2425-MA140 Engineering Calculus

Week 10, Lecture 2 (L29)
Average and Root-Mean-Squared Values of Functions

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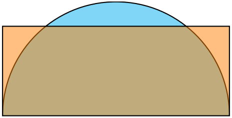


Assignments,etc

1. I'm working on grading Q8 of **Assignment 6** results should be available by Friday.
2. Grades for **Assignment 7** will be posted by Monday,
3. **Assignment 8** is open, and the tutorial sheet is available.

Today, we mean to discuss...

1. Average values of functions

2. Root-Mean-Square Values

3. Exercises

The idea of the "average value" and "RMS" aren't really covered in our textbook.

Average values of functions

In many applications we wish to know the "average" (or **mean**) value of a continuously varying quantity, which is represented by a function.

We are already familiar with this concept when dealing with the mean of a set of n values: $\{x_1, x_2, \dots, x_n\}$. There are two (equivalent) ways of thinking about this:

1. The mean of the set of values is
$$\bar{x} = \frac{x_1 + x_2 + \cdots x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k.$$
2. $\bar{x}$ is the mean of the set of values of $n\bar{x} = x_1 + x_2 + \cdots + x_n$.

That is, if we replaced the x_l with the constant value $\bar{x}$, the sum would not change.

Average values of functions

We can extend both these ideas to defining the "average value of a function", on the interval $[a, b]$, getting the same result.

First, suppose we take n subintervals of $[a, b]$, and denote their end-points $\{x_0, x_1, \dots, x_n\}$. Note that $x_k = x_0 + k\Delta x$, where $\Delta x = (b - a)/n$.

Average values of functions

Now take the average of the n sampled values:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{1}{n} \sum_{k=1}^{n} f(x_k)$$

$$= \frac{\Delta x}{b-a} \sum_{k=1}^{n} f(x_k) = \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

If $n \to \infty$ (or $\Delta x \to 0$), we get the **average value of $f(x)$ on $[a, b]$** is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Average values of functions

The second version (which was mentioned in Week 7, Lecture 1) is more insightful, I think:

Average value of a function

The constant $\bar{f}$ is the **average** value of $f(x)$ on $[a, b]$, if

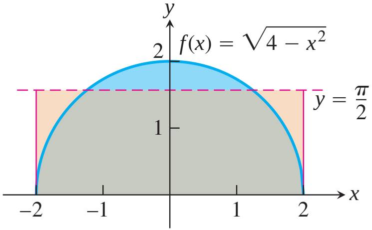
$$\int_a^b \bar{f}\,dx = \int_a^b f(x)\,dx.$$

To see this is equivalent:

Average values of functions

Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.



Average values of functions

Example

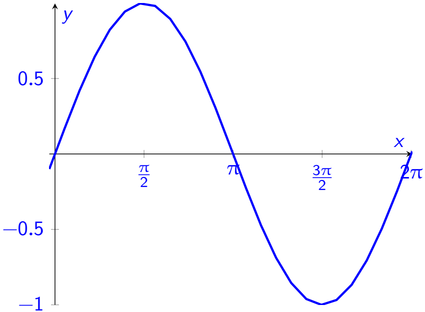
Find the average value of the function $f(x) = x^2 - x - 2$ on $[-3, 3]$.

Average values of functions

Average of $\sin(x)$

Find the average values of $f(x) = \sin(x)$ on

1. $[a, b] = [0, \pi]$
2. $[a, b] = [0, 2\pi]$



Root-Mean-Square Values

In some contexts, the **average value** of a function is a useful summary statistic. But it can be misleading too, as the last example showed.

Notable examples of this include

- The average value of an alternating current is zero;
- The average motion of a piston is zero.

There (especially in power electronics) we need another measure to summarise a function

Root Mean Squared (RMS)

The **root mean square (RMS)** of a function $f(x)$ is

$$f_{\mathrm{RMS}} := \left(\frac{1}{b-a} \int_a^b [f(x)]^2 \, dx \right)^{1/2}$$

Root-Mean-Square Values

Example

An electric current $i(\theta)$ is given by $i(\theta) = I_{\text{peak}} \sin(\theta)$ where I_{peak} is a constant. Find the root mean square of $i(\theta)$ over the interval $[0, 2\pi]$.

(*Hint: use that* $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$).

Root-Mean-Square Values

Exercises

Exer 10.2.1

Find the average value of $f(x) = \dfrac{1}{1-4x^2}$ for $0 \leqslant x \leqslant 1/4$.

Exer 10.2.2

Find $b > 0$ such that the average value of $f(x) = x^2 - 2x + 3/4$ on the interval $[0, b]$ is zero.
Compute the root mean squared of $f(x)$ on the same interval.