CAAM/STAT 31310, Autumn 2024, U Chicago

CAAM 31310: Homework 3

Due Nov 11th, '24 (11:59 pm ET) on Gradescope

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In this homework, we will study a gradient flow on $\mathbb{T}^2$. For $w = [x, y, z] \in \mathbb{R}^3$, define $\ell(w) = x(w)$ to be the first coordinate function. The gradient flow, F^t, of a function $w \to \ell(w)$ is given by $dF^t(w)/dt = -\nabla F(w)$, where ∇ here is the gradient operator induced by the Euclidean metric on $\mathbb{R}^2$. For concreteness, our torus will be as shown below, with $\max_w x(w) = \max_w y(w) = 10 + 1 = 11$ and $\max_w z(w) = 1$. You can play with the script `torus_gradientflow.py` for simulation and visualization of the flow.

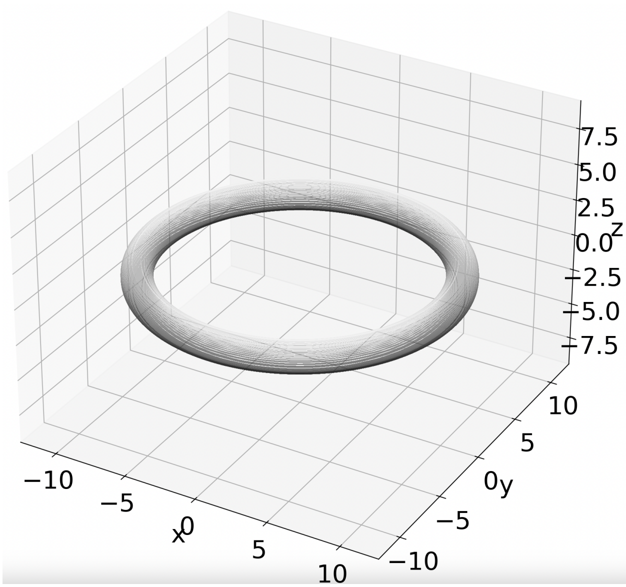

Figure 1: A torus embedded in $\mathbb{R}^3$.

- Give all the fixed points of F^t. (1 point)
- Is $w \to \ell(w)$ geodesically convex on $\mathbb{T}^2$? (1 point)
- Is $w \to x(w) + 11$ a Lyapunov function? If yes, define an appropriate neighborhood around the fixed points for its definition. (3 points)
- Using the Lyapunov function defined above or otherwise, prove the asymptotic stability of all orbits to the set of fixed points. (3 points)
- Change the embedding of $\mathbb{T}^2$, by applying a rotation of the x-y plane by an $\alpha > 0$. How must the Lyapunov function be modified? (2 points)

- Write down a sum of squares optimization problem using polynomial functions that recovers the Lyapunov function for any α. You do not need to submit a code, only the formulation. (3 points)