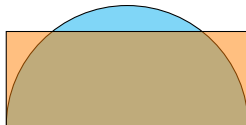


2526-MA140 Engineering Calculus

Week 10, Lecture 2
Cylindrical Shells, and
Mean Values

Dr Niall Madden
University of Galway

Wednesday, 19 November, 2025



Assignments, etc

1. **Assignment 7** has finished and grades posted.
2. **Assignment 8** (last one!) is live and will be covered in tutorials this week. See <https://universityofgalway.instructure.com/courses/46734/assignments/132796>
3. Results for the class test will be available by next Tuesday.
4. There will be a change to the tutorial schedule next week...
5. Those tutorials will focus on revision and a sample paper, to be posted Monday.

2

Today, we mean to discuss

- | | | | |
|---|-----------------------|---|-----------------------------|
| 1 | Assignments,etc | 4 | Average values of functions |
| 2 | Assignment 8, Q2 | ■ | Version 1 |
| | ■ The “washer” method | ■ | Version 1 |
| 3 | Cylindrical Shells | 5 | Root-Mean-Square Values |
| | ■ PS-8, Q2 again | 6 | Exercises |

See also: Sections **6.3** (Volumes of Revolution - Cylindrical Shells) of **Calculus** by Strang & Herman:

[math.libretexts.org/Bookshelves/Calculus/Calculus_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

Assignment 8, Q2

In Q2 of the Tutorial Sheet for PS-8, we are asked:

Assignment 8, Q2

Find the volume of the solid formed by rotating the region enclosed by $x = 0$, $x = 1$, $y = 0$, and $y = 9 + x^6$, about the y -axis.

There are two ways (at least) of solving this:

- ▶ The “washer method” from W09-2;
- ▶ The Method of **Cylindrical Shells**, which we have yet to study.

We'll now do each of these.

Assignment 8, Q2

The “washer” method

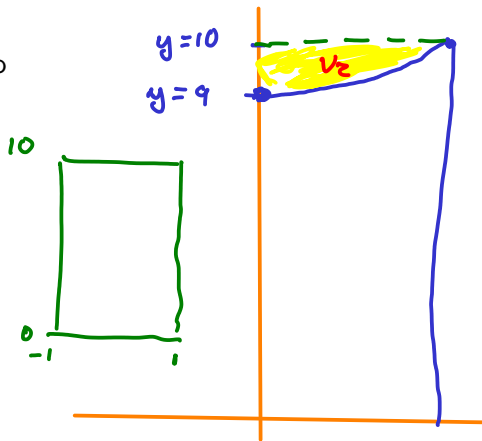
First we'll sketch the object in question: We should be able to convince ourselves that

$$V = V_1 - V_2, \text{ where}$$

- ▶ V_1 is the volume of $f(y) = 1$, from $y = 0$ to $y = 10$, rotated about the y -axis;

- ▶ V_2 is the volume of $f(y) = (y-9)^{1/6}$, from $y = 9$ to $y = 10$, rotated about the y -axis.

$x=0$ ~~$x=0$~~ to ~~$x=1$~~ $x=1$, rotated about the y -axis.



It should be easy to check that $V_1 = 10\pi$, and almost as easy to check that $V_2 = \frac{3}{4}\pi$. This gives that $V = \frac{37}{4}\pi$.

$$y = 9 + x^6 \Rightarrow x^6 = y - 9 \Rightarrow x = (y - 9)^{1/6}$$

Cylindrical Shells

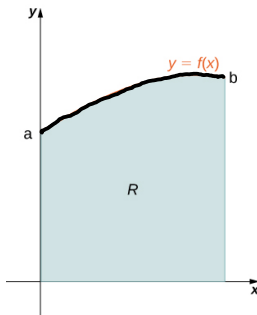
There is another approach, that is arguably easier.

So far, we've used the “disk method” for volumes of rotation. This came from the idea that every “slice” is a disk, whose area we can compute. Then integrating over the domain, we get the “sum” of all the disks: which is the volume.

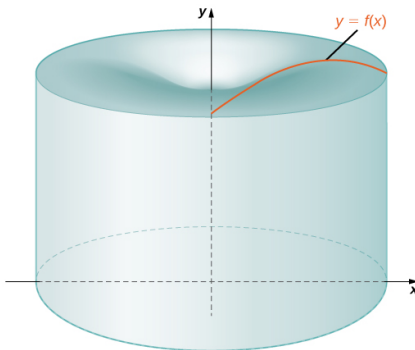
An alternative approach, is to construct think of the solid as an
infinite sum of cylinders...

Cylindrical Shells

We state the problem as follows: *Find the volume of the solid obtained by rotating the region between $y = f(x)$ and $y = 0$, and $x = 0$ and $x = b$, around the y -axis*



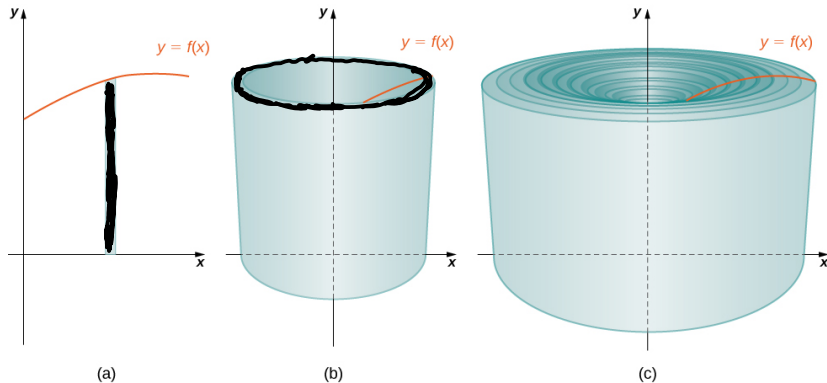
(a)



(b)

Cylindrical Shells

We can think of the region as made up of many small rectangles. When rotated about the y -axis, these become cylinders.



Cylindrical Shells

We can visualise this for a problem related to “Gabriel’s Horn” from earlier, but over a finite region.

Let $f(x) = 1/x$. Construct a solid of revolution by rotating the region between $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$ about the y -axis.

The visualisation from the textbook may be found [at this link](#).

Cylindrical Shells

Very roughly, a cylinder with height $f(x)$, and thickness $\Delta x_i = x_i - x_{i-1}$ has volume

$$\begin{aligned} V_i &= \pi(x_i^2 - x_{i-1}^2)f(x_i) = \pi f(x_i)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i) \frac{x_i + x_{i-1}}{2} \Delta x_i \approx 2\pi f(x_i) x_i \Delta x_i \end{aligned}$$

Then

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi f(x_i) x_i \Delta x_i = 2\pi \int_a^b x f(x) dx.$$

See text-book for more details!

The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Rotate about the y -axis, the region bounded above by $y = f(x)$, below by $y = 0$, on the left by $x = a$, and on the right by $x = b$.

Then the volume of the resulting solid of revolution is

$$V = 2\pi \int_a^b xf(x) dx.$$

Assignment 8, Q2

Use the Method of Cylindrical Shells to find the volume of the solid formed by rotating the region enclosed by $x = 0$, $x = 1$, $y = 0$, and $y = 9 + x^6$, about the y -axis.

$$f(x) = 9 + x^6 \quad a = 0 \quad b = 1$$

$$\begin{aligned} V &= 2\pi \int_a^b x f(x) dx = 2\pi \int_0^1 x (9 + x^6) dx \\ &= 2\pi \int_0^1 9x + x^7 dx = 2\pi \left[\frac{9}{2} x^2 + \frac{x^8}{8} \right]_0^1 \\ &= 2\pi \left[\frac{9}{2} + \frac{1}{8} - 0 - 0 \right] = \pi \left[9 + \frac{1}{4} \right] = \\ &\quad \pi \left[\frac{36}{4} + \frac{1}{4} \right] = \frac{37}{4} \pi \end{aligned}$$

Average values of functions

In many applications we wish to know the “average” (or **mean**) value of a continuously varying quantity, which is represented by a function.

We are already familiar with this concept when dealing with the mean of a set of n values: $\{x_1, x_2, \dots, x_n\}$. There are two (equivalent) ways of thinking about this:

1. The mean of the set of values is

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{1}{n} \sum_{k=1}^n x_k.$$

2. \bar{x} is the mean of the set of values of $\{x_1, x_2, \dots, x_n\}$.

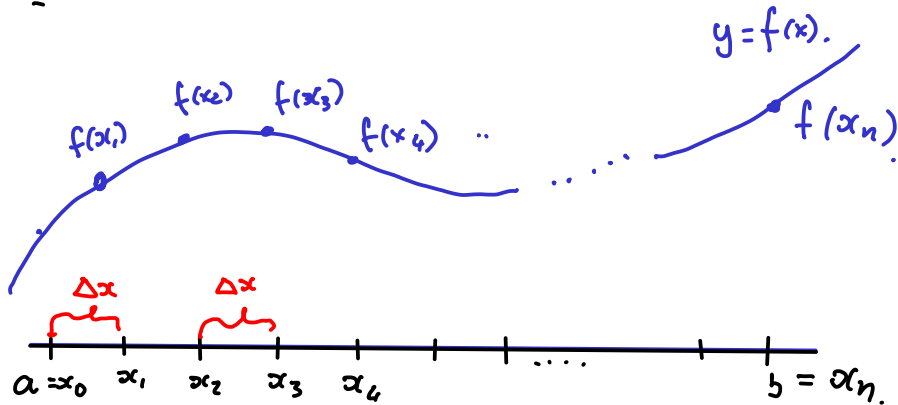
That is, if we replaced each of the x_i with the constant value \bar{x} , the sum would not change.

$$\begin{array}{l} x_1=1, x_2=3, x_3=8 \\ \text{so } \bar{x} = \frac{1+3+8}{3} = \frac{12}{3} = 4. \end{array} \quad \left| \begin{array}{l} 1 + 3 + 8 \\ \text{Also} = 4 + 4 + 4 \\ \hline \sum x_i = n \bar{x} \end{array} \right.$$

We can extend both these ideas to defining the “average value of a function”, on the interval $[a, b]$, getting the same result.

First, suppose we take n subintervals of $[a, b]$, and denote their end-points $\{x_0, x_1, \dots, x_n\}$. Note that $x_k = x_0 + k\Delta x$ where

$$\Delta x = (b - a)/n. \quad \Rightarrow \quad x_1 = a + \Delta x \quad x_2 = x_1 + \Delta x = a + 2\Delta x.$$



Now take the average of the n sampled values:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{1}{n} \sum_{k=1}^n f(x_k)$$

$\Delta x = \frac{b-a}{n}$

$\Rightarrow n = \frac{b-a}{\Delta x}$

$$= \frac{\Delta x}{b-a} \sum_{k=1}^n f(x_k) = \frac{1}{b-a} \left[\sum_{k=1}^n f(x_k) \Delta x \right]$$

$\int_a^b f(x) dx$

If $n \rightarrow \infty$ (or $\Delta x \rightarrow 0$), we get the **average value of $f(x)$ on $[a, b]$** is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

The second version is more insightful, I think:

Average value of a function

The constant \bar{f} is the **average** value of $f(x)$ on $[a, b]$, if

$$\int_a^b \bar{f} \, dx = \int_a^b \underline{f(x)} \, dx.$$

To see this is equivalent:

$$\int_a^b \bar{f} \, dx = [\bar{f} x] \Big|_a^b = \bar{f} (b - a).$$

$$\Rightarrow \bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

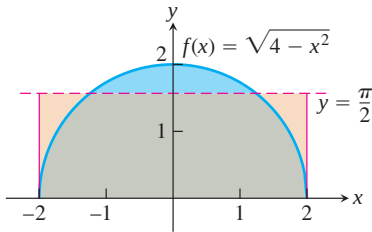
Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Since $\int_a^b f(x) dx$ is the area under the graph of $y = \sqrt{4 - x^2}$ between $x = -2$ to $x = 2$, this is the area of the semicircle of radius 2. : which is $\frac{1}{2} \pi r^2 = \frac{1}{2} \pi (2^2) = 2\pi$.

$$\text{So } \bar{f} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{4} (2\pi) = \frac{\pi}{2}.$$



Example

Find the average value of the function $f(x) = x^2 - x - 2$ on $[-3, 3]$.

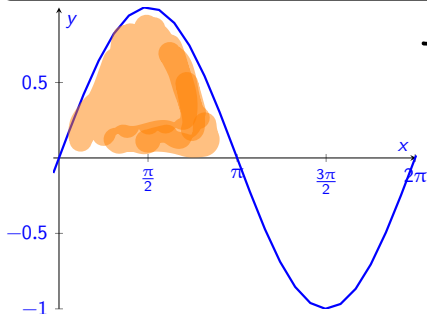
$$\begin{aligned}\bar{f} &= \frac{1}{b-a} \int_a^b f(x) \, dx \\&= \frac{1}{3 - (-3)} \int_{-3}^3 (x^2 - x - 2) \, dx \\&= \frac{1}{6} \left[\frac{1}{3} x^3 - \frac{1}{2} x^2 - 2x \right] \bigg|_{-3}^3 \\&= \frac{1}{6} \left[\frac{27}{3} - \frac{9}{2} - 6 - \left(-\frac{27}{3} - \frac{9}{2} + 6 \right) \right] \stackrel{\text{Check!}}{=} \dots = 1\end{aligned}$$

Average of $\sin(x)$

Find the average values of $f(x) = \sin(x)$ on

1. $[a, b] = [0, \pi]$

2. $[a, b] = [0, 2\pi]$



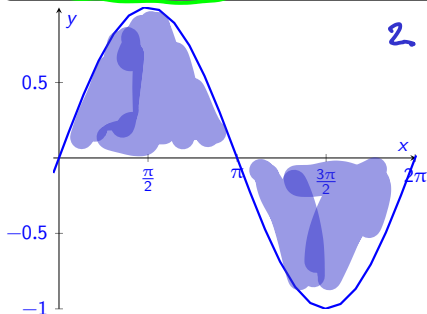
$$\begin{aligned} 1. \bar{f}_1 &= \frac{1}{\pi} \int_0^{\pi} \sin(x) dx \\ &= \frac{1}{\pi} [-\cos(x)]_0^{\pi} \\ &= \frac{1}{\pi} [-(-1) - (-1)] \\ &= \frac{2}{\pi} \end{aligned}$$

Average of $\sin(x)$

Find the average values of $f(x) = \sin(x)$ on

1. ~~$[a, b] = [0, \pi]$~~

2. $[a, b] = [0, 2\pi]$



$$\begin{aligned} 2 \quad \bar{f}_2 &= \frac{1}{2\pi} \int_0^{2\pi} \sin(x) dx \\ &= \frac{1}{2\pi} [-\cos(x)] \Big|_0^{2\pi} \\ &= \frac{1}{2\pi} [-1 + 1] = 0 \end{aligned}$$

Finished here

Root-Mean-Square Values

In some contexts, the **average value** of a function is a useful summary statistic. But it can be misleading too, as the last example showed.

Notable examples of this include

- ▶ The average value of an alternating current is zero;
- ▶ The average motion of a piston is zero.

Therefore (especially in power electronics) we need another measure to summarise a function

Root Mean Squared (RMS)

The **root mean square (RMS)** of a function $f(x)$ is

$$f_{\text{RMS}} := \left(\frac{1}{b-a} \int_a^b [f(x)]^2 dx \right)^{1/2}$$

Root-Mean-Square Values

Example

An electric current $i(\theta)$ is given by $i(\theta) = I_{\text{peak}} \sin(\theta)$ where I_{peak} is a constant. Find the root mean square of $i(\theta)$ over the interval $[0, 2\pi]$.

(Hint: use that $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$).

Root-Mean-Square Values

Exercises

Exer 10.2.1 (from textbook)

Let $f(x) = 1/x$. Find the volume of the solid of revolution by rotating the region between $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$ about the y -axis.

Exer 10.2.2

Find the average value of $f(x) = \frac{1}{1 - 4x^2}$ for $0 \leq x \leq 1/4$.

Exer 10.2.3

Find $b > 0$ such that the average value of $f(x) = x^2 - 2x + 3/4$ on the interval $[0, b]$ is zero.

Compute the root mean squared of $f(x)$ on the same interval.