HW1 : Oct 15

Recap

→ Flows : group of self-maps on M

→ Autonomous

$$F^t = \overset{\text{time-independent}}{F} \circ F \circ \cdots \circ F$$

$$\frac{dF^t(x)}{dt} = v(F^t(x))$$

→ Non autonomous : control / random

$$F_t(x) = x + u_t(x)$$

$$F^t = F_t \cdots \circ F_1 \circ F_0$$

Examples Time-varying PDE solutions

- $\dfrac{\partial u}{\partial t} = \mathcal{N}(u, x)$ $\quad (\underset{\mathbb{R}^3}{x}, t) \to u(x,t) \in \mathbb{R}$

→ $u(x,t) = \sum_k \underline{a_k(t)}\, \phi_k(x)$ (Spectral / Finite-element)

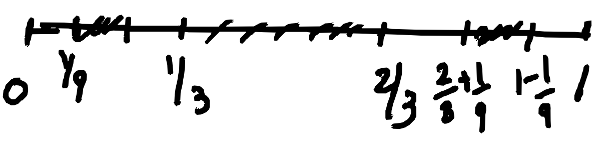
→  Finite-difference

Discretization results in ODEs

$$\frac{d a_k(t)}{dt} = \mathcal{P}(\{a_k\})$$

$k = 1, 2, \ldots, d$

Middle-thirds Cantor set (construction)

- 

$$[0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

Remainder after repeatedly removing middle-thirds → Cantor set.

- $\{0,1,2\} \times 3^{-k}$ (a ternary number)

- $\sum_{k=1}^{\infty} a_k \times 3^{-k}$ $\qquad a_k = \{0,1,2\}$

$= \sum_{k=1}^{\infty} b_k \times 2^{-k}$ $\qquad b_k = \{0,1\}$

$0.\, a_1 a_2 a_3 \ldots$

$= \dfrac{a_1}{3} + \dfrac{a_2}{9} + \dfrac{a_3}{27} + \cdots$

- Cantor set has Lebesgue measure 0

rand()

- but it's uncountable : maps surjectively to $[0,1]$: replace 2 in ternary representation of a number in Cantor set with 1 to get binary representation of number in $[0,1]$.

- Ex: Find F that leaves the Cantor set invariant.

Central limit theorem (classical)

$X_1, \ldots, X_n \ldots$ iid and have finite mean (m) and variance (σ^2),

$$Y_n = \frac{1}{\sqrt{n}\sigma} \sum_{i \leq n} (X_i - m)$$

$$Y_n \xrightarrow{d} \mathcal{N}(0, 1)$$

$\underline{\underline{x}}, \ F(x), \ F^2(x), \ \ldots$: orbit

$F : M \to M$ (self map)

(Markov process)

- Makes sense to think of state as RV even when F is deterministic.

- orbits do not consist of iid RVs in general

- will see a notion of CLT for some dynamical systems later

Convergence in Probability

$X_1, X_2, \dots,$ come from

$(\Omega, \Sigma, \mathbb{P})$

sample space, sigma algebra, Probability measure

$$\lim_{n\to\infty} \mathbb{P}(d(X_n, X) > \varepsilon) = 0$$

$$X_n \xrightarrow{P} X$$

If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{d} X$.

WLLN: $X_1, X_2, \dots$ iid and have $\mathbb{E}X_1 < \infty$

$$Y_n = \frac{1}{n}\sum_{i\le n} X_i$$

WLLN: $Y_n \xrightarrow{P} \mathbb{E}X_1$

$(Y_1, Y_2, \dots$

$(-1)^i X_i \xrightarrow{d} X$ (for some symmetric RV)

but $(-1)^{-i} X_i \not\xrightarrow{P} X$

Almost sure convergence / convergence almost everywhere

$$X_n \xrightarrow{a.s.} X$$

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1$$

$$X_n \xrightarrow{a.s.} X \quad \overset{\not\Leftarrow}{\Longrightarrow} \quad X_n \xrightarrow{P} X$$

- $\mathbb{P}(X_n = 0) = \frac{1}{n}$ $\qquad \mathbb{P}(X_n = 1) = 1 - \frac{1}{n}$

$$\lim_{n\to\infty} \mathbb{P}(\underbrace{|X_n - 1|}_{X_n = 0} > \varepsilon) = 0$$

$$X_n \xrightarrow{P} 1$$

$$\sum_n \mathbb{P}(\{X_n = 0\}) = \sum_n \frac{1}{n}$$

Borel-Cantelli lemma says $\{X_n = 0\}$ has to occur infinitely many times. So, $X_n \not\xrightarrow{a.s.} 1$

By the same logic, $X_n \not\xrightarrow{a.s.} 0$

- Replace $1/n$ with $1/n^2$. Then,

$$X_n \underset{a.s.}{\xrightarrow{P}} 1.$$

Analysis

Compactness:

sequential compactness:
any infinite sequence has a convergent subsequence.

Every open cover has a finite subcover.

→ Heine-Borel : Compact ⟺ closed & bounded on $\mathbb{R}^n$

→ Arzela-Ascoli Theorem:
if $\{f_n\}_{n \in \mathbb{Z}^+}$ is equicontinuous and $\{f_n(x)\}$ has compact closure at every $x \in K$, then, $\{f_n\}$ has a converging subsequence.
(K is compact)

→ Equicontinuity:
For every $\varepsilon > 0$, $x, y \in K$, $\exists$ (there exists) $\delta > 0$ such that
$$d_Y(f_n(x), f_n(y)) < \varepsilon \text{ whenever}$$
$$d_X(x, y) < \delta \quad \text{for all } n \in \mathbb{N}.$$

→ $f_n : X \to Y$

(X, d_X) (Y, d_Y)