

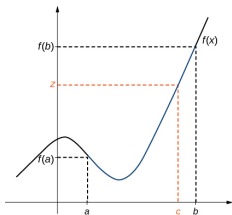
# Week 03, Lecture 3

## Continuity Types; The Intermediate Value Theorem

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*These slides are by Niall Madden. Some content is based on notes by Dr Kirsten Pfeiffer. And some more, such as the figure opposite, taken from Strang & Herman's "Calculus". However, all the typos are Niall's.*

# Today, in MA140...

- 1 Remembering the reminders...
- 2 Types of discontinuity
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- 4 Derivatives: the concept
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For more, see Chapter 2 (Limits) in **Calculus** by Strang & Herman. See [openstax.org/books/calculus-volume-1/pages/2-introduction](https://openstax.org/books/calculus-volume-1/pages/2-introduction). Section 2.4 (Continuity) relates to today's material.

# Remembering the reminders...

## Reminders

- ▶ **Assignment 1** due 5pm, Monday 6 October. You may access it multiple times, by clicking on [Assignments ... Problem Set 1 ...](#) and then, at the bottom of the page:

[Load Problem Set 1 in a new window](#)

- ▶ The **Tutorial Sheet** is available at <https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1>
- ▶ Assignment 2 is also open; deadline is 5pm, 13 Oct.
- ▶ The first (of two) class tests will take place Tuesday, 14th October.
- ▶ If you wish to avail of Reasonable Accommodations for it tests, please complete this form: <https://forms.office.com/e/HaAsrzaE3D> by **10am Thursday 2nd Oct.**

# Types of discontinuity

We have encountered three types of discontinuity.

- ▶ **Removable discontinuity:**  $\lim_{x \rightarrow a} f(x)$  exists but

$$\lim_{x \rightarrow a} f(x) \neq f(a)$$

- ▶ **Jump discontinuity:**  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist (and are finite), but  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$
- ▶ **Infinite discontinuity:** At least one of the one-sided limits does not exist.

# Types of discontinuity

## Example

Each of the following functions has a discontinuity at  $x = 2$ .  
Classify it.

1.  $f(x) = \frac{x^2 - 4}{x - 2}$

2.  $g(x) = \frac{x^2}{x - 2}$

3.  $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$

4.  $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

# Types of discontinuity

# Intermediate Value Theorem

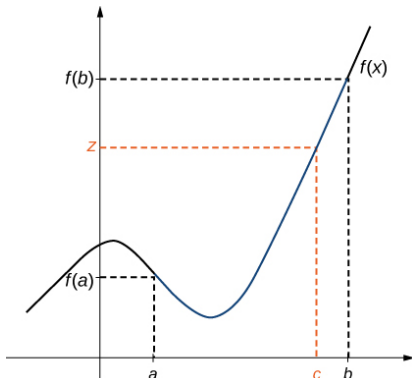
Continuous functions have numerous important properties, many of which we will study in MA140. The first of these is the **Intermediate Value Theorem**.

## Intermediate Value Theorem (IVT)

Suppose that  $f(x)$  is continuous on an interval  $[a, b]$ .

Let  $z$  be any real number between  $f(a)$  and  $f(b)$ .

Then there exists a number  $c \in [a, b]$  such that  $f(c) = z$ .



- ▶ If you travel by train from Galway to Athlone, then there must be a time when you are at Oranmore station, and a time when you are at Athenry, and at Woodlawn, etc.
- ▶ If your car/train/whatever accelerates from  $0\text{km/h}$  to  $100\text{km/h}$ , there was a time when it was travelling at  $30\text{ km/h}$ .
- ▶ This morning my train ticket from Athenry to Galway cost €5.10. Suppose train fares increase next Thursday to €5.50. But there wasn't a day when they cost, say, €5.20, because the price had a jump discontinuity (so the IVT does not apply here).



**Example**

Sketch an example of a function for which the IVT does *not* hold.

One of the main applications of the IVT is in establishing if an equation has a solution:

### Solutions to $f(x) = 0$

If  $f(x)$  defined on  $[a, b]$  is such that  $f(a) < 0$  and  $f(b) > 0$ , then there must be a value  $c \in [a, b]$  such that  $f(c) = 0$ .

More generally, if  $f(a)f(b) \leq 0$ , then  $f(x)$  has at least one zero in  $[a, b]$ .

**Example**

Show that  $f(x) = x - \cos(x)$  has at least one zero.

Given a function  $f(x)$ ,

- ▶ When we say  $c$  is a **zero** of a function,  $f$ , we mean that  $f(c) = 0$ .
- ▶ Many books and website also use the terminology “ $c$  is a **root** of  $f$ ”. This is particularly the case where  $f(x)$  is a polynomial.
- ▶ If  $c$  is a zero of  $f(x)$ , then it is a solution to the equation  $f(x) = 0$ .

**Example**

How many solutions does  $x^3 + 1 = 3x^2$  have?

**Example**

Use the *Intermediate Value Theorem* to show that the equation

$$2x^3 + 3x^2 - 2x - 1 = 0$$

has three solutions in the range  $-2 < x < 1$ .

## Derivatives: the concept

The next section of MA140 is all about **derivatives** of function. The derivative of a function describes how quickly the function is changing.

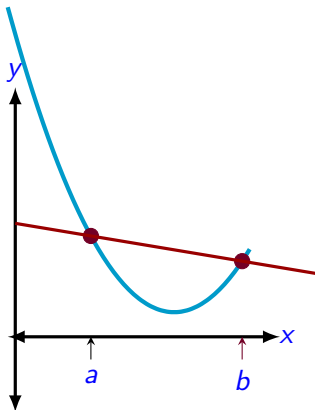
There are many, many applications: derivatives, and equations involving them are used everywhere: **speed/velocity** is the rate of change of displacement; **acceleration** is the rate of change of velocity.

We use derivatives to model how quickly a tumour is growing or shrinking, how pollutants are dispersed in a river, how pressure changes with depth, how inflation is changing in an economy. The list of applications is practically limitless.

Consider the graph opposite. It shows a function,  $f$ , and a secant line that intersects  $f$  at  $a = 1$  and  $b = a + 2$  (the actual values are not important).

If we wanted to summarised how  $f$  is changing between those two values, we could compute it as

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + 2) - f(a)}{2}$$

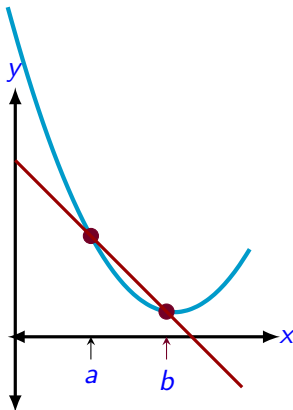




Now we'll consider how  $f$  is changing over a shorter interval: from  $a$  to  $b = a + 1$ . Again, we sketch the secant line that intersects  $f$  at  $x = a$  and  $x = b$ . The rate of change of  $f$  between these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + 1) - f(a)}{1},$$

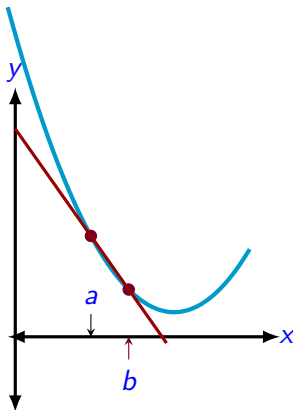
which, of course, is the slope of the secant line.



Next we shorten interval again:  
looking at how  $f$  changes from  
 $a$  to  $b = a + \frac{1}{2}$ , along with the  
secant line that intersects  $f$  at  
 $x = a$  and  $x = b$ .

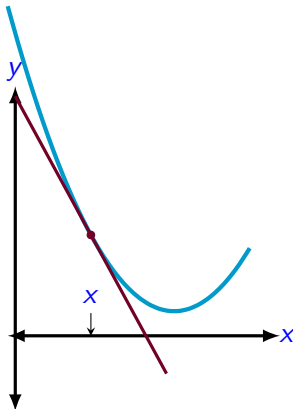
The rate of change of  $f$  between  
these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + \frac{1}{2}) - f(a)}{\frac{1}{2}}.$$



Finally, suppose we want to looking at the **instantaneous** rate of change of  $f$  at  $x = a$ . Hopefully, the preceding images have convinced you we could do this in two (equivalent) ways:

1.  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$
2. or as the slope of the tangent to  $f$  at  $x = a$ .



The slope of the curve  $y = f(x)$  at the point  $P = (a, f(a))$  is given by the number (if it exists)

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, it is called **the derivative of  $f$  at  $x = a$**  and we denote it by  $f'(a)$ .

### Definition: derivative at a point

Let  $f(x)$  be a function that has  $x = a$  in its domain. The **derivative** of the function  $f(x)$  at  $a$ , denoted  $f'(a)$ , is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

**Some terminology**

$f'(a)$  exists then we say that function  $f$  is **differentiable at  $x = a$** .

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the “limit” formula, we are doing “**differentiation from first principles**”.

**Example**

Use the limit definition of a derivative to compute the slope of the tangent to  $f(x) = x^2$  at  $x = 3$ .

## Exercises

### Exercises 3.3.1 (Based on Q1(a), 23/24)

$$\text{Let } g(x) = \begin{cases} 3 & x \leq 0 \\ 2x + 1 & 0 < x < 1 \\ x^2 & x \geq 1. \end{cases}$$

- (i) Sketch the graph of  $g(x)$  on the interval  $[-3, 4]$ , making use of the empty and full circle notation.
- (ii) Compute  $\lim_{x \rightarrow 1^-} g(x)$  and  $\lim_{x \rightarrow 1^+} g(x)$ . Is  $g$  continuous at  $x = 1$ . If not, classify the type of discontinuity.

### Exercise 3.3.2

$$\text{For what values of } b \text{ and } c \text{ is } f(x) = \begin{cases} x^2 + 1 & x \leq -1 \\ x + b & -1 < x < 1 \\ cx^2 & x \geq 1. \end{cases}$$

continuous at  $x = -1$  and  $x = 1$ ?

## Exercises

### Exercise 3.3.3 (23/24 exam)

Use the IVT to show that the equation  $x^3 - 3x + 1 = 0$  has three solutions in the range  $-2 < x < 2$ .

### Exercise 3.3.4

Use the (limit) definition of a derivative to differentiate the function  $f(x) = x^2 + 2$ .

### Exercise 3.3.5

Use the (limit) definition of a derivative to show that the derivative of  $f(x) = \cos(x)$  is  $f'(x) = -\sin(x)$ .