Annotated slides from Friday

MA313 : Linear Algebra I

Week 2: Subspaces ~~and Spans~~

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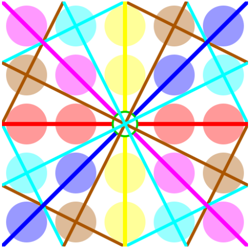

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These slides are based on ones by Tobias Rossmann.

Part 3: Subspaces

One of the key concepts of vector spaces is that, given an example of a vector space, we can usually construct another, smaller one from it. These new smaller ones are called **subspaces**.

Definition (Subspace)

Let V be a vector space. A *subspace* of V is a subset of V which forms a vector space with respect to the same addition and scalar multiplication operations in V.

Example (The boring examples)

The "boring" subspaces of a vector space V are

- $\{0\}$ and
- V itself.

Part 3: Subspaces

Our definition said that, if H is a subspace of V then everything in H is also in V, and also that H is a subspace in its own right. However, we don't have to check if *all* eight axioms hold for H.

Fact

Let H be a subset of V. Then H is a subspace of V if and only if the following conditions are all satisfied:

- $\mathbf{0} \in H$.
- H is closed under addition operation in V, i.e., for all $u, v \in H$, we have $u + v \in H$.
- H is closed under multiplication by scalars, i.e. for all $u \in H$ and $c \in \mathbb{R}$, we have $cu \in H$.

Example

Let

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \text{ with } x_1 + x_2 = 0 \right\}.$$

Then H is a subspace of $\mathbb{R}^2$.

Eg $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in H$ and $\begin{bmatrix} -\pi \\ \pi \end{bmatrix} \in H$, but not $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ since $3-2 = 1 \neq 0$. Nor does $\begin{bmatrix} i \\ -i \end{bmatrix}$ since $i = \sqrt{-1} \notin \mathbb{R}$.

Part 3: Subspaces

Example

Let

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \text{ with } x_1 + x_2 = 0 \right\}.$$

Then H is a subspace of $\mathbb{R}^2$.

To verify we check

① $\vec{0} \in H$. ② if $u, v \in H$ then $u+v \in H$.

③ if $c \in \mathbb{R}$, $u \in H$. then $cu \in H$. [Check!].

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① note that $0+0=0$, so $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in H$. ✓

② If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then $u_1 + u_2 = 0$, $v_1 + v_2 = 0$.

And $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$. And $(u_1 + v_1) + (u_2 + v_2) =$
$(u_1 + u_2) + (v_1 + v_2) = 0 + 0 = 0$

Example

Decide (with justification) whether

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 + x_2 = \mathbf{1} \right\}$$

is a subspace of $\mathbb{R}^2$.

Eg $\begin{bmatrix} 3 \\ -2 \end{bmatrix} \in H$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in H$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$.

But not $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

H is not a subspace of $\mathbb{R}^2$ because..

for example $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin H$, since $0+0=0 \neq 1$.

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Example (MA313 Semester 1 Exam, Q1(a)(ii))

Decide, with justification, whether

$$H_{2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 > 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

No, it is not, since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin H$,

because $0^2 + 0^2 + 0^2 = 0$

Part 3: Subspaces

Example (MA313 Semester 1 Exam, Q1(a)(iii))

Decide, with justification, whether

$$H_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 \geq 0, x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

Eg $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \in H_3$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H_3$. But not $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

However, it is not true that, if $v \in H_3$ and $c \in \mathbb{R}$ then always $cv \in H_3$. Example,

$v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $c = -1$.

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Week 2: Subspaces and Spans

Start of ...

PART 4: More examples of subspaces

Recall

$\mathbb{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n : a_0, \ldots, a_n \in \mathbb{R}\}$ is the vector space of polynomials of degree at most n in the variable t.

Definition (All the polynomials)

$\mathbb{P} := \bigcup_{n=0}^{\infty} \mathbb{P}_n = \{p(t) = a_0 + a_1 t + \cdots + a_n t^n : n \geqslant 0;\ a_0, \ldots, a_n \in \mathbb{R}\}$ is the vector space of *all* polynomials in t (without any bounds on the degree!).

Here, the addition and scalar multiplication in $\mathbb{P}$ are just the usual operations: we add and multiply as expected.

We now see that...

- $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$ if and only if $m \leqslant n$.
- $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all $n \geqslant 0$.

Clearly $\mathbb{P}_m$ is a <u>subset</u> of $\mathbb{P}_n$

since $\mathbb{P}_n$ includes all polys of lower degree.

Also: (1) $0 \in \mathbb{P}_m$; take $a_0 = a_1 = \cdots = a_m = 0$.

(2) If $p, q \in \mathbb{P}_m$ then so too is $p+q$.

(3) If $p \in \mathbb{P}_m$ and $c \in \mathbb{R}$, then $cp \in \mathbb{P}_m$

because, for any $a_i \in \mathbb{R}$, $ca_i \in \mathbb{R}$.

Note that

$$\begin{aligned}\mathbb{P}_0 = \mathbb{R} &= \text{constant polynomials}\\ &= \{p(t) \in \mathbb{P} : p'(t) = 0\}\end{aligned}$$

where the **derivative** of $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ is

$$p'(t) = a_1 + 2a_2 t + \cdots + na_n t^{n-1}.$$

More generally, we can define $\mathbb{P}_n$ by

$$\mathbb{P}_n = \left\{ p(t) \in \mathbb{P} : p^{(n+1)}(t) = 0 \right\}.$$

That is, *another* approach to describing the subspaces $\mathbb{P}_n$ of $\mathbb{P}$ is to look at the space of solutions of certain equations.

Example (Continuous functions)

Let $\mathbb{D} \subseteq \mathbb{R}$ be a subset. Let V be the vector space of all functions $\mathbb{D} \to \mathbb{R}$ from Week 1, where, as we said

- $(f + g)(x) = f(x) + g(x)$ for $f, g \in V$ and $x \in \mathbb{D}$
- $(cf)(x) = cf(x)$ for $f \in V$ and $c \in \mathbb{R}$

Let

$$C(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{R} : f \text{ is continuous}\} \subseteq V.$$

With a little calculus, one can show that $C(\mathbb{D})$ is a subspace of V.

Finished here Friday.