CAAM/STAT 31310, Autumn 2024, U Chicago

# CAAM 31310: Homework 2

Due Oct 27, '24 (11:59 pm ET) on Gradescope

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In this homework, we explore perturbations of contraction maps. Let $F$ be a contraction map on a complete normed space $M$ with contraction coefficient $\lambda \in (0,1)$.

## Part 1 (5 points)

(Variation of Proposition 1.1.5 Katok-Hasselblatt) Prove the following statement. For every $\delta > 0$, there exists an $\epsilon \in (0, 2(1-\lambda))$ such that for any map $G$ with $\|F-G\|_\infty + \|F-G\|_{0,1} < \epsilon$, where $\|F\|_{0,1} := \sup_{x,y\,\in M, x\neq y} \|F(x)-F(y)\|/\|x-y\|$ is a Lipschitz semi-norm, for any $x \in M$, the orbits of $F$ and $G$ are $\delta$-close for all time. That is, $\|F^n(x) - G^n(x)\| \leq \delta$, for all $n$. In particular, $\|x_F^* - x_G^*\| \leq \delta$, where $x_F^*$ is the fixed point of $F$. (Hint: first show that $G$ is a contraction)

## Part 2 (3 points)

Consider a flow $d\varphi^t(x)/dt = v(t, \varphi^t(x))$ in $\mathbb{R}^d$. Assume that the vector field $(t,x) \to v(t,x) \in \mathbb{R}^d$ is continuous in $t$ and differentiable on $M$ for all time $\mathbb{R}$. Show that the flow exists in $M$ for some time interval. This is the Picard-Lindelöf theorem (2 points). Give sufficient conditions on $x \to v(t,x)$ for the flow to exist for all time. (1 point)

## Part 3 (2 points)

Suppose $(t,x) \to v(t,x)$ is not known exactly and should be estimated from data. Let $v_\theta$ be the vector field that is parameterized by $\theta$ to approximate $v := v_{\theta^*}$. Use Parts 1 and 2 to give sufficient conditions on $v_{\theta^*}$ and $\theta$ under which the learned and true flows are arbitrarily close (for any $x_0$) for all time.

## Part 4

1. Let $F$ be a contraction map on the space of continuous functions on $\mathbb{R}$ with values in $M$ whose fixed point is the orbit $t \to \varphi^t(x_0)$. Use your definition of $F$ from Part 2. Define a map $G$ by replacing the integral in the definition of $F$ with a quadrature scheme such that $G$ is a contraction. (2 points)

2. Solve $\varphi^t(x_0)$ numerically for $v(t, [x_1, x_2, x_3]) = [-k_1x_1 + k_2x_2x_3, k_1x_1 - k_2x_2x_3 - k_3x_2^2, k_3x_2^2]^\top$ with $k_1 = 0.04, k_2 = 10^4, k_3 = 3 \times 10^7$. (Source: H. Robertson, "The solution of a set of reaction rate equations," in Numerical Analysis: Introduction (Thompson, 1966), pp. 178–182). Submit your plot of solutions starting from $[1, 0, 0]^\top$ over time upto $10^5$. Explain your ODE integrator and give an estimate of the numerical error as a function of time (5 points).

3. Is your ODE integrator a contraction map on a space of continuous functions/bounded sequences? (1 point)

4. Solve the same equations from 2. now using Picard iteration, wherein you replace time integration with quadrature. Use your $G$ from 1. Submit the plot and explain your observations. (5 points)