14^{th} Proposal

Signup for 10-minute meeting about the project 14^{th}, 19^{th}

Best 4 out of 5 hws

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Recap

Lyapunov function method

→ SDP (+ve def matrices)
(convex optimization)

→ HJB

$$\frac{dx(t)}{dt} = v(x(t), \underset{\text{control function}}{u(t)})$$

$$C(x(0), T) = \int_0^T l(x(t), u(t))\,dt$$

$$\frac{\partial C(x,t)}{\partial t} + \min_u \left(l(x, u) + dC(x,t)\, v(x,u) \right) = 0$$

$$l(x, u^*) = - dC(x,t)\, v(x, u^*)$$
$$= - \frac{d\, C \circ F^t(x, u^*)}{dt}$$

$$\frac{d\, F^t(x, u^*)}{dt} = v(F^t(x, u^*), u^*(t))$$

if l +ve, C (value or cost-to-go function) is a Lyapunov function

Lyapunov analysis

Osceledets multiplicative ergodic theorem

(Arnold : Random Dynamical systems)

Furstenberg - Kesten theorem:

if $A_0, A_1, \ldots \in \mathbb{R}^{d\times d}$ nonsingular

$\frac{1}{t} \log \det A_t \to 0$ as $t \to \infty$,

$\exists$ p distinct real numbers λ_i with multiplicity k_i,

(i) sequence is regular:

$$\lim_{t\to\infty} \frac{1}{t} \log|\det \Phi(t)| = \sum_{i=1}^{p} k_i \lambda_i$$

where $\Phi(t) = A_{t-1} A_{t-2} \cdots A_0$

$\lambda_i \in \mathbb{R}$, $k_i \in \mathbb{N}$.

(ii) for any $v \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t) v\| = \lambda_i$$

$\rightarrow$

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\Phi(t) v\| = \liminf_{t\to\infty} \frac{1}{t} \log \|\Phi(t) v\| = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t) v\|$$

Lyapunov exponents

$\rightarrow$ $p \leq d$ number of distinct LEs.

$$\Phi(t) = A_{t-1} \cdots A_0$$

$\rightarrow \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t) v\|$

$\rightarrow$ 1-dimensional subspaces in $\mathbb{R}^d$

$\rightarrow$ 2-dimensional subspaces in $\mathbb{R}^d$

$V^{(2)} = \text{span}\{v^{(1)}, v^{(2)}\}$

why is $p \leq d$?

$$A_t = A_{t-1} = \cdots = A_0 = A$$

$$\|A^t v\| \sim O(\eta_1^t, v)$$

$\eta_1 > \eta_2 > \cdots > \eta_p$ are the distinct eigenvalues of A

$A^t V^{(2)}$

$$(I - \underset{\substack{\downarrow \\ \text{top eigenvector}}}{V_1} V_1^T) A^t V^{(2)}$$

$\rightarrow$ $\underset{d\times k}{V^{(k)}}$: a k-dimensional subspace of $\mathbb{R}^d$.

$$\left[v^{(1)} \middle| \cdots \middle| v^{(k)} \right]$$

$$\det \left(V^{(k)} \longrightarrow \Phi(t) V^{(k)} \right)$$

$$\det\left(V^{(k)T} \Phi(t) V^{(k)}\right)$$

$\rightarrow$ Regularity: $\Phi(t) = A_{t-1} A_{t-2} \cdots A_0$

$A_0, A_1, \ldots, A_t$ is regular if

$$\lim_{t\to\infty} \frac{1}{t} \log \det \|A_t\| = \sum_{i=1}^{p} k_i \lambda_i$$

$\rightarrow$ dim of subspace $V^{(i)}$ such that

$$\frac{1}{t} \log \det (V^{(i)} \rightarrow \Phi(t) V^{(i)}) \xrightarrow{t\to\infty} \underset{\substack{\uparrow \\ \text{multiplicity}}}{k_i} \lambda_i$$

$$\Phi(-t) = A_{-(t-1)}^{-1} \cdots A_{-1}^{-1} A_0^{-1}$$

LEs: $\lim_{t\to\infty} \frac{1}{t} \log \| \Phi(-t) v \| = \lambda_i^-$

$i = 1, 2, \ldots, p$

$\lambda_1 > \lambda_2 > \cdots > \lambda_p$

$\lambda_1^- > \lambda_2^- \cdots > \lambda_p^-$

Arise from dynamical systems:

$$\lambda_i = -\lambda_{p+1-i}^-$$

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Generalization of eigenspaces. Subspaces corresponding to LEs.

$$\mathbb{R}^d = \{ v : \lambda(v) \le \lambda_1 \}$$

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \log \| \Phi(t) v \|$$

$$\{0\} = \{ v : \lambda(v) < \lambda_p \}$$

$$V^{(k)} = \{ v : \lambda(v) \le \lambda_k \}$$

$$\{0\} \subset V^{(p)} \subset \cdots V^{(2)} \subset V^{(1)} = \mathbb{R}^d$$

(filtration / flag)

$\lambda(\omega)$ for $\omega \in V^{(i)} \setminus V^{(i+1)}$

$= \lambda_i$

$$\dim(V^{(i)} \setminus V^{(i+1)}) = k_i$$

- Subspaces associated with $\Phi(-t)$ given that

$$\lambda_i^- = -\lambda_{p+1-i}$$

$$W^{(i)} = \{ v : \lambda^-(v) \le \lambda_i^- \}$$

$$\lambda^-(v) := \lim_{t\to\infty} \frac{1}{t} \log \| \Phi(-t) v \|$$

$$\{0\} \subset W^{(p)} \subset \cdots W^{(2)} \subseteq W^{(1)} = \mathbb{R}^d$$

$\lambda_p < \cdots \lambda_2 < \lambda_1$

$\lambda_p^- < \cdots < \lambda_1^-$

$v \in V^{(1)} \cap W^{(p)}$

$\lambda_p = -\lambda_1$

$\lambda(v) = -\lambda^-(v) = \lambda_1$

$$E^{(i)} := V^{(i)} \cap W^{(p+1-i)}$$

$\lambda(v) = -\lambda^-(v) = \lambda_i$

$\forall\, v \in E^{(i)}$.

$k_i = \dim(E^{(i)})$

Oseledets subspaces

Ergodic theory

$F^t : M \to M$

(M, Σ, P, F) Dynamical system

F preserves μ or μ is an invariant measure for F if

$$F_{\#}\mu := \mu \circ F^{-1} = \mu.$$

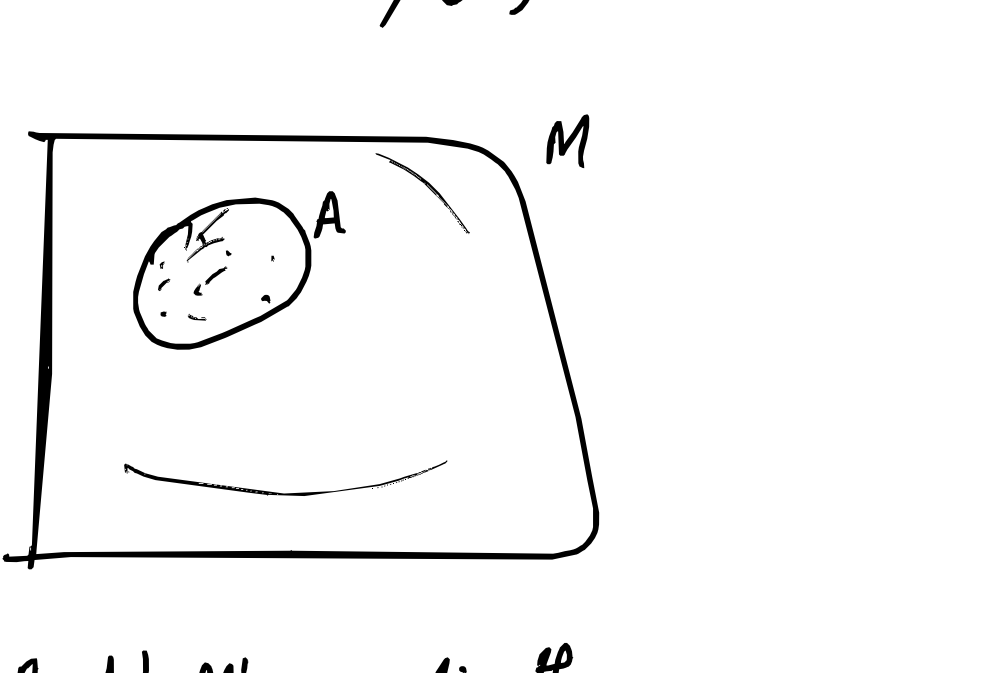
pushforward

$$\mu \circ F^{-1}(A) = \mu(A)$$

A measure μ on M is ergodic

(F, μ) is ergodic

if any F-invariant set A has $\mu(A) = 0$ or 1.



Birkhoff's ergodic theorem.