Proposal : 1 page

HW 3 Friday (delayed to Sunday)

1 problem Lyapunov method

Recap

Linear stability analysis based on tangent equation $\rightarrow$ infinitesimal linear perturbation

$F^t x^* = x^*$ (x^* is fixed pt)

$$\omega_{t+1} = dF(x^*)\,\omega_t \quad \text{(map)}$$

$$\frac{d\omega_t}{dt} = dv(x^*)\,\omega_t \quad \left(\text{Flow: } \frac{dF^t x}{dt} = v(F^t x)\right)$$

$\rightarrow$ characterizing "domain of attraction" to fixed points

$\rightarrow$ Poincare-Bendixon theorem: Existence of closed orbits on a plane.

- IVP Existence & uniqueness

- $\frac{dF^t_x}{dt} = v(F^t_x)$ $\qquad \mathcal{D} \subset \mathbb{R}^d$

Setting: $v \in C^1(\mathcal{D})$
$v(0) = 0 \in \mathbb{R}^d$.

$\exists!$ solution $\{F^t x\}_{t \le t_{\mathcal{D}}}$

$F^{t_{\mathcal{D}}}_x \in \partial\mathcal{D}$. or F^t_x becomes unbounded. as $t \to \infty$.

Asymptotic stability of origin

For some $\delta > 0$, $\|x\| < \delta$, $\lim_{t\to\infty} F^t_x = 0$.

Weaker notion: uniform stability
For every $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.
if $\|x\| < \delta$ then, $\|F^t_x\| < \varepsilon \ \forall\ t \in \mathbb{Z}^+$

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V: Lyapunov function
If there exists a function $x \to V(x) \in \mathbb{R}$, s.t. in a neighborhood $\mathcal{N}$ of the origin,

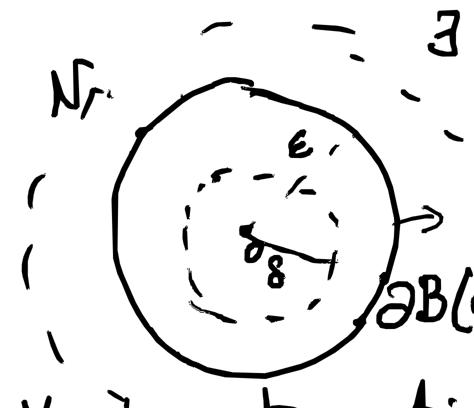
positive definite $\begin{cases} V(x) > 0 & \forall\ x \in \mathcal{N} \\ \text{and } V(x) = 0 & \text{at } x = 0 \end{cases}$

and $\frac{d}{dt} V \circ F^t(x) \le 0$ } negative semi-definite

$$\frac{d}{dt} V \circ F^t(x)\Big|_{x=0} = 0$$

and $V \in C^1(\mathcal{N})$,
then origin is uniformly stable.

Proof: Want: for every $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. if $\|x\| < \delta$ then, $\|F^t x\| < \varepsilon\ \forall\ t \in \mathbb{Z}^+$.



$\partial B(0, \varepsilon)$ compact set.

V has to achieve minimum V_{min} on $\partial B(0, \varepsilon)$ $\qquad V_{min} > 0$

δ can chosen s.t.
whenever $\|x\| < \delta$, $|V(x)| < V_{min}$

$$\frac{d V \circ F^t(x)}{dt} < 0 \quad \forall\ x \in \mathcal{N}$$

$\|F^t_x\| < \varepsilon. \quad \forall\ t > 0$
if $\|x\| < \delta$ because $V(y) \ge V_{min}$ $\forall\ y \in \partial B(0, \varepsilon)$.

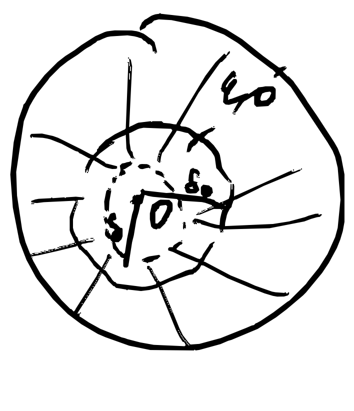
If $\frac{d}{dt} V \circ F^t(x) < 0$ (negative definite)

then asymptotic stability

Proof:

Want: For some $\delta > 0$, if $\|x\| < \delta$, then $\lim_{t\to\infty} F^t x = 0$.

We already showed uniform stability.

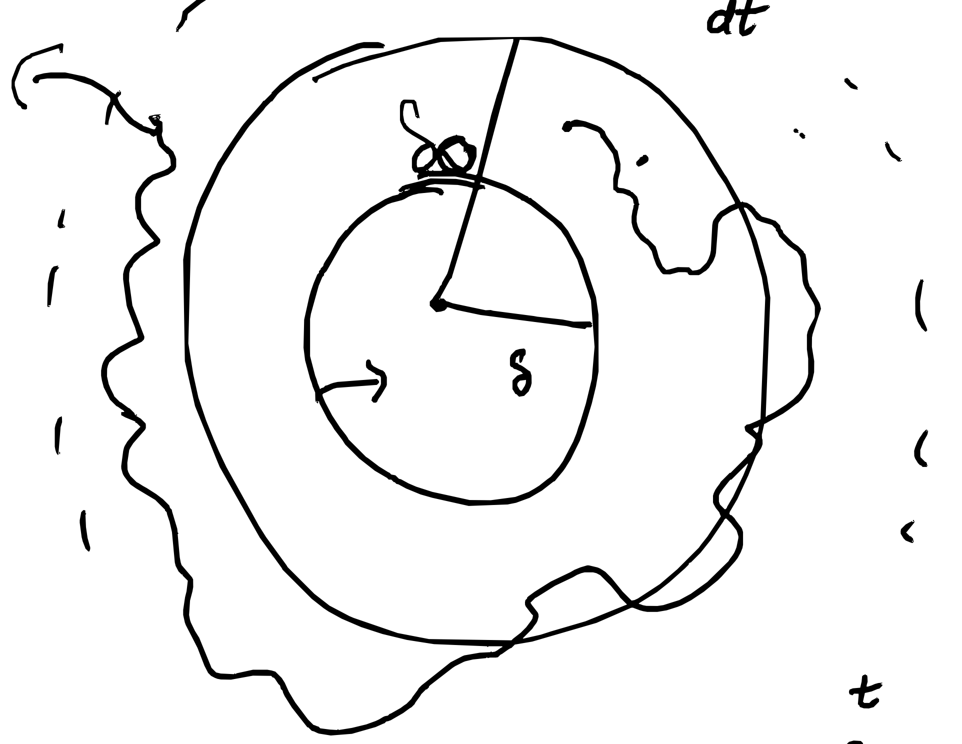


Take $\delta > 0$ arbitrarily small.

$$\left| \frac{d}{dt} V \circ F^t(x) \right| \leq \dot{V}_{max}$$

for all $x \in B(\delta, \varepsilon_0)$

(Continuity of $x \to \frac{d}{dt} V \circ F^t(x)$)



$$0 < V(F^t x) = V(x) + \int_0^t \underbrace{\frac{d}{ds} V \circ F^s(x)}\, ds$$

$$\leq \underline{V(x)} + \dot{V}_{max}\, t$$

($\frac{d}{dt} V \circ F^t$ negative definite, $\dot{V}_{max} < 0$)

Since δ is arbitrarily small, $V(F^t x) > 0$ while $\dot{V}_{max} < 0$, we must have that $\| F^t x \| < \delta$ for some finite t.

$$\lim_{t\to\infty} F^t x = 0$$

Strong Lyapunov function:

$V(x) > 0$

$\frac{d}{dt} V \circ F^t(x) < 0$.

→ How to find V?

→ How big is "domain of attraction"?

Control theory, Robotics, AEV, Reinforcement learning
UAV

Control Lyapunov function

Non-autonomous systems

$$\frac{dF^t x}{dt} = v(F^t x, t, u(t))$$

control function

Optimal control (next class)

$x \to V(x)$

$$\frac{d}{dt} V \circ F^t(x) = dV(F^t x) \cdot \frac{dF^t x}{dt}$$

$$= dV(F^t x)\, v(F^t x, t, u(t))$$

is negative definite for some $t \to u(t)$ at all x.

If there exists a function Λ s.t.

- $\Lambda(x) > 0$ for some x in every neighborhood of origin

- $\frac{d}{dt} \Lambda \circ f^t(x) \geq 0$ is positive semidefinite

then, origin is unstable.

$$\Lambda(F^t x) = \Lambda(x) + \int_0^t \frac{d}{ds} \Lambda \circ F^s(x) ds$$

$$\geq \Lambda(x) + \dot{\Lambda}_{min} t$$

linear dynamics

$$\frac{d x(t)}{dt} = A x(t)$$

Want: Lyapunov function

$$V(x) > 0$$

$$\frac{d}{dt} V(x(t)) < 0$$

$$\parallel$$

$$\underline{\nabla V(x(t))}\, \underline{A}\, x(t) < 0$$

$$V(x) = x^T C x \qquad C > 0$$

$$\frac{d}{dt} V(x(t)) = \left(\frac{d x(t)}{dt}\right)^T C\, x(t) + x(t)^T C \frac{d x(t)}{dt}$$

$$\overset{\parallel}{A x(t)}$$

$$= x(t)^T A^T C x(t) + (x(t))^T C A x(t)$$

$$= (x(t))^T (A^T C + CA) x(t)$$

$$< 0 \qquad \forall t$$

Lyapunov equation

Solve for C s.t. $A^T C + CA = P$

where P is a negative definite matrix

Then, $V(x) = x^T C x$ is a Lyapun

Ex: if A has at least one positive eigenvalue, then show that $\exists\ \Lambda$ (opp. of a Lyap. fn).

$$\Lambda(x) = x^T D^{-1} x$$

where $A = P^{-1} D P$

$$\Lambda(x + e_k) > 0$$

d_k is a positive eigenvalue

"Mildly" nonlinear systems

$$\frac{dx(t)}{dt} = Ax(t) + h(x(t)) \to (1)$$

where $h \in C^1$, $h(0) = 0$

$dh(0) = 0$, $h(x) = h(0) + dh(0)\,x + \|x\|^2$

if $\frac{dx(t)}{dt} = Ax(t)$ is asymptotically stable, then,

(1) is also asymptotically stable.

Remark: There can be "nearby" Lyapunov functions to a Lyapunov functions

$$V(x) = x^T C x$$

$$\frac{d}{dt} V(x(t)) = \left(\frac{dx}{dt}\right)^T C x + x^T C (Ax + h(x))$$

$$= (x^T A^T + h^T) C x + x^T C (Ax + h)$$

$$= x^T \underbrace{(A^T C + CA)} x + 2h^T C x$$

$(\because \exists$ soln to Lyap eq. $A^T C + CA = P < 0)$

whenever $\|x\| < \delta$

$\|h(x)\| < L\|x\|$ (MVT, Con. of dh at 0)

$$\|h^T C x\| \leq \|h(x)\|\,\|C\|\,\|x\|$$

$$\leq L\,\|C\|\,\|x\|^2$$

$$\leq \frac{\|x\|^2}{2}$$

Can be arranged.

$\to$ Convex optimization

$\hookrightarrow$ How to find Lyapunov functions

$\to$ optimal control