2425-MA140 Engineering Calculus

Week 04, Lecture 2
Differentiation Rules

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Calcalas | **Calculus**

Díorthaigh | **Derivatives**

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$

Riail an toraidh — **Product rule**

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Riail an lín — **Quotient rule**

$$y = \frac{u}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Cuingriail — **Chain rule**

$$f(x) = u(v(x)) \quad \Rightarrow \quad f'(x) = \frac{du}{dv}\frac{dv}{dx}$$

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Let's learn about...

See also Section 3.3 of **Calculus** by Strang & Herman:
https://openstax.org/books/calculus-volume-1/pages/3-3-differentiation-rules

And Section 3.6 of the text.

Remember:

Reminders

- Assignment 2 is open; deadline is 5pm, 13 Oct. You can access it at https://universityofgalway.instructure.com/courses/46734/assignments/129715. (Or: go to Canvas, click on `Assignments` ... `Problem Set 2` ... the bottom of the page, click `Load Problem Set 2 in a new window`
- This week's **Tutorial Sheet** is available at https://universityofgalway.instructure.com/courses/46734/files/2883465?module_item_id=943734
- Info on next Tuesday's will follow in an announcement tomorrow morning.

- Assignment 3 – posted tomorrow

Differentiation by rule

Yesterday, we saw how to compute derivatives of some functions using the "limit" definition (i.e., **differentiation from first principles**). However, while it is useful to be able to do that for some examples, almost always we use a set of "**rules**" which makes the process much more efficient.

These rules are themselves derived from the "limit" definition – but we don't have to use that every time.

Differentiation by rule

Yesterday, we looked at two rules:

1. The Constant Rule

If f is a constant function, i.e. $f(x) = c$ for all x, then:

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

2. The Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Eg
$f(x) = x^{72}$
$f'(x) = 72x^{71}$

$f(x) = x^{-3} \Rightarrow f'(x) = (-3)x^{-4}.$

The constant multiple rule

Let $f(x)$ be any differentiable function, and let k be constant, then

$$\frac{d}{dx}\big(\underline{kf(x)}\big) = k\frac{d}{dx}\big(f(x)\big).$$

Example: Find the derivative of $f(x) = 5x^4$.

$$f(x) = 5x^4 \qquad (\text{so } k = 5)$$

$$\frac{df}{dx} = f'(x) = 5\underbrace{(x^4)'}_{\text{ie derivative of } x^4.} = 5(4x^3) = 20x^3.$$

The Sum and Difference Rules

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}\big(u(x)+v(x)\big) = \frac{d}{dx}\big(u(x)\big) + \frac{d}{dx}\big(v(x)\big).$$

Similarly, $\frac{d}{dx}\big(u(x)-v(x)\big) = \frac{d}{dx}\big(u(x)\big) - \frac{d}{dx}\big(v(x)\big).$

Example: Find the derivative of $f(x) = 1 + x + x^2$.

$u = (1+x)$ $v = x^2$

$f(x) = u(x) + v(x)$

$f'(x) = u'(x) + v'(x)$

$= (1+x)' + (x^2)' = (1)' + (x)' + 2x = 0+1+2x$

Apply rule again!

Ans: $2x+1$

Actually, the "**Difference Rule**", which states that

$$\boxed{\frac{d}{dx}\big(u(x) - v(x)\big) = \frac{d}{dx}\big(u(x)\big) - \frac{d}{dx}\big(v(x)\big).}$$

can be combined by combining the **Sum Rule** and the **Constant Multiple Rule**.

$$f(x) = u(x) - v(x)$$
$$= u(x) + (-1)\,v(x)$$

Then $f'(x) = u'(x) + \big[(-1)\,v(x)\big]'$ by Sum Rule

$= u'(x) + (-1)\,v'(x)$ by Constant Multiple Rule

$= u'(x) - v'(x)$

Example

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$;

(b) The slope of the tangent line at $x = 2$; , ie $f'(2)$

(c) The equation of the tangent at $x = 2$.

(a) $f'(x) = \underline{-15x^2 + 6x - 9}$

(b) The slope of the tangent line at $x = 2$ is $f'(2)$:

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57.$$

(c) The equation of the line with slope M and passing through a point (x_1, y_1) is

$$y - y_1 = M(x - x_1)$$

$x_1 = 2$ $y_1 = f(x_2)$

The y coordinate at $x = 2$ is

$$\begin{aligned} f(2) &= -5(2)^3 + 3(2)^2 - 9(2) + 7 \\ &= -5(8) + 3(4) - 18 + 7 \\ &= -40 + 12 - 18 + 7 \\ &= -39. \end{aligned}$$

So the tangent line passes through the point $(2, -39)$ and the slope of the line is -57.

Therefore, the equation of this line is $y + 39 = -57(x - 2)$

Ans: The equation of the tangent line is $x = 2$ is $y = 75 - 57x$.

The Product Rule

We now consider some rules which are a little more complicated and, I think, less obvious.

The first concerns the derivative of the **product** of two functions.

The Product Rule

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Let's check that this gives the current answer when evaluating the derivative of $f(x) = x^3$ when we set $u(x) = x^2$ and $v(x) = x$:

$x^3 = (x^2)(x)$ so $u'(x) = 2x$ $v'(x) = 1$

$$\frac{d}{dx}(f) = \frac{d}{dx}\Big(\underbrace{(x^2)}_{u}\underbrace{(x)}_{v}\Big) = (x^2)(1) + x(2x)$$

$$= x^2 + 2x^2$$

$$= 3x^2 \checkmark$$

The Product Rule

The Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example Use the **product rule** to find the derivative of $f(x) = x^3(x^2+1)$. We'll do this 2 ways:

1. Without Product Rule: $f(x) = x^3(x^2+1)$

$= x^5 + x^3$

$f'(x) = 5x^4 + 3x^2$

2. With Product Rule:

$u(x) = x^3 \qquad v(x) = x^2+1$

$u'(x) = 3x^2 \qquad v'(x) = 2x.$

$f'(x) = uv'(x) + vu'(x) = (x^3)(2x) + (x^2+1)(3x^2)$

$= 2x^4 + 3x^4 + 3x^2 = 5x^4 + 3x^2$

The Product Rule

Example: use the product rule to show that, if $f(x) = x\sin(x)$, then $f'(x) = x\cos(x) + \sin(x)$.

$u(x) = x \qquad v(x) = \sin(x)$

$u'(x) = 1 \qquad v'(x) = \cos(x)$

$f'(x) = u(x)\,v'(x) + v(x)\,u'(x)$

$= x\cos(x) + \sin(x)(1)$ ✓

The Quotient Rule

The Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then $f(x) = \dfrac{u(x)}{v(x)}$ is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example: Use this rule to find the derivative of $f(x) = \dfrac{x+1}{x^2}$

$u(x) = x+1 \qquad v(x) = x^2$

$u'(x) = 1 \qquad v'(x) = 2x.$

$$\frac{df}{dx} = \frac{v(x)\,u'(x) - u(x)\,v'(x)}{(v(x))^2} = \frac{(x^2)(1) - (x+1)(2x)}{(x^2)^2}$$

$$= \frac{x^2 - 2x^2 - 2x}{x^4} = \frac{-x^2 - 2x}{x^4} = -x^{-2} - 2x^{-3}.$$

The Quotient Rule

Example

We know that

- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$,
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

Use these facts, and the Quotient Rule to show that

$$\frac{d}{dx}(\tan(x)) = \left(\frac{1}{\cos(x)}\right)^2.$$

The Quotient Rule

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}.$$

$u(x) = \sin(x)$ $\qquad$ $v(x) = \cos(x)$

$u'(x) = \cos(x)$ $\qquad$ $v'(x) = -\sin(x).$

Apply the Quotient Rule:

$$\frac{df}{dx} = \frac{v\,u'(x) - u\,v'(x)}{(v(x))^2} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

$$=: \sec^2(x)$$

Chain Rule

Of all the differentiation rules, the **chain rule** is the most important: most other rules are actually just special cases of it. It applies to a "function of a function"

The Chain Rule

If $u(x)$ and $v(x)$ are differentiable, and f is the composite function $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

$u(v(x))$ is <u>not</u> $u(x)v(x)$.

Eg, if $f(x) = (2x+1)^3 = u(v(x))$ $(= u \circ v(x))$

then $u(v) = v^3$ $\quad v(x) = 2x+1$.

Chain Rule

The Chain Rule

If $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

Example: What is the derivative of $f(x) = \cos(x^2)$?

$$f(x) = \cos(x^2)$$

$$u(v) = \cos(v) \qquad v(x) = x^2$$

$$\frac{du}{dv} = -\sin(v) \qquad \frac{dv}{dx} = 2x.$$

So

$$\frac{df}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx} = -\sin(v)\cdot 2x.$$

$$= -\sin(x^2)\cdot 2x = -2x\sin(x^2)$$

Chain Rule

Example

Find $\frac{dy}{dx}$ if $y = (x^3 + 4x^4 + 7)^{12}$.

Example: Let $u(v) = v^{12}$ and $v(x) = x^3 + 4x^4 + 7$, then y is $y = u\big(v(x)\big)$.

Note that

$$\frac{du}{dv} = 12v^{11} \quad \text{and} \quad \frac{dv}{dx} = 3x^2 + 16x^3.$$

By the Chain Rule we have

$$\frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dx} = 12v^{11}(3x^2 + 16x^3),$$

and therefore

$$\frac{dy}{dx} = 12(x^3 + 4x^4 + 7)^{11}(3x^2 + 16x^3).$$

 Finished Here.

Example (Not done in detail in class)

Find $\dfrac{dy}{dx}$ if $y = \dfrac{1}{(x^4+2x^2+8)^{40}}$.

We have $y = (x^4+2x^2+8)^{-40}$. We can write y as $y(x) = u(v(x))$ with

- $u(v) = \underline{v}^{-40}$ and so $\dfrac{du}{dv} = -40v^{-41}$; and
- $v(x) = \underline{x^4+2x^2+8}$, so $\dfrac{dv}{dx} = 4x^3+4x$.

Applying the Chain Rule: $\dfrac{dy}{dx} = \dfrac{du}{dv}\dfrac{dv}{dx}$. we get

$$\frac{dy}{dx} = -40v^{-41}(4x^3+4x) = \frac{-40(4x^3+4x)}{(x^4+2x^2+8)^{41}}$$

Often we apply the **Chain Rule** to "functions of functions of functions": if $y(x) = t\Big(u\big(v(x)\big)\Big)$, then

$$\frac{dy}{dx} = \frac{dt}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$ when $y = \sin^4(x^5 + 7)$.

Example

Show that the derivative of $y = \cos^2(1/x)$ is $\dfrac{dy}{dx} = 2\dfrac{\sin(1/x)\cos(1/x)}{x^2}$

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Riail an toraidh	$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	**Product rule**
Riail an lín	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	**Quotient rule**
Cuingriail	$f(x) = u(v(x)) \Rightarrow f'(x) = \frac{du}{dv}\frac{dv}{dx}$	**Chain rule**

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Exercises

Exercises 4.2.1 (Based on Q2(a), 2023/2024)

Find the derivative of $f(x) = \dfrac{\sin(x)}{\sqrt{x}}$.

Exercise 4.2.2 (Based on Q2(b), 2019/2020

Find the derivative of $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$.

Exercise 4.2.3

Find the derivative of

1. $f(x) = x^3 \cos(x^2)$
2. $f(x) = \tan^3\left(\sin^2(x^4)\right)$