Linear Algebra I - Assignment 4

Deadline: 5pm, Monday 7 November

Q1 [25 Marks] Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n, in the variable $t \in \mathbb{R}$. Which of the following are **subspaces** of $\mathbb{P}_3$? Explain your answers.

(a) $H_0 := \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{P}_2$.

(b) $H_1 := \{\mathbf{0}, t, t^2, t^3\}$.

(c) $H_2 := \operatorname{span}\{4t^2\}$

(d) $H_3 := \operatorname{span}\{t, t^3\}$

(e) $H_4 := \{p(t) \in \mathbb{P}_1\}$.

(f) $H_5 := \{p(t) \in \mathbb{P}_2\}$.

(g) $H_6 := \{p(t) \in \mathbb{P}_2 : p'(0) = 0\}$.

(h) $H_7 := \{p(t) \in \mathbb{P}_2 : p(1) = 0\}$.

Tip: in Week 2 we saw that, in order to verify that H is a subspace of a real vector space V, we have to check:

- That every element of H is also an element of V;
- That the zero vector in V is also in H;
- If $u, v \in H$ then $u + v \in H$.
- If $u \in H$ then $cu \in H$ for any scalar $c \in \mathbb{R}$.

Q2 [15 Marks] (This is Q1(c) on the 2021/2022 exam paper). Let

$$A = \begin{bmatrix} -3 & 8 & 19 \\ 1 & -6 & -13 \\ 2 & -2 & -6 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}.$$

Determine, with justification, if $x \in \operatorname{Nul} A$, and if $x \in \operatorname{Col} A$.

Q3 [15 Marks] Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} p+q+r \\ p+q+r \\ p+2q-r \\ p+2q-r \end{bmatrix} : p, q, r \in \mathbb{R} \right\},$$

of $\mathbb{R}^4$ and give a basis for it.

Q4 [20 Marks] (Based on Q3(b) on the 2021/2022 exam paper).

(a) What is the largest possible rank of an 10×5 matrix?

(b) If the null space of a 10×8 matrix A is 1-dimensional, what are the dimensions of its column space, of its row space, and of its left null space?

(c) Give an example of a 4×3 matrix A with nullity $A = 2$.

(d) Suppose a $m \times n$ matrix has $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in both its null and column space. What are m and n?

(e) Give an example of a matrix that has $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in its null space, and $y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in its column space.

Q5 [10 Marks] (This is similar to Q2(a) on the 2021/2022 exam paper). Let $\mathbb{P}_n$ denote the vector space of polynomials of degree at most n. Determine if

$$p_1(t) = 1 - 2t, \quad p_2(t) = 3 + 4t, \quad \text{and} \quad p_3(t) = 5,$$

are linearly independent in $\mathbb{P}_1$. Give a basis for $\operatorname{Span}\{p_1(t), p_2(t), p_3(t)\}$.

[15 Marks] for clarity and correctness of exposition and presentation.