MA385 Part 3: Linear Algebra 1

3.3 LU-factorisation

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November 2025

In these slides,

- *LT means "lower triangular"*
- *UT means "upper triangular"*

1. Outline of Section 3.3

1. A formula for LU-factorisation
2. Existence of an LU-factorisation
3. Exercises

For more, see Section 2.3 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG

1. Outline of Section 3.3

The goal of this section is to demonstrate that the process of Gaussian Elimination applied to a matrix A is equivalent to factoring A as the product of a unit lower triangular and upper triangular matrix.

The Section 3.2 we saw that each elementary row operation in Gaussian Elimination involves replacing A with $(I + \mu_{rs}E^{(rs)})A$.

Example: For the 3×3 case, this involved computing

$$(I + \mu_{32}E^{(32)})(I + \mu_{31}E^{(31)})(I + \mu_{21}E^{(21)})A.$$

1. Outline of Section 3.3

In general we multiply A by a sequence of matrices

$$(I + \mu_{rs} E^{(rs)}),$$

all of which are **unit lower triangular** (=unit LT) matrices.
When we are finished we have reduced A to an **upper triangular** (UT) matrix.
So we can write the whole process as

$$L_k L_{k-1} L_{k-2} \dots L_2 L_1 A = U, \tag{1}$$

where each of the L_i is a unit LT matrix.

$$\underbrace{L_k L_{k-1} \cdots L_2 L_1}_{L}$$

However, we know from Section 3.2 that the product of **unit LT** matrices is itself a unit LT matrix. So we can write the whole process described in (1) as

$$\tilde{L}A = U. \tag{2}$$

Also from Section 3.2, the inverse of a **unit LT** matrix exists and is a **unit LT** matrix. So we can write (2) as

$$A = LU$$

where L is unit lower triangular and U is upper triangular. This is called "LU-**factorisation**".

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Mon 3rd Nov (W8.1)

Definition 3.4.1

The LU-**factorization** of the matrix is a unit lower triangular matrix L and an upper triangular matrix U such that $LU = A$.

Example 3.4.1

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$ then:

$$LU = A \quad \Rightarrow \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix} \Rightarrow \begin{matrix} u_{11} = 3 \\ u_{12} = 2 \end{matrix}$$

and $\quad l_{21}(3) = -1 \quad \Rightarrow \quad l_{21} = -1/3 \quad \ldots \quad u_{22} = -1 - (-\frac{1}{3})(2) = 8/3$

Example 3.4.2

If $A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & -4 \end{pmatrix}$ then:

$LU = A$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} & " & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 0 & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} & A & \end{pmatrix}$$

1. Outline of Section 3.3

You should find

$$\underbrace{\begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & -4 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 0 & 3/7 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & -1 & 1 \\ 0 & 14/3 & 7/3 \\ 0 & 0 & -5 \end{pmatrix}}_{U}.$$

2. A formula for LU-factorisation

We now want to work out formulae for L and U where

$$a_{i,j} = (LU)_{ij} = \sum_{k=1}^{n} l_{ik} u_{kj} \qquad 1 \leq i, j \leq n.$$

Since L and U are triangular.

$$\text{If } i \leq j \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{i} l_{ik} u_{kj} \tag{3a}$$

$$\text{If } j < i \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{j} l_{ik} u_{kj} \tag{3b}$$

2. A formula for LU-factorisation

The first of these equations can be written as

$$a_{i,j} = \sum_{k=1}^{i-1} l_{ik} u_{kj} + \overset{=1}{l_{ii}} u_{ij}.$$

But $l_{ii} = 1$ so:

$$\boxed{u_{i,j} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \quad \begin{cases} i = 1, \dots, j-1, \\ j = 2, \dots, n. \end{cases}} \tag{4a}$$

And from the second:

$$\boxed{l_{i,j} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right) \quad \begin{cases} i = 2, \dots, n, \\ j = 1, \dots, i-1. \end{cases}} \tag{4b}$$

Example 3.4.3

Find the LU-factorisation of

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -2 & -2 & 1 & 4 \\ -3 & -4 & -2 & 4 \\ -4 & -6 & -5 & 0 \end{pmatrix}$$

$\sum_{k=1}^{0} x$ is empty!

Note from the formulae on the previous page:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

So $u_{11} = a_{11} - \sum_{k=1}^{0} l_{ik} u_{kj} = a_{11}$. And $u_{1j} = a_{1j}$

Full details of the example: First, using (4a) with $i = 1$ we have $u_{1j} = a_{1j}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

Then (4b) with $j = 1$ we have $l_{i1} = a_{i1}/u_{11}$:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & l_{32} & 1 & 0 \\ 4 & l_{42} & l_{43} & 1 \end{pmatrix}.$$

Next (4a) with $i = 2$ we have $u_{2j} = a_{2j} - l_{21}u_{2j}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix},$$

2. A formula for LU-factorisation

then (4b) with $j = 2$ we have $l_{i2} = (a_{i2} - l_{i1}u_{12})/u_{22}$:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & l_{43} & 1 \end{pmatrix}$$

Etc....

Not every matrix has an LU-factorisation. So we need to characterise the matrices that do.
To prove the next theorem we need the Cauchy-Binet Formula: $\det(AB) = \det(A)\det(B)$.

Theorem 3.4.1

If $n \geq 2$ and $A \in \mathbb{R}^{n\times n}$ is such that every leading principal submatrix of A is nonsingular for $1 \leq k < n$, then A has an LU-factorisation.

- Recall $A^{(k)}$ is the leading prin. submatrix of order k of A.
  ie the $k \times k$ submatrix in the top left corner of A.
- $A^{(n)} = A$ is nonsingular, as is $A^{(1)} = (a_{11})$

So that means $a_{11} \neq 0$

Proof: Proof is by induction. Let $n=2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note $a \neq 0$

Then $L = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}$ $U = \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}$

is an LU factorization of A.

Now assume the theorm holds for all matrices of order (size) up to $n-1$.

Let A be an $n \times n$ matrix. Partition by the final row & column:

$$\underbrace{\left(\begin{array}{c|c} A^{(n-1)} & \vec{b} \\ \hline \vec{c}^{\,T} & d \end{array}\right)}_{A} = \underbrace{\left(\begin{array}{c|c} L^{(n-1)} & \vec{0} \\ \hline \vec{\omega}^{\,T} & 1 \end{array}\right)}_{L} \underbrace{\left(\begin{array}{c|c} U^{(n-1)} & \vec{x} \\ \hline \vec{0} & z \end{array}\right)}_{U}$$

This gives $L^{(n-1)} U^{(n-1)} = A^{(n-1)}$ so, by the Inductive Hypothesis exist.

Next $L^{(n-1)} \vec{x} + \vec{0} z = \vec{b}$. So $\vec{x} = \left(L^{(n-1)}\right)^{-1} \vec{b}$ exists since $\det(L^{(n-1)}) = 1$.

- Next

$$\vec{w}^{T} U^{(n-1)} + (1)\,\vec{0}^{T} = \vec{c}^{T}$$

So $\vec{w}^{T} = \vec{c}^{T} \left(U^{(n-1)} \right)^{-1}$

But we know $\left(U^{(n-1)} \right)^{-1}$ exists since, From Cauchy - Binet

$$\det\left(U^{(n-1)} \right) = \det\left(U^{(n-1)} \overset{=1}{L^{(n-1)}} \right)$$

$$= \det\left(A^{(n-1)} \right) \neq 0.$$

∎

Exercise 3.4.1

Many textbooks and computing systems compute the factorisation $A = LDU$ where L and U are unit lower and *unit* upper triangular matrices respectively, and D is a diagonal matrix. Show such a factorisation exists, providing that if $n \geq 2$ and $A \in \mathbb{R}^{n \times n}$, then every leading principal submatrix of A is nonsingular for $1 \leq k < n$.