Prof. Ryan Cotterell

Simon Wachter: Assignment 2

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Question 1:

a) Prove that the expectation semiring satisfies the semiring axioms:

- $(\mathbb{R} \times \mathbb{R}, \oplus, \mathbf{0})$ must be a commutative monoid with identity element $\mathbf{0}$:

$$\begin{aligned}
(\langle x, y\rangle \oplus \langle x', y'\rangle) \oplus \langle x'', y''\rangle &= \langle x + x', y + y'\rangle \oplus \langle x'', y''\rangle && (1)\\
&= \langle x + x' + x'', y + y' + y''\rangle && (2)\\
&= \langle x, y\rangle \oplus \langle x' + x'', y' + y''\rangle && (3)\\
&= \langle x, y\rangle \oplus (\langle x', y'\rangle \oplus \langle x'', y''\rangle) && (4)
\end{aligned}$$

$$\begin{aligned}
\mathbf{0} + \langle x, y\rangle &= \langle 0, 0\rangle \oplus \langle x, y\rangle && (5)\\
&= \langle 0 + x, 0 + y\rangle && (6)\\
&= \langle x, y\rangle && (7)\\
&= \langle x + 0, y + 0\rangle && (8)\\
&= \langle x, y\rangle + \mathbf{0} && (9)
\end{aligned}$$

$$\begin{aligned}
\langle x, y\rangle + \langle x', y'\rangle &= \langle x + x', y + y'\rangle && (10)\\
&= \langle x' + x, y' + y\rangle && (11)\\
&= \langle x', y'\rangle + \langle x, y\rangle && (12)
\end{aligned}$$

- $(\mathbb{R} \times \mathbb{R}, \otimes, \mathbf{1})$ must be a monoid with identity element $\mathbf{1}$:

$$\begin{aligned}
(\langle x, y\rangle \otimes \langle x', y'\rangle) \otimes \langle x'', y''\rangle &= \langle x \cdot x', x \cdot y' + y \cdot x'\rangle \otimes \langle x'', y''\rangle && (13)\\
&= \langle x \cdot x' \cdot x'', x \cdot x' \cdot y'' + (x \cdot y' + y \cdot x') \cdot x''\rangle && (14)\\
&= \langle x \cdot x' \cdot x'', x \cdot x' \cdot y'' + x \cdot y' \cdot x'' + y \cdot x' \cdot x''\rangle && (15)\\
&= \langle x, y\rangle \otimes \langle x' \cdot x'', x' \cdot y'' + y' \cdot x''\rangle && (16)\\
&= \langle x, y\rangle \otimes (\langle x', y'\rangle \otimes \langle x'', y''\rangle) && (17)
\end{aligned}$$

$$\begin{aligned}
\mathbf{1} \otimes \langle x, y\rangle &= \langle 1, 0\rangle \otimes \langle x, y\rangle && (18)\\
&= \langle 1 \cdot x, 1 \cdot y\rangle && (19)\\
&= \langle x, y\rangle && (20)\\
&= \langle x \cdot 1, y \cdot 1\rangle && (21)\\
&= \langle x, y\rangle \otimes \mathbf{1} && (22)
\end{aligned}$$

- Multiplication left and right distributes over addition:

$$\begin{aligned}
\langle x, y\rangle \otimes (\langle x', y'\rangle \oplus \langle x'', y''\rangle) &= \langle x, y\rangle \otimes \langle x' + x'', y' + y''\rangle && (23)\\
&= \langle x \cdot x' + x \cdot x'', x \cdot y' + x \cdot y'' + y \cdot x' + y \cdot x''\rangle && (24)\\
&= \langle x \cdot x', x \cdot y' + y \cdot x'\rangle \oplus \langle x \cdot x'', x \cdot y'' + y \cdot x''\rangle && (25)\\
&= (\langle x, y\rangle \otimes \langle x', y'\rangle) \oplus (\langle x, y\rangle \otimes \langle x'', y''\rangle) && (26)
\end{aligned}$$

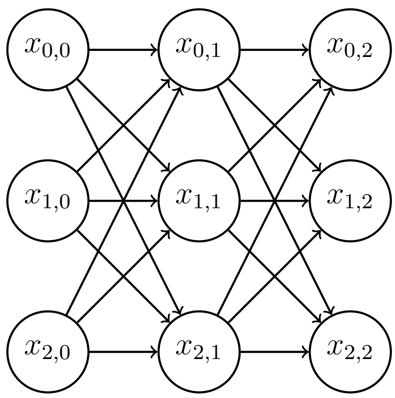
$$\begin{aligned}
(\langle x, y\rangle \oplus \langle x', y'\rangle) \otimes \langle x'', y''\rangle &= \langle x + x', y + y'\rangle \otimes \langle x'', y''\rangle && (27)\\
&= \langle x \cdot x'' + x' \cdot x'', x \cdot y'' + x' \cdot y'' + y \cdot x'' + y' \cdot x''\rangle && (28)\\
&= \langle x \cdot x'', x \cdot y'' + y \cdot x''\rangle \oplus \langle x' \cdot x'', x' \cdot y'' + y' \cdot x''\rangle && (29)\\
&= (\langle x, y\rangle \otimes \langle x'', y''\rangle) \oplus (\langle x', y'\rangle \otimes \langle x'', y''\rangle) && (30)
\end{aligned}$$

- Multiplication by $\mathbf{0}$ annihilates $\mathbb{R} \times \mathbb{R}$:

$$\begin{aligned}
\mathbf{0} \otimes \langle x, y\rangle &= \langle 0, 0\rangle \otimes \langle x, y\rangle && (31)\\
&= \langle 0 \cdot x, 0 \cdot y\rangle && (32)\\
&= \langle 0, 0\rangle && (33)\\
&= \mathbf{0} && (34)\\
&= \langle 0, 0\rangle && (35)\\
&= \langle x \cdot 0, y \cdot 0\rangle && (36)\\
&= \langle x, y\rangle \otimes \langle 0, 0\rangle && (37)\\
&= \langle x, y\rangle \otimes \mathbf{0} && (38)
\end{aligned}$$

b) Our initial graph looks like **??** 1.

Figure 1: The initial graph



Where the columns represent the words in $\mathbf{w}$ and the rows represent different tags. In this non lifted graph, our forward propagation will yield the following values, when using the score functions as edge weights in the graph:

$$z_{i,j} = \sum_{k=0}^{T} (z_{k,j-1} \cdot \text{score}(x_{k,j-1}, x_{i,j}, \mathbf{w})) \tag{39}$$

When we now lift the CRF into the expectation semiring, the forward propagation will yield the following values:

$$z_{i,j} = \sum_{k=0}^{T} (z_{k,j-1} \cdot \langle w, -w \log(w) \rangle) \qquad w = \text{score}(x_{k,j-1}, x_{i,j}, \mathbf{w}) \tag{40}$$

By choosing $w = \exp(\text{score}(x_{k,j-1}, x_{i,j}, \mathbf{w}))$ and summing over the last column of our graph we get:

$$\begin{aligned} z &= \sum_{i=0}^{T} z_{i,N} && (41) \\ &= \sum_{i=0}^{T} \sum_{k=0}^{T} (z_{k,N-1} \cdot \langle w, -w \log(w) \rangle) \qquad w = \exp(\text{score}(x_{k,N-1}, x_{i,N}, \mathbf{w})) && (42) \\ & && (43) \end{aligned}$$