Propositional Logic: Normal Forms

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Semantic vs. Syntax

$\models \phi$	vs.	$\vdash \phi$
Truth		Tools
Semantics		Syntax
Validity		Proof
All Interpretations		Finite Proof Trees
Undecidable (except propositional logic)		Manual Heuristics

Natural Deduction

We have a collection of proof rules. Natural deduction does not have axioms.

- Suppose we have premises $\phi_1, \phi_2, \ldots, \phi_n$ and would like to prove a conclusion $\cancel{A}\psi$. The intention is denoted by $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ We call this expression a *sequent*; it is valid if a proof for it can be found.

Definition 1

A logical formula ϕ with the valid sequent $\vdash \phi$ is theorem.

Proof Rules

	Introduction	Elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\dfrac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \dfrac{\phi \wedge \psi}{\psi} \wedge e_2$

- $\wedge_i$ (and-introduction): to prove $\phi \wedge \psi$, you must first prove ϕ and ψ separately and then use the rule $\wedge i$.
- $\wedge e_1$: (and-elimination) to prove ϕ, try proving $\phi \wedge \psi$ and then use the rule $\wedge e1$. Probably only useful when you already have $\phi \wedge \psi$ somwhere; otherwise, proving $\phi \wedge \psi$ may be harder than proving ϕ.

Proof Rules

	Introduction	Elimination
$\vee$	$\dfrac{\phi}{\phi \vee \psi} \vee i_1 \qquad \dfrac{\psi}{\phi \vee \psi} \vee i_2$	$\dfrac{\phi \vee \psi \quad \begin{array}{c}\phi\\ \vdots\\ \chi\end{array} \quad \begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}{\chi} \wedge e$

- $\vee i_1$ (or-intoduction): to prove $\phi \vee \psi$, try proving ϕ. Again, in general it is harder to prove ϕ than it is to prove $\phi \vee \psi$, so this will usually be useful only if you have already managed to prove ϕ.
- $\vee e$ (or-elimination): has an excellent procedural interpretation. It says: if you have $\phi \vee \psi$, and you want to prove some χ, then try to prove χ from ϕ and from ψ in turn. In those subproofs, of course you can use the other prevailing premises as well.

Proof Rules

	Introduction	Elimination
$\rightarrow$	$\dfrac{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}{\phi \rightarrow \psi} \rightarrow_i$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow_e$
$\neg$	$\dfrac{\begin{array}{c}\phi \\ \vdots \\ \perp\end{array}}{\neg\phi} \neg_i$	$\dfrac{\phi \quad \neg\phi}{\perp} \neg_e$
$\perp$	(No introduction rule for $\perp$)	$\dfrac{\perp}{\phi} \perp_e$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi} \neg\neg_e$

Derived Rules

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ MT \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

$$\begin{array}{c} \neg\phi \\ \vdots \\ \bot \\ \hline \phi \end{array}\ RAA \qquad \frac{}{\phi \vee \neg\phi}\ LEM$$

- Modus Tollens (MT): "If Abraham Lincoln was Ethiopian, then he was African. Abraham Lincoln was not African; therefore, he was not Ethiopian."
- Introduction of double negation.
- Reductio Ad Absurdum, i.e. Proof By Contradiction.
- Tertium Non Datur, or Law of the Excluded Middle.

How Proof Rules Work

Prove that $p \wedge q, r \vdash q \wedge r$.

Proof Tree

$$\cfrac{\cfrac{p \wedge q}{q}\wedge_{e_2} \quad r}{q \wedge r}\wedge_i$$

Linear Form

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge_{e_2}$, 1
4.	$q \wedge r$	$\wedge_i$, 3, 2

Scope Box

We can temporarily make any assumptions, and apply rules to them. We use scope boxes to represent their scope, i.e. to represent which other steps *depend* on them. For example, let us show that $p \rightarrow q \vdash \neg q \rightarrow \neg p$.

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	modus tollens, 1, 2
4.	$\neg q \rightarrow \neg p$	$\rightarrow_i$, 2, 3

- Note that $\neg p$ *depends* on the assumption, $\neg q$. However, step 4 does not depends on step 2 or 3.
- The line immediately following a closed box has to match the pattern of the conclusion of the rule using the box.

Example 1

Prove that $p \land \neg q \to r, \neg r, p \vdash q$.

1.	$p \land \neg q \to r$	premise
2.	$\neg r$	premise
3.	p	premise
4.	$\neg q$	assumption
5.	$p \land \neg q$	$\land_i$, 3, 4
6.	r	$\to_i$, 5, 1
7.	$\bot$	$\neg_e$, 6, 2
8.	$\neg\neg q$	$\neg_i$, 4-7
9.	q	$\neg\neg_e$, 8

Example 2

Prove that $p \rightarrow q \vdash \neg p \vee q$.

1.	$p \rightarrow q$	premise
2.	$\neg p \vee p$	law of eliminated middle
3.	$\neg p$	assumption
4.	$\neg p \vee q$	$\vee_{i_3}$, 3
5.	p	assumption
6.	q	$\rightarrow_i$, 1, 5
7.	$\neg p \vee q$	$\vee_{i_2}$, 6
8.	$\neg p \vee q$	$\vee_e$, 2, 3-4, 5-7

Note that, earlier in the lecture, we also showed $p \rightarrow q \models \neg p \vee q$.
Can you explain the differences?

Example 3: Law of Excluded Middle

Prove the law of excluded middle, i.e. $\dfrac{}{\phi \vee \neg\phi}\ LEM$.

1.	$\neg(\phi \vee \neg\phi)$	assumption
2.	ϕ	assumption
3.	$\phi \vee \neg\phi$	$\vee_{i_1}$, 2
4.	$\bot$	$\neg_e$, 3, 1
5.	$\neg\phi$	$\neg_i$, 2-4
6.	$\phi \vee \neg\phi$	$\vee_{i_2}$, 5
7.	$\bot$	$\neg_e$, 1, 6
8.	$\neg\neg(\phi \vee \neg\phi)$	$\neg_i$, 1-7
9.	$\phi \vee \neg\phi$	$\neg\neg_e$, 8

Proof Tips

- Write down the premises at the top.
- Write down the conclusion at the bottom.
- Observe the structure of the conclusion, and try to fit a rule backward.

Exercises

Prove the following:

- $\neg p \vee q \vdash p \to q$
- $p \to q, p \to \neg q \vdash \neg p$
- $p \to (q \to r), p, \neg r \vdash \neg q$