$\lambda(x) = \lambda_i \qquad x \in V_i \setminus V_{i+1}$

$|g(t,x)| \leq C|x|^{1+\alpha}$

$\alpha > 0$

$$\sum_{i=1}^{p} \lambda_i k_i = \lim_{t\to\infty} \frac{1}{t} \log|\det A(t)|$$

$(A_{t-1} \cdots A_0)^T \qquad \frac{1}{t}\log|\det \Phi(t)|$

$\log|\det e^{tA}|$

$e^{k_1\lambda_1} \cdot e^{k_2\lambda_2} \cdots e^{\lambda_p k_2}$

$\log(e^{k_1(a+ib)}) \qquad A$: periodic

$\log(e^{k_1 a} \cdot e^{k_1 ib}) \qquad A(t) = A(t+\tau)$

$e^{k_1 a} \; e^{k_1 ib}$

$\frac{1}{t}\log\|A_t\| \to 0 \qquad \|A(t)\| \leq e^{ct}$

$$\sum_{i=1}^{p} k_i \lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \det|A(t)|$$

$\Phi(t) = P(t)\, e^{tR}$

$\dot{x}(t) = A(t)x(t)$

$x(t) = P(t)y(t)$

$$\frac{d}{dt}x(t) = \frac{dP(t)}{dt}y(t) + P(t)\dot{y}(t)$$

$$= A(t)P(t)y(t)$$

$$P(t)\dot{y}(t) = \left(-\frac{dP(t)}{dt} + A(t)P(t)\right)y(t)$$

$$= P(t)R\,y(t)$$

$$A(t)P(t) - \frac{dP(t)}{dt} = P(t)R$$

$$\frac{dP(t)}{dt} = A(t)P(t) - P(t)R$$

$\Phi(t) = P(t)\, e^{tR}$

$$A(t) = \frac{dP(t)}{dt}e^{tR} + P(t)e^{tR}$$

Sign-up sheet : 2 days for presentation
11^{th}

→ 10 mins   Participatory presentation
→ broad introduction
→ computational algorithm/analysis
→ interpretation

Note: doesn't have to be new research

5 pages report : last day of class
Check Rubric on canvas.

HW4 Due Sunday

HW5

HW2, HW3 grades

Recap

Introduction to ergodic theory

$F: M \to M \quad (M, \Sigma, \mathbb{P}, F)$

Corollary of Birkhoff's ergodic theorem:

For any $g \in L^1$, and measure-preserving dynamical system F,

$$\frac{1}{T}\sum_{t \le T} g \circ F^t(x) \xrightarrow{T \to \infty} \mathbb{E}_{x \sim \mu}\, g(x) = \langle g, \mu \rangle$$

for x μ-a.e.

- (F, μ) is ergodic

- Ergodic decomposition theorem: any invariant measure can be decomposed into ergodic components

M (viewed as a Lebesgue space) → partitioned into $\{M_\alpha\}_\alpha$

$$\int g \, d\underset{\text{invariant}}{\mu} = \int_\alpha \int_{M_\alpha} g \, d\underset{\text{ergodic measures supported in } M_\alpha}{\mu_\alpha} \, d\alpha$$

- Ergodic measures are extremal points on (convex) sets of invariant measures.

- ℓ^2 sequences shifts translations

- For any continuous map $F: M \to M$ on compact M, $\exists$ at least one invariant ergodic measure.

- dissipative Chaotic systems : an ergodic measure μ may not be "observable".
(may have $\text{leb}(\text{supp}(\mu)) = 0$)
volume measure

Kingman's subadditive ergodic theorem

$(M, \Sigma, \mathbb{P}, F)$ (F, μ) is ergodic

$g_1, g_2, g_3 \ldots$ are a sequence of random variables ($g_i : M \to \mathbb{R}$) and

- $g_i \in L^1 \ \forall i$.

- $\{g_i\}$ is a subadditive sequence

$$g_{m+n}(x) \leq g_n(x) + g_m(F^n x)$$

x μ-a.e.

Then,

- $$\frac{g_n}{n} \xrightarrow{L^1} \bar{g} \in L^1$$

$$\left(\lim_{n\to\infty} \frac{g_n(x)}{n} = \bar{g}(x) \quad \forall x \ \mu\text{-a.e.} \right)$$

- $$\frac{\mathbb{E} g_n}{n} \xrightarrow{L^1} c$$

$$g_n = \sum_{i \leq n} g \circ F^i$$

- HW4 : $\text{Unif}\{x_0, \ldots, x_n\} \xrightarrow{n\to\infty} \mu_1$,

$x_i = F(x_{i-1})$

$\text{Unif}\{y_0, \ldots, y_n\} \xrightarrow{n\to\infty} \mu_2$

Oseledets multiplicative ergodic theorem (OMET)

$(M, \Sigma, \mathbb{P}, F)$ RDS

$A_i(x) \in \mathbb{R}^{d \times d}$

$A_{m+n}(x)$

- $\log^+ \|A_n\|(x) := \max\{0, \log \|A_n(x)\|\}$

  $\log^+ \|A_n\| \in L^1$

- The sequence $\{A_n\}$ is regular for μ-a.e.

$$\sum_{i=1}^{p(x)} \lambda_i(x) k_i(x) = \lim_{t \to \infty} \frac{1}{t} \log \det \|\Phi(x,t)\|$$

$\Phi(x,t) = A_{t-1}(x) \cdots A_0(x)$

if (F, μ) is ergodic,

$\lambda_i(x) = \lambda_i$ μ-a.e.

$p(x) = p$ μ a.e.

$k_i(x) = k_i$ μ a.e.

Lyapunov exponent, subspaces, Oseledets subspaces

OMET says

- $\lim_{t \to \infty} \frac{1}{t} \log \|\Phi(x,t)\, v\|$

  $:= \lambda(x, v) = \lambda(v)$

  (F, μ) ergodic

Filtration

- $V_{p+1}^{(x)} \subset V_p^{(x)} \subset \cdots V_2^{(x)} \subset V_1^{(x)} = \mathbb{R}^d$

  $\shortparallel \{0\}$

  $V_i := \{v \in \mathbb{R}^d : \lambda(x, v) \leq \lambda_i\}$

- $\lambda_1 > \lambda_2 \cdots > \lambda_p$ are distinct LEs.

- "Backward" regular

  $\Phi(x,t)$ $t \in \mathbb{Z}^-$

  $A_{-1}(x)$ $A_{-1}(x) = A(F^{-1}x)$

- $\lambda_1^- > \lambda_2^- > \cdots > \lambda_p^-$

  OMET: $\lambda_i^- = -\lambda_{p+1-i}$

$\{0\} \subset V_p^-(x) \subset \cdots V_2^-(x) \subset V_1^-(x) = \mathbb{R}^d$

$\shortparallel V_{p+1}^-(x)$

- $E_i(x) = V_i(x) \cap V_{p+1-i}^-(x)$ Oseledets subspaces

  $E_i(F^n x) = \Phi(x, n)\, E_i(x)$ $n \in \mathbb{Z}$

  $E_i(x) = \{e \in \mathbb{R}^d : \lambda(x, e) = \pm \lambda_i\}$

Proof idea

- $\lim_{t \to \infty} \frac{1}{t} \log \det |\Phi(x,t)|$ exist

- $\frac{1}{t} \log \|\Phi(x,t)\|$

$\Phi(x, t+u) = \Phi(F^u(x), t)\, \Phi(x, u)$

$\mathbb{R}^{d \times d}$ ($\Phi(x,t)$ is a cocycle)

$\log \|\Phi(x, t+u)\| \leq \log \|\Phi(F^u(x), t)\| + \log \|\Phi(x, u)\|$

Apply Kingman's subadditive ergodic theorem