HW 3 Sunday

1-page proposal (14^{th}) week

Send me an email (18^{th})

also on 14^{th}

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→ dynamical system

→ computational question

- perturbation
  stability
- dimension reduction
- learning / feature extraction
- data assimilation
- inverse problems

Recap

→ Lyapunov function method

$$\frac{dF^t x}{dt} = v(F^t x, t)$$

$$v(0, t) = 0$$

$x \to V(x)$, +ve def, $\frac{d\, V \circ F^t(x)}{dt} < 0$

⇒ asymptotic stability at origin

→ $$\frac{dF^t x}{dt} = \underset{\mathbb{R}^{d\times d}}{\underset{\uparrow}{A}} (F^t x)$$

$$V(x) = x^T K x \quad \text{(ansatz)}$$

$K > 0$

Find K st.

$$AK + KA^T = -C$$

$$C > 0 \quad \text{(Rmk: } AK + KB = C \text{ Sylvester)}$$

$\frac{d}{dt} V \circ F^t x < 0$ for V to be a Lyap fn.

$$\frac{d}{dt} (F^t x)^T K (F^t x)$$

$$\frac{d (F^t x)^T}{dt} K (F^t x) + (F^t x)^T K \frac{d(F^t x)}{dt}$$

$$= (F^t x)^T A^T K (F^t x) + (F^t x)^T K A (F^t x)$$

$$= (F^t x)^T \underbrace{(A^T K + KA)}_{< 0} (F^t x)$$

$x^T B x \Rightarrow (KA)^T = A^T K$

Comp.
Schur decomposition of A

$$Q^T A Q = U$$

$$O(d^2 \times d^2) = O(d^4)$$

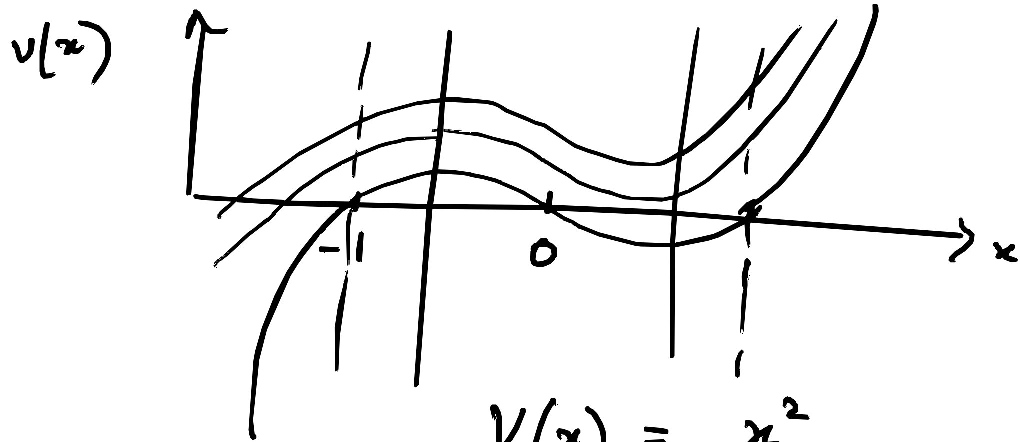
algorithm

$A \in \mathbb{R}^{d\times d}$

→ closed form expression for K

$$K = \int_0^\infty e^{tA^T} e^{tA}\, dt$$

$$\frac{dx}{dt} = -x + x^3 + \alpha$$



$$V(x) = \frac{x^2}{2}$$

By looking at a Lyapunov function we can obtain estimates of "invariant sets".

Barrier function

$B(\text{bad}) = 1$

$B(0) = 0$

if $x \in$ good region, ie $B(x) < 0$

$F^t x \notin$ bad for any t

as long as $\frac{d}{dt} B \circ F^t(x) < 0$ in the good region.

$$\frac{dx(t)}{dt} = Ax(t) + h(x(t)).$$

$$h(0) = 0 \quad , \quad dh(0) = 0$$

Even if one eigenvalue of $A > 0$, then, no Lyapunov fn exists.

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Optimization for <u>finding</u> Lyap fns

$$V(x) > 0$$

$$V(0) = 0$$

and $\frac{d}{dt} V \circ F^t_x < 0 \quad \forall x \in \mathcal{W}$

Polynomial in x.

$$V(x) = a_0 + a_1 x^{(1)} + a_2 x^{(2)} + \cdots + a_{d+1} x^{(1)} x^{(2)} + \cdots$$

$$x = [x^{(1)}, x^{(2)}, \ldots, x^{(d)}]^T \in \mathbb{R}^d.$$

Find a matrix K such that

$$p(x)^T K p(x) > 0$$

and $\frac{d}{dt}\left((p(F^t_x))^T K p(F^t_x)\right) < 0$

and $p(0)^T K p(0) = 0$

$$p(x) = [p_1(x) \ p_2(x) \ \ldots \ p_n(x)]$$

feasible region
cone of SPD matrices K.

$p(x)^T K p(x)$ is SOS

$-\frac{d}{dt} p(F^t_x)^T K p(F^t_x)$ is SOS

$$p(0)^T K p(0) = 0$$

→ need not take polynomials in x

→ change of variables

$$x \to [e^{i\omega_1 \cdot x} \ e^{i\omega_2 \cdot x} \ \ldots \ e^{i\omega_m \cdot x}]$$

$$x \Rightarrow T(x)$$

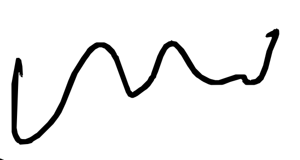
Take $p(T(x))$

$$V(x) = p(T(x))^T K p(T(x))$$

Convex <u>optimization</u> (unique global minimum)

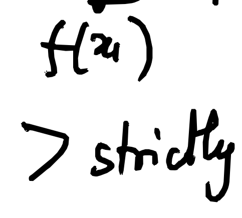
<u>Primal</u>

$$\min_{x \in \mathbb{R}^d} f(x)$$



$$g_i(x) \le 0 \quad i = 1, 2, \ldots, m$$

(add $-g_i(x) \le 0$ to make it an equality constraint



$f(x_1)$ $f(x_2)$

$$\alpha f(x_1) + (1-\alpha) f(x_2) \geqslant f(\alpha x_1 + (1-\alpha) x_2)$$

($>$ strictly convex)

$$\forall \alpha \in [0,1]$$

and $\forall x_1, x_2 \in \mathcal{K}$. convex

$\exists x \in \text{int}(K)$

$g_i(x) < 0 \quad i = 1, 2, \ldots, m$

KKT condition: at x^* (unique global minimum) and λ^* (unique global ~~minimum~~ maximizer)

$$\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

Lagrangian

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$$

$$\nabla_\lambda \mathcal{L}(x^*, \lambda^*) = g_i(x^*) \le 0$$

$$g(x_i^*)\lambda_i^* = 0 \quad \forall i$$

<u>Dual</u>

$$\max_{\lambda} \min_{x} \mathcal{L}(x, \lambda)$$

dual variables

$$\lambda \geqslant 0$$

<u>Thm</u>: $\min_x f(x) \geqslant \max_\lambda \min_x \mathcal{L}(x, \lambda)$

equality when convex

$$\min_{x \in X} \; p(x)^T K \, p(x)$$

$$\Big\downarrow \text{ Dual}$$

Semi-definite program (convex optimization) over cone of SPD matrices.

Optimal control

$t \to u(t)$ control

$$\frac{dx(t)}{dt} = v(x(t), u(t))$$

$$V(x(0), T) = \min_u \int_0^T l(x(t), u(t)) \, dt$$

value function

cost-to-go function

Hamiltonian - Jacobi - Bellman equation

Dynamic programing:

$$V(x, t) = \min_u \left(V(x_1, t+\Delta t) + \int_t^{t+\Delta t} l(x, u(t)) \, dt \right)$$

$$\frac{d}{dt} x(t) = v(x(t), u(t))$$

$$V(F^{\Delta t}(x, u), t + \Delta t)$$

$$= V(x, t) + \frac{\partial V}{\partial t}(x(t), t) + dV(x(t), t) \, v(x(t), u(t))$$

$(\lim \Delta t \to 0) \Downarrow$

$$\frac{\partial V(x_1, t)}{\partial t} + \min_{u(t)} \left(l(x, u(t)) + \underbrace{dV(x_1, t)}_{v(x, u(t))} \right) = 0 \quad \text{(HJB)}$$

(Reinforcement learning)

steady state

$$l(x, u^*) = -dV(x, t) \, v(x, u^*)$$

$$= -\frac{d}{dt} V(F^t(x, u^*))$$

if l is +ve, HJB soln is a Lyapunov function

Oseledets multiplicative ergodic theorem (OMET)

Motivation 1

$$\frac{dx(t)}{dt} = A(t)\,x(t)$$

Motivation 2

$$\frac{dx(t)}{dt} = A(t)\,x(t) + h(t, x(t))$$

$$h(t,0) = 0 \quad \forall\, t$$

$$\|h(t, x)\| \sim O(\|x\|^k)$$

$$(k \in \mathbb{N})$$

Motivation 3

Infinite-dimensional non-autonomous linear cocycles

Take a sequence:

$A_0, A_1, \dots, A_T$

for any $v \in \mathbb{R}^d$

$$\lim_{T \to \infty} \frac{1}{T} \log \frac{\| A_T A_{T-1} \cdots A_0 v \|}{\| v \|}$$

$\longrightarrow ?$