Initial Value Problems

§2.2: Euler's Method

MA385/530 – Numerical Analysis 1

October 2019 (Week 5)

<<<ANNOTATED SLIDES...ANNOTATED SLIDES>>>

Our goal is to generate numerical solutions to initial value differential equations. The solutions to such problems are functions (usually, of one variable that we'll denote t). Our approximation will give estimates of the values of this function at certain points.

We'll denote the points we at which we are seeking approximations as

$$t_0 < t_1 < \cdots < t_n.$$

The methods we'll use are all **one-step** methods, and the first example we'll consider is ***Euler's Method***.

Although it is not too important, we'll make the assumption that the points are equally spaced. So

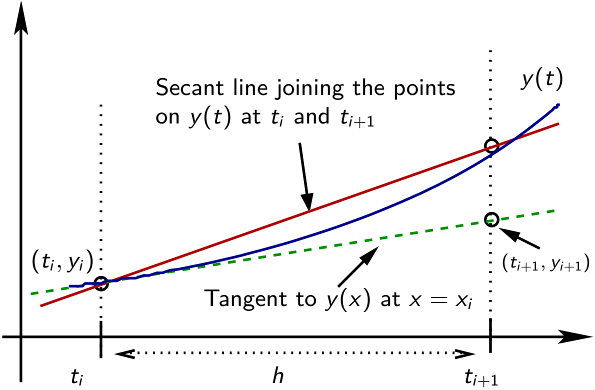
$$t_{i+1} - t_i = \frac{t_n - t_0}{n} = h.$$

The simplest method is ***Euler's Method***. We motivate it as follows.

Motivation

Suppose we know $y(t_i)$, and want to compute $y(t_{i+1})$. From the differential equation we can calculate the slope of the tangent to y at t_i. If this approximates the slope of the line joining $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$, then

$$y'(t_i) = f(t_i, y(t_i)) \approx \frac{y_{i+1} - y_i}{t_{i+1} - t_i}.$$



$$y(t_{i+1}) = y(t_i) + h\,\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}$$

$$\approx y(t_i) + h\,y'(t_i) \approx y(t_i) + h f(t_i, y_i)$$

Euler's Method

Choose equally spaced points $t_0, t_1, \ldots, t_n$ so that

$$t_i - t_{i-1} = h = (t_n - t_0)/n \quad \text{for } i = 0, \ldots, n-1.$$

We call h the "time step". Let y_i denote the approximation for $y(t)$ at $t = t_i$. Set

$$y_{i+1} = y_i + hf(t_i, y_i), \quad i = 0, 1, \ldots, n-1. \tag{3}$$

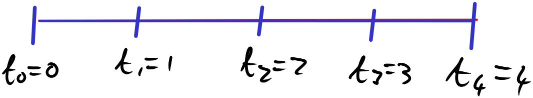
ie y_i is the approximation for $y(t_i)$.

Example 2.4

Taking $h = 1$, estimate $y(4)$ where

$$y'(t) = y/(1+t^2), \qquad y(0) = 1.$$

Choosing $h = 1$ we get

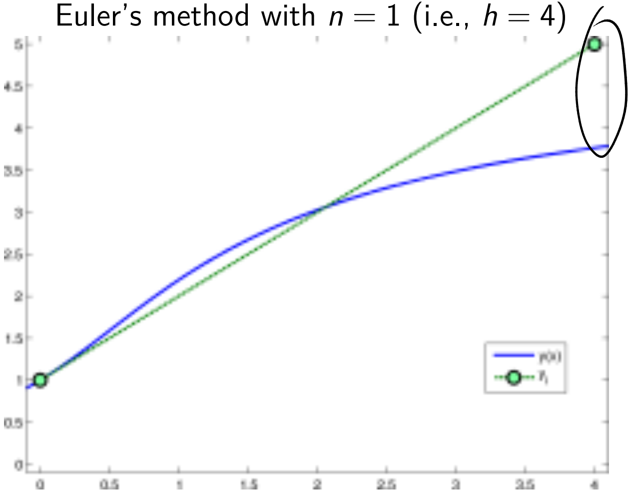


- $i = 0$: $t_0 = 0$, $y_0 = 1$.
- $i = 1$: $t_1 = t_0 + h = 1$.
  $y_1 = y_0 + hf(t_0, y_0) = 1 + \frac{1}{1+0^2} = 2.$
- $i = 2$: $t_2 = t_0 + 2h = 2$.
  $y_2 = y_1 + hf(t_1, y_1) = 2 + 1\frac{2}{1+1^2} = 3.$
- $i = 3$: $t_3 = t_0 + 3h = 3$.
  $y_3 = y_2 + hf(t_2, y_2) = 3 + 1\frac{3}{1+2^2} = 3.6$
- $i = 4$: $t_n = t_4 = t_0 + 4h = 4$.
  $y_n = y_4 = y_3 + hf(t_3, y_3) = 3.6 + \frac{3.6}{1+3^2} = \mathbf{3.96}$

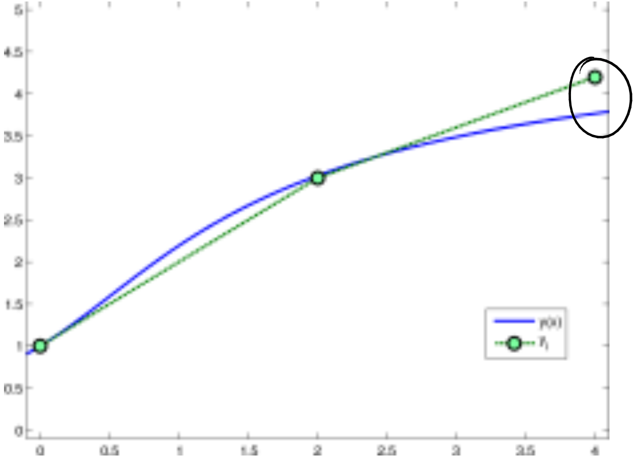
If we had chosen $h = 4$ we would have only required one step: $y_n = y_0 + 4f(t_0, y_0) = \mathbf{5}$. However, this would not be very accurate.

With a little work one can show that the solution to this problem is $y(t) = \mathrm{e}^{\tan^{-1}(t)}$ and so $y(4) = 3.7652$. Hence the computed solution with $h = 1$ is much more accurate than the computed solution when $h = 4$. This is also demonstrated in next figure below, and in the follow table, where we see that the error seems to be proportional to h.

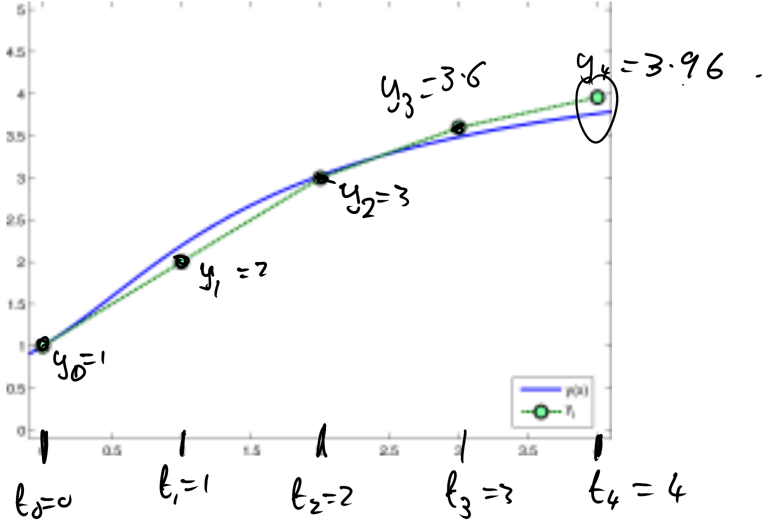
Euler's method with $n = 1$ (i.e., $h = 4$)



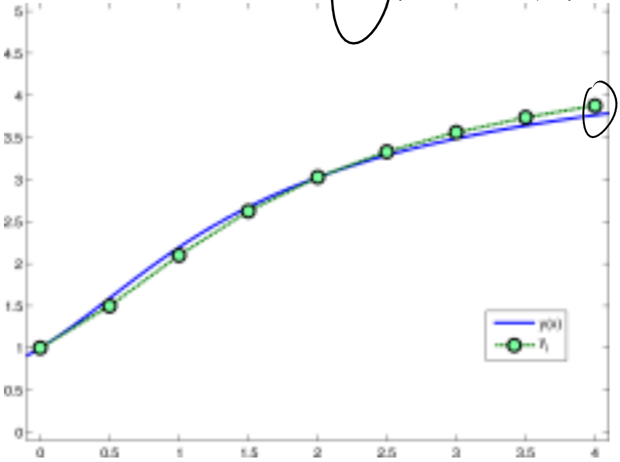
Euler's method with $n = 2$ (i.e., $h = 2$)



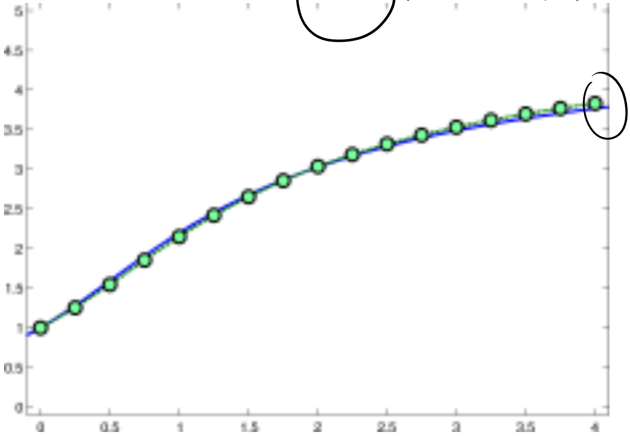
Euler's method with $n = 4$ (i.e., $h = 1$)



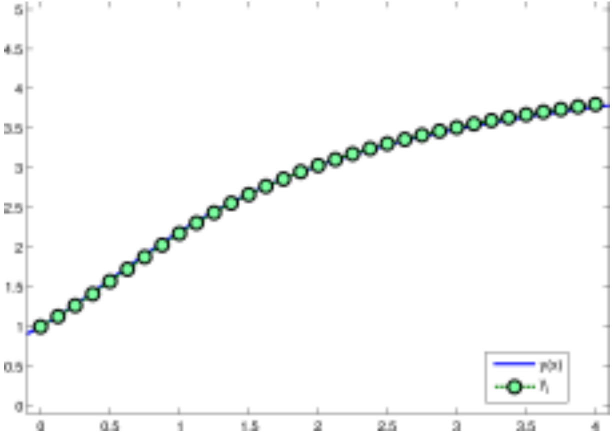
Euler's method with $n = 8$ (i.e., $h = 1/2$)



Euler's method with $n = 16$ (i.e., $h = 1/4$)



Euler's method with $n = 32$ (i.e., $h = 1/8$)



n	h	y_n	$\lvert y(t_n) - y_n \rvert$
1	4	5.0	1.235
2	2	4.2	0.435
4	1	3.960	0.195
8	1/2	3.881	0.115
16	1/4	3.831	0.065
32	1/8	3.800	0.035

Table: Error in Euler's method for Example 2.4

It appears as though,

- if h is halved, so too is the Error,
- As $h \to 0$, Error $\to 0$.

Exercise 2.3

As a special case in which the error of Euler's method can be analysed directly, consider Euler's method applied to

$$y'(t) = y(t), \quad y(0) = 1.$$

The true solution is $y(t) = e^t$.

(i) Show that the solution to Euler's method can be written as

$$y_i = (1 + h)^{t_i/h}, \quad i \geq 0.$$

(ii) Show that

$$\lim_{h \to 0} (1 + h)^{1/h} = e.$$

This then shows that, if we denote by $y_n(T)$ the approximation for $y(T)$ obtained using Euler's method with n intervals between t_0 and T, then

$$\lim_{n \to \infty} y_n(T) = e^T.$$

Hint: Let $w = (1 + h)^{1/h}$, so that $\log w = (1/h) \log(1 + h)$. Now use l'Hospital's rule to find $\lim_{h \to \mathbf{0}} w$.