2526-MA140: Week 02, Lecture 3 (L06)

Limits; The Squeeze Theorem

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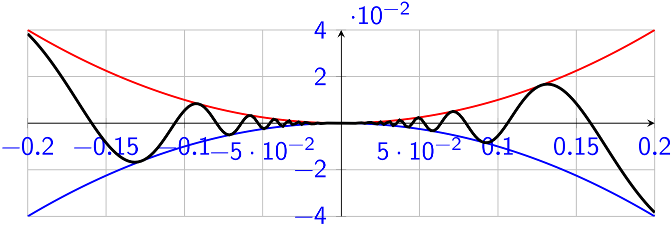


Outline

For more, see Chapter 2 (Limits) of Strang and Herman's Calculus, especially Section and 2.3 (Limit Laws).

Slides are on canvas, and at niallmadden.ie/2526-MA140

Recall... Limits

Yesterday, we learned that

$$\lim_{x \to a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

Properties of Limits

We finish with the following "Limit Laws": *Suppose that*

$$\lim_{x\to a} f_1(x) = L_1 \qquad \text{and} \qquad \lim_{x\to a} f_2(x) = L_2,$$

and $c \in \mathbb{R}$ *is any constant.* Then,

(1) $\lim_{x\to a} c = c, \; c \in \mathbb{R}$

(2) $\lim_{x\to a} x = a$

(3) $\lim_{x\to a} [c f_1(x)] = cL_1$

(4) $$\lim_{x\to a} [f_1(x) + f_2(x)] = L_1 + L_2$$
and
$$\lim_{x\to a} [f_1(x) - f_2(x)] = L_1 - L_2$$

(5) $\lim_{x\to a} \big(f_1(x) f_2(x)\big) = L_1 L_2$

(6) $\lim_{x\to a} \big((f_1(x))^n\big) = (L_1)^n$

(7) $\lim_{x\to a} \left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$, providing $L_2 \neq 0$.

(8) $\lim_{x\to a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$

Evaluating limits

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\boxed{\lim_{x \to a} x^n = a^n}$$

Example

Evaluate the limit $\lim_{x \to 1} (x^3 + 4x^2 - 3)$

Evaluating limits

Example

Evaluate $\lim_{x \to 1} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ using the Properties of Limits.

Limits of rational functions

In many cases, evaluating limits is more complicated. In particular, we'll consider numerous examples where we want to evaluate $\lim_{x \to a} f(x)$ where a is not in the domain of f.

A typical example of this is when we evaluate a rational function:

$$\boxed{\lim_{x \to a} \frac{p(x)}{q(x)}}$$

where **both** $p(a) = 0$ and $q(a) = 0$.

Idea: Since we care about the value of p and q **near** $x = a$, but not actually at $x = a$, it is safe to factor out an $(x - a)$ term from both.

Limits of rational functions

Three examples

Evaluate the limits:

$$(a)\ \lim_{x\to 0}\frac{x}{x} \qquad (b)\ \lim_{x\to 0}\frac{x^2}{x} \qquad (c)\ \lim_{x\to 0}\frac{x}{x^2}$$

Limits of rational functions

Example

Evaluate the limit
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

Limits of rational functions

In that last example, we found that

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x}$$

But these are different functions:

Limits of rational functions

Evaluate the limit

$$\lim_{x\to 2}\left(\frac{\frac{1}{2}-\frac{1}{x}}{x-2}\right)$$

Completing the square

Very often, we'll evaluate limits of the form:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

where f and g are not polynomials. Some of the same ideas still apply.

Example

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

Completing the square

The Squeeze Theorem

There are various approaches to evaluating limits, including...

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that we have three functions f, g and h on some interval $[x_0, x_1]$, with

$$g(x) \leqslant f(x) \leqslant h(x),$$

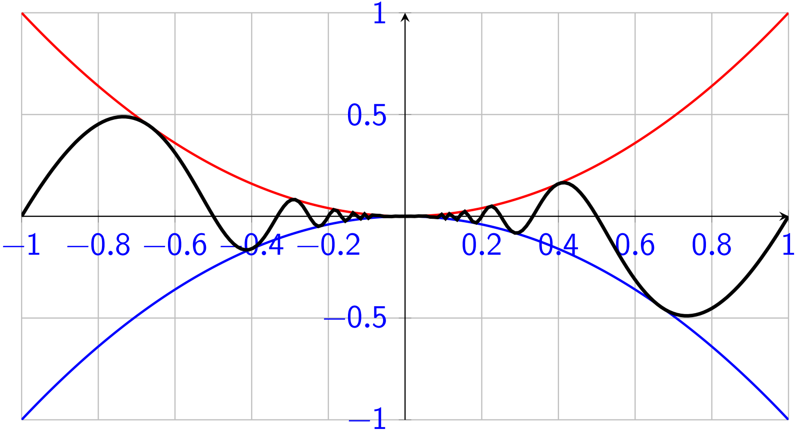
and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L,$$

for some $a \in [x_0, x_1]$. Then $\boxed{\lim_{x \to a} f(x) = L.}$

That is: if $f(x)$ and $g(x)$ have the same limit as $x \to a$, and $h(x)$ is "squeezed" between them, then $h(x)$ has the same limit as $x \to a$.

The Squeeze Theorem



The Squeeze Theorem

Example

Suppose $f(x)$ is a function such that

$$1 - \frac{x^2}{4} \leqslant f(x) \leqslant 1 + \frac{x^2}{2}, \ \forall x \neq 0$$

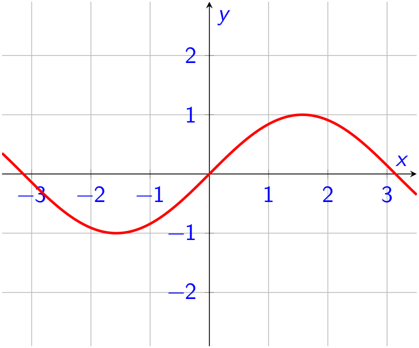
Find $\lim_{x \to 0} f(x)$.

The Squeeze Theorem

Next week, we will use the Squeese Theorem to explain an important limit:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$$

For now, let's just convince ourselves:



Exercises

Exercise 2.3.1

Evaluate the following limits

(a) $\displaystyle\lim_{x\to\frac{1}{2}} \frac{x-\frac{1}{2}}{x^2-\frac{1}{4}}$

(b) $\displaystyle\lim_{x\to -4} \frac{x^2+3x-4}{x^2+x-12}$

Exercise 2.3.2

(From 2023/2024 MA140 exam, Q1(a)) Evaluate the limit

$$\lim_{x\to 4} \frac{x-4}{(\sqrt{x}-2)(x+9)}$$

Exercises

Exercise 2.3.3

Suppose that $g(x) = 9x^2 - 3x + 1/4$, and $f(x)$ is such that $-g(x) \leqslant f(x) \leqslant g(x)$ for all x.

1. Can one use the Squeeze Theorem to determine $\lim_{x \to 1/3} f(x)$? If so, do so. If not, explain why.
2. Can one use the Squeeze Theorem to determine $\lim_{x \to 1/6} f(x)$? If so, do so. If not, explain why.

Exercise 2.3.4 (from 2425-MA140 exam)

Let $f(x) = \dfrac{x^2 - 2x - 15}{3x^3 - 6x^2 - 45x}$. For each of the following, evaluate the limit, or determine that it does not exist.

(i) $\lim_{x\to -3} f(x)$ (ii) $\lim_{x\to 0} f(x)$ (iii) $\lim_{x\to 5} f(x)$