MA385 Part 2: Initial Value Problems

2.1: Introduction

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Emile Picard: his fundamental work on differential equations was only one of his many contributions to mathematics

Olga Ladyzhenskaya: her extensive achievements include providing the first proof of the convergence of finite difference methods for the Navier-Stokes equations

0. Tutorials and Labs

1. Tutorials started next week... tutorial sheet is available at https://www.niallmadden.ie/2526-MA385/MA385-Tutorial-1.pdf.
2. Next week (Week 5) we'll have a lab.
   - That will be based on Python/Jupyter;
   - You will have to submit your work (worth 3.333%) by Monday 13 Oct.
   - Collaboration is encouraged.
   - Lab will take place
     - Mondays at 10 in AC-201
     - Thursday at 2 in ENG-3036.
     - Go to either/both/neither, as you prefer.

Week 4 (29/10) : Tutorial 1 (Monday and Thursday)

Week 5 (06/10) : **Lab 1** (Monday and Thursday)

Week 6 (13/10) : Tutorial 2 (Monday and Thursday)

Week 7 (20/10) : Tutorial 3 (Monday and Thursday).
Assignment due (OK?).

Week 8 (27/10) : No tutorials/labs. **Class test** Thursday at 3pm (OK?).

Week 9 (03/11) : **Lab 2** (Monday and Thursday)

Week 10 (10/11) : Tutorial 4 (Monday and Thursday).

Week 11 (17/11) : **Lab 3** (Monday and Thursday).

Week 12 (24/11) : Tutorial (Monday and Thursday).

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Discuss...

0. Outline of Section 1

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For more details, see Chapter 6 of Süli and Mayers, *An Introduction to Numerical Analysis*, and Chapter 12 of Epperson: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9781118730966

Motivation (See Chap 6 of Epperson)

The growth of some tumours can be modelled as

$$\frac{dR}{dt} = R'(t) = -\frac{1}{3}S_i R(t) + \frac{2\lambda\sigma}{\mu R + \sqrt{\mu^2 R^2 + 4\sigma}},$$

subject to the initial condition $R(t_0) = a$, where R is the radius of the tumour at time t.

Clearly, it would be useful to know the value of R as certain times in the future. Though it's essentially impossible to solve for R exactly, we can accurately estimate it. In this section, we'll study techniques for this.

Initial Value Problems (IVPs)

Initial Value Problems (IVPs) are differential equations of the form: *Find $y(t)$ such that*

differential equation.

$$\boxed{\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = f(t, y)} \text{ for } t > t_0, \qquad \text{and } y(t_0) = y_0. \qquad (1)$$

Here $y' = f(t, y)$ is the *differential equation* and $y(t_0) = y_0$ is the *initial value*.

Some IVPs are easy to solve. For example:

$$y' = t^2 \quad \text{with } y(1) = 1.$$

t_0: initial time.

y_0: initial value, ie the (known) solution at time t_0

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Since $\frac{dy}{dt} = t^2$, we integrate to get

$y(t) = \frac{1}{3}t^3 + C$. Since $y(1) = 1$, $\frac{1}{3}(1)^3 + C = 1 \Rightarrow C = \frac{2}{3}$.

3. Lipschtiz

Most problems are much harder, and some don't have solutions at all. In many cases, it is possible to determine that a giving problem does indeed have a solution, even if we can't write it down. The idea is that the function f should be "Lipschitz", a notion closely related to that of a **contraction**.

Definition 2.1.1 (Lipschitz Condition)

A function f satisfies a **Lipschitz Condition** (with respect to its second argument) in the rectangular region D if there is a positive real number L such that

$$|f(t,u) - f(t,v)| \leq L|u - v| \tag{2}$$

for all $(t,u) \in D$ and $(t,v) \in D$.

"f is Lipschitz".

Example 2.1.1

For each of the following functions f, show that is satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L.

(i) $f(t, y) = y/(1+t)^2$ for $0 \leq t < \infty$.

(ii) $f(t, y) = 4y - e^{-t}$ for all t.

(iii) $f(t, y) = -(1+t^2)y + \sin(t)$ for $1 \leq t \leq 2$.

Eg 1: $f(t,y) = \dfrac{y}{(1+t)^2}$ $\quad 0 \leq t < \infty$.

$$|f(t,u) - f(t,v)| = \left| \frac{u}{(1+t)^2} - \frac{v}{(1+t)^2} \right| =$$

$$\frac{|u-v|}{(1+t)^2} \leq |u-v| \quad \text{since } \frac{1}{(1+t)^2} \leq 1 \quad \forall\, t \geq 0$$

Therefore f is Lipschitz, with $L = 1$.

Example 2.1.1

For each of the following functions f, show that is satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L.

(i) $f(t,y) = y/(1+t)^2$ for $0 \leq t \leq \infty$.

(ii) $f(t,y) = 4y - e^{-t}$ for all t.

(iii) $f(t,y) = -(1+t^2)y + \sin(t)$ for $1 \leq t \leq 2$.

(ii) $|f(t,u) - f(v)| = |4u - e^{-t} - (4v - e^{-t})|$

$= |4u - 4v| = 4|u-v|.$

So f is Lipschitz with $L = 4$.

Example 2.1.1

For each of the following functions f, show that is satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L.

(i) $f(t, y) = y/(1+t)^2$ for $0 \leq t \leq \infty$.

(ii) $f(t, y) = 4y - e^{-t}$ for all t.

(iii) $f(t, y) = -(1+t^2)y + \sin(t)$ for $1 \leq t \leq 2$.

(iii) $|f(t,u) - f(v)| =$

$$\left| -(1+t^2)u + \sin(t) + (1+t^2)v - \sin(t) \right|$$

$$= \left| -(1+t^2)(u-v) \right| \leq (1+t^2)|u-v|$$

$$\leq 5|u-v| \quad \text{since} \quad 1 \leq t \leq 2.$$

Theorem 2.1.1 (Picard's)

Suppose that the real-valued function $f(t, y)$ is continuous for $t \in [t_0, t_M]$ and $y \in [y_0 - C, y_0 + C]$; that $|f(t, y_0)| \leq K$ for $t_0 \leq t \leq t_M$; and that f satisfies the *Lipschitz condition* (2). If

$$C \geq \frac{K}{L}\left(\mathrm{e}^{L(t_M - t_0)} - 1 \right),$$

then (1) has a unique solution on $[t_0, t_M]$. Furthermore

$$|y(t) - y(t_0)| \leq C \qquad t_0 \leq t \leq t_M.$$

You are not required to know this theorem for this course. However, it's important to be able to determine when a given f satisfies a Lipschitz condition.

Exercise 2.1.1

For the following functions show that they satisfy a Lipschitz condition on the corresponding domain, and give an upper-bound for L:

(i) $f(t,y) = 2yt^{-4}$ for $t \in [1, \infty)$,

(ii) $f(t,y) = 1 + t\sin(ty)$ for $0 \leq t \leq 2$.

Exercise 2.1.2

Many text books, instead of giving the version of the Lipschitz condition we use, give the following: *There is a finite, positive, real number L such that*

$$|\frac{\partial}{\partial y} f(t,y)| \leq L \qquad \text{for all } (t,y) \in D.$$

Is this statement *stronger than* (i.e., more restrictive then), *equivalent to* or *weaker than* (i.e., less restrictive than) the usual Lipschitz condition? Justify your answer.

Hint: the Wikipedia article on Lipschitz continuity is very informative.