MA378 Chapter 3: Numerical Integration

§3.5 Orthogonal Polynomials

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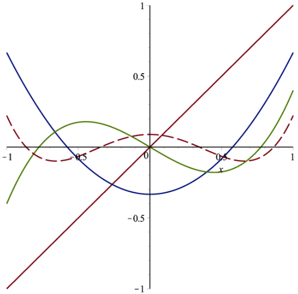

5.1 Orthogonal Polynomials

High order Newton-Cotes methods are of little use because of the problems associated with interpolation be high degree polynomials at equally spaced points. However, high-order Gaussian methods are very useful.

Driving such methods by undetermined coefficients is not practical, however. There is a simpler way, but some mathematical preliminaries are required, including the ideas of **vector spaces** and **inner products**.

5.2 Inner products

Definition 5.1 (Vector Space)

V is a *vector space* (a.k.a., a *linear space*) over a field F (e.g, the real or complex numbers) if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

(i) $\mathbf{u} + \mathbf{v} \in V$ (closed under addition)

(ii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)

(iii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)

(iv) V has a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(v) $-\mathbf{u} \in V$

(vi) $a\mathbf{u} \in V$

(vii) $a(b\mathbf{u}) = (ab)\mathbf{u}$

(viii) F contains 0 and 1 such that $1\mathbf{u} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$.

(ix) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

5.2 Inner products

Examples:

5.2 Inner products

Definition 5.2 (Inner Product)

Let V is a real vector space. An **Inner Product** (IP) is a real-valued function $(\cdot, \cdot)$ on $V \times V$ such that, for all $f, g, h \in V$,

(i) $(f + g, h) = (f, h) + (g, h)$,

(ii) $(\lambda f, g) = \lambda(f, g)$, for $\lambda \in \mathbb{R}$.

(iii) $(f, g) = (g, f)$,

(iv) $(f, f) \geq 0$. $\quad (f, f) = 0 \Leftrightarrow f \equiv 0$.

5.2 Inner products

Example 5.3

Let $\mathbb{R}^n$ be our vector space, with $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$. Then the following is an inner product:

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i y_i,$$

5.2 Inner products

Example 5.4

The set of real-valued functions that are continuous and defined on the ionterval $[a, b]$, denoted $C[a, b]$, is a vector space. And

$$(f, g) := \int_a^b f(x)g(x)dx, \tag{1}$$

is an inner product.

5.3 Sequence of Orthogonal Monic Polynomials

(See Lecture 23 of Stewart's "Afternotes" for more details).

Definition 5.5 (Monic Polynomial)

A polynomial is *monic* if the coefficient of its leading term is 1.

Examples:

5.3 Sequence of Orthogonal Monic Polynomials

Definition 5.6

Two elements a, b, of a vector space are *orthogonal* with respect to a given inner product $(\cdot, \cdot)$ if $(a, b) = 0$.

Example:

5.3 Sequence of Orthogonal Monic Polynomials

Example 5.7

Take the space of polynomials of degree 2 or less and the IP

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$

Let $p(x) \equiv 1$, $q(x) \equiv x$, $r(x) \equiv x^2 - 1/3$, and $f(x) = 3x - 4$

We can check that $(r, p) = 0$, and $(r, q) = 0$. We can the verify that $(r, f) = 0$. **Details:**

5.3 Sequence of Orthogonal Monic Polynomials

As given above, a polynomial is **monic** if the coefficient of the leading term is 1:

$$p_n = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-1} + \cdots + c_1 x + c_0.$$

We'll now look at a sequence of such polynomials

$$\{\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n, \ldots\}$$

that have the property they are orthogonal to each other:

$$(\widetilde{p}_i, \widetilde{p}_j) := \int_a^b \widetilde{p}_i(x)\widetilde{p}_j(x)dx = 0 \quad \text{if } i \neq j.$$

We want to establish some important facts about monic polys:

- A set of monic polys of degrees $1, \ldots, n$, forms a basis for $\mathcal{P}_n$.
- If the members of that set are orthogonal to each other, then they are orthogonal to *all* polynomials of lower degree.
- We can construct such as set.

5.3 Sequence of Orthogonal Monic Polynomials

Theorem 5.8

Let $\{\widetilde{p}_i\}_{i=0}^{n}$ be a sequence of polynomials where each p_i is monic an exactly of degree i. This sequence forms as basis for $\mathcal{P}_n$.

Proof:

5.3 Sequence of Orthogonal Monic Polynomials

Theorem 5.8 means that if q is a polynomial of degree n then it can be written uniquely as a linear combination of the $\widetilde{p}_i$:

$$q(x) = \sum_{i=0}^{n} a_i \widetilde{p}_i(x),$$

for some unique choice of the real coefficients a_i.

5.3 Sequence of Orthogonal Monic Polynomials

Definition 5.9

The sequence $\{\widetilde{p}_i\}_{i=0}^{n}$ is a sequence of *monic*, *orthogonal* polynomials if each $\widetilde{p}_i$ is monic and *exactly* of degree i and

$$(\widetilde{p}_i, \widetilde{p}_j) = 0 \quad \text{if } i \neq j.$$

Theorem 5.10

If $\widetilde{p}_j \in \{\tilde{p}_i\}_{i=0}^{\infty}$ *then* $\widetilde{p}_j$ *is orthogonal to all polynomials of degree less than* j*.*

Proof:

5.4 Constructing the Sequence

Theorem 5.11

The sequence $\{\tilde{p}_i\}_{i=0}^{\infty}$ exists and can be constructed as follows: Let α and β be defined as

$$\alpha_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_n)}{(\widetilde{p}_n, \widetilde{p}_n)}, \quad \text{and} \quad \beta_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_{n-1})}{(\widetilde{p}_{n-1}, \widetilde{p}_{n-1})},$$

then the sequence is given by

$$\widetilde{p}_0(x) \equiv 1, \qquad \widetilde{p}_1(x) = x - \alpha_1$$

and

$$\widetilde{p}_{n+1}(x) = (x - \alpha_{n+1})\widetilde{p}_n(x) - \beta_{n+1}\widetilde{p}_{n-1}(x),$$

for $n \geq 1$.

The proof uses *Gram-Schmidt Orthogonalization*.

5.4 Constructing the Sequence

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Example 5.12

If we use the inner product $(f,g) := \int_{-1}^{1} f(x)g(x)$ then the first 3 polynomials in the sequence are:

$$\widetilde{p}_0 = 1, \quad \widetilde{p}_1 = x, \quad \text{and} \quad \widetilde{p}_2 = x^2 - 1/3.$$

Example 5.13

The zeros of $\widetilde{p}_2$ are ...

5.5 Properties of the sequence

One of the ways of constructing Gaussian Quadrature rule $G_n(\cdot)$ on $n+1$ is to take the quadrature points as the roots of $\widetilde{p}_{n+1}$. We know (from the fundamental theorem of algebra) a polynomial of degree $n+1$ has exactly $n+1$ roots in $\mathbb{C}$ up to multiplicity.

However, the polynomials $\widetilde{p}$ have the special properties, established in the following lemma. (*A slightly different proof of these facts is given in Thm 9.4 of Suli and Mayers.*).

5.5 Properties of the sequence

Theorem 5.14

Let $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\widetilde{p}_0, \widetilde{p}_1, \dots\}$ *be the set of monic polynomials that are orthogonal with respect to the (usual) inner product.*

(i) *The zeros of each* $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ *are simple (not repeated).*

(ii) *All the zeros of* $\widetilde{p}_i$ *are real numbers in the interval* $[a, b]$.

5.5 Properties of the sequence

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5.6 Exercises

Exercise 5.1

$\mathcal{P}_n$, the space of polynomials of degree (at most) n forms a vector space. Is it true that the space of *monic* polynomials of degree n forms a vector space?

Exercise 5.2

(i) Using the Inner Product

$$(f, g) := \int_0^1 f(x)g(x)dx,$$

find $\widetilde{p}_0(x)$, $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$.

(ii) Find the zeros of $\widetilde{p}_2(x)$ and call them x_0 and x_1. Construct a quadrature rule for $\int_{-1}^{1} f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials.