3.0 Started 23/02/2024

MA378 Chapter 3: Numerical Integration

# §3.3 Precision and Composition

Dr Niall Madden

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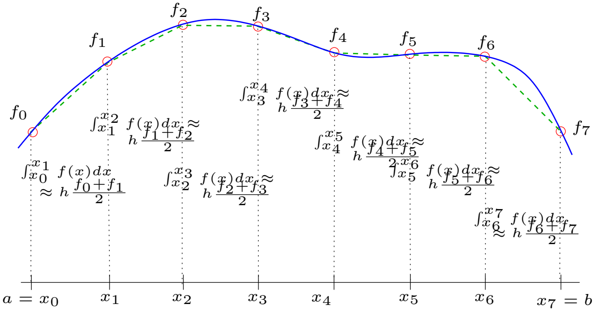

# 3.1 Simpson's, again

In the final example of Section 3.2 ("Simpson's Rule") we noticed that $\boxed{\text{Simpson's Rule}}$ yields exactly the right answer (i.e., the error is zero) when applied to approximating $\int_0^1 x^3 dx$.

In fact, Simpson's Rule is exact for **any** polynomial of degree 3 or less. We'll now explain why, restricting our focus to the interval $[-1, 1]$. (See below for the general case).

1. Suppose that $Q_2(f)$ approximates $\int_{-1}^1 f(x)dx$.
2. We know that, if $p_2$ is any quadratic polynomial, then $Q_2(p_2) = \int_{-1}^1 p_2(x)dx$.
3. Write $p_3$ as

$$p_3(x) = c_3x^3 + c_2x^2 + c_1x + c_0 = c_3x^3 + p_2(x),$$

for some quadratic $p_2(x)$.

# 3.1 Simpson's, again

4. Note that $\displaystyle\int_{-1}^{1} p_3(x)dx = c_3 \int_{-1}^{1} x^3 dx + \int_{-1}^{1} p_2(x)dx.$ $\because$ (i) + (ii)
5. However, $\int_{-1}^{1} x^3 dx = 0$.
6. Therefore, $\displaystyle\int_{-1}^{1} p_3(x)dx = \int_{-1}^{1} p_2(x)dx.$
7. Similarly, $Q_2(p_3)dx = c_3 Q_2(x^3) + Q_2(p_2)$. $\because$ (iii) & (iv)
8. However, $Q_2(x^3) = \frac{2}{6}\big((-1)^3 + 4(0^3) + 1^3\big) = 0$.
9. Therefore $\displaystyle Q_2(p_3)dx = Q_2(p_2) = \int_{-1}^{1} p_2(x)dx.$
10. So, we have that

$$\int_{-1}^{1} p_3(x)dx = \int_{-1}^{1} p_2(x)dx = Q_2(p_3).$$

**Conclude:** $Q_2(\cdot)$ gives exactly the right answer when applied to estimating $\int_{-1}^{1} p_3(x)dx$ for any cubic polynomial, $p_3$.

*Here are the details for the general case. We won't go though this in class, but please read it carefully.*

We now claim that $\boxed{\text{Simpson's Rule}}$ is exact for *any* polynomial of degree 3 or less, and on *any* interval.

Denote by $\boxed{Q_2(f)}$ the approximation of $\int_a^b f(x)dx$ with Simpson's Rule:

$$Q_2(f) = \frac{b-a}{6}\Big(f(a) + 4f\big(\frac{a+b}{2}\big) + f(b)\Big).$$

Since the method can be derived by integrating the quadratic that interpolates $f(x)$ at the three points $a$, $(a+b)/2$, and $b$, it is clearly exact for all quadratics.

Let $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Then

$$\int_a^b f(x)dx = \int_a^b \left(c_0 + c_1 x + c_2 x^2\right)dx + c_3 \int_a^b x^3 dx = \int_a^b \left(c_0 + c_1 x + c_2 x^2\right)dx + c_3 \frac{b^4 - a^4}{4}.$$

Also,

$$Q_2(f) = Q_2(c_0 + c_1 x + c_2 x^2) + Q_2(c_3 x^3) = \int_a^b \left(c_0 + c_1 x + c_2 x^2\right)dx + c_3 \left(\frac{b-a}{6}\right)\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right). \quad (1)$$

With a bit of symbolic manipulation we get that

$$\frac{b^4 - a^4}{4} = (\frac{b-a}{6})\left(a^3 + 4(\frac{a+b}{2})^3 + b^3\right),$$

as required.

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In Section 3.2, we discussed a nïeve attempt to derive an upper bound for the error in Simpson's rule leading to

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3.$$

This is not wrong – just not sharp. For example it does not give that Simpson's Rule is exact for all cubic. The sharp result is

**Theorem 3.1**

$$\mathcal{E}_2 := \left| \int_a^b f(x)dx - Q_2(f) \right| \leq \frac{(b-a)^5}{2880} M_4.$$

For the proof see the text book (Theorem 7.2 of Suli and Mayers). Instead of working through it in class we'll prove a more general version of a consequence it.

# 3.2 Precision

## Definition 3.2 (Precision of a Quadrature Rule)

A quadrature rule has **precision** $n$ if it is exact for all polynomials of degree $n$ or less. That is, the rule $Q(f)$ has precision $n$ if

$$Q(p_n) = \int_a^b p_n(x)dx \quad \text{for all } p_n \in \mathcal{P}_n.$$

## Example 3.3

By construction, the $(N + 1)$-point Newton-Cotes rule has precision $N$

This is because $Q_N(f)$ is defined to be $Q_N(f) = \int_a^b p_N(x)dx$ where $p_N$ is the Lagrange interpolant to $f$.

# 3.2 Precision

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## Example 3.3

By construction, the $(n+1)$-point Newton-Cotes rule has precision $n$.

Eg, $Q_1(f)$ (ie Trapezium Rule) has precision 1.
$Q_2(f)$ (Simpson's Rule) is designed to have precision 2, but in fact, has precision 3.

 That is $Q_N(\cdot)$ for even $N$.

**Theorem 3.4**

*If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k+1$ points, then $Q_{2k}(\cdot)$ has in fact precision $2k+1$.*

Important: $n = 2k$

**Proof:** Let $p_{n+1}$ be a polynomial of degree $n+1$. We wish to show that $Q_n(p_{n+1}) = \int_a^b p_{n+1}(x)dx$. We can take $a=-1$, $b=1$ because a simple linear transformation can be used to map to an arbitrary interval.

Also, since the quadrature points are equally spaced on $[-1,1]$ we have that $x_i = -x_{n-i}$.

Furthermore (see Exercise 3.1) the quadrature weights are symmetric: $q_i = q_{n-i}$.

Hint: $q_i = \int_{-1}^{1} L_i(x)\,dx$ where $L_i$ is a Lagrange Poly.

# 3.2 Precision

Note that

$$\int_{-1}^{1} p_{n+1}(x)\,dx = \int_{-1}^{1} p_n(x) + a_{n+1}\,x^{n+1}\,dx$$

$$= \underbrace{\int_{-1}^{1} p_n(x)\,dx}_{=\,\underline{Q_n}(p_n)} + a_{n+1}\underbrace{\int_{-1}^{1} x^{n+1}\,dx}_{=\,0}$$

$$= \underline{Q_n}(p_n) + 0 \quad \text{since } n+1 \text{ is odd}$$

Similarly

$$Q_n(p_{n\cancel{+1}}) = \underbrace{Q_n(p_n)}_{=\,Q_n(p_n)} + a_{n+1}\underbrace{Q(x^{n+1})}_{=\,0}$$

$$= Q_n(p_n) + 0$$

why? See board!

# 3.3 Composite Rules

Suppose that we want to estimate $\int_a^b f(x)dx$ and the Trapezium rule is not sufficiently accurate. We could try Simpson's Rule, which should be better. Failing that, we could try a 4-point rule, based on integrating the $p_4$ interpolant, or a five-point rule, based on integrating $p_5$.

However, quite apart from the fact that it might be tedious to derive these rules, we know (Runge's example again!) that high-order polynomial interpolation can be very inaccurate.

# 3.3 Composite Rules

It is better to use a **Composite Rule**. This is analogous to the idea behind piecewise linear interpolation.

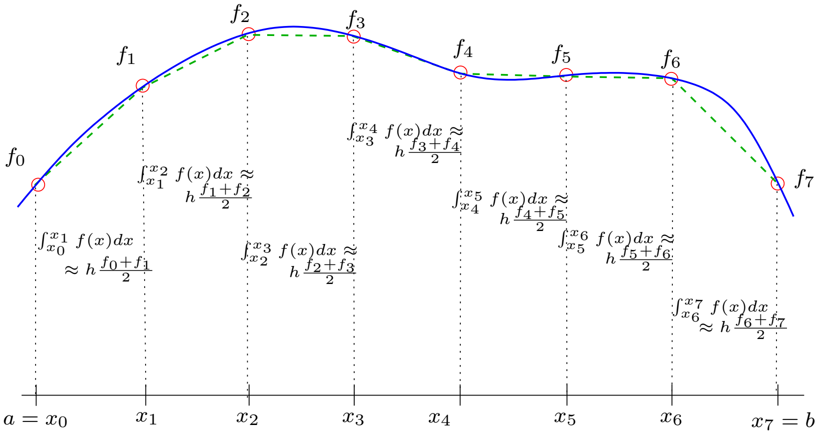
For the **Composite Trapezium Rule** we divide $[a, b]$ into $N$ intervals of size $h = (b-a)/N$. Applying the Trapezium Rule on each interval $[x_{i-1}, x_i]$ we get

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx h\frac{f_{i-1} + f_i}{2}.$$

Summing for the $n$ intervals we get

$$\int_a^b f(x)dx \approx \underbrace{\frac{b-a}{N}}_{= h}\left(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-1} + \frac{f_N}{2}\right). \qquad (2)$$

# 3.3 Composite Rules




Similarly, one can define a composite Simpson's Rule, etc

# 3.4 Exercises

## Exercise 3.1

Explain clearly, with an example, why in general it is not true that $Q_n(f) \to \int_a^b f(x)dx$ as $n \to \infty$.

## Exercise 3.2

(i) Deduce an error estimate for the Composite Trapezium Rule (2).

(ii) Taking $N = 10$, give an upper bound for the error in the Composite Trapezium Rule when approximating $\int_1^2 \ln(x)dx$.

(iii) What value of $n$ would you have to take to ensure that the error was less that $10^{-5}$?

## Exercise 3.3

(i) Deduce the formula for the *composite Simpson's Rule*.

(ii) Derive an error estimate for the *composite Simpson's Rule*.

(iii) What value of $N$ would you have to take to ensure that the error in the estimate of $\int_1^2 \ln(x)dx$ is less that $10^{-6}$?

(iv) Denote the $(N+1)$-point Composite Simpson's Rule by $S_N(f) \approx \displaystyle\int_a^b f(x)dx$. Show that, for sufficiently smooth $f(x)$,

$$\lim_{n\to\infty} S_N(f) = \int_a^b f(x)dx.$$

## Exercise 3.4

Determine the precision of the following schemes for estimating $\int_0^1 f(x)dx$.

(i) $Q(f) = f(\frac{1}{2})$.

(ii) $Q(f) = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.

(iii) $Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{\mathbf{3}})$.

## Exercise 3.5 ($\star$)

Consider the rule:

$$R(f) = q_0 f(1/3) - f\Big(\frac{1}{2}\Big) + q_2 f\Big(\frac{3}{4}\Big)$$

for approximating $\int_0^1 f(x)dx$.

1. Determine values of $q_0$ and $q_2$ that ensure this rule has precision 2.
2. What is the maximum precision of $R(\cdot)$ with the values of $q_1$ and $q_2$ that you have determined?
3. Why is this not, strictly speaking, a Newton-Cotes rule?