Propositional Logic: Deductive Proof & Natural Deduction Part 1

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Shin Yoo

Deductive Proof

In propositional logic, a *valid* formula is a tautology. So far, we could show the validity of a formula ϕ in the following ways:

- Through the truth table for ϕ
- Obtain ϕ as a substitution instance of a formula known to be valid. That is, $q \rightarrow (p \rightarrow q)$ is valid, therefore $r \wedge s \rightarrow (p \vee q \rightarrow r \wedge s)$ is also valid.
- Obtain ϕ through interchange of equivalent formulas. That is, if $\phi \equiv \psi$ and ϕ is a subformula of a valid formula χ, χ' obtained by replacing all occurrences of ϕ in χ with ψ is also valid.

Deductive Proof

Goals of logic: (given U), is ϕ valid?

Theorem 1 (2.38, Ben-Ari)

$U \models \phi$ iff $\models A_1 \wedge \ldots \wedge A_n \rightarrow \phi$ when $U = \{A_1, \ldots, A_n\}$.

However, there are problems in semantic approach.

- Set of axioms may be *infinite*: for example, Peano and ZFC (Zermelo-Fraenkel set theory) theories cannot be finitely axiomatised. Hilbert system, $\mathcal{H}$, uses axiom schema, which in turn generates an infinite number of axioms. We cannot write truth tables for these.
- The truth table itself is not always there! Very few logical systems have decision procedures for validity. For example, predicate logic does not have any such decision procedure.

Semantic vs. Syntax

$\models \phi$	vs.	$\vdash \phi$
Truth		Tools
Semantics		Syntax
Validity		Proof
All Interpretations		Finite Proof Trees
Undecidable (except propositional logic)		Manual Heuristics

Deductive Proof

A deductive proof system relies on a set of proof rules (also *inference* rules), which are in themselves *syntactic transformations* following specific patterns.

- There may be an infinite number of axioms, but only a finite number of axioms will appear on any deductive proof.
- Any particular proof consists of a finite sequence of sets of formulas, and the legality of each individual deduction can be easily and efficiently determined from the syntax of the formulas.
- The proof of a formula clearly shows which axioms, theorems and rules are used and for what purposes.

Soundness and Completeness

- Given a logical system, its proof system is *sound* if and only if: $U \vdash \phi \rightarrow U \models \phi$.
- Given a logical system, its proof system is *complete* if and only if: $U \models \phi \rightarrow U \vdash \phi$.

Proof Calculus

Proof calculus refers to a family of formal systems that use a common style of formal inference for their inference rules. There are three classicl systems:

- Hilbert Systems, $\mathcal{H}$
- Gentzen Systems, $\mathcal{G}$. There are two variants:
  - Natural Deduction: every line has exactly one asserted propositions.
  - Sequent Calculus: every line has zero or more asserted propositions.

Natural Deduction

We have a collection of proof rules. Natural deduction does not have axioms.

- Suppose we have premises $\phi_1, \phi_2, \ldots, \phi_n$ and would like to prove a conclusion ψ. The intention is denoted by $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ We call this expression a *sequent*; it is valid if a proof for it can be found.

Definition 1

A logical formula ϕ with the valid sequent $\vdash \phi$ is theorem.

Proof Rules

	Introduction	Elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge i$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge e_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge e_2$

- $\wedge_i$ (and-introduction): to prove $\phi \wedge \psi$, you must first prove ϕ and ψ separately and then use the rule $\wedge i$.
- $\wedge e_1$: (and-elimination) to prove ϕ, try proving $\phi \wedge \psi$ and then use the rule $\wedge e1$. Probably only useful when you already have $\phi \wedge \psi$ somwhere; otherwise, proving $\phi \wedge \psi$ may be harder than proving ϕ.

Proof Rules

	Introduction	Elimination
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee i_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee i_2$	$\dfrac{\phi \vee \psi \quad \begin{array}{c}\phi\\ \vdots\\ \chi\end{array} \quad \begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}{\chi}\ \wedge e$

- $\vee i_1$ (or-introduction): to prove $\phi \vee \psi$, try proving ϕ. Again, in general it is harder to prove ϕ than it is to prove $\phi \vee \psi$, so this will usually be useful only if you have already managed to prove ϕ.
- $\vee e$ (or-elimination): has an excellent procedural interpretation. It says: if you have $\phi \vee \psi$, and you want to prove some χ, then try to prove χ from ϕ and from ψ in turn. In those subproofs, of course you can use the other prevailing premises as well.

Proof Rules

	Introduction	Elimination
$\rightarrow$	$$\dfrac{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}{\phi \rightarrow \psi} \rightarrow_i$$	$$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow_e$$
$\neg$	$$\dfrac{\begin{array}{c}\phi\\ \vdots\\ \perp\end{array}}{\neg\phi} \neg_i$$	$$\dfrac{\neg\phi \quad \begin{array}{c}\neg\phi\\ \vdots\\ \psi\end{array} \quad \begin{array}{c}\neg\phi\\ \vdots\\ \neg\psi\end{array}}{\phi} \neg_e$$
$\perp$	(No introduction rule for $\perp$)	$$\dfrac{\perp}{\phi} \perp_e$$
$\neg\neg$		$$\dfrac{\neg\neg\phi}{\phi} \neg\neg_e$$

Derived Rules

$\dfrac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ MT$	$\dfrac{\phi}{\neg\neg\phi}\ \neg\neg i$
$\begin{array}{c}\neg\phi \\ \vdots \\ \bot \\ \hline \phi\end{array}\ RAA$	$\dfrac{}{\phi \vee \neg\phi}\ LEM$

- Modus Tollens (MT): "If Abraham Lincoln was Ethiopian, then he was African. Abraham Lincoln was not African; therefore, he was not Ethiopian."
- Introduction of double negation.
- Reductio Ad Absurdum, i.e. Proof By Contradiction.
- Tertium Non Datur, or Law of the Excluded Middle.

How Proof Rules Work

Prove that $p \wedge q, r \vdash q \wedge r$.

Proof Tree

$$\cfrac{\cfrac{p \wedge q}{q}\wedge_{e_2} \quad r}{q \wedge r}\wedge_i$$

Linear Form

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge_{e_2}$, 1
4.	$q \wedge r$	$\wedge_i$, 3, 2

Scope Box

We can temporarily make any assumptions, and apply rules to them. We use scope boxes to represent their scope, i.e. to represent which other steps *depend* on them. For example, let us show that $p \to q \vdash \neg q \to \neg p$.

1.	$p \to q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	modus tollens, 1, 2
4.	$\neg q \to \neg p$	$\to_i$, 2, 3

- Note that $\neg p$ *depends* on the assumption, $\neg q$. However, step 4 does not depends on step 2 or 3.
- The line immediately following a closed box has to match the pattern of the conclusion of the rule using the box.

Example 1

Prove that $p \land \neg q \to r, \neg r, p \vdash q$.

1.	$p \land \neg q \to r$	premise
2.	$\neg r$	premise
3.	p	premise
4.	$\neg q$	assumption
5.	$p \land \neg q$	$\land_i$, 3, 4
6.	r	$\to_i$, 5, 1
7.	$\perp$	$\neg_e$, 6, 2
8.	$\neg\neg q$	$\neg_i$, 4-7
9.	q	$\neg\neg_e$, 8

Example 2

Prove that $p \rightarrow q \vdash \neg p \vee q$.

1.	$p \rightarrow q$	premise
2.	$\neg p \vee p$	law of eliminated middle
3.	$\neg p$	assumption
4.	$\neg p \vee q$	$\vee_{i_3}$, 3
5.	p	assumption
6.	q	$\rightarrow_i$, 1, 5
7.	$\neg p \vee q$	$\vee_{i_2}$, 6
8.	$\neg p \vee q$	$\vee_e$, 2, 3-4, 5-7

Note that, earlier in the lecture, we also showed $p \rightarrow q \models \neg p \vee q$. Can you explain the differences?

Example 3: Law of Excluded Middle

Prove the law of excluded middle, i.e. $\dfrac{}{\phi \lor \neg\phi}\ LEM$.

1.	$\neg(\phi \lor \neg\phi)$	assumption
2.	ϕ	assumption
3.	$\phi \lor \neg\phi$	$\lor_{i_1}$, 2
4.	$\bot$	$\neg_e$, 3, 1
5.	$\neg\phi$	$\neg_i$, 2-4
6.	$\phi \lor \neg\phi$	$\lor_{i_2}$, 5
7.	$\bot$	$\neg_e$, 1, 6
8.	$\neg\neg(\phi \lor \neg\phi)$	$\neg_i$, 1-7
9.	$\phi \lor \neg\phi$	$\neg\neg_e$, 8

Proof Tips

- Write down the premises at the top.
- Write down the conclusion at the bottom.
- Observe the structure of the conclusion, and try to fit a rule backward.

Exercises

Prove the following:

- $\neg p \lor q \vdash p \rightarrow q$
- $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$
- $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$