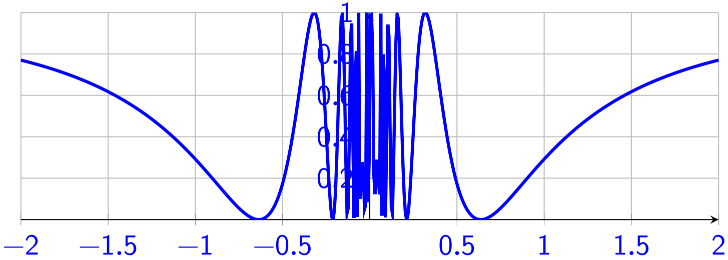
Week 04, Lecture 3
The Chain Rule and Inverse Functions

Dr Niall Madden

University of Galway

Thursday, 09 October, 2025

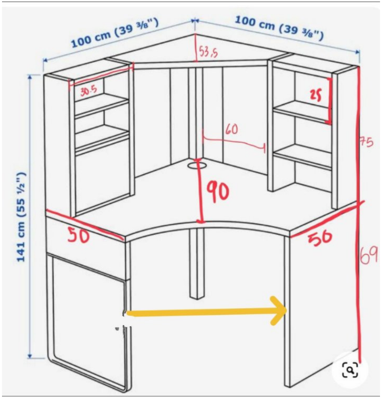


Assignments, etc

Assignments

- **Assignment 2** is open. See https://universityofgalway.instructure.com/courses/35693/assignments/96620.
  Due by 17:00, Monday 13 October.
- The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2065926
- **Assignment 3** is also open. Access through Canvas, or at https://universityofgalway.instructure.com/courses/46734/assignments/130491 Due by 17:00. Monday 20 October.

Warm-up



“Olive” is thinking of buying this desk unit in IKEA. Her (wheel)chain is 55cm. Is the sitting region of the desk indicated by the yellow line, wide enough?

What we'll do today:

See also: Sections 3.6 (The Chain Rule) and 3.8 (Implicit Differentiation) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Chain Rule (again)

Yesterday, we first learned about the *most important* differentiation rule: **chain rule**. It applies to a "function of a function"

The Chain Rule

If $u(x)$ and $v(x)$ are differentiable, and f is the composite function $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

Chain Rule (again)

Example (Ex 3.6.1 in text-book)

Find the derivative of $f(x) = \dfrac{1}{(3x^2+1)^2}$.

Composites of 3 or more functions

One can apply the **Chain Rule** to "functions of functions of functions": if $y(x) = t\Big(u\big(v(x)\big)\Big)$, then

$$\frac{dy}{dx} = \frac{dt}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$ when $y = \sin^4(x^5 + 7)$.

Composites of 3 or more functions

Example

Show that the derivative of $y = \cos^2(1/x)$ is

$$\frac{dy}{dx} = 2\frac{\sin(1/x)\cos(1/x)}{x^2}.$$

Inverse functions

Suppose that $y = f(x)$. That is, f maps x to y.

Then the **inverse** of f is the function, f^{-1}, that maps y back to x.

Example

- The inverse of $f(x) = \frac{1}{2}x$ is $f^{-1}(x) = 2x$.
- The inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$.

Warning: $f^{-1}(x)$ is not the same as $\dfrac{1}{f(x)}$.

Inverse functions

It is often useful to be able to express the derivative (assuming there is one) of an inverse function $f^{-1}(x)$ in terms of the derivative of $f(x)$.

To do this, we use the following rule:

Inverse-Function Rule

If $y = f^{-1}(x)$, then $x = f(y)$ and also

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

Example

If $y = x^{1/3}$, use the **Inverse Rule** to find $\frac{dy}{dx}$.

Note: we can solve this just using the **Power Rule:** $\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$.
But we'll also do this with the **Inverse Rule** for purposes of *exposition*.

If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2\,.$$

By the inverse rule, $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}\,.$$

As $y = x^{\frac{1}{3}}$ we have

$$\frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}}\,.$$

Example

Find the derivative of $\sin^{-1}(x)$

Let $y = \sin^{-1}(x)$, then $x = \sin(y) \quad (\star)$, so

$$\frac{dx}{dy} = \cos(y)\,. \qquad (\star\star)$$

From $\sin^2(y) + \cos^2(y) = 1$, we find $\cos(y) = \sqrt{1 - \sin^2(y)}$ (choosing the positive square root as $\cos(y)$ is positive for y here). Using $(\star)$:

$$\cos y = \sqrt{1 - x^2}\,.$$

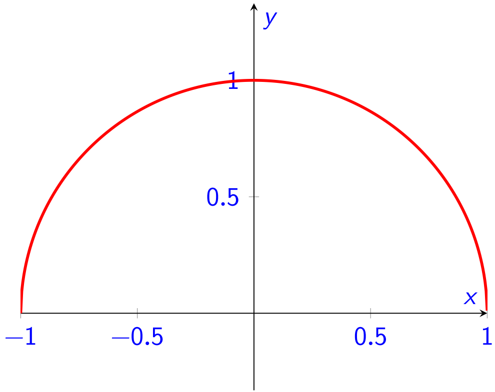
Now using the inverse rule and $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}}\,.$$

Implicit differentiation

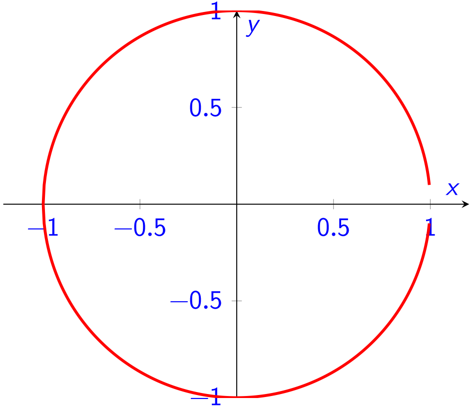
To date, most functions we have studied have been **explicitly** defined. Such functions and be written as $y = f(x)$: given a value of x we can substitute it into $f(x)$ to get the corresponding value of y

Example: $y = \sqrt{1 - x^2}$.



Implicit differentiation

However, sometimes we are given an equation involving x and y where these two terms are not "separated" entirely; e.g, $x^2 + y^2 = 1$. Here y is **implicitly** defined: for any pair (x, y) we can check if it is on the curve described by the equation.



Implicit differentiation

Since **implicit equations** define curves, we can use **implicit differentiation**, for example, finding tangents to these curves.

Method:

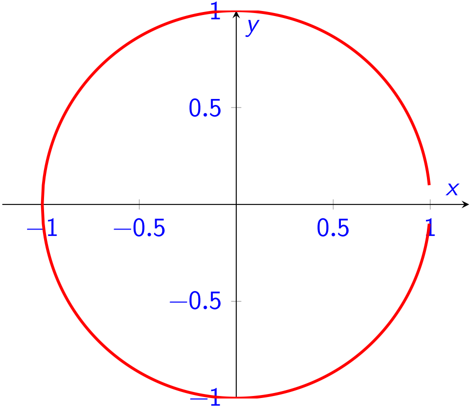
1. Differentiate both size of the equation, with respect to x, , keeping in mind that y is a function of x, using the Chain Rule where needed.
2. Solve for dy/dx.

Implicit differentiation

If y is defined by $x^2 + y^2 = 1$, find $\dfrac{dy}{dx}$.

Implicit differentiation

Now we know that if $x^2 + y^2 = 1$, then $\dfrac{dy}{dx} = -\dfrac{x}{y}$. We can check that this relates to the slope of the tangents to this curve at various places:



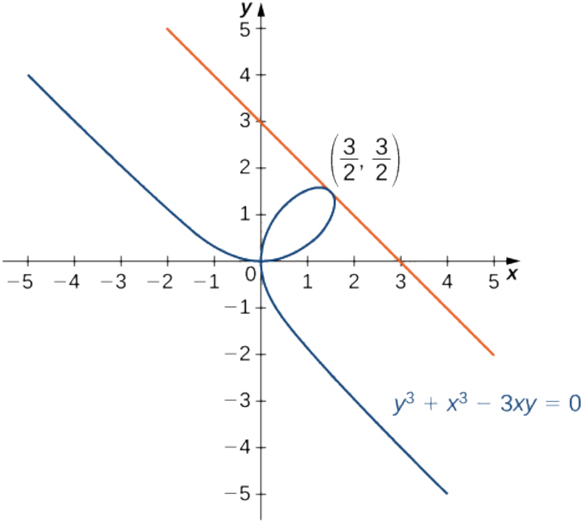
Implicit differentiation

Find the tangent to the curve $x^2 + y^2 = 25$, at the point $(3, -4)$.

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$.

Implicit differentiation



Exponential functions

Earlier in this course we met functions such as $y = x^2$; this is a **power** function.

Now we consider **exponential functions**, such as $\boxed{y = 2^x}$.

Such functions occur in many applications. For example: if I invest €100 with an annual interest rate of 20%, then after x years, I will have €$100 \times (1.2)^x$. **Why?**

Exponential functions

Exponential functions grow quite fast: if my investment is indeed worth $f(x) = 100 \times (1.2)^x$ euros after x years, then...

- After 1 year, I have €120
- After 10 years, I have €619.17
- After 20 years, I have €3,833.80
- After 25 years, I have €9,539.60
- After 50 years, and 190 days, I'll be a millionaire!

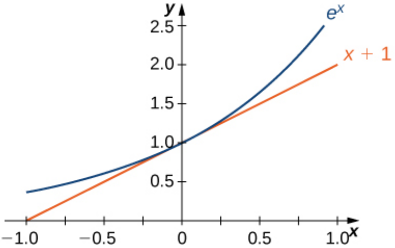
Here I remind you of some properties of exponents that you should already know: for any positive numbers a and b,

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x a^y$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{a^y}$

The number $e \approx 2.7182818284$. It is often called **Euler's Number** after Leonard Euler, who did not discover it: that was (probably) Jacob Bernoulli in 1683 while studying compound interest. Or maybe 100 years earlier by John Napier.

The Natural Exponential Function

The Natural Exponential Function is $f(x) = e^x$. It is special for many reasons, including the its tangent at $x = 0$ has slope 1.



Let's assume that e is the number for which, if $f(x) = e^x$, then $f'(0) = 1$. Using the limit definition of the derivative, this means

$$1 = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

From this can deduce that...

So now we know that

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

That is e^x is the function that is its own derivative!!!

Example

Compute the derivative of $f(x) = e^{\sin(x)}$

Exercises

Exercise 4.3.1

Find the derivative of

1. $f(x) = x^3 \cos(x^2)$
2. $f(x) = \tan^3\left(\sin^2(x^4)\right)$

Exercise 4.3.2

Show that $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$.

Exercise 4.3.3

Find the equation of the tangent to the curve defined by $x^2 - y^2 = 16$ at the point $(5, 3)$.