Triangular numbers, Visual Proofs, and Combinatorics

Senior Mathematics Enrichment Programme

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Can you form all the numbers from 0 to 9 using four 4's, and the usual operations $+$, $-$, $\times$, and $\div$? You can also use $($ and $)$ if needed.

We start with the following examples.

(i) Find a "visual" way of showing that $1 + 3 + \cdots + (2n - 1) = n^2$

(ii) Now extend this to show that $2 + 4 + \cdots + 2n = n^2 + n$

(iii) Combine these to get an expression for $1 + 2 + 3 + 4 + \cdots + n$.

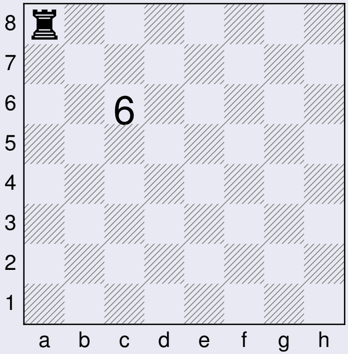
This leads us on to "triangular numbers".

The *Triangular* numbers are $T_n = 1 + 2 + 3 + \cdots + n$.
Here are a few identities they satisfy.

(1) $T_{n-1} + T_n = n^2$.
(2) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1} n^2 = (-1)^{n+1} T_n$.
(3) $8T_n + 1 = (2n+1)^2$.
(4) $T_{2n} = 3T_n + T_{n-1}$.
(5) $T_{2n+1} = 3T_n + T_{n+1}$.
(6) $T_{3n+1} - T_n = (2n+1)^2$.
(7) $T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$.
(8) $T_n T_k + T_{n-1} T_{k-1} = T_{nk}$
(9) $3(T_1 + T_2 + T_3 + \cdots + T_n) = T_n(n+2)$

You can find a “Proof without words” of (9) on the MVP YouTube channel: `https://www.youtube.com/watch?v=N0ETyJ5K6j0&list=PLZh9gzIvXQUtkRlg8-epNxe18SZq70UBr&index=27`

A rook can move only in straight lines (not diagonally). *Fill in each square of the chess board below with the number of different shortest paths the rook in the top left corner can take to get to the square, moving one space at a time.* E.g., there are **six** paths from the rook to the square **c6**: DDRR, DRDR, DRRD, RDDR, RDRD, and RRDD. (R = *right*, D = *down*).



Factorial

If you have n objects then the number of different ways of ranking them is

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n! \quad (\text{“}n \text{ factorial}).$$

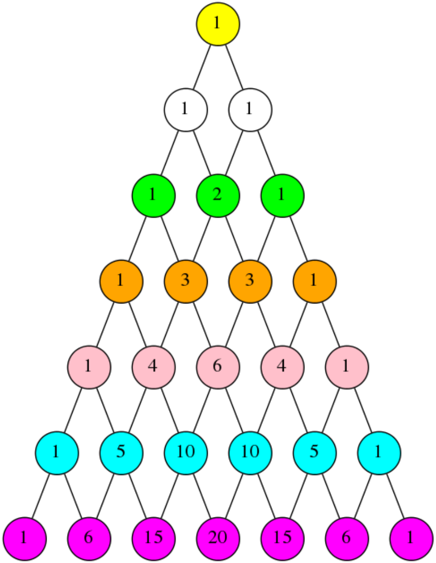
As n increases, $n!$ *increases very quickly. Did you know that the age of the universe is less than 20! seconds old?*.

Permutations

In general, if you have n objects, the number of ways of ranking $k \leqslant n$ of them is

$$\frac{n!}{(n-k)!}.$$

We call this a *permutation*.



This is **Pascal's Triangle**.
What patterns can we spot in the numbers shown here?

Combinations

We use $\binom{n}{k}$ to denote the number of ways of choosing k items from n.
$\binom{n}{k}$ is also the kth entry in row n of Pascal's triangle (where we start counting the rows from zero).

A fact that we will ignore about about the **Binomial coefficient formula**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{1}$$

Just like we were able to prove some facts about triangular numbers, without using mathematical formulae, we can prove some facts about binomial coefficients without using this "factorial" formula.

Example

If 30 people compete in the Irish Mathematics Olympiad, and 6 are chosen to represent Ireland at the IMO, there are

$$\binom{30}{6} = 593,775$$

possible teams.

Binomial coefficient

$\binom{n}{k}$ is also called the “binomial coefficient” because the coefficient of $a^k b^{n-k}$ in $(a+b)^n$ is $\binom{n}{k}$. That is

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{n-1} a^{n-1} b \\ &\quad + \binom{n}{n-2} a^{n-2} b^2 + \cdots + \binom{n}{1} a^1 b^{n-1} + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \qquad (2)\end{aligned}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Another Identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Here are some other identities. *Can you prove the following?*

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.
3. $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

4. $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy.$
5. $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}.$

6. $\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2}1 = \binom{n+3}{5}$

7. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

8. Find an simple expression for $\sum_{k=0}^{n} \binom{n}{k}$.

9. $\displaystyle\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{k-r}\binom{n}{r}.$

10. $\displaystyle\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}.$

11. How many ways can you write n as the sum of r non-negative integers, where order matters? (E.g, three ways of writing $n = 5$ as $r = 3$ integers are $5 = 0+1+4$, $5 = 1+0+4$, $5 = 1+2+2$).