MA378 Chapter 3: Numerical Integration

§3.1 Introduction / Newton-Cotes / The Trapezium Rule

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1.1 Introduction

Problem

Given a real-valued function f that is continuous on $[a, b]$, can we find an estimate for

$$I(f) := \int_a^b f(x)dx?$$

And if we can, can we say how accurate it is?

1.1 Introduction

Why is this an interesting problem?

- Many problems in applicable mathematics require definite integrals to be evaluated. (These methods were originally motivated by problems in astronomy).
- Evaluating them by finding the anti-derivative can be hard, and very hard to automate.
- Some times, although the function is integrable, its anti-derivative doesn't exist in a closed form.

Except when it is impossible.

1.1 Introduction

The process of numerically estimating a definite integral is called **Numerical Integration** or **Quadrature**.

The formulae we'll derive all look like

$$Q_n(f) := q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + \cdots + q_n f(x_n).$$

Here the points x_i are called *quadrature points* and the q_i are *quadrature weights*.

We need a way of choosing these. ie (1) Choose the x_i
(2) Choose the q_i

The simplest approach is to take the points to be equally spaced, i.e., $x_i = a + hi$ where $h = (b - a)/n$.

How to choose the weights?

We've spent quite a while talking and thinking about approximating functions with polynomials. So why not find a polynomial interpolant to f and take the integral of that to be the answer? The appeal of this approach is due to the fact that

- Finding polynomial interpolants is easy.
- Integrating polynomials is easy.
- We can estimate the error easily (yet again, we'll make use of Cauchy's Theorem).

This leads to the **Newton-Cotes** methods, which are the subject of this section, and the next one. Later again, we'll look at more sophisticated methods, called **Gaussian Methods** which use non-uniformly spaced points.

1.2 Newton-Cotes methods

Definition 1.1 (Newton-Cotes quadrature)

The **Newton-Cotes** quadrature rule for $\int_a^b f(x)dx$ with $n+1$ points is derived by integrating exactly the polynomial of degree n that interpolates f at the $n+1$ equally spaced points $a = x_0 < x_1 < \cdots < x_n = b$. The method is written as

$$Q_n(f) := q_0 f_0 + q_1 f_1 + q_2 f_2 + \cdots + q_n f_n,$$

where we use the notation $f_k := f(x_k)$.

That is, the quadrature weights are chosen so that

$$Q_n(f) = \int_a^b p_n(x)dx,$$

where p_n is the polynomial of degree n that interpolates f at the $n+1$ quadrature points...

1.2 Newton-Cotes methods

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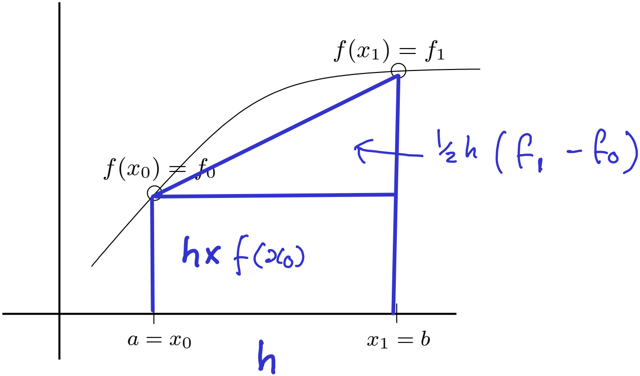
However, it turns out that we can compute the weights q_0, q_1, ..., q_n, **without** knowing p_n.

We'll do this for $n = 1$ in the next section, and $n = 2$ (the most interesting case) in Section 3.2.

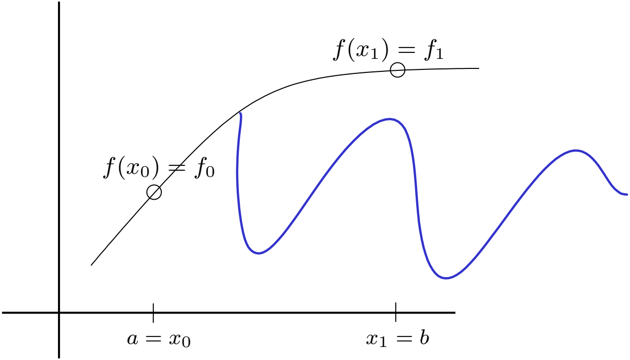
1.3 The Trapezium rule

Suppose we wanted to estimate the integral of a function, f, shown below, on the interval $[a, b]$.

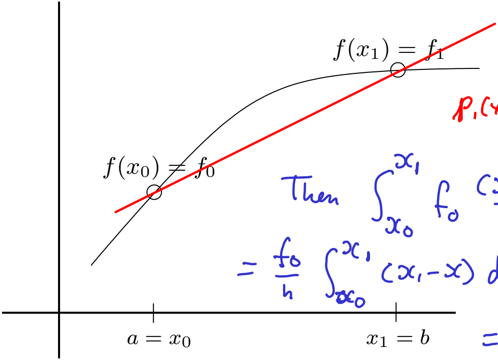


So Area is $h\left(f_0 + \frac{1}{2}f_1 - \frac{1}{2}f_0\right) = \frac{h}{2}\left(f_0 + f_1\right)$

Method 1: We could try to estimate the area of the trapezium that fits under the graph:



Method 2: We could find p_1, the polynomial of degree 1 that interpolates f at $x = a$ and $x = b$:



← The equation of this line is

$$p_1(x) = f_0 \frac{(x_1 - x)}{h} + f_1 \frac{(x - x_0)}{h}$$

Then $\displaystyle\int_{x_0}^{x_1} f_0 \frac{(x_1 - x)}{h} + f_1 \frac{(x - x_0)}{h}\, dx$

$$= \frac{f_0}{h} \int_{x_0}^{x_1} (x_1 - x)\, dx + \frac{f_1}{h} \int_{x_0}^{x_1} (x - x_0)\, dx.$$

$$= \frac{1}{2} h \left(f_0 + f_1 \right)$$

Note that this shows that $q_i = \displaystyle\int_a^b L_i(x) dx$, where, as usual, the L_i are the Lagrange Polynomials.

Method 3: The third approach for generating the Trapezium Rule is called the *Method of Undetermined Coefficients*. Because the method is based on integrating a linear function we expect it to yield an exact solution for any constant or linear function (i.e., there should be no error). To keep the algebra simple, we'll take $a = 0$ and $b = 1$. So,

$$Q_1(f) = q_0 f(0) + q_1 f(1),$$

and, setting $f(x) \equiv 1$, and then $f(x) = x$ we get

$f \equiv 1$ $\quad \int_0^1 f(x)\,dx = 1 \quad \& \quad Q_1(f) = q_0 + q_1$

So $\quad q_0 + q_1 = 1$

$f \equiv x$ $\quad \int_0^1 x\,dx = \frac{1}{2}, \quad Q(f) = q_0(0) + q_1(1) \Rightarrow q_1 = \frac{1}{2}$

$\Rightarrow q_0 = \frac{1}{2}$

Now we need to extend this to estimating $\int_a^b g(x)dx$ as follows:

Define a mapping from $[0,1]$ to $[a,b]$ as

$t = a + (b-a)x$. Note: $\frac{dt}{dx} = b-a$.

Let $f(x) = g(a \underbrace{+ (b-a)x}_{t})$.

So

$$\int_a^b g(t)\,dt = \int_0^1 f(x)(b-a)\,dx = (b-a)\int_0^1 f(x)\,dx$$

So $\underbrace{\frac{1}{2}f(0) + \frac{1}{2}f(1)}_{\text{Trap Rule on } [0,1]} \to \underbrace{\frac{(b-a)}{2}\big(g(a) + g(b)\big)}_{\text{Trap Rule on } [a,b]}$

Example 1.2

Use the trapezoid to estimate

Error is 0.0367.

$$\int_0^{\pi/4} \cos(x)dx.$$

Calculate the (exact) error $|\int_a^b f(x)dx - Q_1(f)|$.

$f(x) = \cos(x)$, and $a = x_0 = 0$, $b = x_1 = \pi/4$.

Also $h = \pi/4$

Then $Q_1(f) = \frac{b-a}{2}\big(f(a) + f(b)\big) = \frac{\pi/4}{2}\big(\cos(0) + \cos(\pi/4)\big)$

$= 0.67038.$

Also $I(f) = \int_0^{\pi/4} \cos(x)dx = \sin(x)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}} = 0.707!$

1.4 Exercises

Exercise 1.1

Let q_0, q_1, ..., q_n be the quadrature weights for the Newton-Cotes rule $Q_n(f)$. Show that $q_i = q_{n-i}$ for $i = 0, \dots n$.

Exercise 1.2

$\star$ Show that $\sum_{i=0}^{n} q_i = b - a$.