Numerically accurate formulation of implicit turbulent bottom stress in an ocean model with barotropic-baroclinic mode splitting†

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Resumé

Bottom drag plays an important role in dissipating tides, and becomes one of the dominant forces in tidal bays and estuaries. One of such places is Galway Bay, Ireland, where tidally induced currents can reach the speed of 2 m/sec, posing challenges in hydrodynamic modeling, essentially due to the interference of different algorithms, which need to work in concert, but originally were not thought to be this way.

The split-explicit time stepping procedure for an oceanic model (e.g. ROMS, Shchepetkin & McWilliams, 2005) implies that the barotropic mode system (vertical integrals of horizontal velocities coupled with contribution of free-surface elevation into vertical integrals of pressure-gradient terms) is solved separately from the rest of the model using smaller time step. This leads to significant computational savings, because the large number of short time steps are applicable only for 2D part of the whole 3D model, but also results in more complicated code, carefully designed to avoid numerical errors and instability.

At the same time, vertical processes—mixing of tracers, viscous exchange of momenta, and, recently, vertical advection (only where it is strictly unavoidable (Shchepetkin, 2015)—are treated implicitly, but only in a one-dimensional manner resulting in a simple and efficient solver.

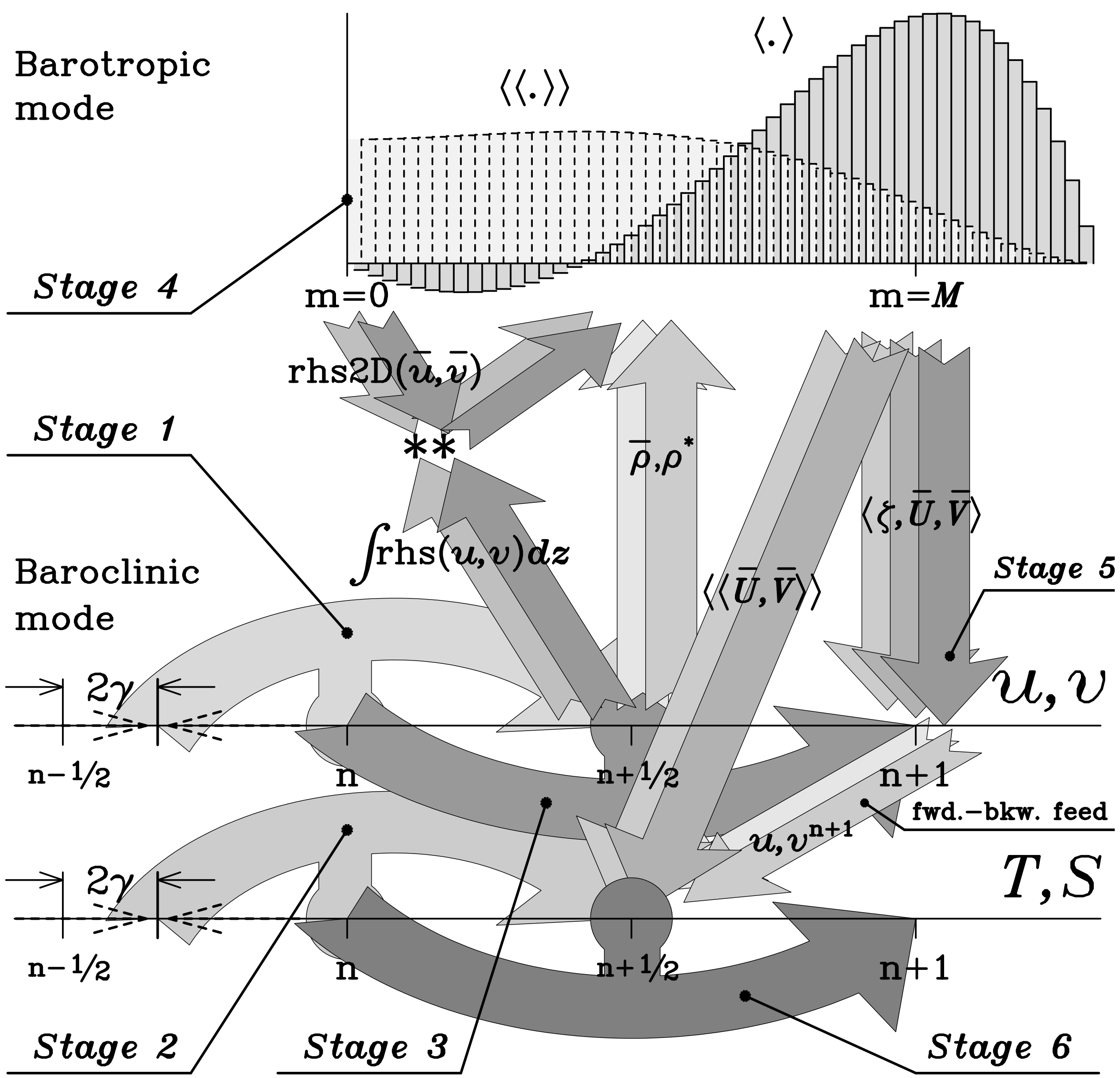
The third ingredient is parameterization of vertical profile of turbulent mixing coefficient along with kinematic stress bottom boundary condition, which is of the no-slip type, but nonlinear in nature due to the fact that both bottom drag coefficient and vertical viscosity profile depend on the magnitude of the current (Soulsby,1983).

Taken separately, these three aspects are well understood at this point. However, combining them in a single computational model requires special care: the barotropic mode needs to know the bottom drag terms in advance (which can be computed only within the 3D part of the code), but when done, the result barotropic mode calculation adjusts the horizontal velocity components in the 3D mode, compromising both the no-slip boundary conditions and the consistency of bottom stress with vertical viscosity profile.

This is reminiscent to the classical dilemma of loosing second-order accuracy of implicit no-slip viscous boundary conditions in the case of projection method for incompressible flows (e.g., Dukowicz & Dvinsky, 1992).

In this study we propose modification to existing algorithms to eliminate the splitting errors associated with bottom drag. Special attention was also paid to bottom boundary condition capable to deal with under-resolved boundary layer in vertical mixing parameterization scheme.

ROMS time stepping algorithm with barotropic mode splitting and coupling during corrector stage



All arrows are drawn in the actual sequence of operations: what takes place later overlaps what took place earlier.

Arched horizontal arrows indicate progression of slow-time variables, u, v and T, S ((a.k.a. baroclinic variables).

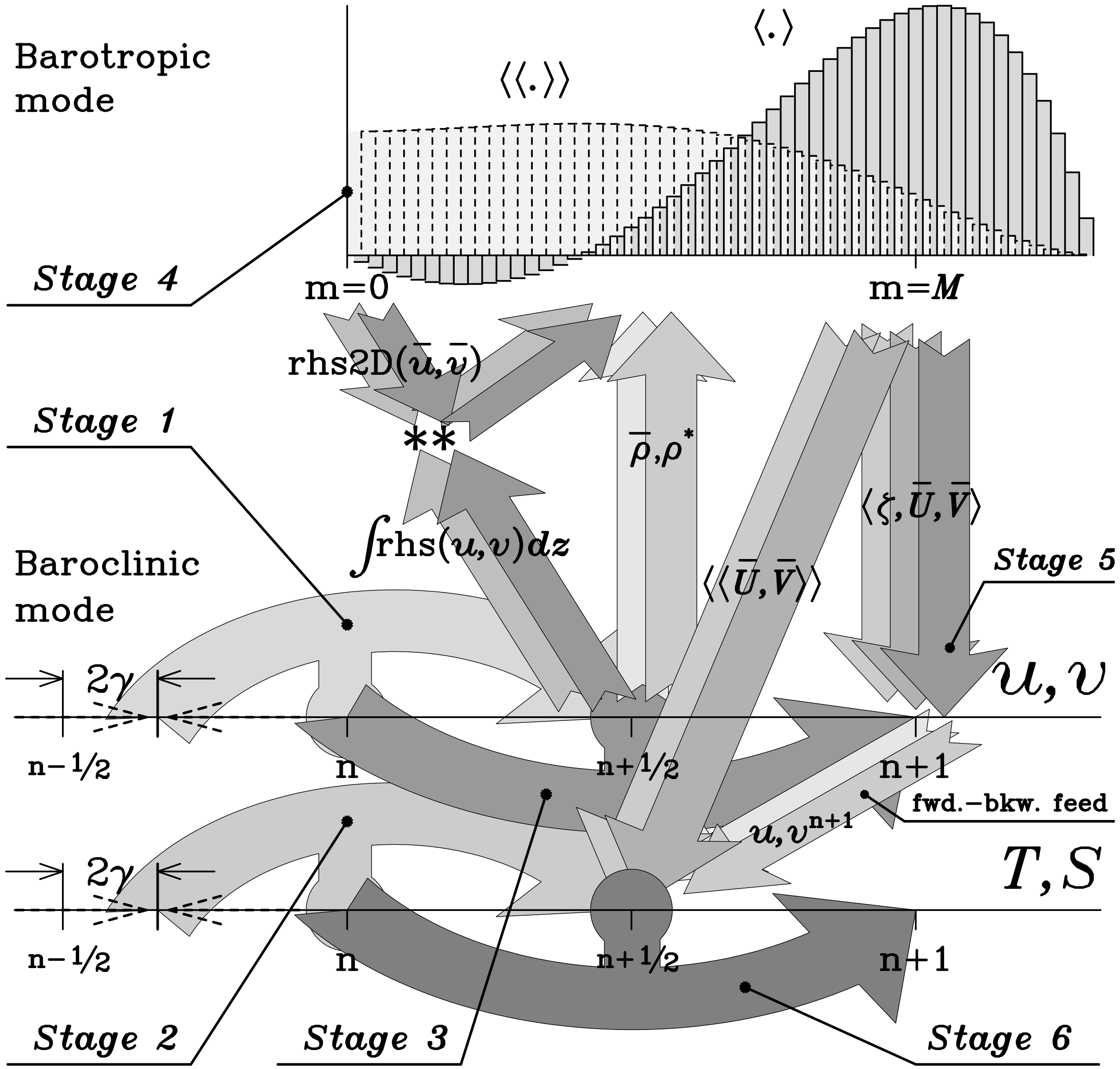
Circles indicate computation of r.h.s. for the respective variables, u, v and T, S.

Four ascending arrows represent computation of 3D→BM forcing terms: vertically integrated r.h.s. terms for 3D momentum equations and 2-way vertically averaged densities, $\overline{\rho}$ and ρ_*. Asterisks ** where the ascending and small descending arrows meet indicate computation of baroclinic-to-barotropic mode forcing (small arrrow ascending to the right). These terms are kept constant during barotropic-mode stepping and are applied at every barotriopic step to barotropic momentum equations as external forcing.

Five descending arrows represent updated barotropic barotropic fluxes (vertically integrated velocities), which are 2-way averaged in fast time. During copuling procedure it is enforced that, at the end of time step, vertical integrals of velocities for 3D mode, u, v, are exactly the same as the ones computed by barotropic mode.

Vertical integrals of inermediate values $u^{n+1/2}, v^{n+1/2}$ (computed during predictor stage) are forced the respective barotropic values at the previous time step n.

Why bottom drag term does not naturally fit into this data flow and requires special attention?



Bottom drag term relates net force applied to the fluid in bottom-most grid box $\Delta z_1 \equiv \Delta z_{k=1}$ with velocity inside it $u_1 = u_{k=1}$,

$$\frac{\partial}{\partial t}(\Delta z_1 \cdot u_1) + ... = -r_D \cdot u_1 \,,$$

thus is to be interpreted in finite-volume sense.

Drag coefficient r_D may be constant, i.e., a finite-difference or finite-volume approximation based on constant viscosity, no-slip boundary condition and assumption about velocity profile (hence its derivative at the bottom), or it may depend on velocity itself (nonlinear, as it happens in turbulent mixing schemes).

Situation when $r_D \cdot \Delta t/\Delta z_1 > 1$, where Δt is baroclinic time step, requires impicit treatment of vertical viscosity, including bottim b.c.: it attempts to remove more momentum from grid-box Δz_1 that available there, leading to oscillations and instability.

During coupling stage barotropic-mode corrections to u, v velocity components of 3D mode are applied uniformly thoughout the entire vertical column, which unavoidably alter the bottom-most u_1, v_1, thus distorting the no-slip bottom b.c.

Recomputing bottom drag term after the barotropic corrections are applied restores the no-slip bottom b.c., but distorts the state of approximate nondivergence of barotropic fluxes.

Using Ekman spiral test problem to illustrate mode splitting error associated with bottom drag

Wind-driven Ekman problem for a layer of finite depth h,

$$\frac{\partial u}{\partial t} - fv = \frac{\partial}{\partial z}\left(A \cdot \frac{\partial u}{\partial z}\right)\,, \qquad \frac{\partial v}{\partial t} + fu = \frac{\partial}{\partial z}\left(A \cdot \frac{\partial v}{\partial z}\right)\,,$$

subject to wind stress forcing, $\quad A \cdot \left.\frac{\partial u}{\partial z}\right|_{z=0} = \tau_x\,, \quad A \cdot \left.\frac{\partial v}{\partial z}\right|_{z=0} = \tau_y\,, \quad$ at surface

and no-slip b.c. $\quad u\big|_{z=-h} = 0\,, \quad v\big|_{z=-h} = 0 \quad$ at bottom $z = -h$.

We are interested in stationary solution with constant vertical viscosity $A = const$, hence

$$-fv = A \cdot \partial^2_{zz} u$$
$$+fu = A \cdot \partial^2_{zz} v$$

which is further combined into a single eqn $\qquad i \cdot f(u + iv) = A \cdot \partial^2_{zz}(u + iv)\,.$

Its general solution

$$u + iv = (\mathscr{U}_+ + i\mathscr{V}_+) \cdot e^{+\sigma z} + (\mathscr{U}_- + i\mathscr{V}_-) \cdot e^{-\sigma z}\,, \qquad \text{where} \quad \sigma = \frac{1+i}{\sqrt{2}}\sqrt{\frac{f}{A}}, \quad \text{hence} \quad \sigma^2 = i \cdot \sqrt{\frac{f}{A}}\,,$$

and the four constants, $\mathscr{U}_+, \mathscr{V}_+, \mathscr{U}_-, \mathscr{V}_-$, are to be determined to satisfy surface and bottom b.c.

No-slip b.c. at bottom $z = -h$ requires $\quad (\mathscr{U}_+ + i\mathscr{V}_+) \cdot e^{-\sigma h} + (\mathscr{U}_- + i\mathscr{V}_-) \cdot e^{+\sigma h} = 0$
hence

$$u + iv = (\mathscr{U} + i\mathscr{V}) \cdot \sinh\big(\sigma(z + h)\big)$$

leaving only two unknowns.

Applying surface b.c. $A \cdot \partial_z(u+iv)\Big|_{z=0} = A\cdot\sigma\cdot(\mathscr{U}+i\mathscr{V})\cdot\cosh\big(\sigma(z+h)\big)\Big|_{z=0} = \tau_x + i\tau_y$ yields

$$\mathscr{U}+i\mathscr{V} = \frac{\tau_x+i\tau_y}{A\cdot\sigma\cdot\cosh(\sigma h)} \qquad \text{and, finally} \qquad u+iv = \frac{\tau_x+i\tau_y}{A\cdot\sigma\cdot\cosh(\sigma h)}\cdot\sinh\big(\sigma(z+h)\big)\,,$$

where the arguments of sinh() and cosh() are still complex-valued. Denote $\gamma=\sqrt{f/(2A)}$, hence $\sigma=\gamma+i\gamma$, the above becomes

$$u+iv = \frac{\tau_x+\tau_y+i\,(\tau_y-\tau_x)}{\sqrt{2Af}}\cdot\frac{\sinh\big(\gamma(z+h)\big)\cdot\cos\big(\gamma(z+h)\big)+i\cdot\cosh\big(\gamma(z+h)\big)\cdot\sin\big(\gamma(z+h)\big)}{\cosh(\gamma h)\cdot\cos(\gamma h)+i\cdot\sinh(\gamma h)\cdot\sin(\gamma h)}$$

where one still needs to separate real and imaginary parts.

Length scale $h_E=\sqrt{A/f}$ is known as Ekman depth. Nondimensional value $h/h_E<1$ or $h/h_E>1$ classifies Ekman layers as *shallow* or *deep*. In the case of $h/h_E\gg 1$ the above asymptotes to the classical Ekman spiral in infinitely deep barotropic fluid, with 45^0 angle to the right between velocity direction at surface and wind stress.

Vertical integration of the above solution yields

$$if\int\limits_{z=-h}^{z=0}(u+iv)\,dz = ifh\cdot(\overline{u}+i\overline{v}) = if\cdot\frac{\tau_x+i\tau_y}{A\cdot\sigma\cdot\cosh(\sigma h)}\cdot\frac{\cosh(\sigma h)-1}{\sigma} = \overbrace{\tau_x+i\tau_y}^{\text{surface wind stress}} \underbrace{-\frac{\tau_x+i\tau_y}{\cosh(\sigma h)}}_{\text{bottom drag}},$$

which means that, from the point of view of split barotropic mode, the stationary solution is a 3-way balance of net Coriolis force (which can be expressed via barotropic variables $\overline{u},\overline{v}$), surface wind stress (an external force), and bottom drag (can be computed by the 3D mode only, and is kept constant during barotropic time stepping).

As $\cosh(\sigma h)$ is complex-valued, bottom drag is rotated relative to $\tau_x+i\tau_y$, and to $\overline{u}+i\overline{v}$ as well.

A split-explicit model, like ROMS, must be able to achieve this stationary state as the result of time stepping under steady wind forcing.

Wind-driven Ekman layer in finite-depth layer of fluid: A test problem for ROMS (courtesy of Yusuke Uchiyama)

$h = 10m$, surface wind stress $u_* = 6 \times 10^{-2} m/s$ ($\approx 5m/s$ wind) $f = 10^{-4}$, $A_v = 2 \times 10^{-3} m^2/s = const$, non-slip b.c. at $z = -h$, $N = 30$; everything is constant in time, running from rest state until stationary solution is reached; analytical solution available

Top: Explicit, CFL-limited, bottom drag term computed **before** Barotropic Mode (BM) for **both** r.h.s. 3D and for BM forcing **(⇒ no splitting error)**; implicit step for vertical viscosity **after** with bottom drag excluded (⇒ undisturbed coupling of 2D and 3D); however needs $r_D < \Delta z_{\text{bottom}}/\Delta t_{\text{3D}}$ for stability

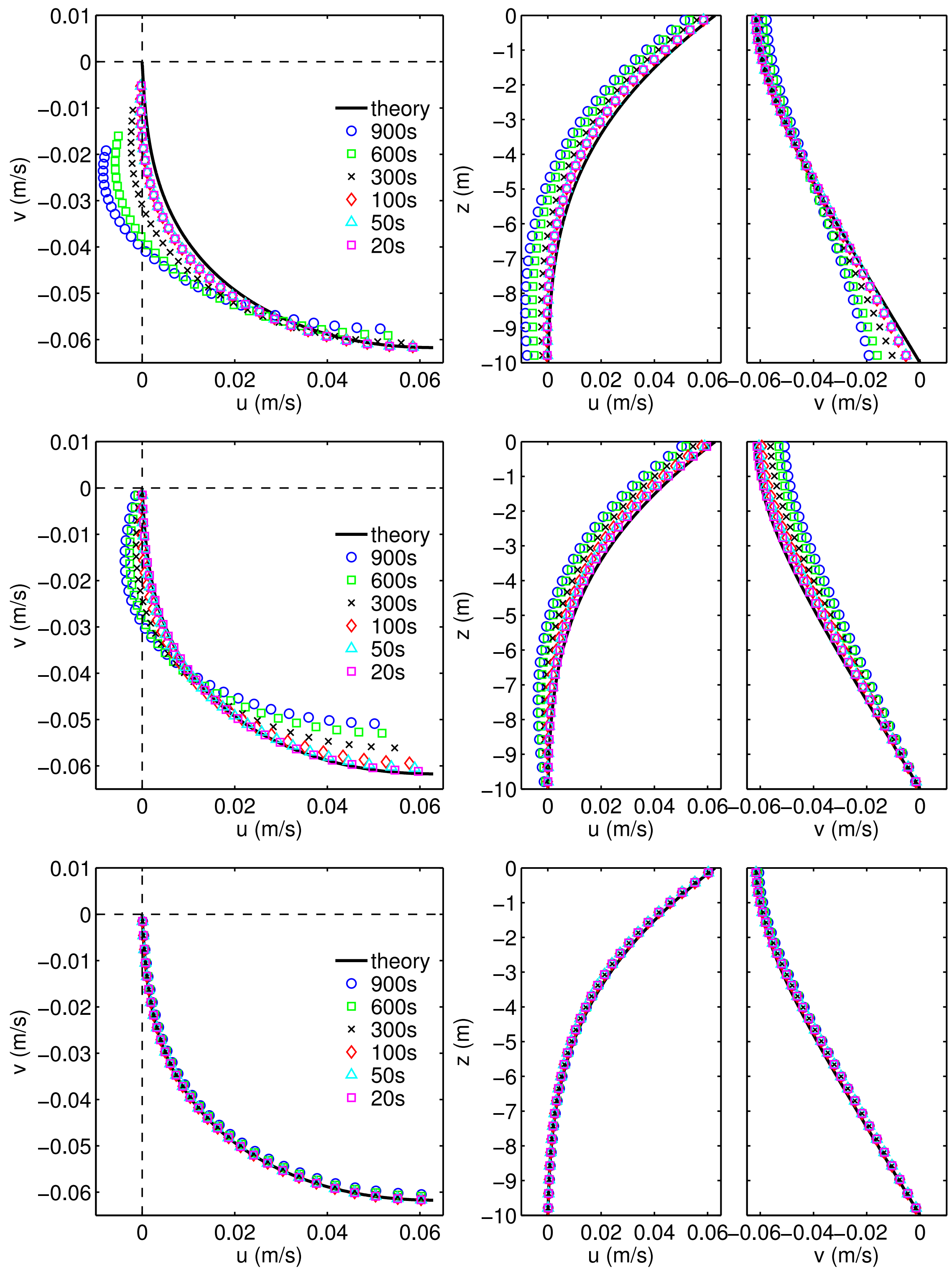
Middle: Unlimited implicit drag **before** BM applies for BM forcing **only**; implicit vertical viscosity **after** with drag included into implicit vertical solver **(⇒ no-slip bottom b.c. is OK)**, however the drag term is recomputed relative to what BM got before as input (⇒ splitting error)

Bottom: Bottom drag term is computed as a part of implicit vertical viscosity step **before** BM and for **both** 3D and BM forcing

In all cases BM has bottom drag term which captures its tendency in fast time

$$\partial_t \overline{\mathbf{U}} = \ldots \underbrace{[\underbrace{-r_D \cdot \mathbf{u}_{\text{bottom}}}_{\text{drag from 3D mode}} + r_D \cdot \overline{\mathbf{u}}^{m=0}]}_{\text{3D}\rightarrow\text{BM forcing}} - r_D \cdot \overline{\mathbf{u}}$$

so when $\mathbf{u}_{\text{bottom}}$ is updated/corrected by BM, so does the $-r_D \cdot \mathbf{u}_{\text{bottom}}$ term computed from it; above $\overline{\mathbf{U}} = (h + \zeta)\overline{\mathbf{u}}$



Why stationationary solution depends on time step size Δt?

- Classical operator splitting dilemma

$$\partial_t \mathbf{u} = \mathscr{R}(\mathbf{u}) \qquad \text{where} \qquad \mathscr{R}(\mathbf{u}) = \mathscr{R}_1(\mathbf{u}) + \mathscr{R}_2(\mathbf{u})$$

  both are stiff, and single-stage solution

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \cdot \mathscr{R}\left(\mathbf{u}^{n:n+1}\right)$$

  is not practical because of complexity (implicitness, nonlinearity), so instead

$$\mathbf{u}' = \mathbf{u}^n + \Delta t \cdot \mathscr{R}_1(\mathbf{u}^{n:\prime}) \qquad \text{followed by} \qquad \mathbf{u}^{n+1} = \mathbf{u}' + \Delta t \cdot \mathscr{R}_2(\mathbf{u}^{\prime:n+1})$$

  resulting in

$$\mathbf{u}^{n+1} = [1 + \Delta t \cdot \mathscr{R}_2(.)] \cdot [1 + \Delta t \cdot \mathscr{R}_1(.)]\, \mathbf{u}^n$$

  where operators do not commute

$$[1 + \Delta t \cdot \mathscr{R}_2(.)] \cdot [1 + \Delta t \cdot \mathscr{R}_1(.)] \neq [1 + \Delta t \cdot \mathscr{R}_1(.)] \cdot [1 + \Delta t \cdot \mathscr{R}_2(.)]$$

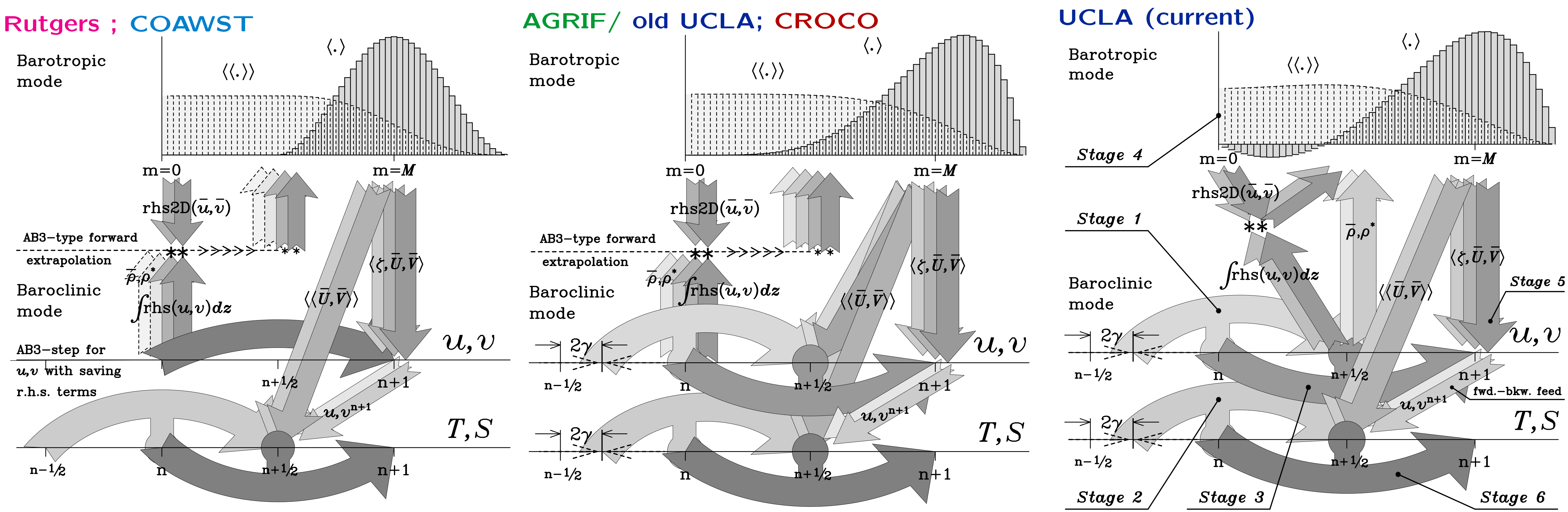
  resulting in $\mathcal{O}(\Delta t)$ operator splitting error. Especially inaccurate in near cancellation situation $R_1 \approx -R_2$ situation (balance).

- reminiscent of implicit no-slip boundaries + pressure-Poisson projection method for incompressible flows, e.g. Dukowicz & Dvinsky (1992)

- requires substantial redesign of ROMS kernel

- somewhat encourages **anti-modular** code design: focus on interference, rather than individual algorithms

Making things worse: there are other variants of ROMS time stepping algorithm, which involve Adams-Bashforth-type extrapolation forward in time of 3d→BM forcing terms.



Nonlinear bottom drag: a modeler prospective

model needs $\quad \Delta z_1 \cdot \dfrac{u_1^{n+1} - u_1^n}{\Delta t} = A_{3/2} \cdot \dfrac{u_2^{n+1} - u_1^{n+1}}{\Delta z_{3/2}} - r_D \cdot u_1^{n?} \qquad r_D = ?$

where $u_1 \equiv u_{k=1}$ is understood in finite-volume sense $u_1 = \dfrac{1}{\Delta z_1} \displaystyle\int_{\text{bottom}}^{\text{bottom}+\Delta z_1} u\left(z'\right)\, \mathrm{d}z'$

from physics $\quad \text{STRESS} = F(u), \qquad F = ?$

duality of u_*: it controls **both** bottom stress and vertical viscosity profile

$$\text{STRESS} = u_*^2\,, \qquad \text{and} \qquad A = A(z) = \kappa u_* \cdot (z_* + z)\,, \qquad z \to 0$$

roughness length z_* = statistically averaged scale of unresolved of topography

constant-stress boundary layer $\quad A(z) \cdot \partial_z u = \text{STRESS} = const = u_*^2$

$$\kappa u_* \,(z_* + z)\, \partial_z u = u_*^2 \qquad \text{hence} \qquad u(z) = \frac{u_*}{\kappa} \ln\left(1 + \frac{z}{z_*}\right)$$

$$u_1 = \frac{u_*}{\kappa}\left[\left(\frac{z_*}{\Delta z_1} + 1\right)\ln\left(1 + \frac{\Delta z_1}{z_*}\right) - 1\right] \qquad \text{hence} \qquad u_* = \kappa \cdot u_1/[...]$$

$$-r_D \cdot u_1 = -\kappa^2 \,|u_1| \cdot \left[\left(\frac{z_*}{\Delta z_1} + 1\right)\ln\left(1 + \frac{\Delta z_1}{z_*}\right) - 1\right]^{-2} \cdot u_1$$

hence $$r_D = \kappa^2\,|u_1| \Bigg/ \left[\left(\frac{z_*}{\Delta z_1}+1\right)\ln\left(1+\frac{\Delta z_1}{z_*}\right)-1\right]^2$$

function in denominator, $f\left(\Delta z_1/z_*\right) = f(x)$, is well-behaving when $x \to 0$,

$$f(x) = \left(1+\frac{1}{x}\right)\cdot\ln\left(1+x\right)-1 = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot x^n}{n(n+1)} = \frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{12}-\frac{x^4}{20}+\ldots$$

in **well-resolved** asymptotic limit for $\Delta z_1/z_* \ll 1$ behaves as $\quad r_D \sim 4\kappa^2\,|u_1|\cdot\dfrac{z_*^2}{\Delta z_1^2}$

however in this case $\quad u(z) = \dfrac{u_*}{\kappa}\ln\left(1+\dfrac{z}{z_*}\right) \sim \dfrac{u_*}{\kappa}\cdot\dfrac{z}{z_*} \quad$ hence $\quad u_1 = \dfrac{u_*}{\kappa}\cdot\dfrac{\Delta z_1}{2z_*}$

resulting $\quad r_D \sim \kappa^2 u_*\cdot\dfrac{2z_*}{\Delta z_1} = \dfrac{A_{\text{bottom}}}{\Delta z_1/2} \quad$ in line with no-slip with laminar viscosity

in **unresolved** case, $\Delta z_1/z_* \gg 1$, $\quad r_D \sim \kappa^2\,|u_1| \Bigg/ \ln^2\left(\dfrac{\Delta z_1}{z_*}\right) \quad$ known as "log-layer"

- smooth transition between resolved and unresolved limits **...for the first time in oceanic modeling practice**
- in 3D $\quad r_D = \kappa^2\,|\mathbf{u}_1| \Big/ \left[\left(z_*/\Delta z_1+1\right)\ln\left(1+\Delta z_1/z_*\right)-1\right]^2$, $\quad$ where $|\mathbf{u}_1| = \sqrt{u_1^2+v_1^2}$
- avoids introduction of *ad hoc* “reference height” z_a, e.g., Soulsby (1995) $\quad \mathrm{STRESS} = \left[\kappa\,/\ln\left(\mathrm{z_a}/\mathrm{z_*}\right)\right]^2\cdot \mathrm{u}^2\Big|_{\mathrm{z}=\mathrm{z_a}}$, where $u\big|_{z=z_a}$ is hard (or impossible) to estimate from discrete variables
- in practice this differs by a factor of 2 from published formulae, e.g., Blaas (2007), with $z_a = \Delta z_1/2$, due to finite-volume vs. finite-difference interpretation of discrete model variables
- near-bottom vertical grid-box height Δz_1 is an **inherent control parameter** of r_D, making it impossible to specify “physical” quadratic drag coefficient, $r_D = C_{\mathrm{D}}\cdot|u|$

How large is $\dfrac{\Delta t \cdot r_D}{\Delta z_1}$?

$$\frac{\Delta t \cdot r_D}{\Delta z_1} = \underbrace{\frac{\Delta t \cdot |u_1|}{\Delta x}}_{\text{horizontal advective Courant number}} \cdot \underbrace{\kappa^2 \cdot \frac{\Delta x}{\Delta z_1} \Bigg/ \left[\left(\frac{z_*}{\Delta z_1} + 1 \right) \ln \left(1 + \frac{\Delta z_1}{z_*} \right) - 1 \right]^2}_{\text{purely geometric criterion}}$$

in unresolved case $\quad \dfrac{\Delta x}{\Delta z_1} \cdot \left[\kappa \Big/ \ln \left(\dfrac{\Delta z_1}{z_*} \right) \right]^2$

Typical high-resolution ROMS practice for basin-scale and regional configurations $h_{\text{min}} \sim 25m$, $N = 30...50$, hence $\Delta z \sim 1m$, $\Delta x = 1km$, and $z_* = 0.01m$, $\kappa = 0.4$ estimates the above as 7.5.

- It is somewhat mitigated by the bottom-most horizontal velocity Courant number ~ 0.1 but, still exceeds the limit of what explicit treatment can handle.
- This situation becomes radically more aggressive in tidally-forced shallow waters, coastal areas and topographically constricted flows.
- sigma-models are the most affected, but they are the ones which are mostly used when bottom drag matters
- vertical grid refinement toward the bottom makes this condition stiffer
- overall an *a priori* estimate is hard to make, can be diagnosed from the results of practial simulations

Galway Bay model setup:

ROMS code (UCLA branch)

orthogonal curvilinear grid in horizontal directions, variable $50 < \Delta x_{i,j} < 450$, locally $\Delta x_{i,j} = \Delta y_{i,j}$, $\forall\, i, j$, hence it is called "`izogrid`" dimensions $1150 \times 322 \times 32$

modified vertical S-coordinate: SM09 ("new" type a.k.a. Rutgers `Vtransform=2`, `Vstretching=4`)

`theta_s` $= 6$, `theta_b` $= 2$, `hc` $= 30m$

(free of limitation `hc` $> h_{\text{min}} = 3.5m$)

time step for 3D mode `dt` $= 20\,sec$

mode splitting ratio `ndtfast` $= 14$

`adaptively-implicit, Courant-number-dependent, vertical advection for both momentum and tracer equations`

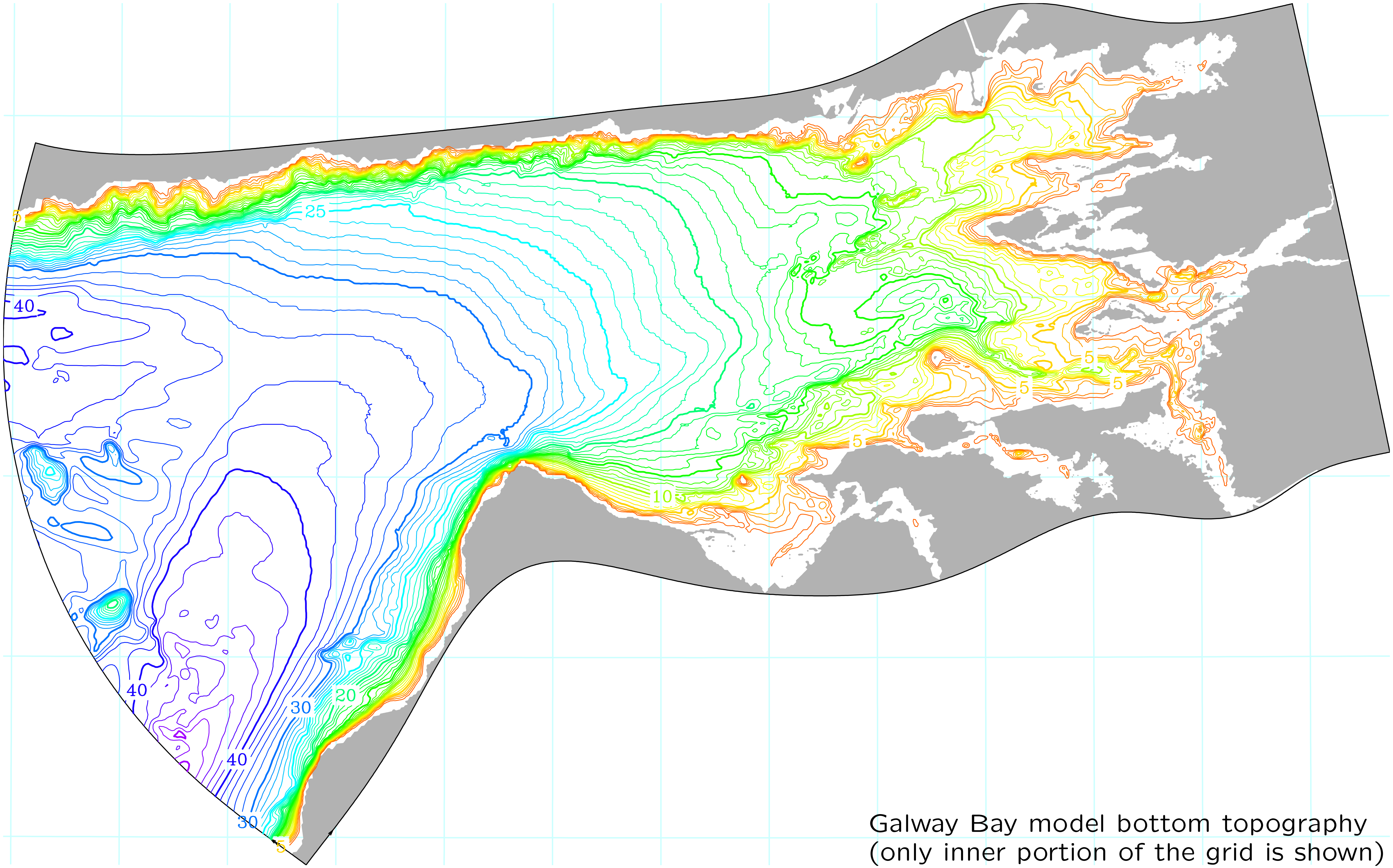
horizontal viscosity/diffusion 0 (except in sponge zone), relying on 3rd-order upstream-biased advection scheme in horizontal directions) maximum sponge zone horizontal viscosity/diffusion $20\,m^2/sec$ at the open boundary, smoothly goes to 0 into the interior

vertical mixing parameterization: KPP `"ifless" variant`, both surface and `bottom boundary layer`, $z_* = 0.01\,m$, bottom drag determined internally;

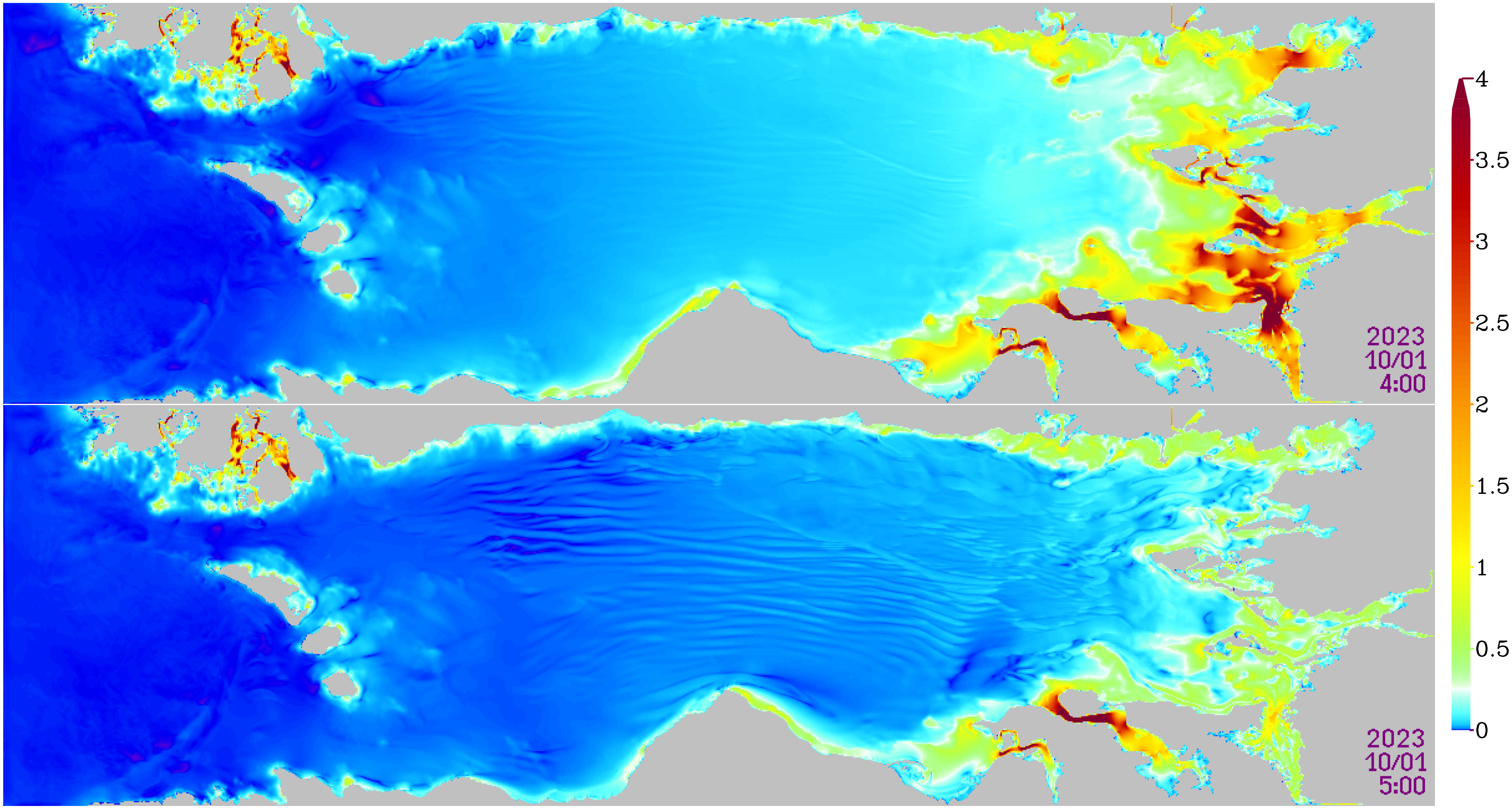
implicit bottom drag for both 3D and 2D

open boundary forcing: `BRY mechanism only` (no 3D nudging within a band near open boundaries or anywhere else); data input every hour, extracted from Marine Institute North-Eastern Atlantic solution;

Flather B.C. for barotropic mode; radiation / advective intake for 3D mode (this code does not rely on user-specified `tau_in`/`tau_out`)



Galway Bay model bottom topography
(only inner portion of the grid is shown)



Maps of $\Delta t \cdot r_D/\Delta z_1$ diagnosed from Galway Bay model run. Two consecutive snapshots, one hour difference. The colors are saturated. The maximum value of $\Delta t \cdot r_D/\Delta z_1$ achieved during this run is 12 (the upper panel).

Turbulent Ekman spiral test problem

$$\frac{\partial u}{\partial t} - fv = \frac{\partial}{\partial z}\left(A\cdot\frac{\partial u}{\partial z}\right), \quad \frac{\partial v}{\partial t} + fu = \frac{\partial}{\partial z}\left(A\cdot\frac{\partial v}{\partial z}\right), \qquad A = A(z) \neq const$$

subject to wind forcing, $A\cdot\left.\frac{\partial u}{\partial z}\right|_{z=0} = \tau_x\,,\quad A\cdot\left.\frac{\partial v}{\partial z}\right|_{z=0} = \tau_y\,,$ and no-slip b.c. $u\big|_{z=-h} = 0\,,\ v\big|_{z=-h} = 0$ at bottom.

Vertical viscosity $A = A(z)$ is set by a vertical mixing parameterezition scheme.

KPP (Large, McWilliams, & Doney, 1994, LMD94) treats planetary boundary layer (PBL, a.k.a. ocean surface b.l.) as integral, determining its thickness h_{bl} via bulk Richardson number criterion,

$$Ri_b(z) = \frac{(z_r - z)\cdot(g/\rho_0)\cdot[\rho(z) - \rho_r]}{|\mathbf{u}_r - \mathbf{u}(z)|^2 + V_t^2(z)}\,, \qquad Ri_b\big|_{z=-h_{\text{bl}}} = \mathtt{Ri}_{\text{cr}} = 0.3 \quad \Rightarrow \text{value of } h_{\text{bl}}\,,$$

where $z_r, \rho_r, \mathbf{u}_r$ are reference values slightly below the surface (usually in the uppermost gridbox). Subsequently h_{bl} is checked against Monin-Obukhov $h_{\text{MO}} = u_*^3/(\kappa\cdot B_{\text{f}})$, and Ekman $h_{\text{Ek}} = 0.7u_*/f$ depth and is limited by both of them in the case of stable buoyancy forcing $B_{\text{f}} > 0$. Once h_{bl} is known, vertical viscosity and mixing coefficients for T, S are computed as $K_{m,s}(z) = w_{m,s}\cdot h_{\text{bl}}\cdot G(z/h_{\text{bl}})$, where $w_{m,s} = \kappa u_*\cdot\psi_{m,s}\left(zB_{\text{f}}/u_*^3\right)$, $u_* = \sqrt{\text{wind stress}}$ is friction velocity, $\psi_{m,s}(.)$ are universal emirical stability functions, and $G(.)$ is universal non-dimensional shape function.

"Ifless" KPP replaces the above with an integral criterion for finding h_{bl}: define it as an integral layer within which net production of turbulence due to shear-layer instability is balanced by dissipation,

$$Cr(z) = \int_z^{\text{surface}} \mathscr{K}(z)\left\{\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2 - \frac{N^2}{\mathtt{Ri}_{\text{cr}}} - C_{\text{Ek}}\cdot f^2\right\}dz' + \frac{V_t^2(z)}{z}$$

and search for crossing point $Cr(z) = 0$. Above $N^2 = -(g/\rho_0)\cdot(\partial\rho/\partial z)\big|_{ad}$ is B-V frequency ($ad \equiv$ adiabatic); f is Coriolis parameter; C_{Ek} is a nondimensional constant; $V_t^2(z)$ is unresolved turbulent velocity shear (same as in LMD94); integration Kernel $\mathscr{K}(z) = (\zeta - z)/(\epsilon h_{\text{bl}} + \zeta - z)$ is to ignore contribution from near-surface sublayer ϵh_{bl} where M-O similarity law is not valid (plays the same role as to distinguish between ρ_{ref} vs. ρ_{surf} in Ri_b of LMD94). ζ is free surface; $\epsilon = 0.1$.

Like the original KPP, "Ifless" KPP is tuned to yield correct behavior in pure physical limits:

	destabilizing	vs.	stabilizing factor	
balance $\int\limits_{z=-h_{\mathsf{bl}}}^{\text{surface}} ...dz$ of	$\left\lvert\dfrac{\partial \mathbf{u}}{\partial z}\right\rvert^2$	vs.	$\dfrac{N^2}{\mathtt{Ri}_{\mathsf{cr}}} > 0$	$\Rightarrow$ shear layer instability
	$\left\lvert\dfrac{\partial \mathbf{u}}{\partial z}\right\rvert^2$	vs.	$C_{\mathsf{Ek}} \cdot f^2$	$\Rightarrow$ turbulent Ekman layer, yield $h_{\mathsf{bl}} = h_{\mathsf{Ek}} = 0.7 u_*/f$
negatively forced	$\dfrac{N^2}{\mathtt{Ri}_{\mathsf{cr}}}$	vs.	$V_t^2(z)$	$\Rightarrow$ free convection, 0.2 empirical entrainment rule, LMD94

but without relying on imposing after-the-fact logical checks and limiters (hence it is "ifless").

Coriolis effect is incorporated into $Cr(z)$ as on of the stabilizing factors.

Bottom boundary layer: Computing $Cr = Cr(z)$ all the way to the bottom yields value Cr_{bot}. Applying the same rationale, but starting search from the bottom and progressing upward, find the first location where $Cr(z) - Cr_{\mathsf{bot}}$ changes its sign from negative to positive—this is the extent of bottom boundary layer, h_{bbl}—as within the portion of water column below it and all the way to the bottom, turbulence production by velocity shear is balanced by one or more stabilizing factors. Because $Cr = Cr(z)$ is the same for both surface and bottom BLs, this approach naturally allows parameterization for their merging (mixing all the way from surface to the bottom).

Once h_{bbl} is found, turbulent viscosity within BBL, $-h < z < -h + h_{\mathsf{bbl}}$ is set $A = \kappa \cdot u_* \cdot h_{\mathsf{bbl}} \cdot G\big((z + h + z_*)/h_{\mathsf{bbl}}\big)$, similarly to surface KPP, with the exception that (*i*) friction velocity is u_* is associated with bottom drag, thus is to be determined internally by he model; (*ii*) z_* is length scale of unresolved bottom roughness; and (*iii*) there is no need for $u_* \to w_*$ translation via stability functions, as the is no boyancy forcing at the bottom.

For the subsequent numerical tests we focus only on computation of bottom stress u_*, while surface and bottom BLs are expected to be merged, making our test be insensitive to algorithmic detail of computation of h_{bl} and h_{bbl}.

Turbulent Ekman spiral: Convergence with respect to vertical resolution

The relationship between the magnitude of velocity at the bottom most grid box (to be interpreted as grid-box averaged value) and the bottom stress coefficient,

$$r_D = |\mathbf{u}_1| \cdot \kappa^2 \bigg/ \left[\left(\frac{z_*}{\Delta z_1} + 1 \right) \ln \left(1 + \frac{\Delta z_1}{z_*} \right) - 1 \right]^2$$

is radically different from commonly used in the other models, e.g., in Princeton ocean Model POM,

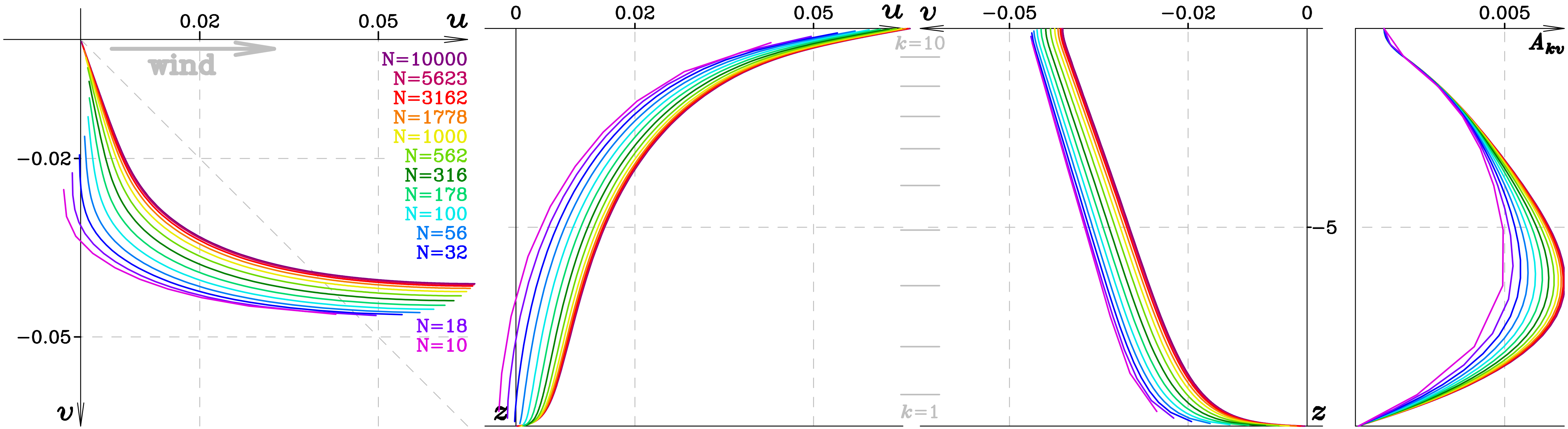
$$r_D = C_D \cdot |\mathbf{u}_1| \ , \quad \text{where} \quad C_D = \max \left\{ C_{D\text{min}}, \ \min \left\{ \left[\kappa \bigg/ \ln \left(\frac{\Delta z_1 / 2}{z_*} \right) \right]^2 , \ C_{D\text{max}} \right\} \right\}$$

canonically restricting $C_{D\text{min}} = 0.0025 \le C_D \le 1 = C_{D\text{max}}$.

Similar expressions appear in many modern codes—Rutgers ROMS, COAWST, CROCO, GETM, and others. All what they have in common, including our, is the asymptotic behavior $\sim \kappa^2 / \ln^2 (\Delta z_1 / z_*)$ when $\Delta z_1 / z_* \gg 1$. Variations include various measures and restrictions to prevent division by zero, if $\Delta z_1 / z_* = 1$

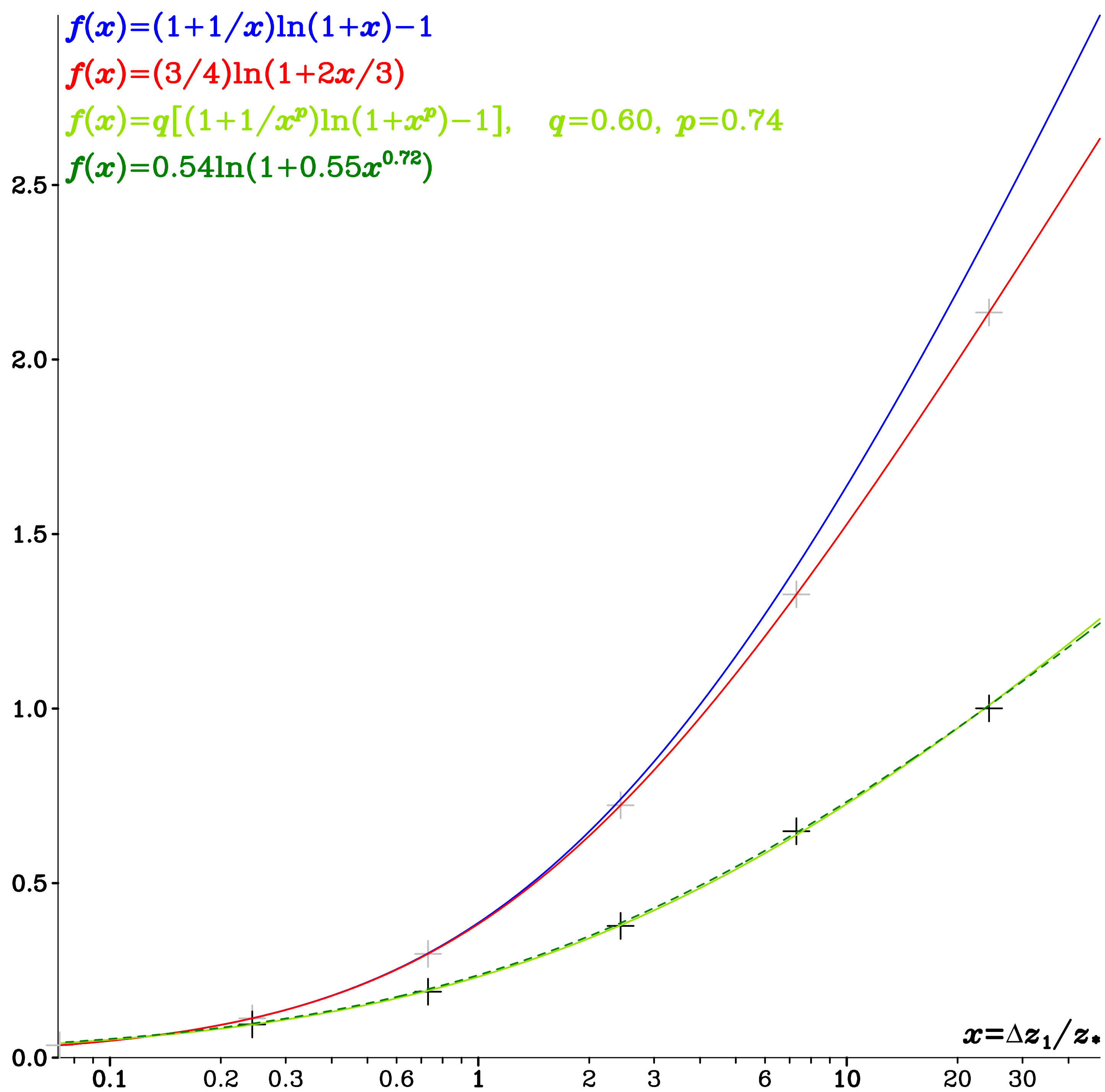
Our relationship allows smooth transition from $\Delta z_1 / z_* > 1$ to $\Delta z_1 / z_* < 1$ and has meaningful limit when $\Delta z_1 / z_* \to 0$.

This encourages us to perform convergence study with respect vertical grid resolution.



convergence study for turbulent Ekman spiral test problem with respect to vertical resolution, for $N \to \infty$

$$r_D = |\mathbf{u}_1| \cdot \kappa^2 / [(1 + 1/x + 1) \ln(1 + x) - 1]^2 \quad \text{where} \quad x = \Delta z_1 / z_*$$



While this result may seem to be quite disappointing—eventually the second order convergence takes place, but at the resolutions way beyond any practical value—it can be traced back of the inability of the finite differences to capture sharp changes in vertical derivative within the log layer—there are simply not enough points there.

Can we modify f in $u_*/|u_1| = \kappa^2/f(x)^2$ to compensate for the lack of resolution?

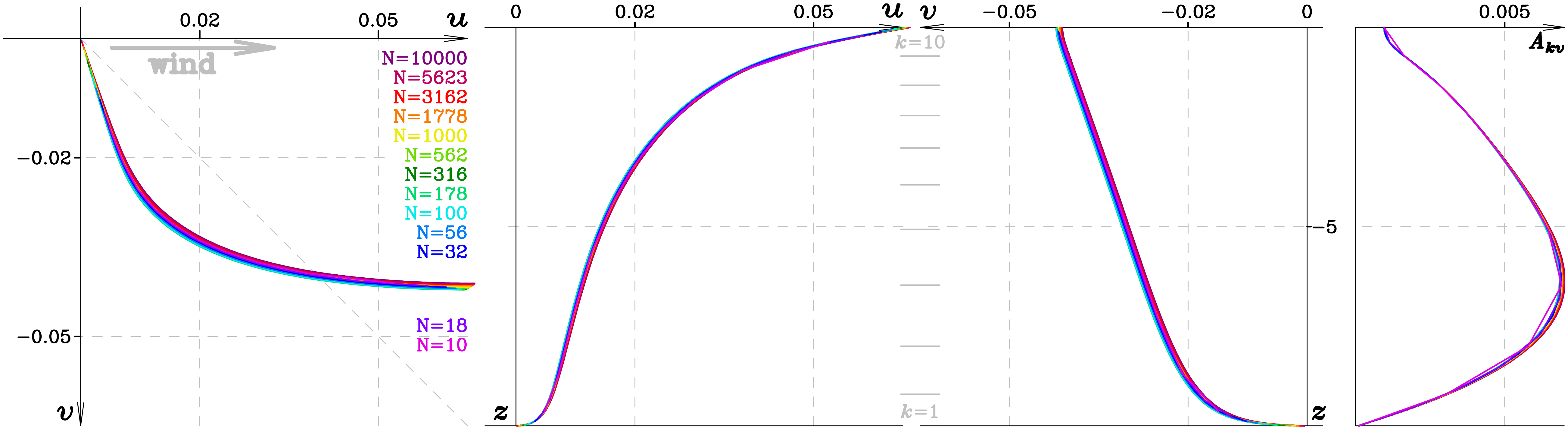
All solutions on the previous page were obtained with $z_* = 0.01$. It is worth to explore the sensitivity of them with respect to varying z_*, trying to match the solution for given N with the one with the highest resolution available.

This results in a set of adjusted z'_*:

N	10^4	3000	1000	300	100	30
z'_*	.01	.012	.017	.0248	.0354	.058

Computing $f = f(x')$, with $x' = \Delta z_1(N)/z'_*(N)$ using the adjusted values $z'_* = z'_*(N)$ from the table yields the set of crosses below the normal $f = f(x)$ curve.

Finally, **fit a function**, which passes through all the crosses as closely as possible (there are multiple ways to do it).



convergence study for turbulent Ekman spiral problem for $N \to \infty$ using $f = f(x)$ from the previous page

$$r_D = q \cdot |\mathbf{u}_1| \cdot \kappa^2 \big/ [(1/x^p + 1)\ln(1 + x^p) - 1]^2 \quad \text{where} \quad x = \Delta z_1/z_*, \quad q = 0.60 \quad p = 0.74$$

Conclusion:

Implicit treatment of $-\Delta t \cdot r_D \cdot u_1^{n+1}$ term: include it into implicit solver for vertical viscosity terms, however this interferes with Barotropic Mode (BM) splitting:

- Bottom drag can be computed only from full 3D velocity, but not from the vertically averaged velocities alone.
- Barotropic Mode must know the bottom drag term **in advance** as a part of 3D→2D forcing for consistency of splitting. This places computing vertical viscosity before BM, however, later when BM corrects the vertical mean of 3D velocities, it **destroys** the consistency of (no-slip like) bottom boundary condition.
- If BM receives bottom drag based on the most recent state of 3D velocity **before** BM, but the implicit vertical viscosity terms along with (the final) bottom drag are computed **after** BM is complete (hence accurately respecting the bottom boundary condition), this changes the state of vertical integrals of 3D velocities, interfering with BM in keeping the vertically integrated velocities in nearly non-divergent state.
- Current ROMS practice is to split bottom drag term from the rest of vertical viscosity computation. This limits the time step (or r_D itself) by the explicit stability constraint.
- Possible only in corrector-coupled and Generalized FB variants of ROMS kernels
- Incompatible (or at least hard to implement) in Rutgers kernel because of forward extrapolation of r.h.s. terms for 3D momenta (AB3 stepping) and extrapolation of 3D→BM forcing terms which is not compatible with having stiff terms there
- Incompatible with predictor-coupled kernel (currently used by CROCO), because of extrapolation of 3D→BM forcing, and because overall having BM too early the computing sequence (implicit vertical viscosity step is done only after predictor step for tracers which is after BM)

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