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Question 1:

a) We show that under Definition 1, the following holds:

$$
\begin{aligned}
a^* &\overset{!}{=} 1 \oplus a \otimes a^* && (1)\\
&= 1 \oplus a \otimes \bigotimes_{n=0}^{\infty} a^{\otimes n} && (2)\\
&= 1 \oplus \bigotimes_{n=0}^{\infty} a^{\otimes n+1} && (3)\\
&= 1 \oplus \bigotimes_{n=1}^{\infty} a^{\otimes n} && (4)\\
&= \bigotimes_{n=0}^{\infty} a^{\otimes n} && (5)\\
&= a^* && (6)
\end{aligned}
$$

b) First we show that $\log a \oplus a = \log(2) + a$:

$$
\begin{aligned}
a \oplus \log a &= \log(e^a + e^a) && (7)\\
&= \log(2e^a) && (8)\\
&= \log(2) + a && (9)
\end{aligned}
$$

Then we calculate the Kleene start:

$$
\begin{aligned}
\bigoplus_{\log\ n=0}^{\infty} a^{\oplus n} &= a^{\oplus 0} \oplus_{\log} \left(\bigoplus_{\log\ n=1}^{\infty} a^{\oplus n} \right) && (10)\\
&= 0 \oplus_{\log} \left(\bigoplus_{\log\ n=1}^{\infty} a^{\oplus n} \right) && (11)\\
&= 0 \oplus_{\log} a \oplus_{\log} 2a \oplus_{\log} 3a \oplus_{\log} \cdots && (12)\\
&= \log\left(e^0 + e^a\right) \oplus_{\log} 2a \oplus_{\log} 3a \oplus_{\log} \cdots && (13)\\
&= \log\left(e^{\log(e^0+e^a)} + e^{2a}\right) \oplus_{\log} 3a \oplus_{\log} \cdots && (14)\\
&= \log\left(e^0 + e^a + e^{2a}\right) \oplus_{\log} 3a \oplus_{\log} \cdots && (15)\\
&= \log\left(\sum_{n=0}^{\infty} e^{a^{\oplus n}}\right) && (16)
\end{aligned}
$$

We have two cases here, either $a > 0$ or $a \leq 0$. In the first case, the sum diverges and we get ∞. In the second case, the sum converges:

$$\sum_{n=0}^{\infty} e^{a^{\oplus n}} = \frac{1}{1 - e^a} \qquad \text{limit geometric series} \tag{17}$$

Therefore, we get:

$$\begin{aligned}
\log\left(\sum_{n=0}^{\infty} e^{a^{\oplus n}}\right) &= \log\left(\frac{1}{1 - e^a}\right) && (18)\\
&= \log(1) - \log(1 - e^a) && (19)\\
&= \log(1 - e^a) && (20)
\end{aligned}$$

c) First we derive a closed form solution for $(x, y)^{\oplus n}$:

$$\begin{aligned}
\langle x, y\rangle^{\oplus 1} &= \langle x, y\rangle && (21)\\
\langle x, y\rangle^{\oplus 2} &= \langle x^2, 2xy\rangle && (22)\\
\langle x, y\rangle^{\oplus 3} &= \langle x^3, 3x^2y\rangle && (23)\\
\langle x, y\rangle^{\oplus 4} &= \langle x^4, 4x^3y\rangle && (24)\\
\langle x, y\rangle^{\oplus n} &= \langle x^n, nx^{n-1}y\rangle && (25)
\end{aligned}$$

Then we derive a closed form for a^*:

$$\begin{aligned}
\bigoplus_{n=0}^{\infty} \langle x, y\rangle^{\oplus n} &= \bigoplus_{n=0}^{\infty} \langle x^n, nx^{n-1}y\rangle && \text{eq. (25)} \quad (26)\\
&= \left\langle \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} nx^{n-1}y \right\rangle && (27)
\end{aligned}$$

Both parts only converge for $|x| < 1$. The left parts is a geometric series and has limit $\frac{1}{1-x}$. And the right parts, which is a power series:

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^{n-1}y &= y \sum_{n=0}^{\infty} nx^{n-1} && (28)\\
&= y\left(0 + \sum_{n=1}^{\infty} nx^{n-1}\right) && (29)\\
&= y \sum_{n=0}^{\infty} (n+1)x^n && (30)\\
&= y \frac{1}{(x-1)^2} && \text{limit power series} \quad (31)\\
&= \frac{y}{(x-1)^2} && (32)
\end{aligned}$$

Hence our closed form with inserted limits is given by:

$$\left\langle \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} nx^{n-1}y \right\rangle = \left\langle \frac{1}{1-x}, \frac{y}{(x-1)^2} \right\rangle \qquad |x| \leq 1 \tag{33}$$

If $|x| > 1$ both parts diverge.

d)

$$\mathcal{W}_{\text{lang}} = \left\langle 2^{\Sigma^*}, \bigcup, \otimes, \{\}, \{\epsilon\} \right\rangle \tag{34}$$

We first show that $\mathcal{W}_{\text{lang}}$ is a semiring:

- $(2^{\Sigma^*}, \oplus, \mathbf{0})$ must be a commutative monoid with identity element $\mathbf{0}$:

$$\begin{aligned}
(x \oplus y) \oplus z &= (x \cup y) \cup z && (35)\\
&= \{x, y\} \cup z && (36)\\
&= \{x, y, z\} && (37)\\
&= x \oplus \{y, z\} && (38)\\
&= x \oplus (y \oplus z) && (39)
\end{aligned}$$

$$\begin{aligned}
\mathbf{0} \oplus x &= \{\} \oplus x && (40)\\
&= \{\} \cup x && (41)\\
&= \oplus && (42)
\end{aligned}$$

$$\begin{aligned}
x \oplus y &= x \cup y && (43)\\
&= \{x, y\} && (44)\\
&= y \cup x && (45)\\
&= y + x && (46)
\end{aligned}$$

- $(2^{\Sigma^*}, \otimes, \mathbf{1})$ must be a monoid with identity element $\mathbf{1}$:

$$\begin{aligned}
(x \otimes y) \otimes z &= xy \otimes z && (47)\\
&= xyz && (48)\\
&= x \otimes yz && (49)\\
&= x \otimes (y \otimes z) && (50)
\end{aligned}$$

$$\begin{aligned}
\mathbf{1} \otimes x &= \{\epsilon\} \otimes x && (51)\\
&= x && (52)\\
&= x \otimes \{\epsilon\} && (53)\\
&= x \otimes \mathbf{1} && (54)
\end{aligned}$$

- Multiplication left and right distributes over addition:

$$\begin{aligned}
x \otimes (y \oplus z) &= x \otimes \{y, z\} && (55)\\
&= \{xy, xz\} && (56)\\
&= \{xy\} \cup \{xz\} && (57)\\
&= (x \otimes y) \oplus (x \otimes z) && (58)
\end{aligned}$$

$$\begin{aligned}
(x \oplus y) \otimes z &= \{x, y\} \otimes z && (59)\\
&= \{xz, yz\} && (60)\\
&= \{xz\} \cup \{yz\} && (61)\\
&= (x \otimes z) \oplus (y \otimes z) && (62)
\end{aligned}$$

- Multiplication by $\mathbf{0}$ annihilates $\mathbb{R} \times \mathbb{R}$:

$$\begin{aligned}
\mathbf{0} \otimes x &= \{a \circ b \,|\, a \in A, b \in \{\}\} && (63)\\
&= \{\} && \text{by definition of } \circ \text{, because no } b \text{ exists} \quad (64)\\
&= x \otimes \mathbf{0} && (65)\\
& && (66)
\end{aligned}$$

The Kleene start for $\mathcal{W}_{\text{lang}}$ given by:

$$\begin{aligned}
A^{\otimes n} &= \{\bigotimes_{i=0}^{n} a_i \mid a_i \in A\} && (67)\\
A^* &= \bigoplus_{n=0}^{\infty} A^{\otimes n} && (68)\\
&= \bigoplus_{n=0}^{\infty} \left\{\bigotimes_{i=0}^{n} a_i \mid a_i \in A\right\} && (69)\\
&= \left\{\bigotimes_{i=0}^{n} a_i \mid a_i \in A, n \in \mathbb{Z}\right\} && (70)
\end{aligned}$$

Question 2:

a) Tropical semiring is 0-closed:

$$\begin{aligned}
a \oplus \mathbf{0} &= \min(a, \mathbf{0}) && (71)\\
&= \min(a, 0) && (72)\\
&= 0 && \text{because } a \in \mathbb{R}_{\geq 0} \quad (73)
\end{aligned}$$

Arctic semiring is 0-closed:

$$\begin{aligned}
a \oplus \mathbf{0} &= \max(a, \mathbf{0}) && (74)\\
&= \max(a, 0) && (75)\\
&= 0 && \text{because } a \in \mathbb{R}_{\leq 0} \quad (76)
\end{aligned}$$

b) Proof by induction:
Base case $i = 1$:

$$\begin{aligned}
M^1 &= M && (77)\\
M_{ij} &= w_{ij} && \text{by def of } M \quad (78)
\end{aligned}$$

w_{ij} is exactly the semiring-sum over all paths from i to j of length 1. This holds because there is only one path of length 1 from i to j.
Induction hypothesis: M_{ij}^i is the semiring-sum over all paths from i to j of length i.
Induction step $i \to i + 1$:

$$\begin{aligned}
M^{i+1} &= M^i \otimes M && (79)\\
M_{kj}^{i+1} &= \sum_{l=0}^{n} M_{kl}^i \otimes M_{lj} && \text{def. matrix mult.} \quad (80)
\end{aligned}$$

In eq. (80) we sum over the product of all possible paths of length i from k to another node l and all possible paths of length 1 from nodes l to j. This sum is exactly the semiring-sum over all possible paths of length $i+1$ from k to j.

c) s

d) Per definition of the Kleene start, we have:

$$M^* = \bigoplus_{i=0}^{\infty} M^{\otimes i} \tag{81}$$

In b) we have shown that $M^{\otimes i}$ is the semiring-sum over all paths of length i and in c) we have shown that the shortest path depends only on paths of length $l \leq N-1$. We also showed that under the $\bigoplus$ operation, only paths of length $l \leq N-1$ are considered. Therefore, we know that M^* depends only on:

$$M^* = \bigoplus_{i=0}^{N-1} M^{\otimes i} \tag{82}$$

e) We define a simple algorithm:

Algorithm 1: Matrix multiplication for Kleene star

M;
$M' \leftarrow M \;\; M^* \leftarrow \mathbf{0}$;
for $i = 0$ *to* $N-1$ **do**
    for $j = 0$ *to len(M)* **do**
        for $k = 0$ *to len(M)* **do**
            for $l = 0$ *to len(M)* **do**
                $M'_{j,k} \leftarrow M'_{j,k} \oplus (M_{j,l} \otimes M_{l,k})$
            $M^*_{j,k} \leftarrow M^*_{j,k} \oplus M'_{j,k}$
    $M \leftarrow M'$;

The inner for loops calculate the matrix multiplication, $M^{\otimes n}$, and the outer loop iterates $N-1$ times to calculate the Kleene star. Since each loop has $\mathcal{O}(N)$ iterations, the algorithm has a runtime of $\mathcal{O}(N^4)$.