HW3 : Tuesday (11/05)

Project proposal - 1 pager ($^{\text{Due}}$) soon

Recap

→ Topological transitivity of irrational rotations on $\mathbb{T}^n$.

→ Linear perturbation analysis

(e.g. Limit cycles (closed isolated orbit)
chaos

$$F : M \to M$$

$$dF : TM \to TM$$

Evolution of infinitesimal linear perturbations

__Maps:__

$$v(x) \in T_x M$$

$$v(f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon v(x)) - f(x)}{\varepsilon}$$

$f \in C^\infty(M)$

$$(dF)_i = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}$$

(matrix representation)

$$\to \quad F(x + \varepsilon v(x)) = F(x) + \varepsilon\, dF(x) v(x) + O(\varepsilon^2)$$

$$(dF\, v)(F(x)) = dF(x)\, v(x)$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(F(x + \varepsilon v(x)) - F(x)\big) = dF(x)\, v(x)$$

$$(dF)_\# v$$

$$v_n(F^n x) = dF(F^{n-1}x)\, v_{n-1}(F^{n-1}x) = dF^n(x)\, v_0(x) = (dF)^n_\# v_0(x)$$

$$\to \quad \underset{T_{x_n}M}{\underset{\Uparrow}{v_n}} = dF(x_n)\, \underset{T_{x_{n-1}}M}{\underset{\Uparrow}{v_{n-1}}} \in \mathbb{R}^d$$

Fix $x_0, x_1, \ldots$

linear dynamical system, non-autonomous (no)

$$\to \quad x \to g(x) \in \mathbb{C}.$$

$$g \in C^\infty(M)$$

$$v(g)(x) = \lim_{\varepsilon \to 0} \frac{g(x + \varepsilon v(x)) - g(x)}{\varepsilon}$$

$$v(g \circ F^t) = ?$$

$$\text{Loss} = \frac{1}{T} \sum_{t=0}^{T-1} v(g \circ F^t)$$

$$= \frac{1}{T} \sum_{t=0}^{T-1} d(g \circ F^t)(x)\, v(x)$$

$$= \frac{1}{T} \sum_{t=0}^{T-1} (dg)(F^t x)\, v_t(F^t x)$$

$$v_t(F^t x) = (dF)^t_\# \underline{v}(x) \leftarrow \text{Tangent equation}$$

$$x \in \mathbb{R}^d \qquad d \gg 1$$

→ Efficient matrix-vector product

→ automatic differentiation

function returns $F(x)$

function return $x \to g(x)$

$$AD_\varepsilon \big((\varepsilon, x, v_0) \to \underset{\in \mathbb{R}^d}{F(x + \varepsilon v_0)} \big)$$

$$\frac{dF^t(x)}{dt} = v(F^t(x))$$

Tangent equation

$$\frac{d}{dt}\big(F^t(x) + \varepsilon \underset{T_{F^t x}M}{\underset{\Uparrow}{\omega(F^t x)}}\big) = \frac{dF^t(x)}{dt} + \varepsilon \frac{d\omega(F^t x)}{dt}$$

$$= v(F^t(x) + \varepsilon\,\omega(x))$$

$$= v(F^t(x)) + \varepsilon\, \underset{\mathbb{R}^{d\times d}}{\underset{\Uparrow}{dv}}(F^t x)\, \omega(x) + O(\varepsilon^2)$$

$$\frac{d\omega_0 \circ F^t}{dt} = dv \circ F^t\, \omega_0$$

$$\frac{d}{dt}\omega_t = (dv)^t_\# \omega_0$$

__Note__ (computational)

- $d \gg 1$ time integration

$$\omega_0, \omega_{\delta t}, \underset{T_{x_{2\delta t}}M}{\underset{\Uparrow}{\omega_{2\delta t}}} \ldots$$

- $F^t x \qquad dv(F^t x) \qquad \omega_0, \omega_{\delta t} \ldots \qquad x \quad F^{\delta t} x \ldots$

- $\tilde{F}^n(x) = F^{n\delta t}(x)$ (time δt maps of flow)

$$\tilde{F}(x) = F^{\delta t}(x)$$

$$\omega_{n+1} = d\tilde{F}(x_n)\, \omega_n \quad \text{(tangent equation for maps)}$$

$$\tilde{F}(x) = F^{\delta t}(x) = \underbrace{x + \delta t\, v(x)}_{\text{Forward Euler time discretization}} + \cdots$$

Linear perturbation evolution

Maps	Flows
$w_{n+1} = dF(x_n)\, w_n$	$\frac{dw(x_t)}{dt} = dv(x_t)\, w(x_t)$
Fix: $x_0, x_1, \dots$	$F^t(x_0)$

x_0 : fixed point

Maps	Flows
$F(x_0) = x_0$	$v(x_0) = 0$ $F^t x_0 = x_0 \quad \forall t$

Behavior completely understood from

Maps	Flows
$\mathrm{eig}(dF(x_0))$	$\mathrm{eig}(dv(x_0))$

→ Local behavior / behavior of inf linear perturbations

→ Not stability around fixed points

Lyapunov method for asymptotic stability of fixed points

→ what do orbits look like around fixed pts?

→ Are fixed pts asymptotically stable?

Setting: $\dfrac{dF^t(x)}{dt} = v(F^t x, t)$

$v \in C^1(M \times I_t)$

$\exists$ unique orbit $F^t(x)$ in some open set on $M \times I_t$.

$I_t \equiv (0, \infty)$, As $t \to \infty$, $F^t(x)$ maybe unbounded

or

for t finite, $F^t(x)$ approaches ∂M.