MA378: Tutorial Sheet 1

These exercises are for tutorials. You do not have to submit solutions to these questions.

Q1. (a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

   (b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

   (c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

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Q2. For each of the following interpolation problems, determine (with explanation) if there is no solution, exactly one solution, or more than one solution. In all cases p_n denotes a polynomial of degree (at most) n. You are not required to determine p_n where it exists.

   (a) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

   (b) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = -2$.

   (c) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

   (d) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

   (e) Find $p_2(x)$ that interpolates (x_0, y_0) and (x_1, y_1) where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$.

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Q3. (From the 24/25 exam)

   (a) Let $f(x) = xe^x$. Calculate the Lagrange form of p_2, the polynomial interpolant to f at the interpolation points $\{x_0, x_1, x_2\} = \{0, 1, 2\}$.

   (b) Use Cauchy's Theorem to find an upper bound for $|f(x) - p_2(x)|$, at $x = 1.5$.

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Q4. Let's suppose there is something called a *clamped* polynomial interpolant, $\hat{p}_{n+2}(x)$, associated with a set of distinct points $\{x_0, x_1, \ldots, x_n\}$, which has the properties that, for a given function f,

- $\hat{p}_{n+1}$ has degree at most $n + 1$.
- $\hat{p}_{n+1}(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.
- $\hat{p}'_{n+1}(x_0) = f'(x_0)$ and $\hat{p}'_{n+1}(x_n) = f'(x_n)$.

Show that, if such a polynomial exists, then it is unique.

Q5. Write down the linear spline l that interpolates $f(x) = \ln(x)$ at the points $x_0 = 1$, $x_1 = 1.5$ and $x_2 = 2$. Use this to estimate $\ln(x)$ at $x = 1.2$. How does this compare to the true value?

Give an estimate for the maximum error, $\max\limits_{1 \leqslant x \leqslant 2} |f(x) - l(x)|$, using Theorem 1.3 from Section 2.1 (Linear Splines).