

This is a sample paper for 2526-MA385. It is similar to the final Semester 1 exam paper in the following ways:

- It features 5 questions; all to be attempted.
  - Questions 1 and 2 are based on material from Section 1 (may have some over-lapping content. E.g., Newton's Method, or FPI could feature on both).
  - Questions 3, 4 and 5 are based on Sections 2, 3, and 4 respectively with minimal overlap (and only in so far as Sections 3 and 4 overlap a little)
  - Questions feature a mixture of definitions, theory and calculations.
  - The questions on the exam will, of course, be different. However, if you can attempt this paper unseen, you are well prepared.
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Q1. Suppose we wish to find  $\tau \in [a, b]$  such that  $f(\tau) = 0$  for some nonlinear function  $f(x)$ .

- (a) State the **Secant Method** for this problem. Provide a justification for it.
  - (b) Suppose that  $f(x) = 2x^2 - 5$ . Show that  $f(x) = 0$  has a solution in  $[1, 2]$ .
  - (c) Taking  $x_0 = 1$  and  $x_1 = 2$ , carry out **three** iterations of the Secant Method to estimate the solution to  $2x^3 - 4 = 0$ . Show your calculations to 4 decimal places.
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Q2. (a) What does it mean for a function,  $g$ , to be a contraction on an interval  $[a, b]$ ?

- (b) Suppose that we have a fixed point iteration (FPI) method  $x_{k+1} = g(x_k)$ , and that  $g$  is known to be a contraction, with a fixed point  $\tau$ . Show that the sequence generated by the method,  $\{x_0, x_1, x_2, \dots\}$  converges *at least linearly* to  $\tau$ .
- (c) Suppose that we want to solve  $2x^2 - 5 = 0$  using FPI, in order to approximate  $\tau = \sqrt{10}/2$ . That is, we choose a function  $g = g(x)$ , and initial guess  $x_0 \in [1, 2]$ , and set  $x_{k+1} = g(x_k)$  for  $k = 0, 1, 2, \dots$ . Consider the following functions:

$$g_1(x) = 2x^2 + x - 5, \quad g_2(x) = x/2 + 5/(4x), \quad g_3(x) = x^2/5 - 1/2.$$

For each of these, determine whether or not it is a suitable choice of  $g$  in the FPI.

- (d) Show that Newton's method for solving  $f(x) = 0$  can be considered as a FPI method. What FPI method does it yield when we use it to solve  $f(x) = 2x^2 - 5 = 0$ ?
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Q3. Consider the general two-stage Runge-Kutta (RK2) method:  $y_{i+1} = y_i + f\Phi(t_i, y_I; h)$ , where

$$k_1 = f(t_i, y_i), \quad k_2 = f(t_i + \alpha h, y_i + \beta h k_1)$$

and

$$\Phi(t_i, y_i; h) = ak_1 + bk_2.$$

For a specific method we can take

$$\alpha = 2, \text{ and } a = 1/4. \tag{1}$$

- (a) What does it mean for a one-step method to be *consistent*? Determine the value of  $b_2$  for the method to be consistent.
- (b) Suppose  $y$  solves the initial value problem

$$y(1) = 1, \quad y'(t) = 2t \quad \text{for } t > 1.$$

Explain why the RK2 method should compute the exact solution. Use this fact to determine the value for  $\alpha$ .

- (c) Suppose that we attempt to solve

$$y'(t) = \lambda y(t) \quad y(0) = 1,$$

with a RK2 method. Use the fact that the RK2 solution should agree with the Taylor series for  $y(t_{i+1})$  about  $t_i$ , up to terms of order  $h^2$ , to find a value of  $\beta$ .

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- Q4. (a) Let  $L \in \mathbb{R}^{n \times n}$  be a non-singular lower triangular matrix, and  $\mathbf{b} \in \mathbb{R}^n$  be such that  $b_i = 0$  for  $i = 1, \dots, k \leq n$ . If  $\mathbf{y}$  solves  $L\mathbf{y} = \mathbf{b}$ , show that  $y_i = 0$  for  $i = 1, \dots, k \leq n$ . Hence or otherwise, show that the inverse of a nonsingular lower triangular matrix is also lower triangular.
- (b) Define the *LU* factorization of a matrix. What assumptions must be made on the matrix to ensure that such a factorization exists?
- (c) Find the *LU*-factorisation of

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix}.$$

Use this factorization to solve  $Ax = b$ , where  $b = (4, -4, 0, 8)^T$ .

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- Q5. (a) Recall the definition of the Euclidean norm on  $\mathbb{R}^n$ :  $\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}^T \mathbf{u}}$ . Prove the Cauchy-Schwarz inequality:

$$\left| \sum_{i=1}^n u_i v_i \right| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Hence show that  $\|\cdot\|_2$  satisfies the triangle inequality.

- (b) Let  $A$  be any matrix in  $\mathbb{R}^{n \times n}$ . What are the *singular values* of  $A$ ? Show that they are real and non-negative.

Define the *subordinate matrix norm* on  $\mathbb{R}^{n \times n}$  associated with  $\|\cdot\|_2$  and show that  $\|A\|_2$  is the largest singular value of  $A$ .

- (c) State the Gershgorin First Circle Theorem, and use it to find an upper bound on  $\|A\|_2$  when

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix}.$$