2425-MA140 Engineering Calculus

Week 08, Lecture 3 (L24)
Improper Integrals

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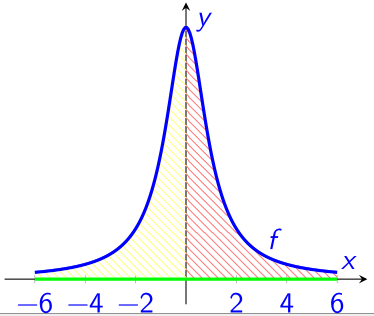


Today, we'll take areas to the limit:

1. Areas Between Curves (again)

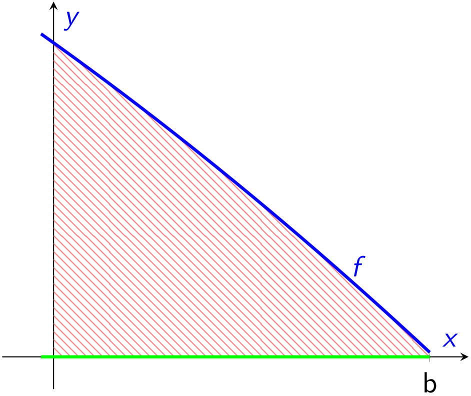
2. Improper Integrals
   - Last example

3. Exercises

For more reading, see Section **7.7** (Improper Integrals) in **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

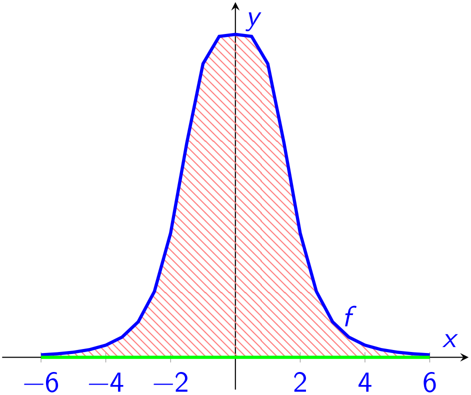
Areas Between Curves (again)

Yesterday, we riffed on the idea that $\int_a^b f(x)\,dx$ evaluates as the area of the region between $y = f(x)$ and $y = 0$, and between $x = a$ and $x = b$.



Areas Between Curves (again)

But what if we wanted the area of the region between $y = f(x)$ and $y = 0$, and between (say) $x = -\infty$ and $x = \infty$?



Improper Integrals

So far we have dealt with the definite integral $\int_a^b f(x)\,dx$ for a continuous function f on a finite interval $[a, b]$, i.e. where a and b are both real numbers.

But sometimes the region in which we are interested is over an **infinite** interval, i.e. an interval of the form $[a, \infty)$, $(-\infty, b]$ or $(-\infty, \infty)$.

Let's consider how we might try to define an **improper integral** such as

$$\int_a^\infty f(x)\,dx\,.$$

Improper Integrals

Definition (Improper Integral)

Let $f(x)$ be a continuous function on $[a, \infty)$. Then the **improper integral of f over** $[a, \infty)$ is defined by

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$

provided that the limit exists. In this case, we say that the integral $\int_a^\infty f(x)\,dx$ is **convergent**. If the limit does not exist, we say that $\int_a^\infty f(x)\,dx$ is **divergent**.

Similarly, if $g(x)$ is a continuous function on $(-\infty, b]$, we say that the improper integral $\int_{-\infty}^b g(x)\,dx$ is **convergent** and given by

$$\int_{-\infty}^b g(x)\,dx = \lim_{t\to-\infty}\int_t^b g(x)\,dx$$

provided that the limit exists, and it is **divergent** otherwise.

Furthermore:

If f is a continuous function on $\mathbb{R} = (-\infty, \infty)$ and the improper integrals

$$\int_{-\infty}^{0} f(x)\,dx \quad \text{and} \quad \int_{0}^{\infty} f(x)\,dx$$

are both convergent, then the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx\,.$$

is also **convergent**. If not, we say it is **divergent**.

Example

Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

Idea: Use the definition:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

That is: set $g(t) = \int_1^t f(x)\,dx$ and then evaluate $\lim_{t\to\infty} g(t)$

Improper Integrals

Many improper integrals are divergent. Example: $\int_1^{\infty} x\, dx$.

If $f(x)$ is a positive function, for $\int_a^\infty f(x)\,dx$ to exist, at the very least we need $f(x)$ to be a decreasing function. But often that alone is not enough!

- We know that $\int_1^\infty x^{-2}\,dx$ is convergent.
- From that we can deduce that $\int_1^\infty x^{-n}\,dx$ is convergent for any $n \geqslant 2$. (Why?)
- And we know $\int_1^\infty x^0\,dx$ is divergent.
- But what about $\int_1^\infty x^{-1}\,dx$?

Example

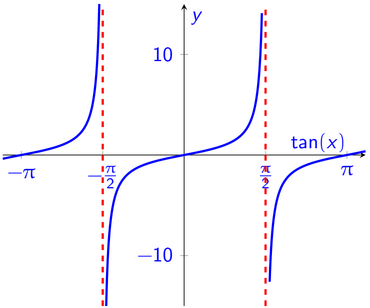
Determine whether the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\, dx$ is convergent or divergent.

For $t \geqslant 1$, we have

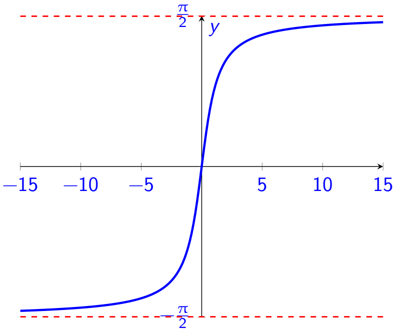
$$\int_1^t \frac{1}{x}\, dx = \ln(x)\,\Big|_1^t = \ln(t)\,.$$

Since $\lim_{t\to\infty} \ln(t)$ does not exist, it follows that $\int_1^\infty \frac{1}{x}\, dx$ is divergent.

[*This slide, and the next one, were vadded after the lecture*]
In our next, and final example, we'll try to integrate
$f(x) = \dfrac{1}{1+x^2}$. To follow the solution, you might find it useful to revise the fundamentals of **inverse trigonometric functions**. You can find that in Section 1.4 of the textbook:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)



In the figure opposite, we see the graph of $\tan(x)$. Notice that it has vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$.



And now we show the **inverse of the** $\tan(x)$ function, which is often written as either $\tan^{-1}(x)$ or $\arctan(x)$. Notice that it has **horizontal** asymptotes at $y = -\pi/2$ and $y = \pi/2$.

This means that

$$\lim_{x \to -\infty} tan^{-1}(x) = -\frac{\pi}{2},$$

and

$$\lim_{x \to \infty} tan^{-1}(x) = \frac{\pi}{2}.$$

Example

Evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Exercises

Exer 8.3.1 (From 23/24 exam)

Evaluate $\int_0^\infty \frac{x}{1+x^4}\, dx$ (*Hint: try substitution with* $u = x^2$).