MA378 Chapter 2: Splines

Any question marked with a $\star$ may feature on the class test and/or Assignment 2, and so won't be covered in tutorials.

Exercise 1.1. Page 28 of the Department of Education's old Mathematics Tables ("The *Log Tables*") reports that $\ln(1) = 0$, $\ln(1.5) = 0.4055$ and $\ln(2) = 0.6931$.

(i) Write down the linear spline l that interpolates $f(x) = \ln(x)$ at the points $x_0 = 1$, $x_1 = 1.5$ and $x_2 = 2$.

(ii) Use this to estimate $\ln(x)$ at $x = 1.2$. How does this compare to the value in the tables, which is 0.1823?

(iii) Give an estimate for the maximum error:

$$\max_{1 \leqslant x \leqslant 2} |f(x) - l(x)|.$$

(iv) What value of n would you choose to ensure that $|f(x) - l(x)| \leqslant 0.001$ for all $x \in [1, 2]$.

Exercise 1.2. As an alternative to the definition given in class, one can define the linear spline interpolant to a function is as a linear combination of a set of piecewise linear basis functions $\{\psi_i\}_{i=0}^N$:

$$\psi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

They are depicted in **??**.

(i) Write down a formula for the $\psi_i(x)$;

(ii) derive a formula for $l(x)$ in terms of the ψ_i.

This exercise is useful: we'll use these basis functions (called "hat" functions) in the final section of the course.

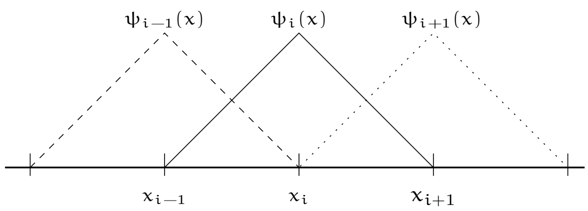

Fig. 1.1: Some hat functions

Exercise 2.1. [Equation numbers given here refer to those for the slides for Section 2.2.] When deducing the system of equations for the natural cubic spline, we showed how to construct the formulation in (1). and the relationship between σ_i, α_i and β_i in (2). Now carefully show to deduce the system (3).

Exercise 2.2. (For students who did MA385). Write the equations in (3) as a matrix-vector equation $A\boldsymbol{\sigma} = \mathbf{b}$, where A is an $n \times n$ matrix. Show that A is nonsingular, and hence that the system as a unique solution.

Exercise 2.3. Find the natural cubic spline interpolant to $f(x) = \sin(\pi x/2)$ at the nodes $\{x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3\}$. Calculate value of the interpolant at $x = 2.5$. What is the error at this point?

Exercise 2.4 ($\star$). Take $f(x) = \ln(x)$, $x_0 = 1$, $x_N = 2$. What value of N would you have to take to ensure that $|\ln(x) - S(x)| \leqslant 10^{-4}$ for all $x \in [1, 2]$, where S is the natural cubic spline interpolant to f.

Answer: In Theorem 2.3 we learned that $\|f - S\|_\infty \leqslant \dfrac{5}{384} M_4 h^4$, where $M_4 = \max\limits_{1 \leqslant x \leqslant 2} |f^{(iv)}(x)|$. So we need to ensure that $\frac{5}{384} M_4 h^4 \leqslant 10^{-4}$. First calculate that $M_4 = \max\limits_{1 \leqslant x \leqslant 2} |-6/x^4| = 6$. With this, we see we need h such that $h^4 \leqslant 10^{-4} \times \frac{384}{30} = 1.28 \times 10^{-3}$. That gives $h \leqslant 0.1891$ Using that, in this case $N = 1/h$, we get the requirement that $N \geqslant 5.2869$. Since N must be an integer, the answer is **we must take** $N = 6$.

Exercise 2.5. Suppose that S is a natural cubic spline on $[0, 2]$ with

$$S(x) = \begin{cases} -3x + 2(1-x) + a(1-x)^3 + \frac{2}{3}x^3, & x \in [0, 1), \\ b(2-x) + c(2-x)^3 + d(x-1)^3, & x \in [1, 2]. \end{cases}$$

Find a, b, c, and d.

Exercise 2.6 ($\star$). Suppose that S is a natural cubic spline on $[0, 2]$ with

$$S(x) = \begin{cases} 3x + a(1-x)^3 + bx^3, & \text{for } 0 \leqslant x < 1, \\ c(2-x) - (2-x)^3 + d(x-1)^3, & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

Find a, b, c, and d.

Answer: First note that

$$S'(x) = \begin{cases} 3 - 3a(1-x)^2 + 3bx^2, & \text{for } 0 \leqslant x < 1, \\ -c + 3(2-x)^2 + 3d(x-1)^2, & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

and

$$S''(x) = \begin{cases} 6a(1-x) + 6bx, & \text{for } 0 \leqslant x < 1, \\ -6(2-x) + 6d(x-1), & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

A natural spline has $S''(0) = 0$, so that gives $a = 0$. Similarly, requiring that $S''(2) = 0$ gives that $d = 0$.
Next use that S must be continuous at $x = 1$, to get that $3 + b = c - 1$, and
S' must be continuous at $x = 1$, which gives $3 + 3b = -c + 3$
Solving these equations gives $a = 0, b = -1, c = 3$ and $d = 0$.

Exercise 3.1. Recall Exercise 2.3. Calculate the value to the PCHIP interpolant to $f(x) = \sin(\pi x/2)$ at the nodes $\{x_i\}_{i=0}^{3} = \{0, 1, 2, 3\}$ at the point $x = 2.5$. What is the error at this point?

Answer: This is a somewhat tedious question, and I should probably change it in future years. Here is a partial solution. The PCHIP interpolant can be written as

$$S(x) = \begin{cases} S_1(x) & 0 \leqslant x \leqslant 1 \\ S_2(x) & 1 \leqslant x \leqslant 2 \\ S_3(x) & 2 \leqslant x \leqslant 3 \end{cases}$$

Here I'll give the formula just for S_3.

$$S_3(x) = c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3,$$

where

$$c_0 = f_2, c_1 = f_2', c_2 = \frac{3}{h^2}(f_3 - f_2) - \frac{1}{h}(f_3' + 2f_2'), c_3 = \frac{1}{h^2}(f_3' + f_2') - \frac{2}{h^3}(f_3 - f_2).$$

For this problem $h = 1$, $f_2 = f(2) = 0$, and $f_3 = f(3) = -1$. Also $f'(x) = \frac{\pi}{2}\cos(\pi x/2))$. So $f_2' = f'(2) = -\Pi/2$, and $f_3' = f'(3) = 0$. With a bit of calculation,

$$c_0 = -1, c_1 = -\pi/2, c_2 = 3(f_3 - f_2) - (f_3' + 2f_2') = \pi - 3, c_3 = (f_3' + f_2') - 2(f_3 - f_2) = 2 - \pi/2.$$

That gives

$$S_3(x) = -\frac{\pi}{2}(x-2) + (\pi - 3)(x-2)^2 + (2 - \pi/2)(x-2)^3.$$

Next, use that $S(2.5) = S_3(2.5) \approx -0.69635$. Then the error at $x = 2.5$ is $|f(2.5) - S_3(2.5)| = -0.70711 + -0.69635 = 0.0108$.

Exercise 3.2. Let $f(x) = \ln(x) - x^4$. Let l and S be the piecewise linear and Hermite cubic spline interpolants (respectively) to f on $N+1$ equally spaced points $1 = x_0 < x_1 < \cdots < x_N = 2$. What value of N would you have to take to ensure that

(i) $\max_{1 \leqslant x \leqslant 2} |f(x) - l(x)| \leqslant 10^{-4}$?

Answer: From Thm 1.3 of Chapter 3, the error is bounded as

$$\|f - l\|_\infty \leqslant \frac{h^2}{8}\|f''\|_\infty.$$

Since $f''(x) = -2(6x^2 + x^{-2})$ is negative and decreasing for on $1 \leqslant x \leqslant 2$, $\|f - l\|_\infty = -f''(2) = 97/2 = 48.5$. So we need to choose h so that $(h^2)(48.5)/8 \leqslant 10^{-6}$. That gives $h \leqslant \sqrt{8 \times 10^{-6}/48.5} = 4.06 \times 10^{-4}$. Since $N = 1/h$, this gives $N \geqslant 2462.2$. As N must be an integer, we choose $N = 2463$.

(ii) $\max_{1 \leqslant x \leqslant 2} |f(x) - S(x)| \leqslant 10^{-4}$?

Answer: From Thm 3.2 of Chapter 3, the error is bounded as

$$\|f - S\|_\infty \leqslant \frac{h^4}{384}\|f^{(iv)}\|_\infty.$$

Since $f^{(iv)}(x) = -12(2 + x^{-4})$ is negative but increasing for on $1 \leqslant x \leqslant 2$, $\|f - l\|_\infty = -f''(1) = 36$. So we need to choose h so that $(h^4)(36)/384 \leqslant 10^{-6}$. That gives $h \leqslant \left(384 \times 10^{-6}/36\right)^{1/4} = 5.715 \times 10^{-2}$. Since $N = 1/h$, this gives $N \geqslant 17.498$. As N must be an integer, we choose $N = 18$.

Exercise 3.3. There are ways of constructing the PCHIP, other than that shown in (1) of Section 2.2. For example, let $s = x - x_{k-1}$, then

$$S(x) = \frac{h^3 - 3hs^2 + 2s^3}{h^3} f_{k-1} + \frac{3hs^2 - 2s^3}{h^3} f_k + \frac{s(s-h)^2}{h^2} f'_{k-1} + \frac{s^2(s-h)}{h^2} f'_k,$$

Show that this is the same as the PCHIP.

Exercise 3.4 (Note: this exercise is really just the same as Exer 3.2; I've included it here because I had solutions prepared!). Let $f(x) = \ln(x^2) - x^4$. Let l and S be the piecewise linear and Hermite cubic spline interpolants (respectively) to f on $N + 1$ equally spaced points $1 = x_0 < x_1 < \cdots < x_N = 2$. What value of N would you have to take to ensure that

(i) $\max\limits_{1 \leqslant x \leqslant 2} |f(x) - l(x)| \leqslant 10^{-6}$?

> **Answer:** From Thm 1.3 of Chapter 3, the error is bounded as
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> $$\|f - l\|_\infty \leqslant \frac{h^2}{8} \|f''\|_\infty.$$
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> Since $f''(x) = -2(6x^2 + x^{-2})$ is negative and decreasing for on $1 \leqslant x \leqslant 2$, $\|f - l\|_\infty = -f''(2) = 97/2 = 48.5$. So we need to choose h so that $(h^2)(48.5)/8 \leqslant 10^{-6}$. That gives $h \leqslant \sqrt{8 \times 10^{-6}/48.5} = 4.06 \times 10^{-4}$. Since $N = 1/h$, this gives $N \geqslant 2462.2$. As N must be an integer, we choose $N = 2463$.

(ii) $\max\limits_{1 \leqslant x \leqslant 2} |f(x) - S(x)| \leqslant 10^{-6}$?

> **Answer:** From Thm 3.2 of Chapter 3, the error is bounded as
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> $$\|f - S\|_\infty \leqslant \frac{h^4}{384} \|f^{(iv)}\|_\infty.$$
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