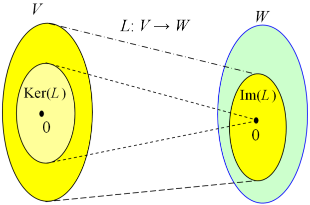
MA313 : Linear Algebra I

Week 3: Spanning set; the Null and Column Spaces

Dr Niall Madden

20 and 23 September, 2022



https://commons.wikimedia.org/wiki/File:KerIm_2015Joz_L2.png.

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details,

- LinAlg for Data Science: Chapter 7 for Linear Independence and Span
- Lay et al: Sections 4.1 and 4.2.

Assignment 1

Deadline is Tuesday, 20 Sept at 5pm.

Assignment 2

- Opened Monday, 19 Sep 2022.
- **Deadline:** 5pm, Friday 30 Sep 2022.
- It contributes 5% to the final grade for MA313.
- Topics: ...

Communication Skills

1. Topics and Info posted on Blackboard. Also at https://www.niallmadden.ie/teaching/2223-MA313/22_23_Communication_Skills.pdf
2. Select one that is not crossed out, or propose one of your own.
3. Confirm your topic by this Friday (23 September); do that by first emailing Niall with your choice and, if agreed, entering in on Blackboard.

Tutorials start this week.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12					
12 – 1				Tutorial IT206	Lecture
1 – 2		Lecture			
2 – 3					
3 – 4					
4 – 5					

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Start of ...

PART 1: Linear combinations

Part 1: Linear combinations

A question

Last week we learned how to check if a given space is indeed a subspace of some other vector space.

It is natural to wonder: *how can we make those subspaces in the first place?*

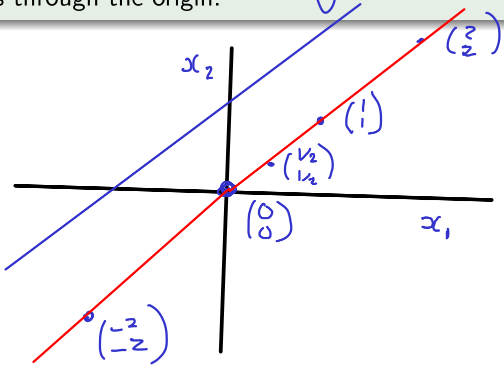
Equivalently: *How can we describe all subspaces of a given vector space?*

Example (Subspaces of $\mathbb{R}^2$)

There are precisely three *types* of subspaces of $\mathbb{R}^2$:

- $\{0\}$,
- $\mathbb{R}^2$,
- lines through the origin.

Points on this line not a vector space.



How we build subspaces?

There are two possible approaches.

- **Top down:** start with the full space, and look at all vectors that have "suitable properties".
- **Bottom up:** start with some collection of vectors and consider the subspace that they "span".

Definition (Linear combinations)

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form

$$c_1 u_1 + \cdots + c_p u_p$$

for scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

Example

In $\mathbb{R}^2$, $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

we want to show $\begin{bmatrix} 2 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for some c_1, c_2.

That is $\begin{bmatrix} c_1 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. So can take $c_1 = 2$ and then $c_2 = -1$.

Example

Show that $\begin{bmatrix}1\\1\end{bmatrix}$ is **not** linear combination of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}-4\\-6\end{bmatrix}$ in $\mathbb{R}^2$.

Suppose there are numbers c_1, c_2 with

$$c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}-4\\-6\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

So $\begin{bmatrix}2c_1 - 4c_2\\3c_1 - 6c_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \begin{matrix}2c_1 - 4c_2 = 1\\3c_1 - 6c_2 = 1\end{matrix} \Rightarrow \begin{matrix}c_1 - 2c_2 = 1/2\\c_1 - 2c_2 = 1/3.\end{matrix}$

But $1/2 \neq 1/3$, so this is not possible.

Example (Quadratic polynomials)

Which vectors in $\mathbb{P}_2$ (over t) are linear combinations of the vectors $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$?

Ans: all of them.

Any poly in $\mathbb{P}_2$ can be written as

$$p_2(t) = c_0 + c_1 t + c_2 t^2$$

$$= c_0 p_0 + c_1 p_1 + c_2 p_2 .$$

Example (Polynomials again)

Which vectors in $\mathbb{P}_2$ (over t) are linear combinations of the vectors $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = 2t$?

Ans: we can make any vector in $\mathbb{P}_1$

$p_1 = c_0 + c_1 t$ ✓

Example

Define the 2×3 matrix

$A = [a_1 \mid a_2 \mid a_3]$

$$A = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \end{array}.$$

For any vector

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

the vector Ax is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a - 3b - 2c \\ -5a + 9b + c \end{bmatrix}$$

$$= a \begin{bmatrix} 1 \\ -5 \end{bmatrix} + b \begin{bmatrix} -3 \\ 9 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

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Start of ...

PART 2: Spans

Part 2: Spans

Definition (SPAN)

Given vectors $u_1, \ldots, u_p$ in some vector space V, their **span** is

$$\operatorname{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

In other words, $\operatorname{span}\{u_1, \ldots, u_p\}$ is the set of all linear combinations of $u_1, \ldots, u_p$ within V.

Eg: span$\{u_1\}$ is the set of all mutiples of u_1

span$\{u_1, u_2\}$ is the set of all linear combinations of u_1 & u_2.

Part 2: Spans

Theorem

$\mathrm{span}\{u_1, \ldots, u_p\}$ *is a subspace of* V.

In fact, more than this is true: one can show that $\mathrm{span}\{u_1, \ldots, u_p\}$ is the "smallest" subspace of V which contains each of $u_1, \ldots, u_p$.

Why: ① clearly $0 \in \mathrm{span}\{u_1, \ldots, u_p\}$ – just take $c_1 = c_2 = \cdots = c_p = 0$.

② If v & w are in $\mathrm{span}\{u_1, \ldots, u_p\}$ so

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p$$
$$w = d_1 u_1 + d_2 u_2 + \cdots + d_p u_p$$

So $v + w = (c_1 + d_1) u_1 + (c_2 + d_2) u_2 + \cdots + (c_p + d_p) u_p$, is in $\mathrm{span}\{u_1 \ldots, u_p\}$.

Immediate consequences

- Every choice of vectors $u_1, \ldots, u_p$ provides us with an example of a subspace of V.
  (However, ***different*** sequences of vectors may well span the ***same*** subspace!)
- If we can show a *subset* of V is the a **span of some set of vectors**, then we we have shown it is a subspace!

Eg $\quad \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$

Finished here Tuesday