MA313 : Linear Algebra I

Week 6: More about bases, and coordinates

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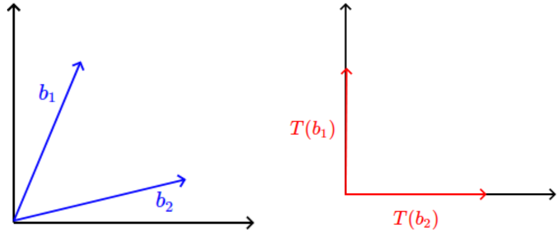

Two vectors and their image under a linear transformation

These slides were produced by Niall Madden, based on ones by Tobias Rossmann.

Outline

For more details, see

- Chapter 7 (Linear Independence) of Linear Algebra for Data Science: `https://shainarace.github.io/LinearAlgebra/linind.html`
- Section 4.3 of the Lay: `https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=5174425`

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PART 1: Bases

1: Bases (again)

Recall from last week ...

Definition (Basis for a vector space)

A sequence of vectors $(v_1, \ldots, v_p)$ in some vector space V is a **BASIS** for V if

- $v_1, \ldots, v_p$ are linearly independent.
- $V = \mathrm{span}\,\{v_1, \ldots, v_p\}$.

Basically:

- Every element of the vector space is some linear combination of the vectors in $\{v_1, \ldots, v_p\}$ (i.e., it is a spanning set).
- The set $\{v_1, \ldots, v_p\}$ is linearly independent.
- We treat it as a sequence (rather than a set) because it is useful to have the vectors in the basis ordered.

As mentioned before, a vector space can have many bases. (We say "the basis is not unique".)

Example

Show that $\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$ is a basis of $\mathbb{R}^2$.

The answer this, we have to show two things:

1. These two vectors are linearly independent;
2. They span $\mathbb{R}^2$.

Remark

We only considered *finite* spanning sets and bases of vector spaces and we only defined linear independence for finite collections of vectors.
All of these notions admit infinite generalisations. We will not pursue this (that is for a longer course).
Infinite bases are mathematically interesting, but they quickly lead to tricky foundational issues of **set theory**.

Questions

- Does every vector space have a basis?
- How can we find bases?
- What are bases good for?

Bases of null spaces

Let A be an $m \times n$ matrix.

Using row reduction, beginning with A, we obtain a (unique!) matrix A' in reduced row echelon form.

Recall:

- $\mathrm{Nul}\, A = \mathrm{Nul}\, A'$.
- We can read off a spanning set of $\mathrm{Nul} A$ from A'. (See Week 4).

FACT

This method always produces a basis of $\mathrm{Nul}\, A$.

Example

Suppose that $A \rightsquigarrow A' = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ via row reduction.

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PART 2: Finitely generated vector spaces

2: Finitely generated vector spaces

Question: Does every vector space have a basis?

The way we defined them, bases are always **finite**.

It turns out that some vector spaces are so "large" that they don't admit (finite) bases.

Examples include:

- $\mathbb{P}$–the space of polynomials of arbitrary degree,
- $C(\mathbb{R})$–the space of continuous functions $\mathbb{R} \to \mathbb{R}$.

Rather than extend our concept of a basis to include such examples, we will now study those vector spaces that have (finite) bases in detail.

Definition (FINITELY GENERATED VECTOR SPACE)

A vector space V is **finitely generated** (or **finite-dimensional**) if

$$V = \mathrm{span}\,\{v_1, \ldots, v_p\}$$

for some $p \geqslant 0$ and some sequence $v_1, \ldots, v_p \in V$.
(Here, for $p = 0$, we write $\mathrm{span}\,\{\} := \{0\}$.)

Lemma (The "Casting out" lemma)

Suppose that $V = \mathrm{span}\{v_1, \dots, v_p\}$ and that some v_k is a linear combination of the other vectors

$$v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p.$$

Then

$$V = \mathrm{span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p\}.$$

What this means is

1. The original set $\{v_1, \dots, v_p\}$ was *not* linearly independent.
2. So we can write some v_k in terms of the other vectors.
3. Removing (casting out) v_k from the sequence, we still have a spanning set for V.

2: Finitely generated vector spaces

If a vector space has a basis, then it is spanned by that basis. So that means it is finitely generated. The converse is also true!

Theorem (A finitely generated vector space has a basis)

Let V be a finitely generated vector space with $V \neq \{0\}$.
Then V has a basis.

2: Finitely generated vector spaces

There is a method for constructing a basis of V:

- Write $V = \operatorname{span}\{v_1, \ldots, v_p\}$.
- If no v_k belongs to $\operatorname{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\}$, then $v_1, \ldots, v_p$ are linearly independent. In that case,

  $$(v_1, \ldots, v_p)$$

  is a basis of V and we stop.
- Otherwise, for some k, we have $v_k \in \operatorname{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\}$. We then discard v_k from our spanning set (this lowers p!) and repeat our procedure for the resulting smaller spanning set.
- After finitely many iterations, we will have found a basis of V.

(This is not especially practical. But it will do for now).

Example

Let $p_1(t) = 2t - t^2$, $p_2(t) = 2 + 2t$, and $p_3(t) = 6 + 16t - 5t^2$. Let $V = \mathrm{span}\{p_1(t), p_2(t), p_3(t)\}$, a subspace of $\mathbb{P}_2$.

Find a basis of V.

3: Basis of a column space

Recall ...

COLUMN SPACE

Let A be a $m \times n$ matrix, with column $a_1, a_2, \ldots, a_n$. That is

$$A = [a_1 \ a_2 \ \cdots \ a_n].$$

The **COLUMN SPACE** of A is the space spanned by the $a_1, \ldots, a_n$. That is

$$\operatorname{Col} A := \operatorname{span}\{a_1, \ldots, a_n\}.$$

3: Basis of a column space

QUESTION: How can we find a basis of $\operatorname{Col} A$?

Note

- We get a spanning set for free: the columns of A.
- We could then use the "casting out method" to find a basis of $\operatorname{Col} A$, but there is a better approach.

3: Basis of a column space

FACT

Let A be an $m \times n$ matrix with associated matrix A' in reduced row echelon form. Then the columns of A and A' satisfy the "same linear dependence relations".

Formally: $Ax = 0$ if and only if $A'x = 0$ for each $x \in \mathbb{R}^n$.

In particular, the ith column of A is a linear combination of some other columns if and only if the same is true for A'.

Example 3.1

Consider the matrix

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is already in reduced row echelon form.

We observe:

- Non-pivot columns are linear combinations of pivot columns.
- Pivot columns are linearly independent.

Theorem

The pivot columns of a matrix A form a basis of $\operatorname{Col} A$.

That is:

- Given A, compute it's reduced row echelon form, A';
- The pivot columns of A', which are the ones with a single non-zero entry, are also the pivot columns of A.
- The pivot columns of A for a basis for $\operatorname{Col} A$.

Example 3.2

Let

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

Goal: find a basis of $\mathrm{Col}\, A$. *Hint: the reduced row echelon form for this matrix is the one in Example 3.1.*

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PART 4: Coordinates

4: Coordinates

Question

Why should we care about bases? What can we do with them?

Answer...

Each choice of a basis of a vector space provides us with a "coordinate system" for it.

4: Coordinates

Theorem (Unique Representation Theorem)

Let $(b_1, \ldots, b_n)$ be a basis of a vector space V.
Then for each $x \in V$, there exists a unique sequence $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$x = c_1 b_1 + \cdots + c_n b_n.$$

Proof...

Definition (COORDINATE VECTOR)

Let $\mathcal{B} = (b_1, \dots, b_n)$ be a basis of V.
The **coordinate vector** of $x \in V$ relative to $\mathcal{B}$ is

$$[x]_{\mathcal{B}} := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

where $c_1, \dots, c_n \in \mathbb{R}$ is the unique sequence with

$$x = c_1 b_1 + \cdots + c_n b_n$$

from the Unique Representation Theorem.

The function $V \to \mathbb{R}^n, \quad x \mapsto [x]_{\mathcal{B}}$ is the **coordinate mapping** determined by $\mathcal{B}$.

Example

Let

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

be the standard basis of $\mathbb{R}^n$.
Then $[x]_{\mathcal{B}} = x$ for all $x \in \mathbb{R}^n$.

Hence, taking coordinate vectors *generalises* extracting the components of a vector in $\mathbb{R}^n$.

Example

Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

1. $\mathcal{B}$ is a basis of $\mathbb{R}^2$.

Example

Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

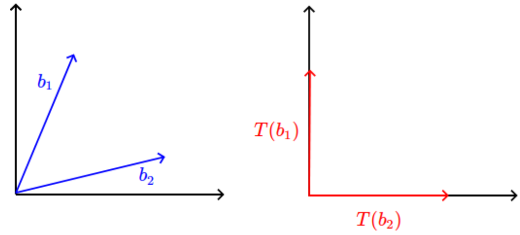
2. Write down the coordinate mapping determined by $\mathcal{B}$. It is a linear transformation, so also write down the matrix of the linear transformation.

Suppose that $\mathcal{B} = (b_1, b_2)$ is a basis of $\mathbb{R}^2$.

Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto [x]_{\mathcal{B}}$ be the associated coordinate mapping.

Then $T(b_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(b_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note that $\mathcal{B}$ defines a parallelogram. The coordinate mapping T "stretches", "rotates", and perhaps "reflects" it into a square!



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PART 5: Isomorphisms

INVERTIBLE FUNCTIONS

Let X and Y be sets and let $f\colon X \to Y$ be a function.
Then the following are equivalent:

- f is invertible, i.e., there exists $f^{-1}\colon Y \to X$ such that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.
- f is *one-to-one* and ***onto***. (Also called "*injective*" and "***surjective***").

Moreover, if f is invertible, then the function f^{-1} is **uniquely** determined.

5: Isomorphisms

Definition (ISOMORPHISM)

An **isomorphism** from a vector space V to a vector space W is an invertible linear transformation $V \to W$.

We say that V and W are **isomorphic** if there exists an isomorphism between them.

Example

- For any vector space V, the **identity map**

  $$\mathrm{id}_V \colon V \to V, \quad x \mapsto x$$

  is an isomorphism.
  Hence, every vector space is isomorphic to itself.
- Given any basis $\mathcal{B} = (b_1, \ldots, b_n)$ of V, the coordinate mapping

  $$V \to \mathbb{R}^n, \quad x \mapsto [x]_{\mathcal{B}}$$

  is an isomorphism.
  (We saw in Part 3 that this is an invertible linear transformation).

Theorem

Let U, V, and W be vector spaces.
Let $S: U \to V$ and $T: V \to W$ be linear transformations.
Then:

- $T \circ S: U \to W, x \mapsto T(S(x))$ *is a linear transformation.*
- *If S and T are isomorphisms, then so is $T \circ S$.*

That is: if U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.

Theorem

If $T\colon V \to W$ is an isomorphism of vector spaces, then so is $T^{-1}\colon W \to V$.

Hence, if V is isomorphic to W, then W is isomorphic to V.

Question

Can we relate this to matrices and vectors?

Let A be an $n \times n$ matrix.

Then the function

$$T\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto Ax$$

is a linear transformation.

It is invertible if and only if A is an invertible matrix. In that case, T^{-1} is the function

$$\mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto A^{-1}y.$$

Summary

- $m \times n$ matrices correspond to linear transformations $\mathbb{R}^n \to \mathbb{R}^m$.
- An $n \times n$ matrix is invertible if and only if the corresponding linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. In that case, the inverse of the linear transformation corresponds to the inverse matrix.

Question...

Can there be an isomorphism $\mathbb{R}^n \to \mathbb{R}^m$ when $m \neq n$?

Let $\mathcal{B} = (b_1, \ldots, b_n)$ be a basis of $\mathbb{R}^n$.

Then

$$T\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto [x]_{\mathcal{B}}$$

and its inverse $T^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are both linear transformations from $\mathbb{R}^n$ to itself.

Question...

What are the matrices corresponding to T and T^{-1}?

By definition:

$$T(x) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \iff x = c_1 b_1 + \cdots + c_n b_n \iff T^{-1}\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = x.$$

Hence, for $i = 1, \ldots, n$,

$$T^{-1}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = b_i$$

so the matrix of T^{-1} is $A := [b_1 \cdots b_n]$, and the matrix of T is therefore A^{-1}.

Exercises

Q1. Find a basis for the null space of

$$\begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix}.$$

Q2. Find a basis for the null space of

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -8 & 0 & 16 \end{bmatrix}.$$

Q3. Find a basis for the subspace of $\mathbb{R}^3$ consisting of those vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with $x - 3y + 2z = 0$.

Exercises

Q4. Find bases for $\mathrm{Nul}\, A$ and $\mathrm{Col}\, A$, where

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}.$$

Q5. Find bases for $\mathrm{Nul}\, A$ and $\mathrm{Col}\, A$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}.$$

Q6. Find a basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Exercises

Q7. Let $\mathcal{B} = \left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right)$. Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$ and find the vector $x \in \mathbb{R}^2$ with coordinate vector $[x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Q8. Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right)$. Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$ and find the vector $x \in \mathbb{R}^3$ with coordinate vector $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

Q9. Show that

$$\mathcal{B} = \left(1 + t^2,\ t + t^2,\ 1 + 2t + t^2\right)$$

is a basis of $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to $\mathcal{B}$.