HW2: Due 27th

→ Rotations/ Translations on $\mathbb{T}^n$ and S^1.

→ Topological transitivity: $\exists$ a dense orbit
Minimality: all orbits are dense

→ M $\quad O_x = \{x, F(x), F^2(x), \dots\}$

O_x is dense on M $\Leftrightarrow$ every point on M is in O_x or is arbitrarily close to a point in O_x.

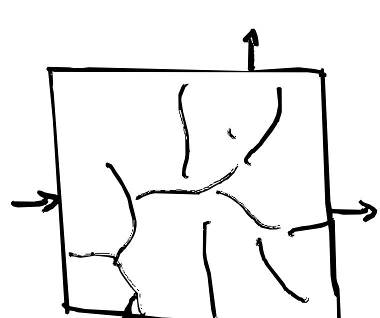
→ $x = (a_1, a_2, \dots, a_d)$

$$F(x) = \begin{bmatrix} a_1 + \theta_1 \\ a_2 + \theta_2 \\ \vdots \\ a_d + \theta_d \end{bmatrix} \bmod 1$$

Side note: $\mathbb{T}^d \sim [0, 2\pi]^d$ (periodic boundary conditions)

$0 \approx 2\pi$.

$F : [0, 2\pi]^d \to [0, 2\pi]^d$ by taking mod 2π



PACMAN

$\mathbb{T}^2 \approx [0,1]^2$ or $\mathbb{R}^2/\mathbb{Z}^2$

→ Irrational linearly independent rotations:
$\theta_1, \theta_2, \dots, \theta_d$

For any $k_1, k_2, \dots, k_d \in \mathbb{Z}$,
$\sum_{i=1}^{d} k_i \theta_i \notin \mathbb{Z}$ except when all $k_i = 0$.

(Rationals θ: periodicity)

<u>Theorem</u>: (KH)
→ Irrational linearly independent rotations on torus are minimal

<u>Proposition</u> For any pair of open sets $U, V \subset M$, there exists some $N(U,V) \in \mathbb{N}$ s.t. $F^N(U) \cap V \neq \emptyset$ iff F is topologically transitive.

Lemma. there are no disjoint open sets that are F-invariant

<u>Corollary</u>:
there is no non-constant continuous function on M that is F-invariant, if F is topologically transitive.
$(f = f \circ F)$

(A is an F-invariant set if $f^{-1}(A) = A$)

Construct $x = (a_1, \dots a_d)^T \in M$

$$f(x) = \sin\left(2\pi \sum_{i=1}^{d} k_i a_i\right)$$

$$f \circ F(x) = \sin\left(2\pi \sum_{k=1}^{d} k_i (a_i + \theta_i)\right)$$

$$= \sin\left(2\pi \sum_{i=1}^{d} k_i a_i + \sum_{i=1}^{d} 2\pi \theta_i k_i\right)$$

$$= \sin\left(2\pi \sum_{i=1}^{d} k_i a_i\right)$$

(if $\sum k_i \theta_i$ is an integer)

$$= f(x)$$

f is invariant
f is non-constant
f is continuous.
$\Rightarrow F$ is not topologically transitive.

Proposition: Let F be a continuous open map on compact M.

For any pair of open sets U, V, $\exists$ some $N(U,V) \in \mathbb{N}$ s.t.

$$F^N(U) \cap V \neq \emptyset$$

$$\iff \text{iff}$$

F is topologically transitive on M

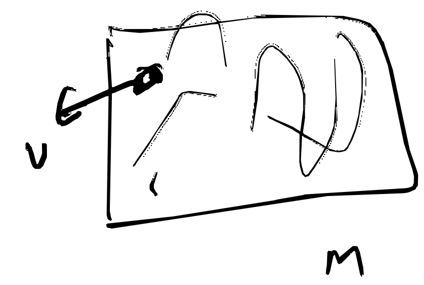
Proof:

$\exists$ a dense orbit.

Want: U, V, $\exists$ N s.t. $F^N(U) \cap V \neq \emptyset$.

Let O_x be dense on M.

For any two sets U, V, there is some N_U, N_V s.t. $x_{N_U} \in U$ and $x_{N_V} \in V$.



WLOG, $N_U < N_V$

$$F^{N_V - N_U}(x_{N_U}) \subset V$$

$$\Rightarrow F^{N_V - N_U}(U) \cap V \neq \emptyset.$$

Given: For any U, V, $\exists$ N s.t. $F^N(U) \cap V \neq \emptyset$

Want: $\exists$ dense orbit

$U_1, U_2, \ldots$ be a countable cover of M.

WKT for some N_1, $F^{N_1}(U_1) \cap U_2 \neq \emptyset$.

Let E_1 be an open set such that $\bar{E}_1 \subset U_1 \cap F^{-N_1}(U_2)$.

For some N_2, $F^{N_2}(E_1) \cap U_3 \neq \emptyset$

E_2 s.t. $\bar{E}_2 \subset E_1 \cap F^{-N_2}(U_3)$

$\vdots$

$E = \bigcap_{i=1}^{\infty} \bar{E}_i$ is non-empty

($E_{i+1} \subseteq E_i$; nested sequence of compact sets)

$x \in E$, $F^{N_1}(x) \in U_2$, $F^{N_2}(x) \in U_3$

(so, $x \in U_1$) $\ldots$

O_x is dense.

Linear perturbation analysis

Local behavior / behavior under perturbations

$F : M \to M$

differential / Jacobian of F : dF

linear maps on TM

$dF : TM \to TM$

- $\underset{d \times d}{dF(x)_{ij}} = \partial_j F_i(x)$

- $dF(x)\, v(x) \in T_{F(x)} M$

  $v(x) \in T_x M \equiv \mathbb{R}^d$

  $$= \lim_{\varepsilon \to 0} \frac{F(x + \varepsilon v(x)) - F(x)}{\varepsilon}$$

- $dF : TM \to TM$

  $(dF v)(x) \in T_x M$

  $$= \lim_{\varepsilon \to 0} \frac{F(F^{-1}x + \varepsilon \underbrace{v(F^{-1}x)}_{T_{F^{-1}x} M}) - F(F^{-1}x)}{\varepsilon}$$

Lyapunov function method

(for perturbations of autonomous systems)

Before that:

- $$\frac{dF^t(x)}{dt} = v(F^t x)$$ $\qquad$ $\underset{\substack{\text{d×d} \\ \text{matrix}}}{F}(x_t) = x_{t+1}$

$$\frac{dF^t(x)}{dt} = \underset{\substack{\text{d×d} \\ \text{matrix}}}{v}\, F^t(x)$$

if eig(v) are all negative, 0 fixed pt is stable.

if v has any positive eigen value, 0 is unstable.

At fixed point x^*, stability of map $F(x_t) = x_{t+1}$ is determined by the eigenvalues of $dF(x^*)$

- if there is an eigenvalue of $dF(x^*)$ of norm > 1, x^* is unstable

Linear dynamics of infinitesimal linear perturbations

$x \to v(x)$ (vector field) $\qquad v_t = v(x_t) \in T_{x_t}M$

$$F(x_t + \varepsilon v_t) = F(x_t) + \varepsilon\, dF(x_t) v_t + O(\varepsilon^2)$$

$$\lim_{\varepsilon \to 0} \frac{F(x_t + \varepsilon v_t) - F(x_t)}{\varepsilon} = dF(x_t) v_t$$

$$\Downarrow$$

$$(dF\, v)(x_{t+1}) \in T_{x_{t+1}}M$$

dynamics on $\mathbb{R}^d$:

$$G_t(v) = dF(x_t)\, v$$

non-autonomous linear

Infinitesimal
Linear perturbation evolve along orbits of flows

$$\frac{dF^t(x)}{dt} = \omega(F^t(x))$$

$$\frac{d}{dt} F^t(x + \varepsilon v(F^t_x)) = F^t(x) + \varepsilon\, d\omega(F^t_x)$$

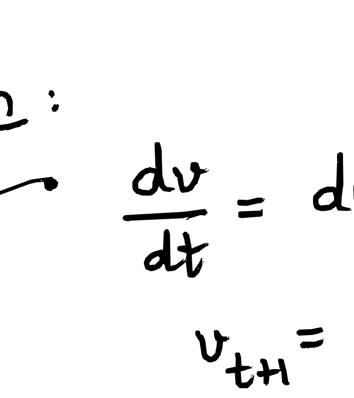
$$\omega(F^t(x)) + \varepsilon v(F^t_x)) = \omega(F^t_x) + \varepsilon\, d\omega(F^t(x))\, v(F^t_x) + O(\varepsilon^2)$$

$$\rightarrow \frac{dv(F^t_x)}{dt} = \underbrace{d\omega(F^t(x))}_{\text{non-autonomous linear dynamical}} v(F^t_x)$$

Fixed points

Maps	flows
$x_{t+1} = F(x_t)$	$\frac{dF^t_x}{dt} = \omega(F^t_x)$
$v_{t+1} = dF(x_t)\, v_t$	$\frac{dv(F^t_x)}{dt} = d\omega(F^t_x)\, v(F^t_x)$
$x_t = x^*$ $\quad F(x_t) = x^* = x_{t+1}$	$F^t_x = x^* \ \forall t.$
$\underbrace{v_{t+1} = dF(x^*)\, v_t}_{\text{autonomous linear dynamics}}$	$\frac{dv}{dt} = d\omega(x^*)\, v$

Evolution of infinitesimal "linear" perturbation



$\mathbb{R}^d$

Lyapunov function
→ nonautonomous as well as autonomous

→ stability of fixed points

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Plan :

$$\left.\begin{aligned} \frac{dv}{dt} &= d\omega(x_t)\, v \\ v_{t+1} &= dF(x_t)\, v_t \end{aligned}\right] \text{Linear non-autonomous systems} \rightarrow \text{Oseledets theorem}$$

Infinitesimal linear perturbations evolve along orbits

• Stability around fixed points] → Control theory