MA313 : Linear Algebra I
Week 2: Subspaces

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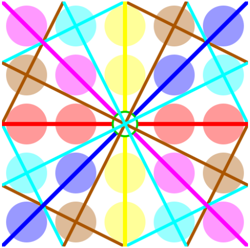
13 and 16 September, 2022



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These slides are based on ones by Tobias Rossmann.

Outline

Assignment 1

- The first assignment has opened. Deadline is Monday, 19 September.
- It contributes 3% to the final grade for MA313.
- Topics: addition of vectors, matrix-vector multiplication, matrix-matrix multiplication, solving linear systems by row-reduction.
- System still has a few glitches. We are working to fix them. Don't worry about time-out errors.

Communications skills

DRAFT: will be edited later

1. Later this week: list of topics will be posted.
2. End of Week 3: confirm your topic.
3. End of Week 7: progress report due. Will include scope, outline of structure, and major sources.
4. Week 12: submission of essay and slides and presentations.

Announcements

Tutorials start in Week 3. When:
https://forms.office.com/r/0ya9Bp8qBU

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12					
12 – 1					Lecture
1 – 2		Lecture			
2 – 3					
3 – 4					
4 – 5					

Everyone who attended Friday's class was available

- Wednesday at 11.00
- Friday at 11.00

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Week 2: Subspaces

Start of ...

PART 1: Definition of a Vector Space

See Section 4.1 of the text-book:
https://search.library.nuigalway.ie/permalink/f/1pmb9lf/353GAL_ALMA_DS51920676300036 26

Definition of a vector space (1/2)

A **vector space** consists of

- a (non-empty!) set V, whose elements we call **vectors**,
- an operation called **addition** which assigns a vector

$$u + v \in V$$

  to any two vectors $u, v \in V$, and
- an operation called **scalar multiplication** which assigns a vector

$$cu \in V$$

  to each scalar $c \in \mathbb{R}$ and vector $u \in V$

such that the axioms on the following slides are satisfied.

Definition of a vector space (2/2)

We require that the following conditions **V1–V8** are satisfied for all vectors $u, v, w \in V$ and scalars $c, d \in \mathbb{R}$:

V1. $u + v = v + u$ (**commutativity** of addition)

V2. $(u + v) + w = u + (v + w)$ (**associativity** of addition)

V3. There exists $\mathbf{0} \in V$, called the **zero vector** such that $u + \mathbf{0} = u$ for all $u \in V$,

V4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$

V5. $c(u + v) = cu + cv$ (**distributivity I**)

V6. $(c + d)u = cu + du$ (**distributivity II**)

V7. $c(du) = (cd)u$

V8. $1u = u$

Example ($\mathbb{R}^n$ is a vector space.)

We define

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}$$

with addition defined as $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$ and

scalar multiplication defined as $c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$.

Then $\mathbb{R}^n$ is a vector space. The proof is a quite tedious, but quite easy.

It would take too long to show that $\mathbb{R}^n$ satisfies each of the 8 axioms. So we'll just verify the three of them.

V1. $u + v = v + u$ (**commutativity** of addition)

V3. There exists $\mathbf{0}$, called the **zero vector**, such that $u + \mathbf{0} = u$ for all $u \in V$.

V4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$.

For an integer $n \geqslant 0$, let $\mathbb{P}_n$ consist of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

of degree at most n, where $a_0, \ldots, a_n \in \mathbb{R}$.

We can add polynomials in $\mathbb{P}_n$ in the usual way:

$$\begin{aligned}(a_0 + a_1 t + \cdots + a_n t^n) \\ + (b_0 + b_1 t + \cdots + b_n t^n) \\ = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n.\end{aligned}$$

Also,

$$cp(t) = ca_0 + ca_1 t + \cdots + ca_n t^n,$$

where $c \in \mathbb{R}$.

Claim: These operations turn $\mathbb{P}_n$ into a vector space.

The reasoning again just boils down to to properties of real numbers.

Example

Function spaces Let $\mathbb{D}$ be an arbitrary set.
Let V be the set of **all** functions $f : \mathbb{D} \to \mathbb{R}$.
Given $f, g \in V$ and $c \in \mathbb{R}$, we define $f + g \in V$ and $cf \in V$ via

$$(f + g)(x) := f(x) + g(x)$$

and

$$(cf)(x) := cf(x)$$

for $x \in \mathbb{D}$.
Claim: These operations turn V into a vector space.

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Week 2: Subspaces

Start of ...

PART 2: **Not everything is a vector space**

Part 2: Not everything is a vector space

So far, all of the examples we have looked at correspond to vector spaces. But not every set equipped with addition and scalar multiplication is a vector space.

Here are a few examples of things that are not vector spaces.

1. The set of vectors in $\mathbb{R}^2$ with strictly positive entries.

2. The set of vectors in $\mathbb{R}^2$ with non-negative entries.

Part 2: Not everything is a vector space

3. The set of polynomials of degree **exactly** 3.

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Week 2: Subspaces

Start of ...

PART 3: Subspaces

Part 3: Subspaces

One of the key concepts of vector spaces is that, given an example of a vector space, we can usually construct another, smaller one from it. These new smaller ones are called **subspaces**.

Definition (Subspace)

Let V be a vector space. A *subspace* of V is a subset of V which forms a vector space with respect to the same addition and scalar multiplication operations in V.

Example (The boring examples)

The "boring" subspaces of a vector space V are

- $\{0\}$ and
- V itself.

Part 3: Subspaces

Our definition said that, if H is a subspace of V then everything in H is also in V, and also that H is a subspace in its own right. However, we don't have to check if *all* eight axioms hold for H.

Fact

Let H be a subset of V. Then H is a subspace of V if and only if the following conditions are all satisfied:

- $\mathbf{0} \in H$.
- H is closed under addition operation in V, i.e., for all $u, v \in H$, we have $u + v \in H$.
- H is closed under multiplication by scalars, i.e. for all $u \in H$ and $c \in \mathbb{R}$, we have $cu \in H$.

Part 3: Subspaces

Example

Let

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \text{ with } x_1 + x_2 = 0 \right\}.$$

Then H is a subspace of $\mathbb{R}^2$.

Example

Decide (with justification) whether

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 + x_2 = \mathbf{1} \right\}$$

is a subspace of $\mathbb{R}^2$.

Part 3: Subspaces

Example (MA313 21/22 Semester 1 Exam, Q1(a)(i))

Decide, with justification, whether

$$H_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 = x_2, x_3 = 0 \right\} \text{ is a subspace of} \mathbb{R}^3.$$

(Note: we didn't do this one in class). Answer: Yes it is. To check

(i) $\mathbf{0}$ is in H_1, since, if $x_1 = x_2 = x_3 = 0$ then we have $x_1 = x_2$ and $x_3 = 0$ as needed.

(ii) If $u \in H_1$ and $v \in H_1$, we can write $u = \begin{bmatrix} u_1 \\ -u_1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ -v_1 \\ 0 \end{bmatrix}$. Then, if

$w = u + v$ we get $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ -u_1 - v_1 \\ 0 \end{bmatrix}$. So

$w_1 + w_2 = (u_1 + v_1) + (-u_1 - v_1) = (u_1 - u_1) + (v_1 - v_1) = 0$, and $w_3 = 0$.

(iii) Similarly, can show $cu \in H_1$.

Part 3: Subspaces

Example (MA313 Semester 1 Exam, Q1(a)(ii))

Decide, with justification, whether

$$H_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 > 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

Example (MA313 Semester 1 Exam, Q1(a)(iii))

Decide, with justification, whether

$$H_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 \geq 0, x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

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Week 2: Subspaces

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PART 4: More examples of subspaces

Recall

$\mathbb{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n : a_0, \ldots, a_n \in \mathbb{R}\}$ is the vector space of polynomials of degree at most n in the variable t.

Definition (All the polynomials)

$\mathbb{P} := \bigcup_{n=0}^{\infty} \mathbb{P}_n = \{p(t) = a_0 + a_1 t + \cdots + a_n t^n : n \geqslant 0;\ a_0, \ldots, a_n \in \mathbb{R}\}$ is the vector space of *all* polynomials in t (without any bounds on the degree!).

Here, the addition and scalar multiplication in $\mathbb{P}$ are just the usual operations: we add and multiply as expected.

We now see that...

- $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$ if and only if $m \leqslant n$.
- $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all $n \geqslant 0$.

Note that

$$\begin{aligned}\mathbb{P}_0 = \mathbb{R} &= \text{constant polynomials}\\ &= \{p(t) \in \mathbb{P} : p'(t) = 0\}\end{aligned}$$

where the **derivative** of $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ is

$$p'(t) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}.$$

More generally, we can define $\mathbb{P}_n$ by

$$\mathbb{P}_n = \left\{ p(t) \in \mathbb{P} : p^{(n+1)}(t) = 0 \right\}.$$

That is, *another* approach to describing the subspaces $\mathbb{P}_n$ of $\mathbb{P}$ is to look at the space of solutions of certain equations.

Example (Continuous functions)

Let $\mathbb{D} \subseteq \mathbb{R}$ be a subset. Let V be the vector space of all functions $\mathbb{D} \to \mathbb{R}$ from Week 1, where, as we said

- $(f+g)(x) = f(x) + g(x)$ for $f, g \in V$ and $x \in \mathbb{D}$
- $(cf)(x) = cf(x)$ for $f \in V$ and $c \in \mathbb{R}$

Let

$$C(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{R} : f \text{ is continuous}\} \subseteq V.$$

With a little calculus, one can show that $C(\mathbb{D})$ is a subspace of V.

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Week 2: Subspaces

Start of ...

PART 5: Linear combinations

Part 5: Linear combinations

A question

So far we have been able to check if a given space is indeed a subspace of some other vector space.
It is natural to wonder: *how can we make those subspaces in the first place?*
Equivalently: *How can we describe all subspaces of a given vector space?*

Part 5: Linear combinations

Example (Subspaces of $\mathbb{R}^2$)

There are precisely three *types* of subspaces of $\mathbb{R}^2$:

- $\{0\}$,
- $\mathbb{R}^2$,
- lines through the origin.

How we build subspaces?

There are two possible approaches.

- **Top down:** start with the full space, and look at all vectors that have "suitable properties".
- **Bottom up:** start with some collection of vectors and consider the subspace that they "span".

Definition (Linear combinations)

A **linear combination** of vectors $u_1, \dots, u_p$ in some vector space is a vector of the form

$$c_1 u_1 + \cdots + c_p u_p$$

for scalars $c_1, c_2, \dots, c_p \in \mathbb{R}$.

Example

In $\mathbb{R}^2$, $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example

Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is **not** linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -6 \end{bmatrix}$ in $\mathbb{R}^2$.

Example

Which vectors in $\mathbb{P}_2$ (over t) are linear combinations of the vectors $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$?

Example

Which vectors in $\mathbb{P}_2$ (over t) are linear combinations of the vectors $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = 2t$?

Example

Define the 2×3 matrix

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

For any vector

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

the vector Ax is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

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PART 6: Spans

Part 6: Spans

Definition (SPAN)

Given vectors $u_1, \ldots, u_p$ in some vector space V, their **span** is

$$\operatorname{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

In other words, $\operatorname{span}\{u_1, \ldots, u_p\}$ is the set of all linear combinations of $u_1, \ldots, u_p$ within V.

Part 6: Spans

Theorem

$\operatorname{span}\{u_1, \ldots, u_p\}$ *is a subspace of* V*.*

In fact, more than this is true: one can show that $\operatorname{span}\{u_1, \ldots, u_p\}$ is the "smallest" subspace of V which contains each of $u_1, \ldots, u_p$.

Immediate consequences

- Every choice of vectors $u_1, \ldots, u_p$ provides us with an example of a subspace of V.
  (However, ***different*** sequences of vectors may well span the ***same*** subspace!)
- If we can show a *subset* of V is the a **span of some set of vectors**, then we we have shown it is a subspace!

Example

Show that $$H = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$ is a subspace of $\mathbb{R}^4$.

Example (From 2018/2019 exam paper)

Find vectors $u, v, w \in V$ with $V = \mathrm{span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$.

Example: Care is required!

Is $H = \left\{ \begin{bmatrix} 3s \\ 2+5s \end{bmatrix} : s \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$.

We now know that the span of any subset of vectors in a vectors space is itself a subspace (and, so, is a vector space). But...

Question

Is every subspace the span of some (collection of) vectors?

Part 7: Exercises

Q1. Let $M_{m\times n}$ be the set of $m \times n$ matrices with real entries. Then $M_{m\times n}$ is a vector space with respect to the following operations:

$$\begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \ddots & \ddots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & \dots & b_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ b_{m1} & \ddots & \ddots & b_{mn} \end{bmatrix}$$

$$:= \begin{bmatrix} a_{11}+b_{11} & \dots & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1}+b_{m1} & \ddots & \ddots & a_{mn}+b_{mn} \end{bmatrix}$$

$$c\begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \ddots & \ddots & a_{mn} \end{bmatrix} := \begin{bmatrix} ca_{11} & \dots & \dots & ca_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ ca_{m1} & \ddots & \ddots & ca_{mn} \end{bmatrix}$$

Part 7: Exercises

These are just the usual operations of adding matrices and multiplying a matrix by a scalar. *Verify that the axioms **V1–V8** are really satisfied.*

Q2. For each of the following sets, determine, with justification, if it is a subspace of $\mathbb{R}^2$.

- $H_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 x_2 \geq 0 \right\}$
- $H_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 x_2 \geq 0 \right\}$
- $H_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 = 0 \right\}$