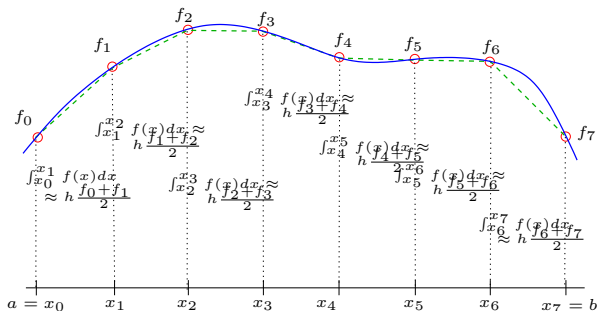


MA378 Chapter 3: Numerical Integration

§3.3 Precision and Composition

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3.1 Simpson's, again

In the final example of Section 3.2 (“Simpson’s Rule”) we noticed that **Simpson’s Rule** yields exactly the right answer (i.e., the error is zero) when applied to approximating $\int_0^1 x^3 dx$.

In fact, Simpson’s Rule is exact for **any** polynomial of degree 3 or less. We’ll now explain why, restricting our focus to the interval $[-1, 1]$. (See below for the general case).

1. Suppose that $Q_2(f)$ approximates $\int_{-1}^1 f(x)dx$.
2. We know that, if p_2 is any quadratic polynomial, then $Q_2(p_2) = \int_{-1}^1 p_2(x)dx$.
3. Write p_3 as

$$p_3(x) = c_3x^3 + c_2x^2 + c_1x + c_0 = c_3x^3 + p_2(x),$$

for some quadratic $p_2(x)$.

3.1 Simpson's, again

4. Note that $\int_{-1}^1 p_3(x)dx = c_3 \int_{-1}^1 x^3 dx + \int_{-1}^1 p_2(x)dx.$
5. However, $\int_{-1}^1 x^3 dx = 0.$
6. Therefore, $\int_{-1}^1 p_3(x)dx = \int_{-1}^1 p_2(x)dx.$
7. Similarly, $Q_2(p_3)dx = c_3 Q_2(x^3) + Q_2(p_2).$
8. However, $Q_2(x^3) = \frac{2}{6}((-1)^3 + 4(0^3) + 1^3) = 0.$
9. Therefore $Q_2(p_3)dx = Q_2(p_2) = \int_{-1}^1 p_2(x)dx.$
10. So, we have that

$$\int_{-1}^1 p_3(x)dx = \int_{-1}^1 p_2(x)dx = Q_2(p_3).$$

Conclude: $Q_2(\cdot)$ gives exactly the right answer when applied to estimating $\int_{-1}^1 p_3(x)dx$ for any cubic polynomial, p_3 .

Here are the details for the general case. We won't go through this in class, but please read it carefully.

We now claim that **Simpson's Rule** is exact for *any* polynomial of degree 3 or less, and on *any* interval.

Denote by **$Q_2(f)$** the approximation of $\int_a^b f(x)dx$ with Simpson's Rule:

$$Q_2(f) = \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right).$$

Since the method can be derived by integrating the quadratic that interpolates $f(x)$ at the three points a , $(a+b)/2$, and b , it is clearly exact for all quadratics.

Let $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. Then

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b (c_0 + c_1x + c_2x^2)dx + c_3 \int_a^b x^3dx = \\ &= \int_a^b (c_0 + c_1x + c_2x^2)dx + c_3 \frac{b^4 - a^4}{4}.\end{aligned}$$

Also,

$$\begin{aligned}Q_2(f) &= Q_2(c_0 + c_1x + c_2x^2) + Q_2(c_3x^3) = \\ &= \int_a^b (c_0 + c_1x + c_2x^2)dx \\ &\quad + c_3 \left(\frac{b-a}{6} \right) \left(a^3 + 4 \left(\frac{a+b}{2} \right)^3 + b^3 \right). \quad (1)\end{aligned}$$

With a bit of symbolic manipulation we get that

$$\frac{b^4 - a^4}{4} = \left(\frac{b-a}{6}\right) \left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right),$$

as required.

In Section 3.2, we discussed a naïve attempt to derive an upper bound for the error in Simpson's rule leading to

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3.$$

This is not wrong – just not sharp. For example it does not give that Simpson's Rule is exact for all cubic. The sharp result is

Theorem 3.1

$$\mathcal{E}_2 := \left| \int_a^b f(x) dx - Q_2(f) \right| \leq \frac{(b-a)^5}{2880} M_4.$$

For the proof see the text book (Theorem 7.2 of Suli and Mayers). Instead of working through it in class we'll prove a more general version of a consequence it.

3.2 Precision

Definition 3.2 (Precision of a Quadrature Rule)

A quadrature rule has **precision** n if it is exact for all polynomials of degree n or less. That is, the rule $Q(f)$ has precision n if

$$Q(p_n) = \int_a^b p_n(x) dx \quad \text{for all } p_n \in \mathcal{P}_n.$$

Example 3.3

By construction, the $(n + 1)$ -point Newton-Cotes rule has precision n .

3.2 Precision

Theorem 3.4

If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k + 1$ points, then $Q_{2k}(\cdot)$ has in fact precision $2k + 1$.

Proof: Let p_{n+1} be a polynomial of degree $n + 1$. We wish to show that $Q_n(p_{n+1}) = \int_a^b p_{n+1}(x)dx$. We can take $a = -1$, $b = 1$ because a simple linear transformation can be used to map to an arbitrary interval.

Also, since the quadrature points are equally spaced on $[-1, 1]$ we have that $x_i = -x_{n-i}$.

Furthermore (see Exercise 3.1) the quadrature weights are symmetric: $q_i = q_{n-i}$.

3.2 Precision

3.3 Composite Rules

Suppose that we want to estimate $\int_a^b f(x)dx$ and the Trapezium rule is not sufficiently accurate. We could try Simpson's Rule, which should be better. Failing that, we could try a 4-point rule, based on integrating the p_4 interpolant, or a five-point rule, based on integrating p_5 .

However, quite apart from the fact that it might be tedious to derive these rules, we know (Runge's example again!) that high-order polynomial interpolation can be very inaccurate.

3.3 Composite Rules

It is better to use a **Composite Rule**. This is analogous to the idea behind piecewise linear.

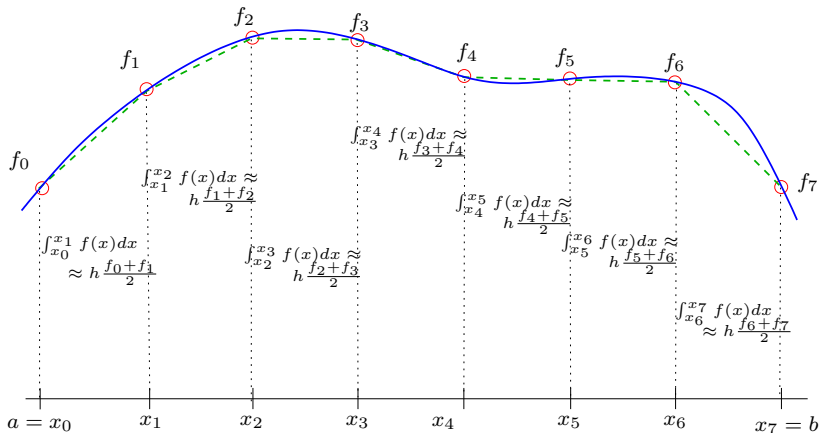
For the **Composite Trapezium Rule** we divide $[a, b]$ into N intervals of size $h = (b - a)/N$. Applying the Trapezium Rule on each interval $[x_{i-1}, x_i]$ we get

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx h \frac{f_{i-1} + f_i}{2}.$$

Summing for the n intervals we get

$$\int_a^b f(x)dx \approx \frac{b-a}{N} \left(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-1} + \frac{f_N}{2} \right). \quad (2)$$

3.3 Composite Rules



3.4 Exercises

Exercise 3.1

Explain clearly, with an example, why in general it is not true that

$$Q_n(f) \rightarrow \int_a^b f(x)dx \text{ as } n \rightarrow \infty.$$

Exercise 3.2

- (i) Deduce an error estimate for the Composite Trapezium Rule.
- (ii) Taking $N = 10$, give an upper bound for the error in the Composite Trapezium Rule when approximating $\int_1^2 \ln(x)dx$.
- (iii) What value of n would you have to take to ensure that the error was less than 10^{-5} ?

3.4 Exercises

Exercise 3.3

- (i) Deduce the formula for the *composite Simpson's Rule*.
- (ii) Derive an error estimate for the *composite Simpson's Rule*.
- (iii) What value of N would you have to take to ensure that the error in the estimate of $\int_1^2 \ln(x)dx$ is less than 10^{-6} ?
- (iv) Denote the $(N + 1)$ -point Composite Simpson's Rule by $S_N(f) \approx \int_a^b f(x)dx$. Show that, for sufficiently smooth $f(x)$,

$$\lim_{n \rightarrow \infty} S_N(f) = \int_a^b f(x)dx.$$

3.4 Exercises

Exercise 3.4

Determine the precision of the following schemes for estimating $\int_0^1 f(x)dx$.

- (i) $Q(f) = f(\frac{1}{2})$.
- (ii) $Q(f) = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.
- (iii) $Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{3})$.

Exercise 3.5 (★)

Consider the rule:

$$R(f) = q_0 f(1/3) - f(\frac{1}{2}) + q_2 f(\frac{3}{4})$$

for approximating $\int_0^1 f(x)dx$.

1. Determine values of q_0 and q_2 that ensure this rule has precision 2.
2. What is the maximum precision of $R(\cdot)$ with the values of q_1 and q_2 that you have determined?
3. Why is this not, strictly speaking, a Newton-Cotes rule?