2425-MA140 Engineering Calculus

Week 09, Lecture 2 (L26)
Volumes and Arc Length

Dr Niall Madden

School of Maths, University of Galway

Wednesday, 13 November, 2024

Assignments (reminder)

1. **Assignment 6:** Everyone should (re)do Q8 in **Assignment-6-Q8**.
2. On Canvas, `Assignment-6-Q1-Q7` records your score for Assignment 6 as it was before my error. A score of 89% means you had a perfect score for Questions 1-7; so you only need to complete the ungraded Q8.
3. If you have a score of less than 89%, you can choose to keep it, or redo **Assignment-6**. Tedious, but you have a whole extra week to improve your grade!
4. **Assignment 7** is open.

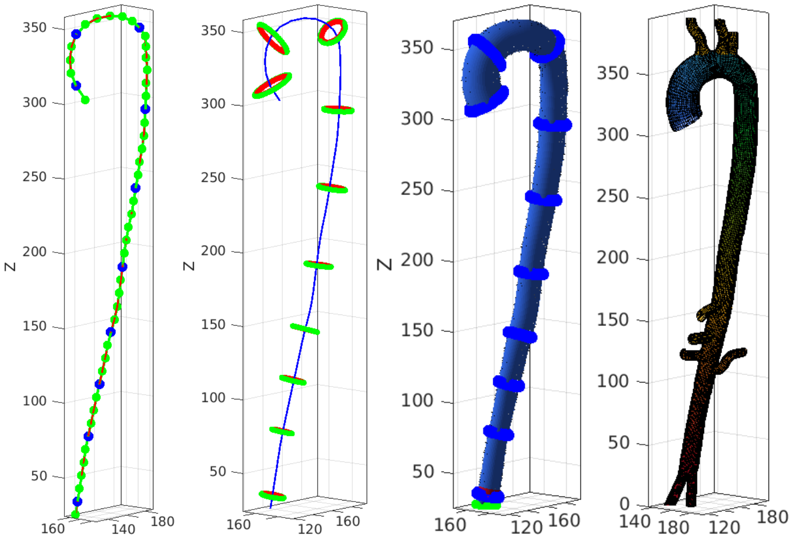
Talking about a (solid of) revolution:

See also: Section **6.2** (Determining Volumes by Slicing) and Section **6.4** (Arc Length of a Curve and Surface Area) in **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Some motivation

- The following images representing a human aorta. But the data are artificially generated (this is not from a real person).
- The images are generated by Kevin Moerman (biomechanical engineering)
- It is part of a project involving Dr Niamh Hynes (look her up!), and one of your tutors, Sean Tobin.
- The meaning of the images on the following slides, and significance to MA140, was discussed in class, but is not detailed in these notes.

Some motivation



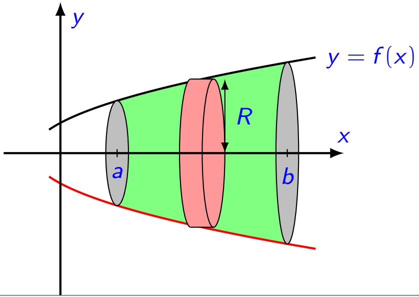
Recall "Solids of Revolution"

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**.

Often that region is bounded:

- above by $y = f(x)$, where y is some given nonnegative function;
- below by $y = 0$ (i.e., the x-axis)
- on the left by $x = a$, and on the right by $x = b$

The whole region is then **rotated about the x-axis**.



Recall "Solids of Revolution"

Since, every cross-section of for a solids of revolution, is a disk with area $A(x) = \pi\big(f(x)\big)^2$, we can directly compute the volume, we get the following formula.

Solids of revolution: disk method

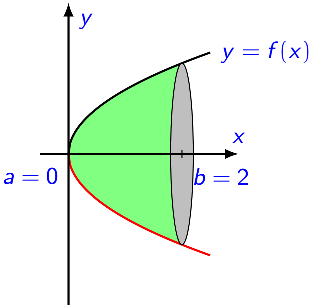
Let $f(x)$ be continuous and nonnegative. The volume of region formed by revolving the region between $f(x)$ and the x-axis, and between $x = a$ and $x = b$, about the x-axis is

$$V = \pi \int_a^b \big(f(x)\big)^2 dx.$$

Recall "Solids of Revolution"

Example:

Find the **volume** of the solid of revolution obtained by rotating $y = \frac{3}{\sqrt{2}}\sqrt{x}$, between $x = 0$ and $x = 2$, about the x-axis.



Solids of revolution: the "washer method"

There are numerous other variations on this type of problem, such as

- Rotating the function about the y-axis; (easy: just give a function for x in terms of y).
- Rotating about a line that is not an axis (a little trickier: need to transform the problem).
- **rotating a region bounded by two functions**.

We'll look at the last of these, the method for which is sometimes called the "**washer method**".

However, it is not too hard: we apply the "disk" method to both functions, and then subtract.

Solids of revolution: the "washer method"

Washer Method

Let $f(x)$ and $g(x)$ be continuous functions on $[a, b]$, with $f(x) \geqslant g(x) \geqslant 0$ for any $x \in [a, b]$. The volume of the solid obtained by rotating the region between $f(x)$ and $g(x)$, and $x = a$ and $x = b$, is

$$V = \pi \int_a^b \big(f(x)\big)^2 - \big(g(x)\big)^2 \, dx.$$

Solids of revolution: the "washer method"

Example (from textbook: see Figure 6.2.12)

Consider the region in the plane bounded above by $y = \sqrt{x}$, below by $y = 1$, left by $x = 1$ and right by $x = 4$. If this region is rotated about the x-axis, show that the volume of the resulting solid of rotation is $\frac{9\pi}{2}$.

First we visualise: **the animation**

Solids of revolution: the "washer method"

Example (finding a and b)

Find the volume of the solid obtained by rotating, around the x-axis, the region bounded by $f(x) = x$ and $g(x) = x^2$.

Note: for this example, we have to determine the values of a and b.

Solids of revolution: the “washer method”

Arc Length

"How long is a piece of string?"

In this section we will work out the **length of a curve**.

The method for doing this is a little surprising, since it involves both differentiation and integration.

However, it is easy, if the function is linear (so it's graph is a straight line):

Arc Length

Given a curve, we can use this idea, to get an estimate for its length, by approximating it by a straight line:

We can improve upon that by (say), approximating the curve by two straight lines:

Arc Length

The more intervals we take, the better approximation we get. If we take n, the approximation is

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

$$= \sum_{k=1}^{n} \sqrt{(\Delta x)^2 \left(1 + \left[\frac{\Delta y_k}{\Delta x}\right]^2\right)}.$$

Arc Length

Hence,

$$L \approx \sum_{k=1}^{n} \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x}\right]^2} \cdot \Delta x\,.$$

If we take an infinite number of intervals, then

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x}\right]^2} \cdot \Delta x\,.$$

Using the "limit" definition of the derivative, and the integral, we get that the **arc length**, L, of a curve of $y = f(x)$, from $x = a$ to $x = b$, is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx\,.$$

Arc Length

[These next slides were added after class, in response to a question I was asked].

In the last step of the previous slide, there are several things going on.

- Since $\delta x = \dfrac{b-a}{n}$, as $n \to \infty$, so $\delta x \to 0$.
- Back in Week 4, Lecture 1, we defined

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Here, $\delta x = h$, and $\delta y = f(x_i + \delta x) - f(x_i)$ where x_k is the start point of interval k. That is,

$$\lim_{n \to \infty} \frac{\delta y_k}{\delta x} = \lim_{h \to 0} \frac{f(x_k + h)}{h} = f'(x_k).$$

Arc Length

- In Week 7, Lecture 1, (Slide 15) we defined

$$\int_a^b g(x)\,dx = \lim_{n\to\infty} hg(x_k).$$

Here, we let $g(x_k) = \sqrt{1 + (f'(x_k))^2}$, our expression is

$$\lim_{n\to\infty} (\Delta x) g(x_k) = \int_a^b g(x)\,dx$$

where again we are swapping the notation h and δx.

Arc Length

Arc length of a curve

If $f(x)$ is a differentiable function on the interval $[a, b]$, then the **arc length**, L, of the graph of $f(x)$, from $x = a$ to $x = b$, is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,.$$

Arc Length

Example

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}\, x^{\frac{3}{2}} - 1, \quad 0 \leqslant x \leqslant 1\,.$$

Exercises

Exer 9.2.1

Use the "washer" method to find the volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = x^2$ and $g(x) = x$, for $1 \leqslant x \leqslant 2$, about the x-axis.

Exer 9.2.2

The volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = x^2$ and $g(x) = 1$, for $1 \leqslant x \leqslant b$, about the x-axis, is $4/3$. Find b. (Hint: b is an integer; this information will help you find b by inspection).

Exer 9.2.3

Find the volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = 2-x^2$ and $g(x) = x^2$ about the x-axis. (Hint: you need to find where the graphs of f and g intersect: these will be the points a and b).