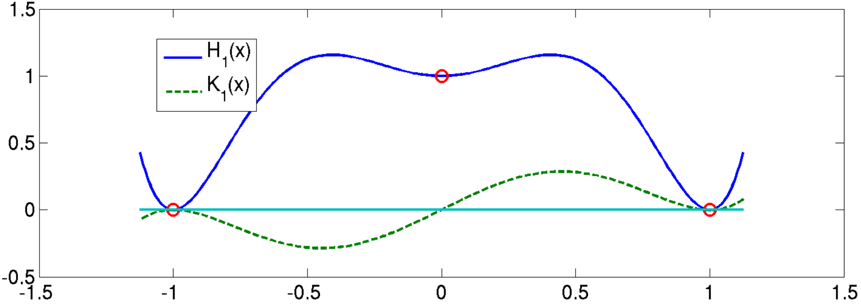
# MA378 Chapter 1: Interpolation

## §1.4 Hermite Interpolation

Dr Niall Madden

January 2024



## Charles Hermite

Charles Hermite, France, 1822–1901. Apart from this form of interpolation, his contributions to mathematics included the first proof that $e$ is transcendental.

His methods were later used to show that $\pi$ is transcendental.

Hermite interpolation is a variant on the standard Polynomial Interpolation Problem: we seek a polynomial that not only agrees with a given function $f$ at the interpolation points, but its first derivative also matches $f'$ at those points.

We are not that interested in this problem for its own sake, but the idea recurs again in the sections in piecewise polynomial interpolation and Gaussian quadrature.

Formally, the problem is

**The Hermite Polynomial Interpolation Problem** (HPIP) *Given a set of interpolation points* $x_0 < x_1 < \cdots < x_n$ *and a continuous, differentiable function* $f$, *find* $p_{2n+1} \in \mathcal{P}_{2n+1}$ *such that*

$$p_{2n+1}(x_i) = f(x_i) \quad \textit{and} \quad p'_{2n+1}(x_i) = f'(x_i).$$

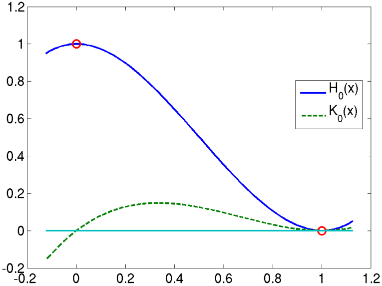
One can prove that if there is a solution to this problem, then it is unique (see exercise).

# 4.2 Constructing Hermite Interpolants

It is possible to solve this problem using an extension of the Lagrange Polynomial approach. Given the usual Lagrange Polynomials, $\{L_i\}$, for $i = 0, \dots, n$, let

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

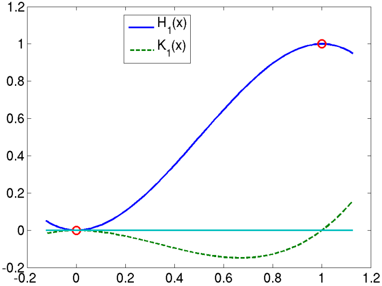



Hermite bases functions $H_0$ and $K_0$ for $n = 1$, $x_0 = 0$ and $x_1 = 1$

# 4.2 Constructing Hermite Interpolants

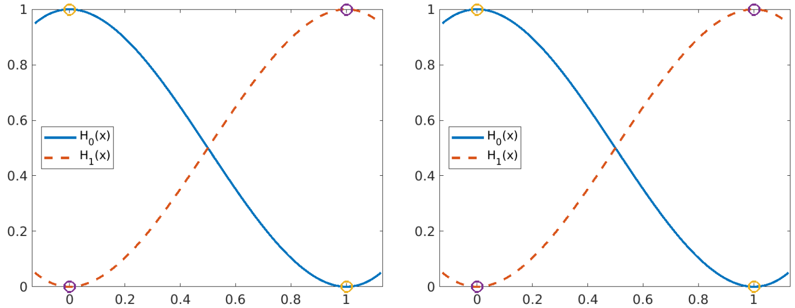
$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$




Hermite bases functions $H_1$ and $K_1$ (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

# 4.2 Constructing Hermite Interpolants




Hermite bases functions $H_0$, $H_1$ (left) and $K_0$, $K_1$ (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

## 4.2 Constructing Hermite Interpolants

**The Hermite basis functions**

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We can show that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad\qquad H_i'(x_k) = 0 \ \forall k$$

# 4.2 Constructing Hermite Interpolants

**The Hermite basis functions**

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Also, for $i, k = 0, 1, \dots n$,

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

**This part is left to Exercise 4.3(a)**.

# 4.2 Constructing Hermite Interpolants

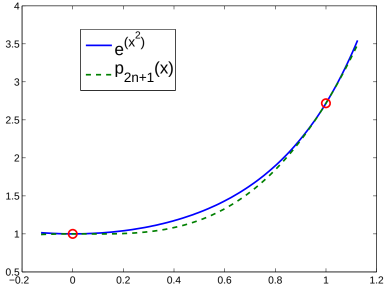
One can now show that the solution to the HPIP exists and is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big( f(x_i) H_i(x) + f'(x_i) K_i(x) \big).$$

**This part is left to Exercise 4.3(b)**.

## Example 4.1

Find the polynomial of degree 3 that interpolates $\exp(x^2)$, and its first derivative, at $x_0 = 0$ and $x_1 = 1$. (See below).



# 4.2 Constructing Hermite Interpolants

# 4.3 Error estimates

## Theorem 4.2

*Let be a real-valued function that is continuous and defined on $[a, b]$, such that the derivatives of $f$ of order $2n + 2$ exist and are continuous on $[a, b]$. Let $p_{2n+1}$ be the Hermite interpolant to $f$. Then, for any $x \in [a, b]$ there is an $\tau \in (a, b)$ such that*

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!}[\pi_{n+1}(x)]^2.$$

We won't do a proof of this in class. However, later in this course we'll be interested in the particular example of finding $p_3$ the cubic Hermite Polynomial Interpolant to a function $f$ at the points $x_0$ and $x_1$.

# 4.4 Exercises

## Exercise 4.1

For *just* the case $n = 1$, state and prove an appropriate version of Theorem 4.2 (i.e., error in the Hermite interpolant). Use this to find a bound for

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_3(x)|$$

in terms of $f$ and $h = x_1 - x_0$.

## Exercise 4.2

Let $n = 2$ and $x_0 = -1$, $x_0 = 0$ and $x_1 = 1$. Write out the formulae for $H_i$ and $K_i$ for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

# 4.4 Exercises

## Exercise 4.3

Let $L_0$, $L_1$, ..., $L_n$ be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \dots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

and

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad\qquad H_i'(x_k) = 0.$$

(a) Show that $K_i(x_k) = 0$, for $k = 0, 1, \dots n$, and $K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

(b) Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$