

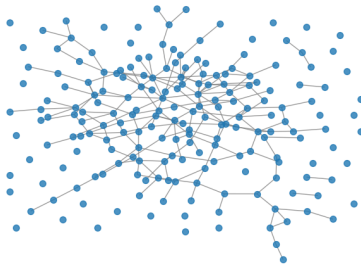
CS4423: Networks

Week 8, Part 1: Introduction to Random Networks

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(05 March 2025)



Class Test!

Answers (but not solutions)
posted by 11 today.

Class Test 2pm tomorrow!

Details:

- ▶ LENS reports: email Niall today!
- ▶ Locations: see announcement!
- ▶ Content:

- ▶ Similar to Problem Set 2
- ▶ Nothing from this week.
- ▶ No `networkx`
- ▶ Focus on skills, rather than theory.

- ▶ Bring a pen. And maybe a calculator (?).

- ▶ If you miss the test, for any reason, your grade will be based on the assignments (20%) and the final exam (80%).

4FMZ : CA116a

Everyone else:
Larmor.

nothing on Betweenness
Centrality.

Outline

Today, all notes will be based on these slides (no Jupyter).

- | | | | |
|---|---------------------------------|---|----------------------------|
| 1 | Random Models of Networks | 3 | Random samples |
| 2 | Erdős-Rényi Random Graph Models | 4 | The two Erdős-Rényi Models |
| | ■ Some examples | | ■ Model A: $G_{ER}(n, m)$ |
| | | | ■ Model B: $G_{ER}(n, p)$ |

Slides are at:

<https://www.niallmadden.ie/2425-CS4423>



Random Models of Networks

Random Models of Networks

One of the remaining “big” ideas for us to study in CS4423 is that of **Random Networks**. In a sense, we are not so interested in their randomness. It is more like we decide on the general structure of networks, but then choose a particular example by tossing a coin, or rolling dice.

What we are interested in:

- ▶ The **statistical properties** of very large networks, such as average degree, the number of 3-cycles, or the size of component.
- ▶ How well our random networks share these properties.

Erdő-Rényi Random Graph Models

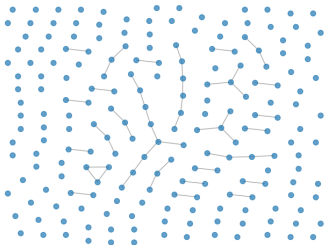
A **Random Graph**¹ is a *mathematical model* of a family of networks, where certain parameters (like the number of nodes and edges) have fixed values, but other aspects (like the actual edges) are randomly assigned.

The simplest example of a random graph is in fact a network with fixed numbers n of nodes and m of edges, randomly placed between the vertices.

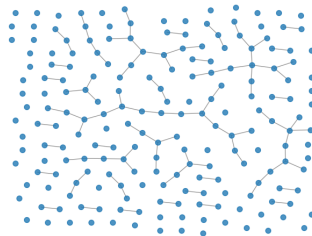
Although a random graph is not a specific object, many of its properties can be described precisely in the form of **expected values** or **probability distributions**.

¹https://en.wikipedia.org/wiki/Random_graph

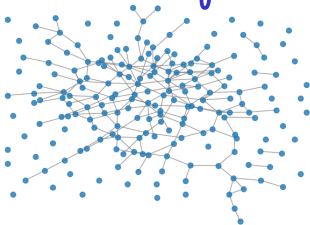
20 edges



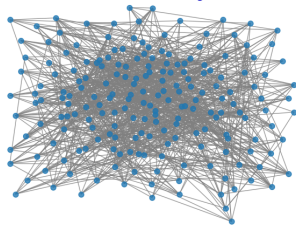
50 edges



100 edges



1000 edges



All have $n = 200$ nodes.

Random samples

$$N = \binom{n}{2}$$

Suppose our network $G = (X, E)$ has $|X| = n$ nodes. Then we know the most number of edges E can have is:

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}. \quad (\text{roughly } \frac{1}{2}n^2)$$

- Our goal is to randomly select edges on the vertex set X . That is, pick at random elements from the set $\binom{X}{2}$ of pairs of nodes.
- So we need a procedure for selecting m from N objects randomly, in such a way that each of the $\binom{N}{m}$ subsets of the N objects is an equally likely outcome.

- We first discuss sampling m values in the range $\{0, 1, \dots, N-1\}$.
Recall $\binom{X}{2}$ is the set of pairs drawn from X . So $|\binom{X}{2}| = \binom{|X|}{2}$

Random samples

1. Suppose we choose a natural number N , and real number $p \in [0, 1]$
2. Then iterate over each element of the set $\{0, 1, \dots, N - 1\}$.
3. For each, we pick a random number $x \in [0, 1]$.
4. If $x < p$, we keep that number. Otherwise remove it from the set.

When we are done, how many elements do we expect in the set if $p = m/N$ for some chosen m ?

And what is the likelihood of there being, say k elements in the set?

Random samples

We are creating random samples. The size of each is a random number, k .

Claim: Expected value: $E[k] = Np = m$.

Proof: This is a **binomial distribution**²

- ▶ The probability of a specific subset of size k to be chosen is $p^k(1-p)^{N-k}$.
- ▶ There are $\binom{N}{k}$ subsets of size k . So the probability $P(k)$ of the sample to have size k is $P(k) = \binom{N}{k} p^k (1-p)^{N-k}$.

We use the following facts

- (i) $j \binom{N}{j} p^j = Np \binom{N-1}{j-1} p^{j-1}$,
- (ii) $(1-p)^{N-j} = (1-p)^{(N-1)-(j-1)}$,
- (iii) $(p + (1-p))^r = 1$ for all r .

²https://en.wikipedia.org/wiki/Binomial_distribution

Random samples

Expected value:

$$\begin{aligned} E[k] &= \underbrace{\sum_{j=0}^N jP(j)}_{\text{weighted average of } j} = \sum_{j=0}^N j \underbrace{\binom{N}{j} p^j (1-p)^{N-j}}_{\text{Formula for } P(j)} \\ &= Np \underbrace{\left[\sum_{l=0}^{N-1} \binom{N-1}{l} p^l (1-p)^{(N-1)-l} \right]}_{\text{From (i),(ii),(ii)}} = Np, \quad (1) \end{aligned}$$

Handwritten notes: A green bracket under the sum in the second line is labeled "From (i),(ii),(ii)". A green arrow points from the expression in brackets to "1." on the right. Another green arrow points from the final result "Np" to the right.

substituting $l = k - 1$,

↑
Exer: check this!

Random samples

Next week, we'll look at some computational examples, as well as an algorithm for choosing exactly m numbers from a set of N .

For now, we'll just assume it can be done...

Important:
nodes are labelled

Uniformly selected edges

ER Model $G_{ER}(n, m)$: Uniform Random Graphs

Let $n \geq 1$, let $N = \binom{n}{2}$ and let $0 \leq m \leq N$.

The model $G_{ER}(n, m)$ consists of the ensemble of graphs G on the n nodes $X = \{0, 1, \dots, n-1\}$, and m randomly selected edges, chosen uniformly from the $N = \binom{n}{2}$ possible edges.

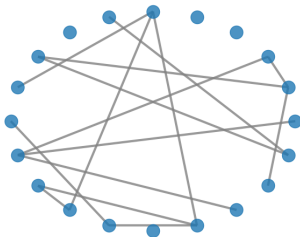
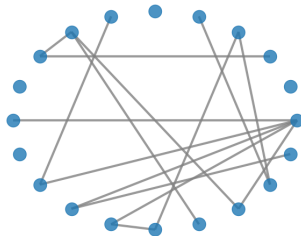
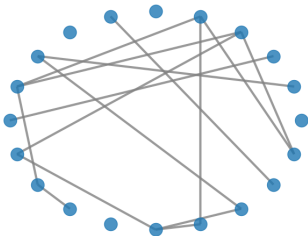
Equivalently, one can choose uniformly at random one network in the **set** $\mathcal{G}(n, m)$ of *all* networks on a given set of n nodes with *exactly* m edges.

The two Erdős-Rényi Models

Model A: $G_{ER}(n, m)$

N

$n=20, m=20(?)$



Finished here
Wed at 10,
with an example
of $G(3,2)$
on the board.

Randomly selected edges

ER Model $G_{ER}(n, p)$: Random Edges

Let $n \geq 1$, let $N = \binom{n}{2}$ and let $0 \leq p \leq 1$.

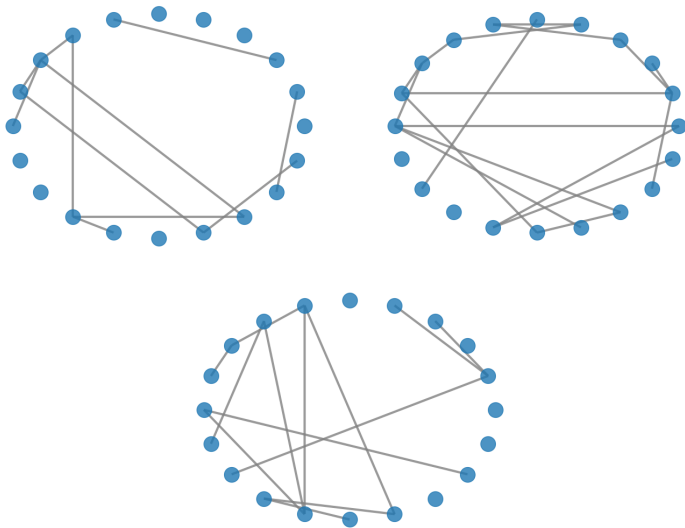
The model $G_{ER}(n, p)$ consists of the ensemble of graphs G on the n nodes $X = \{0, 1, \dots, n-1\}$, with each of the possible $N = \binom{n}{2}$ edges chosen with probability p .

The probability $P(G)$ of a particular graph $G = (X, E)$ with $X = \{0, 1, \dots, n-1\}$ and $m = |E|$ edges in the $G_{ER}(n, p)$ model is

$$P(G) = p^m(1 - p)^{N-m}.$$

The two Erdős-Rényi Models

Model B: $G_{ER}(n, p)$



Of the two models, $G_{ER}(n, p)$ is the more studied. They are many similarities, but do differ. For example:

1. $G_{ER}(n, m)$ will have m edges with probability 1.
2. A graph in $G_{ER}(n, p)$ will have m edges with probability $\binom{N}{m} p^m (1-p)^{N-m}$.

