Annoated slides

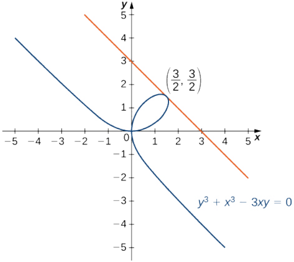
2526-MA140 Engineering Calculus

Week 05, Lecture 2
Implicit Differentiation; Exponential and Logarithmic Functions

Dr Niall Madden

University of Galway

Wednesday, 15 October, 2025



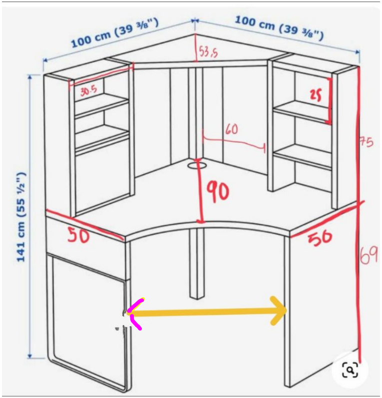
Assessment Schedule for the Rest of the Semester

- Week 6 (next week): **Assignment 3** due 17:00, Monday, 20 Oct.
- Week 7: **Assignment 4** (which just opened) due 17:00, Tuesday, 28 Oct.
- Week 8: **Assignment 5** due 17:00, Monday, 3 Nov.
- Week 9: **Assignment 6** due 17:00, Monday, 10 Nov.
- Week 10: **Assignment 7** due 17:00, Monday, 17 Nov.
- Week 10: **Class Test 2** 10:00, Tuesday 18 Nov.
- Week 11: **Assignment 8** due 17:00, Monday, 24 Nov.

About the online assignments

- The purpose of the assignments is to maintain engagement and develop confidence.
- When you submit an answer, you get immediate feedback on if your answer is correct or not.
- If not, you usually have up to 4 more attempts (except for "true/false" type questions).
- Each assignment is worth (only) about 1.25% of your final grade.
- Most assignments have 6 questions, so each is worth about 0.2%.
- Tutorial sheets are based on the same assignments. Solutions to those are posted after the deadline (usually).
- Collaboration is encouraged. Engagement with SUMS is especially encouraged.
- The system is very well established, and bugs are rare. However, integration with Canvas is not perfect. When I allow you to see your "live" grade, it sends a message saying the assignment has been graded.

Remember "Olive's desk"?

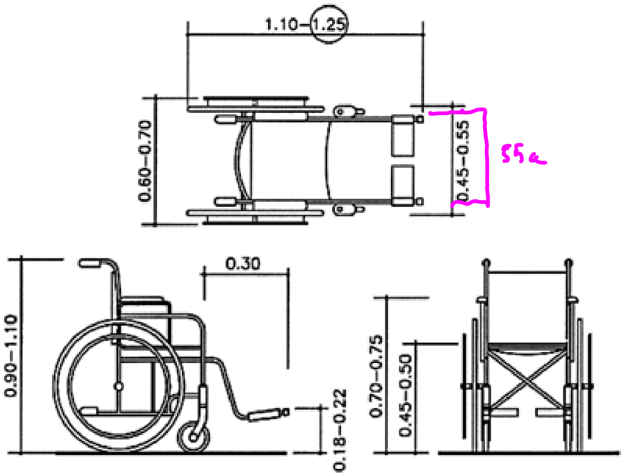


Source: IKEA catalogue

Last week, I told you that "Olive" was thinking of buying this "MICKE" corner desk unit in IKEA. Her (wheel)chain is 55cm wide. Is the sitting region of the desk indicated by the yellow line, wide enough?

1. What do you think the answer is?
2. But actually..

Remember "Olive's desk"?



Source: *https://www.un.org/esa/socdev/enable/designm/AD5-02.htm*

Today, in Engineering Calculus...

1. Implicit differentiation
2. Exponential functions
   - Properties
   - The number e
   - The derivative of e^x
3. Logarithms
   - Properties
   - The natural logarithm
   - Derivative of $\ln(x)$
   - Logarithmic differentiation
4. Exercises

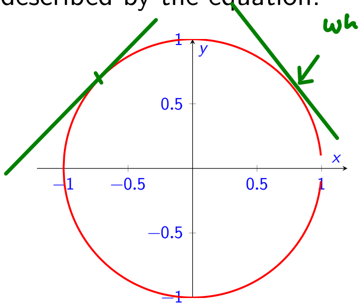
See also: Sections 3.8 (Implicit Differentiation) and 3.9 (Derivatives of Exponential and Logarithmic Functions) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Implicit differentiation

Last week, we introduced the idea of an *implicitly defined function*:

- **Explicit**: given a value of x, we have a formula for computing the (single) corresponding value of y;
- **Implicit**: the formula relates the variables, without giving an explicit value of one (y) in terms of the other (x).

Classic example: $x^2 + y^2 = 1$. For any pair (x, y) we can check if it is on the curve described by the equation.



what is the slope of the tangent?

Implicit differentiation

Since **implicit equations** define curves, we can use **implicit differentiation**, for example to find a tangent to an implicitly defined curve.

Method:

1. Differentiate both size of the equation, with respect to x, keeping in mind that y is a function of x, using the **Chain Rule** where needed.
2. Solve for dy/dx.

Eg: If y depends on x, what is the derivative, with Respect to x, of (for Example) y^2?

Chain Rule: if $f(x) = u(v(x))$, then $\frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}$.

so $\frac{d}{dx}[y^2] = 2y \cdot \frac{dy}{dx}$ where we have $u(v) = v^2$, $v(x) = y(x)$

Implicit differentiation

If y is defined by $x^2 + y^2 = 1$, find $\dfrac{dy}{dx}$.

1. The curve is defined by

$$x^2 + y^2 = 1$$

2. Differentiate (both sides of) the equation

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[1]$$

$$\Rightarrow \frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$ (from prev slide)

3. Solve for $\dfrac{dy}{dx}$: $2y\dfrac{dy}{dx} = -2x \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$ ✓

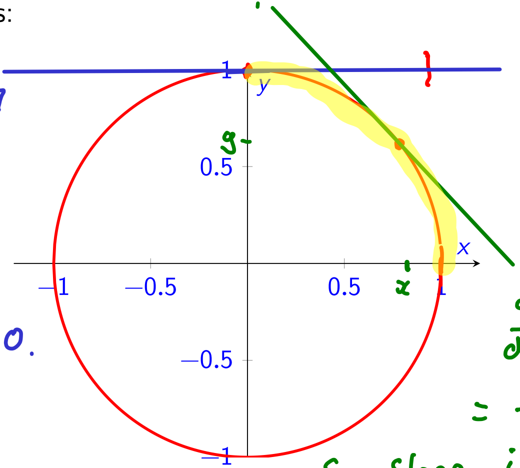
Implicit differentiation

Now we know that if $x^2 + y^2 = 1$, then $\dfrac{dy}{dx} = -\dfrac{x}{y}$. We can check that this relates to the slope of the tangents to this curve at various places:

Check:

tangent at $(x, y) = (0, 1)$

So slope is $-\dfrac{x}{y} = -\dfrac{0}{1} = 0$.



Suppose $x = y$.

So $x^2 + x^2 = 1$

$\Rightarrow 2x^2 = 1$

$\Rightarrow x = \dfrac{1}{\sqrt{2}}$

$y = \dfrac{1}{\sqrt{2}}$.

$\dfrac{dy}{dx} = -\dfrac{x}{y} = -\dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{1} = -1$

So slope is -1.

Implicit differentiation

Find the tangent to the curve $x^2 + y^2 = 25$, at the point $(3, -4)$.

Niall will add details later.

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$. "Folius of Descartes".

First, check "is $x = \frac{3}{2}$, $y = \frac{3}{2}$ on the curve?"

$$\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^3 - 3\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) =$$

$$\frac{27}{8} + \frac{27}{8} - \frac{27}{4} = \frac{27}{8} + \frac{27}{8} - 2\frac{27}{8} = 0 \quad \checkmark$$

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$.

Now differentite

$$\frac{d}{dx}\left[y^3\right] + \frac{d}{dx}\left[x^3\right] - 3\frac{d}{dx}\left[xy\right] = 0.$$

$$3y^2 \cdot \frac{dy}{dx} + 3x^2 - 3\frac{d}{dx}\left[xy\right] = 0$$

For $\frac{d}{dx}[xy]$ use the product rule

$$x \cdot y = u(x) \cdot v(x)$$ where $u(x) = x$, $v(x) = y(x)$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = \frac{dy}{dx}$.

So $$\frac{d}{dx}[x \cdot y] = x \cdot \frac{dy}{dx} + (1) \cdot y$$

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$.

So now we have

$$3y^2 \cdot \frac{dy}{dx} + 3x^2 - 3\left(x\frac{dy}{dx} + y\right) = 0.$$

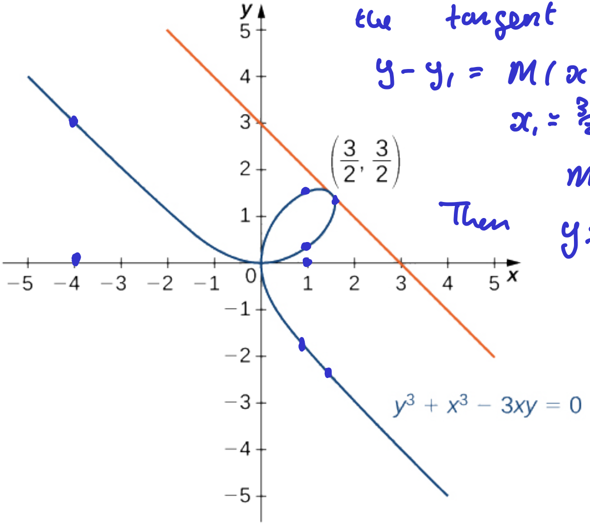
Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(3y^2 - 3x\right) + 3x^2 - 3y = 0$$

So $\displaystyle \frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}.$

Now evaluate at $x = 3/2$, $y = 3/2$, get $\frac{dy}{dx} = -1$. (Check!)

Implicit differentiation



To get the equation of the tangent use

$y - y_1 = m(x - y_1)$ with

$x_1 = \frac{3}{2}$, $y_1 = \frac{3}{2}$

$m = -1$.

Then $y = -x + 3$

Exponential functions

Earlier in this course we met functions such as $y = x^2$; this is a **power** function.

Now we consider **exponential functions**, such as $\boxed{y = 2^x}$.

Such functions occur in many applications. For example: if I invest €100 with an annual interest rate of 20%, then after x years, I will have €$100 \times (1.2)^x$. **Why?**

$f(x)$ = "value after x years".

So $f(0) = 100$

$f(1) = 100(1 + 0.2) = \boxed{100(1.2)} = 120$

$f(2) = 120(1.2) = 100(1.2)(1.2) = 100 \cdot (1.2)^2$

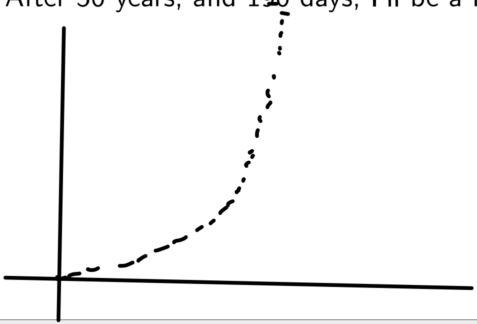
$f(3) = [100(1.2)^2](1.2) = 100(1.2)^3$

$f(x) = 100(1.2^x)$

Exponential functions

Exponential functions grow quite fast: if my investment is indeed worth $f(x) = 100 \times (1.2)^x$ euros after x years, then...

- After 1 year, I have €120
- After 10 years, I have €619.17
- After 20 years, I have €3,833.80
- After 25 years, I have €9,539.60
- After 50 years, and 190 days, I'll be a millionaire!



Here I remind you of some properties of exponents that you should already know: for any positive numbers a and b,

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x \cancel{a^y}\ b^y$
5. $\left(\frac{a}{b}\right)^x = \cancel{\dfrac{a^x}{a^y}}\ \dfrac{a^x}{b^x}$

$$(2 \cdot 10)^3 = 2^3 \cdot 10^3 = 8{,}000$$

Check:

take $b = 10$,
and (say) $x = 3$, $y = 2$.

So $b^x = 1{,}000$ $\quad b^y = 100$

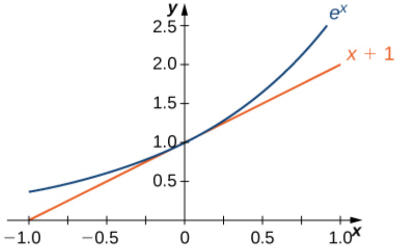
$$b^x b^y = b^{(x+y)} = b^5 = 100{,}000.$$

Finish here.

The number $e \approx 2.7182818284$. It is often called **Euler's Number** after Leonard Euler, who did not discover it: that was (probably) Jacob Bernoulli in 1683 while studying compound interest. Or maybe 100 years earlier by John Napier.

The Natural Exponential Function

The Natural Exponential Function is $f(x) = e^x$. It is special for many reasons, including the its tangent at $x = 0$ has slope 1.



Let's assume that e is the number for which, if $f(x) = e^x$, then $f'(0) = 1$. Using the limit definition of the derivative, this means

$$1 = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

From this can deduce that...

So now we know that

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

That is e^x is the function that is its own derivative!!!

Example

Compute the derivative of $f(x) = e^{\sin(x)}$

Logarithms

Suppose that $y = f(x)$ is an **exponential** function; that is: $y = b^x$ for some $b > 0$ (and excluding $x = 1$).

Its **inverse** is called a **logarithmic function**, denoted $\log_b$

$$\text{If } y = b^x \quad \text{then} \quad \log_b(y) = x.$$

Examples

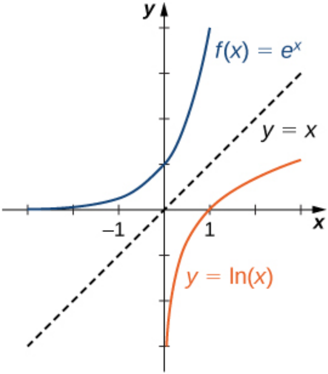
- $log_2(8) = 3$
- $\log_{10}(100) = 2$
- $\log_e(e^x) = x$

Properties of Logarithms

If $a, b, c > 0$ and $b \neq 1$m then

- $\log_b(ac) = \log_b(a) + \log_b(c)$
- $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$
- $\log_b(a^r) = r\log_b(a)$

We denote $\boxed{\log_e(x)}$ as $\boxed{\ln(x)}$



$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Why?

Example:

Find the derivative of $f(x) = \ln(x^2 + 2x + 3)$.

To finish we introduce the idea of **logarithmic differentiation**, which helps us differentiate functions with x, or a function of x in the exponent, such as $y = (2x)^{\sin(x)}$ or $y = x^x$.

Strategy:

- Take ln of both sides
- Simplify, using properties of logarithms.
- Differentiate.
- Solve for $\dfrac{dy}{dx}$

Example

Differentiate $f(x) = x^x$.

Exercises

Exercise 5.2.1

Find the derivative of

1. $f(x) = x^3 \cos(x^2)$
2. $f(x) = \tan^3\left(\sin^2(x^4)\right)$

Exercise 5.2.2

Show that $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$.

Exercise 5.2.3

Find the equation of the tangent to the curve defined by $x^2 - y^2 = 16$ at the point $(5, 3)$.

Exercises

Exercise 5.2.3

Find the equation of the tangent to the curve defined by $x^2 - y^2 = 16$ at the point $(5, 3)$.