MA438: PROBLEM SET 3
DUE DATE: FRIDAY MARCH 31

Please upload your answers to (any four of) Exercises 3.1–3.7 as a PDF file, and any one of Exercises 3.8–3.11 (which may be MATLAB files, as appropriate) through the `Assignment 3` *link in the "*Assignments*" area of the Blackboard page. Upload your work for Lab 3 to the "Labs" area. Include a note of any print or online sources that significantly inform your work on these problems. Please think about the quality of your explanations for a reader, as well as about the mathematical content. Full marks will not be awarded for explanations that are poorly organized or unclear. All questions carry the same marks.*

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Exercise 3.1. Let A be a non-negative square matrix with the property that A^k is positive for some positive integer k. A matrix with this property is called *primitive*, and the least k for which A^k is positive is called the *exponent* of A.

- If the exponent of A is k, show that A^m is positive for all $m \geqslant k$.
- By adapting the first part of the proof of the Perron-Frobenius Theorem or otherwise, show that A has a positive real eigenvalue with a corresponding positive eigenvector.

Exercise 3.2. Let S be a set with n elements. For any integer $k \leqslant n$, let $\mathcal{I}_k$ denote the set of all subsets of S with at most k elements. Show that $(S, \mathcal{I}_k)$ is a matroid. (It is referred to as the *uniform matroid* $U(n, k)$).

Exercise 3.3. Let Γ be a graph with vertex set V and edge set E. Let $\mathcal{I}$ be a set of all subsets of E that include the edges of at most one cycle. Show that $(E, \mathcal{I})$ is a matroid.

Remark: As usual with matroids, the augmentation property is the tricky one to show. For this example, this can be achieved more or less in the same way that we did it for the acyclic case. If Y and Z are independent sets with $|Y| < |Z|$, then the edges in Y form a graph that is either a forest or has exacly one comonent with exactly one cycle. A connected graph on n vertices has exactly one cycle if and only if it has n edges.

Exercise 3.4. Show, by means of an example, that if "at most one cycle" is changed to "at most two cycles" in Problem 3.3, then the sets involved do *not* form a matroid.

Remark: The difference in this case is that adding one edge to a graph with only one cycle can create a graph with three or more cycles.

Exercise 3.5. Let Γ be a bipartite graph with vertex set partitioned as $X \cup Y$, and every edge involving one vertex from X and one from Y. Let $\mathcal{I}$ be the set of all subsets W of X with the property that there is a matching in Γ that matches all the elements of W. Show that $(X, \mathcal{I})$ is a matroid.

Remark: If $|W_1$ and $|W_2|$ are independent sets with $|W_1| < |W_2|$, consider matchings in the subgraph of Γ induced on the vertices of $W_1 \cup W_2$ and their neighbours in Y. Use the fact that if a matching does not have maximum cardinality, then there is an augmenting path for it.

Exercise 3.6. Let A be a symmetric positive definite matrix.

(a) Give two different reasons for why A^{-1} must exist.

(b) Is A^{-1} be symmetric positive definite? Either prove this is true, or give a counter example.

(c) Is it possible for a matrix $A \in M_n(\mathbb{R})$ to have the property that $\vec{x}^T A x > 0$ for all $x \neq 0$, and yet not be symmetric? (Hint: take $A \in \mathbb{R}^{2\times 2}$ to be of the form $A = \begin{pmatrix} 0 & b \\ c & 1 \end{pmatrix}$, and try to find b and c such that $\vec{x}^T A x > 0$).

Exercise 3.7. (a) Is there any nonsingular matrix, A, in $M_2(\mathbb{R})$ that is *not* a Z-matrix, but $A^{-1} \geqslant 0$? Give an example, or show it is not possible?

(b) Give an example of a nonsingular matrix, A, in $M_3(\mathbb{R})$ that is not a Z-matrix, but $A^{-1} \geqslant 0$. (Hint: try an upper triangular matrix).

(c) Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$. Find the smallest $s^\star$ such that $A = sI - B$ is an M-matrix for all $s > s^\star$.

(d) Show that if A is an M-matrix, and D is any non-negative diagonal matrix, then $(A + D)^{-1}$ exists.

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Submit solutions to any *one* of Exercises 3.8–3.11

Exercise 3.8. See exercise at the end of Lab 5: See
`https://www.niallmadden.ie/2223-MA438/Lab5_GraphMatching.html`

Exercise 3.9. The rank r nonnegative matrix factorisation of an $m \times n$ matrix, A, may be estimated using the following algorithm.

- Set w to be any $m \times r$ matrix, and h to be any $r \times n$ matrix, both non-negative and of full rank.
- Iteratively compute
  $h = h \cdot\star (w^T A) \cdot / (w^T w h)$ and $w = w \cdot\star ((A h^T) \cdot / (w h h^T)$,
  where here we use MATLABesque notation, and denote the entry-wise matrix multiplication and division operators as $\cdot\star$ and $\cdot/$

(a) Give example of a situation where, due to the initial choices of w and h, this algorithm would fail.

(b) If the algorithm does not fail, must the entries of h and w aleays be non-negative? Explain your answer.

(c) Use the algorithm to compute a nonnegative matrix factorisation of $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$.

Exercise 3.10. (This question relates to Lab 6. Provide your solution to this problem as a MATLAB Live script). Consider the graph in Figure 1.

(a) Write down the weighted adjacency matrix, A, for this graph, and store it as a matrix in MATLAB.

(b) Using MATLAB's `nnmf()` function, compute the rank 2 nonnegative matrix factorization of A.

(c) Use the information represented by the nonnegative matrix factorization to classify the vertices in the graph in Figure 1 into one of two types. Give an interpretation of this classification.

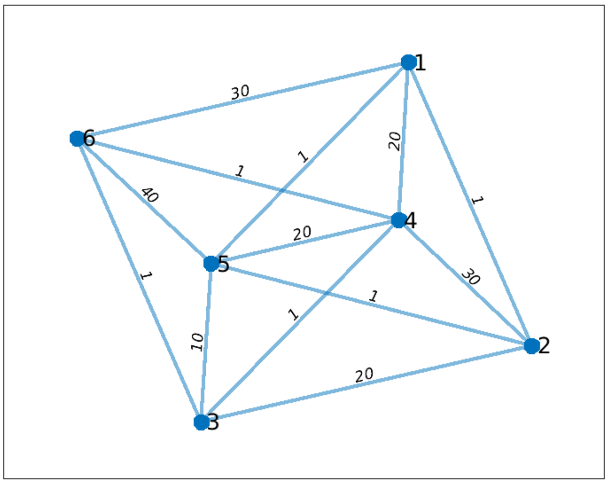

Figure 1: A weighted graph for Exercises 3.10 and 3.11.

Exercise 3.11. (This question relates to Lab 7. Provide your solution to this problem as a MATLAB Live script). Consider (again) the graph in Figure 1.

(a) Write a MATLAB script that that defines and plots this graph with that adjacency matrix.

(b) Find and plot a minimum spanning for the graph.

(c) How many different matchings are possible of this graph? (You don't have to list them all). Plot the graph, highlighting the matching with the maximum weight.