MA313 : Linear Algebra I

Week 8: The 4 Fundamental Subspaces

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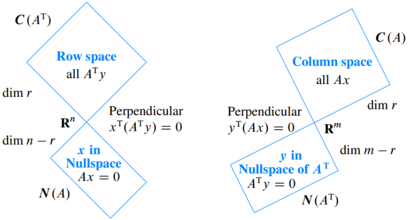
25th and 28th of October, 2022



Taken from Gil Strang's "The Four Fundamental Subspaces" https://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf

These slides were produced by Niall Madden, based on ones by Tobias Rossmann.

Outline

For more details, see Sections 4.5 (Dimension of a Vector Space) and 4.5 (Rank) of the text-book. To access that online:

- Go to `https://nuigalway-primo.hosted.exlibrisgroup.com/permalink/f/1pmb9lf/353GAL_ALMA_DS5192067630003626`
- Go to "**Full text available at:**" and click the link.
- That will redirect to Blackboard. Login using the 3rd Party STO.

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PART 1: Announcements and Preview of Week 8

Some reminders on assignments, and a very short review of material to date that are needed this week.

Assignment 4

Assignment 4 was posted last week. Deadline is 5pm, Monday, 7th November.

Upload you solutions, in PDF, to blackboard. If you prefer, you can give them to me in class on Tuesday, 8th Nov.

Communication Skills : Progress Report

Thanks for the progress reports. Jim and I will give feedback, and update you on the next steps next week.

~~The big ideas from this week will be~~

- ~~Row Rank = Column Rank~~
- ~~Matrices of Linear Transformations~~
- ~~Towards change of basis~~

- A **vector space** consists of a set V together with operations $+$ (vector addition) and $\cdot$ (scalar multiplication) that resemble the usual operations in $\mathbb{R}^n$.
- A **basis** of a vector space V is a sequence $(v_1, \ldots, v_n)$ of vectors which is linearly independent and which spans V.
- The **dimension** of V is the number of vectors in any basis for V.
- Isomorphic vector spaces have the same dimension.
- The **COLUMN SPACE** of A is the space spanned by the vectors that make up its columns. That is, it is the space of all vectors v, such that $Ax = v$ for any vector x. Note that $x \in \mathbb{R}^m$.
- The **RANK** of a $m \times n$ matrix, A, is the dimension of its column space: $\operatorname{rank} A := \dim \operatorname{Col} A$. It is equal to the number of pivot columns in the reduced row echelon form of A.

- The **NULL SPACE** of a $m \times n$ matrix, A, is the set of all vectors such that $Ax = 0$. Note that $x \in \mathbb{R}^n$.
- The **NULLITY** of A is the dimension of its null space: $\operatorname{nullity} A := \dim \operatorname{Nul} A$.
- (Rank-Nullity Theorem) $\operatorname{rank} A + \operatorname{nullity} A = n$.

ie $(\dim \operatorname{Col} A) + (\dim \operatorname{Nul} A) = n$

$=$ number of cols of A.

Invertible Matrix Theorem

Let A be an $n \times n$ matrix.
Then the following are equivalent:

1. A is invertible, i.e. there exists an $n \times n$ matrix B such that $AB = I_n = BA$. (We usually write B as A^{-1}.
2. $\mathrm{rank}\, A = n$.
3. $\mathrm{nullity}\, A = 0$.
4. The columns of A form a basis of $\mathbb{R}^n$.

Suppose we have a linear system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Here the (a_{ij}) and (b_i) are given, and we want to solve for the vector x.

We write this system as

$$Ax = b$$

.

Sometimes here is a solution to this equation, and sometimes not...

Another view of the column space

There is a solution to $Ax = b$ exactly when $b \in \operatorname{Col} A$.

This links directly to the idea of the **rank** of A.

- If A is a $n \times n$ matrix, of rank n then, as we just saw, the matrix A^{-1} exists. Therefore, $Ax = b$ has a solution: it is $x = A^{-1}b$.
- Alternatively, if A is a $n \times n$ matrix, of rank n then its columns form a basis for $\mathbb{R}^n$. So, in fact $\operatorname{Col} A = \mathbb{R}^n$. So for every possible b, there is a solution to $Ax = b$.
- Suppose $\operatorname{rank} A < n$, and $b \in \operatorname{Col} A$. Then there are multiple solutions to $Ax = b$...

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PART 2: Rank of the transpose

Part 2: Rank of the transpose

Definition (TRANSPOSE)

Let A be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix, denoted $A^\top$, whose (j, i)-entry is the (i, j)-entry of A.

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Also $(A^T)^T = A$

If $A = A^T$ we say A is symmetric.

Eg $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is symetric. Note if A is symmetric, then A is square $(m = n)$,

Part 2: Rank of the transpose

Here is a very important result, though we are not going to prove it...

Theorem

$$\operatorname{rank} A = \operatorname{rank} A^\top.$$

This is surprising since it is true for any $m \times n$ matrix, even with $m \neq n$.

Also $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$,

but $\mathrm{Col}(A^T)$ is a subspace of $\mathbb{R}^n$.

Part 2: Rank of the transpose

Example

Find the rank of A and $A^\top$, when $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$

Rank (A) = number of pivot columns of the reduced row echelon form of A. To compute that:

$$\begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 1 & 0 & -1/3 \\ 1 & 0 & 5/4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 19/12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 19/12 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Two pivot cols, so Rank $(A) = 2$.

Example

Find the rank of A and $A^\top$, when $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$

Similarly $A^T = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & 0 \\ -1 & -1 & 5 \end{bmatrix}$.

The reduced Row echelon form of A^T is

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Again 2 pivot Cols.

So $\text{Rank}(A^T) = 2$.

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PART 3: Row space

Part 3: Row space and Left Null Space

So far, we have always thought of vectors in $\mathbb{R}^n$ is being column vectors. But it is just as reasonable to think to them as *row* vectors:

We think of vectors in $\mathbb{R}^n$ as column vectors.

Eg $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^4$. But we could write

them as row vectors, eg $[0, 1, 2, 3]$.

With this view we can think of a matrix consisting of row vectors.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \underbrace{\left[\begin{array}{c:c} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}\right]}_{\text{Cols.}} = \underbrace{\left[\begin{array}{cc} 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \end{array}\right]}_{\text{Rows.}}$$

Part 3: Row space and Left Null Space

Definition (ROW SPACE)

The **ROW SPACE** of an $m \times n$ matrix A is the space spanned by the (row) vectors that make up its rows.
That is, it is the space of all (row) vectors $v \in \mathbb{R}^n$, such that $xA = v$ for any (row) vector $x \in \mathbb{R}^m$.

Eg $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad x = [x_1, \; x_2, \; x_3]$.

Then $\quad xA = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} =$

$$[x_1 + 3x_2 + 5x_3 \, , \quad 2x_1 + 4x_2 + 6x_3]$$

$$\Rightarrow \quad x_1[1, 2] + x_2[3, 4] + x_3[5, 6].$$

row vector

 $(AB)^T = B^T A^T$

Definition (ROW SPACE)

The **ROW SPACE** of an $m \times n$ matrix A is the space spanned by the (row) vectors that make up its rows.
That is, it is the space of all (row) vectors $v \in \mathbb{R}^n$, such that $xA = v$ for any (row) vector $x \in \mathbb{R}^m$.

So Row A = span (rows of A).

It is the set of row vectors that can be written as $xA = v$.

But, if we prefer column vectors, this is

$$(x^T A)^T = v^T$$

$$\Rightarrow A^T x = v^T.$$

Part 3: Row space and Left Null Space

From what we learned in Part 2, we know that the dimension of the row space is always the same as the dimension of the column space.

For proof, see video!

(Finished here Wed)

Part 3: Row space and Left Null Space

We know that the vector x is in the null space of A if $Ax = 0$. But now that we know we can multiply a matrix on the left by a (row) vector, we have the idea of...

Definition (LEFT NULL SPACE)

The *Left Null Space* of a matrix, A is the space of row vectors x for which $xA = 0$. Equivalently, it is the set of row vectors x whose transpose is in the (usual) null space of $A^\top$.

If we prefer to think just in terms of column vectors then

$x^T A = 0^T$ $\quad$ x is a col vector

$\Rightarrow (x^T A)^T = (0^T)^T \Rightarrow A^T (x^T)^T = 0$

$\Rightarrow A^T x = 0$

Part 3: Row space and Left Null Space

Example

Show that $x = [1, -2, 1]$ is in the left null space of $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$.

$$[1, -2, 1] \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = [1-2+1, \quad 1-4+3]$$

$$= [0, 0].$$

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PART 4: More About Rank

Example, from 2018/2019 exam paper

Q2(c) Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p, q \in \mathbb{R} \right\} \text{ of } \mathbb{R}^4.$$

Q2(d) Show that $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right)$ is a basis of $\mathbb{R}^3$.

Moreover, find the coordinate vector of $y = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ relative to $\mathcal{B}$.

Find the dimension of the subspace $H = \left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p, q \in \mathbb{R} \right\}$ of $\mathbb{R}^4$.

Any vector in H can be written as

$$p \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix} + q \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

So $H = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. Since these vectors are not multiples of each other, they are linearly indep, so are a basis.

Find the dimension of the subspace $H = \left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p, q \in \mathbb{R} \right\}$ of $\mathbb{R}^4$.

What is the dimension of H?

Sime there ore $\underline{2}$ vector is the basis, $\dim(H) = 2$.

Show that $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right)$ is a basis of $\mathbb{R}^3$. Moreover, find the coordinate vector of $y = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ relative to $\mathcal{B}$.

Augmented matrix is $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 8 & 3 & -2 \end{array} \right)$

Row reduce to get (details)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 8 & 3 & -2 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

Show that $\mathcal{B} = \left(\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\8\end{bmatrix}, \begin{bmatrix}1\\-1\\3\end{bmatrix} \right)$ is a basis of $\mathbb{R}^3$. Moreover, find the coordinate vector of $y = \begin{bmatrix}0\\0\\-2\end{bmatrix}$ relative to $\mathcal{B}$.

Since this has 3 pivots it means $\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\8\end{bmatrix}, \begin{bmatrix}1\\-1\\3\end{bmatrix}$ are linearly independent, and so are a basis. Also $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is $y_{\mathcal{B}}$

Check $(1)\begin{bmatrix}1\\1\\3\end{bmatrix} + (-1)\begin{bmatrix}2\\0\\8\end{bmatrix} + (1)\begin{bmatrix}1\\-1\\3\end{bmatrix} = \begin{bmatrix}1-2+1\\1-1\\3-8+3\end{bmatrix} = \begin{bmatrix}0\\0\\-2\end{bmatrix}$

What are the possible ranks of an $m \times n$ matrix A?

- Because $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$, we have

$$\mathrm{rank}\, A = \dim\ \mathsf{Col}\ A \leqslant \dim \mathbb{R}^m = m.$$

- On the other hand,

$$\mathrm{rank}\, A = \mathsf{rank}\ A^\top = \mathsf{dim}\, \mathrm{Col}\, A^\top \leq \mathsf{dim}\, \mathbb{R}^n = n.$$

Hence: $\mathrm{rank}\, A \leq \min(m, n)$.

$$A = \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{n \text{ cols.}} \Big\} \, m \text{ rows}$$

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Hence: $\text{rank}\, A \leq \min(m, n)$.

Note: can think of a vector as a rank 1 matrix.

Example (From Q3(c), 2018/2019 exam paper)

(i) What is the largest possible rank of a 4×7 matrix?

(ii) What is the largest possible rank of a 7×4 matrix?

(iii) If the null space of a 4×7 matrix A is 3-dimensional, what is the dimension of its column space?

(i) $\text{Rank}(A) \leq \min(m, n) = \min(4, 7) = \underline{\underline{4}}$

(ii) Also 4.

(iii) $\dim(\text{Col } A) + \dim(\text{Nul } A) = n$.

So $\dim(\text{Col } A) + 3 = 7$.

So $\dim(\text{Col } A) = 4$.

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PART 5: The 4 Fundamental Subspaces

5: The 4 Fundamental Subspaces

Given an matrix $A \in \mathbb{R}^{m \times n}$ we can define four subspaces, two associated with $\mathbb{R}^m$, and two with $\mathbb{R}^n$:

- The **column space** of A, denoted $\operatorname{Col} A$, which is the space of all vectors $b = Ax$ for some $x \in \mathbb{R}^n$. It is a subspace of $\mathbb{R}^m$.
- The **null space** of A, denoted $\operatorname{Nul} A$, which is the space of all vectors x for which $Ax = 0$. It is a subspace of $\mathbb{R}^n$.
- The **row space** of A, denoted $\operatorname{Row} A$, which is the space of all vectors b for which $A^\top y = b$ for some $y \in \mathbb{R}^m$. It is a subspace of $\mathbb{R}^n$.
- The **null space** of $A^\top$, denoted $\operatorname{Nul} A^\top$, which is the space of all vectors x for which $A^\top x = 0$. It is a subspace of $\mathbb{R}^m$.

5: The 4 Fundamental Subspaces

There is a celebrated theorem, known as the *Fundamental Theorem of Linear Algebra* which links these spaces. It has 3 parts.

Fundamental Theorem: Part 1

- The dimensions of the column space and row space are both. equal to r, the rank of A.
- $\dim \operatorname{Nul} A = n - r$
- $\dim \operatorname{Nul} A^\top = m - r$

We already know these facts. **Part 2** is new (to us), but is related to the idea of *orthogonality*, which is the next big topic.

Part 3 concerns the Singular Value Decomposition, which (unfortunately) is beyond this module.

Exercises

1. Find the rank of $\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$.
2. If the null space of a 4×6 matrix A is 3-dimensional, what is the dimension of the column space of A?
3. If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A?
4. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A?
5. Find the matrix of the linear transformation

$$T : \mathbb{P}_2 \to \mathbb{P}_2, \quad p(t) \mapsto p(t) + p'(t) + p''(t)$$

relative to the basis $(1, t, t^2)$ of $\mathbb{P}_2$.