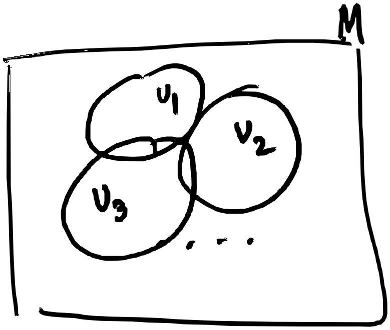


if

$$\bar{E}_1 \subseteq U_1 \cap f^{-N_1}(U_2)$$

$$f^{N_2}(E_1) \cap U_3$$

$$\bar{E}_2 \subseteq f^{-N_2}(U_3) \cap E_1$$

- HW2 27^{th}
- Zoom lecture make up for Wed 10/23
- Canvas ann. of time & day

Recap

Linear hyperbolic dynamics

$$F^t(x) = A^t x \qquad A \in \mathbb{R}^{d\times d}$$

if $|\mathrm{eig}(A)| \neq 1$, F^t hyperbolic

(flows: $F^t(x) = e^{tA}x \qquad \mathrm{eig}(A) \neq 0$)

$$\mathbb{R}^d = E^u \oplus E^s$$

$$E^s = \{ v \in \mathbb{R}^d : v \in \ker(A - \lambda I)^k \text{ for some } |\lambda| < 1,\ k \in \mathbb{N} \}$$

Asymptotic behavior

$\rightarrow v \in \mathbb{R}^d \setminus E^s$

$\lim_{t\to\infty} \|F^t v\|$ diverges

$$E^s := \{ v \in \mathbb{R}^d : \|F^t(v)\| \xrightarrow{t\to\infty} 0 \}$$

$$E^u \cancel{:=} \{ v \in \mathbb{R}^d : \|F^t(v)\| \xrightarrow{t\to\infty} \infty \}$$

non-hyperbolic: $\mathbb{R}^d = E^s \oplus E^u \oplus E^c$

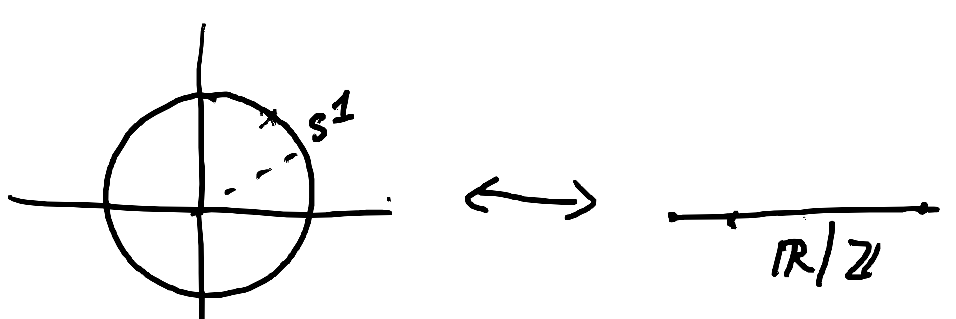
subexponential growth/decay

$$E^c := \{ v \in \mathbb{R}^d : v \in \ker(A - \lambda I)^k \text{ for some } k \in \mathbb{N} \text{ and } |\lambda| = 1 \}$$

$$E^{cu} = E^u \oplus E^c$$

$$E^{sc} = E^s \oplus E^c$$

Rotations/translations on manifold



(equivalence relation $x \sim y$ if $x - y \in \mathbb{Z}$)

Practically speaking, $[0,1]$ with periodic boundaries $(0 \equiv 1)$.

$$x \longrightarrow e^{2\pi i x}$$

$x \in \mathbb{R}/\mathbb{Z}$, $\quad e^{2\pi i x} \in S^1$

$$F(x) = \begin{cases} x+\theta-1 & x+\theta \geq 1 \\ x+\theta & 0 \leq x+\theta < 1 \end{cases}$$

$F(x) = x + \theta \pmod 1$ (additive)

$$\underset{\text{on } S^1}{F}(x) = e^{2\pi i (x+\theta)} = e^{2\pi i\theta} e^{2\pi i x}$$

(multiplicative)

Rational

$\theta = \frac{p}{q}$

$$F^q(x) = \left(x + \frac{qp}{q}\right) \bmod 1$$
$$= x$$

Irrational θ

$\{x, F(x), F^2(x), \dots \} = O(x)$

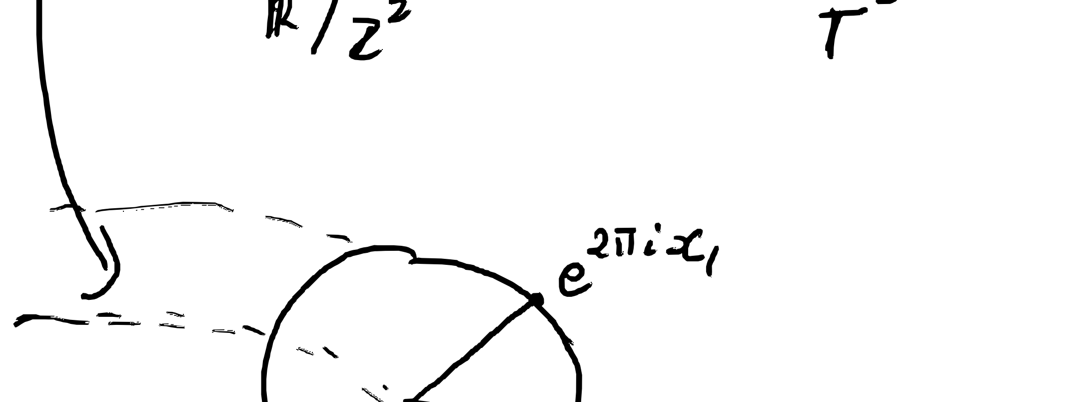
is dense in S^1.

Minimal: Every orbit is dense.

Ex: Prove "Irrational rotation on S^1 is minimal".

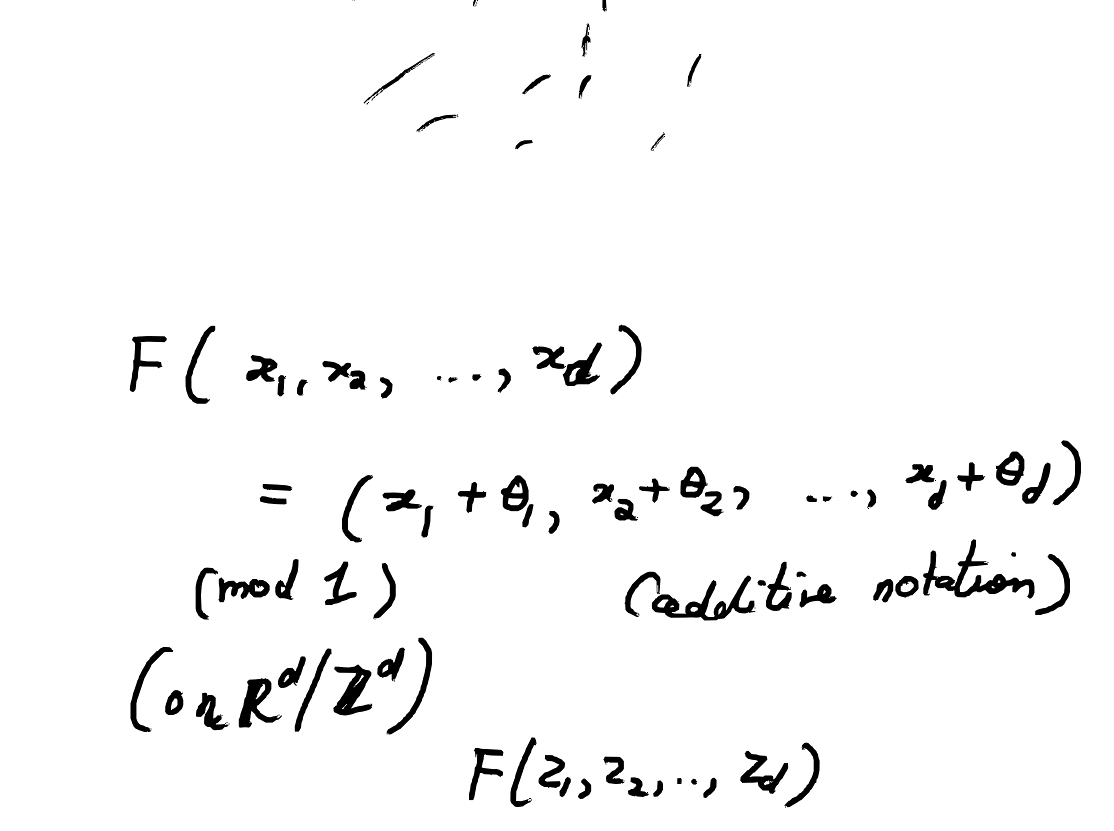
$$3^n = 10^{n(\log_{10} 3)}$$

Rotations on tori



$$(x_1, x_2) \rightarrow (e^{2\pi i x_1}, e^{2\pi i x_2})$$

$(x_1, x_2) \in \mathbb{R}^2/\mathbb{Z}^2$, $\quad (e^{2\pi i x_1}, e^{2\pi i x_2}) \in T^2$



$$F(x_1, x_2, \dots, x_d)$$
$$= (x_1 + \theta_1, x_2 + \theta_2, \dots, x_d + \theta_d)$$

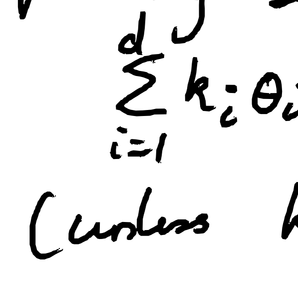
(mod 1) (additive notation)

(on $\mathbb{R}^d/\mathbb{Z}^d$)

$$F(z_1, z_2, \dots, z_d)$$
$$= (e^{2\pi i \theta_1} z_1, \dots, e^{2\pi i \theta_d} z_d)$$

(on T^d)

if for some $i \in [d]$, θ_i is irrational,



$x_1 = x_2$
$\theta_1 = \theta_2$

necessary & sufficient condition for minimality on T^d:
for any set of integers $k_1, k_2, \dots, k_d$
$\sum_{i=1}^{d} k_i \theta_i$ is not an integer
(unless $k_1 = k_2 = \dots = k_d = 0$).

Let M be compact.

Theorem (KH): A continuous open map $F: M \to M$ is topologically transitive iff for every pair of open subsets U, V, $\exists\, N(U,V) \in \mathbb{N}$ s.t. $F^N U \cap V \neq \emptyset$

Topologically transitive: $\exists$ a dense orbit

$\Rightarrow$ linearly independent irrational rotations on tori are minimal

Corollary: F is topologically transitive $\Leftrightarrow$ there are no disjoint invariant open sets.

U, V are open sets.
Then, $\exists N$ s.t. $F^N(U) \cap V \neq \emptyset$

$\bigcup_{n \in \mathbb{Z}} F^n(U) = U_0$ and $\bigcup_{n \in \mathbb{Z}} F^n(V) = V_0$ are invariant open set

($\because$ A is invariant if $F^{-1}(A) = A$)
$\{x \in M : F(x) \in A\}$

$\exists k$ $F^k(U_0) \cap V_0$ is nonempty

Corollary 2:
There exist no non-constant invariant continuous functions.

$f: M \to \mathbb{R}$ is invariant if
$f \circ F = f$.

$U = \{x \in M : f(x) < c\}$
$V = \{x \in M : f(x) > c\}$
$\exists k$, $F^k(U) \cap V$ is nonempty
$f \circ F^k(x) = f(x)$

Proof of minimality of linearly independent irrational rotation

$$g(x) = \sin\left(2\pi \left(\sum_{i=1}^{d} k_i x_i\right)\right)$$

$$g \circ F(x) = \sin\left(2\pi \sum_{i=1}^{d} k_i (x_i + \theta_i)\right)$$

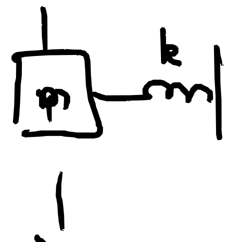
$\sin(2\pi + x) = \sin x$ $\quad$ if $\sum_{i=1}^{d} k_i \theta_i$ is an integer

then,
$$g \circ F = g$$

F is not topologically transitive
if $\sum_{i=1}^{d} k_i \theta_i$ is an integer
and $k_i \neq 0$ for all i.

Hamiltonian systems (Conserve Hamiltonian).

Simple Harmonic oscillator:



$$H(q, p) = \frac{1}{2} k q^2 + \frac{1}{2} m p^2$$

$x = (q, p)$ velocity

$$\frac{d}{dt} F^t(x) = \begin{bmatrix} p \\ -\frac{k}{m} q \end{bmatrix}$$

Lagrangian dynamics

$H(x)$ is conserved

$H \circ F = H$.

$$\frac{d}{dt} F^t(x) = \begin{bmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{bmatrix}$$

Linear flows on $\mathbb{T}^{2d}$ $\Big\}$ Conjugate $\Longleftrightarrow$ "locally" $F^t(x)$

H is such that there exist some change of coordinates

$$x \to \varphi(x) = (\underbrace{c_1, c_2, \dots,}\ c_{2d})$$

$\underbrace{c_1, c_2, \dots, c_d}$ are constants

action-angle coordinates