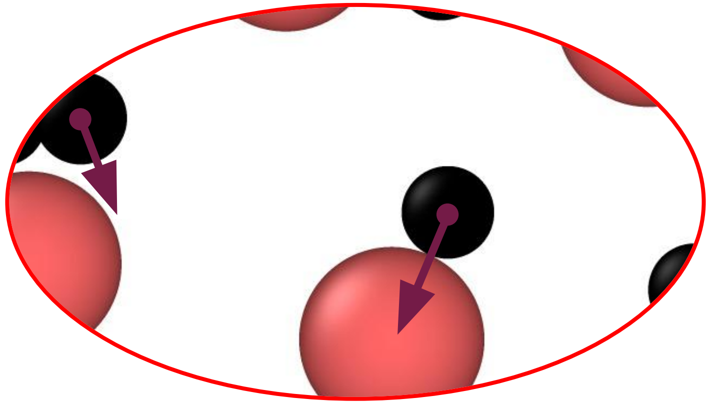
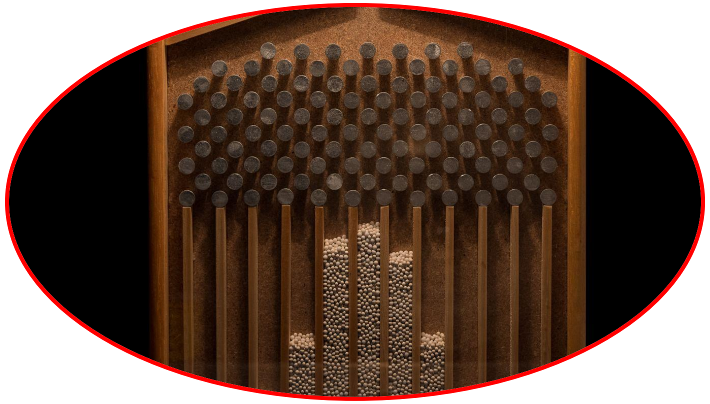
Galton Board

Dropping a set of beads on a board with evenly distributed pegs results in a binomial distribution. Is it possible to generate other kinds of distributions by varying some parameters (Pegs size, pegs distribution, bead format, etc.)? Is it possible to achieve a distribution that does not obey the central limit theorem in an i.i.d. scenario? What happens to the distribution when one makes the board vibrate?

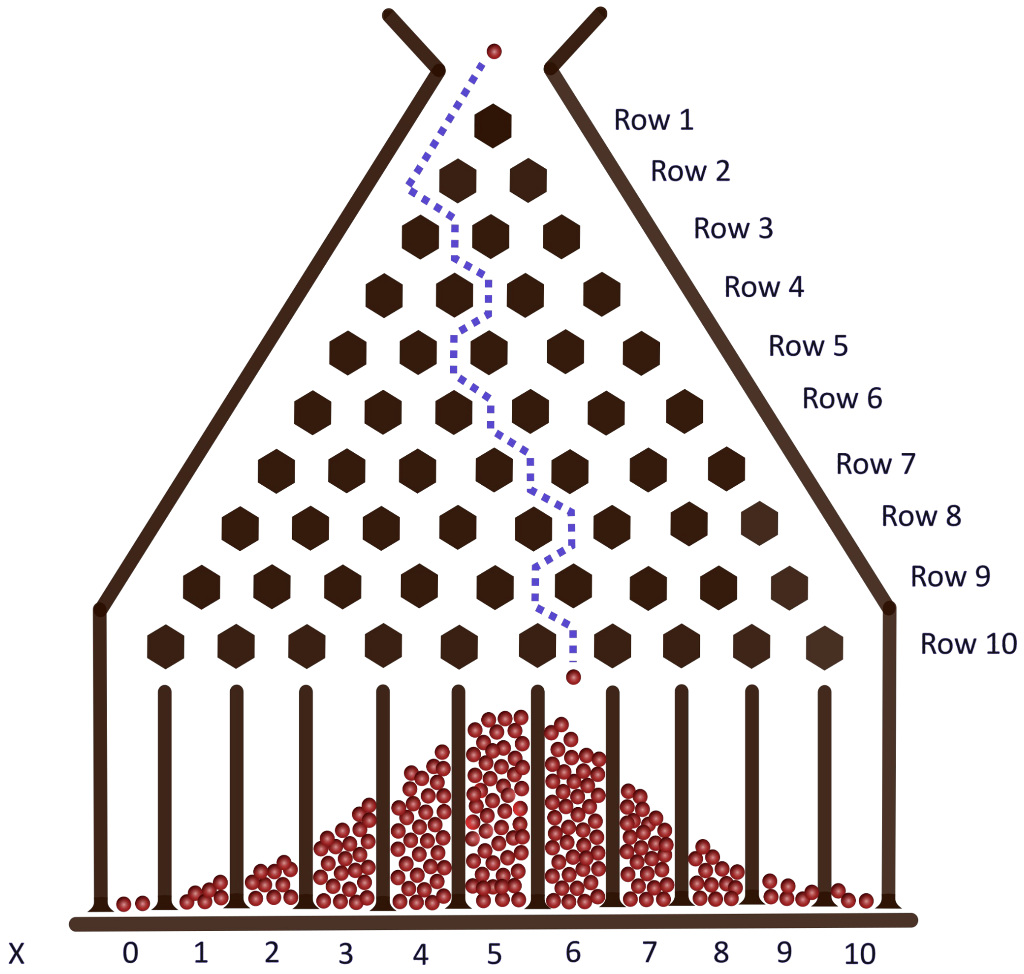




Objectives:.

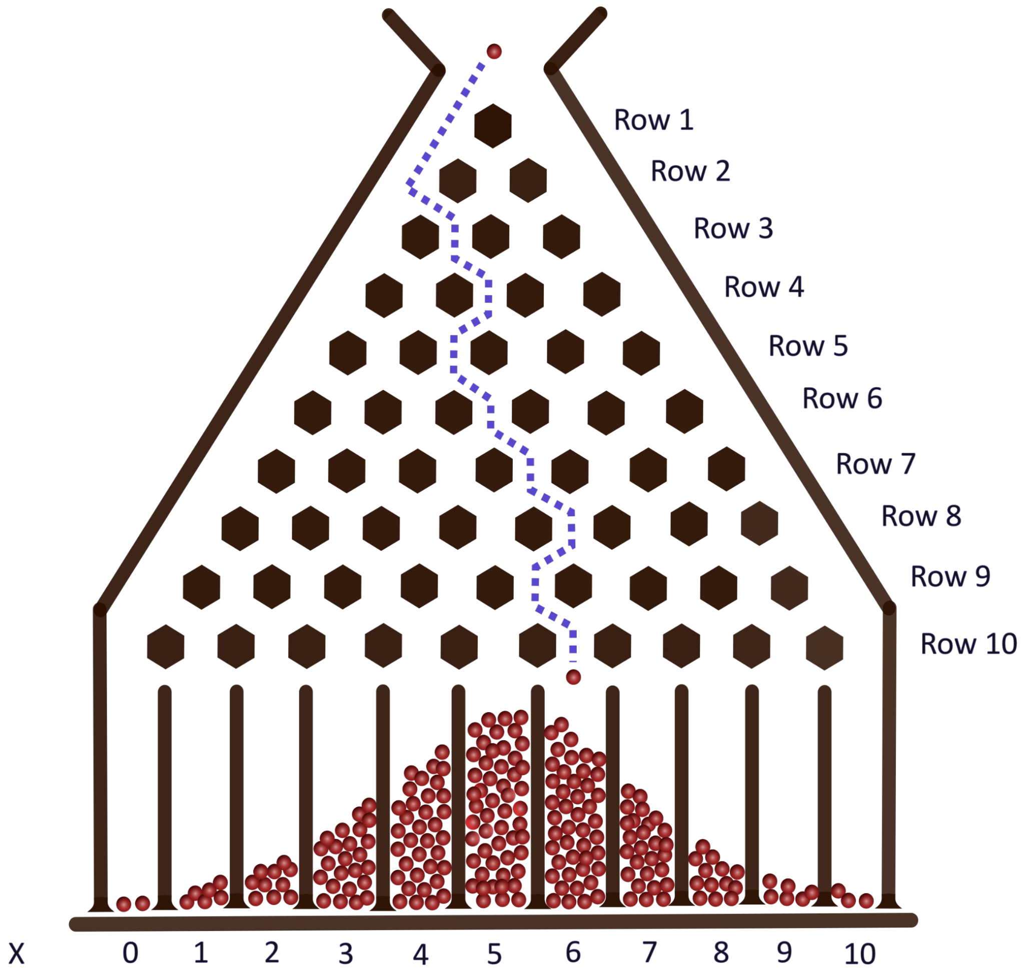
- Describe the emergence of the binomial distribution in a typical Galton Board.
- Characterize the limit distribution of the sum of independent random variables.
- Create a simulation that allows us to vary the parameters of interest in the physical system in order to obtain deviations from the normal law.
- Analyze the effects of vibration of the board.

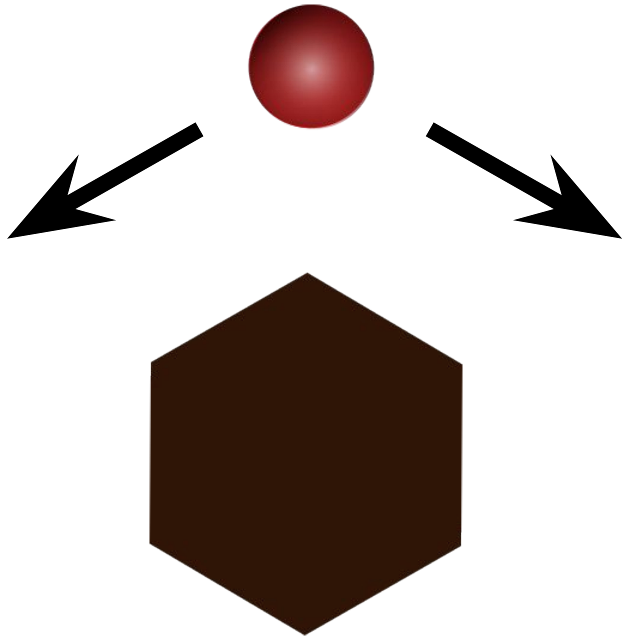
Random walk and Binomial distribution



All graphics and illustrations presented above were generated by the research group IPT Univalle.

Random walk and Binomial distribution





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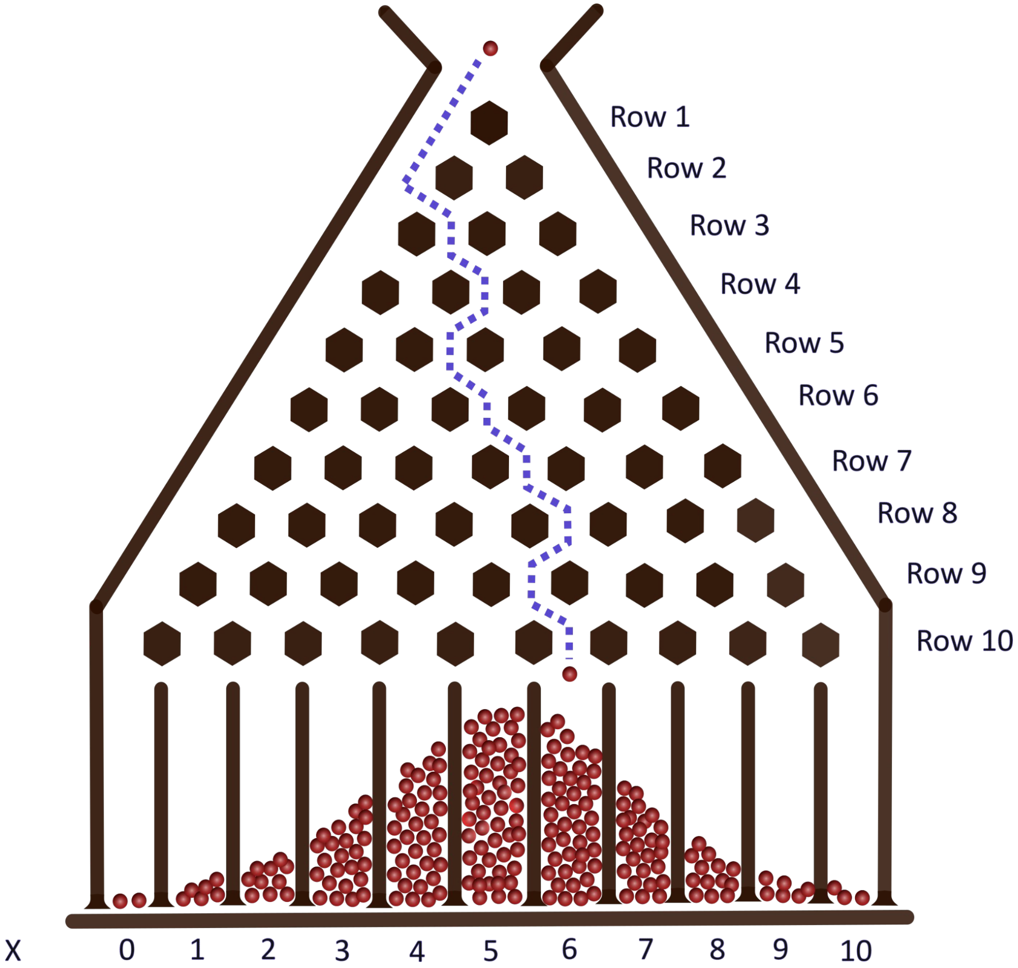

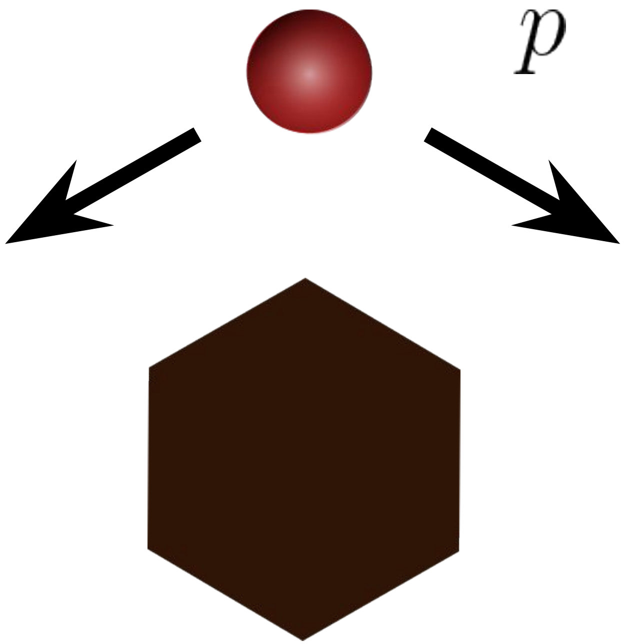

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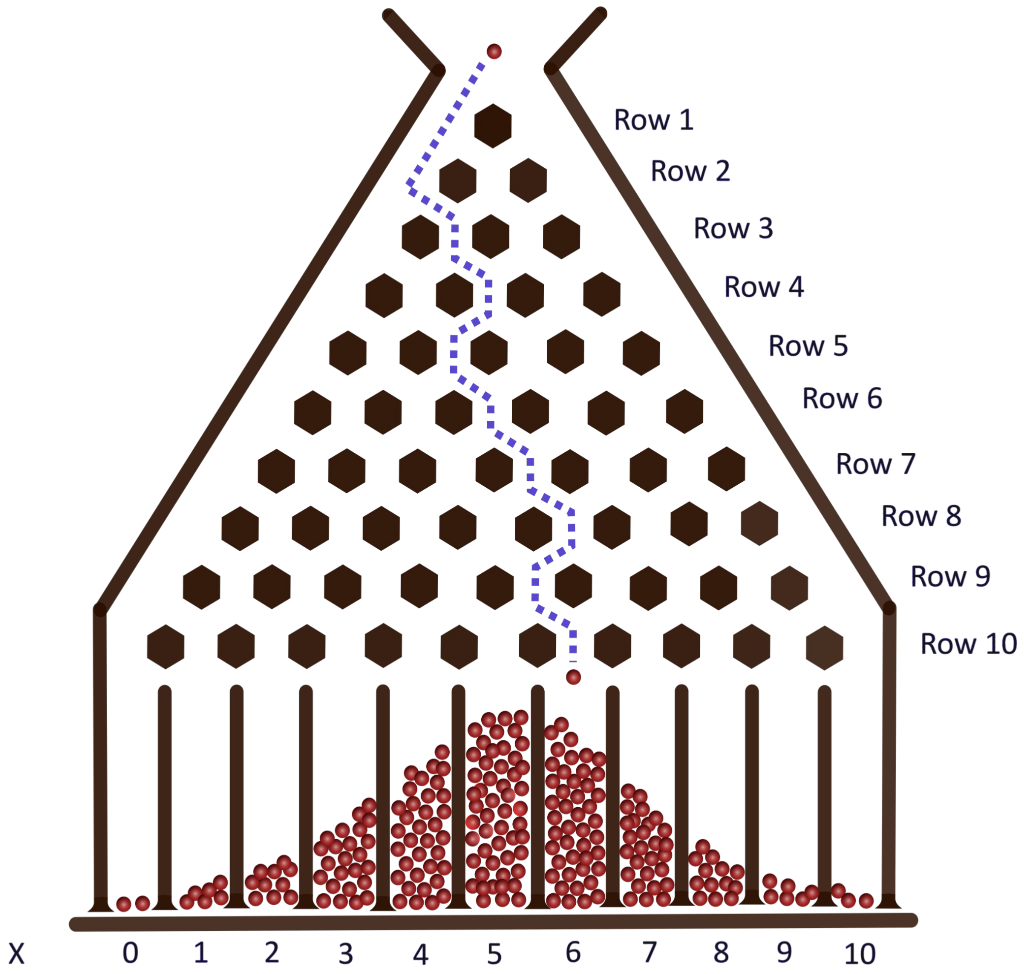
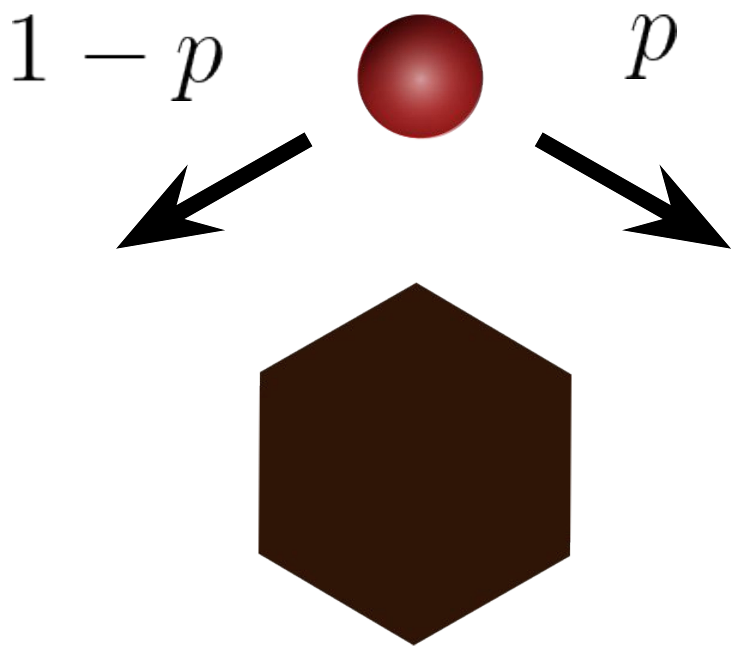
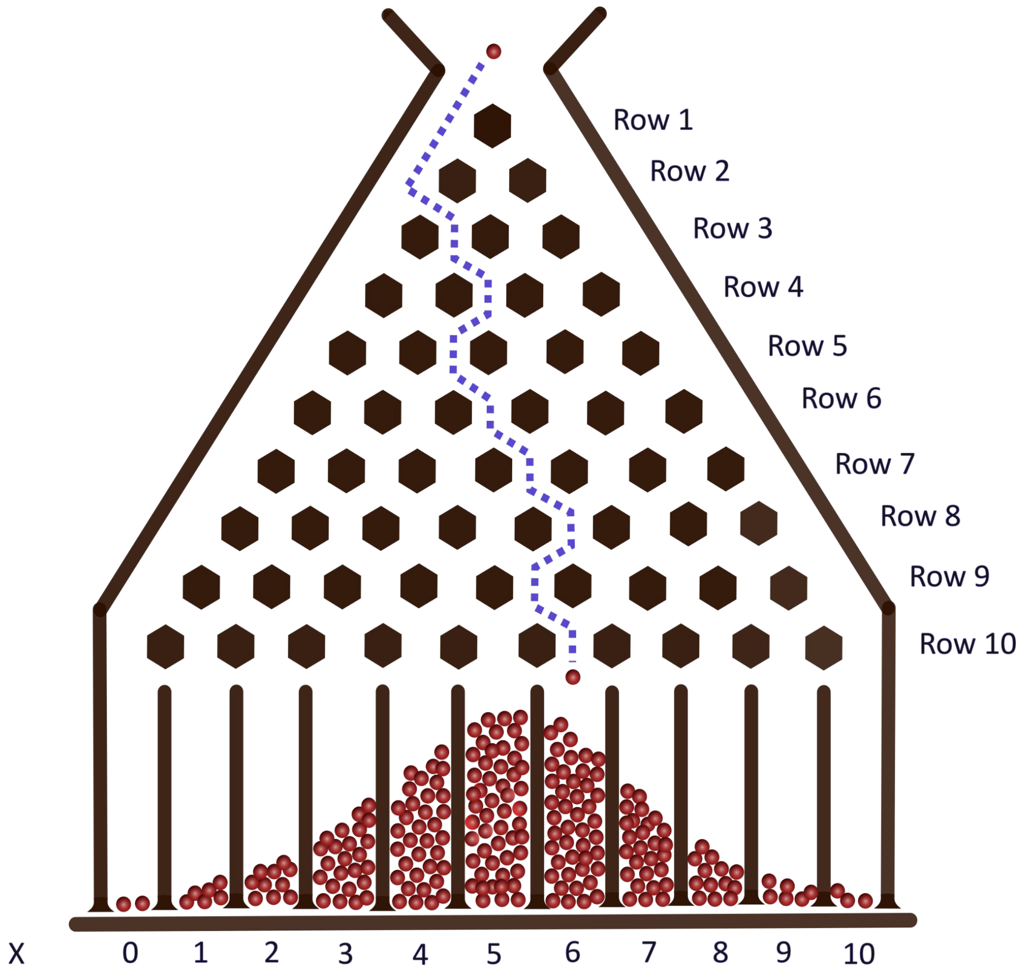
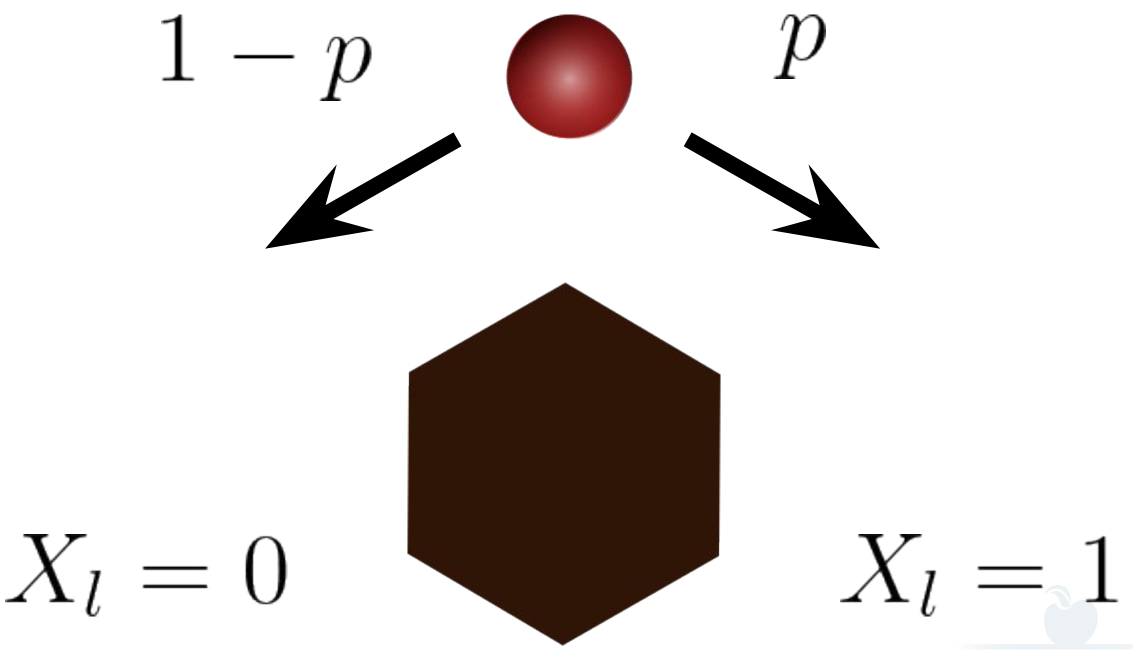




Random walk and Binomial distribution





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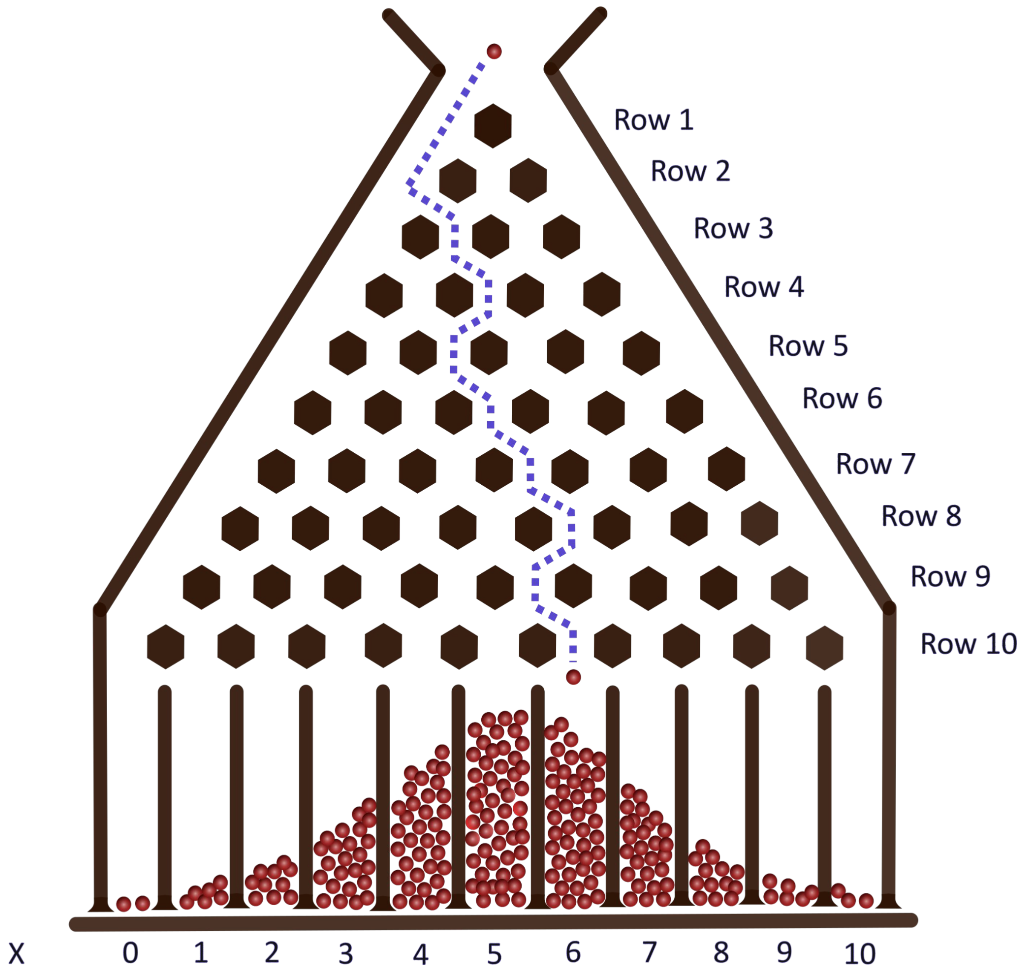
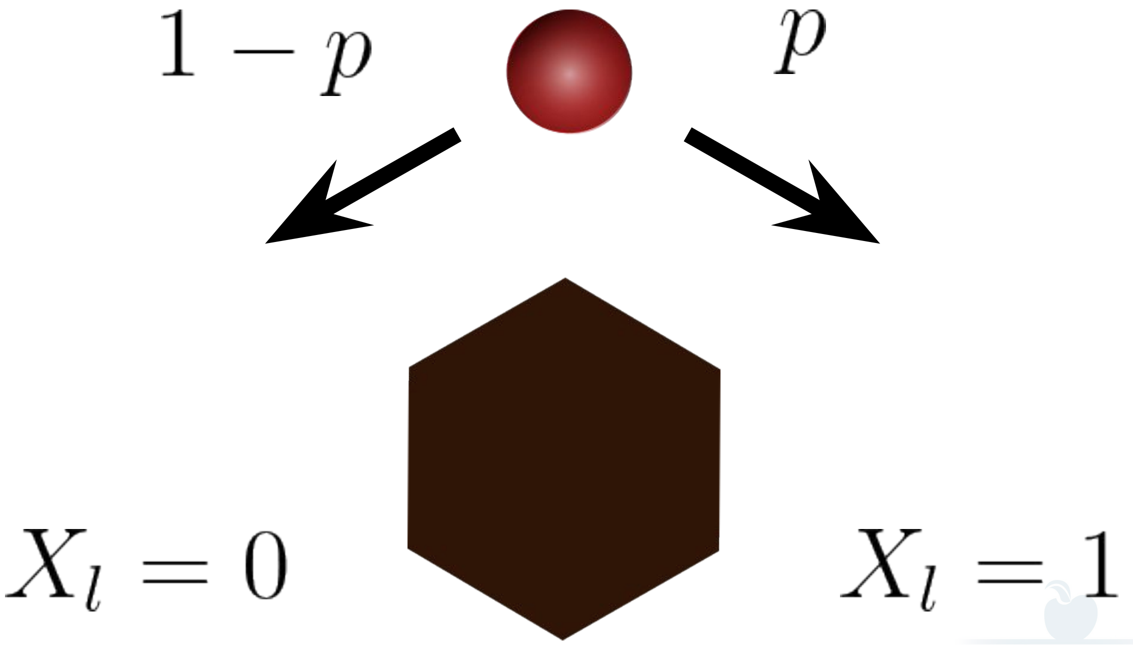
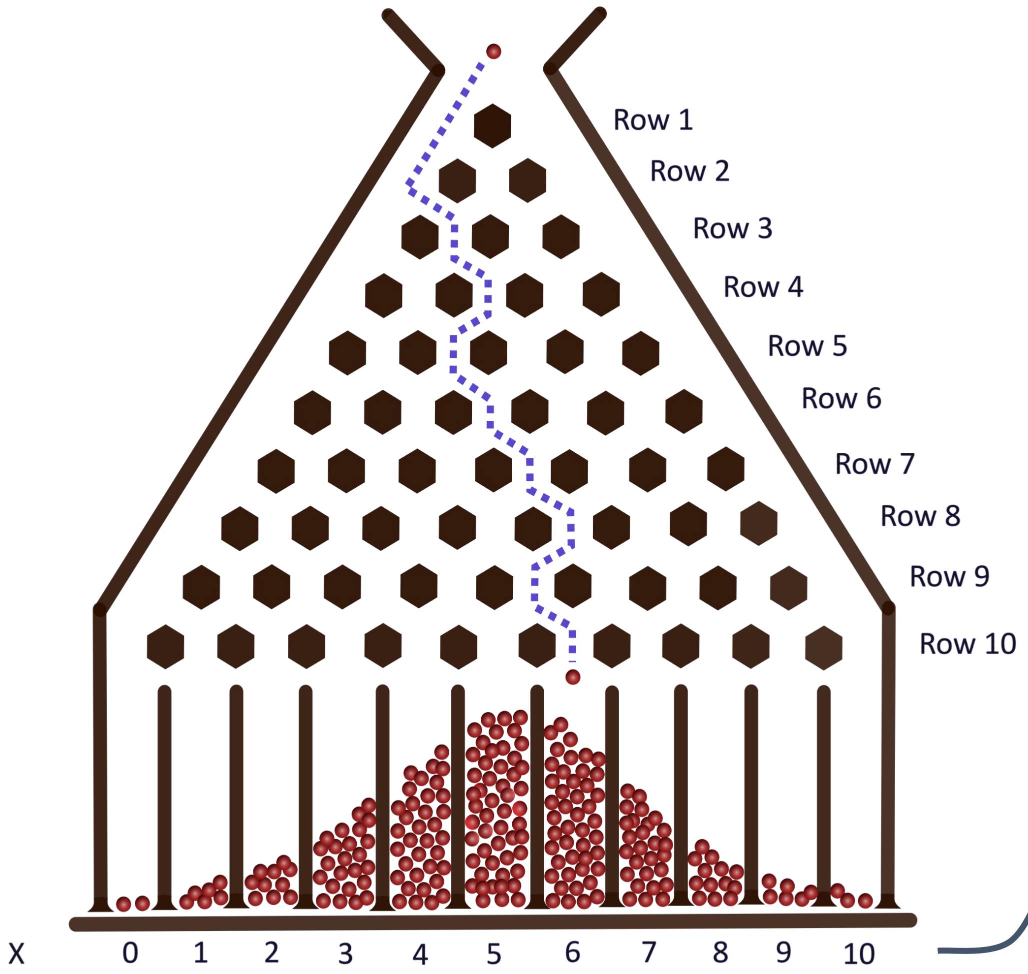




$$X_l \sim p(l) = \begin{cases} p & \text{for } X_l = 1 \\ 1-p & \text{for } X_l = 0 \end{cases}$$

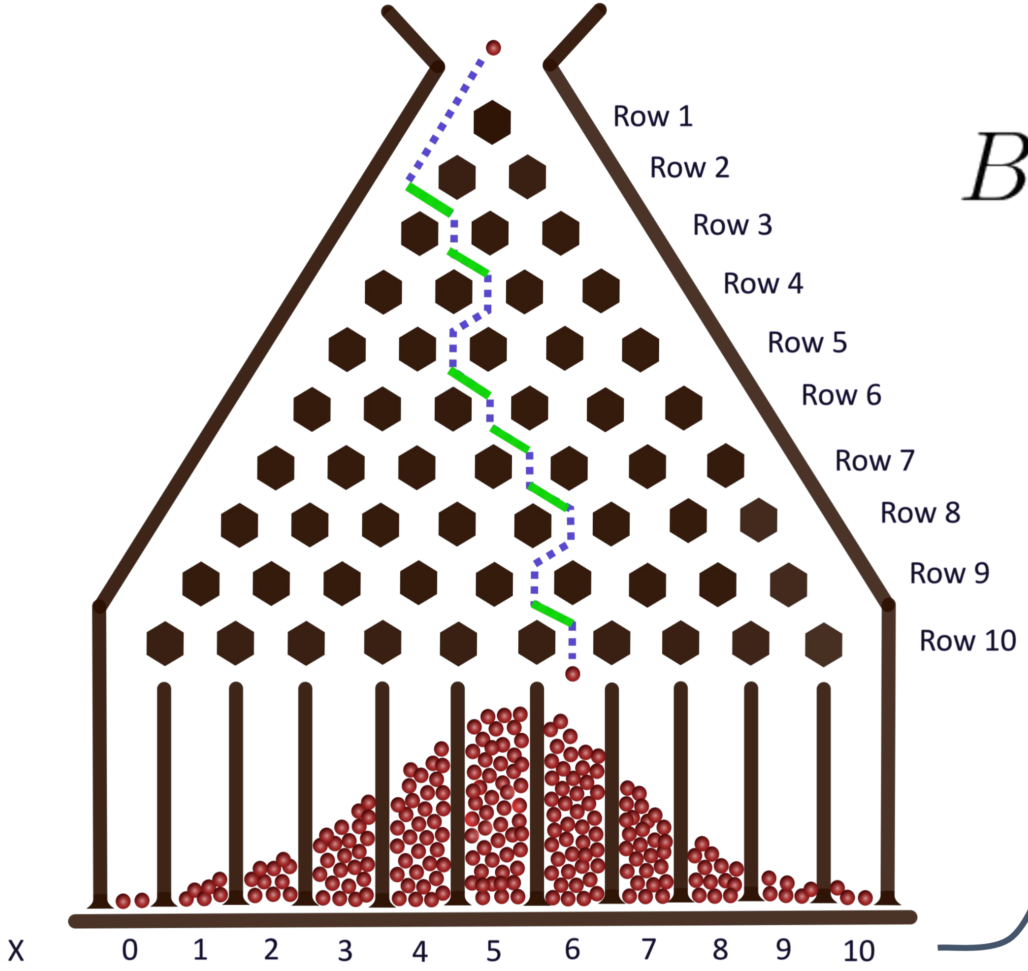
Random walk and Binomial distribution



Random variable

$$X = \sum_{l=1}^{n} X_l$$

Random walk and Binomial distribution

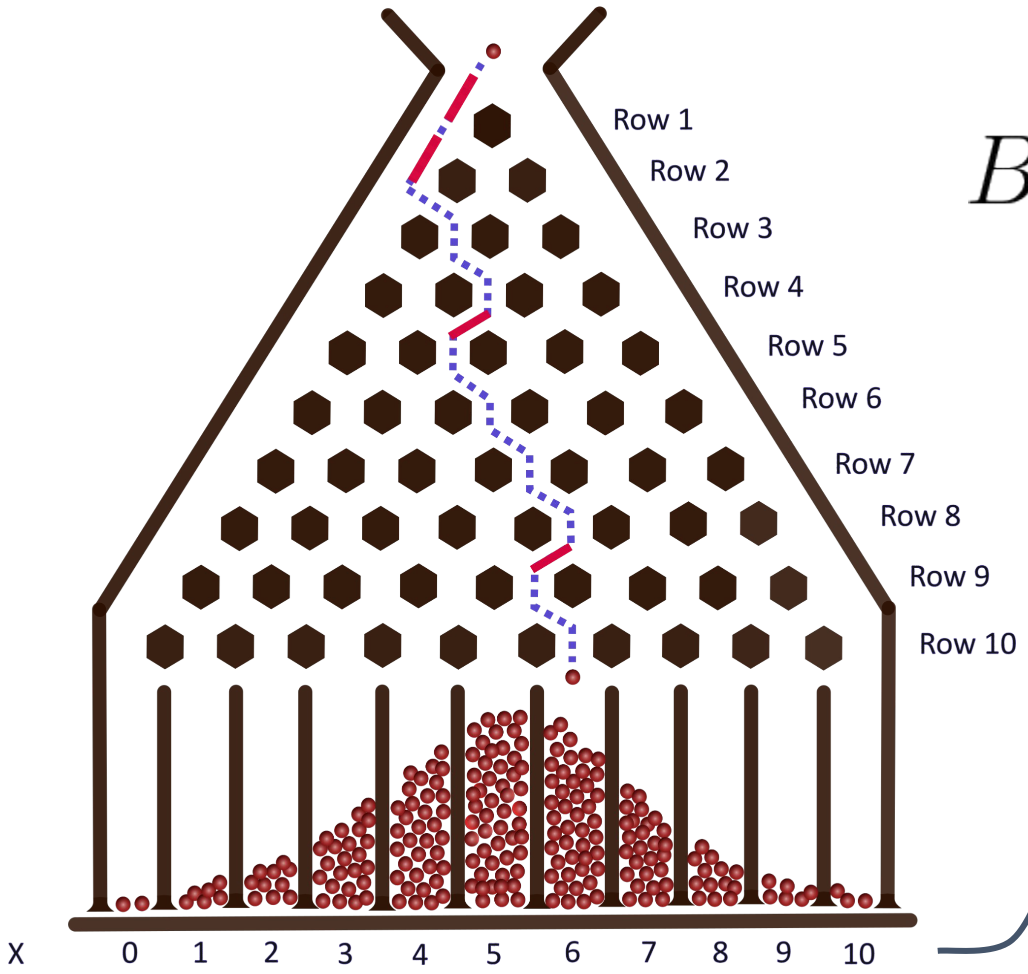


$$B(X = k) = p^k$$

Random variable

$$X = \sum_{l=1}^{n} X_l$$

Random walk and Binomial distribution

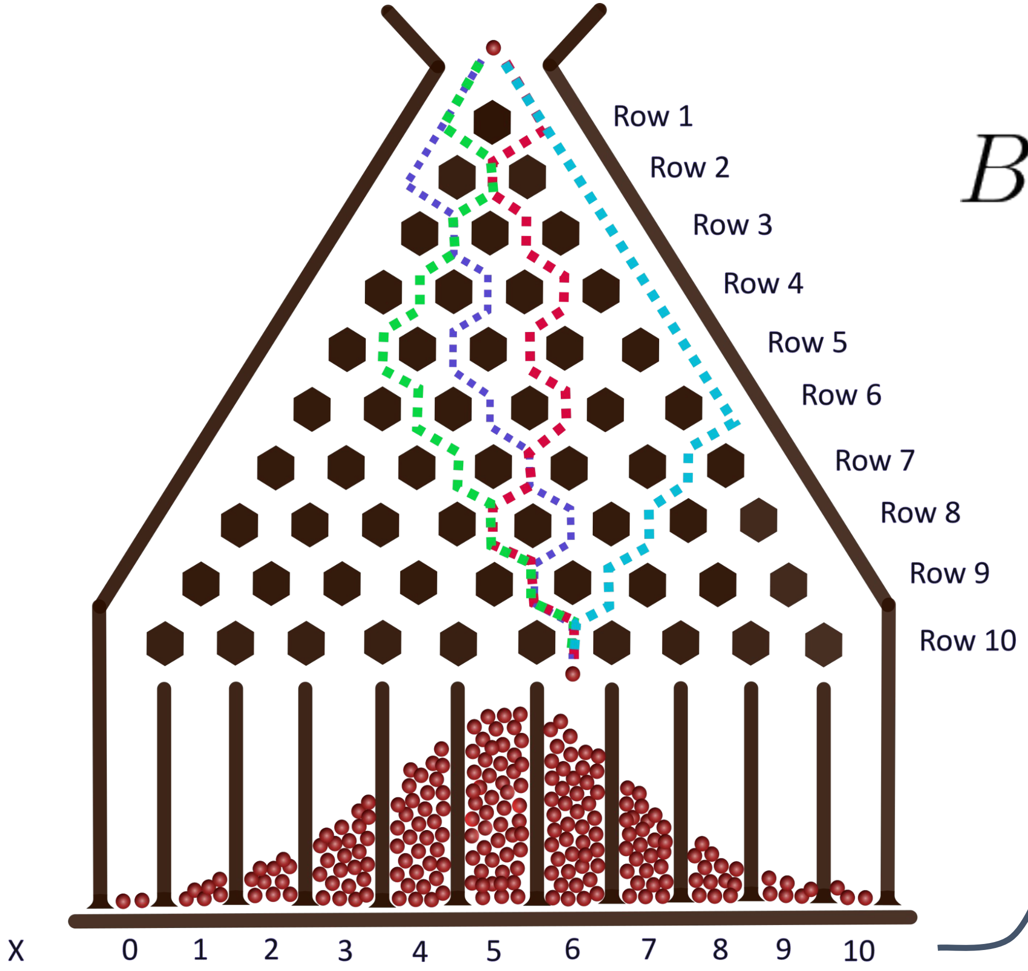


$$B(X=k)=p^k(1-p)^{n-k}$$

Random variable

$$X=\sum_{l=1}^{n} X_l$$

Random walk and Binomial distribution



$$B(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Random variable

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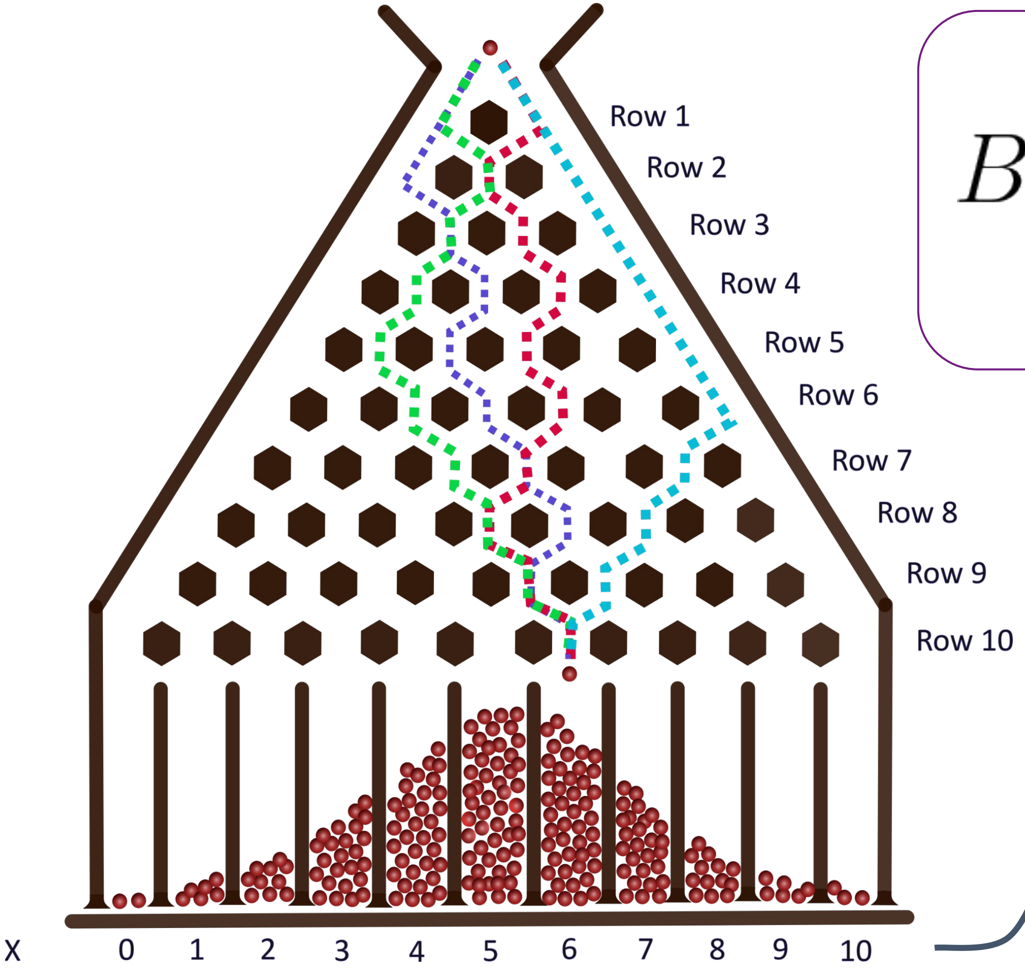
Random walk and Binomial distribution



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Binomial distribution

Random variable

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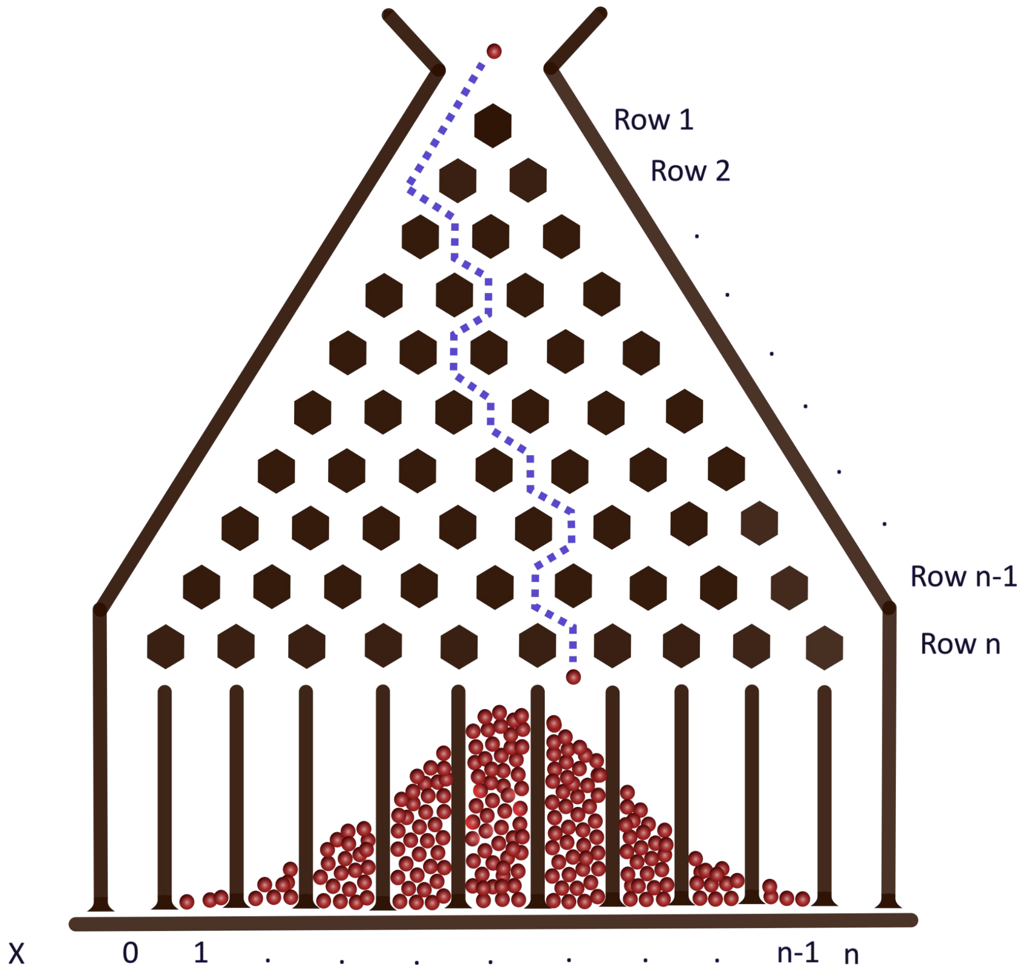
General characterization of the attraction basin



$$X = \sum_{l=1}^{n} X_l$$

1. How one must rescale the variable X, in order to obtain a limit distribution?

2. What is the limit distribution?

General characterization of the attraction basin

THEOREM. Stable law (Levy, 1925):

For $P(X)$ to be a possible **limiting distribution** for the reduced variable, there must exist, for all a_1, $a_2 > 0$, b_1 and b_2, two quantities $a > 0$ and b, such that:

$$P(a_1 X + b_1) * P(a_2 X + b_2) = P(aX + b)$$

[1] Bouchaud, J., & Georges, A. (1990). Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Physics Reports, 195, 127-293.

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DEFINITION. Characteristic function:

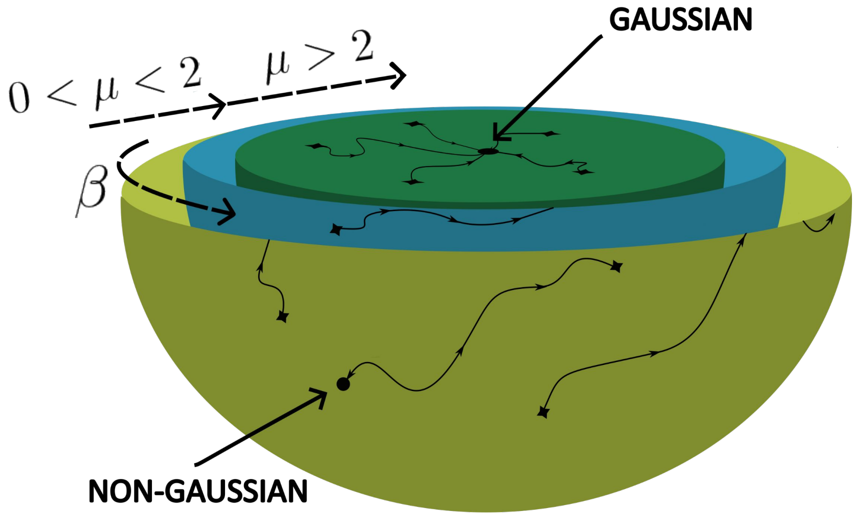
We define the **characteristic function** of a probability distribution as the Fourier transform of the probability density function.

$$\hat{P}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikX} p(X) dX$$

[1] Bouchaud, J., & Georges, A. (1990). Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Physics Reports, 195, 127-293.

General characterization of the attraction basin

THEOREM. Canonical representation of stable laws (Levy, Khintchine):



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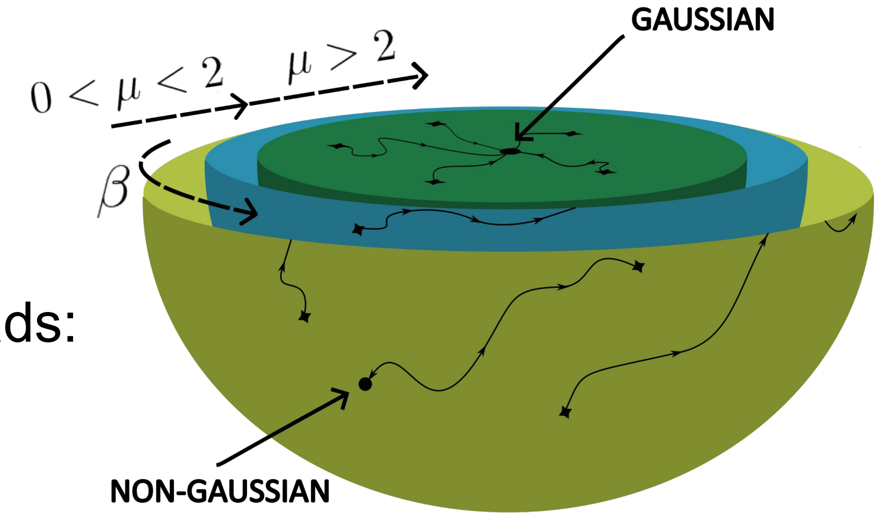

$P(X)$ is stable if and only if its characteristic function reads:

$$\ln \hat{P}(k) = i\gamma k - c|k|^{\mu}[1 + i\beta\, \mathrm{sign}(k)\omega(k,\mu)]$$

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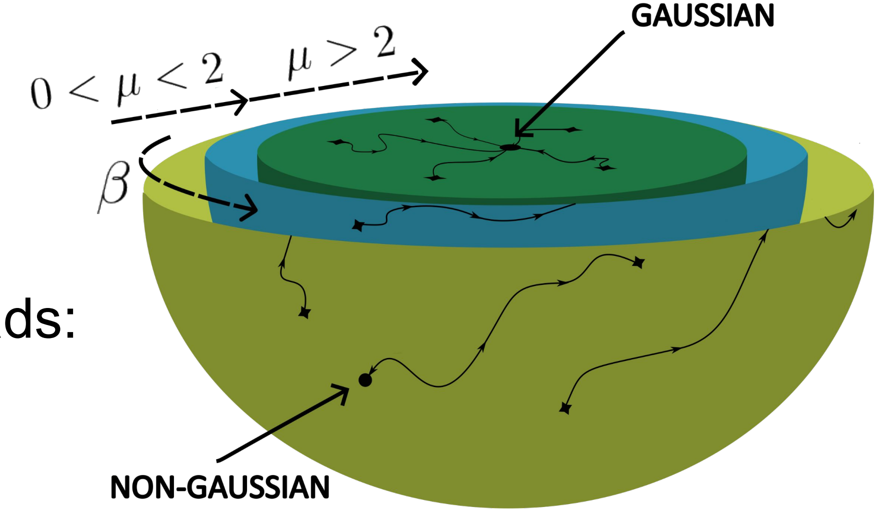

$P(X)$ is stable if and only if its characteristic function reads:

$$\ln \hat{P}(k) = i\gamma k - c|k|^{\mu}[1 + i\beta \,\mathrm{sign}(k)\omega(k,\mu)]$$

Where μ, β, γ and C are real numbers such that

$-1 \leq \beta \leq 1$, $0 < \mu \leq 2$, $C \geq 0$ and

$\omega(k,\mu) = tan(\pi\mu/2)$ for $\mu \neq 1$ $\omega(k,\mu) = \frac{2}{\pi}\ln|k|$ for $\mu = 1$

[1] Bouchaud, J., & Georges, A. (1990). Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Physics Reports, 195, 127-293.

General characterization of the attraction basin

THEOREM (Gnedenko)

$P(X)$ belongs to the **attraction basin** $L_{\mu\beta}(Z)$, with

$$Z = \begin{cases} \dfrac{X}{N^{\frac{1}{\mu}}} & \text{for } 0 < \mu < 1 \\ \dfrac{X - <X_l> N}{N^{\frac{1}{\mu}}} & \text{for } 1 < \mu < 2 \end{cases}$$

and

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}$$

If and only if

$$p(l) \simeq \begin{cases} c_1 |x|^{-(1+\mu)} & \text{for } x \to -\infty \\ c_2 x^{-(1+\mu)} & \text{for } x \to \infty, \end{cases}$$

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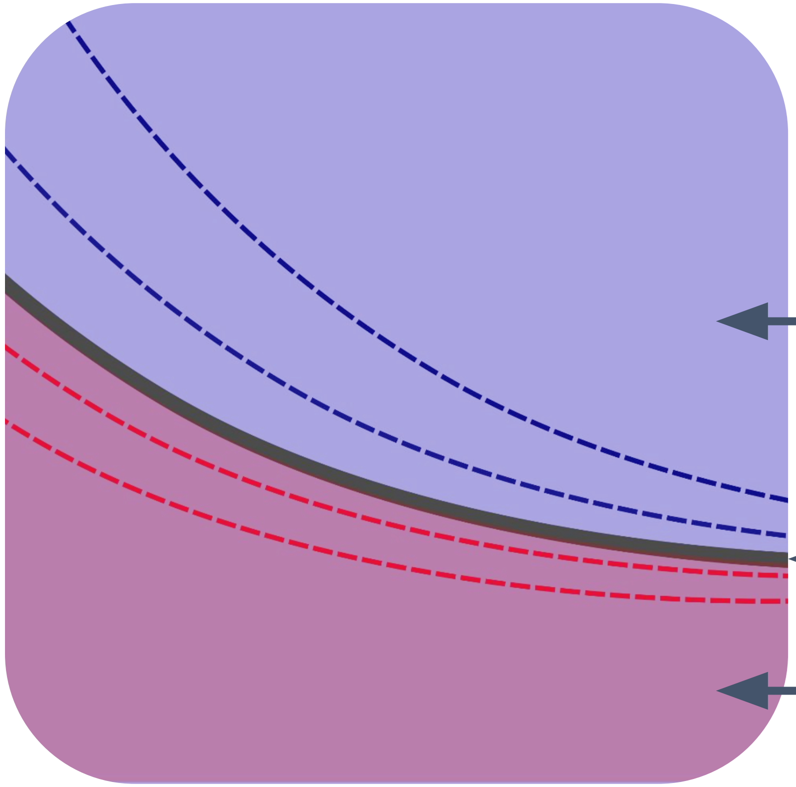
General characterization of the attraction basin



Furthermore

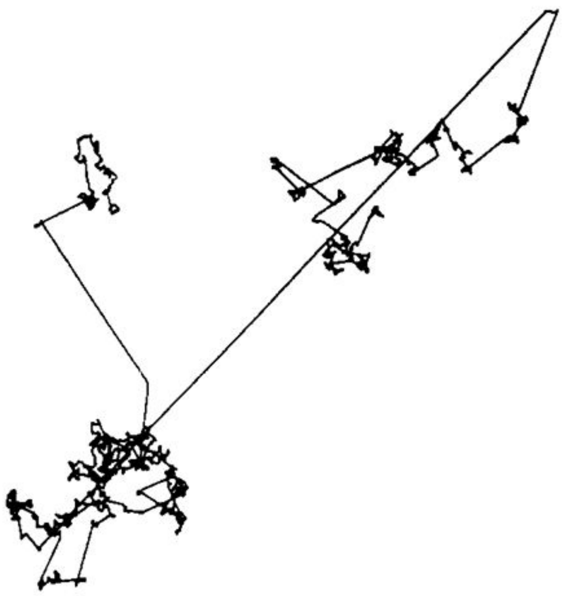
Non-Gaussian large-x behaviour (Heavy Tail)

$$L_{\mu\beta} \sim Z^{-(1+\mu)} \quad \text{for} \quad Z \to \infty$$

Gaussian large-x behaviour

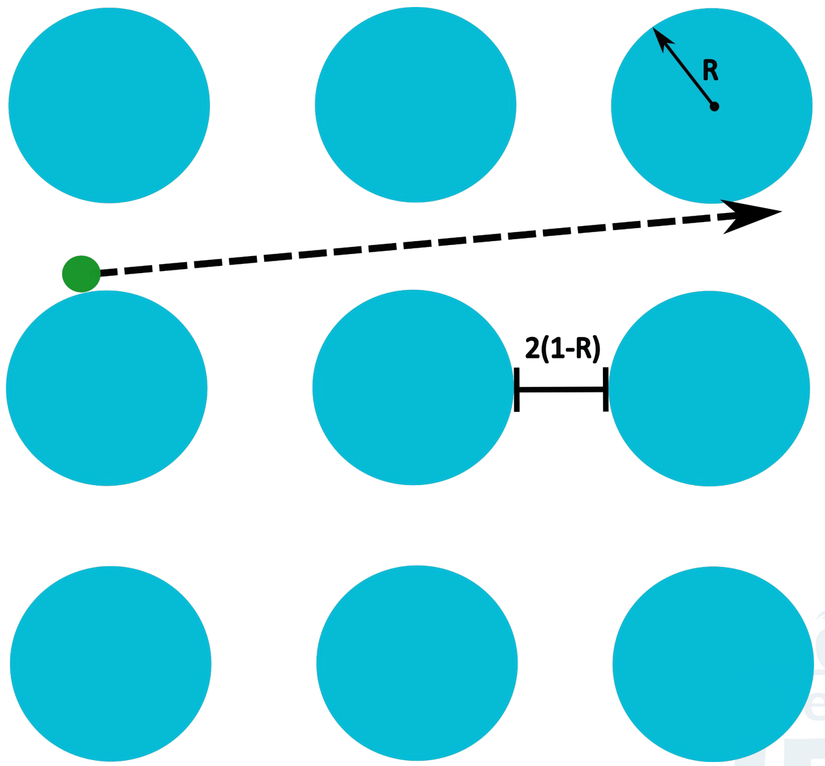
[1] Bouchaud, J., & Georges, A. (1990). Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Physics Reports, 195, 127-293.

Physical Motivation



Levy walks [1] .

Sinai billiard on a square lattice [2].



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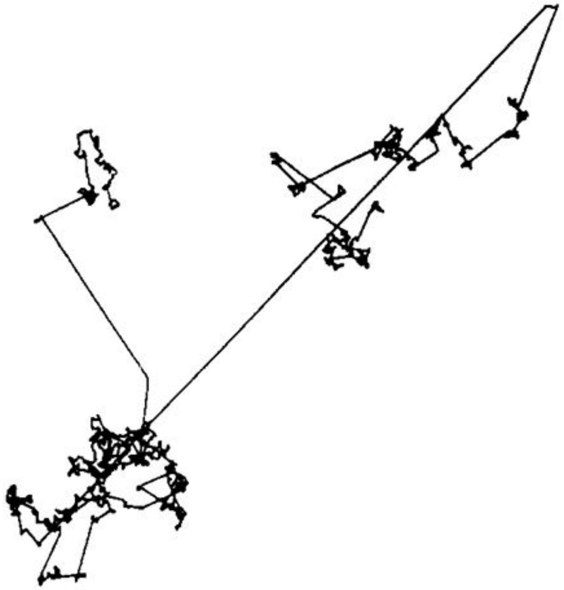

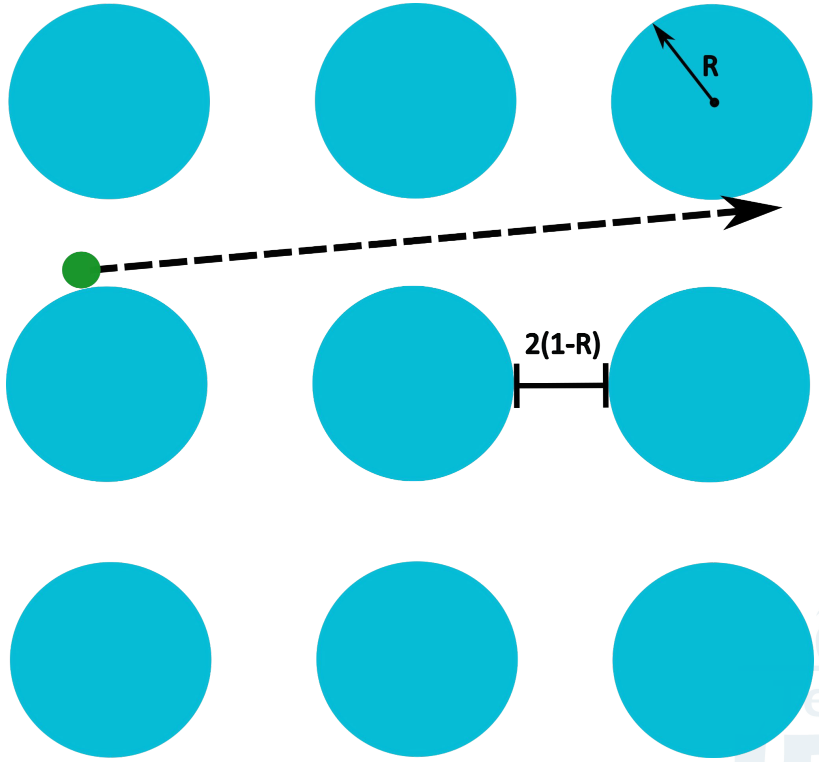

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Physical Motivation

$$p(l) \sim \frac{1-R}{X_l^3} \quad as \quad X_l \to \infty$$

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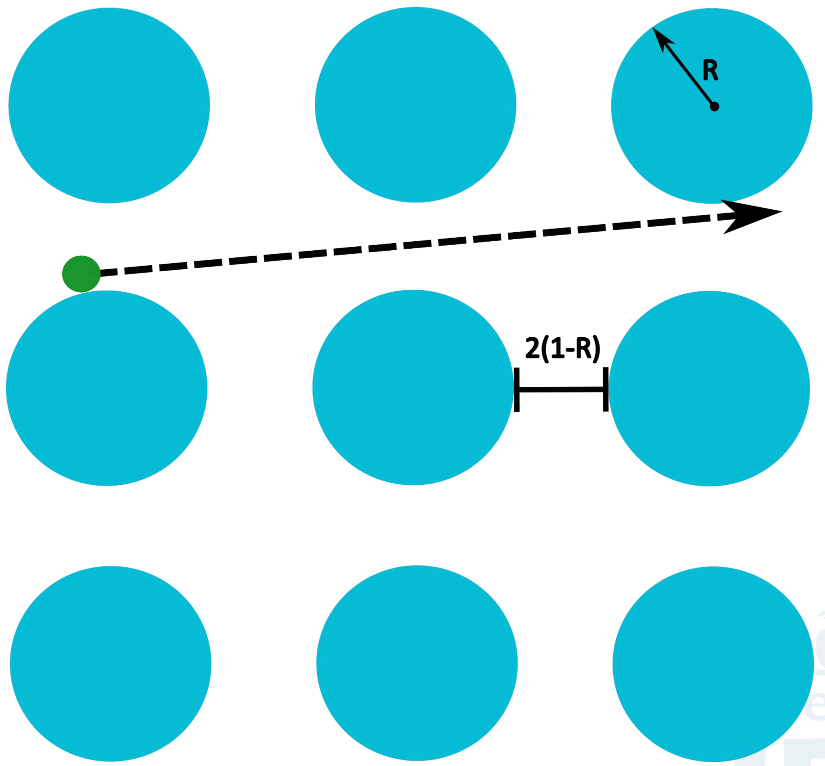

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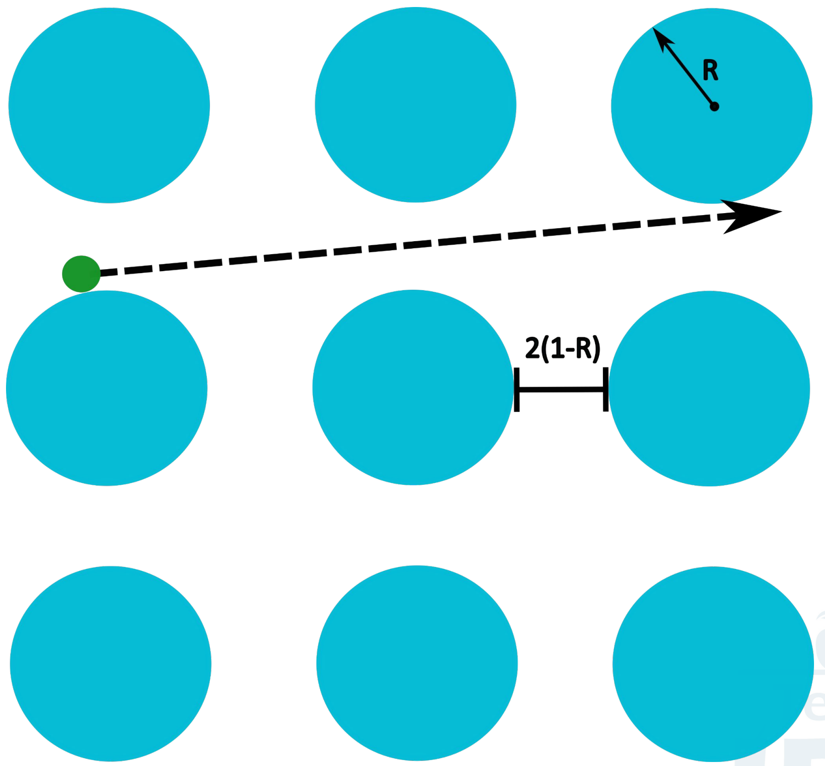


$$p(l) \sim \frac{1-R}{X_l^3} \quad as \quad X_l \to \infty$$

Anomalous diffusion regime,

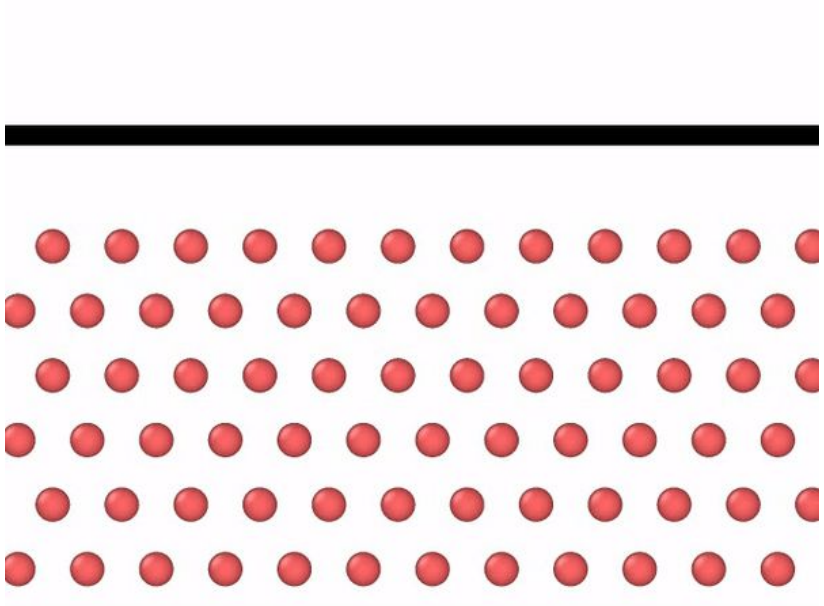
$$p(l) \sim \frac{1}{X_l^{5/2}} \quad for \quad l \leq l^* \sim R^{-1}$$

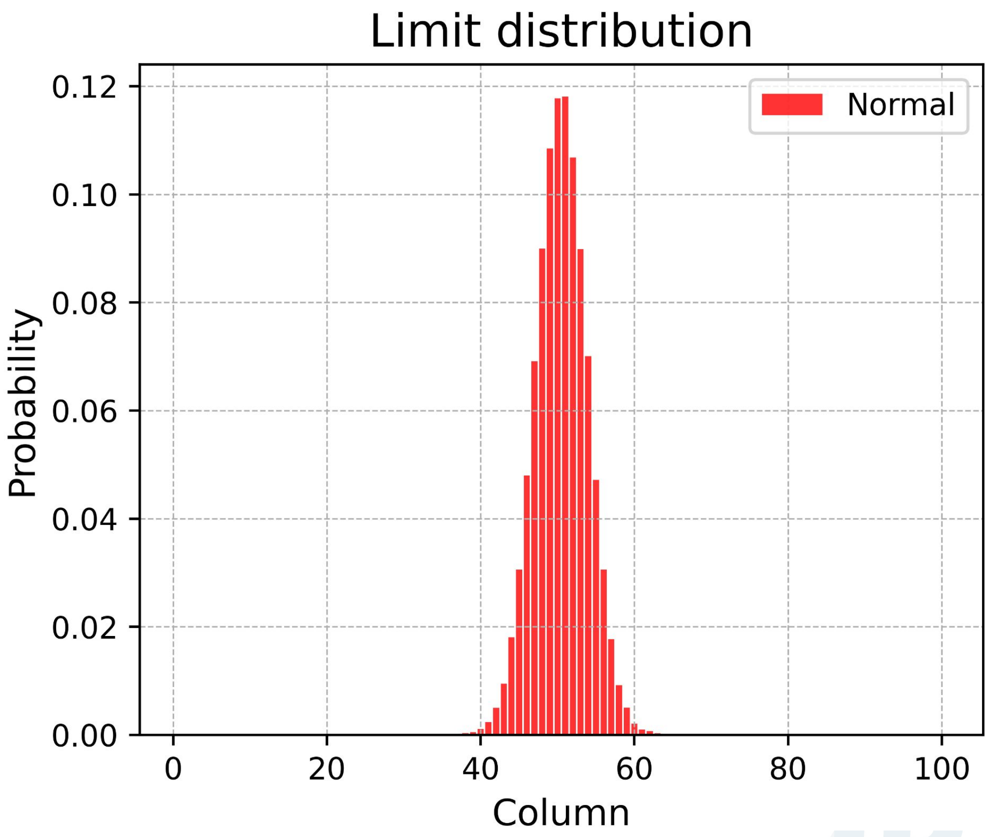
[2] Bouchaud, JP., Le Doussal, P. (1985). Numerical study of a D-dimensional periodic Lorentz gas with universal properties. J Stat Phys 41, 225–248.

Galton Board Simulation

- Interactions among the beads are always neglected.
- Elastic collisions are implemented.
- The beads are under the action of the gravitational force.
- Particles can be released with a given initial velocity
- The bin at which it is collected is defined by the last horizontal position of the bead.

Galton Board Simulation

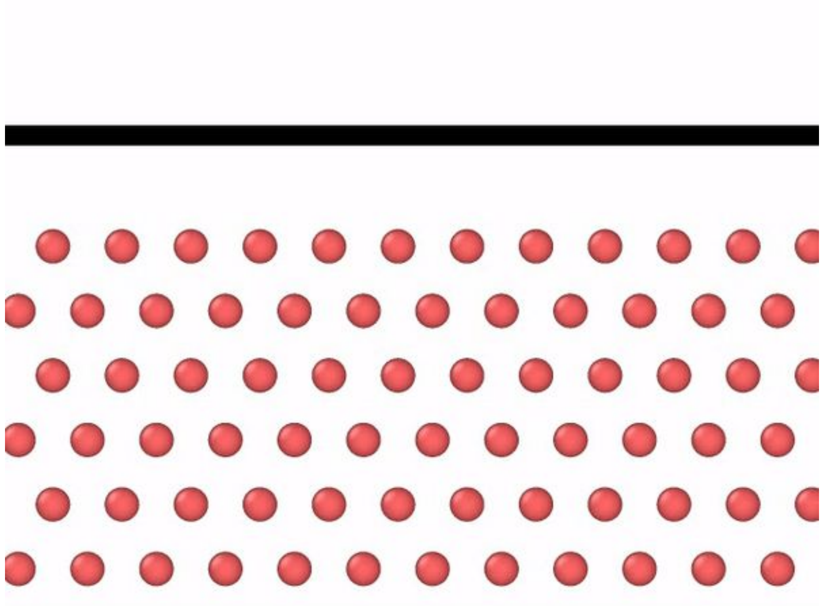


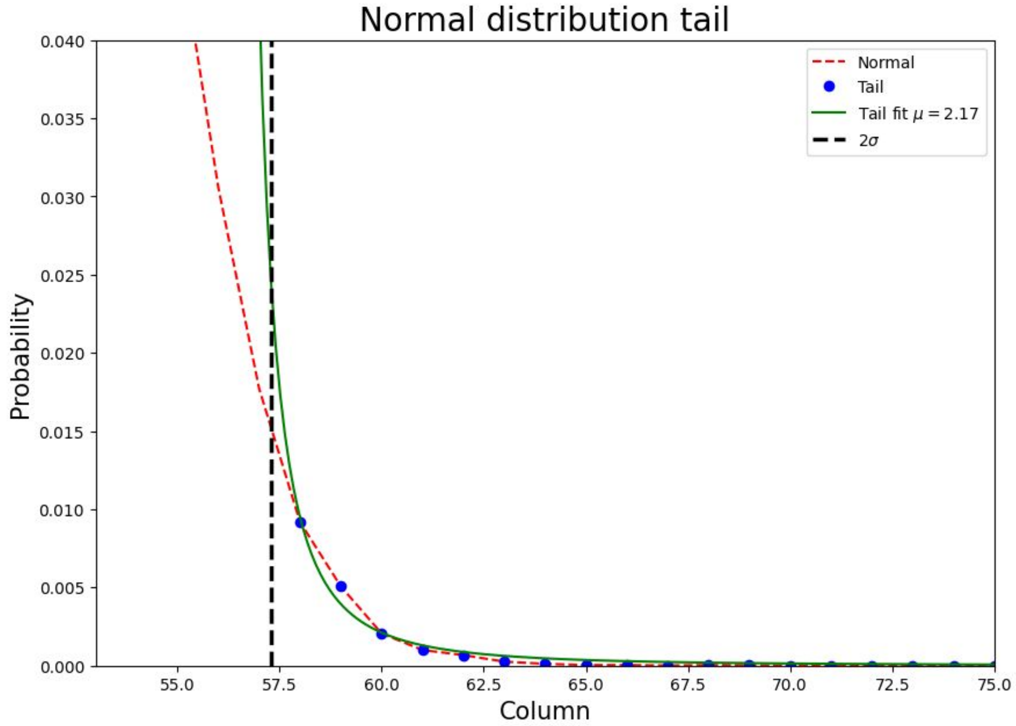


Fixed parameters:

- **number of Beads** → 100,000
- **number of Rows** → 30
- **relative sizes of pegs and beads** → 1 and 0.5
- **Time interval** → 0.1s with 5 substeps.
- **Damping coefficient** → 0.2
- **Velocity** → 30 as standard

Galton Board Simulation





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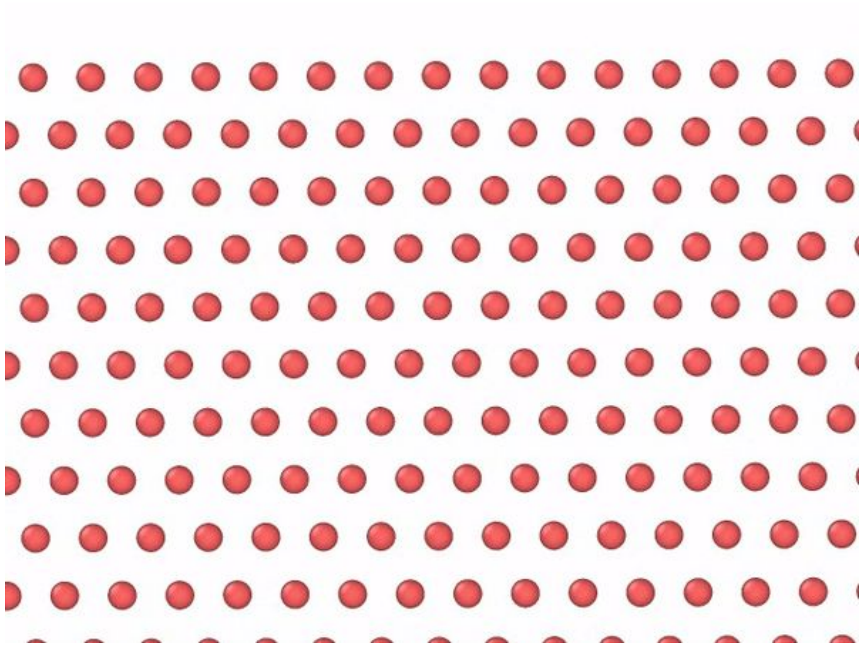
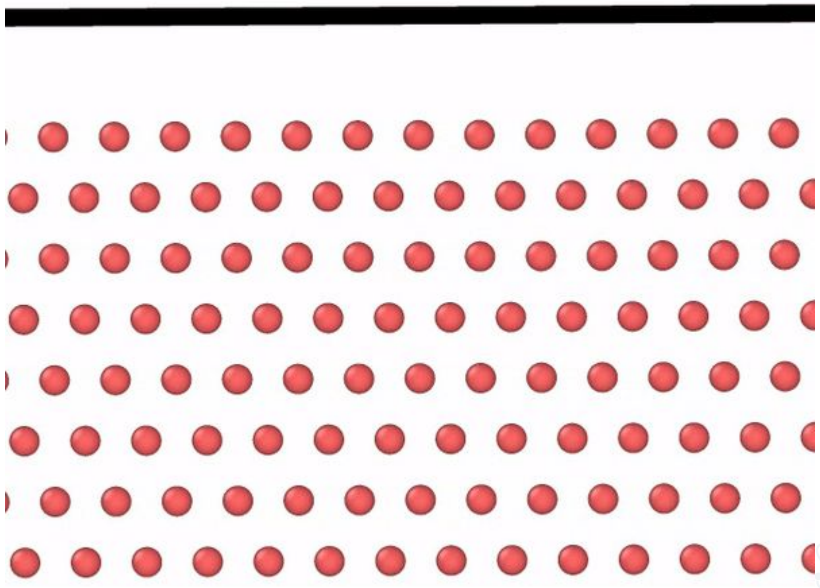
Deviations from the Normal Law

Relevant parameters: 1 Angle of incidence of the beads.

30°

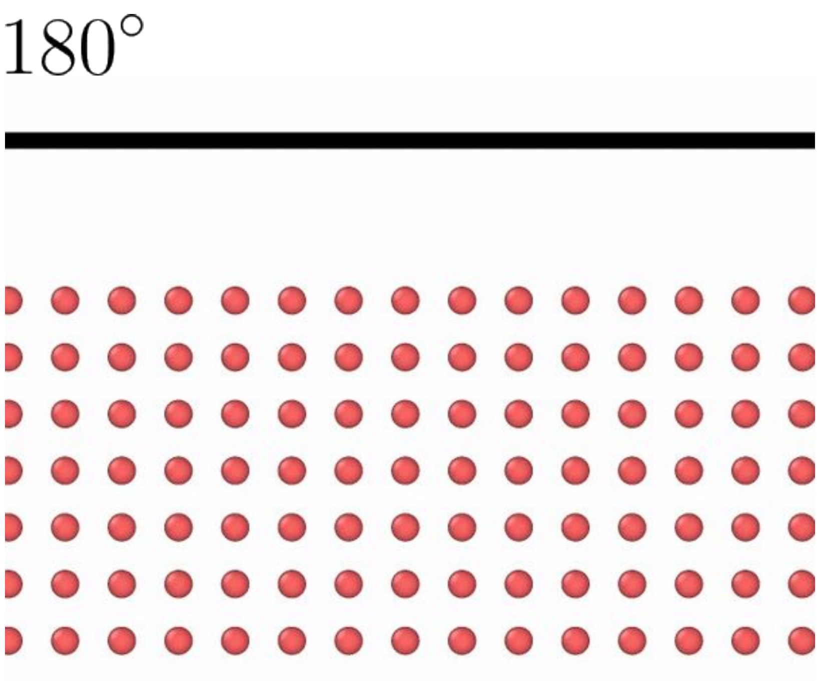


180°

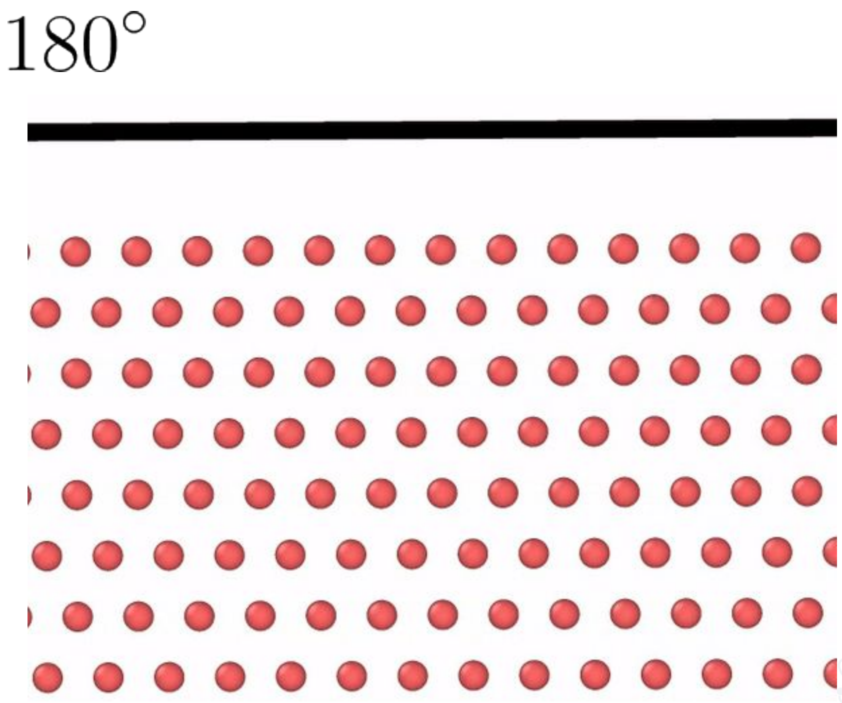


Deviations from the Normal Law

Relevant parameters: 2 Pegs distribution.

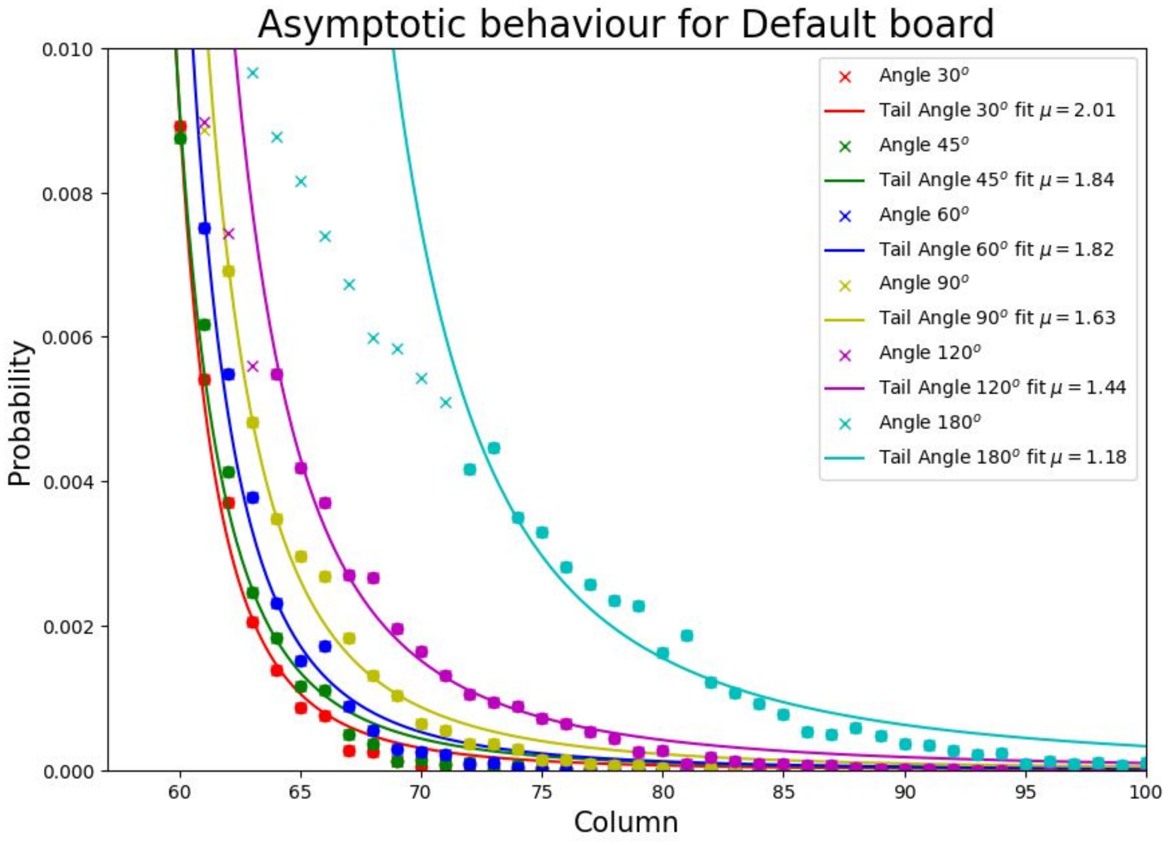


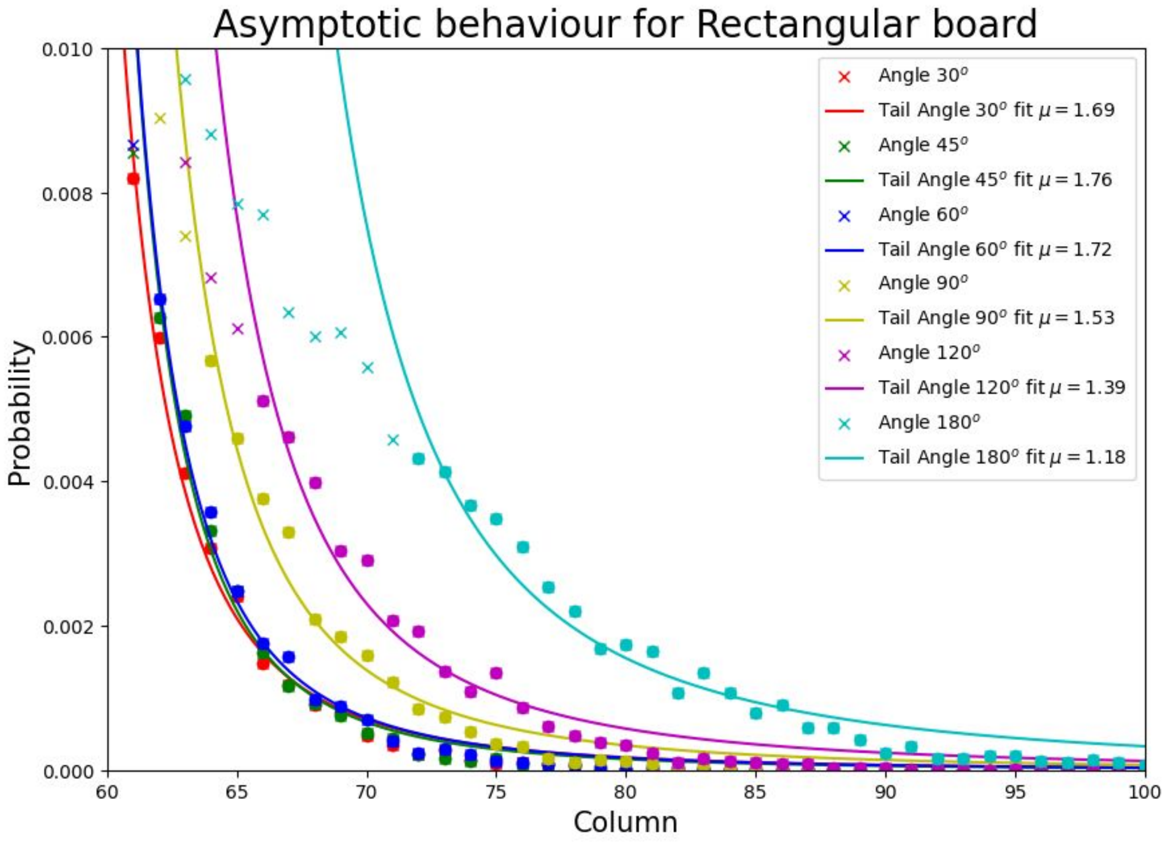
Rectangular lattice.



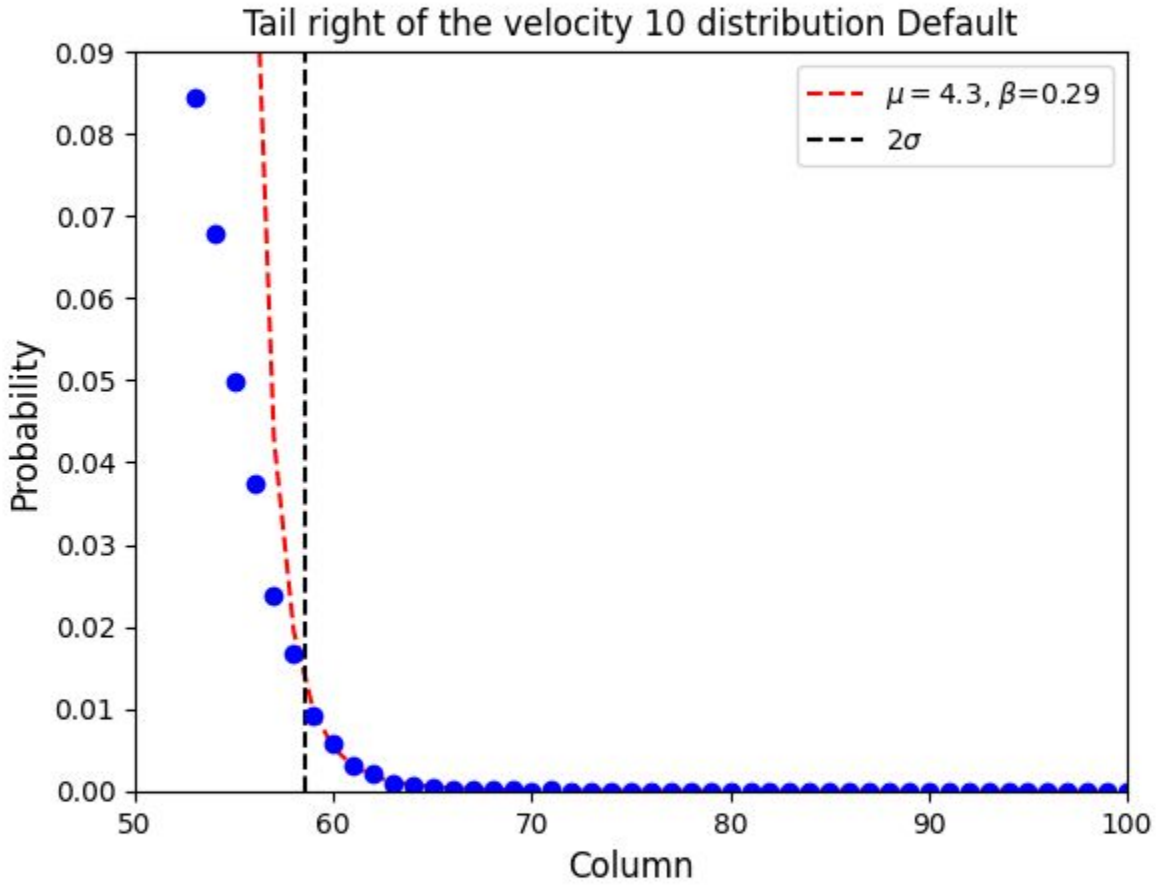
Hexagonal lattice.

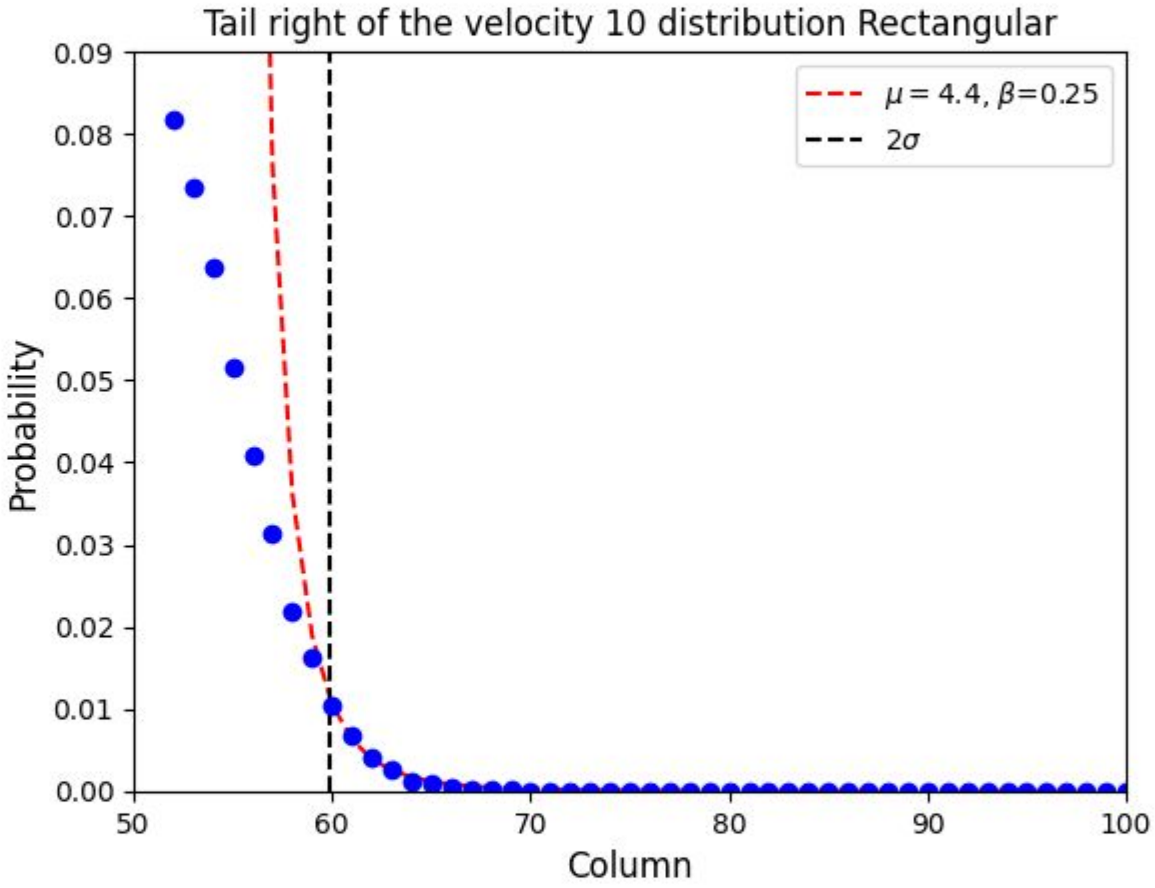
Deviations from the Normal Law





Effects of Velocity of the particles





With taking random values between 0 and 180

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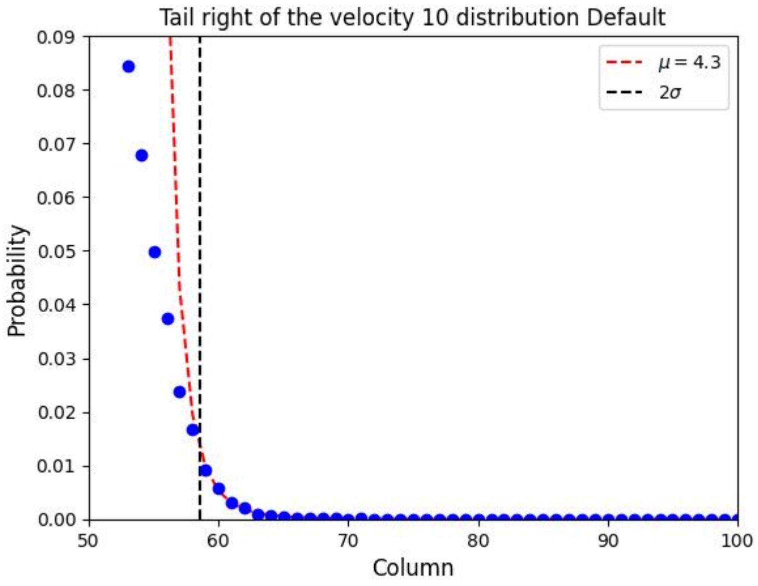
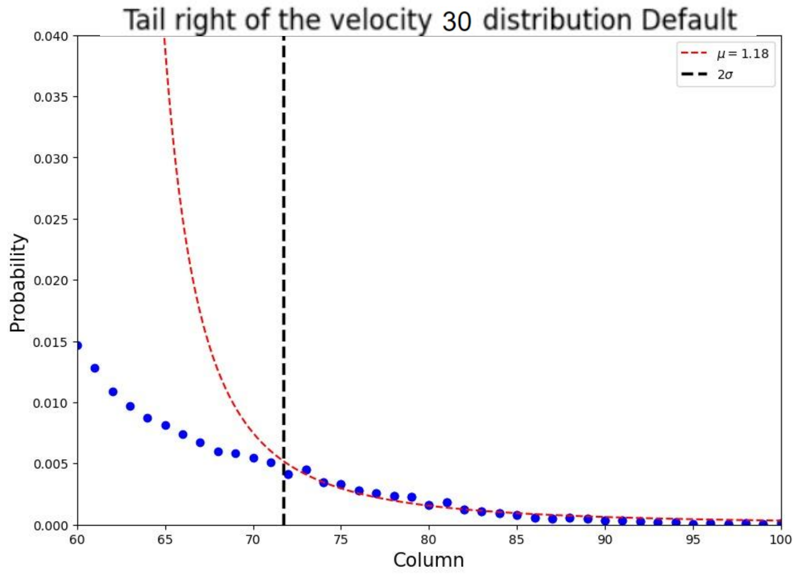
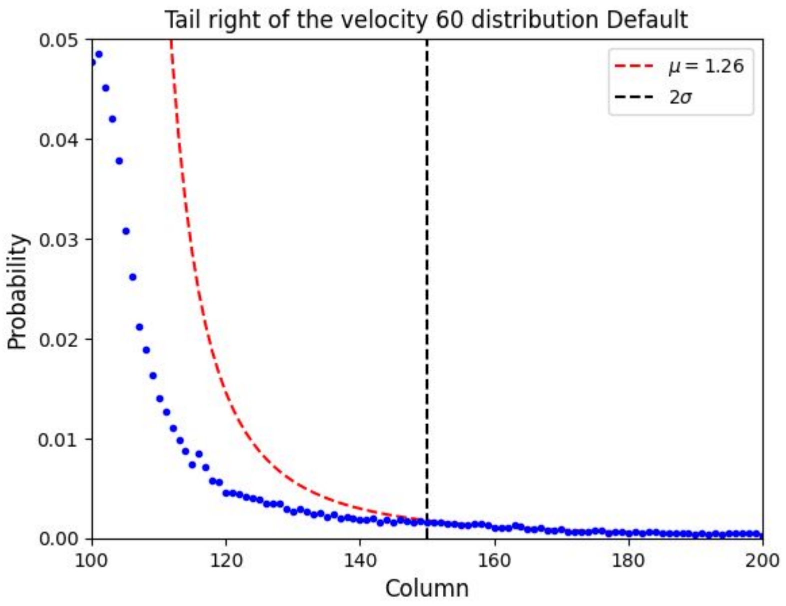
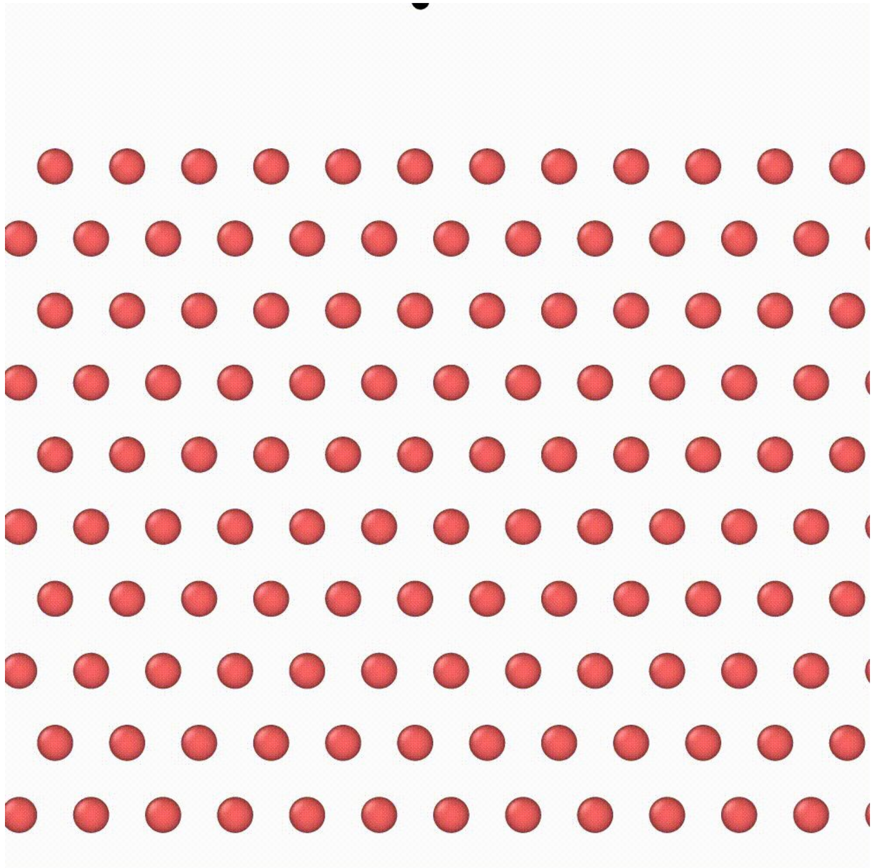




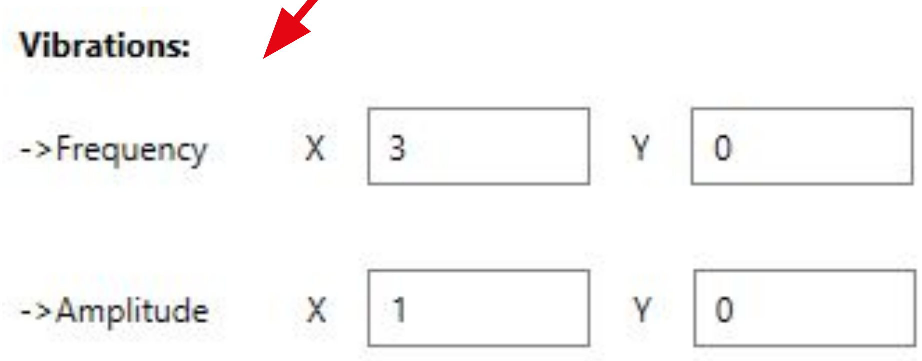


With θ taking random values between 0 and 180.

Effects of board vibration

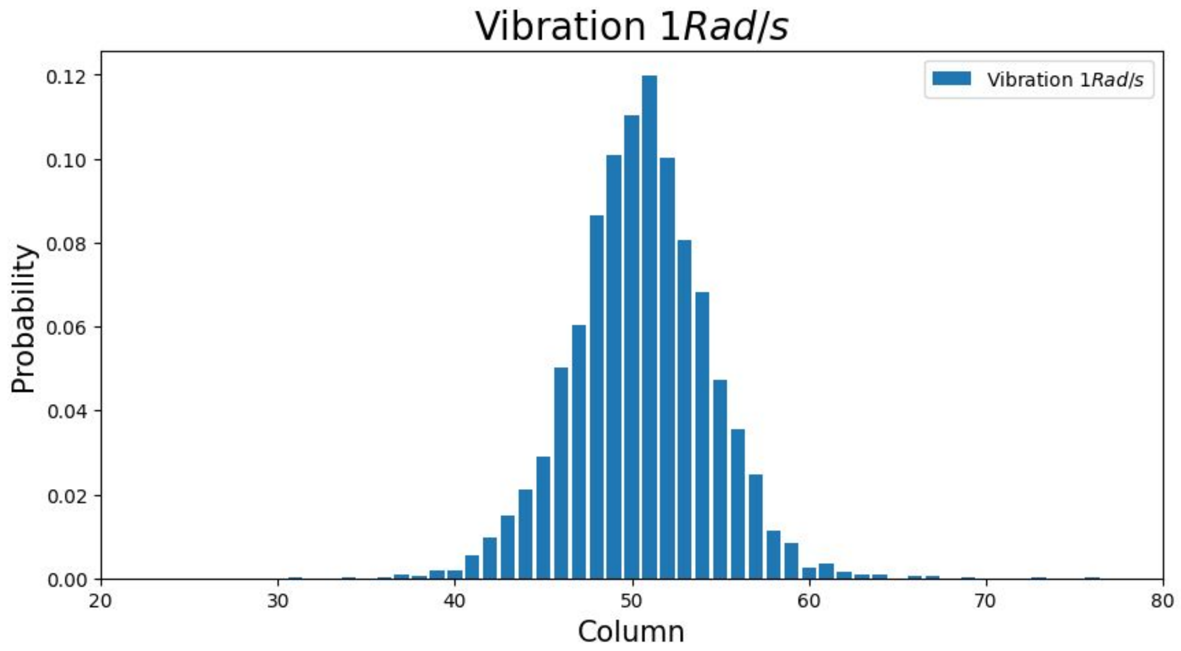


Angular Frequency

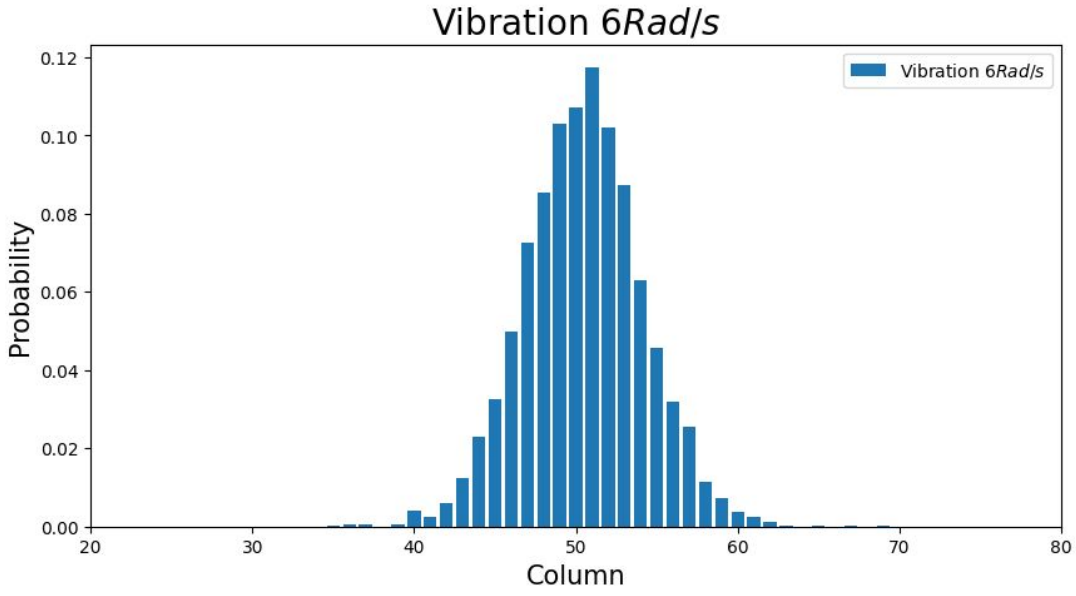


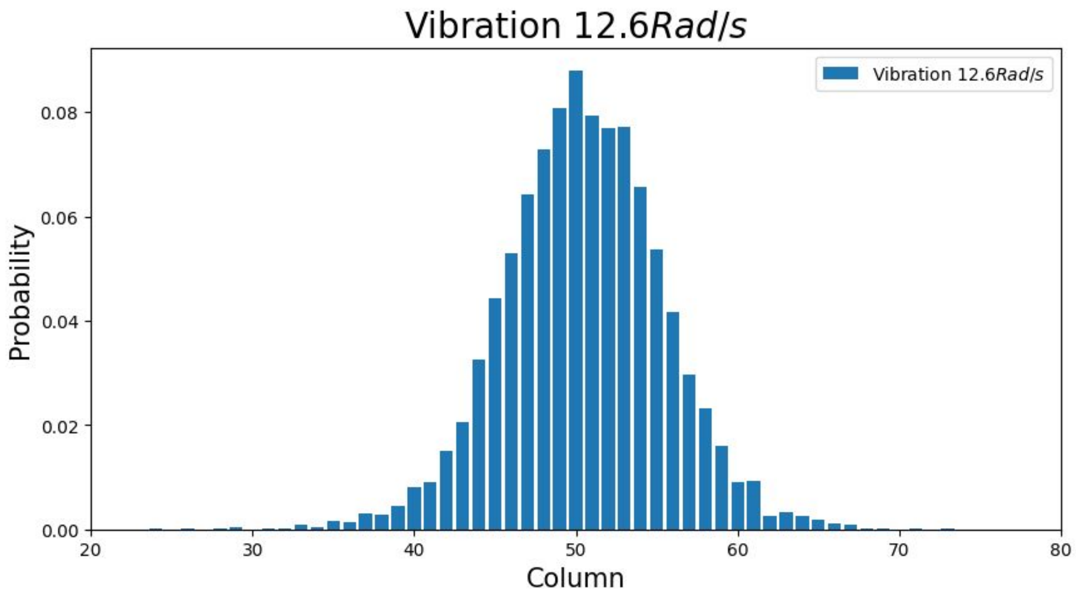
Allows to simulate sinusoidal behavior on the pegs

Effects of board vibration



It is found that for values greater than 16 Rad/s the vibration prevents the uninterrupted passage of particles.



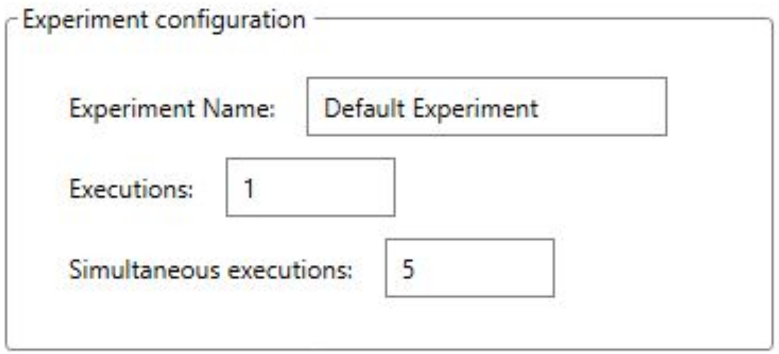


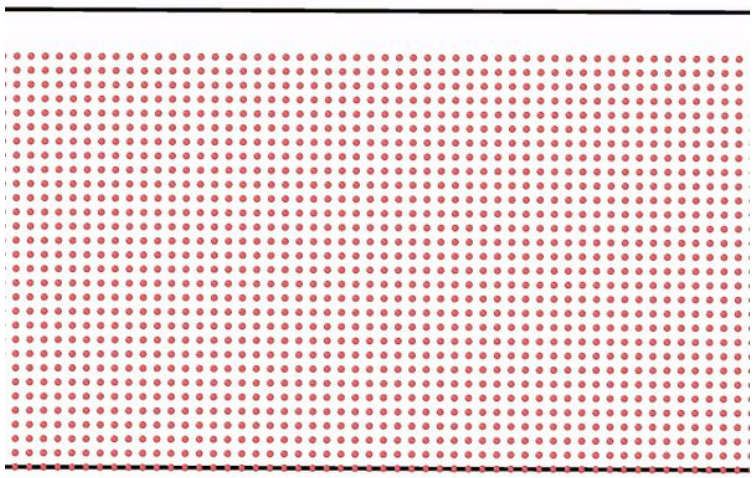
Social Commitment

SUSTAINABLE DEVELOPMENT GOALS

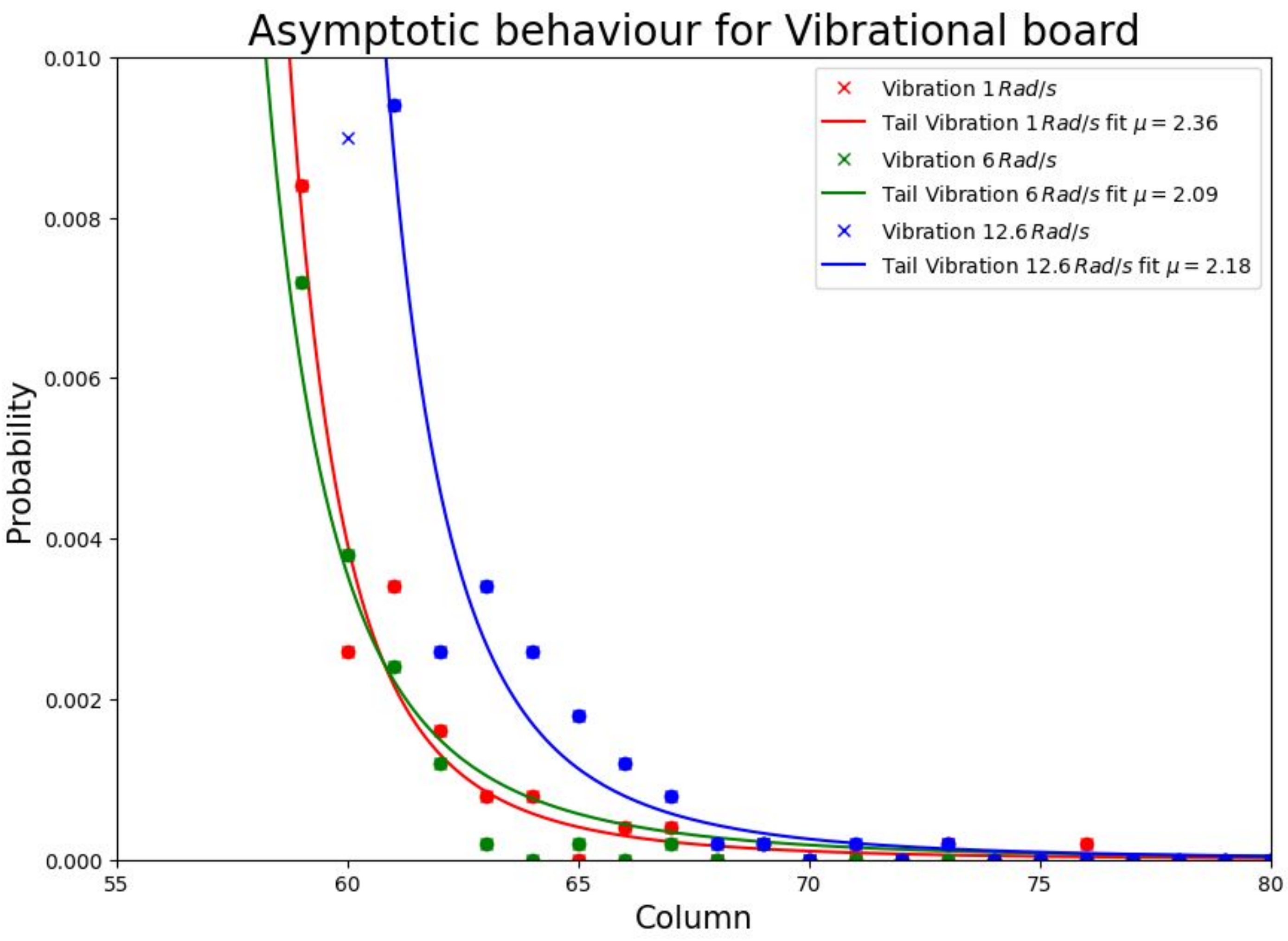
1. Interactive Learning.

2. Visual Representation of Concepts.





Effects of board vibration



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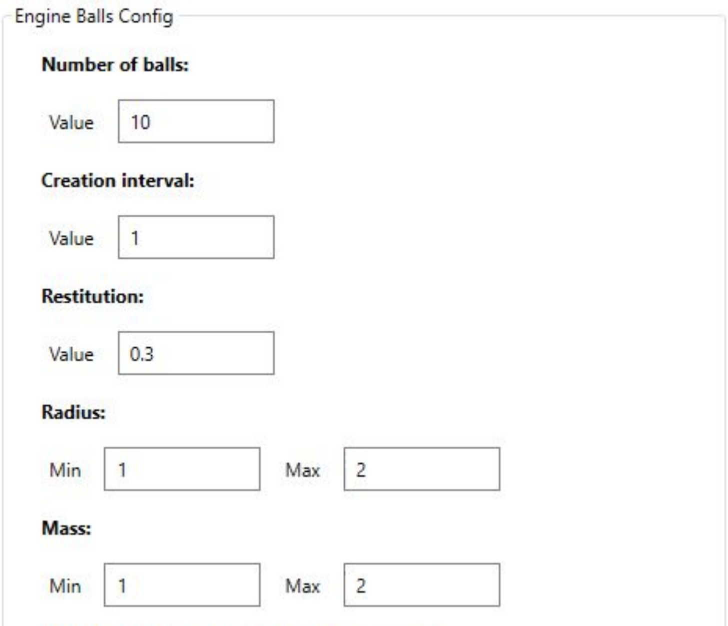
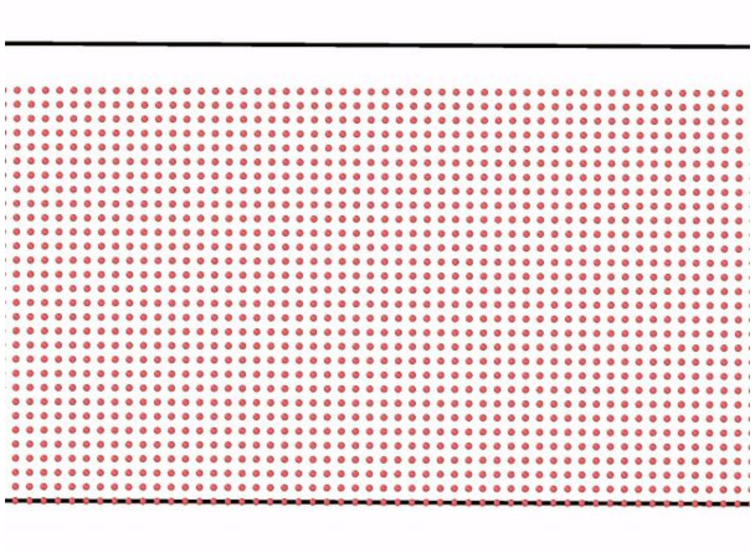
2 Visual Representation of Concepts.

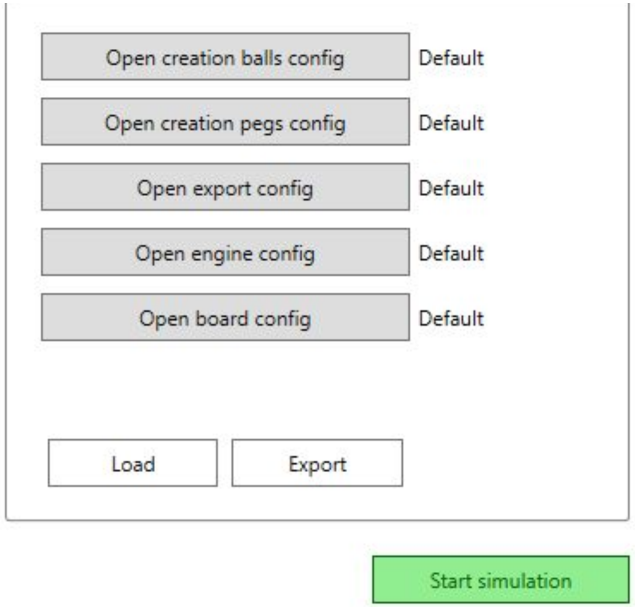




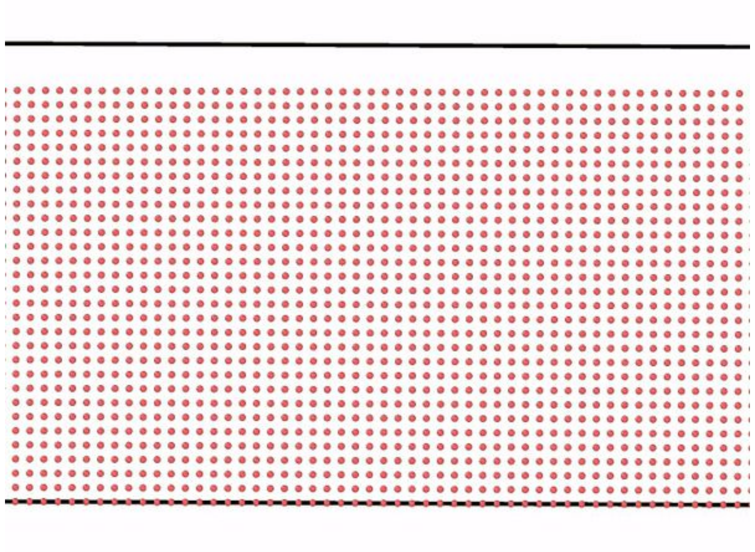
Social Commitment

SUSTAINABLE DEVELOPMENT GOALS

1 Interactive Learning.



2 Visual Representation of Concepts.



Social Commitment

SUSTAINABLE DEVELOPMENT GOALS

3 Accessible Learning Resources.

Github: https://github.com/niaggar/GaltonBoard

Summary

- We explained why the Binomial distribution is the limiting function for the Galton Board with evenly distributed pegs, by treating the system as a random walk.
- We characterized the limit distribution of the sum of independent random variables as being stable functions completely defined by the parameters μ and β.
- We created a simulation that allows us to vary the parameters of interest, in particular; the angle of incidence and the distribution of pegs.

Conclusions and Perspectives

- It is possible to produce other kinds of distributions by varying the physical parameters of the Board. This asymptotic distributions belong to the family of Stable functions.

- It is possible to obtain a limit distribution with parameter $\mu \neq 2$, deviating from the normal law and therefore violating the Central Limit Theorem in its conventional form (i.i.d scenario).

- Vibration of the board at a fixed frequency induced a change in the angles of incidence of the beads, equivalent to the random variation performed in the simulation. The vibration thus increased the deviation from the normal law.

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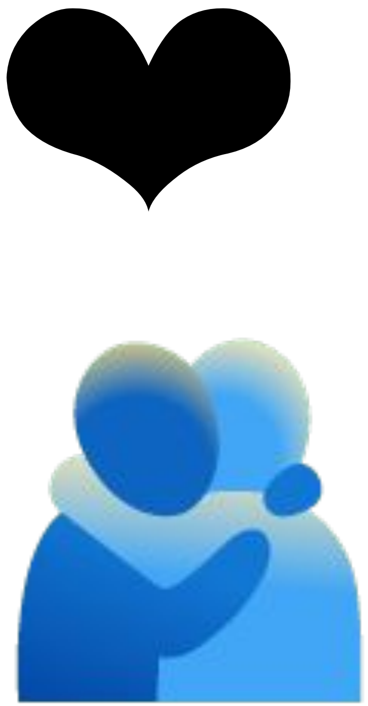
Team IPT UniValle

Thank you!

Acknowledgments

Galton Board Colombian Team



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