MA378 Chapter 1: Polynomial Interpolation

Submit carefully written solutions to Exercise $4.7\star$

Deadline: 5pm, Friday 10 February.

Your solutions must be clearly written, and neatly presented. You can submit an electronic copy, through blackboard, or a hard copy. If submitting a hard copy, please do so at the 10am lecture in the 10th. Also, make sure pages should be stapled together. Marks will be given for quality and clarity of exposition. Collaboration policy will be discussed in class.

Exercise 1.1. Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p+q$?

(b) What is the minimum possible degree of $p-q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

Exercise 1.2. Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space.

Exercise 1.3. (a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

Exercise 2.1. The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{0 \leqslant i < j \leqslant n} (x_j - x_i). \qquad (2.0.1)$$

Verify this for the 2×2 and 3×3 cases.

(Note that from Formula (2.0.1) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \ldots, x_n$ are all distinct.)

Exercise 2.2. Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute

$$\max_{0 \leqslant x \leqslant a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayers.

Exercise 2.3. Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \quad \text{for all } x.$$

Exercise 2.4. Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0, 3)$, $(1, 2)$ and $(2, 4)$.

Source: Chapter 2 of Stoer and Bulirsch.

Exercise 2.5. Show that all the following represent the same polynomial (usually called the "Chebyshev Polynomial of Degree 3"), $T_3(x) = 4x^3 - 3x$.

(a) Horner form: $\big((4x + 0)x - 3\big)x + 0$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3} \Bigg(\prod_{j=0, j \neq k}^{3} \frac{x - x_j}{x_k - x_j} \Bigg) (-1)^{k+1}$, where $x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$.

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and $T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

(d) Trigonometric form: $T_3(x) = \cos\big(3 \cos^{-1}(x)\big)$.

Exercise 3.1. Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0 \leqslant x \leqslant x_2} |f(x) - p_2(x)| \leqslant \frac{1}{6} \frac{2}{3\sqrt{3}} h^3 M_3 = \frac{1}{9\sqrt{3}} h^3 M_3.$$

Hint: simplify the calculations by taking $t = x - x_1$, *writing* $(x - x_0)(x - x_1)(x - x_2)$ *in terms of* h *and* t.

Exercise 4.1. For *just* the case $n = 1$, state and prove an appropriate version of Theorem 4.2 (i.e., error in the Hermite interpolant). Use this to find a bound for $\|f - p_3\|_{[x_0,x_1]}$ in terms of f and $h = x_1 - x_0$. (Here $\|g\|_{[x_0,x_1]}$ is short-hand for $\max_{x_0 \leqslant x \leqslant x_1} |g(x)|$.)

Exercise 4.2. Let $n = 2$ and $x_0 = -1$, $x_0 = 1$ and $x_1 = 1$. Write out the formulae for H_i and K_i for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 4.3. Do Exercise 6.6 from from Süli and Mayers, *An Introduction to Numerical Analysis*.

Exercise 4.4. Let L_0, L_1, ..., L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \dots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

and

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0.$$

Show that: $K_i(x_k) = 0$, for $k = 0, 1, \dots n$, and $K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$

Exercise 4.5. Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$.

Exercise 4.6. (This exercise is based on Exer 6.5 from Süli and Mayers' *Introduction to Numerical Analysis*). Consider the following problem.

> Take $n + 1$ distinct interpolation points $x_0 < x_1 < \cdots < x_n$. Let p_{2n+1} be the polynomial of degree $2n+1$ with the property that
>
> $$p_{2n+1}(x_i) = f(x_i),$$
>
> and
>
> $$p''_{2n+1}(x_i) = f''(x_i).$$

In general this problem does *not* have a unique problem.

(i) Explain briefly but carefully why the arguments, based on Rolle's Theorem, used to prove **uniqueness** of solutions to the HPIP, will not work here.

(ii) Show that there is no $p_5(x)$ that solves this problem when
- $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.
- $f(-1) = 1$, $f(0) = 0$, $f(1) = 1$.
- $f''(-1) = 0$, $f''(0) = 0$, $f''(1) = 0$.

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Homework exercise

Exercise 4.7 ($\star$). Take $f(x) = x^3$ and $\{x_0, x_1, x_2\} = \{-1, 0, 1\}$.

(a) Write down the Lagrange form of p_2, the polynomial of degree two that interpolates f at x_0, x_1, and x_2. Simplify the expression for $p_2(x)$ as much as possible.

(b) Use Corollary 3.5 to give an upper bound for
$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_2(x)|.$$

(c) Using calculus, give a sharper bound for $|f(x) - p_2(x)|$ on the interval $[-1, 1]$. That is, find the maxima/minima of the function $g(x) = f(x) - p_2(x)$ on $[-1, 1]$, and thus compute exactly
$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_2(x)|.$$

(d) Suppose we have $\{x_0, x_1, x_2\} = \{-a, 0, a\}$ for some number a, which we can choose. What is the largest value of a that can be permitted if we require that
$$\max_{-a \leqslant x \leqslant a} |f(x) - p_2(x)| \leqslant 10^{-3}?$$
You may use the result in Exercise 3.1 (**without** proof).

(e) Write down the formula for the polynomial that is the Hermite interpolant to $f(x) = x^3$ at $x_0 = -1$ and $x_1 = 1$. (Hint: be lazy; you can do this without figuring out what $H_i(x)$ and $K_i(x)$ are).