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Exit distribution function crossover in a Galton board

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Abstract Here we present an experimental and numerical study concerning the problem of the influence of walls on a granular flow through obstacles. The system based on a Galton Board is a vertical bi-dimensional array of equally spaced obstacles arranged to form a triangular lattice. Lateral walls, whose relative separation can be varied, are set. Disks of equal diameters are launched at the top of the system. During the fall, the disks collide inelastically with the obstacles of the lattice and, eventually, with lateral walls. At the exit, at the bottom of the board, disks are collected in separate bins depending on their final horizontal positions. The aim of the present paper is to study the dependence of the exit distributions of small disks on the separation between the lateral walls of the board. We found that there exists a crossover from a Gaussian-like to a uniform-like behaviour when the separation between walls is less than a critical value. A systematic study of this crossover is performed through numerical simulations and the results are also approximated by a theoretical model of the problem.

Keywords Galton board · Exit distribution · Mixing · Diffusion · Crossover

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1 Introduction

Obstacles arranged in a periodic lattice are often used to study transport and diffusion in many phenomena ranging from crystals to grain mixing.

The Galton's board is an important example of a two dimensional system of small pins forming a regular triangular lattice. It has been used intensively for many purposes, even for a simple illustration of the central limit theorem. A typical sketch of the board is shown in Fig. 1.

Usually, the board is arranged in a vertical or nearly vertical position and the experiments performed on this board mainly consist in launching small balls or small disks (compared to the spacing between pins of the lattice) from the top row of the array.

Many studies have been done using this set up due to the fact that it presents an interesting dynamics. To provide a short review of the relevance of studying the passage of small particles down a Galton array, we will cite some of the works performed before. For instance,

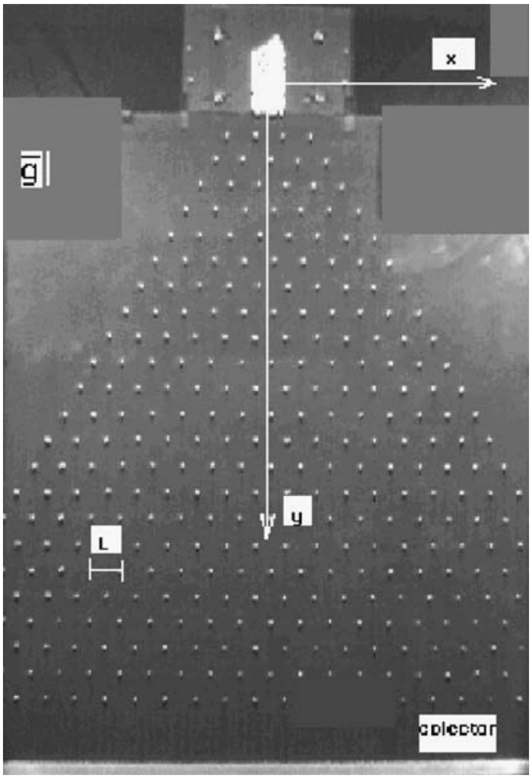

Fig. 1 Scheme of the Galton's board used in the experiments

several years ago, diffusion in a periodic Lorentz gas has been modelled through a regular Galton board lattice of fixed particles arranged in a dense triangular-lattice structure, where the moving scattering particle travels through the lattice at constant kinetic energy, undergoing elastic hard-disk collisions with the fixed particles [1]. On the same line of reasoning, Klages and Korabel [2] studied the fractal properties of diffusion coefficients and current through numerical analysis of dimensions from box-counting and from the autocorrelation function. In particular, fractal chaotic behaviour is analysed in detail in [3,4] and chaos in viscous flow of charged particles, instabilities and phase flow topology are treated in [5,6], proving that a Galton board is a system that presents a complex behaviour while still being computationally amiable. Recursively defined combinatorial functions extended to Galton's boards have also been studied [7]. Keeping in mind the flow scenario but now in a "macro" scale, dispersive flow of particles have extensively been studied by Bruno et al. [8]. In fact, the idea of describing the percolation of small particles in bi-dimensional lattices of obstacles has attracted the attention of many researchers [9–12]. In particular, the possibility of the implementation of a Galton board like a particle mixer has been developed and successfully proved in [13,14].

In spite of the extensive work performed in the past and the interesting and rich behaviour already obtained, many questions still remain to be answered in the context of Galton boards. For instance, most of the experiments, numerical simulations and theory concerning these systems are performed without the presence of lateral walls. Particles "diffuse" in a gravity driven flow but never find a wall on their way. Thus, Gaussian exit distributions are found most of the time. This set up is adequate for the characterization of percolation and diffusion problems, but this situation is in fact not always common in industrial mixing problems. For that reason the aim of the present work is to study the influence of walls in the percolation of small disks through a vertical Galton board where obstacles are very small pins arranged in a regular triangular lattice. Our study will basically contain three parts: experiments, numerical simulations and a theoretical approach. We are especially interested in measuring the exit distributions of particles at the bottom of the board as a function of the separation between lateral walls. Because a large range of size parameters is not easily attainable experimentally, simulations using a very simple random model will allow us to recover the main features of experiments and will provide a valuable extrapolation of the behaviour found for larger systems. A theoretical approach of our scenario will confirm both experimental and simulation results and will allow to propose a criteria to select the dimensions of a Galton board for practical purposes and applications.

The following section will explain the experimental set up and the obtained results; the next one will develop the numerical model, will present the results of simulations and will compare and discuss both results; after that, we will develop the referred theoretical approximation to explain the behaviour found and finally we will give our main conclusions.

2 Experimental set-up and results

The experimental device (Fig. 1) is the same as that used in earlier experiments by Bruno et al. (For a detailed description see [8,9]). The Galton board consists of two parallel plates, a smooth wood one and a glass one, of 120 cm wide and 80 cm of height separated by a small gap of 1.5 mm in which particles fall down. In our experiments we use discs of diameter $d = 8$ mm and of different materials: rubber, aluminum and lead of densities $\rho = 1.6, 2.7$ and 11.3 g/cm^3, respectively.

The obstacles, styrene discs (4 mm diameter, 1.2 mm thick), are glued onto the wooden wall forming a hexagonal network with lattice spacing of 2.7 cm. This distance

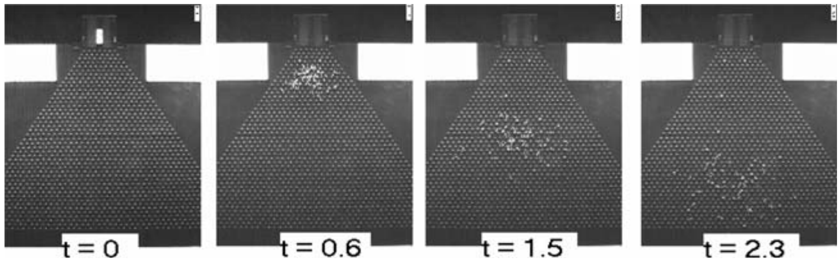

Fig. 2 A time sequence of an experiment performed with 10 mm styrene discs. Initially, discs are released together from the top. In the second frame we can see that collisions between discs are important. However after four or five obstacle rows (third frame) collisions disc–obstacles become dominant and collisions disc–disc are quite neglected. Finally, for $t = 1.2$ s, particles are completely decorrelated and a one-body approximation can be used

was chosen to be larger than the diameter of the bigger flowing discs in order to avoid the formation of arcs [18]. The number of rows of obstacles N is fixed and equal to 46.

A box is located just at the top of the Galton board and it can be filled with a large number of discs (typically 100). After passing through the obstacles, at the bottom, where the exit of the device is located, the discs accumulate orderly (heap up) into different 70 bins of 15 mm width. The final spatial distribution is recorded with a digital camera. An image processing software allows then to obtain the final position (x and y coordinates) and to build the corresponding histogram in the horizontal (x) direction.

In order to study the influence of the presence of lateral walls we have placed two lateral PVC rods of 1 m height and 1 cm width, at equal distance from the center $x = 0$. The distance W between the rods can be varied. In our experiments, $W = 70, 60, 50, 38, 26$ and 16, measured in number of bins contained between the walls.

The discs can be released simultaneously, or one by one. While the discs are falling, they collide inelastically with the obstacles and other falling discs, loosing energy and changing their local velocity direction. Bruno et al. [8,9] analyzed the influence of the collective interactions, that is the influence of other falling particles on a tracer disc, by studying the velocity distribution of tracer particle and comparing it to the one obtained when the discs are released one by one and they noticed no significant influence. Even more, they have performed numerical simulations of the spatial frequency of disc–disc and disc–obstacle collisions. They have seen that the spatial frequency of disc–obstacle collisions remains constant in average during all the chute and disc–disc collisions occur mainly close to the entrance where the particles are clustered together (region of high disc–disc interaction density) corresponding to the first four or five rows of obstacles. Afterwards, disc–disc collisions became less probable, when the particles have spread sufficiently due to the presence of obstacles, and the system will behave as a single particle one. In Fig. 2, one can see a time sequence done by Bruno et al. [8,9] where one can clearly see that after a few obstacle rows, collisions disc–disc can be neglected and only disc–obstacle collisions dominate. For this reason, the results presented and discussed from now on, will refer to the one particle-at-a-time problem.

From a practical point of view, we fill the box at the top with the discs. A blocking horizontal bar closes the container. Suddenly, we pull out the blocking bar and the discs are gravity driven until arriving at the collector. The procedure is repeated in order to have significant statistics (from 1,500 to 3,000 discs are released for each kind of material and for a given W). In Fig. 3 we show the final spatial distributions obtained at the exit of the Galton's Board for the aluminium discs and for $W = 70, 60, 50, 38, 26$ and 16. In all figures we have superimposed the corresponding Gaussian density function calculated from the experimental frequency.

For $W = 70$, we observe that there is no influence of the presence of the walls and the distribution can be well described by a Gaussian characterized by a standard deviation σ and a mean value which corresponds to the mean initial position of the box launcher. The motion of a disc in the flow can be pictured by a driven random walk produced by the fluctuations in the local direction of velocity induced by the collisions with the obstacles of the network [8,9]. For devices that are long enough, particle dispersion displays a diffusive behavior.

In Fig. 3b–d we see that the influence of lateral walls on the distributions becomes important as W decreases: The distributions are truncated and keep the bell-like shape. In Fig. 3c and d two small peaks close to lateral wall (bin 11 and 53, respectively) indicate that some discs found a "preferential channel", that is a channel with a small density of obstacles due to to the arrangement.

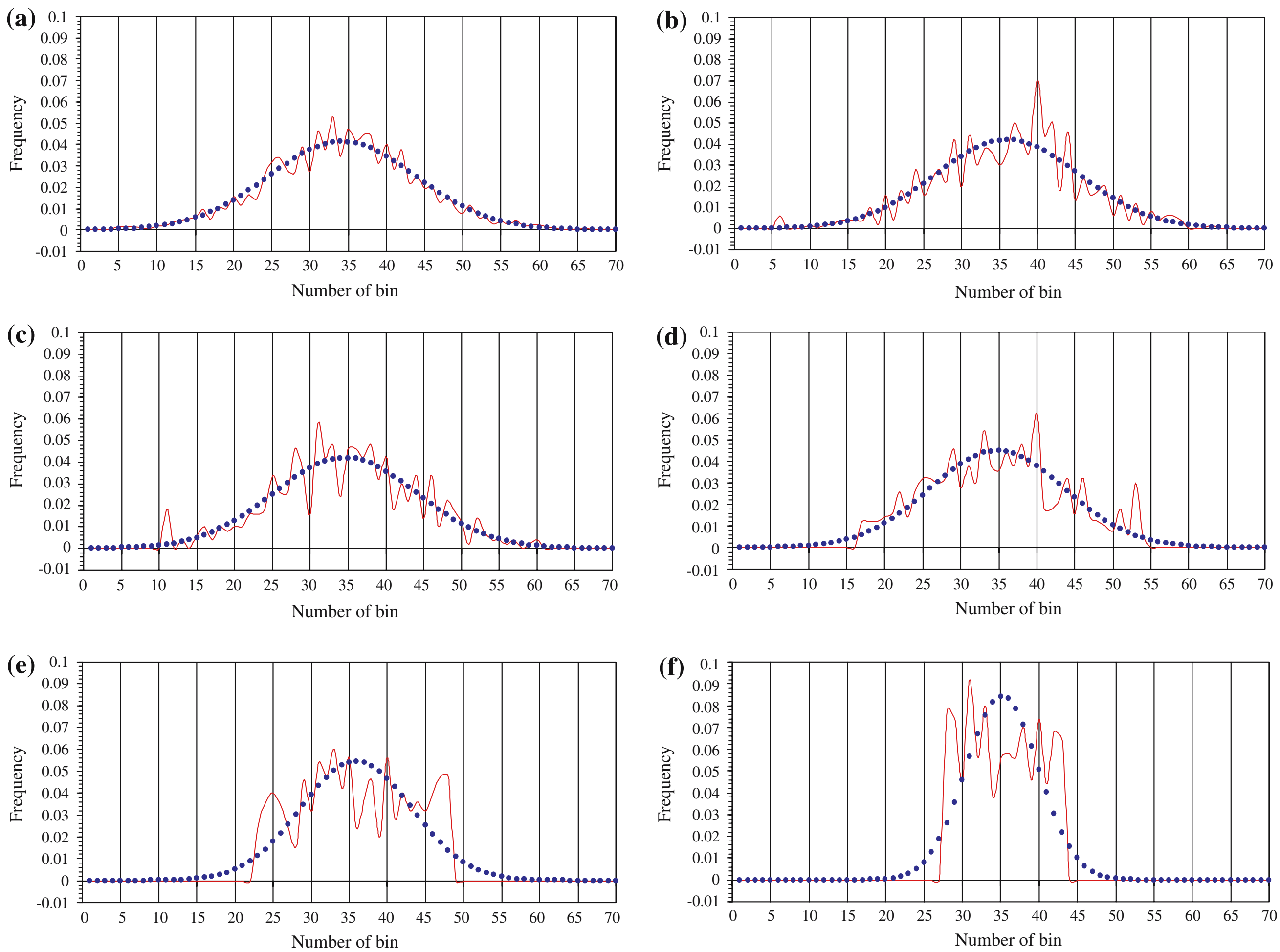

Fig. 3 Experimental histograms obtained for aluminium discs with horizontal vertical walls separated by a distance W of: **a** 70; **b** 60; **c** 50; **d** 38; **e** 26 and **f** 16. We have superimposed the corresponding Gaussian curves computed from the experimental frequencies

Figure 3e and f shows distributions that tend to uniform ones and that cannot be described by Gaussians any more. In fact, particles are now re-injected after collisions with the walls.

The qualitative behaviour displayed in Fig. 3 when W decreases, that is an evolution from a Gaussian behaviour toward a uniform one, was found also for other materials.

We have calculated the standard deviations for each distribution achieved from experiments carried out with all kind of materials and all values of W.

In Fig. 4 we have plotted σ as a function of W, the distance between lateral walls, for rubber, aluminium and lead discs. We can see two different behaviours. First, σ increases up to $W \approx 40$ and then, tends to a constant. Up to $W \approx 40$, the σ values are those corresponding to uniform distributions; for $W > 40$, the influence of the walls becomes less significant and the distributions are quite normal, so σ tends to a constant.

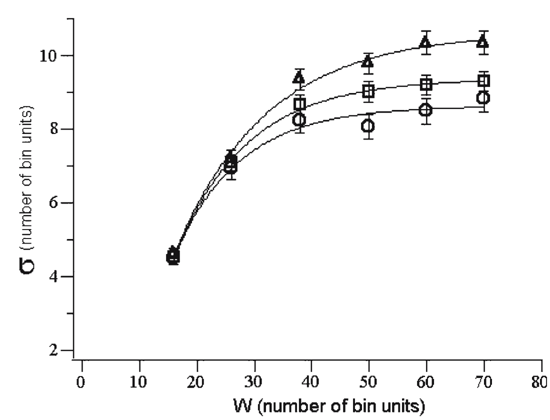

Fig. 4 Variation of σ as a function of W for *circle* lead, *square* aluminium and *triangle* rubber discs. We have superimposed exponential fittings

The same variation is found for all kinds of materials used in the experiments, so that we can say that this effect is due to the geometry of the system.

However, we can observe that σ depends on the material clearly in the region where it tends to a constant. In fact, in this region, the discs attains a diffusive gravity driven regime characterized by a constant velocity of the centre of mass, which depends essentially on the collisions dynamics which directly involves the coefficient of restitution. We want to remark here that the density does not take part in the problem: the energy dissipated during the collisions is gained during the chute at the scale of the distance between the obstacles, and this gives rise to a flow in steady state characterized by a mean velocity.

In order to characterize the elastic properties of the collisions we have performed experiments to determine the coefficient of restitution for binary collisions between a disc and an obstacle of the network. Accordingly we have built a small cell with the same materials as used in the Galton Board: two flat plates (wood and glass) separated by the same distance and dimensions of 20 cm × 20 cm. We have glued between the plates a styrene disc of 4 mm of diameter (obstacle). We launch a 8 mm disc at a distance of 2.7 cm (typical lattice parameter of the Galton Board). With a digital video camera we record the collision. Afterwards by image processing we can measure the velocity of the disc just before and after the collision takes place. In this way, we can compute the coefficient of restitution as the ratio between the velocity after and before the collision. We have carried out around 25 measurements for each kind of materials. We have found $e = 0.71 \pm 0.01$ for lead; 0.83 ± 0.02 for aluminium and (0.89 ± 0.02) for rubber discs impacting on a styrene disc. In Fig. 4 the asymptotic constant values of σ are 8.6, 9.37 and 10.6 for lead, aluminium and rubber discs, respectively, and increase when the coefficient of restitution increases. The spreading of the curves is closely related to the variations in local velocity direction, and this is related to the collision properties: the rubber discs suffer more elastic collisions inducing more fluctuations in their local velocities. All the above explains the different limiting values for σ found for each material for large W, but, the behaviour of σ with W is qualitatively the same, independently of the restitution coefficient of the percolating disks.

In order to better understand the influence of the geometry of the system and the separation between walls on the exit distributions we will present a numerical study in the following section.

3 Simulations and results

The experimental results obtained, encouraged us to explore a larger range of parameters in order to corroborate if the crossover found in the exit distribution functions is still sustained for other sizes of the board and if there exists a critical curve where such crossover can be correlated with the number of columns (width) and the number of rows (height) of the board.

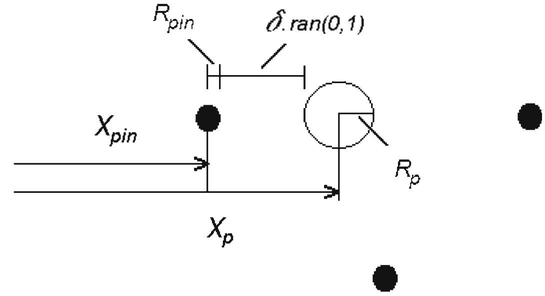

Fig. 5 Schematic representation showing how the horizontal position of the percolating particle (*open circle*) is attained in simulations. The case shown corresponds to $a = +1$ in Eq. 1

Basically, simulations reproduce experiments in what regards to the geometry. We define the sites arranged in a triangular lattice (obstacles) keeping experimental distances. Disks of the same size like those of the experiment are launched one at a time from above, at the middle of the top row of the board.

Trajectories followed by small particles are calculated in this way: a disk falls down by gravity until it encounters a pin in its trajectory. At this moment and depending on the relative positions of the centres, the small disk will roll over the pin to the right (the x-coordinate of its centre is to the right of that of the pin) or to the left (the x-coordinate of its centre is to the left of that of the pin). At this stage, we have to decide which will be the new x-position of the particle, i.e., we have to introduce the effect of bouncing without calculating any force (we are not interested here in the dynamics of the problem). Thus, we just choose a random number uniformly distributed between 0 and 1 (ran(0,1)) and put the particle at the corresponding x-position given by:

$$x_\mathrm{p} = x_\mathrm{pin} + aR_\mathrm{pin} + aR_\mathrm{p} + \delta\ \mathrm{ran}(0,1) \tag{1}$$

Here, x_pin is the x-coordinate of the centre of the pin; R_pin is the radius of the pin (the same for all of them in the lattice), R_p is the particle radius, δ is a fixed distance proportional to the separation between two neighbouring pins minus two times their radius and a is a factor equal to +1 (-1) if the x-coordinate of the centre of the particle is to the right (*left*) of that of the pin. For this set of simulations the proportionality factor will be set to 1, see Fig. 5 for a better understanding. Here, the election of δ is totally arbitrary and this parameter will represent some real bouncing parameter adequately scaled for the problem. The y-coordinate is taken to be equal to the one of the pins in that row.

We performed simulations for different values of δ in order to introduce a pseudo-bouncing effect, i.e., the smaller the value of δ, the smaller the hardness of the percolating particle. This simple approach will be adequate for our purposes given that our main objective here is to study the "re-orientation" of the disks in their way down to the exit due to the presence of lateral walls, and no detailed analysis of the dynamics is needed.

When the new position is attained, a new calculation for the trajectory of the particle is performed, following the procedure explained above, and so on. When the particle leaves the last row of the Galton, it is collected in different bins according to its final x-positions, like in real experiments. We perform series of 10^5 equal experiments in order to build the exit distribution of percolating particles. This exit distribution function is fitted with a suitable curve in order to characterize the final distribution of the particles through its dispersion σ.

First, we inspected the exit distributions for boards with walls separated the same distance as in experiments. Then, we systematically reduced the distance between the walls in order to sample all the possible separation distances given the number of total rows of the board. This means that walls extremely separated would have no sense because we will always recover just the same Gaussian behaviour (with the same probability distribution function). On the other hand, walls too close to each other would always give the same uniform exit distribution function, as we will see below. Figure 6 shows the results obtained for σ as a function of the separation range studied with a board with 45 rows, resembling experiments. There, circles represent simulations results. As we can see, results show a fair qualitative agreement with experimental data reported above in a similar format. A clear crossover is observed near $W \approx 25$. The difference with experimental data is basically due to the arbitrary bouncing coefficient, δ, selected for our runs. If one would select a higher bouncing coefficient, results would better represent the experiments; however, the aim of this work is not to fit experimental data, but to explain the basic nature of experimental findings. We will discuss the role of the bouncing coefficient on the exit distributions below.

Besides the simulation results (symbols) in Fig. 6, we have also plotted a straight line representing the theoretical dispersion, σ_u, for a uniform distribution of events, $P(x)$, for the variable x, in the interval (a,b), i.e.:

$$\sigma_u = \int_a^b (x - x_{av})P(x)dx = \frac{b-a}{\sqrt{12}} \tag{2}$$

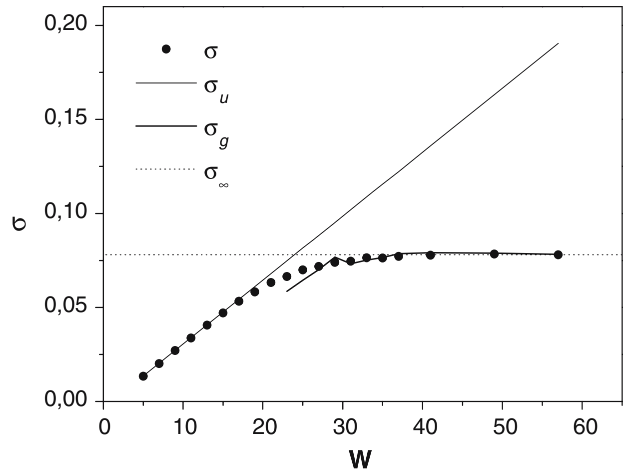

Fig. 6 Results for the dispersion σ as a function of W, for a board with 45 rows. *Filled circle* simulation results; *dot line* normal distribution dispersion centred at x_{av}, for the interval $(-\infty, +\infty)$; *thick line* dispersion obtained from a Gaussian fitting of the exit distribution simulation results and, finally, *straight line* representing the theoretical dispersion, σ_u, for a uniform distribution of events

where x_{av} is the average value of x and is equal to $\frac{b+a}{2}$ and $P(x) = \frac{1}{b-a}$. Here, $(b-a)$ is the separation between walls, thus, when separation grows, σ grows linearly with it. From now on, we will always use W instead of $(b-a)$ and will always measure it in number of columns of the Galton board. On the same figure, we have drawn a line (dotted horizontal line) representing the dispersion of a normal distribution centred at x_{av}, for the interval $(-\infty, +\infty)$. Finally, we have also drawn a curve (thick line) that represents the dispersion obtained from a Gaussian fitting of the simulation results for the exit distribution. As seen, this fitting gets worst as W decreases.

As one can see, the presence of walls does affect the exit distributions. For W larger than 33, simulation data fall over the horizontal line corresponding to a Gaussian behaviour. For smaller W, a Gaussian fit fails to represent the exit distribution (look at the dashed line in the figure). On the other hand, when W is even smaller, simulation points fall over the limiting straight line representing a uniform behaviour for the exit distribution function ($W \approx 19$). There is a crossover zone where no fitting have proven to be suitable. To characterize in some sense this zone, we counted the number of particles that touch the walls at least once and plot their exit distribution for three different values of W in Fig. 7: above the crossover, at the crossover and below it. We chose a separation W_c to represent a typical case belonging to the crossover zone. The aim is to visualize the "tails" of particles that are redirected into the board because of the presence of walls. These particles are responsible of

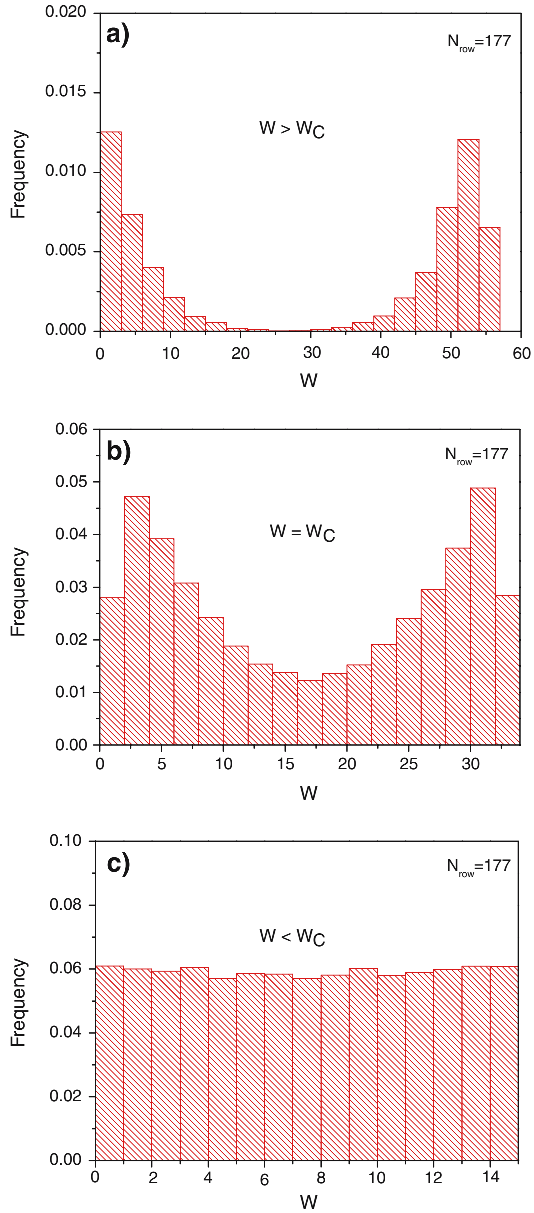

Fig. 7 Exit distribution functions for three different W. Here, W_c is a typical separation belonging to the crossover zone. **a** For W greater than W_c, particles distribute like exponential tails. **b** For W close to W_c, the exit distribution is not easy to characterize. **c** For W smaller than W_c, particles at the exit present a quite uniform distribution

the observed change in the behaviour. Characterization of such "tails" is not straightforward. Although for W greater than W_c particles seem to distribute like exponential tails, the shape of the exit distribution for W close to W_c is not easy to fit. Even more, for W smaller than W_c, particles at the exit present a quite uniform distribution, going back to the same initial question. We will overcome this problem after the theoretical approach is presented below.

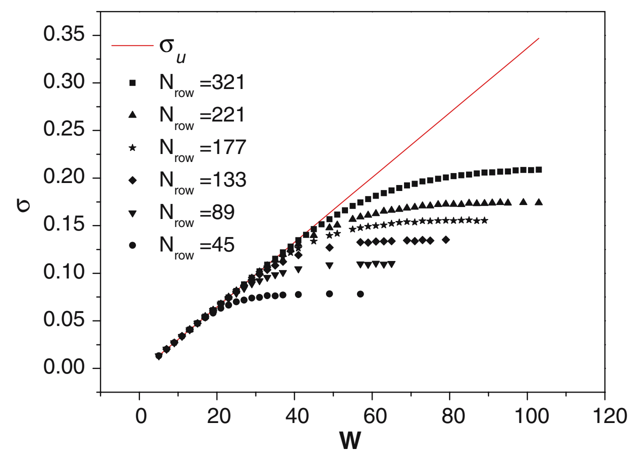

Fig. 8 Simulation results for σ as a function of W for six different numbers of rows, as indicated. Results already pictured on Fig. 3 are also plotted. The *line* represents the behaviour of dispersion for a uniform exit distribution as expressed in Eq. (2)

At this stage, we wondered about the behaviour of the percolating particles as the number of rows of the Galton was increased. As explained above, this cannot be easily accomplished in an experiment. Consequently, we performed simulations for different number of rows, scanning all the separation values needed to build up Fig. 8. In this figure, we report the results for σ as a function of W for six different numbers of rows, including the results already pictured in Fig. 6. Besides the simulation results, we plot again in this figure the line representing the uniform behaviour for the exit distribution. As clearly seen and expected, all results present a similar behaviour. For large W, Gaussian exit distributions are obtained; as W is decreased, a crossover is observed until all curves fall on the limiting line corresponding to a linear behaviour. The extent of the crossover depends on the number of rows of the board and the limiting values for large W (Gaussian behaviour) depend on the number of rows because of diffusion effects, as it is well known [17]. The inception point of each curve on the straight line also depends on the height of the board.

Going back to our main concern, what is the relationship between the crossover found and the size of the Galton board? To answer this question we first have to define, following some criterion, the critical value of W (W_c) at which the exit distribution might start to be considered as uniform. By looking at the curves of Fig. 8, we choose W_c such that σ falls for the first time on the limiting line of uniform behaviour. In Fig. 9 we plot the number of rows of the board, N_{row}, as a function of W_c values obtained from this criterion. The fitting curve is a second order polynomial function. As clearly seen, the correlation between W_c and N_{row}, is a parabola. Thus, if one is searching for a uniform exit distribution

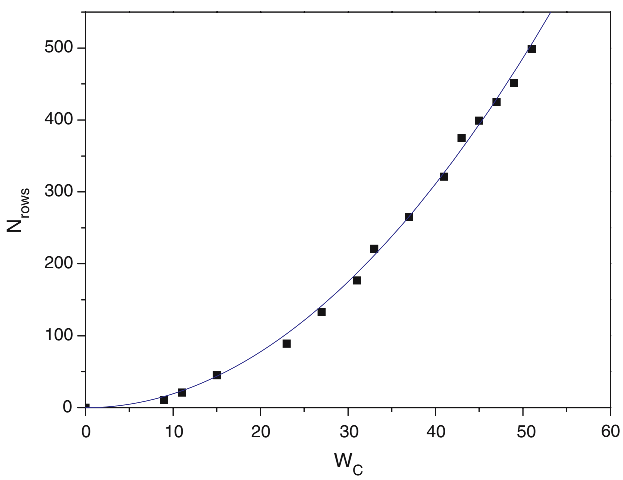

Fig. 9 Number of rows of the board, N_{row}, as a function of W_{c} values obtained from a ad hoc criterion. The *curve line* is a fitting with a second order polynomial function

of particles flowing down through a Galton board of N_{row}, one needs to have, at least, a number of columns proportional to $\sqrt{N_{\text{row}}}$, where the constant of proportionality could be determined from the fitting function in Fig. 9. Even so, this way is pretty much dependent on the initial criterion for choosing W_{c}. Consequently, an independent method to measure the departure from uniform exit distribution behaviour would be of great help. For that reason we looked for a theoretical approximated model of the present system. In the following section we will briefly develop and explain that model.

4 Theoretical approach

We consider the one dimensional diffusion of a particle in the finite range $-L/2 \leq x \leq L/2$ [19] to approximate the solution of our discrete problem.

Let us denote by $P(x,t)$ the conditional probability density of finding the particle between x and $x+\mathrm{d}x$ given that it started at the centre of the interval $[-L/2, L/2]$. Solving the diffusion equation with reflecting boundary conditions and the initial condition $P(x,t=0)=\delta(x)$, we obtain the solution

$$P(x,t) = \frac{1}{L} + \frac{2}{L}\sum_{j=1}^{\infty} \exp\left(-\frac{tD}{L^2}4j^2\pi^2\right)\cos\left(\frac{2j\pi x}{L}\right) \tag{3}$$

where D is the diffusion coefficient. It can be seen from Eq. 3 that for a long enough time the probability density is nearly a constant (an uniform distribution): $P(x, t\rightarrow\infty) = 1/L$.

The probability density $P(x,t)$ of Eq. 3 gives an approximation to the probability for the position in a discrete time random walk in a finite one dimensional lattice [20] with reflecting boundary conditions.

Let us consider a RW on a finite one dimensional lattice of spacing a. Assuming $W+1$ lattice sites the size of the space domain is $L = aW$. Choosing the coordinate origin at the centre of the lattice the walker position is $x = a.s$, with s an integer number with values $-W/2 < s < W/2$. Assuming reflecting boundary conditions, the probability of finding the walker at lattice site s after N steps is approximated by Eq. 3, expressed as

$$P(s,N) = \frac{1}{W} + \frac{2}{W}\sum_{j=1}^{\infty} \exp\left(-\frac{N}{W^2}2j^2\pi^2\right)\cos\left(\frac{2j\pi s}{W}\right) \tag{4}$$

with the usual identification

$$D = \frac{a^2}{2\tau} \tag{5}$$

where τ is the waiting time between steps ($t = N\tau$).

From Eq. 4 we could establish a criterion for choosing W_{c} by imposing a finite limit to the terms entering the sum above, in such a way that the dominant part of it be the first "uniform-distribution" term. Thus, given that $|\cos(x) \leqslant 1|$, we may ask for the first correction ($j = 1$) to be less than ε, obtaining:

$$\exp\left(-\frac{N}{W^2}2\pi^2\right) < \varepsilon \tag{6}$$

or, equivalently:

$$\frac{N}{W^2} > -\frac{\ln(\varepsilon)}{2\pi^2} \tag{7}$$

The remaining terms will be smaller than that given in Eq. 6.

If one wants a uniform exit distribution to within some specified relative error, Eq. 7 provides a criterion, independent of the size of the system, where the obtained W will represent W_{c} and N will represent the number of rows of the Galton board. In summary, one can introduce the desired value for ε and obtain the necessary size ratio of the Galton board. On the other hand, if one has a fixed dispersion system, this equation can provide an idea of the departure from uniform behaviour.

In Fig. 10 we plot the phase diagram corresponding to the points in Fig. 9 that could be considered as representing σ values for a uniform exit distribution. We also show a set of parabolas corresponding to different values of ε, as indicated in the caption. As it can be seen, the number of points belonging to the uniform exit distribution phase increases as the requirements on ε are less strong. This criterion seems to be more straightforward and clear that any possible analysis that could be performed on the tails of the distributions of Fig. 7. From

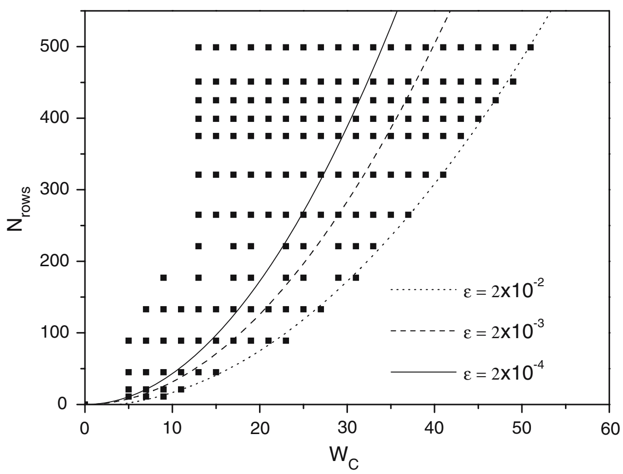

Fig. 10 Points corresponding to Fig. 7 that could be considered as representing σ values for a uniform exit distribution, depending on the desired departure. The three parabolas correspond to plots using Eq. 7 for $\varepsilon = 2 \times 10^{-2}$, $\varepsilon = 2 \times 10^{-3}$ and $\varepsilon = 2 \times 10^{-4}$, as indicated. The lowest value of ε coincides with the preceding plot

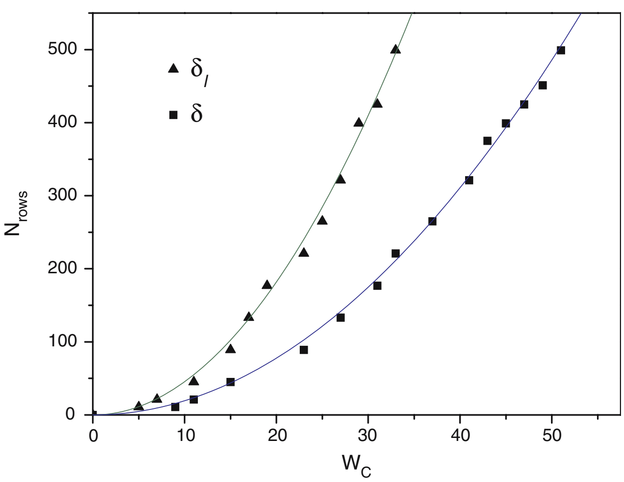

Fig. 11 Number of rows of the board, N_{row}, as a function of W_{c} values obtained from the same ad hoc criterion used in Fig. 7 and showing the results for two different bouncing parameters: δ (corresponding to Fig. 7) and δ_l. The *curve lines* are fittings with a second order polynomial function

this theoretical approach, the observed crossover is well characterized and described.

Finally, we have performed a similar set of simulations but with a different "bouncing" parameter. As explained above, our model introduces artificially a bouncing effect through Eq. 1 and parameter δ. In this way, high δ values imply large bouncing properties in collisions. Thus, for our second set of simulations, we chose a lower value for δ that will be denoted as δ_l.

Figure 11 shows the obtained results along with the previous ones obtained for greater δ. As seen, the new results fit a more closed parabola, i.e., the coefficient of the second order term is greater for smaller bouncing and, as a result, a board with a fixed number of rows will need closer walls to distribute uniformly particles with lower bouncing coefficient. On the other hand, as this coefficient is higher, the percolating particles explore even farther regions of the board, noting the presence of walls with greater likelihood. Thus, the crossover region from Gaussian-like to uniform-like behaviour shows up earlier, giving higher values for W_{c}.

5 Conclusions

We have performed experiments and simulations of small particles falling down through Galton boards with different number of columns and rows. We inspected the behaviour of the exit density distributions of particles as a function of the separation between walls.

The results shown in this work prove the existence of a crossover from a Gaussian-like to a uniform-like behaviour in the exit distributions as walls are closer to each other.

We characterize the crossover by defining some ad hoc criterion and found that the relationship between the number of rows and the number of columns of the board was non linear, presenting a parabolic dependence.

The behaviour found both in experiments and simulations was then modelled by a simple discrete theoretical approximation to the problem of diffusion through a one dimension medium. This model helped to set up a criterion to discern among uniform and Gaussian behaviour of exit distributions.

In this way, given the dimensions of a Galton board that will be used as a scatter device, one can find the departure from uniform exit behaviour for the dispersed particles. On the other hand, if one needs to build up a board to ensure a specified exit distribution within a desired error, the necessary number of columns and rows can be obtained from the above analysis.

Finally, simulations for a different bouncing parameter were performed and results showed that the crossover to uniform exit distribution behaviour shows up at greater W_{c} as bouncing is increased because particles explore a wider board space with greater likelihood. The limiting σ values for walls well separated, were also higher for the same reason. Given that the coefficient of the second order term (inequality 8) is $-\frac{\ln(\varepsilon)}{2\pi^2}$, a relationship between $\ln(\varepsilon)$ and bouncing parameters could be found by performing a set of simulations like those made above for different δ. In this way, the function ε vs. δ could be established and from this function one could calculate the departure from uniform behaviour of the exit distributions. To find this function could be the aim of future work.

At this stage, the influence of walls in a Galton scatter has been well characterized. Future efforts will focus on the problem of uniform mixing of two species of particles using these devices with variable wall separation.

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