MA211

Lecture 13: Nonhomogeneous DEs

Monday, 20^{th} October 2008

This morning's class:

1. Non-homogeneous Problems
   - General Technique
   - Examples
2. $f(x)$ is a polynomial
3. $f(x) = Me^{Tx}$

For further details and examples, look at the section on *Nonhomogeneous Linear Equations*, Section 17.2 of Stewart *Calculus: early transcendentals*.

Non-homogeneous Problems

So far this week, we have looked at differential equations that have a zero right-hand side:

Homogeneous

$$ay'' + by' + cy = \mathbf{0}.$$

Now we'll look at non-homogeneous problems:

Non-Homogeneous

$$ay'' + by' + cy = \mathbf{f(x)}.$$

The key idea will be to

- First compute the general solution to the complimentary Homogeneous problem
- Work out what we need to add to this to get the solution to the Non-homogeneous Problems

Non-homogeneous Problems

Non-Homogeneous

$$ay'' + by' + cy = \mathbf{f(x)}.$$

The cases we'll consider are

1. f is a polynomial.
2. $f = Me^{Tx}$ where M and T are constant.
3. f is a trig function, such as sin and cos
4. Some combination of the above.

The technique we shall use is sometimes called the *method of undetermined coefficients*.

Suppose we want to solve $ay''(x) + by'(x) + cy(x) = \mathbf{f(x)}$.

Step 1

Solve the corresponding *homogeneous* problem:

$$ah''(x) + bh'(x) + ch(x) = \mathbf{0}.$$

Step 2

Chose a suitable function u and substitute it into the DE

$$au''(x) + bu'(x) + cu(x) = \mathbf{f}$$

to determine its coefficients. This is called a *particular* solution.

Step 3

To get the general solution to the original problem, set

$$y(x) = h(x) + u(x).$$

Theorem

If h is the general solution to:
$ah''(x) + bh'(x) + ch(x) = 0$
and u is a particular solution of
$u''(x) + bu'(x) + cu(x) = f(x)$,
then $y(x) = h(x) + u(x)$ is the general solution of
$ay''(x) + by'(x) + cy(x) = f(x)$.

$f(x)$ is a polynomial

f is a polynomial

When solving the Non-homogeneous DE

$$ay'' + by' + cy = \mathbf{f(x)}.$$

where f is a polynomial of degree n:

$$f(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n.$$

1. Solve the homogeneous DE $ah'' + bh' + ch = \mathbf{0}$.
2. Let u a polynomial of the same degree as f:
$$u(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n.$$
3. Substitute u into the DE and solve (in order) for q_n, q_{n-1}, $\ldots$, q_1, q_0.
4. The general solution is then $y(x) = h(x) + u(x)$.

Example ($f \equiv 1$)

Find the general solution to the non-homogeneous problem:

$$y'' + y' - 2y = 1.$$

$f(x)$ is a polynomial

Example ($f(x) = x + 2$)

Find the general solution to the non-homogeneous problem:

$$y'' + y' - 2y = x + 2.$$

$f(x)$ is a polynomial

Example ($f(x) = x^3 + 1$)

Find the general solution to the non-homogeneous problem:

$$y'' - y = x^3 + 1.$$

Exercise (Q13.1)

Find general solutions to the following differential equations:

1. $y'' + y' - 2y = 1$.
2. $y'' - 6y' + 9y = x$.
3. $y'' - 2y' = x^2 + 4$.
4. $y'' = 4x^3$.

$f(x) = Me^{Tx}$

If the right-hand side of the DE is an exponential function:

$f = Me^{Tx}$

When solving the Non-homogeneous DE

$$ay'' + by' + cy = \mathbf{f(x)}.$$

where $f = Me^{Tx}$:

1. Solve the homogeneous DE $ah'' + bh' + ch = \mathbf{0}$.
2. Check if term $e^T x$ appears in h
   - If it doesn't, set $u = Me^{Tx}$.
   - If it does, set $u = Mxe^{Tx}$, or $u = Mx^2e^{Tx}$.

   (More about this later)
3. Substitute u into the DE, divide by e^{Tx} and solve for M.
4. The general solution is then $y(x) = h(x) + u(x)$.

$f(x) = Me^{Tx}$

Example ($f(x) = e^{2x}$)

Find the general solution to the non-homogeneous problem:

$$y'' - y = e^{2x}.$$

Example ($f = e^{-3x}$)

Find the general solution to the non-homogeneous problem:

$$y'' - \sqrt{7}y' + 2y = e^{-3x}.$$