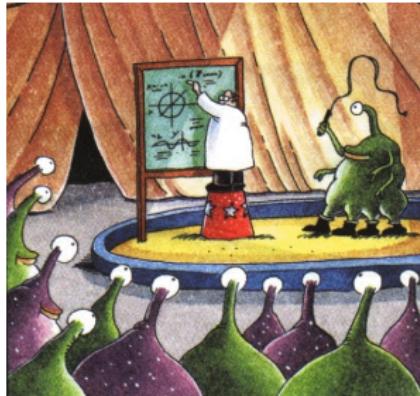


# Week 08, Lecture 1 (REVISED) Techniques of Integration

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## Assignments, etc

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- ▶ **Problem Set 6** is open, and will be covered in tutorials this week. Deadline is 5pm next Monday (10 November).
- ▶ **Problem Set 7** opens latex this week.
- ▶ The finally weekly assignment, will open next week.
- ▶ Reminder: The second **class test** takes place November 18.

# This morning, I think we will think about...

- 1 Substitution
  - Definite Integrals
- 2 Rational functions
  - Partial Fractions
- 3 Integration by Parts
  - When to use
  - The Rule
  - Why it works
  - Examples
- 4 Exercises

See also Section 5.5 (Substitution) of **Calculus** by Strang & Herman:  
[math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)) Maybe also  
Section 7.1 (Integration by Parts).

## Substitution

Suppose we want to evaluate an integral of the form

$$\int e^{x^3+x^2} (3x^2 + 2x) \, dx.$$

At first, this looks tricky: there is nothing like this in our table of integrals.

However, there is something a little unusual about it: it features both the function  $x^3 + x^2$ , and its derivative  $3x^2 + 2x$ .

It turns out that such problems are quite common (at least in textbooks and on exams!). Moreover, there is a handy technique called **substitution** for evaluating them. In this case:

# Substitution

## Method of substitution

When integrating an *integrand* of the form  $\int f(g(x))g'(x)dx$ , set  $u = g(x)$ , and then use that

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

**Equivalently:**  $\int f(u) \frac{du}{dx} dx = \int f(u) du$ . (For a proof of why this works, see Section 5.5 of the textbook).

After this substitution, our task is reduced to evaluating  $\int f(u) du$  which, we hope, is easier.

# Substitution

## Example

Evaluate the integral  $\int 3x^2 \sin(x^3) dx$ .

Notice that  $3x^2$  is the derivative of  $x^3$ .

Let's try integration by substitution with  $u = x^3$ .

If  $u = x^3$ , then  $\frac{du}{dx} = 3x^2$ , so

$$du = \frac{du}{dx} dx = 3x^2 dx.$$

Thus,

$$\begin{aligned}\int \sin(x^3)3x^2 dx &= \int \sin(u) du \\&= -\cos(u) + C \\&= -\cos(x^3) + C.\end{aligned}$$

# Substitution

## Example

Evaluate  $\int 2x\sqrt{1+x^2} dx$ .

Notice that  $2x$  is the derivative of  $1+x^2$ .

Let's try integration by substitution with  $u = 1+x^2$ :

# Substitution

## Example

Evaluate  $\int \cos(4x - 7) dx$ .

Idea: think of this as  $\frac{1}{4} \int \cos(4x - 7)4 dx$ .

# Substitution

## Example

Show that  $\int \sin^3(x) \cos(x) dx = \frac{1}{4} \sin^4(x) + C.$

**Substitution** can be used with **definite integrals**. However, this may require a change to the limits of integration.

### Substitution with Definite Integrals

Let  $u = g(x)$ , with  $g'$  continuous on  $[a, b]$ , and  $f$  continuous over the range of  $u = g(x)$ . Then,

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u) du.$$

This allows us to apply the FTC2, without having to invert the substitution.

**Example**

$$\text{Evaluate } I = \int_0^1 x^2(1 + 2x^3)^2 dx$$

**Example**

$$\text{Evaluate } I = \int_{-1}^0 xe^{x^2} dx$$

## Recall: Rational Functions

A *rational function* is a function of the form  $f(x) = \frac{p(x)}{q(x)}$ , where  $p(x)$  and  $q(x)$  are polynomials.

Before trying to find an antiderivative of a rational function

$$f(x) = \frac{p(x)}{q(x)}$$
:

Step 1: If  $\deg(p(x)) \geq \deg(q(x))$ , divide  $p(x)$  by  $q(x)$ .

Step 2: Check if integration by substitution might work.

Step 3: Factorise the denominator as far as possible.

Step 4: Write the rational function as sum of **partial fractions** to simplify.

## Example

Evaluate the integral

$$\int \frac{x}{x^2 + 1} dx.$$

In this case we can use substitution.

**Example**

Evaluate the integral

$$\int \frac{3x + 4}{x^2 + 7x + 12} dx.$$

In this case, we must factorise the denominator, and express the integrand as **partial fractions**. Factorise:

$$x^2 + 7x + 12 = (x + 4)(x + 3).$$

Express as Partial Fractions:

$$\frac{3x + 4}{x^2 + 7x + 12} = \frac{A}{x + 4} + \frac{B}{x + 3}$$

With a little work, we can find that  $A = 8$  and  $B = -5$ . Therefore,

$$\frac{3x + 4}{x^2 + 7x + 12} = \frac{8}{x + 4} - \frac{5}{x + 3}.$$

We can now express the integral as

$$\begin{aligned}\int \frac{3x+4}{x^2+7x+12} dx &= \int \left( \frac{8}{x+4} - \frac{5}{x+3} \right) dx \\ &= \underbrace{\int \frac{8}{x+4} dx}_{I_1} - \underbrace{\int \frac{5}{x+3} dx}_{I_2}.\end{aligned}$$

First, we evaluate  $I_1$ .

$$I_1 = \int \frac{8}{x+4} dx = 8 \int \frac{1}{x+4} dx.$$

If we let  $u = x + 4$ , then  $du = dx$  and, hence,

$$I_1 = 8 \int \frac{1}{x+4} dx = 8 \int \frac{1}{u} du = 8 \ln|u| + C_1 = 8 \ln|x+4| + C_1.$$

Similarly, we find that:  $I_2 = \int \frac{5}{x+3} dx = 5 \ln|x+3| + C_2$ .

To conclude:

$$\begin{aligned}\int \frac{3x+4}{x^2+7x+12} dx &= \int \left( \frac{8}{x+4} - \frac{5}{x+3} \right) dx \\&= \int \frac{8}{x+4} dx - \int \frac{5}{x+3} dx \\&= I_1 - I_2 \\&= (8 \ln|x+4| + C_1) - (5 \ln|x+3| + C_2) \\&= 8 \ln|x+4| - 5 \ln|x+3| + C.\end{aligned}$$

We use **substitution** when integrating the product of two functions,  $f(x)g'(x)$ , where  $f(x) = f(g(x))$ , which is a very specialised situation.

**Integration by parts** is also applied when we are integrating the **product** of a function and a derivative  $f(x) = u(x)v'(x)$ , but with no other constraint.

**Integration by Parts** (IBP) is, by some distance, the most important technique for integration, in both theory and practice.

## Integration by Parts

If  $u$  and  $v$  are differentiable functions in the variable  $x$ , then

$$\int u(x) dv = u(x)v(x) - \int v(x) du.$$

It can also be written as

$$\int uv' dx = uv - \int vu' dx.$$

The link between the two formulations is given by

$$dv = v' dx \quad \text{and} \quad du = u' dx.$$

The Product Rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

We can use this rule to develop another integration technique after a little rearrangement. From the above we have

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

and integrating both sides gives

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

**Example**

Evaluate  $I = \int x \cos(x) dx$

We'll take  $u = x$  and  $dv = \cos(x)dx$ .

## Exercises

### Exer 8.1.1

Evaluate the follow integrals using a suitable substitution.

$$1. \int \sin(\ln x) \frac{1}{x} dx.$$

$$2. \int x^2(x^3 + 5)^9 dx$$

$$3. \int \frac{\sin(x)}{\cos^3(x)} dx$$

### Exer 8.1.2

Evaluate  $\int_0^1 x^2(x^3 + 5)^9 dx$