Solving Linear Systems

§3.3 LU-factorisation

MA385/530 – Numerical Analysis 1

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<<< Annotated slides >>>

In these slides,

- *LT means "lower triangular"*
- *UT means "upper triangular"*

The goal of this section is to demonstrate that the process of Gaussian Elimination applied to a matrix A is equivalent to factoring A as the product of a unit lower triangular and upper triangular matrix.

The Section 3.2 we saw that each elementary row operation in Gaussian Elimination involves replacing A with $(I + \mu_{rs}E^{(rs)})A$.

Example: For the 3×3 case, this involved computing

$$(I + \mu_{32}E^{(32)})(I + \mu_{31}E^{(31)})(I + \mu_{21}E^{(21)})A.$$

In general we multiply A by a sequence of matrices

$$(I + \mu_{rs} E^{(rs)}),$$

all of which are unit lower triangular matrices.

When we are finished we have reduced A to an upper triangular matrix.

So we can write the whole process as

$$\underbrace{L_k L_{k-1} L_{k-2} \dots L_2 L_1}_{\tilde{L}} A = U, \tag{3}$$

where each of the L_i is a unit LT matrix.

$$\tilde{L} = L_K L_{K-1} \cdots L_1$$

But from Theorem 3.2.6, we know that the product of unit LT matrices is itself a unit LT matrix. So we can write the whole process described in (3) as

$$\tilde{L}A = U. \quad \text{Set } L = (\tilde{L})^{-1} \tag{4}$$

But Theorem 3.2.6 also tells us that the inverse of a unit LT matrix exists and is a unit LT matrix. So we can write (4) as

$$A = LU$$

where L is unit lower triangular and U is upper triangular.

This is called "***LU*-factorisation**".

ie $L = (L_k \cdot L_{k-1} \cdots L_1)^{-1}$

Definition 3.9

The LU-**factorization** of the matrix is a unit lower triangular matrix L and an upper triangular matrix U such that $LU = A$.

Example 3.10

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$ then:

$LU = A$, ie,
$$\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$$

mulipying L and U, we get
$$\begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}.$$

So $u_{11} = 3$ $\quad u_{12} = 2$

$l_{21} = -1/3$, $\quad u_{22} = 8/3$

Example 3.11

If $A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & -4 \end{pmatrix}$ then: $LU = A$, ie

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & & \\ 0 & 0. & \end{pmatrix}$$

[get row 1 of U first.

Then, . . .

col 1 of L next.

Exer: finish this, alternating rows of U & cols of L.

We now want to work out formulae for L and U where

$$a_{i,j} = (LU)_{ij} = \sum_{k=1}^{n} l_{ik}u_{kj} \qquad 1 \leq i, j \leq n.$$

Since L and U are triangular,

$$\text{If } i \leq j \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{i} l_{ik}u_{kj} \tag{5a}$$

$$\text{If } j < i \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{j} l_{ik}u_{kj} \tag{5b}$$

The first of these equations can be written as

$$a_{i,j} = \sum_{k=1}^{i-1} l_{ik} u_{kj} + \overset{=1}{l_{ii}} u_{ij}.$$

But $l_{ii} = 1$ so:

$$u_{i,j} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \quad \begin{cases} i = 1, \ldots, j-1, \\ j = 2, \ldots, n. \end{cases} \tag{6a}$$

And from the second:

$$l_{i,j} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right) \quad \begin{cases} i = 2, \ldots, n, \\ j = 1, \ldots, i-1. \end{cases} \tag{6b}$$

Example 3.12

Find the LU-factorisation of

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -2 & -2 & 1 & 4 \\ -3 & -4 & -2 & 4 \\ -4 & -6 & -5 & 0 \end{pmatrix}$$

Try the formula just defined, first for

u_{11}, u_{12}, u_{13} u_{14}

Then l_{21}, l_{31}, l_{41}. Then

u_{22}, u_{23}, u_{24},

etc.

Should get...

Full details of Example 3.12: First, using (6a) with $i = 1$ we have $u_{1j} = a_{1j}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

Then (6b) with $j = 1$ we have $l_{i1} = a_{i1}/u_{11}$:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & l_{32} & 1 & 0 \\ 4 & l_{42} & l_{43} & 1 \end{pmatrix}.$$

Next (6a) with $i = 2$ we have $u_{2j} = a_{2j} - l_{21}u_{2j}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix},$$

then (6b) with $j = 2$ we have $l_{i2} = (a_{i2} - l_{i1}u_{12})/u_{22}$:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & l_{43} & 1 \end{pmatrix}$$

Etc....

Not every matrix has an LU-factorisation. So we need to characterise the matrices that do.

To prove the next theorem we need the Cauchy-Binet Formula: $\det(AB) = \det(A)\det(B)$.[1]

Theorem 3.13

If $n \geq 2$ and $A \in \mathbb{R}^{n \times n}$ is such that every leading principal submatrix of A is nonsingular for $1 \leq k < n$, then A has an LU-factorisation.

Eg if $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$, LU-factorisation would fail since $A^{(1)} = (0)$ & $\det(0) = 0$.

[1] Wikipedia disagrees with this attribution

Proof: Let $n=2$. So $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since

$\det(A^{(1)}) \neq 0$ we know $a \neq 0$.

Then $L = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}$

is an LU-factorisation of A.

Next assume the theorem is true for any A

of order $< n$, ie, up to $(n-1) \times (n-1)$.

For an $n \times n$ matrix A, partition by the

final row and column.

$$LU = A$$

$$\left(\begin{array}{c|c} L^{(n-1)} & \vec{0} \\ \hline \vec{\omega}^T & 1 \end{array}\right)\left(\begin{array}{c|c} U^{(n-1)} & \vec{x} \\ \hline \vec{0}^T & z \end{array}\right) = \left(\begin{array}{c|c} A^{(n-1)} & \vec{b} \\ \hline \vec{c}^T & d \end{array}\right)$$

where $L^{(n-1)}$, $U^{(n-1)}$ and $A^{(n-1)}$ are $(n-1)\times(n-1)$ matrices, and $\vec{0} = (0,0,0\ldots,0)^T$, $\vec{x}$, $\vec{b}$, $\vec{\omega}$ and $\vec{c}$ are $(n-1)$ vectors and z, d are scalars.

This gives

$$L^{(n-1)} U^{(n-1)} = A^{(n-1)}.$$

But the inductive hypothesis, $L^{(n-1)}$ $U^{(n-1)}$ exist.

Next $L^{(n-1)} \vec{x} = \vec{b}$. So $\vec{x} = \left(L^{(n-1)}\right)^{-1} \vec{b}$ exists, since $\det\left(L^{(n-1)}\right) = 1 \neq 0$.

Then use $\vec{w}^T U^{(n-1)} = \vec{c}^T$. So $\vec{w}^T = \vec{c}^T \left(U^{(n-1)}\right)^{-1}$ exists because $0 \neq \det\left(A^{(n-1)}\right) = \det\left(L^{(n-1)}\right) \det\left(U^{(n-1)}\right) = \det\left(U^{(n-1)}\right)$

Finally, we know

$$\vec{w}^T \vec{x} + z = d.$$

So $z = d - \vec{w}^T \vec{x}$ exists.

And we are done

Finished here 14/11/19.

Exercise 3.9

Many textbooks and computing systems compute the factorisation $A = LDU$ where L and U are unit lower and *unit* upper triangular matrices respectively, and D is a diagonal matrix. Show such a factorisation exists, providing that if $n \geq 2$ and $A \in \mathbb{R}^{n \times n}$, then every leading principal submatrix of A is nonsingular for $1 \leq k < n$.