CAAM/STAT 31310, Autumn 2024, U Chicago

CAAM 31310: Homework 0

Due Oct 2, '24 (11:59 pm ET) on Gradescope

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Problem 1

This problem is about interchanging limits and the problems this can lead to. Source: Rudin Chapter 7.

1. Prove Theorem 7.13 from Rudin: Suppose $f_n, n \in \mathbb{N}$ is a sequence of continuous functions on a compact set E. Assume that i) f_n converge pointwise to a continuous function f and ii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in E$. Then, $f_n \to f$ uniformly on E. (5 points)
2. State in mathematical terms what uniform convergence of a sequence of continuous functions with respect to the supremum norm is. (1 point)
3. Consider $f_n = \sin nx$. Does this sequence converge uniformly on a compact subset of $\mathbb{R}$? (1 point) Does it satisfy the assumptions i) and ii) of the Theorem in Part I.1? (1 point)
4. Consider $f_n = \sum_{k=0}^{n} x^2/(1+x^2)^k$. Does this sequence converge uniformly on $[0,1]$? (1 point) Does it satisfy the assumptions i) and ii) of the Theorem in Part I.1? (1 point)

Problem 2

Consider independent tosses of unbiased coins, and set random variable $X_i = 1$ when the ith coin lands on heads, and $X_i = -1$, when the ith coin lands on tails, with $i = 1, 2, 3, \cdots$. Let $Y_i = \sum_{j \leq i} X_j$ and $Z_i = X_{i+1} - X_i$. Are the following statements true or false, and why?

1. Exactly one out of the three random variables, $A_n := (1/\sqrt{n})\, Y_n$, $B_n := (1/\sqrt{n})\, \sum_{i \leq n} Y_i$, and $C_n := (1/\sqrt{n})\, \sum_{i \leq n} Z_i$, converges in distribution to a random variable as $n \to \infty$. (4 points)
2. Two out of the three sequences $\{X_i\}, \{Y_i\}, \{Z_i\}$ consist of pairwise uncorrelated random variables (2 points)
3. There exists a non-constant, converging sequence c_n such that all three of $c_n A_n$, $c_n B_n$ and $c_n C_n$ converge in distribution. (2 points)