HW2 : Contraction maps
Due in 10 days

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Recap

Applications of Contraction mapping principle / Banach Fixed point theorem

→ Inverse function theorem
→ Conjugacies

$F : M \to M$

$G : M \to M$ is near F

in $\|F - G\|_\infty < \varepsilon$

F is "near" G if $\exists\ h : M \to M$ which is close to the identity and is invertible s.t. $\underline{F} = h^{-1} \circ \underline{G} \circ h$

Why contraction?

$h = Id + v$

. Write $\phi : B_V(0, \delta) \to B_V(0, \delta)$

(δ-ball around C^0 vector fields on M)

and show that ϕ is a contraction

- $\{x_n\}$: orbit of F

$x_0, x_1, x_2 \ldots$ $x_n = F(x_{n-1})$

$y_0, y_1, \ldots$ $y_n = G(y_{n-1})$

$$\underline{h^{-1} \circ F \circ h}(x_n) = F(x_n + v(x_n)) - v(x_{n+1}) = G(x_n)$$

- $x_0, x_1, \ldots\ldots x_T$ (e.g. option prices you observe)

$$\| G(x_t) - x_{t+1} \| < \varepsilon \quad \forall\ t \in [1, T]$$

"finite-time orbit equivalence"

ϕ contraction defined on $\{T_{x_0}M, T_{x_1}M, \ldots, T_{x_T}M\}$

Linear systems

Map	Flow
$x_{t+1} = F x_t$	$\frac{dx}{dt} = Ax$
$F : \mathbb{R}^d \to \mathbb{R}^d$	$A : \mathbb{R}^d \to \mathbb{R}^d$
	$\frac{d F^t x}{dt} = A(F^t x)$
$F^t x = F \dots F x$	$F^t x = e^{tA} x$
	$e^{tA} = Id + tA + \frac{t^2 A^2}{2!} + \dots$

F = time-1 map of the flow $e^{tA} x$

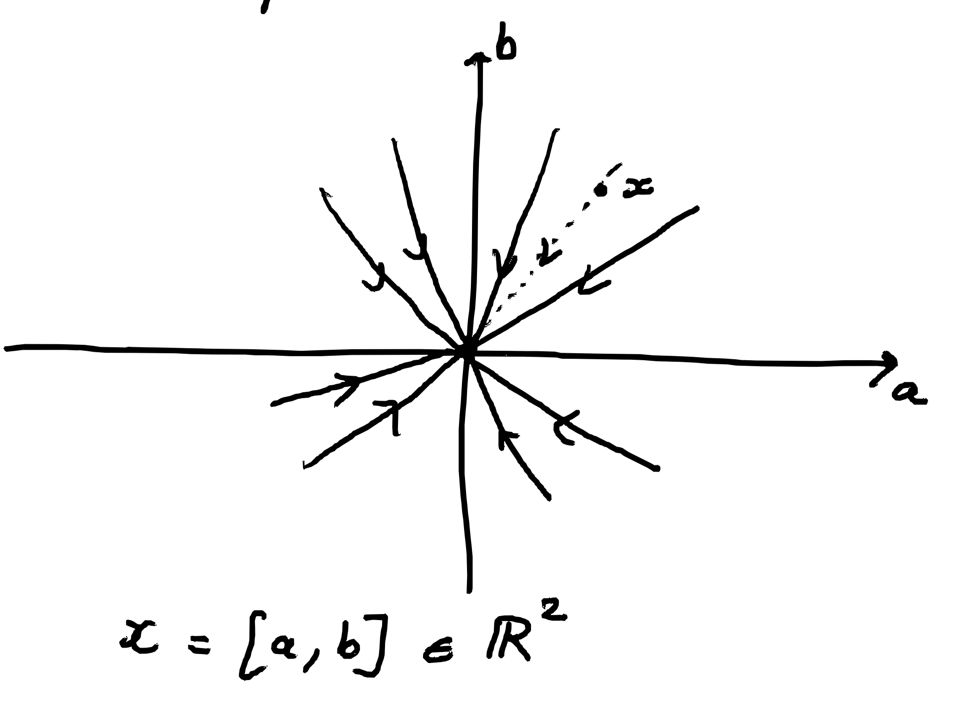
$$F = e^A$$

Map | Flow
- $F = 0$ $\leftarrow$ Identity flow

ID : $x(t) = e^{tA} x(0)$ (nonautonomous)

Strogatz

Phase portrait



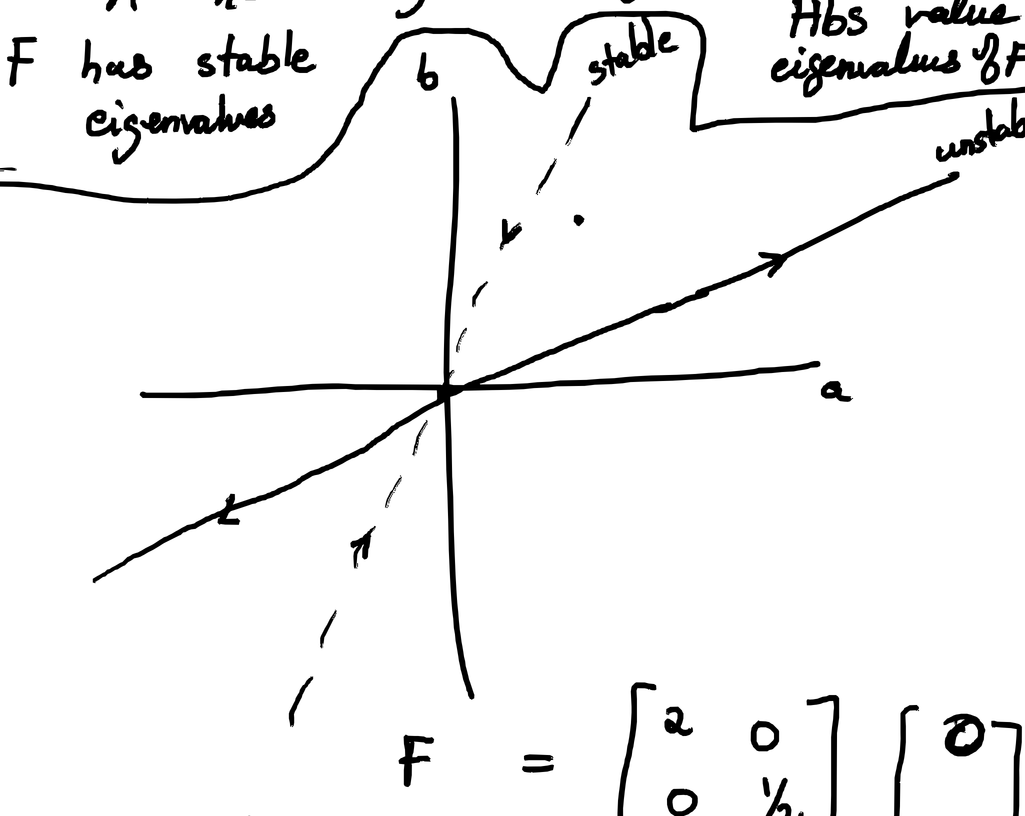
$x = [a, b] \in \mathbb{R}^2$

$$\frac{dF^t x}{dt} = 0 \quad (0 \text{ is a fixed point})$$

A has negative eigenvalues

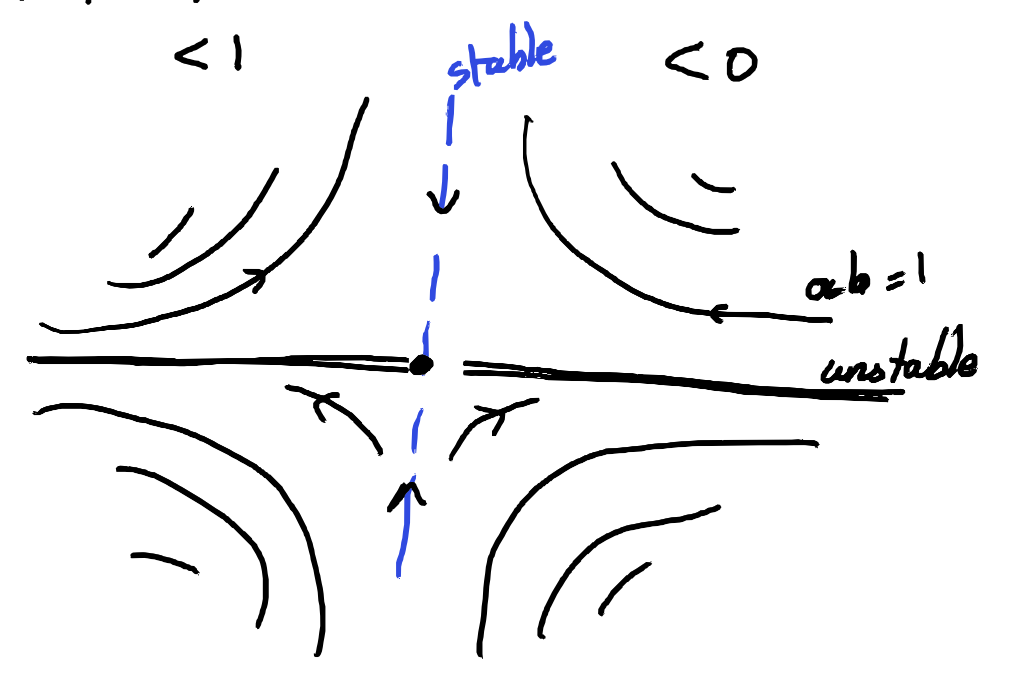
F has stable eigenvalues

Abs value of eigenvalues of $F < 1$



$$F = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}$$

	Eigs		eigs
F :	> 1	A	> 0
	< 1		< 0



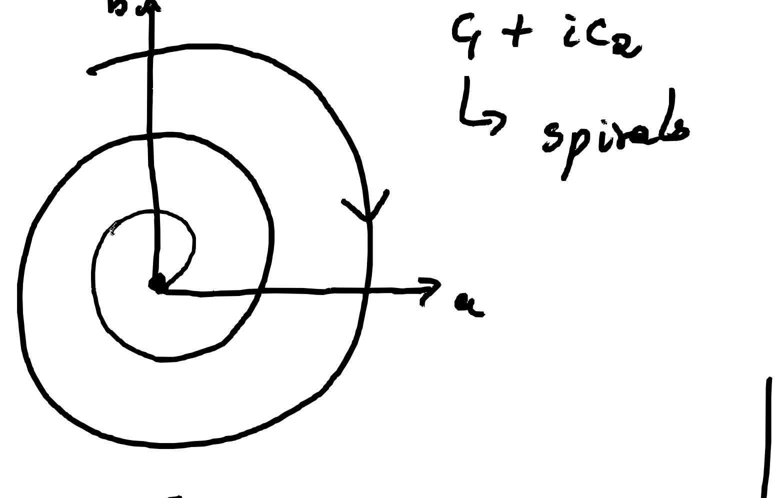
$$F v = v_1 F e_1 + v_2 F e_2$$

$$A = \begin{bmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{bmatrix} \quad \begin{matrix} \lambda_u > 0 \\ \lambda_s < 0 \end{matrix}$$

Invariant curves : $\frac{a^{\lambda_s}}{b^{\lambda_u}} = \text{const}$

$$\frac{d}{dt} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\frac{d}{dt} \left(\frac{a^{\lambda_s}}{b^{\lambda_u}} \right) = 0$$

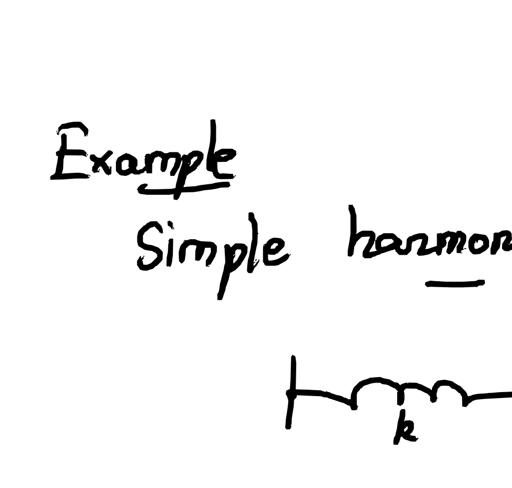


Numerically simulate orbits

A has 2 negative eigs.
O : stable (node)

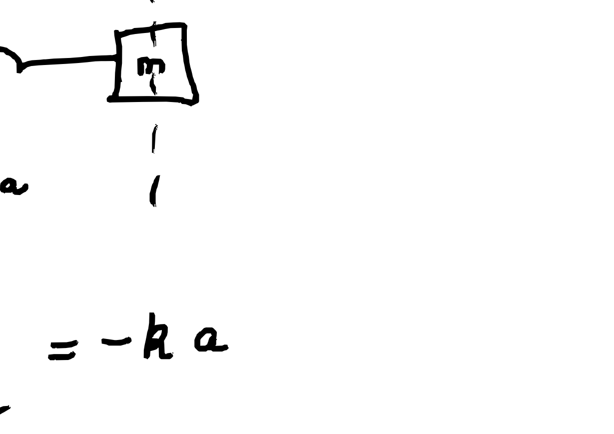
A has 1 +ve 1 -ve eig
O: saddle point

A has complex eigenvalues
$c_1 + i c_2$
↳ spirals $c_1 \neq 0$

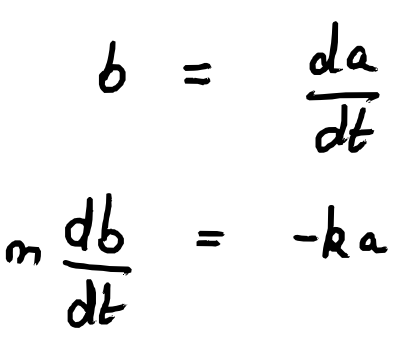


$c_1 < 0$

$c_1 = 0$



Example
Simple harmonic oscillator

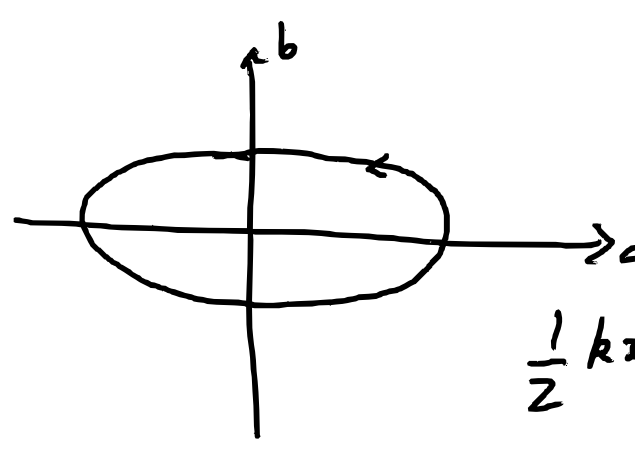


$$m \frac{d^2 a}{dt^2} = -k a$$

$$b = \frac{da}{dt}$$

$$m \frac{db}{dt} = -k a$$

$$\frac{d}{dt} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$



$$\frac{1}{2} k a^2 + \frac{1}{2} m b^2 = \text{const}$$

Linear dynamics in $\mathbb{R}^d$

$$x_{t+1} = F x_t \qquad (F = e^A)$$

Define $\lambda \in (0,1)$ $\quad c > 0$

stable subspace

$$E^s = \{ v \in \mathbb{R}^d : \|F^t v\| \leq c\lambda^t \|v\| \ \forall t \in \mathbb{Z}^+ \}$$

unstable subspace

$$E^u = \{ v \in \mathbb{R}^d : \|F^t v\| \leq c\lambda^{|t|} \|v\| \ \forall t \in \mathbb{Z}^- \}$$

neutral / center subspace

$$E^c \oplus E^s = E^{cs} = \{ v \in \mathbb{R}^d : \text{sub-expon. growth/decay} \}$$

For any $v \in E^c$,

$$\lim_{t\to\infty} \frac{\|F^t v\|}{\|v\| \, \|t^k\|} = c$$

for some k

$$\mathbb{R}^d = E^s \oplus E^u \oplus E^c$$

Alternative characterization of (un)stable subspaces

$$E^s = \{ v \in \mathbb{R}^d : v \in \ker(F - \lambda I)^m \text{ for some } |\lambda| < 1, \ m \in \mathbb{N} \}$$

$$E^u = \{ v \in \mathbb{R}^d : v \in \ker(F - \lambda I)^m, \ |\lambda| > 1 \}$$

$$E^c = \{ v \in \mathbb{R}^d : v \in \ker(F - \lambda I)^m, \ |\lambda| = 1 \}$$

$$E^c = \{ 0 \in \mathbb{R}^d \} \quad : \quad F \text{ is linear hyperbolic}$$

→ F is not diagonalizable

- eigenvectors are not linearly dependent
- algebraic multiplicity > geometric multiplicity for some eigenvalue

(Eigenvalues are continuous functions on matrices)

Ex: how small are the set of defective matrices?

→ Jordan canonical form

$\ker((F - \lambda I)^m)$ for some $m \in \mathbb{N}$.

Generalized eigenvectors

$$J = P^{-1} F P$$

$$\begin{bmatrix} \lambda & & \\ & \mu & 1 \\ & & \mu \end{bmatrix}$$

$$x(t) = t^k x(0)$$

$$x(t+1) = (t+1)^k x(0)$$

$$= \frac{(t+1)^k}{t^k} x(t)$$