MA378 Chapter 1: Interpolation

§1.5 Wrap-up: Convergence & Runge's Example

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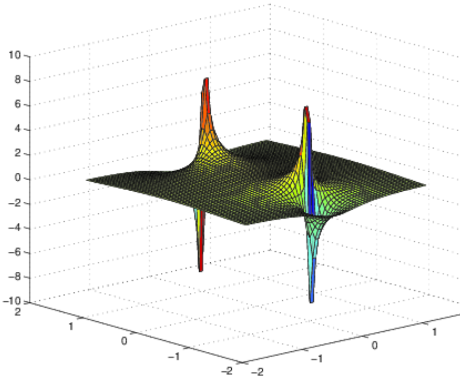


5.1 Convergence

The celebrated Weierstrass approximation theorem states that, given f and a positive number ε, there is a polynomial p such that

$$\max_{x \in [a,b]} |f(x) - p(x)| := \|f - p\|_\infty \leq \varepsilon.$$

That means: "*for any function, f, you can find a polynomial that approximates it as accurately as you would like*".

Now suppose that f is a continuous function on $[a, b]$ and that $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials that interpolate f at $n+1$ **equally spaced** points. One might be inclined to believe that

$$\lim_{n \to \infty} \|f - p_n\|_\infty = 0.$$

5.1 Convergence

Another way of thinking about this is recalling the error bound:

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|$$

we might expect that

$$\lim_{n \to \infty} \max_{x \in [a,b]} \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)| = 0.$$

In order words, we might think that, in order to find an interpolating polynomial that is as accurate as we would like, we just need to choose large enough n.

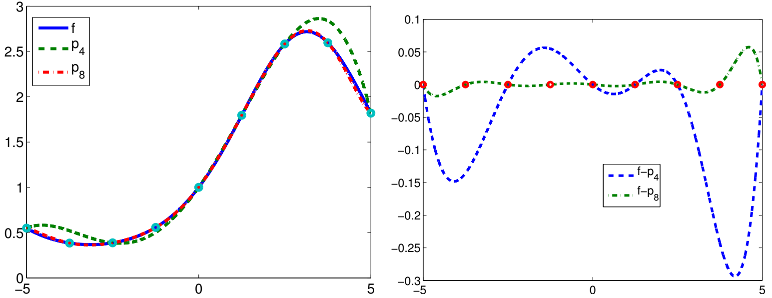
5.1 Convergence

And **some times** this is true. For example, suppose that $a = -5$, $b = 5$, and $f(x) = \mathrm{e}^{\sin(x/2)}$. In Table 1 the errors for successive interpolants are shown.

Table: Errors in polynomial interpolants to $\mathrm{e}^{\sin(x/2)}$ on $[-5, 5]$

n	$\|f - p_n\|_\infty$
2	1.27e-00
4	2.94e-01
6	8.39e-02
8	5.75e-02
16	1.07e-03

5.1 Convergence



Polynomial interpolants, p_4 and p_8, to $\mathrm{e}^{\sin(x/2)}$ on $[-5, 5]$, and their errors (right)

5.1 Convergence

However, there is a famous example of a simple function that cannot be successfully interpolated in this manner.

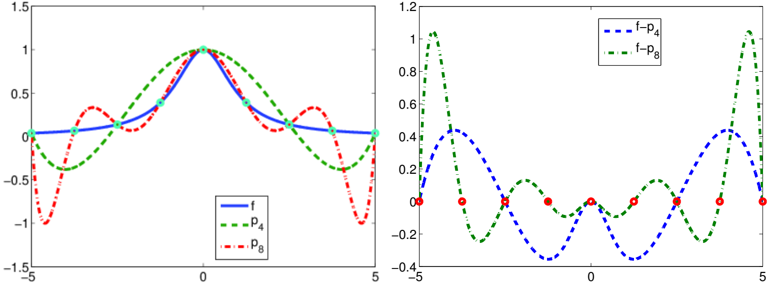
Runge's Example

$$f(x) = \frac{1}{1+x^2} \qquad \text{on } [-5,5].$$

Errors for some n are shown below. Notice they *increase* with n.

n	$\|f - p_n\|$
2	0.65
4	0.44
6	0.62
8	1.05
16	14.39
20	59.66
22	122.91
24	257.21

5.1 Convergence



Polynomial interpolants to $\dfrac{1}{1+x^2}$ *on* $[-5, 5]$

5.1 Convergence

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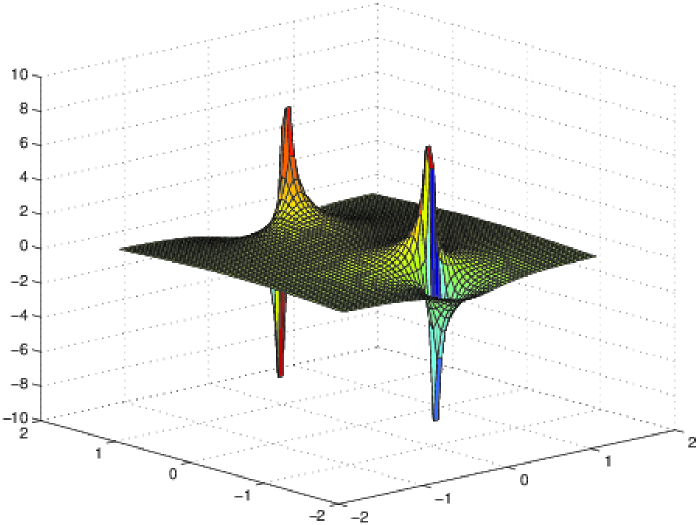

5.2 Where to from here?

So now it looks like polynomial interpolation is bad, at least on equidistant points.

However, in Lab 1 we'll do an exercise that might lead us to be more optimistic: it is possible to find a set of points that made the approximation as good we wanted (until round-off error dominated).

5.2 Where to from here?

Unfortunately, just because we have a good set of points for interpolating one particular function, it does not follow that that set is good for every continuous function: this is **Faber's Theorem**. This has often led numerical analysts to abandon the idea of interpolation by high-order polynomials completely.

However, there is a set of points that are useful, if f is smooth enough: the **Chebyshev** points of Lab 1. If you are interested, read the essay **Inverse Yogiisms** by Lloyd N. (Nick) Trefethen, Notices AMS, Dec. 2016. To investigate this numerically in MATLAB, try exploring the Chebfun toolbox.

5.2 Where to from here?

The approach we will take is different. We say that if p_1 is the polynomial of degree 1 that interpolates the function f at the points x_0 and x_1, with $h = x_1 - x_0$, then

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

So, assuming M_2 is bounded (which is reasonable), we can make p_1 as close to f as we would like by taking a small enough interval $[x_0, x_1]$. The next section of this module is devoted to seeing how this can be used in theory and practice.