

Figure 1: Vortex Torus with ring axis in blue

# Simulation of Toroidal Vortex Ring(From Ghoniem's paper[1])

## 0.1 Geometry and Initial Vorticity Distribution

Here we look at a toroidal vortex ring with an initial vorticity distribution modeled by a third order Gaussian function:

$$\boldsymbol{\omega}(\mathbf{x}) = \frac{1}{a\sigma^2} \exp(-r_{\mathbf{x}}^3/\sigma^3) \mathbf{e}_{\theta} \quad (1)$$

The vorticity is along the direction  $\mathbf{e}_{\theta}$  which is tangential to the ring axis(shown in the figure). The ring axis is the circle,  $x^2 + y^2 = R^2$  on the plane  $z = z_i$ . The magnitude of the total vorticity is independent of  $\theta$  where  $(\rho, \theta, z)$  represent coordinates in the cylindrical coordinate system.

The cross section of the ring, which is circular, has a radius,  $\sigma = 0.275R$ . In the figure,  $R = 1$  and  $z_i = 0.5$ .

In a particular cross section,

- $r$  is the radial distance measured from the centre of the core of the ring or the centre of a cross section. For all  $\mathbf{x}$  within the ring,  $0 \leq r_{\mathbf{x}} \leq \sigma$ .

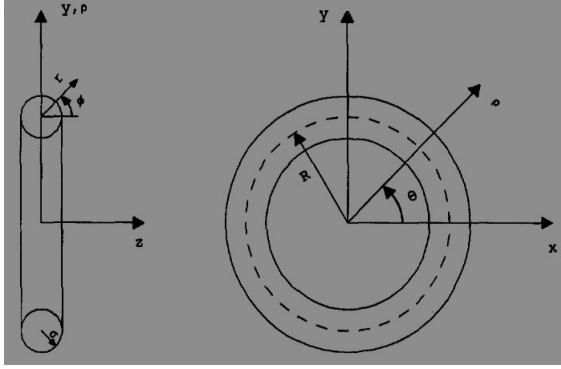


Figure 2: From [1]

- $\phi$  is the angle measured, at every cross section, from the line passing through the centre of the cross section and parallel to the  $z$  axis.

## 0.2 Discretization of the Ring

The toroidal ring is divided into many small vortex tubes. Here, we make an equi-spaced mesh found in the third position on the third row of Fig.11 b of the paper[1]. The ring is discretized at  $N_c$  cross sections and each cross-section is discretized at  $N_{per\_theta}$  locations in total.

Of these  $N_{per\_theta}$  locations,  $N_\theta$  points are at each radial location and there are  $N_r$  radial locations per cross section.

In the code,  $N_r = 3$  and  $N_\theta$  is a multiple of 6. Hence, there are  $N_{per\_theta} = 1 + 6 + 12 + 18 = 37$  elements (including one at the centre) in total per cross section.

Therefore, the ring is divided into  $N = N_c \times N_{per\_theta}$  vortex tubes in total. In the code,  $N_c = 120$ . Hence, we have 4440 vortex elements which are part of  $N_{per\_theta}$  different vortex tubes.

## 0.3 Evaluating Strength Vectors

In a regularized particle vortex method, the total vorticity field at a given  $\mathbf{x}$  at time  $t$  is written as:

$$\boldsymbol{\omega}(\mathbf{x}, t) = \sum_{i=1}^N \boldsymbol{\alpha}_i(t) \zeta_\delta(\mathbf{x} - \mathbf{x}_i(t)) \quad (2)$$

where,  $\boldsymbol{\alpha}_i$  is the strength vector associated with the vortex element  $i$  whose center is at  $\mathbf{x}_i$ . And, the regularized core function,

$$\zeta_\delta(\mathbf{x} - \mathbf{x}_i) = \frac{1}{\delta^3} \zeta(|\mathbf{x} - \mathbf{x}_i|/\delta) \quad (3)$$

where

$$\zeta(s) = \frac{15}{8\pi} \frac{1}{(1 + s^2)^{7/2}} \quad (4)$$

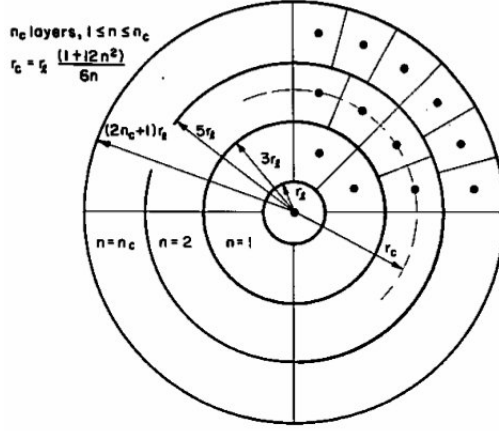


Figure 3: From [2]

In the paper[1], a Gaussian core function is used. With the above mentioned core function that provides higher order algebraic smoothing[2], the accuracy in the total circulation is maintained as well, justifying its selection.

The Strength Vector,  $\alpha_i$  is actually the contribution to the vorticity of the small vortex tube element  $i$  or the product of the element's circulation and length. That is,

$$\alpha_i(t) = \omega_i dV_i = \Gamma_i \delta\chi_i \quad (5)$$

Here,  $\delta\chi_i$  is a small length along the vorticity vector (or along a material line). It can be thought of the length of the small vortex tube elements in the Lagrangian system. Therefore, the initial vorticity field, as in Eq:2 can be expressed as :

$$\omega(\mathbf{x}, 0) = \sum_{i=1}^N \Gamma_i \delta\mathbf{x}_i \zeta_\delta(\mathbf{x} - \mathbf{x}_i) \quad (6)$$

$$= \frac{1}{a\sigma^2} \exp(-r_{\mathbf{x}}^3/\sigma^3) \mathbf{e}_\theta \quad (7)$$

where,  $\delta\mathbf{x}_i = \delta\chi_i(T = 0)$ .

The above equations are expressed as a system of linear equations for  $\mathbf{x}$  being each of the  $\mathbf{x}_i$  s or the centres of the elements. Hence, the system of equations

solved for the circulation in the  $N_{per}\theta$  vortex tubes can be written as:

$$\begin{aligned}
& \begin{bmatrix} \sum_{j=1}^{N_c} \zeta_\delta(\mathbf{x}_1 - \mathbf{x}_j) \delta \mathbf{x}_j & \sum_{j=N_c+1}^{2 \times N_c} \zeta_\delta(\mathbf{x}_1 - \mathbf{x}_j) \delta \mathbf{x}_j & \cdot & \cdot & \sum_{j=N-N_r+1}^{N_c * N_r} \zeta_\delta(\mathbf{x}_1 - \mathbf{x}_j) \delta \mathbf{x}_j \\ \sum_{j=1}^{N_c} \zeta_\delta(\mathbf{x}_2 - \mathbf{x}_j) \delta \mathbf{x}_j & \sum_{j=N_c+1}^{2 \times N_c} \zeta_\delta(\mathbf{x}_2 - \mathbf{x}_j) \delta \mathbf{x}_j & \cdot & \cdot & \sum_{j=N-N_r+1}^N \zeta_\delta(\mathbf{x}_2 - \mathbf{x}_j) \delta \mathbf{x}_j \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum_{j=1}^{N_c} \zeta_\delta(\mathbf{x}_N - \mathbf{x}_j) \delta \mathbf{x}_j & \sum_{j=N_c+1}^{2 \times N_c} \zeta_\delta(\mathbf{x}_N - \mathbf{x}_j) \delta \mathbf{x}_j & \cdot & \cdot & \sum_{j=N-N_r+1}^N \zeta_\delta(\mathbf{x}_N - \mathbf{x}_j) \delta \mathbf{x}_j \end{bmatrix} \\
& = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \cdot \\ \cdot \\ \Gamma_N \end{bmatrix} \begin{bmatrix} \frac{1}{a\sigma^2} \exp(-r_{\mathbf{x}_1}^3/\sigma^3) \\ \frac{1}{a\sigma^2} \exp(-r_{\mathbf{x}_2}^3/\sigma^3) \\ \cdot \\ \cdot \\ \frac{1}{a\sigma^2} \exp(-r_{\mathbf{x}_N}^3/\sigma^3) \end{bmatrix} \quad (8)
\end{aligned}$$

These  $N_r$  equations are solved to give the circulations of the  $N_r$  vortex tubes. It must be noted that  $\delta \mathbf{x}_i$  or the initial length of the vortex tube  $i$  depends both on the radial and azimuthal positions of the tube within a cross section or  $r$  and  $\phi$  respectively.

$$\delta \mathbf{x}_i = 2\pi \times (R + r_{\mathbf{x}_i} \cos \phi) \quad (9)$$

( $\mathbf{x}_i$  is the centre of the  $i$ th vortex element ).

For the discretization described above, the total circulation of the ring is close to the theoretical value of 2.

$$\sum_{i=1}^{N_r} \Gamma_i = 2.0546 \quad (10)$$

## 0.4 Evolution of Ring

For the first iteration, with 4400 particles (or vortex elements), direct evaluation and FMM computation were performed for the velocity and the stretching term of each particle.

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{u}(\mathbf{x}_i(t), t) \quad (11)$$

$$= \frac{-1}{4\pi} \sum_{j=1}^N \frac{(\mathbf{x}_i - \mathbf{x}_j)^2 + 5/2\delta^2}{((\mathbf{x}_i - \mathbf{x}_j)^2 + \delta^2)^{5/2}} (\mathbf{x}_i - \mathbf{x}_j) \times \alpha_j \quad (12)$$

$$\frac{d\delta\chi_i}{dt} = \delta\chi_i \cdot \nabla \mathbf{u}(\mathbf{x}_i(t), t) \quad (13)$$

$$= \frac{1}{4\pi} \sum_{j=1}^N \left( \frac{|\mathbf{x}_i - \mathbf{x}_j|^2 + 5/2\delta^2}{(|\mathbf{x}_i - \mathbf{x}_j|^2 + \delta^2)^{5/2}} \delta\chi_i \times \delta\chi_j \right. \quad (14)$$

$$+ 3 \frac{(|\mathbf{x}_i - \mathbf{x}_j|^2 + 7/2\delta^2)}{(|\mathbf{x}_i - \mathbf{x}_j|^2 + \delta^2)^{7/2}} \times \quad (15)$$

$$\left. (\delta\chi_i \cdot ((\mathbf{x}_i - \mathbf{x}_j) \times \delta\chi_j)(\mathbf{x}_i - \mathbf{x}_j) \right) \quad (16)$$

The FMM was accurate upto the 5th decimal place when compared to the direct evaluation.

# Bibliography

- [1] *Omar M. Knio and Ahmed F. Ghoniem*, Numerical Study of a Three Dimensional Vortex Method, Journal of Computational Physics, 1990
- [2] *G.S. Wincklemans and A. Leonard* , Contributions to Vortex Particle Methods for the Computation of Three-Dimensional Incompressible Flows, Journal of Computational Physics, 1993.