MA211

Lecture 14: Nonhomogeneous DEs (continued)

Wed 22$^{\mathrm{nd}}$ October 2008

Today...

1. Non-homogeneous Problems
2. $f(x) = Ke^{Tx}$
3. f is a trigonometric function
4. f is the sum of two functions

For further details and examples, look at the section on *Nonhomogeneous Linear Equations*, Section 17.2 of Stewart *Calculus: early transcendentals*.

Non-homogeneous Problems

On Monday we began the section of the course that deals with solving problems of the form

Non-Homogeneous

$$ay'' + by' + cy = \mathbf{f(x)}.$$

where

1. f is a polynomial. (done in Lecture 13)
2. $f = Ke^{Tx}$ for some numbers K and T (started in Lecture 13).
3. f is a trig function, such as sin and cos
4. Some combination of the above.

The technique we introduced is sometimes called the *method of undetermined coefficients*.

$f(x) = Ke^{Tx}$

If the right-hand side of the DE is an exponential function:

$f = Ke^{Tx}$

When solving the Non-homogeneous DE

$$ay'' + by' + cy = \mathbf{f(x)}. \qquad \text{where } f = Ke^{Tx}:$$

1. Solve the homogeneous DE $ah'' + bh' + ch = \mathbf{0}$.
2. Check if term e^Tx appears in h, and choose u to be one of Me^{Tx}, Mxe^{Tx} or Mx^2e^{Tx} accordingly. Specifically:
   - If T is *not* a solution to the auxiliary equation, set $u = Me^{Tx}$.
   - If the auxiliary equation has *two* distinct solutions, and one of them is T, set $u = Mxe^{Tx}$.
   - If the auxiliary equation has just *one* solution, and that is T, set $u = Mx^2e^{Tx}$.
3. Substitute u into the DE, divide by e^{Tx} and solve for M.
4. The general solution is then $y(x) = h(x) + u(x)$.

$f(x) = Ke^{Tx}$

In the two example that we did at the end on Monday's lecture, f did not appear in the solution to the complementary homogeneous equation.

Now we'll do two examples where it does.

Example

Solve the following DE

$$2y'' + y' - y = 3e^{x/2}.$$

Example

Solve the following DE

$$y'' + 2y' + y = 4e^{-x}.$$

$f(x) = Ke^{Tx}$

Exercise (Q14.1)

Find general solutions to the following non-homogeneous differential equations:

1. $y'' + y' - 2y = e^{-x}$.
2. $y'' + y' - 2y = 3e^{x}$.
3. $y'' + 5y' + 6y = 4e^{-2x}$.
4. $-3y'' + 3y' - y = \frac{1}{2}e^{-x/2}$.

Exercise (Q14.2)

Suppose the solution to $ah'' + bh' + ch = 0$, where $D = b^2 - 4ac > 0$ so h is of the form $h = Ae^{R_1 x} + Be^{R_2 x}$.
Show that, if u is a *particular* solution to $au'' + bu' + cu = Ke^{R_1 x}$, then $u = \frac{K}{\sqrt{D}} x e^{R_1 x}$.

f is a trigonometric function

f is $\sin(Tx)$ or $\cos(Tx)$

When solving the Non-homogeneous DE

$$ay'' + by' + cy = \mathbf{f(x)}.$$

where f is $\sin(Tx)$ or $\cos(Tx)$:

1. Solve the homogeneous DE $ah'' + bh' + ch = \mathbf{0}$.
   - If f does *not* appear as part of h, set $u = z_1 \sin(Tx) + z_2 \cos(Tx)$, where z_1 and z_2 are some numbers.
   - If f *does* appear as part of h, set $u = z_1 x \sin(Tx) + z_2 x \cos(Tx)$.
2. Substitute u into the DE. Get 2 equations: one for sin and one for *cos*.
3. Solve the 2 equations for z_1 and z_2.
4. The general solution is then $y(x) = h(x) + z_1 \sin(x) + z_2 \cos(x)$.

f is a trigonometric function

Example

Find the general solution to the non-homogeneous problem:

$$4y'' + 12y' + 9y = \sin(\frac{x}{2}).$$

f is a trigonometric function

Example

Find the general solution to the non-homogeneous problem:

$$y'' + 4y = \cos(2x).$$

f is a trigonometric function

Exercise (Q14.3)

Find general solutions to the following differential equations:

1. $y'' - y = \cos(x)$.
2. $y'' + y' - 2y = 5\sin(-2x)$.

See also Exer 14.4 on Problem Set 3.

f is the sum of two functions

Now we can solve each of the following problems:

$$ay'' + by' + cy = P(x).$$
$$ay'' + by' + cy = e^{Tx}.$$
$$ay'' + by' + cy = \sin(Tx).$$

To solve a problem like:

$$ay'' + by' + cy = P(x) + e^{Tx},$$

proceed by solving each of

$$ah'' + bh' + ch = 0;$$
$$au'' + bu' + cu = P(x);$$
$$av'' + bv' + cv = e^{Tx}.$$

Then the general solution will be

$$y(x) = h(x) + u(x) + v(x).$$

f is the sum of two functions

Example (Q4 (c), Semester 1, 06/07)

Find the general solution to the non-homogeneous problem:

$$y'' + 4y' + 4y = e^x + x.$$

f is the sum of two functions

Exercise (Q14.5)

Find general solutions to the following differential equations:

1. $y'' + 4y' + y = e^x + \cos(x)$.
2. $y'' + y' - 2y = 2 + 2\sin(x)$.