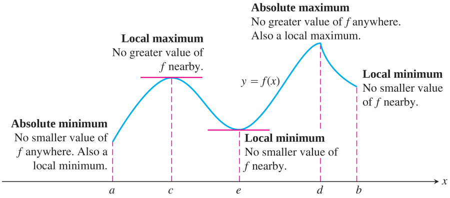
Week 05, Lecture 3
Maxima and Minima

Dr Niall Madden

School of Maths, University of Galway

Thursday, 17 October, 2024



Survey, Assignments, etc

- The module survey for MA140 has started. Please take a few minutes to complete it. See https://universityofgalway.instructure.com/courses/35693/discussion_topics/127822
- **Assignment 3** is due Monday at 5pm.
- If you take a break while doing the assignment, click on Pause.
- Make sure any completed question shows Answer saved
- When you've finished, click End Exam
- Click Print this results summary and save the PDF.
- Don't re-attempt the assignment.
- Don't worry if you get a message saying your assignment is not graded: that just means the deadline has not yet passed.
- Do worry if the deadline passes, and your grade is incorrect. In that case, email Niall with a copy of your results summary.

Today, we'll max out on...

See also: Sections 3.8 (Implicit Differentiation) and 4.3 (Maxima and Minima) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

This section of MA140 is concerned with using techniques of differentiation to finding where a function is

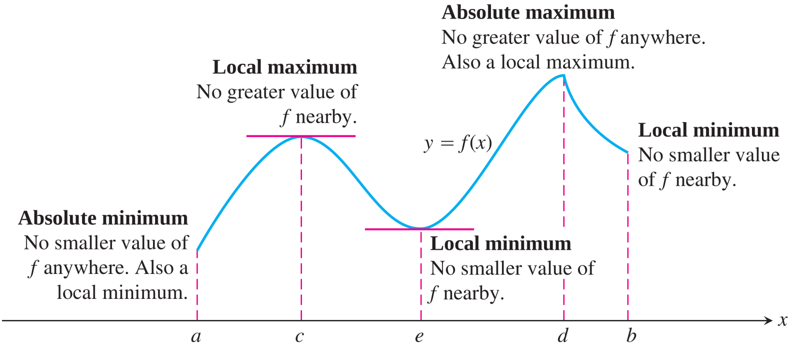
- Increasing
- Decreasing
- Has its maximum value
- Has its minimum value

Along the way we'll learn about **critical values** and the **first derivative test**.

Mathematical English

- The plural of **maximum** is **maxima**;
- The plural of **minimum** is **minima**;
- An **extremum** a maximum or a minimum.
- The plural of **extremum** is **extrema**.

Given an interval $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$, consider the function $f : [a, b] \to \mathbb{R}$ whose graph is given below. It illustrates local and absolute (="global") maxima and minima. Collectively, these are called **extrema**.



Definition: critical points

Let c in an point in the domain of a function f. We say that $x = c$ is a **critical point** of $f(x)$ if either

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist.}$$

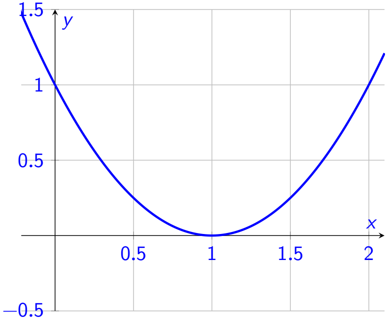
Important: If f has am extremum at $x = c$, then c must be a critical point of f (This is called "Fermat's Theorem").

So, to find a maximum or minimum of f, it is enough to check at the critical points.

Warning: All extrema are at critical points, but not all critical points correspond to a extrema.

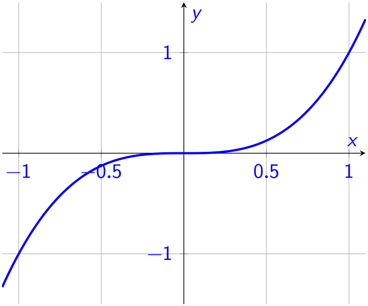
Example

$f(x) = x^2 - 2x + 1$ has one critical point. Find it. Does it correspond to an extremum?



Example

Find all critical points of $f(x) = x^3$. Do they correspond to extrema?



Definition (Increasing/Decreasing)

Let f be a function whose domain includes the interval $[a, b]$. Let let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$.

- If $\boxed{f(x_1) < f(x_2)}$, then f is said to be *increasing* on $[a, b]$.
- If $\boxed{f(x_1) > f(x_2)}$, then f is said to be *decreasing* on $[a, b]$.

Example

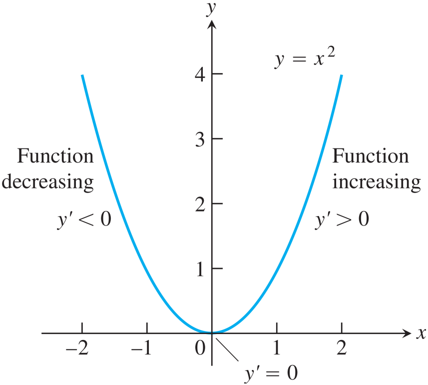
The function $f(x) = x^2$ is decreasing on $(-\infty, 0]$, and increasing on $[0, \infty)$.

The First Derivative Test

Theorem

Suppose that f is differentiable on an interval $[a, b]$.

- If $f'(x) > 0$ at each point $x \in [a, b]$, then f is increasing.
- If $f'(x) < 0$ at each point $x \in [a, b]$, then f is decreasing.



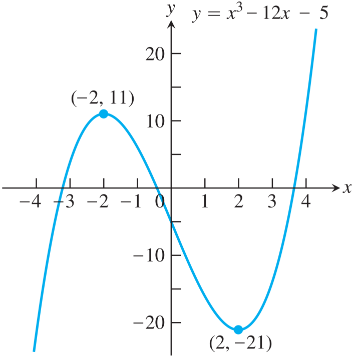
Example

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing

Idea: find $f'(x)$ and then solve for $f'(x) = 0$.

The critical points $c = -2$ and $c = 2$ of $f(x) = x^3 - 12x - 5$ subdivide the domain of f into intervals $(-\infty, -2), (-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval.

		-2		2	
$3(x+2)$	$-$	$\bullet$	$+$		$+$
$x-2$	$-$		$-$	$\bullet$	$+$
$f'(x)$	$+$	$\bullet$	$-$	$\bullet$	$+$



Important:

- If $f(x)$ has a local minimum of $f(x)$ at $x = c$, then it switches from **decreasing** to **increasing**. That means, $f'(x)$ changes sign at $x = 2$. Therefore, $f'(c) = 0$.
- If $f(x)$ has a local maximum at $x = c$, we have that $f'(c) = 0$.

First Derivative Test for local maxima and minima

Suppose that c is a critical point of a differentiable function f.

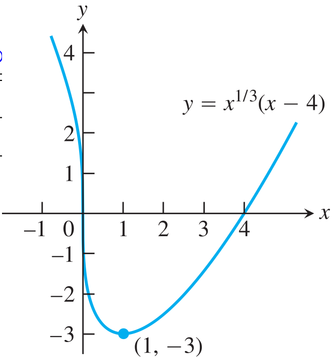
- If f' changes from negative to positive through c, then f has a local minimum at c.
- If f' changes from positive to negative through c, then f has a local maximum at c.
- If f' does not change sign through c (that is, f' is positive on both sides of c or negative on both sides), then f does not have a local maximum or minimum at c.

Example

Find the critical points of $f(x) = x^{\frac{1}{3}}(x-4)$. Identify the local maxima and minima (if any).

First find $f'(x)$, and then where it is either zero or undefined:

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty$
$4(x-1)$	$-$	$-$	$+$
$3x^{2/3}$	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$+$



The First Derivative Test

Review

If a function g is differentiable on an interval $[a, b]$, then

- $g'(x) > 0$ for all $x \in [a, b] \Leftrightarrow g$ increasing on $[a, b]$.
- $g'(x) < 0$ for all $x \in [a, b] \Leftrightarrow g$ decreasing on $[a, b] I$.

Similarly, if g' is also differentiable on $[a, b]$, then

- $(g')'(x) = g''(x) > 0$ for all $x \in [a, b] \Leftrightarrow g'$ increasing on I.
- $(g')'(x) = g''(x) < 0$ for all $x \in [a, b] \Leftrightarrow g'$ decreasing on I.

Exercise 5.3.1 : 23/24 Exam, Q3(a)

Let $f(x) = \ln(x^2 + 1)$.

(i) Find all critical point(s) of f and determine whether f has a local minimum, local maximum or neither.

(ii) Determine the interval on which f is increasing.

(iii) Determine the interval on which f is decreasing.

(iv) Find all point(s) of inflection of f, justifying your answer.