2425-MA140 Engineering Calculus

Week 03, Lectures 3
Continuity and The Intermediate Value Theorem

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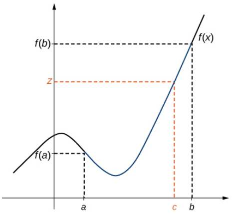
Thursday, 3 October, 2024



These slides are by Niall Madden, with some content based on notes by Dr Kirsten Pfeiffer, and some, such as the figure opposite, taken from Strang & Herman's "Calculus". The typos are Niall's.

Outline

For more, see Section 7.9 (Continuity) in *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

And I *highly* recommend Chapter 2 (Limits) in **Calculus** by Strang & Herman. See openstax.org/books/calculus-volume-1/pages/2-introduction. Section 2.4 (Continuity) covers today's material.

Reminder

- **Assignment 1** has a deadline of 5pm, Friday. You can access it on Canvas... 2425-MA140... Assignments.
- The **Tutorial Sheet** is available at https://universityofgalway.instructure.com/files/2040359/download?download_frd=1
- A new assignment will be posted later this week.

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For help with the assignment, attend a tutorial. The schedule is on the Canvas "Course Information" page: https://universityofgalway.instructure.com/courses/35693/pages/2425-ma140-information. Note the change of venue for the Irish language tutorials (Tue at 1, AMB-G021).

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Support is also available at tutorials and **SUMS**.

*Exercises are now at the end of each set of slides
* These are not for homework - they are for exam prep.
* Solutions will be posted each week.

Recall... continuity

Definition

A function f is **continuous at** $x = a$ if

1. $f(a)$ is defined, i.e., a is in the domain of f,
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$ we say it is **discontinuous** at $x = a$.

If f is continuous **at every point** in its domain, we say f **is continuous.**

f is "discontinuous" means it is discontinuous at at least one point.

Many functions are continuous, e.g. all polynomial functions, most trigonometric functions (not tan), $|x|$, and so on.

Example

Consider the function

$$f(x) = \begin{cases} x+1, & x<2 \\ b\,x^2, & x \geqslant 2 \end{cases}$$

For what value of b is f continuous at $x=2$?

First, note that 2 is in the domain of f. ✓

Next: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2} (x+1) = 3.$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} bx^2 = 4b.$

So we need $4b=3$. That is $b = 3/4$ ☒

So $\lim_{x\to 2} f(x) = 3 = f(2)$ ✓

Example

For what values of x is $f(x) = \dfrac{2x+1}{2x-2}$ continuous?

$f(x)$ is continuous for all x, except, possibly, where $2x-2=0$. That is when $x=1$. Note that the numerator is $2(1)+1=3 \neq 0$. So, f is not defined at $x=1$. So it is not continuous at $x=1$.

Ans: f is continuous for all x: $x<1$ or $x>1$

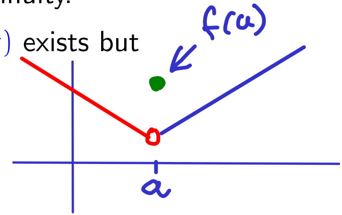
Same as: $x \in (-\infty,1) \cup (1,\infty)$ or $\mathbb{R}/\{1\}$

Types of discontinuity

We have encountered three types of discontinuity.

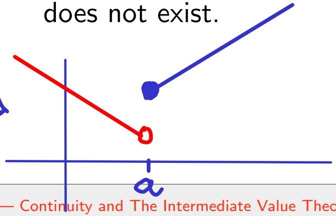
- **Removable discontinuity**: $\lim_{x\to a} f(x)$ exists but

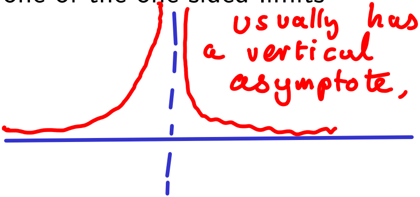
$$\lim_{x\to a} f(x) \neq f(a)$$



- **Jump discontinuity**: $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+ f(x)}$ both exist (and are finite), but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

- **Infinite discontinuity**: At least one of the one-sided limits does not exist.





Types of discontinuity

Example

Each of the following functions has a discontinuity at $x = 2$. Classify it.

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$

   $f(2) = \frac{0}{0}$. But $f(x) = \frac{(x-2)(x+2)}{x-2}$

   $= x+2$ for all x except $x = 2$.

   So this is a removable discontinuity.

2. $g(x) = \dfrac{x^2}{x - 2}$
3. $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$
4. $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

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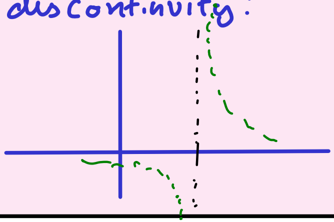
Note that $g(2) = \frac{4}{0}$

So $\lim_{x \to 2} g(x)$ is not defined. We have an infinite discontinuity:



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$\lim_{x \to 2^-} h(x) = \lim_{x \to 2} \frac{x}{2} = 1.$

$\lim_{x \to 2^+} h(x) = \lim_{x \to 2} (x^2 - 3) = 1.$

So $\lim_{x \to 2} h(x) = 1.$

But $h(2) = -2$

So this is a removable discontinuity.

Types of discontinuity

Example

Each of the following functions has a discontinuity at $x = 2$. Classify it.

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4. $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x \geqslant 2. \end{cases}$

$\lim\limits_{x \to 2^-} h(x) = \lim\limits_{x \to 2} \left(\frac{x}{2}\right) = 1$

$\lim\limits_{x \to 2^+} h(x) = \lim\limits_{x \to 2} (x^2 - 2) = 2.$

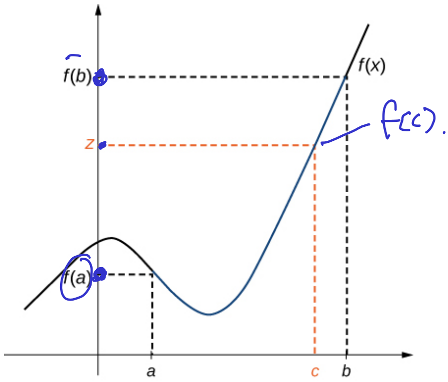
Jump Discontinuity.

Intermediate Value Theorem

Continuous functions have numerous important properties, many of which we will study in MA140. The first of these is the **Intermediate Value Theorem**.

Intermediate Value Theorem (IVT)

Suppose that $f(x)$ is continuous on an interval $[a, b]$.
Let z be any real number between $f(a)$ and $f(b)$.
Then there exists a number $c \in [a, b]$ such that $f(c) = z$.

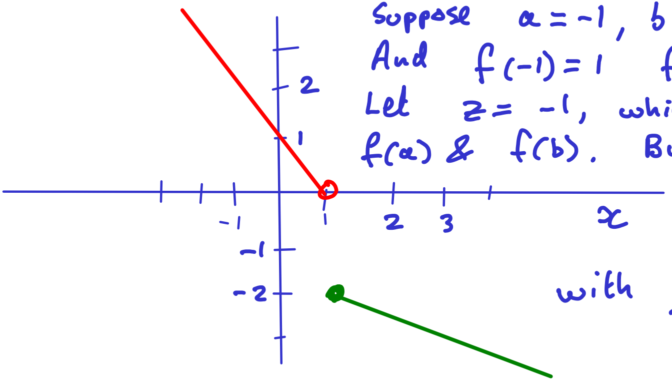


- If you travel by train from Galway to Athlone, then there must be a time when you are at Oranmore station, and a time when you are at Athenry, and at Woodlawn, etc.
- If your car is stopped, and then accelerates to 100km/h, there was a time when it was travelling at 30 km/h.
- Last week, a packet of 20 cigarettes cost €17. Since the budget on Tuesday, they cost €18. But there wasn't a day when they cost, say, €17.50, because the price had a jump discontinuity (so the IVT does not apply here).

Example

Sketch an example of a function for which the IVT does *not* hold.

Eg Draw a function with a jump discontinuity



Suppose $a = -1$, $b = 2$.
And $f(-1) = 1$ $f(2) = -3$.
Let $z = -1$, which is between $f(a)$ & $f(b)$. But there is no $c \in [-1, 2]$ with $f(c) = -1$

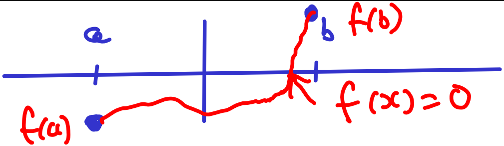
One of the main applications of the IVT is in establishing if an equation as a solution:

Solutions to $f(x) = 0$

If $f(x)$ defined on $[a, b]$ is such that $f(a) < 0$ and $f(b) > 0$, then there must be a value $c \in [a, b]$ such that $f(x) = 0$. More generally, if $f(a)f(b) \leqslant 0$, then $f(x)$ has at least one zero in $[a, b]$.

Example

So that $f(x) = x - \cos(x)$ has at least one zero.



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Example

So that $f(x) = x - \cos(x)$ has at least one zero.

idea: $f(0) = 0 - 1 = -1 < 0$.
$f(2) = 2 - \cos(2) > 0$
since $-1 \leq \cos(x) \leq 1$

So $f(x)$ must have a zero in $[0, 2]$

Given a function $f(x)$,

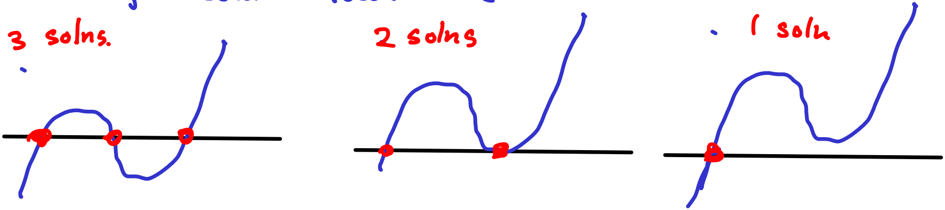
- When we say c is a **zero** of a function, f, we mean that $f(c) = 0$.
- Many books and website also use the terminology "c is a **root** of f". This is particularly the case where $f(x)$ is a polynomial.
- If c is a zero of $f(x)$, then it is a solution to the equation $f(x) = 0$.

Example

How many solutions does $x^3 + 1 = 3x^2$ have?

Step 1: Write this as $x^3 - 3x + 1 = 0$, or $f(x) = 0$ with $f(x) = x^3 - 3x + 1$.

Note f could look like



Example

How many solutions does $x^3 + 1 = 3x^2$ have?

$f(x) = x^3 - 3x^2 + 1$

To use the IVT.

$f(-1) = (-1) - 3 + 1 = -3 < 0$

$f(0) = 0 - 0 + 1 = 1 > 0$

$f(2) = 8 - 12 + 1 = -3 < 0$

$f(3) = \quad = 1 > 0$

So this has 3 solutions!

between $[-1, 0]$

between $[0, 2]$

between $[2, 3]$.

Q 1(c) from 2019 Exam

Use the *Intermediate Value Theorem* to show that the equation

$$2x^3 + 3x^2 - 2x - 1 = 0$$

has three solutions in the range $-2 < x < 1$.

These notes were added after class

Let $f(x) = 2x^3 + 3x^2 - 2x - 1$

$f(-2) = -16 + 12 + 4 - 1 = -1 < 0$

$f(-1) = -2 + 3 + 2 - 1 = 2 > 0$

$f(0) = 0 + 0 + 0 - 1 = -1 < 0$

$f(1) = 2 + 3 - 2 - 1 = 2 > 0$

So, by the IVT there are zeros in $(-2,-1)$, $(-1,0)$ and $(0,1)$ – three in total.

Exercises

Exercises 3.3.1 (Based on Q1(a), 23/24)

Let $g(x) = \begin{cases} 3 & x \leqslant 0 \\ 2x+1 & 0 < x < 1 \\ x^2 & x \geqslant 1. \end{cases}$

(i) Sketch the graph of $g(x)$ on the interval $[-3, 4]$, making use of the empty and full circle notation.

(ii) Compute $\lim_{x \to 1^-} g(x)$ and $\lim_{x \to 1^+} g(x)$. Is g continuous at $x = 1$. If not, classify the type of discontinuity.

Exercise 3.3.2

For what values of b and c is $f(x) = \begin{cases} x^2+1 & x \leqslant -1 \\ x+b & -1 < x < 1 \\ cx^2 & x \geqslant 1. \end{cases}$

continuous at $x = -1$ and $x = 1$?

Exercise 3.3.3 (23/24 Q(1)(c)(ii)

Use the IVT to show that the equation $x^3 - 3x + 1 = 0$ has three solutions in the range $-2 < x < 2$.

Exercises

Exercise 3.3.3 (23/24 Q(1)(c)(ii)

Use the IVT to show that the equation $x^3 - 3x + 1 = 0$ has three solutions in the range $-2 < x < 2$.