Oseledets MET: Today
+ computation

Dimension reduction: 2 lectures

Chaotic systems: nonlinear systems that are aperiodic and have "high" sensitivity to infinitesimal perturbations

Periodic systems

- → Meet me about the project!
- → HW5
- → HW2 - HW3 grades

} This week

Recap

Linear cocycles associated with deterministic ergodic DS

- $M, \mathcal{E}, \mathbb{P}, F$ with ergodic measure μ on M

Fix $x \in M$. Let $A_n = dF(x_n) \in \mathbb{R}^{d \times d}$

- $A(0,n) = A_{n-1} \cdots A_1 A_0$ $\quad x_n = F^n x$

- Cocycle:

$$A(0, m+n) = A_{m+n-1} \cdots A_0 = dF^{m+n}(x)$$

($F^n(x) = F \circ F \circ \cdots \circ F(x)$

$dF^n(x) = dF(x_{n-1}) \cdots dF(x_1) dF(x)$ chain rule)

$$= dF^n(x_m) \, dF^m(x)$$

$$= (A_{m+n-1} \cdots A_m)(A_{m-1} \cdots A_0)$$

$$= A(m, m+n) \, A(0, m)$$

- Take $B_n = (dF(x_n))^T$

$C_n = dF^{-1}(x_n)$ (inverse fn thm)

$= (dF(x_{n-1}))^{-1}$

$D_n = (dF(x_n))^{-T}$

$\cdots \; F^{-2}(x). \; F^{-1}(x) \; x \; F(x) \; F^2(x) \cdots\cdots$

- $A(0,n)$ $\quad$ $B(0,n)$ "adjoint"

$T_x M \to T_{x_n} M$ $\quad$ $T^*_{x_n} M \to T^*_x M$

- $C(0,n)$: $T_{x_n} M \to T_x M$
- $D(0,n)$: $T^*_x M \to T^*_{x_n} M$

<u>SVD</u>

- $A = U \Sigma V^T$ $\qquad U = \begin{bmatrix} u_1 | \cdots | u_d \end{bmatrix}$

$A v_i = \sigma_i u_i$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_d \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 | v_2 | \cdots v_d \end{bmatrix}$$

- $A(t, t+s) : T_{x_t} M \to T_{x_{t+s}} M$

$$A(t, t+s) = U(t, t+s) \Sigma(t, t+s) V(t, t+s)$$

$$A(t, t+s)\, v_i(t, t+s) = \sigma_i(t, t+s)\, u_i(t, t+s)$$

- Kuptsov Parlitz 2012

$$F(t) = \lim_{s \to \infty} \left(A(t, t+s)^T A(t, t+s) \right)^{\frac{1}{2s}}$$

$$= \lim_{s \to \infty} \left(V(t, t+s)^T \Sigma^2(t, t+s) V(t, t+s) \right)^{\frac{1}{2s}}$$

$(U^T U = Id)$ $\displaystyle = \lim_{s \to \infty} \left(V(t, t+s)^T \underline{\Sigma}(t, t+s) V(t, t+s) \right)^{\frac{1}{s}}$

$\displaystyle \lim_{s \to \infty} \frac{1}{s} \log \sigma_i(t, t+s) = \lambda_i$ $\quad \mu$-a.e. x_t.

(independent of t)
OMET to regular linear cocycle
when (F, μ) is ergodic

$$Q(t) = \lim_{s \to \infty} U(t, t+s)$$

$$\underline{P}(t) = \lim_{s \to \infty} V(t, t+s)$$

(OMET)

<u>Cocycle</u>
$A(t, t+s)$ (ergodic)

OMET: $\exists$ $\lambda_1 > \lambda_2 > \cdots > \lambda_p$

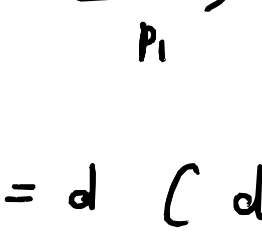
and $\{0\} \subset W_p(t) \cdots W_2(t) \subset W_1(t) = T_{x_t} M \equiv \mathbb{R}^d$
$\{0\} = W_{p+1}$

s.t. for any $v \in W_i(t)$,

$$\lim_{s \to \infty} \frac{1}{s} \log \| A(t, t+s) v \| \leq \lambda_i$$

$P(t)$ = limit of right singular vectors of far future matrix $F(t)$

- $\mathrm{span}\{p_1\} \oplus \mathrm{span}\{p_2\}$
almost surely grows/decays @ λ_1

- 

Volumes grow/decay @
$(\sigma_1)^{k_1} (\sigma_2)^{k_2} \to k_1 \lambda_1 + \lambda_2 k_2$

$p = d$ (distinct LEs)

$A(t, t+s)^{-1}$

<u>LEs</u>: $\lambda_1^- > \lambda_2^- > \cdots > \lambda_d^- = -\lambda_1$
$\lambda_1^- = -\lambda_d$, $\lambda_2^- = -\lambda_{d-1}$

$\subset W_d^-(t) \subset \cdots W_2^-(t) \subset W_1^-(t) = \mathbb{R}^d$
$\{0\} = W_{d+1}^-(t)$

λ

$W^\perp$

LE (forward) $\quad 3 > 2 > 1 > 0 > -5$
(backward) $\quad 5 > 0 > -1 > -2 > -3$

- $v \in W_i(t) \cap W_{d+1-i}^-(t)$
$\lambda(v)$?

$\displaystyle \lim_{s \to \infty} \frac{1}{s} \log \| A(t, t+s) v \| = \lambda_i$

Backwards <u>in</u> time $\qquad$ growth/decay rate
- any $v \in W_{p+1-i}^-(t)$ $\qquad \leq \lambda_{p+1-i}^- = -\lambda_i$

growth/decay rate $\geq \lambda_i$

$i = 2$ $\quad$ g/d rate ≤ 2 (forward)
W_4^- $\quad$ g/d rate ≤ -2 (backward)
(≥ 2)

$O(e^{2t}) \qquad o(e^{2t})$

$$E_i = W_i \cap W_{d+1-i}^- \quad \text{(Oseledets spaces)}$$

$v \in E_i$, $\lambda(v) = \pm \lambda_i$

- $A(t, t+s) P(t) = \Sigma(t)\, Q(t)$

backward Lyapunov vectors

- $A(0, t)\, Y(0) \approx Q(t)\, R(t)$

Far-past operator

$$P_a(t) = \lim_{s \to \infty} \left((A(t-s, t))^{-T} (A(t-s, t))^{-1} \right)^{\frac{1}{2s}}$$

$$A(t-s, t) = U(t-s, t) \Sigma(t-s, t) V(t-s, t)^T$$

$$(A(t-s, t))^{-1} = V(t-s, t) \Sigma^{-1}(t-s, t) U(t-s, t)$$

$$(A(t-s, t))^{-T} = U(t-s, t) \Sigma^{-1}(t-s, t) V(t-s, t)$$

$Q(t)$: eigenvectors of far-past matrix $P_a(t)$

Chaos: F is chaotic if $\lambda_1 > 0$.
$\lambda_1 = 0$
$\lambda_1 < 0$