Propositional Logic: Semantics (2/3)

CS402, Spring 2016

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Overview

- Logical Equivalence and Substitution
- Satisfiability, Validity, and Consequence

Logical Equivalence

Definition 1 (2.13)

Let $A_1, A_2 \in \mathcal{F}$. If $\nu(A_1) = \nu(A_2)$ for *all/every* interpretation ν, then A_1 is *logically equivalent* to A_2, denoted $A_1 \equiv A_2$.

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p	q	$\nu(p \vee q)$	$\nu(q \vee p)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Logical Equivalence: we can extend the result of the previous example from atomic propositions to general formulas.

Theorem 1 (2.15)

Let A_1 and A_2 be any formulas. Then $A_1 \lor A_2 \equiv A_2 \lor A_1$.

Proof.

1. Let ν be an arbitrary interpretation for $A_1 \lor A_2$. Then, ν is an interpretation for $A_2 \lor A_1$, too.
2. Similarly, ν is an interpretation for A_1 and A_2.
3. Therefore, $\nu(A_1 \lor A_2) = T \leftrightarrow (\nu(A_1) = T \lor \nu(A_2) = T) \leftrightarrow \nu(A_2 \lor A_1) = T$.

$\square$

Logical Equivalence

Definition 2 (2.22)

A binary operator, o, is *defined from* a set of operators, $O = \{o_1, \ldots, o_n\}$ iff there is a logical equivalence $A_1 \ o \ A_2 \equiv A$ where A is a formula constructed from occurrences of A_1, A_2, and operators in O.

Similarly, an unary operator o is *defined from* a set of operators, $O = \{o_1, \ldots, o_n\}$ iff there is a logical equivalence $o \ A_1 \equiv A$ where A is a formula constructed from occurrences of A_1, and operator o.

Example 1

- $\leftrightarrow$ is defined from $\{\rightarrow, \wedge\}$ because $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.
- $\rightarrow$ is defined from $\{\neg, \vee\}$ because $A \rightarrow B \equiv \neg A \vee B$.
- $\wedge$ is defined from $\{\neg, \vee\}$ because $A \wedge B \equiv \neg(\neg A \vee \neg B)$.

Theorem 2 (2.16)

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What is the difference between $\leftrightarrow$ and $\equiv$?

- **Material Equivalence** ($\leftrightarrow$): just another statement in the object language; truth value depends on the specific interpretation.
- **Logical Equivalance** ($\equiv$): semantic statement, i.e. if p is logically equivalent to q, it means that under every possible interpretation, p and q logically means the same thing. This is a statement in the metalanguage.

Logical Substitution: logical equivalance justifies *substitution* of one formula for another that is equivalent.

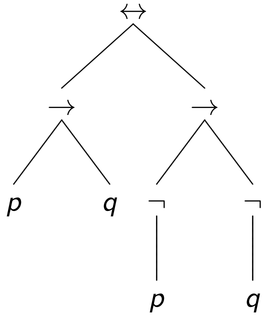
Let us present the intermediate steps first.

Definition 3 (2.17)

A is subformula of B if the formation tree for A occurs as a subtree of the formation tree for B. A is proper subformation of B if A is a subformation of B, but A is not identical to B.

Example 2 (2.18)

The formula $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$ contains the following proper subformulas: $p \rightarrow q, \neg p \rightarrow \neg q, \neg p, \neg q, p$ and q



Definition 4 (2.19)

If A is a subformula of B, and A' is an arbitrary formula, then B', the *substitution* of A' for A in B, denoted $B\{A \leftarrow A'\}$, is the formula obtained by replacing all occurrences of the subtree for A in B by the tree for A'.

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Theorem 3

Let A be a subformula of B and let A' be a formula such that $A \equiv A'$. Then $B \equiv B\{A \leftarrow A'\}$.

Substitution can be naturally used to *simplify* formulas.

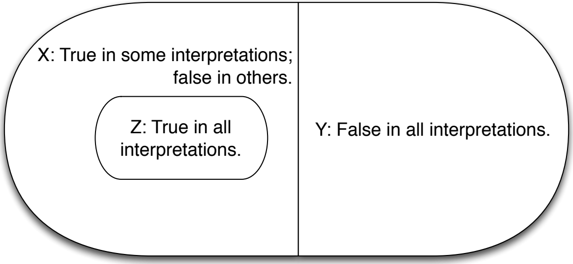
$$p \wedge (\neg p \vee q) \equiv (p \wedge \neg p) \vee (p \wedge q) \equiv \mathit{false} \vee (p \wedge q) \equiv p \wedge q$$

Definition 5 (2.24)

A propositional formula A is *satisfiable* iff $\nu(A) = T$ for *some* interpretation ν. A satisfying interpretation is called a *model* for A. A is *valid*, denoted $\models A$, iff $\nu(A) = T$ for *all* interpretation ν. A valid propositional formula is also called a *tautology*.

Theorem 4 (2.25)

A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is falsifiable.



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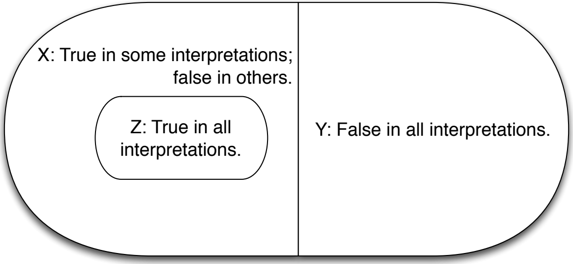

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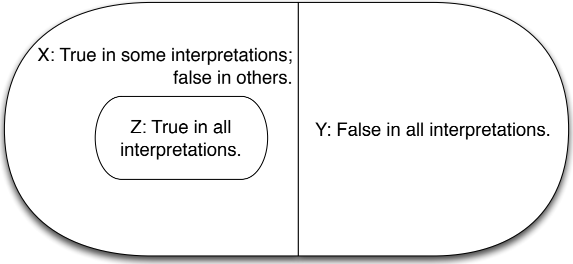

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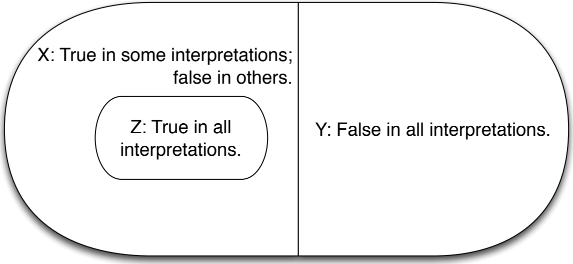

- X (and, therefore, Z): Satisfiable.
- Y: Unsatisfiable.
- Z: Valid.
- $(X - Z) \bigcup Y$: Falsifiable (i.e. can be shown to be false).

Definition 6 (2.26)

Let $\mathcal{V}$ be a set of formulas. An algorithm is a *decision procedure* for $\mathcal{V}$ if given an arbitrary formula $A \in \mathcal{F}$, it terminates and return the answer 'yes' if $A \in \mathcal{V}$ and the answer 'no' if $A \notin \mathcal{V}$.

By Theorem 2.25, a decision procedure for satisfiability can be used as a decision procedure for validity. Let $\mathcal{V}$ be the set of all satisfiable formulas. To decide the validity of A, we can apply the decision procesure for satisfiability of $\neg A$. This decision procedure is called a *refutation procedure*.

Example 3 (2.27)

Is $(p \to q) \leftrightarrow (\neg p \to \neg q)$ valid?

Example 4 (2.28)

$p \vee q$ is satisfiable but not valid.

Logical Consequence

Definition 7 (2.30)

Extension of satisfiability from a single formula to a set of formulas: a set of formulas $U = A_1, \ldots, A_n$ is (simultaneously) satisfiable iff there exists an interpretation ν such that $\nu(A_1) = \ldots = \nu(A_n) = T$. The satisfying interpretation is called a *model* of U. U is *unsatisfiable* iff for every interpretation ν, there exists an i such that $\nu(Ai) = F$.

Logical Consequence: let U be a set of formulas and A a formula. If A is *true* in every model of U, then A is a logical consequence of U, i.e. $U \models A$.

Theorem 5 (2.38)

$U \models A$ iff $A_1 \wedge A_2 \ldots \wedge A_n \rightarrow A$, where $U = \{A_1, \ldots, A_n\}$.

- If $U = \emptyset$, the logical consequence is the same as the validity.
- Note Theorem 2.16: $A_1 \equiv A_2$ if and only if $A_1 \leftrightarrow A_2$ is *true* in every interpretation.

Theories

Logical consequence is the central concept in the foundations of mathematics; validity is often trivial and not very interesting. For example, Euclidean geometry is an extensive set of logical consequences, all deduced from the five axioms.

Definition 8 (2.41)

A set of formulas $\mathcal{T}$ is a theory if and only if it is *closed under logical consequence*, i.e. if $\mathcal{T} \models A$ then $A \in \mathcal{T}$. Elements of $\mathcal{T}$ are called *theorems*.

Let U be a set of formulas. $\mathcal{T}(U) = \{A | U \models A\}$ is called the theory of U. The formulas of U are called *axioms* and the theory $\mathcal{T}(U)$ is *axiomatizable*.

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Is $\mathcal{T}(U)$ a theory?

Examples of Theory

	p	q	r	$p \vee q \vee r$	$q \rightarrow r$	$r \rightarrow p$
ν_1	T	T	T	T	T	T
ν_2	T	T	F	T	F	T
ν_3	T	F	T	T	T	T
ν_4	T	F	F	T	T	T
ν_5	F	T	T	T	T	F
ν_6	F	T	F	T	F	T
ν_7	F	F	T	T	T	F
ν_8	T	F	F	F	T	T

- $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$
- Interpretation ν_1, ν_3, ν_4 are models of U (i.e. interpretations that make all formulas in U true, see Def. 2.30).
- Which of the following are true?
  1. $U \models p$
  2. $U \models q \rightarrow r$
  3. $U \models r \vee \neg q$
  4. $U \models p \wedge \neg q$

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Theory of $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$, i.e. $\mathcal{T}$ (U):

- $U \subseteq \mathcal{T}(U)$

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ν_8	T	F	F	F	T	T

Theory of $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$, i.e. $\mathcal{T}$ (U):

- $U \subseteq \mathcal{T}(U)$ because for all formula $A \in \mathcal{F}$, $A \models A$.
- $p \in \mathcal{T}(U)$

Examples of Theory

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- $U \subseteq \mathcal{T}(U)$ because for all formula $A \in \mathcal{F}$, $A \models A$.
- $p \in \mathcal{T}(U)$ because $U \models p$.
- $(q \rightarrow r) \in \mathcal{T}(U)$

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- $(q \rightarrow r) \in \mathcal{T}(U)$ because $U \models (q \rightarrow r)$.
- $p \wedge (q \rightarrow r) \in \mathcal{T}(U)$

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- $p \in \mathcal{T}(U)$ because $U \models p$.
- $(q \rightarrow r) \in \mathcal{T}(U)$ because $U \models (q \rightarrow r)$.
- $p \wedge (q \rightarrow r) \in \mathcal{T}(U)$ because $U \models p \wedge (q \rightarrow r)$.

Theory of Euclidean Geometry is based on the set of 5 axioms, $U = A_1, A_2, A_3, A_4, A_5$ such that:

- A_1: Any two points can be joined by a unique straight line.
- A_2: Any straight line segment can be extended indefinitely in a straight line.
- A_3: Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- A_4: All right angles are congruent.
- A_5: For every line l and for every point P that does not lie on l, there exists a unique line m through P that is parallel to l.

The ancient Greeks suspected whether A_5 is a logical consequence of the other four. For about 2,000 years, various mathematicians tried to show $\{A_1, \dots, A_4\} \models A_5$. Only in 1868, Beltrami showed that A_5 is independent from the rest. In other words, we accept A_5 as an axiom.

Beltrami also showed that non-Euclidean geometry (i.e. U with A_5 replaced with alternatives) is *consistent*.