2526-MA140 Engineering Calculus

Week 07, Lecture 2
Introduction to Integration

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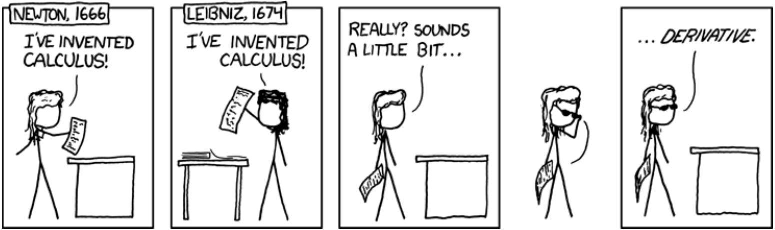
Wednesday, 29 October, 2025



https://imgs.xkcd.com/comics/newton_and_leibniz.png

Assignments

- **Assignment 5** is open. Deadline is 5pm next Monday (3 November). You have 5 attempts for each question.
- And **Assignment 6** is also open. Due 5pm, 10 November. It is based on material we'll cover in the next 3 classes.

Survey Feedback

Many thanks to the 84 of you who completed the survey (in an average time of 2:35).

Thanks for all the nice comments (very much appreciated).

Lots of positive comments about the weekly assignments, including appreciation that the are low-stakes, with collaboration encouraged.
~~that came up, and what I'll do about it:~~ Also

- "*Get Niall a better laptop*" (and variants). Action: Sorry if I failed to hide that my laptop is over-the-hill. I'll pass your comments on to my boss!
- "*Would like if the tutors were more interested...*" I'm not sure if this relates or one or both tutors, but will discuss their class persona with them.

Survey Feedback

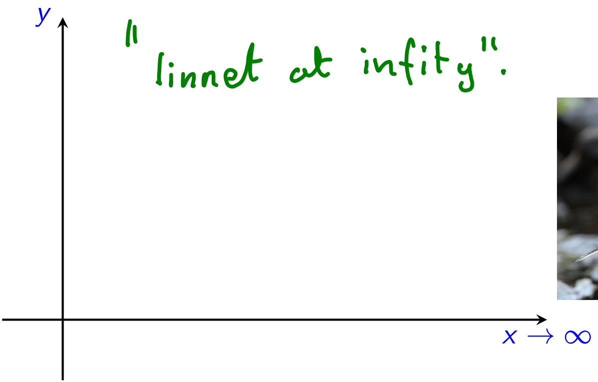
- “*More past exam paper questions, please*”. Action: OK. Thanks to COVID, there are not many – before 2023 and 2024, we only had 2019. But I’ll make a better effort to include them as either examples, or exercises with solutions provided.
- “*More Maths movie quiz*” Action: I’m guessing this is inspired by MP120. Sorry, but I don’t do movie quizzes. But I can other ~~terrible~~ jokes! dad .
- “*Maybe have more worked examples*”. Action: There is a lot of material to fit into this module, but I will try to include some “extra” examples in the notes, even if not covered in class.

Survey Feedback

- "*Smaller tutorial classes*". Action: Sorry – we are constrained to just two per week. However, I understand from the tutors this is not a problem on Thursday or, especially, Friday. Feel free to attend any of those.
- "*Sometimes examples are over complicated*". Action: True, especially for some of the more "detailed" (exam-type) examples. Will work on providing extra material.
- "*Sample Exam Papers*". Action: Yes, I will provide one (but just one), with solutions.
- "*Recommendations for videos/ extra tutoring texts*". Action: Excellent suggestion. Will do. Also: if you find some good ones, let me know!
- "*stuff about Jonathon*". Action: ???

What does this represent?

"linnet at infity".



Credit: https://birdwatchireland.ie/birds/linnet

The area we'll cover today:

See also: Sections **5.1** (Approximating Areas), **5.2** (The Definite Integral) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Introduction

Over the next few weeks, we'll learn all about *integration*.
It is an ancient topic: much, much older than the study of differentiation.

At its heart, integration is the study of **averages**, **areas** and **volumes**.

Versions of what we'll learn was known the Greeks (or, at least Archimedes) over 2000 years ago, and in China at least 1,700 years ago. The formulae we'll learn for integrating polynomials were known to scholars is the Middle East and Africa over 1,000 years ago.

The link between integration and differentiation was developed in the 17th century (Newton and Leibniz).

We'll first learn about **integration** as a means for computing the area of regions is space ("area under the curve"). One approach to estimating areas is the divide the region into much smaller ones, whose areas we can compute, and then add those up. So we need notation for writing down long sums. For that we'll use the "**Sigma**" notation.

For example,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

can be written as $\sum_{i=1}^{10} i$.

"The sum, as i goes from 1 to 10 of i."

Sums

More generally,

Sigma ($\sum$) notation

Given a list of numbers, a_1, a_2, ..., a_n,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

Examples:

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=1}^{5} (-1)^i = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5$$
$$= -1 + 1 - 1 + 1 - 1 = -1.$$

Sums

Although we don't need to derive or prove the following facts, you should check they are correct for some small values of n

Check $n = 4$

$$\sum_{i=1}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n.$$

$$\sum_{i=1}^{4} i = \frac{1}{2}(16) + \frac{1}{2}(4) = 10$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

(These results are not so significant, but we'll have reason to recall them tomorrow...).

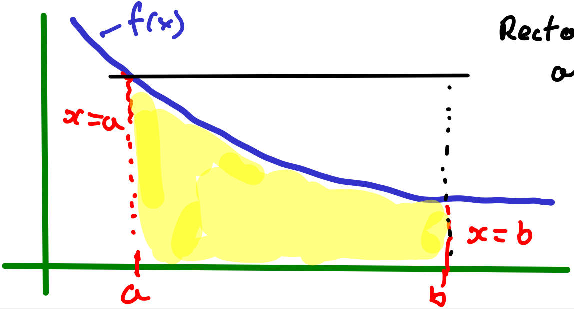
Important: in the following slides, you should read to the "annotated" version because important details will be added in class.

Approximating area

Suppose we had a positive function $f(x)$, and wanted to estimate the area bounded by

- top: $y = f(x)$
- bottom: $y = 0$
- left: $x = a$, and
- right: $x = b$.

Then $A \approx (b-a)f(a)$ would be a crude approximation.



Rectangle has area $(b-a)\,f(a)$.

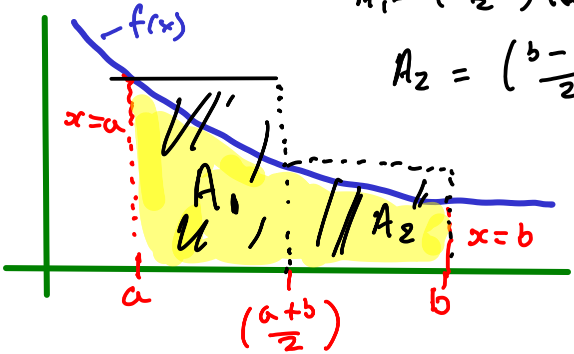
Approximating area

In the previous example, we approximated the area by the area of a single rectangle. We might be able to get a better estimate by using two smaller rectangles: $A \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f\left(\frac{a+b}{2}\right)$

Note $\left(\frac{a+b}{2}\right) - a = \frac{b-a}{2}$

$A_1 = \left(\frac{b-a}{2}\right) f(a)$

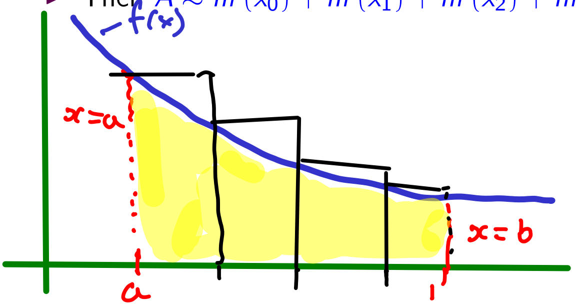
$A_2 = \left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right).$



Approximating area

We could do even better with, for example, 4 rectangles. However, the formulae would get more complicated, unless we introduce some notation:

- Let $h = \frac{b-a}{4}$. (Note: later, we'll call this quantity δx).
- Define four points: $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, and $x_3 = a + 3h$.
- Then $A \approx hf(x_0) + hf(x_1) + hf(x_2) + hf(x_3) = \sum_{i=0}^{3} hf(x_i)$



Approximating area

Now we can see that we can approximate the area using n rectangles, for any natural number n that you like:

- Let $h = \dfrac{b-a}{n}$.
- Define the points: $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $\ldots$ $x_i = a + ih$, $\ldots$
- Then $\boxed{A \approx hf(x_0) + hf(x_1) + \cdots + hf(x_{n-1}) = \sum_{i=0}^{n-1} hf(x_i)}$

The thing we've just been studying is called the **Riemann sum** (after Bernhard Riemann: look him up).

Riemann sum (kinda)

Let $f(x)$ be defined on the interval $[a, b]$. Let n be a positive integer, and set $h = (b - a)/n$. A **Riemann sum** for $f(x)$ is

$$h \sum_{i=0}^{n-1} f(x_i).$$

(The actual definition is a bit more general: see the text book for details). $\Delta_i = x_i - x_{i-1}$

What is important is: the larger n is, the better the approximation we get. And, as $n \to$ *infinity* our estimate tends to the true value of the area!

Area under the curve

Let $f(x)$ be a non-negative function be defined on the interval $[a, b]$. Let n be a positive integer, and set $h = (b - a)/n$, Let $x_i = a + ih$. Then the area between $y = 0$ and $y = f(x)$, from $x = a$ to $a = b$ is

$$A = \lim_{n\to\infty} h \sum_{i=0}^{n-1} f(x_i)$$

Equivalently: $A = \lim_{h\to 0} h \sum_{i=0}^{n-1} f(x_i)$

The Definite Integral

It transpires that the concept we have just studied has many more applications that the previous discussion might suggest. These include

- Computing the distance travelled by an object knowing only its speed.
- The volume of a wine barrel (which involves the origin story of one of my areas of research: **quadrature**).
- Finding the centre of mass/gravity of an object.
- The chances of rolling a natural 20.
- Etc, etc. $\rightsquigarrow$ lenght of a hanging rope.

The applications are so many and varied, that we eventually have to get go of the "area under the curve" idea. But we can keep the notation, to get the idea of a **definite integral**. ✓

The Definite Integral

Definition: definite integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{i=0}^{n-1} f(x_i),$$

where $h = (b-a)/n$ and $x_i = a + ih$, provided the limit exists.

Notes:

- The $\int$ symbol is a stylised "S", meaning "sum"
- When reading "$\int_a^b f(x)dx$" out loud, we do so as "the integral of f of x, from a to b, with respect to x".
- We could use this definition to compute some definite integrals, but we won't!

The Definite Integral

Properties

1. $\int_a^a f(x)dx = 0.$ "area under f of a region of length zero"

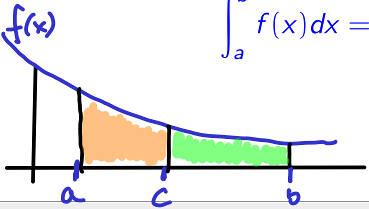
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

4. For a constant, K we have $\int_a^b Kf(x)dx = K\int_a^b f(x)dx$

5. For any points a, b and c,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$



Suppose we had a function $f(x)$ on $[a, b]$, and we wanted to find a simpler, constant function $g(x) = c$ so that area under $y = f(x)$ and $y = c$ are the same. Then...

Average value of a function

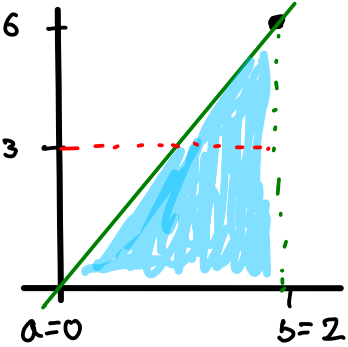
Let $f(x)$ be a continuous function on $[a, b]$. Then its **average value** is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\int_a^b f(x)\,dx = \int_a^b f_{ave}\,dx.$$

Example

Find the average value of the function $f(x) = 3x$ on the interval $[0, 2]$



$f(0) = 0$

$f(2) = 6$

Area under f is

$\frac{1}{2}(2)(6) = 6.$

So

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{1}{2}(6) = 3.$$

Towards anti-derivatives

For the rest of this week, we'll focus on being able to compute definite integrals of various functions. To do that, we'll need to see the link between integration and differentiation, given by the **fundamental theorem of calculus**.

First, we notice that, given a function, f, we can define another, F, as

$$F(\underline{x}) = \int_a^{x} f(t)dt.$$

That is, the variable in F is the upper limit of integration on the right.

Towards anti-derivatives

Example: Let $F(x) = \int_0^x 1 dt$. Then ...

Finished here after skipping to

https://www.geogebra.org/m/ugTmVRHj

Summary

If $f(x)$ is a function defined on an interval $[a, b]$

- The **definite integral** of f from a to b is

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=0}^{n-1} hf(x_i),$$

  where $h = (b-a)/n$ and $x_i = a + ih$.
- This is the area of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$, and $x = b$.
- Given a function, f, we can define another, F as

$$F(x) = \int_a^x f(t)dt.$$

  That is, the variable in F is the upper limit of integration on the right. For an nice illustration, see https://www.geogebra.org/m/ugTmVRHj

Summary

Upper limit of integration

The function is the integrand.

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

Exercises

Exer 7.2.1

Evaluate the following, by calculating the areas under the curve.

1. $\displaystyle\int_{-2}^{3} 3dx$
2. $\displaystyle\int_{0}^{2} (3-x)dx$
3. $\displaystyle\int_{-3}^{3} (3-|x|)dx$