Annotated slides from class.

MA313 : Linear Algebra I

Week 7: Dimension and Rank

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18th and 21st of October, 2022

These slides were produced by Niall Madden, based on ones by Tobias Rossmann.

Outline

For more details, see

- Chapter 7 (Linear Independence) of Linear Algebra for Data Science: `https://shainarace.github.io/LinearAlgebra/linind.html`
- Sections 4.5 (Dimension of a Vector Space) of Lay: `https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=5174425`

Assignment 3

There was a technical issue with WeBWorK over the weekend, only resolved yesterday evening. So I've extended the deadline by 48 hours, to 5pm, Wednesday 19 Oct 2022.

Communication Skills : Progress Report

Your progress report is due 5pm, Friday 21 Oct.
Information on the content and structure are on Blackboard.

New assignment.... see Blackboard.

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Week 7: Dimension and Rank

Start of ...

PART 1: Coordinates

This is continued from the end of last week's classes

1: Coordinates (again)

Last week we learned that, if the sequence $\mathcal{B} = (b_1, b_2, \dots, b_n)$ be a basis of V, then we can write any vector $x \in V$ as a unique linear combination of the vectors in $\mathcal{B}$. That is:

- For any x, there is a set of real numbers c_1, c_2, ..., c_n, such that

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n.$$

- Furthermore, there is only one set of numbers c_1, c_2, ..., c_n for which this is true.

1: Coordinates (again)

Since this collection of numbers is so important, it has a name: the **coordinate vector** of $x \in V$ relative to $\mathcal{B}$ is

$$[x]_{\mathcal{B}} := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

where $c_1, \ldots, c_n \in \mathbb{R}$ is the unique sequence with

$$x = c_1 b_1 + \cdots + c_n b_n$$

The function $V \to \mathbb{R}^n, \quad x \mapsto [x]_{\mathcal{B}}$ is the **coordinate mapping** determined by $\mathcal{B}$.

1: Coordinates (again)

Example

Let

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

be the standard basis of $\mathbb{R}^n$.
Then $[x]_{\mathcal{B}} = x$ for all $x \in \mathbb{R}^n$.

Hence, taking coordinate vectors *generalises* extracting the components of a vector in $\mathbb{R}^n$.

Example

Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

1. $\mathcal{B}$ is a basis of $\mathbb{R}^2$.

First, note $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent : $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for any c.

Second: show $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$. That is, if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we can solve for c_1 & c_2. Here $x_1 = c_1 + c_2$ & $x_2 = 2c_2$.

So $c_2 = x_2/2$ and then $c_1 = x_1 - x_2/2$.

Example

Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

2. Write down the coordinate mapping determined by $\mathcal{B}$. It is a linear transformation, so also write down the matrix of the linear transformation.

We have that

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} x_1 - x_2/2 \\ x_2/2 \end{bmatrix}}_{\text{Coordinate mapping}} = \overset{A}{\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

matrix of the linear trans.

Example

Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

2. Write down the coordinate mapping determined by $\mathcal{B}$. It is a linear transformation, so also write down the matrix of the linear transformation.

Note: $\det(A) = 1/2 \neq 0$

So A^{-1} exists.

Also: $\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - 2(1/2) \\ 0 + (1/2)(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

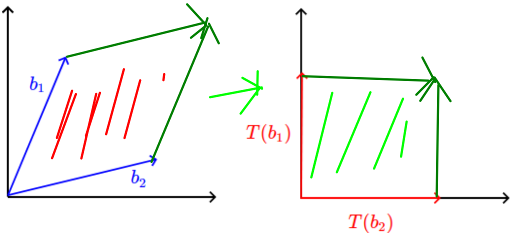
Standard basis!

Suppose that $\mathcal{B} = (b_1, b_2)$ is a basis of $\mathbb{R}^2$.

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto [x]_{\mathcal{B}}$ be the associated coordinate mapping.

Then $T(b_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(b_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note that $\mathcal{B}$ defines a parallelogram. The coordinate mapping T "stretches", "rotates", and perhaps "reflects" it into a square!



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Start of ...

PART 2: Isomorphisms

INVERTIBLE FUNCTIONS

Let X and Y be sets and let $f\colon X \to Y$ be a function.
Then the following are equivalent:

- f is invertible, i.e., there exists $f^{-1}\colon Y \to X$ such that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.
- f is *one-to-one* and ***onto***. (Also called "*injective*" and "***surjective***").

Moreover, if f is invertible, then the function f^{-1} is **uniquely** determined.

2: Isomorphisms

Definition (ISOMORPHISM)

An **isomorphism** from a vector space V to a vector space W is an invertible linear transformation $V \to W$.

We say that V and W are **isomorphic** if there exists an isomorphism between them.

Example

- For any vector space V, the **identity map**

$$\mathrm{id}_V : V \to V, \quad x \mapsto x$$

  is an isomorphism.
  Hence, every vector space is isomorphic to itself.

- Given any basis $\mathcal{B} = (b_1, \ldots, b_n)$ of V, the coordinate mapping

$$V \to \mathbb{R}^n, \quad x \mapsto [x]_{\mathcal{B}}$$

  is an isomorphism.
  (We saw in Part 3 that this is an invertible linear transformation).

Theorem

Let U, V, and W be vector spaces.
Let $S\colon U \to V$ and $T\colon V \to W$ be linear transformations.
Then:

- $T \circ S\colon U \to W, x \mapsto T(S(x))$ *is a linear transformation.*
- *If S and T are isomorphisms, then so is $T \circ S$.*

That is: if U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.

Theorem

If $T: V \to W$ is an isomorphism of vector spaces, then so is $T^{-1}: W \to V$.

Hence, if V is isomorphic to W, then W is isomorphic to V.

Question

Can we relate this to matrices and vectors?

Let A be an $n \times n$ matrix.

Then the function

$$T\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto Ax$$

is a linear transformation.

It is invertible if and only if A is an invertible matrix. In that case, T^{-1} is the function

$$\mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto A^{-1}y.$$

Summary

- $m \times n$ matrices correspond to linear transformations $\mathbb{R}^n \to \mathbb{R}^m$.
- An $n \times n$ matrix is invertible if and only if the corresponding linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. In that case, the inverse of the linear transformation corresponds to the inverse matrix.

Question...

Can there be an isomorphism $\mathbb{R}^n \to \mathbb{R}^m$ when $m \neq n$?

Let $\mathcal{B} = (b_1, \dots, b_n)$ be a basis of $\mathbb{R}^n$.

Then

$$T \colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto [x]_{\mathcal{B}}$$

and its inverse $T^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ are both linear transformations from $\mathbb{R}^n$ to itself.

Question...

What are the matrices corresponding to T and T^{-1}?

By definition:

$$T(x) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \iff x = c_1 b_1 + \cdots + c_n b_n \iff T^{-1}\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = x.$$

Hence, for $i = 1, \ldots, n$,

$$T^{-1}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = b_i$$

so the matrix of T^{-1} is $A := [b_1 \cdots b_n]$, and the matrix of T is therefore A^{-1}.

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PART 3: Dimension

3: Dimension

In many parts of mathematics, the concept of "dimension" can be very difficult and subtle... and even counter-intuitive.

Example (From analysis)

There exists a continuous function from the unit interval $[0, 1]$ onto the unit square $[0, 1]^2$. (In fact, there exists a continuous surjection $[0, 1] \to [0, 1]^n$ for each n.)
Such functions are called space-filling curves

Fortunately, such issues don't arise in linear algebra...

3: Dimension

What should "dimension" mean/imply?

1. The **dimension**, $\dim V$ is a number associate with to each vector space V.
2. If V and W are isomorphic, then we want that $\dim V = \dim W$.
3. We want that $\dim \mathbb{R}^n = n$ for each n.
4. In fact, if $(b_1, \dots, b_n)$ is a basis of V, then we want that $\dim V = n$.

So, would it make sense to take that as a definition, at least when V is finitely generated? It would require that any two bases of V contain the same number of vectors...

~~3: Dimension~~

Theorem

Let $(b_1, \ldots, b_n)$ be a basis of V. Then every sequence consisting of at least $n+1$ vectors in V is linearly dependent.

Clearly, If the set of vectors is

$\{b_1, b_2, \ldots, b_n, d\}$, we can write d as

a linear combination of $b_1, \ldots, b_n$. So set

is linearly dependent.

Otherwise, if the set is

$\{u_1, u_2, \ldots, u_n, u_{n+1}\}$. Write each

$u_1, \ldots, u_n$ in terms of $b_1, \ldots, b_n$,

Theorem

Let $(b_1, \ldots, b_n)$ be a basis of V. Then every sequence consisting of at least $n+1$ vectors in V is linearly dependent.

For a full proof, see Theorem 9 in Chap 4 of the text-book by Lay et al.

Corollary

If V has some basis consisting of precisely n vectors, then every basis of V consists of precisely n vectors.

This is an immediate consequence of the previous theorem.

Example: $\mathbb{R}^3$ has a basis $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

- Any other basis for $\mathbb{R}^3$ has 3 vectors too.
- Eg $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right)$ could <u>not</u> be a basis, since it has too few vectors

Definition (DIMENSION)

The **dimension** of V ~~is~~, a vector space, is

$$\dim V = \begin{cases} 0, & \text{if } V = \{0\}, \\ n, & \text{if } V \text{ has a basis } (b_1, \dots, b_n), . \\ \infty, & \text{if } V \text{ is not finitely generated.} \end{cases}$$

Example

- $\dim \mathbb{R}^n = n$: the standard basis consists of n vectors.
- $\dim \mathbb{P}_n = n+1$: the sequence $(1, t, t^2, \ldots, t^n)$ is a basis.
- $\dim \mathbb{P} = \infty$ because this space is not finitely generated. (Why?)

Eg $\mathbb{P}_2$ has as a basis $(1, t, t^2)$.

So $\dim \mathbb{P}_2 = 3$.

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PART 4: Spaces with the same dimension

FACT

Isomorphic vector spaces have the same dimension.

Isomorphic means that, there is an invertible mapping between the spaces. Call this $T: V \to W$.
So, if $(b_1, b_2, \dots, b_n)$ is a basis for V, then $(T(b_1), T(b_2), \dots, T(b_n))$ is a basis for W.

PTO

FACT

Isomorphic vector spaces have the same dimension.

In more detail: suppose $\dim(V) = n$, and $\dim(W) = m$ and V & W are isomorphic. We want to show $m = n$.

Since $\dim(V) = n$, denote a basis for it as $(b_1, b_2, \dots, b_n)$. Call the isomorphic mapping T. Since $\{b_1, b_2, \dots, b_n\}$ are linearly independent in V, so $\{T(b_1), T(b_2), \dots, T(b_n)\}$ are linearly independent in W. So $m \geq n$.

PTO.

4: Spaces with same dimension

FACT

Isomorphic vector spaces have the same dimension.

Similarly, If $(w_1, w_2, \ldots, w_m)$ is a basis for W, then $\{T(w_1), T(w_2) \ldots, T(w_m)\}$ are linearly indep. in V. So $n \geq m$

But if $m \geq n$ and $n \geq m$,

it must be that $m = n$.

Question

How are the concepts "subspace" and "dimension" related?

Example

The subspaces of $\mathbb{R}^3$, sorted by dimension, are:

- 0-dimensional: just $\{0\}$.
- 1-dimensional: subspaces spanned by a single non-zero vector. That is, such subspaces are lines through the origin.
- 2-dimensional: planes passing through the origin.
- 3-dimensional: just $\mathbb{R}^3$.

Theorem

Let V be a finitely generated vector space. Let H be a ***subspace*** *of V. Then:*

- *H is also finitely generated.*
- $\dim H \leqslant \dim V$.
- *Any linearly independent sequence of vectors in H can be extended to a basis of V.*
- $\dim H = \dim V$ *if and only if* $H = V$.

Theorem (The Basis Theorem)

Let $n = \dim V$ satisfy $1 \leqslant n < \infty$. Let $v_1, \ldots, v_n \in V$. Then following are equivalent:

1. $(v_1, \ldots, v_n)$ *is a basis of V.*
2. $v_1, \ldots, v_n$ *are linearly independent.*
3. $V = \operatorname{span}\{v_1, \ldots, v_n\}$.

So, if we have a space of known dimension, n, and n vectors $(v_1, .., v_n)$ we can check if that is a basis by checking any <u>one</u> of 2 or 3.

Theorem (The Basis Theorem)

Let $n = \dim V$ satisfy $1 \leqslant n < \infty$. Let $v_1, \ldots, v_n \in V$. Then following are equivalent:

1. $(v_1, \ldots, v_n)$ *is a basis of V.*
2. $v_1, \ldots, v_n$ *are linearly independent.*
3. $V = \operatorname{span}\{v_1, \ldots, v_n\}$.

Eg $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$

Not a basis.

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PART 5: **Rank and Nullity**

Definition (RANK and NULLITY)

Let A be an $m \times n$ matrix.

- The **rank** of A is the dimension of its column space: $\operatorname{rank} A := \dim \operatorname{Col} A$.
- The **nullity** of A is the dimension of its null space: $\operatorname{nullity} A := \dim \operatorname{Nul} A$.

When we were finding bases for the column space and null space of a matrix, we found that, if a matrix has p pivot columns then

$$\operatorname{rank} A = p$$

and

$$\operatorname{nullity} A = n - p.$$

5: Rank and Nullity

Theorem

Rank-Nullity Theorem $\operatorname{rank} A + \operatorname{nullity} A = n.$

In particular, $\operatorname{rank} A$ and $\operatorname{nullity} A$ determine one another.

This is one of the most famous and important results of linear algebra!

Example

Confirm that $\operatorname{rank} A + \operatorname{nullity} A = n$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The reduced row echelon for of A is

$$A' = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\uparrow$ $\uparrow$ $\uparrow$

Pivot cols

$\boxed{\text{A has rank 3.}}$

Since cols 1, 3 & 4 of A' are linearly indep in A', so too col 1, 3 & 4 of A are linearly indep. So they are a basis for A's column space. That is

$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Also $\dim \text{Col } A = 3$.

5: Rank and Nullity (This slide added after class)

Recall, if $x \in \text{Nul } A$ (ie $Ax=0$) then $x \in \text{Nul } A'$ (ie $A'x=0$). Thus

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 - 3x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned}$$

Non pivot cols: associated variables are "free".

Solving the system we get

- x_5 is free
- $x_4 = 0$
- $x_3 = -x_5$
- x_2 is free
- $x_1 = -2x_2 + 3x_5$

So, any vector of the form $\begin{bmatrix} -2x_2 + 3x_5 \\ x_2 \\ -x_5 \\ 0 \\ x_5 \end{bmatrix}$ is

in the null space of A' (and A).

So $\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So $\dim(\text{null}(A)) = 2$.

5: Rank and Nullity

In summary, this calculation suggests
* dim Col A = (number of pivot columns of A')
* dim Nul A = (number of non-pivot columns of A')
* so dim Col A   + dim Nul A = (number of columns of A' = n).

There is a version of this for vector spaces.

Theorem

Rank-Nullity Theorem (abstract form) Let $T: V \to W$ be a linear transformation between vector spaces. Then

$$\dim \mathrm{Ker}\, T + \dim \mathrm{Ran}\, T = \dim V.$$

(We'll return later to have a closer look at the required translation between matrices and linear transformations).

Finished here, but some details added after class.

5: Rank and Nullity

Returning the matrices...

Theorem (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix.
Then the following are equivalent:

1. *A is invertible, i.e. there exists an $n \times n$ matrix B such that $AB = I_n = BA$.*
2. $\operatorname{rank} A = n$.
3. $\operatorname{nullity} A = 0$.
4. *The columns of A form a basis of $\mathbb{R}^n$.*

Exercises

Q1. Let $\mathcal{B} = \left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right)$. Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$ and find the vector $x \in \mathbb{R}^2$ with coordinate vector $[x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Q2. Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right)$. Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$ and find the vector $x \in \mathbb{R}^3$ with coordinate vector $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

Q3. Find the dimension of this subspace of $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Exercises

Q4. Find the dimension of this subspace of $\mathbb{R}^4$.

$$\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Q5. Find the dimension of the subspace of $\mathbb{R}^2$ spanned by

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 15 \end{bmatrix}.$$

Q6. Find the dimensions of Nul A and Col A, where

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercises

Q7. Find the rank of these matrices:

$$\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}.$$

Q8. If the null space of a 4×6 matrix A is 3-dimensional, what is the dimension of the column space of A?

Q9. If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A?