Simon Wachter: Assignment 3

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13/12/2022 - 17:45h

Question 1:

a) We show that under Definition 1, the following holds:

$$\begin{aligned}
a^* &\stackrel{!}{=} 1 \oplus a \otimes a^* && (1)\\
&= 1 \oplus a \otimes \bigotimes_{n=0}^{\infty} a^{\otimes n} && (2)\\
&= 1 \oplus \bigotimes_{n=0}^{\infty} a^{\otimes n+1} && (3)\\
&= 1 \oplus \bigotimes_{n=1}^{\infty} a^{\otimes n} && (4)\\
&= \bigotimes_{n=0}^{\infty} a^{\otimes n} && (5)\\
&= a^* && (6)
\end{aligned}$$

b)

$$\mathbb{R}^* \tag{7}$$

c)

$$(\mathbb{R} \times \mathbb{R})^* \tag{8}$$

d)

$$\mathcal{W}_{\text{lang}} = \left\langle 2^{\Sigma^*}, \bigcup, \otimes, \{\}, \{\epsilon\} \right\rangle \tag{9}$$

- $(2^{\Sigma^*}, \oplus, \mathbf{0})$ must be a commutative monoid with identity element $\mathbf{0}$:

$$\begin{aligned}
(x \oplus y) \oplus z &= (x \cup y) \cup z && (10)\\
&= \{x, y\} \cup z && (11)\\
&= \{x, y, z\} && (12)\\
&= x \oplus \{y, z\} && (13)\\
&= x \oplus (y \oplus z) && (14)
\end{aligned}$$

$$
\begin{align}
\mathbf{0} \oplus x &= \{\} \oplus x \tag{15}\\
&= \{\} \cup x \tag{16}\\
&= \oplus \tag{17}
\end{align}
$$

$$
\begin{align}
x \oplus y &= x \cup y \tag{18}\\
&= \{x, y\} \tag{19}\\
&= y \cup x \tag{20}\\
&= y + x \tag{21}
\end{align}
$$

- $(2^{\Sigma^*}, \otimes, \mathbf{1})$ must be a monoid with identity element $\mathbf{1}$:

$$
\begin{align}
(x \otimes y) \otimes z &= xy \otimes z \tag{22}\\
&= xyz \tag{23}\\
&= x \otimes yz \tag{24}\\
&= x \otimes (y \otimes z) \tag{25}
\end{align}
$$

$$
\begin{align}
\mathbf{1} \otimes x &= \{\epsilon\} \otimes x \tag{26}\\
&= x \tag{27}\\
&= x \otimes \{\epsilon\} \tag{28}\\
&= x \otimes \mathbf{1} \tag{29}
\end{align}
$$

- Multiplication left and right distributes over addition:

$$
\begin{align}
x \otimes (y \oplus z) &= x \otimes \{y, z\} \tag{30}\\
&= \{xy, xz\} \tag{31}\\
&= \{xy\} \cup \{xz\} \tag{32}\\
&= (x \otimes y) \oplus (x \otimes z) \tag{33}
\end{align}
$$

$$
\begin{align}
(x \oplus y) \otimes z &= \{x, y\} \otimes z \tag{34}\\
&= \{xz, yz\} \tag{35}\\
&= \{xz\} \cup \{yz\} \tag{36}\\
&= (x \otimes z) \oplus (y \otimes z) \tag{37}
\end{align}
$$

- Multiplication by $\mathbf{0}$ annihilates $\mathbb{R} \times \mathbb{R}$:

$$
\begin{align}
\mathbf{0} \otimes x &= \{a \circ b \,|\, a \in A, b \in \{\}\} \tag{38}\\
&= \{\} && \text{by definition of } \circ\text{, because no } b \text{ exists} \tag{39}\\
&= x \otimes \mathbf{0} \tag{40}\\
& \tag{41}
\end{align}
$$

Question 2:

a) Tropical semiring is 0-closed:

$$\begin{aligned} a + \infty &= a + \infty && (42) \\ &= \infty && (43) \end{aligned}$$

Arctic semiring is 0-closed:

$$\begin{aligned} a + \infty &= a - \infty && (44) \\ &= -\infty && (45) \end{aligned}$$

b) Proof by induction:
Base case $i = 1$:

$$\begin{aligned} M^1 &= M && && (46) \\ M_{ij} &= w_{ij} && \text{by def of } M && (47) \end{aligned}$$

w_{ij} is exactly the semiring-sum over all paths from i to j of length 1. This holds because there is only one path of length 1 from i to j.
Induction hypothesis: M^i_{ij} is the semiring-sum over all paths from i to j of length i.
Induction step $i \to i + 1$:

$$\begin{aligned} M^{i+1} &= M^i \otimes M && && (48) \\ M^{i+1}_{kj} &= \sum_{l=0}^{n} M^i_{kl} \otimes M_{lj} && \text{def matrix mult.} && (49) \\ & && && (50) \end{aligned}$$

In eq. (49) we sum over the product of all possible paths of length i from k to another node l and all possible paths of length 1 from nodes l to j. This sum is exactly the semiring-sum over all possible paths of length $i + 1$ from k to j.