Propositional Logic: Deductive Proof & Natural Deduction Part 2

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Shin Yoo

Basis for Inference Rules

	Introduction	Elimination	
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\dfrac{\phi \wedge \psi}{\phi} \wedge e_1$	$\dfrac{\phi \wedge \psi}{\psi} \wedge e_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi} \vee i_1$	$\dfrac{\psi}{\phi \vee \psi} \vee i_2$	$\dfrac{\phi \vee \psi \quad \begin{array}{c}\phi \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}{\chi} \wedge e$
$\rightarrow$	$\dfrac{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}{\phi \rightarrow \psi} \rightarrow_i$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow_e$	

How do we know the validity of these rules?

Truth tables. In other words, $\bigwedge$ *premise* $\models$ *consequent*

Basis for Inference Rules

How about the following?

	Introduction	Elimination
$\neg$	$$\dfrac{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}{\neg\phi}\ \neg_i$$	$$\dfrac{\neg\phi \quad \begin{array}{c}\neg\phi\\ \vdots\\ \psi\end{array} \quad \begin{array}{c}\neg\phi\\ \vdots\\ \neg\psi\end{array}}{\phi}\ \neg_e$$

These cannot be justified by truth tables. Rather, these are justified by the Reductio Principle of propositional logic.

Theorem 1 (Reductio Principle)

Let Γ be a set of formulas, ϕ and ψ a formula. If $\Gamma \cup \{\varphi\} \vDash \psi$ and $\Gamma \cup \{\varphi\} \vDash \neg\psi$, then $\Gamma \vDash \neg\varphi$. If $\Gamma \cup \{\neg\varphi\} \vDash \psi$ and $\Gamma \cup \{\neg\varphi\} \vDash \neg\psi$, then $\Gamma \vDash \varphi$.

Basis for Inference Rules

A related question: prove that $\wedge$ and $\vee$ cannot define $\neg$.

On Derived Rules

Repetition: since $\models P \rightarrow P$, we can derive an inference rule based on it.

$$\frac{\phi}{\phi}\ \textit{Repetition}$$

For example, this rule can be used to prove $Q \rightarrow (P \rightarrow Q)$:

1.	Q	Assumption
2.	P	Assumption
3.	Q	Repetition, 1
4.	$P \rightarrow Q$	$\rightarrow_i$, 2-3
5.	$Q \rightarrow (P \rightarrow Q)$	$\rightarrow_i$, 1-4

Now, introduction of double negation:

$$\frac{\phi}{\neg\neg\phi}$$

1.	ϕ	Assumption
2.	$\neg\phi$	Assumption
3.	ϕ	Repetition, 1
4.	$\neg\neg\phi$	$\neg_i$, 2-3 (i.e. $\neg\phi \rightarrow \neg\phi \wedge \neg\phi \rightarrow \phi$)
5.	$\phi \rightarrow \neg\neg\phi$	$\rightarrow_i$, 1-4

On Derived Rules

Elimination of double negation:

$$\frac{\neg\neg\phi}{\phi}$$

1.	$\neg\neg\phi$	Assumption
2.	$\neg\phi$	Assumption
3.	$\neg\neg\phi$	Repetition
4.	ϕ	$\neg_e$, 2-3
5.	$\neg\neg\phi \rightarrow \phi$	$\rightarrow_i$, 1-4

More Exercises

Prove the validity of the following sequents.

- $(s \to p) \lor (t \to q) \vdash (s \to q) \lor (t \to p)$
- $\vdash (p \to q) \lor (q \to r)$