2425-MA140 Engineering Calculus

Week 04, Lecture 3
The Chain Rule and Inverse Functions

Dr Niall Madden

School of Maths, University of Galway

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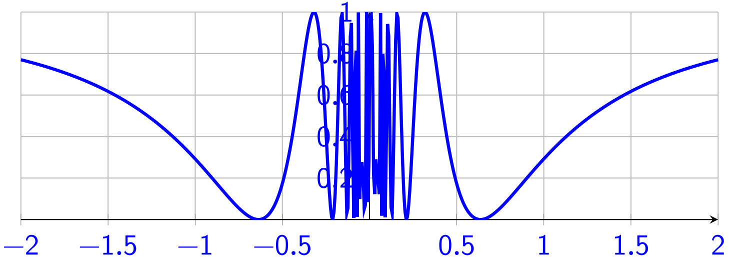
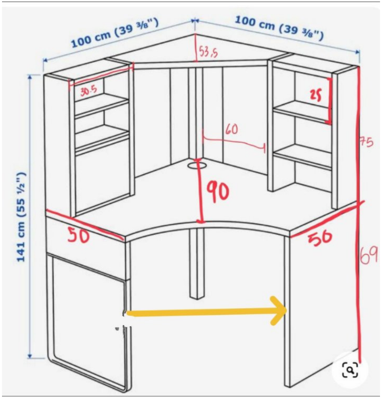


Assignments, etc

Assignments

- **Assignment 2** is open. See https://universityofgalway.instructure.com/courses/35693/assignments/96620.
  Deadline is 5pm, Friday, 11 October.
- The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2065926
- **Assignment 3** opens tomorrow morning.

Warm-up



"Olive" is thinking of buying this desk unit in IKEA. Her (wheel)chain is 55cm. Is the sitting region of the desk indicated by the yellow line, wide enough?

What we'll do today:

See also:

- Sections 3.6 (The Chain Rule) and 3.8 (Implicit Differentiation) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)
- Section 8.3 of *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Chain Rule

Of all the differentiation rules, the **chain rule** is the most important: most other rules are actually just special cases of it. It applies to a "function of a function"

The Chain Rule

If $u(x)$ and $v(x)$ are differentiable, and f is the composite function $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

Example: What is the derivative of $f(x) = \cos(x^2)$?

First note that this is a composite function...

Chain Rule

The Chain Rule

If $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

Example: What is the derivative of $f(x) = \cos(x^2)$?

Chain Rule

Example

Find $\frac{dy}{dx}$ if $y = (x^3 + 4x^4 + 7)^{12}$.

Example: Let $u(v) = v^{12}$ and $v(x) = x^3 + 4x^4 + 7$, then y is $y = u\big(v(x)\big)$.

Note that

$$\frac{du}{dv} = 12v^{11} \quad \text{and } \frac{dv}{dx} = 3x^2 + 16x^3.$$

By the Chain Rule we have

$$\frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dx} = 12v^{11}(3x^2 + 16x^3),$$

and therefore

$$\frac{dy}{dx} = 12(x^3 + 4x^4 + 7)^{11}(3x^2 + 16x^3).$$

Chain Rule

Example (Skimmed this in class)

Find $\dfrac{dy}{dx}$ if $y = \dfrac{1}{(x^4 + 2x^2 + 8)^{40}}$.

We have $y = (x^4 + 2x^2 + 8)^{-40}$. We can write y as $y(x) = u(v(x))$ with

- $u(v) = v^{-40}$ and so $\dfrac{du}{dv} = -40v^{-41}$; and
- $v(x) = x^4 + 2x^2 + 8$, so $\dfrac{dv}{dx} = 4x^3 + 4x$.

Applying the Chain Rule: $\dfrac{dy}{dx} = \dfrac{du}{dv}\dfrac{dv}{dx}$. we get

$$\frac{dy}{dx} = -40v^{-41}(4x^3 + 4x) = \frac{-40(4x^3 + 4x)}{(x^4 + 2x^2 + 8)^{41}}$$

Often we apply the **Chain Rule** to "functions of functions of functions": if $y(x) = t\Big(u\big(v(x)\big)\Big)$, then

$$\frac{dy}{dx} = \frac{dt}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$ when $y = \sin^4(x^5 + 7)$.

Example (changed from draft version)

Show that the derivative of $y = \cos^2(1/x)$ is $\dfrac{dy}{dx} = 2\dfrac{\sin(1/x)\cos(1/x)}{x^2}$

Inverse functions

Suppose that $y = f(x)$. That is, f maps x to y.

Then the **inverse** of f is the function, f^{-1}, that maps y back to x.

Example

- The inverse of $f(x) = \frac{1}{2}x$ is $f^{-1}(x) = 2x$.
- The inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$.

Warning: $f^{-1}(x)$ is not the same as $\dfrac{1}{f(x)}$.

It is often useful to be able to express the derivative (assuming there is one) of an inverse function $f^{-1}(x)$ in terms of the derivative of $f(x)$.

To do this, we use the following rule:

Inverse-Function Rule

If $y = f^{-1}(x)$, then $x = f(y)$ and also

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}.$$

Alternatively: If f and f^{-1} are inverse and differentiable, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example

If $y = x^{1/3}$, use the Inverse Rule to find $\frac{dy}{dx}$.

Note: We can solve this just using the **Power Rule:** $\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$. So we are just doing this because it is *instructive* If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2\,.$$

By the inverse rule, $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}\,.$$

As $y = x^{\frac{1}{3}}$ we have

$$\frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}}\,.$$

Example

Find the derivative of $\sin^{-1}(x)$

Let $y = \sin^{-1}(x)$, then $x = \sin(y) \quad (\star)$, so

$$\frac{dx}{dy} = \cos(y)\,. \qquad (\star\star)$$

From $\sin^2(y) + \cos^2(y) = 1$, we find $\cos(y) = \sqrt{1 - \sin^2(y)}$ (choosing the positive square root as $\cos(y)$ is positive for y here).
Using $(\star)$:

$$\cos y = \sqrt{1 - x^2}\,.$$

Now using the inverse rule and $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}}\,.$$

Exercises

Exercise 4.3.1

Find the derivative of

1. $f(x) = x^3 \cos(x^2)$
2. $f(x) = \tan^3\left(\sin^2(x^4)\right)$

Exercise 4.3.2

Show that $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$.