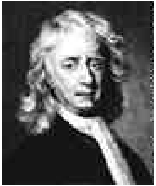
Solving nonlinear equations

§1.3: Newton's Method

MA385 – Numerical Analysis

September 2019

Annotated slides from class. Because of problems with the projector, some were added after class.



Sir Isaac Newton, 1643 - 1727, England. Easily one of the greatest scientist of all time. The method we are studying appeared in his celebrated *Principia Mathematica* in 1687, but it is believed he had used it as early as 1669.

Motivation

Secant method can be written as

$$x_{k+1} = x_k - f(x_k)\phi(x_k, x_{k-1}),$$

where the function ϕ is chosen so that x_{k+1} is the root of the secant line joining the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$.

A closely related idea leads to **Newton's Method**: set $x_{k+1} = x_k - f(x_k)\lambda(x_k)$, where we choose λ so that x_{k+1} is the zero of the tangent to f at $(x_k, f(x_k))$.

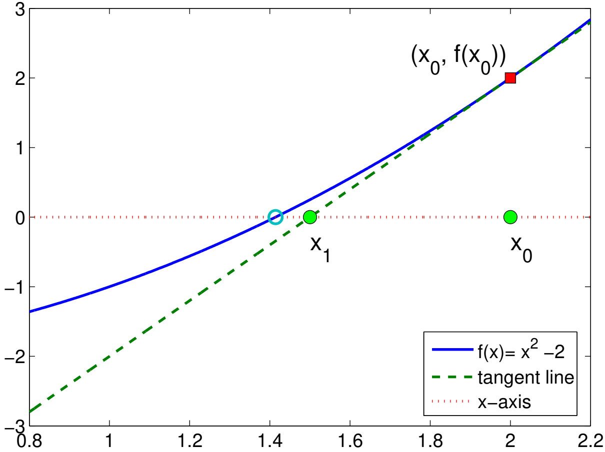


Figure: Estimating $\sqrt{2}$ by solving $x^2 - 2 = 0$ using Newton's Method

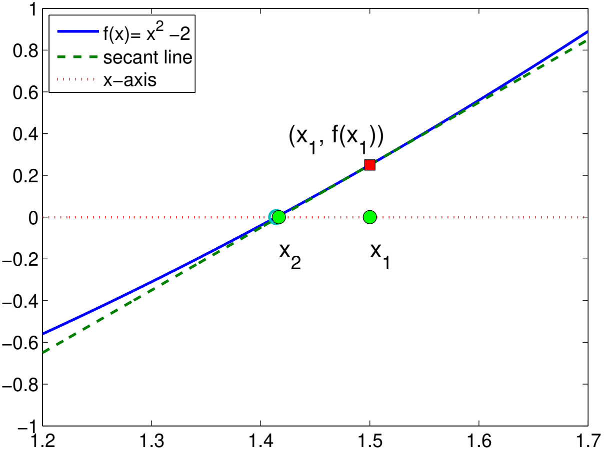


Figure: Estimating $\sqrt{2}$ by solving $x^2 - 2 = 0$ using Newton's Method

The formula for Newton's method may be deduced writing down the equation for the line at $(x_k, f(x_k))$ with slope $f'(x_k)$, and setting x_{k+1} to be its zero; see Exercise 1.8-(i)).

Theorem 1.1 (Newton's Method)

1. Choose any x_0 in $[a, b]$,
2. For $k = 0, 1, \ldots,$ set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{5}$$

Example 1

Use bisection, secant, and Newton's Method to solve $x^2 - 2 = 0$ *in* $[0, 2]$. i.e., $f(x) = x^2 - 2$, so $f'(x) = 2x$

For this case, Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k},$$

which simplifies as

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

Taking $x_0 = 2$, we get $x_1 = 3/2$.
Then $x_2 = 2x_1 + 1/x_1 = 17/12 = 1.46667$.
Then $x_3 = 1.4142$, etc.

Iter	Bisection	Secant	Newton
k	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$
0	1.41	1.41	5.86e-01
1	5.86e-01	5.86e-01	8.58e-02
2	4.14e-01	4.14e-01	2.45e-03
3	8.58e-02	8.09e-02	2.12e-06
4	1.64e-01	1.44e-02	1.59e-12
5	3.92e-02	4.20e-04	2.34e-16
6	2.33e-02	2.12e-06	—
7	7.96e-03	3.16e-10	—
8	7.66e-03	4.44e-16	—
9	1.51e-04	—	—
10	3.76e-03	—	—
11	1.80e-03	—	—
⋮	⋮	⋮	⋮
22	5.72e-07	—	—

Deriving Newton's method geometrically certainly has an intuitive appeal. However, to analyse the method, we need a more abstract derivation based on a **Truncated Taylor Series**.

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(x_k) + \dots$$
$$+ \frac{(x - x_k)^n}{n!}f^{(n)}(x_k) + \frac{(x - x_k)^{n+1}}{(n+1)!}f^{(n+1)}(\eta_k)$$

where $\boxed{\eta_k \in (x, x_k)}$. Now truncate at the second term (i.e., take $n = 1$):

$$f(x) = f(x_K) + (x - X_K) f'(x_K) + \frac{(x-x_K)^2}{2} f''(\eta_K)$$

If we take x to be τ, we get

$$\cancel{f(\tau)}^{\,0} = f(X_K) + (\tau - X_K) f'(x_K) + \tfrac{1}{2}(\tau - X_K)^2 f''(\eta_k)$$

Using $f(\tau)=0$ & neglecting $(\tau - X_K)^2$, and solve for τ:

$$\tau - X_K \approx -f(x_K)/f'(X_K).$$

We now want to show that Newton's converges ***quadratically***, that is, with *at least order* $q = 2$. To do this, we need to

1. Write down a recursive formula for the error.
2. Show that it converges.
3. Then find the limit of $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^2}$.

Step 2 is usually the crucial part.

There are two parts to the proof. The first involves deriving the so-called "***Newton Error formula***".

We'll assume that the functions f, f' and f'' are defined and continuous on the an interval $I_\delta = [\tau - \delta, \tau + \delta]$ around the root τ.

The following proof is essentially the same as the above derivation (see also Theorem 3.2 in Epperson).

Theorem 2 (Newton Error Formula)

If $f(\tau) = 0$ and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point η_k between τ and x_k such that

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2} \boxed{\frac{f''(\eta_k)}{f'(x_k)}}, \qquad (6)$$

Proof: From slide 37,

$$\overset{=0}{\cancel{f(\tau)}} = f(x_k) + (\tau - x_k) f'(x_k) + \tfrac{1}{2}(\tau - x_k)^2 f''(\eta_k)$$

So $-(\tau - x_k) f'(x_k) - f(x_k) = \tfrac{1}{2}(\tau - x_k)^2 f''(\eta_k)$.

If $f'(x_k) \neq 0$, this is $-\tau + \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{x_{k+1}} = \tfrac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}$.

Applying the Newton Error Formula

The Newton Error Formula is important in theory (for proving that Newton's Method converges) and practice (to estimate the error when it is applied to specific problems).

In practical applications, we can use the (6) as follows. So suppose we are applying Newton's Method to solving $f(x) = 0$ on $[a, b]$. Denote the error at Step k by $\varepsilon_k = |\tau - x_k|$. Then we can deduce that

$$\varepsilon_{k+1} \leq \varepsilon_k^2 \frac{\max\limits_{a \leq x \leq b} |f''(x)|}{2|f'(x_k)|}. \tag{7}$$

Then, using that $\varepsilon_0 \leq |b - a|$, (7) can be used repeatedly to bound ε_1, ε_2, etc.

We'll now complete our analysis of this section by proving the convergence of Newton's method.

Theorem 3

Let us suppose that f is a function such that

- *f is continuous and real-valued, with continuous f'', defined on some closed interval $I_\delta = [\tau - \delta, \tau + \delta]$,*
- *$f(\tau) = 0$ and $f''(\tau) \neq 0$,*
- *there is some positive constant A such that*

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all} \quad x, y \in I_\delta.$$

Let $h = \min\{\delta, 1/A\}$. If $|\tau - x_0| \leq h$ then Newton's Method converges quadratically.

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From the NEF,

$$|\tau - x_{k+1}| \leq \frac{1}{2}|\tau - x_k| \cdot |\tau - x_k| \cdot \left| \frac{f''(\eta_k)}{f'(x_k)} \right|.$$

Since $|\tau - x_0| \leq h$, and $\left|\frac{f''(\eta_0)}{f'(x_0)}\right| \leq A$,

$$|\tau - x_1| \leq \frac{1}{2}|\tau - x_0| \cdot h \cdot A \leq \frac{1}{2}|\tau - x_0|,$$

since $h \leq 1/A$.

Similarly $|\tau - x_2| \leq \frac{1}{2}|\tau - x_1| \leq \left(\frac{1}{2}\right)^2 |\tau - x_0|$.

And, in general, $|\tau - x_k| \leq \left(\frac{1}{2}\right)^k |\tau - x_0|$.

So, since $|\tau - x_0| < \infty$, $\lim_{k\to\infty} |\tau - x_k| = 0$.

PTO

So the method converges, at least linearly.

Now to see it converges quadratically, note that, in the NEF, $\eta_k \in [\tau, x_k]$. So as $k \to \infty$, $\eta_k \to \tau$. And, of course, $x_k \to 0$.

So

$$\lim_{k \to \infty} \frac{|\tau - x_{k+1}|}{|\tau - x_k|^2} = \mu$$

where $\mu = \frac{1}{2} \frac{|f''(\tau)|}{|f'(\tau)|}$ is a constant.

$\boxtimes$.

Exercise 1.8 ($\star$)

Write down the equation of the line that is tangential to the function f at the point x_k. Give an expression for its zero. Hence show how to derive Newton's method.

Exercise 1.9

(i) Is it possible to construct a problem for which the bisection method will work, but Newton's method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which Newton's method will work, but bisection will fail? If so, give an example.

Exercise 1.10 ($\star$ Homework problem)

(i) Let q be your student ID number. Find k and m where $k-2$ is the reminder on dividing q by 4, and $m-2$ is the remainder on dividing q by 6.

(ii) Show how Newton's method can be applied to estimate the postive real number $\sqrt[k]{m}$. Simplify the computation as much as possible.

(iii) Do three iterations by hand of Newton's Method for this problem.

Exercise 1.11 ($\star$ Homework problem)

Suppose we want apply Newton's method to solving $f(x)=0$ where f is such that $|f''(x)| \leq 10$ and $|f'(x)| \geq 2$ for all x. How close must x_0 be to τ for the method to converge?

Exercise 1.12 ($\star$ Homework problem)

Here is (yet) another scheme called *Steffenson's Method*: Choose $x_0 \in [a, b]$ and set

$$x_{k+1} = x_k - \frac{\big(f(x_k)\big)^2}{f\big(x_k + f(x_k)\big) - f(x_k)} \text{ for } k = 0, 1, 2, \ldots.$$

It is remarkable because it is quadratic, like Newton's, but does not require derivatives of f.

(a) How how the method can be derived from Newton's Method, using the formal definition of the derivative.

(b) ... [This bit will change when I get a chance to write it up. Sorry!]