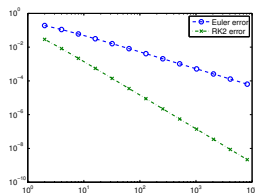
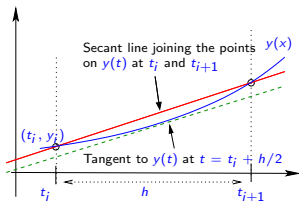


MA385 Part 2: Initial Value Problems

2.4: Runge-Kutta 2 (RK2)

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1 Modified Euler Method

2 General RK2

- Using consistency
- Ensuring 2nd-order

3 Exercises

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For more details, see Chapter 6 of [Süli and Mayers, *An Introduction to Numerical Analysis*](#). In particular:

- ▶ Section 12.3 deals with Consistency; pay attention to the discussion leading to (12.21).
- ▶ Section 12.4 (Runge-Kutta methods)

Recall our original motivation of Euler's method: use the slope of the tangent to y at t_i as an approximation for the slope of the secant line joining the points $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$.

One could argue, given the diagram on the next slide, that the slope of the tangent to y at $t = (t_i + t_{i+1})/2 = t_i + h/2$ would be a better approximation. This would give

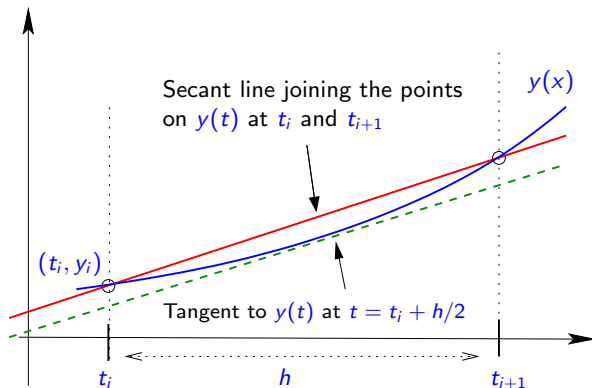
$$y(t_{i+1}) \approx y_i + hf\left(t_i + \frac{h}{2}, y(t_i + \frac{h}{2})\right). \quad (1)$$

However, we don't know $y(t_i + h/2)$, but can approximate it using Euler's Method: $y(t_i + h/2) \approx y_i + (h/2)f(t_i, y_i)$.

1. Modified Euler Method

Modified (Midpoint) Euler's Method

$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)\right). \quad (2)$$



1. Modified Euler Method

Example 2.4.1

Use the Modified Euler Method to approximate $y(1)$ where

$$y(0) = 1, \quad y'(t) = y \log(1 + t^2).$$

This has the solution $y(t) = (1 + t^2)^t \exp(-2t + 2 \tan^{-1} t)$.

1. Modified Euler Method

n	Euler		Modified	
	\mathcal{E}_n	$\mathcal{E}_n/\mathcal{E}_{n-1}$	\mathcal{E}_n	$\mathcal{E}_n/\mathcal{E}_{n-1}$
1	3.02e-01		7.89e-02	
2	1.90e-01	1.59	2.90e-02	2.72
4	1.11e-01	1.72	8.20e-03	3.54
8	6.02e-02	1.84	2.16e-03	3.79
16	3.14e-02	1.91	5.55e-04	3.90
32	1.61e-02	1.95	1.40e-04	3.95
64	8.13e-03	1.98	3.53e-05	3.98
128	4.09e-03	1.99	8.84e-06	3.99

Clearly we get a much more accurate result using the Modified Euler Method. Even more importantly, we get a higher *order of accuracy*: if h is reduced by a factor of **two**, the error in the Modified method is reduced by a factor of **four**.

1. Modified Euler Method

We can also make a direct comparison of the two methods by using a log-log plot of the errors.

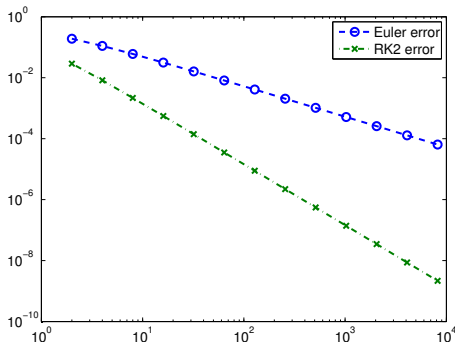


Figure 1: Log-log plot of the errors when Euler's and Modified Euler's methods are applied to the problem in Example 2.4.1

2. General RK2

The “*Modified Euler Method*” is an example of one of the (large) family of 2nd-order *Runge-Kutta* (RK2). Recall that that one-step methods are written as $y_{i+1} = y_i + h\Phi(t_i, y_i; h)$

The general RK2 method is

$$\begin{aligned} k_1 &= f(t_i, y_i) & k_2 &= f(t_i + \alpha h, y_i + \beta h k_1). \\ \Phi(t_i, y_i; h) &= (a k_1 + b k_2) \end{aligned} \tag{3}$$

Example: take $a = 1, b = 0$.

2. General RK2

The general RK2 method is

$$\begin{aligned}k_1 &= f(t_i, y_i) & k_2 &= f(t_i + \alpha h, y_i + \beta h k_1). \\ y_{i+1} &= y_i + h(ak_1 + bk_2)\end{aligned}$$

Example 2: take $\alpha = \beta = 1/2, a = 0, b = 1$.

Our aim now is to deduce general rules for choosing a , b , α and β . We'll see that if we pick any one of these four parameters, then the requirement that the method be consistent and second-order determines the other three.

As well now see, by demanding that RK2 be **consistent** we get that $a + b = 1$:

Next we need to know how to choose α and β . The formal way is to use a two-dimensional Taylor series expansion. However, it is quite technical, involving long calculations.

So, instead, we'll take an approach based on applying the method to a simple, but representative problem.

Because we expect that, for a second order accurate method,
 $|\mathcal{E}_n| \leq Kh^2$ where K depends on $y'''(t)$, if we choose a problem for
which $y'''(t) \equiv 0$, we expect no error...

In the above example, the right-hand side of the differential equation, $f(t, y)$, depended only on t . Now we'll try the same trick: using a problem with a simple known solution (and zero error), but for which f depends explicitly on y .

Consider the DE $y(1) = 1, y'(t) = y(t)/t$. It has a simple solution: $y(t) = t$. We now use that any RK2 method should be exact for this problem to deduce that $\alpha = \beta$.

Now we collect the above results all together and show that the second-order Runge-Kutta (RK2) methods are:

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(t_i, y_i), \quad k_2 = f(t_i + \alpha h, y_i + \beta h k_1),$$

where we choose any $b \neq 0$ and then set

$$a = 1 - b, \quad \alpha = \frac{1}{2b}, \quad \beta = \alpha.$$

It is easy to verify that the Modified method satisfies these criteria.

3. Exercises

Exercise 2.4.1

A popular RK2 method, called the *Improved Euler Method*, is obtained by choosing $\alpha = 1$.

- (i) Use the Improved Euler Method to find an approximation for $y(4)$ when

$$y(0) = 1, \quad y' = y/(1 + t^2),$$

taking $n = 2$. (If you wish, use Python.)

- (ii) Using a diagram similar to the one used to motivate the Modified Euler Method, justify the assertion that the Improved Euler Method is more accurate than the basic Euler Method.
- (iii) Show that the method is consistent.
- (iv) Write out what this method would be for the problem: $y'(t) = \lambda y$ for a constant λ . How does this relate to the Taylor series expansion for $y(t_{i+1})$ about the point t_i ?

3. Exercises

Exercise 2.4.2 (Assignment!)

In his seminal paper of 1901, Carl Runge gave an example of what we now call a *Runge-Kutta 2 method*, where

$$\Phi(t_i, y_i; h) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\right).$$

- (i) Show that it is consistent.
- (ii) Show how this method fits into the general framework of RK2 methods. That is, what are a , b , α , and β ? Do they satisfy the conditions

$$\beta = \alpha, \quad b = \frac{1}{2\alpha}, \quad a = 1 - b?$$

- (iii) Use it to estimate the solution at the point $t = 2$ to $y(1) = 1$, $y' = 1 + t + y/t$ taking $n = 2$ time steps.