Chap. 3: Numerical Linear Algebra

§3.2 Gaussian Elimination (and triangular matrices)

MA385/MA530 – Numerical Analysis 1

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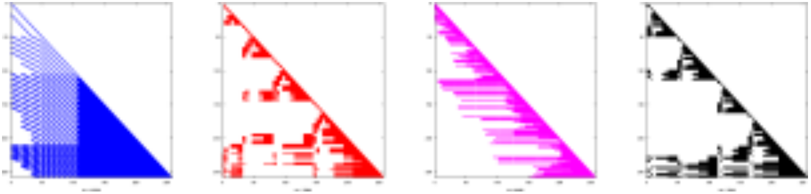

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Gaussian Elimination is an **exact** method for solving linear systems (we replace the problem with one that is easier to solve *and* has the same solution.)
This is in contrast to **approximate** methods studied earlier in the module.

There are approximate methods for solving linear systems, but they are not part of this module.

Carl Freidrich Gauß, Germany, 1777-1855.

Although he produced many very important original ideas, this wasn't one of them. The Chinese knew of "Gaussian Elimination" about 2000 years ago. His actual contributions included major discoveries in the areas of number theory, geometry, and astronomy.

Example 3.2

Consider the problem:

$$\begin{pmatrix} -1 & 3 & -1 \\ 3 & 1 & -2 \\ 2 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}$$

$\Rightarrow$ $-x_1 + 3x_2 - x_3 = 5$

$\Rightarrow$ $2x_1 - 2x_2 + 5_3 = 0$

We can perform a sequence of elementary row operations to yield the system:

$$\begin{pmatrix} -1 & 3 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -5 \end{pmatrix} \Rightarrow$$

Back substitution

$5x_3 = -5 \Rightarrow x_3 = -1$

$2x_2 = 3 + x_3 \Rightarrow x_2 = 1$

Row ops = matrix multiplication

Gaussian Elimination: perform elementary row operations such as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

being replaced by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \underbrace{a_{21} + \mu_{21} a_{11}}_{=0} & a_{22} + \mu_{21} a_{12} & a_{23} + \mu_{21} a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A + \mu_{21} \begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix}$$

where $\mu_{21} = -a_{21}/a_{11}$, so that $a_{21} + \mu_{21} a_{11} = 0$.

Note that

$E^{(21)}$

$$\begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

so we can write the row operation as $(I + \mu_{21} E^{(21)})A$, where $E^{(pq)}$ is the matrix of all zeros, except for $e_{pq} = 1$.

In general each of the row operations in Gaussian Elimination can be written as

$$(I + \mu_{pq} E^{(pq)})A \quad \text{where } 1 \leq q < p \leq n, \tag{2}$$

and $(I + \mu_{pq} E^{(pq)})$ is an example of a ***Unit Lower Triangular Matrix***.

We can conclude that each step of the process will involve multiplying A by a unit lower triangular matrix, resulting in an upper triangular matrix.

Definition 3.3 (Lower Triangular)

$L \in \mathbb{R}^{n \times n}$ is a *Lower Triangular (LT) Matrix* if the only non-zero entries are on or below the main diagonal, i.e., if $l_{ij} = 0$ for $1 \le i < j \le n$.
It is a *unit Lower Triangular matrix* if, in addition, $l_{ii} = 1$.

Examples: The following matrices are Lower Triangular:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & 0 & 0 & 0 \\ c & d & 0 & 0 \\ e & f & g & 0 \\ h & i & g & K \end{pmatrix}, \quad I \text{ (identity)}$$

$E^{(pq)}$ where $p > q$, and the zero matrix.

Of these, A is Unit LT, as are I and B if $a = d = g = K = 1$. Not the zero matrix.

Definition 3.4

Upper Triangular] $U \in \mathbb{R}^{n \times n}$ is an *Upper Triangular (UT) matrix* if $u_{ij} = 0$ for $1 \leq j < i \leq n$. It is a *Unit Upper Triangular Matrix* if $u_{ii} = 1$.

Examples:

U is upper triangular if U^T

is Lower triangular.

Eg
$$\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix}$$

This is Unit UT

if $a = e = h = j = 1$.

Triangular matrices have many important properties. A very important one is: **the determinant of a triangular matrix is the product of the diagonal entries**. For a proof, see Exercise 3.5.

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There are other important properties of triangular matrices, but first we need the idea of **matrix partitioning**.

Definition 3.5 (Submatrix)

X is a *submatrix* of A if it can be obtained by deleting some rows and columns of A.

Example: if $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the

each of the following are submatrices of A:

(a), (e), (f), $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$, $\begin{pmatrix} a & c \\ d & f \\ g & i \end{pmatrix}$

But not $\begin{pmatrix} a & b \\ h & i \end{pmatrix}$.

Definition 3.6 (Leading Principal Submatrix)

The ***Leading Principal Submatrix*** of order k of $A \in \mathbb{R}^{n\times n}$ is $A^{(k)} \in \mathbb{R}^{k\times k}$ obtained by deleting all but the first k rows and columns of A. (Simply put, it's the $k \times k$ matrix in the top left-hand corner of A).

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

$$A^{(1)} = (1), \quad A^{(2)} = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} \quad A^{(3)} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{pmatrix}, \quad A^{(4)} = A$$

Matrix partitioning

To ***partition a matrix*** means to divide it into contiguous blocks that are submatrices.

Example:

$$A = \left(\begin{array}{cc|c} a & b & c \\ d & e & f \\ \hline g & h & i \end{array}\right)$$

can be partition into

$$A^{(2)} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \quad B = \begin{pmatrix} c \\ f \end{pmatrix} \quad C = \begin{pmatrix} g & h \end{pmatrix} \quad D = \begin{pmatrix} i \end{pmatrix}.$$

Next recall that if A and V are matrices of the same size, and each are partitioned

$$A = \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right), \qquad \text{and} \qquad V = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right),$$

where B is the same size as W, C is the same size as X, etc. Then

$$AV = \left(\begin{array}{c|c} BW + CY & BX + CZ \\ \hline DW + EY & DX + EZ \end{array}\right).$$

Theorem 3.7 (Properties of Lower Triangular Matrices)

For any integer $n \geq 2$:

(i) If L_1 and L_2 are $n \times n$ *Lower Triangular* (LT) Matrices that so too is their product L_1L_2.

(ii) If L_1 and L_2 are $n \times n$ *Unit Lower Triangular* matrices, then so too is their product L_1L_2.

(iii) L_1 is nonsingular if and only if all the $l_{ii} \neq 0$. In particular all Unit LT matrices are nonsingular.

(iv) The inverse of a LT matrix is an LT matrix. The inverse of a unit LT matrix is a unit LT matrix.

We restate Part (iv) as follows:

> *Suppose that $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix with $n \geq 2$, and that there is a matrix $L^{-1} \in \mathbb{R}^{n\times n}$ such that $L^{-1}L = I_n$. Then L^{-1} is also a lower triangular matrix.*

The proof is by induction. First suppose that $n=2$. We write L as

$$L = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

Then $L^{-1} = \frac{1}{ad}\begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$ which is a lower triangular matrix.

(note: from Part (iii) $a \neq 0$, $d \neq 0$).

Next assume the Theorem is true for any matrix up to size (="order") $(n-1)\times(n-1)$.

Let L be a $n\times n$ Lower Triangular matrix, and partition it by the last row and column:

$$L = \left(\begin{array}{c|c} L^{(n-1)} & \vec{0} \\ \hline c^T & d \end{array}\right)$$

where $L^{(n-1)}$ is an $(n-1)\times(n-1)$ LT matrix, $\vec{0}$ is zero vector, c^T is a $(n-1)$-vector (transposed), and d is a scaler. Note $d \neq 0$, since L^{-1} exists.

Next write L^{-1} as

$$L^{-1} = \left(\begin{array}{c|c} W & x \\ \hline y^T & z \end{array}\right)$$

where we have portioned in the same way.

Using

$LL^{-1} = I$, we get $\left(\begin{array}{c|c} L^{(n-1)} & \vec{0} \\ \hline c^T & d \end{array}\right)\left(\begin{array}{c|c} W & x \\ \hline y^T & z \end{array}\right)$

$$= \left(\begin{array}{c|c} L^{(n-1)}W + \vec{0}y^T & L^{(n-1)}x + \vec{0}z \\ \hline c^T W + d y & c^T x + d z \end{array}\right) = \left(\begin{array}{c|c} I^{(n-1)} & \vec{0} \\ \hline \vec{0}^T & 1 \end{array}\right)$$

This gives that

(i) $L^{(n-1)} w = I^{(n-1)}$. So

$w = \left(L^{(n-1)}\right)^{-1}$. By induction

this is a LT matrix.

(ii) $L^{(n-1)} x = \vec{0}$. So $x = \vec{0}$.

(iii) $c^T w + d y = 0$ and

$\cancel{c^T x} + d z = 1$

imply that z exists (since $d \neq 0$)

and thus so too does y.

Theorem 3.8 (Properties of Upper Triangular Matrices)

Statements that are analogous to those concerning the properties of lower triangular matrices hold for upper triangular and unit lower triangular matrices. (For proof, see the exercises at the end of this section).

Exercise 3.3

Every step of Gaussian Elimination can be thought of as a left multiplication by a unit lower triangular matrix. That is, we obtain an upper triangular matrix U by multiplying A by k unit lower triangular matrices: $L_k L_{k-1} L_{k-2} \dots L_2 L_1 A = U$, where each $L_i = I + \mu_{pq} E^{(pq)}$, and $E^{(pq)}$ is the matrix whose only non-zero entry is $e_{pq} = 1$. Give an expression for k in terms of n.

Exercise 3.4 ($\star$)

Let L be a lower triangular $n \times n$ matrix. Show that $\det(L) = \prod_{j=1}^{n} l_{jj}$. Hence give a necessary and sufficient condition for L to be invertible. What does that tell us about *Unit* Lower Triangular Matrices?

Exercise 3.5

Let L be a lower triangular matrix. Show that each diagonal entry of L, l_{jj} is an eigenvalue of L.

Exercise 3.6

Prove Parts (i)–(iii) of Theorem 3.7.

Exercise 3.7

Prove Theorem 3.8.

Exercise 3.8

Construct an alternative proof of the first part of 3.7 (iv) as follows: Suppose that L is a non-singular lower triangular matrix. If $\boldsymbol{b} \in \mathbb{R}^n$ is such that $b_i = 0$ for $i = 1, \dots, k \leq n$, and $\boldsymbol{y}$ solves $L\boldsymbol{y} = \boldsymbol{b}$, then $y_i = 0$ for $i = 1, \dots, k \leq n$. (Hint: partition L by the first k rows and columns.)
Now use this to give a alternative proof of the fact that the inverse of a lower triangular matrix is itself lower triangular.