2526-MA140: Week 02, Lecture 2 (L05)

Introduction to Limits

Dr Niall Madden

University of Galway

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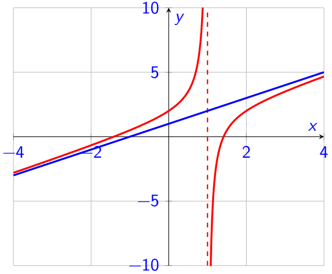


Slides by Niall Madden, with some material adapted from textbooks, and original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Chapter 2 (Limits) of Strang and Herman's Calculus, especially Sections 2.2 (Limit of a Function) and 2.3 (Limit Laws).

Slides are on canvas, and at
niallmadden.ie/2526-MA140

Reminders

- Tutorials started **this** week.
- Current assignment (for this week's tutorials) is PS-0. Just for practice. See https://universityofgalway.instructure.com/courses/46734/assignments/128373
- **Assignment 1** (PS-1) due 5pm, Monday 5 October. Will be covered in tutorials next week.
- Two class tests planned for this module, each worth 10% of the final grade.
  - Test 1: **Tuesday, 14 October** (Week 5)
  - Test 2: **Tuesday, 18 November** (Week 10)
  - Contact Niall if you have any concerns, or wish to avail of alternative arrangements, as provided by LENS reports.

Towards Limits

When we were considering the domain of a function, we looked at those x-values for which the function was not defined.

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1}$$

Neither f nor g are defined at $x = 1$.

But what happens if x gets very closed to 1?

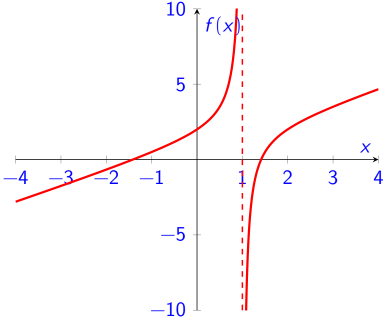
x	0.900	0.990	0.999	1	1.001	1.010	1.100
$f(x)$							
$g(x)$							

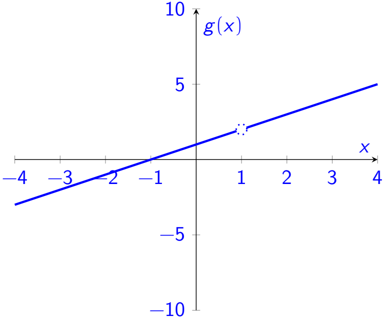
Let's look at the graphs of f and g.

Towards Limits

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad\qquad g(x) = \frac{x^2 - 1}{x - 1}$$





Towards Limits

In the previous example, we saw that, although neither f nor g was defined at $x = 1$, they behaved very differently as x approaches 1.

To discuss this we'll need the concept of a **limit** which, roughly, relates to the value of function as it **approaches** a point (but not actually at that point).

$$\lim_{x \to a} f(x) = L$$

The concept of a limit is a prerequisite for a proper understanding of calculus, and numerical methods.

Definition of a Limit

Some conventions and terminology we'll use:

- x is a variable.
- a is a fixed number.
- ϵ is a **small** positive number (that we get to choose).
- δ is another **small** positive number (determined by ϵ).
- $|x - a| < \delta$ means that the distance between x and a is less than δ, i.e. very small.
- As x approaches a, so $f(x)$ approaches a number L.

When we write

$$\lim_{x \to a} f(x) = L,$$

we read

"The limit of f, as x goes to a, is L".

Definition of a Limit

LIMIT: formal definition

$$\lim_{x \to a} f(x) = L,$$

means that, for every number $\epsilon > 0$, it is possible to find a number $\delta > 0$, such that

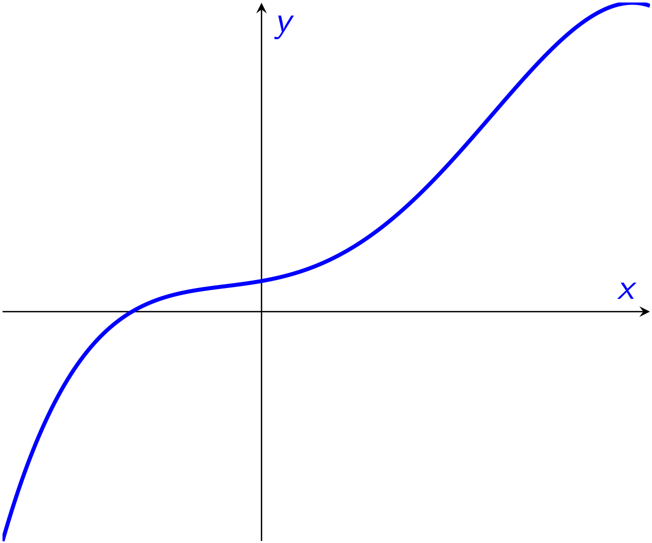
$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

LIMIT: Informal explanation

$$\lim_{x \to a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

Definition of a Limit



Definition of a Limit

Example

Prove formally that $\lim_{x \to 3} (4x - 5) = 7$.

That is, for arbitrary ϵ, find a δ such that

$$|(4x - 5) - 7| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta.$$

Definition of a Limit

The approach we just used is technically correct, but not very practical in many cases.

Fortunately, there are other methods that can be used

- to show that a limit exists;
- find the limit of a function, $f(x)$ as $x \to a$.

Properties of Limits

See also...

... Section 2.3 of the textbook: Limit Laws

Suppose that $\lim_{x \to a} f_1(x) = L_1$, and $\lim_{x \to a} f_2(x) = L_2$ and $c \in \mathbb{R}$ is any constant. Then,

(1) $\lim_{x \to a} c = c, \ c \in \mathbb{R}$

(2) $\lim_{x \to a} x = a$

Properties of Limits

(3) $\lim_{x \to a} [c f_1(x)] = cL_1$

(4) $\lim_{x \to a} [f_1(x) + f_2(x)] = L_1 + L_2$ and
$\lim_{x \to a} [f_1(x) - f_2(x)] = L_1 - L_2$

Properties of Limits

(5) $\lim_{x \to a} \left(f_1(x) f_2(x) \right) = L_1 L_2$

(6) $\lim_{x \to a} \left((f_1(x))^n \right) = (L_1)^n$

Properties of Limits

(7) $\lim_{x \to a} \left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$, providing $L_2 \neq 0$.

(8) $\lim_{x \to a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\boxed{\lim_{x \to a} x^n = a^n}$$

Example

Evaluate the limit $\lim_{x \to 1} (x^3 + 4x^2 - 3)$

Example

Evaluate $\lim_{x \to 1} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ using the Properties of Limits.