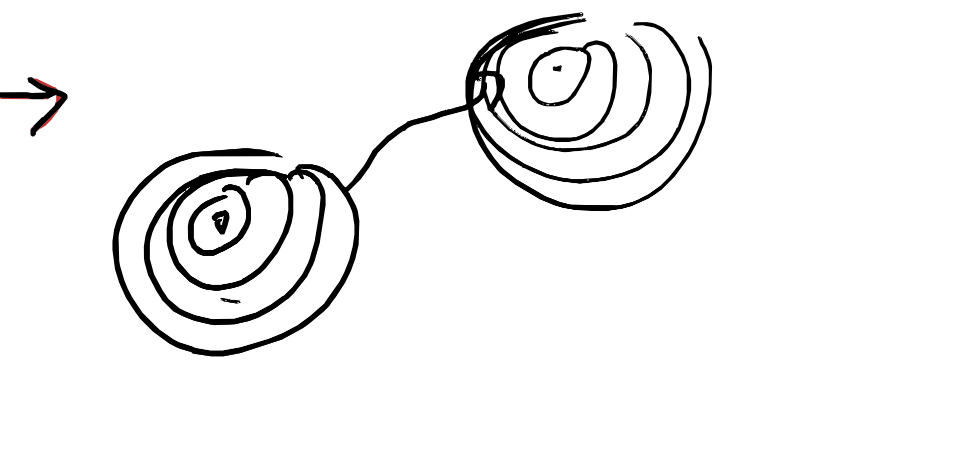
- $\bar{f}$ is an F-invariant function
$$\bar{f}\circ F^{-1} = \bar{f} = \bar{f}\circ F$$
- $\bar{f}$ is constant along μ-almost every orbit
- Lemma: If (F,μ) is ergodic and f is F-invariant, then f is constant μ-a.e.
$$\langle f, \mu\rangle \overset{F\text{-invariance of } F}{=} \langle f\circ F, \mu\rangle \quad \text{change of measure}$$
$$= \langle f, F^{-1}_{\#}\mu\rangle \quad \text{invariance of } \mu$$
$$= \langle f, \mu\rangle$$

Birkhoff's ET:
$$\bar{f}(x) := \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f\circ F^j(x)$$
is independent of x μ-a.e.



$$\text{Unif}\{x_0, x_1, \dots\, x_n\} \xrightarrow{n\to\infty}$$

$$\text{Unif}\{y_0, y_1, \dots\, y_n\}$$



Ergodic decomposition theorem:
If μ is an invariant measure, then $\exists$ $M = \{M_\alpha\}_\alpha$ with $(M_\alpha, F, \mu_\alpha)$ being ergodic such that
$$\langle f, \mu\rangle = \int f \, d\mu$$
$$= \iint f \, d\mu_\alpha \, d\alpha$$
(Total probability)

(M_α, μ_α)

+ countable number of "atoms"

Lebesgue space.

Kingman subadditive ergodic theorem

Random dynamical system

(M, Σ, P, F)

$f_1, f_2, \ldots f_n \ldots$

Subadditive :

$$f_{m+n}(x) \leq f_m(x) + f_n(F_x^m)$$

then,

- $\lim_{n\to\infty} \frac{1}{n} f_n(x) = \bar{f}(x) \in L^1$

- $\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{x\sim P} f_n(x) = \alpha$

$$f_n(x) = \sum_{i=0}^{n-1} g(F_x^i)$$

Furstenberg–Kesten

Oseledets Multiplicative Ergodic Theorem

(M, Σ, P, F)

$$A_i = A(F^i_x) \quad \text{generator}$$

$$\Phi(n) = A_{n-1} A_{n-2} \cdots A_0$$

$$= A(F^{n-1}_x) \cdots A(F(x)) A(x)$$

$\Phi(x,n)$ is regular if

- $$\lim_{n\to\infty} \frac{1}{n} \log |\det \Phi(x,n)| = \sum_{i=1}^{p} k_i \lambda_i(x)$$

λ_i are Lyapunov exponents

k_i multiplicity of λ_i

For any k-dimensional subspace $L_k \subseteq \mathbb{R}^d$

$$\lim_{n\to\infty} \frac{1}{n} \log \det | L_k \to \Phi(x,n) L_k |$$

exists (OMET)

for x μ-a.e.

For any $v \in R^d$

$$\lim_{n\to\infty} \frac{1}{n} \log \| \Phi(x,n) v \| = \lambda(x, v)$$

(Lyapunov exponent)

$$\lambda_1(x) > \lambda_2(x) \ldots > \lambda_p(x)$$

$$V_i(x) = \{ v \in \mathbb{R}^d : \lambda(x, v) \leq \lambda_i(x) \}$$

$$\{0\} \subset V_p(x) \ldots V_2(x) \subseteq V_1(x) = \mathbb{R}^d$$

For any $v \in V_i(x) \setminus V_{i+1}(x)$

$$\lambda(x, v) = \lambda_i(x)$$

(F, μ) is ergodic

$$\lambda(F(x), v) = \lambda(x, v)$$

i.e. $x \to \lambda(x, v)$ is constant μ a.e.

(see lyapunov_exponents.py)

$$\frac{1}{n} \log \| \underline{\Phi}(n) v \|$$

$$\Phi(n) = A_{n-1} A_{n-2} \cdots A_0 .$$

$$\frac{1}{n} \log \| A_{n-1} \cdots A_0 v \|$$

$$= \frac{1}{n} \log \left(\frac{\| A_{n-1} \cdots A_0 v \|}{\| A_{n-2} \cdots A_0 v \|} \right) + \log \left(\frac{\| A_{n-2} \cdots A_0 v \|}{\| A_{n-3} \cdots A_0 v \|} \right) + \cdots + \log \frac{\| A_0 v \|}{\| v \|}$$

<u>OMET</u>: Corollary

$$\lim_{n \to \infty} \left(\underline{\Phi}^T(x, n) \underline{\Phi}(x, n) \right)^{1/2n} .$$

$$= \begin{bmatrix} e^{\lambda_1(x)} & & \\ & \ddots & \\ & & e^{\lambda_p(x)} \end{bmatrix}$$