

Optimal transport - different formulations

Kantorovich problem

$$\min_{\gamma \in \Gamma} \int c(x, y) d\gamma(x, y)$$

where c = cost function on $X \times Y$ as before

Γ : space of joint distributions of random variables X and Y with marginals μ & ν respectively.

Recall that a marginal distribution of a random variable X at x

$$= \int_Y d\gamma(x, y)$$

↙ where γ is the joint distribution of X & Y
"integrating out"
or "marginalizing over Y ."

In the language of transport maps and pushforward distributions,

if $\pi_{X\#}$ is a map from $X \times Y$ into X such that

$$\pi_{X\#}(x, y) = x$$

then $\pi_{X\#} \gamma = \mu$

and $\pi_{Y\#} \gamma = \nu$

if γ is the joint distribution of X & Y
and μ is the distribution (marginal)
of X

and ν is the marginal of Y .

where both X and Y are absolutely
continuous, i.e. have densities,

$$\frac{d\mu(x)}{d\text{Leb}} =: p_X(x) = \int_Y \frac{d\gamma(x, y)}{d\text{Leb}}$$

↓
joint density of
 X & Y .

$$= \int p_{X|Y}(x/y) p_Y(y) dy$$

↓
conditional
density of X given Y

Going back to the Kantorovich problem,

$$KP^* = \min_{\gamma \in \Gamma} \int c(x, y) d\gamma(x, y)$$

$$\Gamma = \left\{ \gamma: X \times Y \rightarrow \mathbb{R}^+ \text{ with } \pi_X \gamma = \mu \text{ and } \pi_Y \gamma = \nu \right\}$$

First we note that this constraint set Γ consists of joint distributions of the form $(Id_X, T)_\# \mu$

with Id_X being the identity function on X
 $T: X \rightarrow Y$ being a transport map from μ to ν
ie $T_\# \mu = \nu$

So, the minimum of the Monge problem
 $\geq KP^*$

When μ is absolutely continuous and $c: X \times Y \rightarrow \mathbb{R}^+$ is continuous* and bounded
*(technically, lower semi continuous, which means supremum of a family of continuous functions, is enough)

from below, then a solution exists
for the KP.

Additionally when $X = Y \subset \mathbb{R}^d$ compact
then, the solution of the Monge
& Kantorovich problems coincide.

In other words, KP admits a
solution of the form
 $(Id, T)_{\#} \mu$