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MA385 Part 1: Solving nonlinear equations

## 1.3: The Secant Method

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# 0. Outline

1 Motivation

2 Order of Convergence

3 Side quest: The Mean Value  
Theorem

4 Analysis of the Secant Method

5 Exercises

■ Solution to Exer 1.3.3

For more details, see Section 1.5 (The secant method) of Süli and Mayers, *An Introduction to Numerical Analysis*

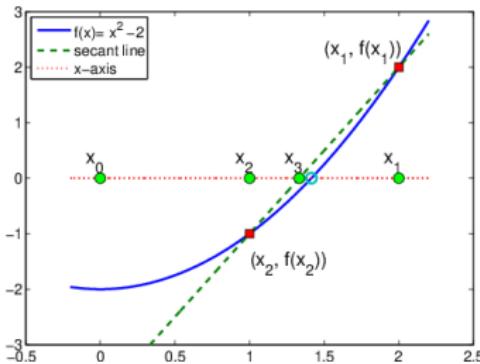
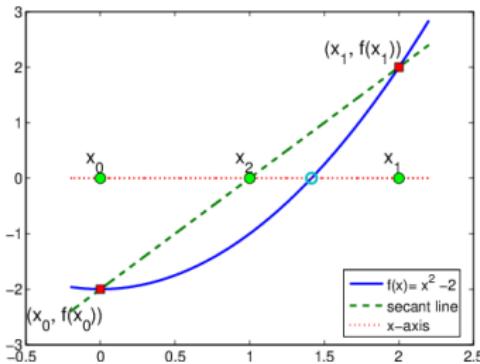
# 1. Motivation

We'll start with considering, heuristically, how we could improve upon bisection:

# 1. Motivation

Idea:

- ▶ Choose two points,  $x_0$  and  $x_1$ .
- ▶ Take  $x_2$  to be the zero of the line joining  $(x_0, f(x_0))$  to  $(x_1, f(x_1))$ .
- ▶ Take  $x_3$  to be the zero of the line joining  $(x_1, f(x_1))$  to  $(x_2, f(x_2))$ .
- ▶ Etc.



# 1. Motivation

## The Secant Method

Choose  $x_0$  and  $x_1$  so that there is a solution in  $[x_0, x_1]$ . Then define

$$x_{k+1} = x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}. \quad (1)$$

# 1. Motivation

## Example 1.3.1

Use the Secant Method to solve  $x^2 - 2 = 0$  in  $[0, 2]$ . Results are shown below. We see that, not only does the method appear to converge to the true solution, it seems to do so *much* more efficiently than Bisection. We'll return to why this is later.

k	Secant		Bisection	
	$x_k$	$ x_k - \tau $	$x_k$	$ x_k - \tau $
0	0.000000	1.41	0.000000	1.41
1	2.000000	5.86e-01	2.000000	5.86e-01
2	1.000000	4.14e-01	1.000000	4.14e-01
3	1.333333	8.09e-02	1.500000	8.58e-02
4	1.428571	1.44e-02	1.250000	1.64e-01
5	1.413793	4.20e-04	1.375000	3.92e-02
6	1.414211	2.12e-06	1.437500	2.33e-02
7	1.414214	3.16e-10	1.406250	7.96e-03

## 2. Order of Convergence

To compare different methods, we need the following concept.

### Definition 1.3.1 (Linear Convergence)

Suppose that  $\tau = \lim_{k \rightarrow \infty} x_k$ . We say that the sequence  $\{x_k\}_{k=0}^{\infty}$  converges to  $\tau$  at least linearly if there is a sequence of positive numbers  $\{\varepsilon_k\}_{k=0}^{\infty}$ , and  $\mu \in (0, 1)$ , such that

$$\lim_{k \rightarrow \infty} \varepsilon_k = 0, \quad (2a)$$

and

$$|\tau - x_k| \leq \varepsilon_k \quad \text{for } k = 0, 1, 2, \dots \quad (2b)$$

and

$$\lim_{k \rightarrow \infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu. \quad (2c)$$

## 2. Order of Convergence

For Example 6, the bisection method converges at least linearly. As we have seen, the Secant Method converge more quickly than bisection. Now we'll give a more precise description of what “*more quickly*” means.

### Definition 1.3.2 (Order of Convergence)

Let  $\tau = \lim_{k \rightarrow \infty} x_k$ . Suppose there exists  $\mu > 0$  and a sequence of positive numbers  $\{\varepsilon_k\}_{k=0}^{\infty}$  such that (2a) and (2b) both hold. Then we say that the sequence  $\{x_k\}_{k=0}^{\infty}$  converges with at least order  $q$  if

$$\lim_{k \rightarrow \infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu.$$

## 2. Order of Convergence

$$\lim_{k \rightarrow \infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu.$$

Two particular values of  $q$  are important to us:

- (i) If  $q = 1$ , and we have that  $0 < \mu < 1$ , then the rate is **linear**.
- (ii) If  $q = 2$ , the rate is **quadratic** for any  $\mu > 0$ .

### 3. Side quest: The Mean Value Theorem

The **Mean Value Theorem** (aka “MVT”) is simple, and incredibly useful. The modern version is due to Cauchy (1823), but there is a version called Rolle’s Theorem which applies to polynomials only from 1691, and one for the sin function to Parameshvara or Kerala (1380–1460).

#### Theorem 1.3.1 (Mean Value Theorem (MVT))

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and differentiable, then there is a point  $c \in (a, b)$ , such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Informally: *there is a point between  $a$  and  $b$  where the tangent to  $f$  is parallel to the line joining  $(a, f(a))$  at  $(b, f(b))$ .*

### 3. Side quest: The Mean Value Theorem

A visual proof:

### 3. Side quest: The Mean Value Theorem

From a numerical analysis point of view, the MVT tells us “how different  $f(b)$  might be from  $f(a)$ ”:

$$f(b) = f(a) + f'(c)(b - a)$$

gives that

$$|f(b) - f(a)| \leq \max_{a \leq x \leq b} |f'(x)|(b - a).$$

That is, in a computation, if we replace  $f(a)$  with  $f(b)$ , the error we introduce depends on  $b - a$  and also the magnitude of  $f'$ .

## 4. Analysis of the Secant Method

### Theorem 1.3.2

Suppose that  $f$  and  $f'$  are real-valued functions, continuous and defined in an interval  $I = [\tau - h, \tau + h]$  for some  $h > 0$ . If  $f(\tau) = 0$  and  $f'(\tau) \neq 0$ , then the sequence (1) converges at least linearly to  $\tau$ .

## 4. Analysis of the Secant Method

- ▶ We wish to show that  $|\tau - x_{k+1}| < |\tau - x_k|$ .
- ▶ From the MVT, there is a point  $w_k \in [x_{k-1}, x_k]$  such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(w_k). \quad (3)$$

- ▶ Also by the MVT, there is a point  $z_k \in [x_k, \tau]$  such that

$$\frac{f(x_k) - f(\tau)}{x_k - \tau} = \frac{f(x_k)}{x_k - \tau} = f'(z_k). \quad (4)$$

Therefore  $f(x_k) = (x_k - \tau)f'(z_k)$ .

## 4. Analysis of the Secant Method

- ▶ Using (3) and (4), we can show that

$$\tau - x_{k+1} = (\tau - x_k) \left( 1 - f'(z_k)/f'(w_k) \right).$$

Therefore

$$\frac{|\tau - x_{k+1}|}{|\tau - x_k|} = \left| 1 - \frac{f'(z_k)}{f'(w_k)} \right|.$$

- ▶ Suppose that  $f'(\tau) > 0$ . (If  $f'(\tau) < 0$  just tweak the arguments accordingly). Saying that  $f'$  is continuous in the region  $[\tau - h, \tau + h]$  means that, for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$|f'(x) - f'(\tau)| < \varepsilon \text{ for any } x \in [\tau - \delta, \tau + \delta].$$

#### 4. Analysis of the Secant Method

Take  $\varepsilon = f'(\tau)/4$ . Then  $|f'(x) - f'(\tau)| < f'(\tau)/4$ . Thus

$$\frac{3}{4}f'(\tau) \leq f'(x) \leq \frac{5}{4}f'(\tau) \quad \text{for any } x \in [\tau - \delta, \tau + \delta].$$

Then, so long as  $w_k$  and  $z_k$  are both in  $[\tau - \delta, \tau + \delta]$

$$\frac{f'(z_k)}{f'(w_k)} \leq \frac{5}{3}.$$

## 4. Analysis of the Secant Method

Given enough time and effort we *could* show that the Secant Method converges faster than linearly. In particular, that the order of convergence is

$$q = (1 + \sqrt{5})/2 \approx 1.618.$$

This number arises as the only positive root of  $q^2 - q - 1$ . It is called the **Golden Mean**, and arises in many areas of Mathematics, including finding an explicit expression for the Fibonacci Sequence:

$$f_0 = 1,$$

$$f_1 = 1,$$

$$f_{k+1} = f_k + f_{k-1} \text{ for } k = 2, 3, \dots$$

That gives,  $f_0 = 1, f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13,$

....

## 4. Analysis of the Secant Method

The connection here is that it turns out that  $\varepsilon_{k+1} \leq C\varepsilon_k\varepsilon_{k-1}$ . Repeatedly using this we get:

- ▶ Let  $r = |x_1 - x_0|$  so that  $\varepsilon_0 \leq r$  and  $\varepsilon_1 \leq r$ ,
- ▶ Then  $\varepsilon_2 \leq C\varepsilon_1\varepsilon_0 \leq Cr^2$
- ▶ Then  $\varepsilon_3 \leq C\varepsilon_2\varepsilon_1 \leq C(Cr^2)r = C^2r^3$ .
- ▶ Then  $\varepsilon_4 \leq C\varepsilon_3\varepsilon_2 \leq C(C^2r^3)(Cr^2) = C^4r^5$ .
- ▶ Then  $\varepsilon_5 \leq C\varepsilon_4\varepsilon_3 \leq C(C^4r^5)(C^2r^3) = C^7r^8$ .
- ▶ And in general,  $\varepsilon_k = C^{f_k-1}r^{f_k}$ .

## 5. Exercises

### Exercise 1.3.1

Suppose we define the Secant Method as follows.

*Choose any two points  $x_0$  and  $x_1$ .*

*For  $k = 1, 2, \dots$ , set  $x_{k+1}$  to be the point where the line through  $(x_{k-1}, f(x_{k-1}))$  and  $(x_k, f(x_k))$  that intersects the  $x$ -axis.*

Show how to derive the formula for the secant method.

## 5. Exercises

### Exercise 1.3.2

Another interpretation of the Secant Method, is that it computes  $x_{k+1}$  as a weighted average:

$$x_{k+1} = (1 - \sigma_k)x_k + \sigma_k x_{k-1},$$

where  $\sigma_k$  chosen to obtain fast convergence to the true solution. (This is something called a “relaxation method”).

Derive a formula for  $\sigma_k$  in terms of  $f(x_{k-1})$  and  $f(x_k)$ . Can you give a physical interpretation to  $\sigma_k$  and  $1 - \sigma_k$  that might justify the claim that the method should be more efficient than bisection?

## 5. Exercises

### Exercise 1.3.3

- (i) Is it possible to construct a problem for which the bisection method will work, but the secant method will fail? If so, give an example.
- (ii) Is it possible to construct a problem for which the secant method will work, but bisection will fail? If so, give an example.

**Exer 1.3.3** This question is really intended to prompt discussion, e.g., in a tutorial. This is because we near a careful understanding regarding what constitutes a solution. (See also Exer 1.5.2).

**Exer 1.3.3 Part(i):** If we take the most limited definition of the problem: "*there is a continuous function  $f$ , and real numbers  $a$  and  $b$  such that  $f(a)f(b) \leq 0$* ", then we can answer (i) as "yes". In this situation, bisection is proven to work. However, the Secant method requires, for any  $i$ , that  $f(x_k) \neq f(x_{k-1})$ . Otherwise we divide by zero when trying to calculate  $x_{k+1}$ . As an example, try  $f(x) = x^2 - 3x + 1$ , with  $a = 0$  and  $b = 2$ . Bisection will work. However, with  $x_0 = 2$  and  $x_1 = 2$ , we should get  $x_2 = 1$ . Then  $f(x_1) = f(x_2) = -1$ , and the method fails.

**Exer 1.3.2 (ii):** Very pedantically, if we insist that the problem is defined a function,  $f$  and two points  $a$  and  $b$ , such that  $f(a)f(b) \leq 0$ , (we say "a abd  $b$  bracket the solution") then bisection cannot fail. However, suppose we just state that there is a function  $f$  and two points  $a$  and  $b$  such that there is  $\tau \in [a, b]$  such that  $f(\tau) = 0$ , then bisection can fail. This could happen if, for example,  $f(x) = 0$  has a solution in  $[a, b]$ , but does not change sign. For example, take  $f(x) = x^2$ , and  $[x_0, x_1] = [-1, 0.5]$ . Clearly

## 5. Exercises

## Solution to Exer 1.3.3

$f(0) = 0$ . But since  $f(x_0) = f(x_1) > 0$ , bisection can't work. However, Secant will converge with these values.