CAAM/STAT 31310, Autumn 2024, U Chicago

# CAAM 31310: Homework 4

Due Nov 24th, '24 (11:59 pm ET) on Gradescope

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In this homework, we will explore Lyapunov analysis and computation of Lyapunov vectors and exponents. We will consider the three-variable Lorenz '63 system, which was introduced as a reduced order model for atmospheric convection. Let $x = [a, b, c]$ be a phase point, and $a(x)$ refers to the first component/coordinate at $x$. The system is given by the ODEs:

$$\frac{d\varphi^t(x)}{dt} = v(\varphi^t(x)) = \begin{bmatrix} \sigma(b-a) \\ a(\rho - c) - b \\ ab - \beta c \end{bmatrix} \circ \varphi^t(x). \tag{1}$$

Fix $\sigma = 10, \beta = 8/3$ and $\rho = 14$.

I Enumerate all the fixed point attractors and indicate their stability. (2 points)

II (3 points) Compute the three Lyapunov exponents. Submit a code snippet, explaining each line. Do they depend on the initial condition?

III (3 points) Use the linearized system around a fixed point to define a Lyapunov function (i.e., check that your definition satisfies the properties of a Lyapunov function).

IV (3 points) Write down a sum-of-squares optimization problem (you don't need to solve it) for the region of stability of a fixed point.

III (5 points) Are the following statements true or false for the Lorenz '63 system? Provide justification.

(a) The adjoint covariant Lyapounov vector is the same as the covariant Lyapunov vector for the top LE

(b) The top (backward) Lyapunov vector is always covariant

(c) There is a zero Lyapunov exponent for the ODE system

(d) The stable adjoint Lyapunov vector is perpendicular to the unstable Lyapunov vector

(e) There is a dense set of points on the attractor that result in different LEs than the ones computed

IV (5 points) Compute and plot the top adjoint Lyapunov vector. Submit a code snippet/algorithm.