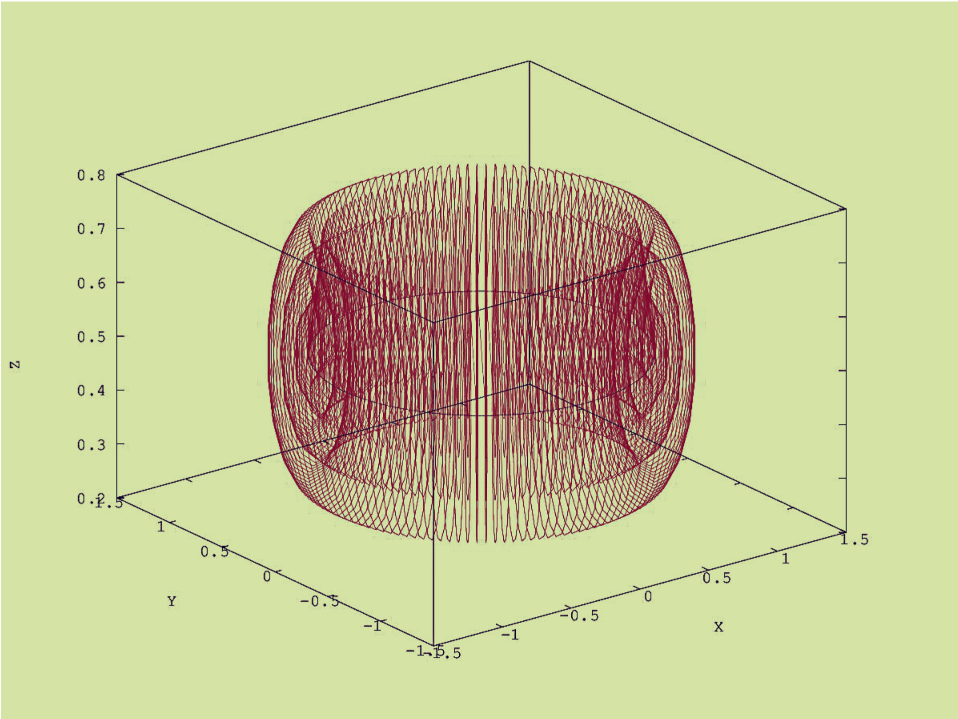

Figure 1: Vortex Torus with ring axis in blue

Simulation of Toroidal Vortex Ring(From Ghoniem's paper[1])

0.1 Geometry and Initial Vorticity Distribution

Here we look at a toroidal vortex ring with an initial vorticity distribution modeled by a third order Gaussian function:

$$\boldsymbol{\omega}(\mathbf{x}) = \frac{1}{a\sigma^2} exp\left(-r_{\mathbf{x}}^3/\sigma^3\right) \mathbf{e}_\theta \tag{1}$$

The vorticity is along the direction $\mathbf{e}_\theta$ which is tangential to the ring axis(shown in the figure). The ring axis is the circle, $x^2 + y^2 = R^2$ on the plane $z = z_i$. The magnitude of the total vorticity is independent of θ where (ρ, θ, z) represent coordinates in the cylindrical coordinate system.

The cross section of the ring, which is circular, has a radius, $\sigma = 0.275R$. In the figure, $R = 1$ and $z_i = 0.5$.

In a particular cross section,

- r is the radial distance measured from the centre of the core of the ring or the centre of a cross section. For all $\mathbf{x}$ within the ring, $0 \leq r_{\mathbf{x}} \leq \sigma$.

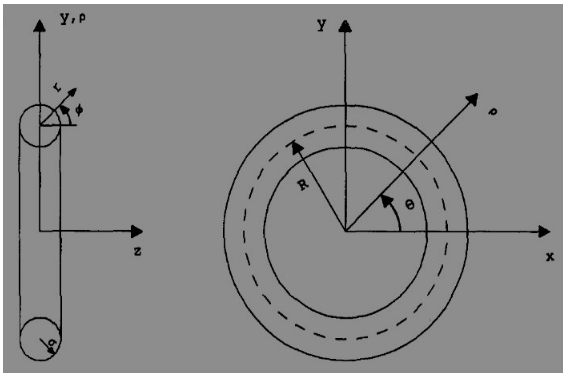

Figure 2: From [1]

- ϕ is the angle measured, at every cross section, from the line passing through the centre of the cross section and parallel to the z axis.

0.2 Discretization of the Ring

The toroidal ring is divided into many small vortex tubes. Here, we make an equi-spaced mesh found in the third position on the third row of Fig.11 b of the paper[1]. The ring is discretized at N_c cross sections and each cross-section is discretized at N_per_θ locations in total.

Of these N_per_θ locations, N_θ points are at each radial location and there are N_r radial locations per cross section.

In the code, $N_r = 3$ and N_θ is a multiple of 6. Hence, there are $N\_per\_\theta = 1 + 6 + 12 + 18 = 37$ elements (including one at the centre) in total per cross section.

Therefore, the ring is divided into $N = N_c \times N\_per\_\theta$ vortex tubes in total. In the code, $N_c = 120$. Hence, we have 4440 vortex elements which are part of N_per_θ different vortex tubes.

0.3 Evaluating Strength Vectors

In a regularized particle vortex method, the total vorticity field at a given $\mathbf{x}$ at time t is written as:

$$\boldsymbol{\omega}(\mathbf{x}, t) = \sum_{i=1}^{N} \boldsymbol{\alpha}_i(t)\zeta_\delta(\mathbf{x} - \mathbf{x}_i(t)) \tag{2}$$

where, $\boldsymbol{\alpha}_i$ is the strength vector associated with the vortex element i whose center is at $\mathbf{x}_i$. And, the regularized core function,

$$\zeta_\delta(\mathbf{x} - \mathbf{x}_i) = \frac{1}{\delta^3}\zeta(|\mathbf{x} - \mathbf{x}_i|/\delta) \tag{3}$$

where

$$\zeta(s) = \frac{15}{8\pi}\frac{1}{(1+s^2)^{7/2}} \tag{4}$$

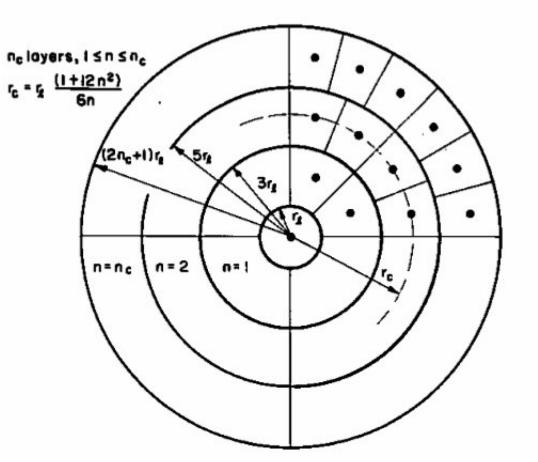

Figure 3: From [2]

In the paper[1], a Gaussian core function is used. With the above mentioned core function that provides higher order algebraic smoothing[2], the accuracy in the total circulation is maintained as well, justifying its selection.

The Strength Vector, $\boldsymbol{\alpha}_i$ is actually the contribution to the vorticity of the small vortex tube element i or the product of the element's circulation and length. That is,

$$\boldsymbol{\alpha}_i(t) = \boldsymbol{\omega}_i dV_i = \Gamma_i \delta \boldsymbol{\chi}_i \tag{5}$$

Here, $\delta \boldsymbol{\chi}_i$ is a small length along the vorticity vector (or along a material line). It can be thought of the length of the small vortex tube elements in the Lagrangian system. Therefore, the initial vorticity field, as in Eq:2 can be expressed as :

$$\boldsymbol{\omega}(\mathbf{x}, 0) = \sum_{i=1}^{N} \Gamma_i \delta \mathbf{x}_i \; \zeta_\delta(\mathbf{x} - \mathbf{x}_i) \tag{6}$$

$$= \frac{1}{a\sigma^2} exp\left(-r_{\mathbf{x}}^3/\sigma^3\right) \mathbf{e}_\theta \tag{7}$$

where, $\delta \mathbf{x}_i = \delta \boldsymbol{\chi}_i(T = 0)$.

The above equations are expressed as a system of linear equations for $\mathbf{x}$ being each of the $\mathbf{x}_i$ s or the centres of the elements. Hence, the system of equations solved for the circulation in the N_per_θ vortex tubes can be written as:

$$\begin{bmatrix} \sum_{j=1}^{N_c} \zeta_\delta(\mathbf{x}_1 - \mathbf{x}_j)\delta\mathbf{x}_j & \sum_{j=N_c+1}^{2\times N_c} \zeta_\delta(\mathbf{x}_1 - \mathbf{x}_j)\delta\mathbf{x}_j & . & . & \sum_{j=N-N_c+1}^{N} \zeta_\delta(\mathbf{x}_1 - \mathbf{x}_j)\delta\mathbf{x}_j \\ \sum_{j=1}^{N_c} \zeta_\delta(\mathbf{x}_2 - \mathbf{x}_j)\delta\mathbf{x}_j & \sum_{j=N_c+1}^{2\times N_c} \zeta_\delta(\mathbf{x}_2 - \mathbf{x}_j)\delta\mathbf{x}_j & . & . & \sum_{j=N-N_c+1}^{N} \zeta_\delta(\mathbf{x}_2 - \mathbf{x}_j)\delta\mathbf{x}_j \\ . & . & . & . & . \\ . & . & . & . & . \\ \sum_{j=1}^{N_c} \zeta_\delta(\mathbf{x}_N - \mathbf{x}_j)\delta\mathbf{x}_j & \sum_{j=N_c+1}^{2\times N_c} \zeta_\delta(\mathbf{x}_N - \mathbf{x}_j)\delta\mathbf{x}_j & . & . & \sum_{j=N-N_c+1}^{N} \zeta_\delta(\mathbf{x}_N - \mathbf{x}_j)\delta\mathbf{x}_j \end{bmatrix} \tag{8}$$

$$\times \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ . \\ . \\ \Gamma_N \end{bmatrix} = \begin{bmatrix} \frac{1}{a\sigma^2} exp\left(-r_{\mathbf{x}_1}^3/\sigma^3\right) \\ \frac{1}{a\sigma^2} exp\left(-r_{\mathbf{x}_2}^3/\sigma^3\right) \\ . \\ . \\ \frac{1}{a\sigma^2} exp\left(-r_{\mathbf{x}_N}^3/\sigma^3\right) \end{bmatrix}$$

These N_r equations are solved to the give the circulations of the N_r vortex tubes. It must be noted that $\delta\mathbf{x}_i$ or the initial length of the vortex tube i depends both on the radial and azimuthal positions of the tube within a cross section or r and ϕ respectively.

$$\delta\mathbf{x}_i = 2\pi \times (R + r_{\mathbf{x}_i} \cos\phi) \tag{9}$$

($\mathbf{x}_i$ is the centre of the ith vortex element).

For the discretization described above, the total circulation of the ring is close to the theoretical value of 2.

$$\sum_{i=1}^{N_r} \Gamma_i = 2.0546 \tag{10}$$

0.4 Evolution of Ring

For the first iteration, with 4400 particles (or vortex elements), direct evaluation and FMM computation were performed for the velocity and the stretching term of each particle.

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{u}(\mathbf{x}_i(t), t) \tag{11}$$

$$= \frac{-1}{4\pi} \sum_{j=1}^{N} \frac{(\mathbf{x}_i - \mathbf{x}_j)^2 + (5/2)\delta^2}{((\mathbf{x}_i - \mathbf{x}_j)^2 + \delta^2)^{5/2}} (\mathbf{x}_i - \mathbf{x}_j) \times \alpha_j \tag{12}$$

$$\frac{d\delta\boldsymbol{\chi}_i}{dt} = \delta\boldsymbol{\chi}_i.\nabla\mathbf{u}(\mathbf{x}_i(t),t) \tag{13}$$

$$= \frac{1}{4\pi}\sum_{j=1}^{N}\left(\frac{|\mathbf{x}_i-\mathbf{x}_j|^2+5/2\delta^2}{(|\mathbf{x}_i-\mathbf{x}_j|^2+\delta^2)^{5/2}}\delta\boldsymbol{\chi}_i\times\delta\boldsymbol{\chi}_j\right. \tag{14}$$

$$+\,3\frac{(|\mathbf{x}_i-\mathbf{x}_j|^2+7/2\delta^2)}{(|\mathbf{x}_i-\mathbf{x}_j|^2+\delta^2)^{7/2}}\times \tag{15}$$

$$\left.(\delta\boldsymbol{\chi}_i.((\mathbf{x}_i-\mathbf{x}_j)\times\delta\boldsymbol{\chi}_j)(\mathbf{x}_i-\mathbf{x}_j)\right) \tag{16}$$

The FMM was accurate upto the 5th decimal place when compared to the direct evaluation.

Bibliography

[1] *Omar M. Knio and Ahmed F. Ghoniem*,Numerical Study of a Three Dimensional Vortex Method, Journal of Computational Physics,1990

[2] *G.S. Wincklemans and A. Leonard* ,Contributions to Vortex Particle Methods for the Computation of Three-Dimensional Incompressible Flows, Journal of Computational Physics, 1993.