2425-MA140 Engineering Calculus

Week 03, Lectures 2
Vertical Asymptotes and Continuity

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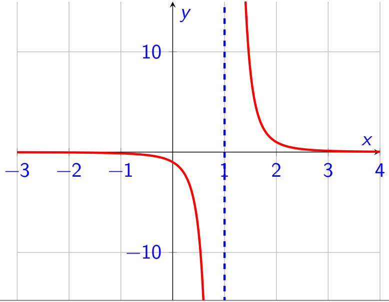
Wednesday, 2 October, 2024



This slides are by Niall Madden, with some content based on notes by Dr Kirsten Pfeiffer, and from Strang & Herman's "Calculus".

Outline

For more, see Section 7.8.1 (Limit of a function of a real number) in *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

However, I *highly* recommend Chapter 2 (Limits) in **Calculus** by Strang & Herman. See openstax.org/books/calculus-volume-1/pages/2-introduction

Reminder

- **Assignment 1** has a deadline of 5pm, Friday. You can access it on Canvas... 2425-MA140... Assignments.
- The **Tutorial Sheet** is available at `https://universityofgalway.instructure.com/files/2040359/download?download_frd=1`
- A new assignment will be posted later this week.

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For help with the assignment, attend a tutorial. The schedule is on the Canvas "Course Information" page: `https://universityofgalway.instructure.com/courses/35693/pages/2425-ma140-information`. Note the change of venue for the Irish language tutorials (Tue at 1, AMB-G021).

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Support is also available at tutorials and **SUMS**.

Yesterday we met the concept of **one-sided limits**:

$\boxed{\lim_{x\to a^-} f(x)}$ is: **limit of f as x approaches a from the left**

$\boxed{\lim_{x\to a^+} f(x)}$ is: **limit of f as x approaches a from the right**

$x \longrightarrow a$

These mean that

- if $\boxed{\lim\limits_{x\to a^-} f(x) = L}$, then we can make $f(x)$ as close to L as we would like by taking x as close to a as needed, and that $\boxed{x < a}$.

  $a \quad \longleftarrow x$

- If $\boxed{\lim \lim\limits_{x\to a^+} f(x) = L}$, then we can make $f(x)$ as close to L as we would like by taking x as close to a as needed, with $\boxed{x > a}$.

Note: One-sided limits can be introduced formally by using the ϵ/δ approach, but we won't do that.

Existence of a limit

$\lim_{x\to a} f(x)$ **exists** if and only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

"finite" means not ∞ or $-\infty$

So if $\lim_{x\to a} f(x) = L$ exists, we have

$$\lim_{x\to a^-} f(x) = \lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = L$$

though it is not necessary that $f(a) = L$

If there is no such L, which is a (finite) real number, we say "the Limit Does not Exist".

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if $\lim_{x \to 2} f(x)$ exists.

First note $\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (3 - x) = 1$

equal!

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \left(\frac{x}{2}\right) = 1$

So the limit exist, and is 1.

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

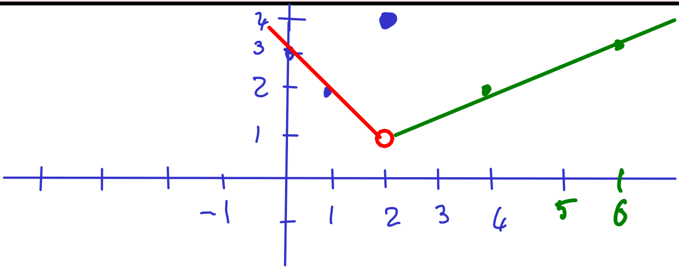
$f(0) = 3$

$f(1) = 2$

$f(4) = 2$

$f(6) = 3$

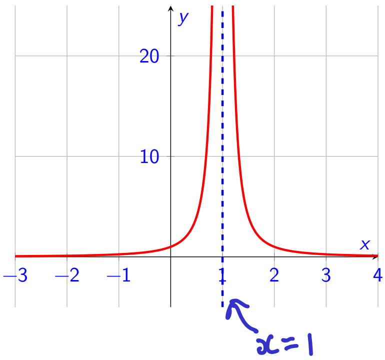
Determine if $\lim_{x \to 2} f(x)$ exists.



Let's revisit the following example from yesterday:

∞

$$\boxed{f(x) = \frac{1}{(x-1)^2}}$$



Note that the points on the graph having x-coordinates very near to 1 are very close to the vertical line $x = 1$. That is, as x approaches 1, the points on the graph of $f(x)$ are closer to the line $x = 1$.

We call the line $x = 1$ a **vertical asymptote** of the graph.

Vertical Asymptotes

Definition: Vertical Asymptote

The vertical line $x = a$ is a **vertical asymptote** of $f(x)$ if any of $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, or $\lim_{x \to a} f(x)$ are ∞ or $-\infty$.

To find a vertical asymptote of a function $f(x) = \dfrac{p(x)}{q(x)}$, we find a value, a for which $p(a) \neq 0$ but $q(a) = 0$.

ie where the function appears to evaluate as $f(a) = \frac{c}{0}$ where $c \neq 0$

(If $f(a) = \frac{0}{0}$, it may have a vertical asymptote – more work is needed).

Example

Find any vertical asymptotes of

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

First note that, if $f(x) = \frac{p(x)}{q(x)}$ then $q(-1) = 0$. So it may have an asymptote at $x = -1$. To check:

$$p(-1) = (-1)^2 - (-1) - 6 = -4 \neq 0$$

So $f(x)$ does have a vertical asymptote at $x = -1$.

Example

Find all vertical asymptotes of the graph of

$$g(x) = -\frac{8}{x^2-4}.$$

difference of two squares

Here $g(x) = -\frac{8}{(x-2)(x+2)}. = \frac{p(x)}{q(x)}$

so we have a vertical asymptote when $(x-2)(x+2)=0$, i,e., $x=2$, $x=-2$.

Example

Find all vertical asymptotes of the graph of

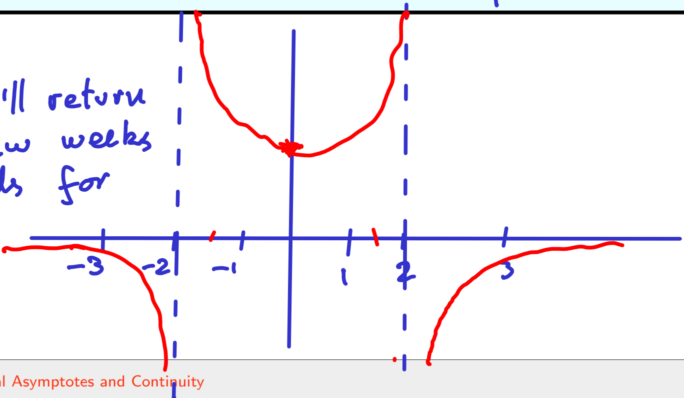
$$g(x) = -\frac{8}{x^2 - 4}.$$

$g(0) = 2$

$g(1) = -\frac{8}{-3} = 2.66.$

Sketch.

Note: we'll return in a few weeks to methods for sketcing!



There is a related concept of a **horizontal asymptote**, but we'll save that for later, when we cover "limits at infinity".

Continuity

Many functions have the property that you can trace their graphs with pen and paper, without lifting the pen from the page. Such functions are called **continuous**.

Some other functions have points were you have to lift the pen occasionally. We say they have a **discontinutity** at such points.

Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

or jump.

More formally, we define continuity in terms of **limits**

Continuity

Definition

A function f is **continuous at** $x = a$ if

1. $f(a)$ is defined, i.e., a is in the domain of f,
2. $\lim_{x \to a} f(x)$ exists. (so left & Right limits are the same)
3. $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$ we say it is **discontinuous** at $x = a$.

If f is continuous **at every point** in its domain, we say f **is continuous.**

"f is discontinuous" means it is discontinuous at at least one point.

Many functions are continuous, e.g. all polynomial functions, most trigonometric functions (not tan), $|x|$, and so on.

Example 1

Determine if $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$.

1 Is 2 in the domain of f?

$f(2) = \dfrac{2^2 - 4}{2 - 2} = \dfrac{0}{0}$ which is not defined. So $f(x)$ is **not** continuous at $x = 2$.

Example 2

Determine if $f(x) = \begin{cases} 1 - x & x \leqslant 0 \\ 2 + x & x > 0 \end{cases}$ is continuous at $x = 0$.

1. $f(0) = 1 - (0) = 1$. So 0 is in the domain of f. It passes the first test!

2. Check if $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

Here $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} 1 - x = 1$.

But $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (2 + x) = 2$

But $1 \neq 2$ so f is not continuous

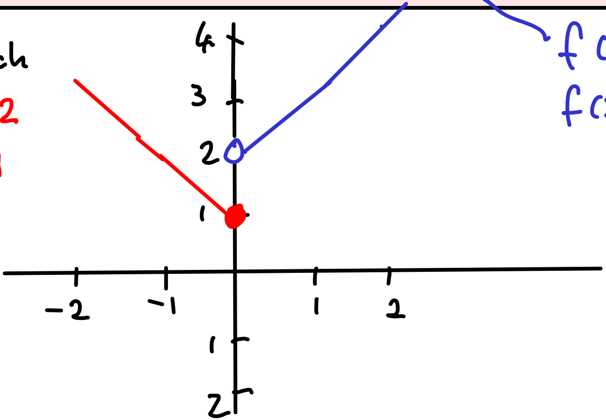
Example 2

Determine if $f(x) = \begin{cases} 1 - x & x \leqslant 0 \\ 2 + x & x > 0 \end{cases}$ is continuous at $x = 0$.

Sketch

$f(-1) = 2$

$f(0) = 1$



$f(1) = 3$

$f(2) = 4$

Finished here Wednesday

Example 3

Determine if $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is continuous at $x = 0$.

1. Yes, 0 is in the domain of f ✓
2. Yesterday, we saw that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

   So the limit does exist

3. Finally $f(0) = 1$ so $\lim_{x \to 0} f(x) = f(0)$

   So it is continuous!!

Continuity

Example

Consider the function

$$f(x) = \begin{cases} x+1, & x < 2 \\ b\,x^2, & x \geqslant 2 \end{cases}$$

For what value of b is f continuous at $x = 2$?

Continuity

Example

For what values of x is $f(x) = \dfrac{2x+1}{2x-2}$ continuous?

Types of discontinuity

We have encountered three types of discontinuity.

- **Removable discontinuity**: $\lim_{x\to a} f(x)$ exists but
$$\lim_{x\to a} f(x) \neq f(a)$$
- **Jump discontinuity**: $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+ f(x)}$ both exist (and are finite), but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$
- **Infinite discontinuity**: At least one of the one-sided limits does not exist.

Types of discontinuity

Example

Each of the following functions has a discontinuity at $x = 2$. Classify it.

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$
2. $g(x) = \dfrac{x^2}{x - 2}$
3. $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$
4. $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

Exercises

Exercises 3.2.1 (Based on Q1(a), 23/24)

Let $g(x) = \begin{cases} 3 & x \leqslant 0 \\ 2x+1 & 0 < x < 1 \\ x^2 & x \geqslant 1. \end{cases}$

(i) Sketch the graph of $g(x)$ on the interval $[-3, 4]$, making use of the empty and full circle notation.

(ii) Compute $\lim_{x \to 1^-} g(x)$ and $\lim_{x \to 1^+} g(x)$. Is g continuous at $x = 1$. If not, classify the type of discontinuity.

Exercise 3.2.2

Find all the vertical asymptotes of $f(x) = \dfrac{x+2}{x^2+2x-8}$.

Exercises

Exercise 3.2.3

For what values of b and x is $f(x) = \begin{cases} x^2 + 1 & x \leqslant -1 \\ x + b & -1 < x < 1 \\ cx^2 & x \geqslant 1. \end{cases}$

continuous at $x = -1$ and $x = 1$?