Week 08, Lecture 1
Techniques of Integration

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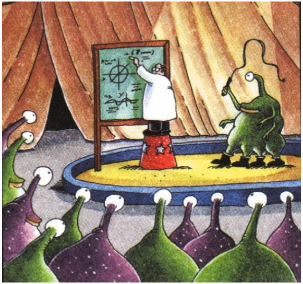

Assignments, etc

- **Problem Set 6** is open, and will be covered in tutorials this well. Deadline is 5pm next Monday (10 November).
- **Problem Set 7** opens alter this week.
- The finally weekly assignment, will open next week.
- Reminder: The second **class test** takes place November 18.

This morning, I think we will think about...

See also Section **5.5** (Substitution) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax) Maybe also Section **7.1** (Integration by Parts).

Substitution

Suppose we want to evaluate an integral of the form

$$\int e^{x^3+x^2}(3x^2+2x)\,\mathrm{d}x.$$

At first, this looks tricky: there is nothing like this in our table of integrals.
However, there is something a little unusual about it: it features both the function x^3+x^2, and its derivative $3x^2+2x$.
It turns out that such problems are quite common (at least in textbooks and on exams!). Moreover, there is a handy technique called **substitution** for evaluating them. In this case:

Substitution

Method of substitution

When integrating an *integrand* of the form $\int f(g(x))g'(x)dx$, set $u = g(x)$, and then use that

$$\int f\big(g(x)\big)\, g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,.$$

Equivalently: $\int f(u)\,\frac{du}{\mathrm{d}x}\mathrm{d}x = \int f(u)\,\mathrm{d}u$. (For a proof of why this works, see Section 5.5 of the textbook).

After this substitution, our task is reduced to evaluating $\int f(u)\,\mathrm{d}u$ which, we hope, is easier.

Substitution

Example

Evaluate the integral $\int 3x^2 \sin(x^3)\, dx$.

Notice that $3x^2$ is the derivative of x^3.

Let's try integration by substition with $u = x^3$.

If $u = x^3$, then $\frac{du}{dx} = 3x^2$, so

$$du = \frac{du}{dx}\, dx = 3x^2\, dx\,.$$

Thus,

$$\begin{aligned}\int \sin(x^3) 3x^2\, dx &= \int \sin(u)\, du \\ &= -\cos(u) + C \\ &= -\cos(x^3) + C\,.\end{aligned}$$

Substitution

Example

Evaluate $\int 2x\sqrt{1+x^2}\,dx$.

Notice that $2x$ is the derivative of $1+x^2$.
Let's try integration by substition with $u = 1 + x^2$:

Substitution

Example

Evaluate $\int \cos(4x-7)\,dx$.

Idea: think of this as $\dfrac{1}{4}\int \cos(4x-7)4\,dx$.

Substitution

Example

Show that $\int \sin^3(x)\,\cos(x)\,\mathrm{d}x = \frac{1}{4}\sin^4(x) + C$.

Substitution can be used with **definite integrals**. However, this may requires a change to the limits of integration.

Substitution with Definite Integrals

Let $u = g(x)$, with g' continuous on $[a, b]$, and f continuous over the range of $u = g(x)$. Then,

$$\int_{x=a}^{x=b} f\big(g(x)\big)g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du.$$

This allows us to apply the FTC2, without having to invert the substitution.

Example

Evaluate $\mathrm{I} = \int_0^1 x^2(1+2x^3)^2 dx$

Example

Evaluate $\mathrm{I} = \int_{-1}^{0} xe^{x^2}\,dx$

Rational functions

Recall: Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Before trying to find an antiderivative of a rational function $f(x) = \dfrac{p(x)}{q(x)}$:

Step 1: If $\deg(p(x)) \geqslant \deg(q(x))$, divide $p(x)$ by $q(x)$.

Step 2: Check if integration by substitution might work.

Step 3: Factorise the denominator as far as possible.

Step 4: Write the rational function as sum of **partial fractions** to simplify.

Rational functions

Example

Evaluate the integral
$$\int \frac{x}{x^2+1}\,dx\,.$$

In this can we can use substitution.

Example

Evaluate the integral

$$\int \frac{3x+4}{x^2+7x+12}\,dx\,.$$

In this case, we must factorise the denominator, and express the integrand as **partial fractions**. Factorise:
$x^2+7x+12=(x+4)(x+3)$.
Express as Partial Fractions:

$$\frac{3x+4}{x^2+7x+12}=\frac{A}{x+4}+\frac{B}{x+3}$$

With a little work, we can find that $A=8$ and $B=-5$. Therefore,

$$\frac{3x+4}{x^2+7x+12}=\frac{8}{x+4}-\frac{5}{x+3}\,.$$

We can now express the integral as

$$\int \frac{3x+4}{x^2+7x+12}\,dx = \int \left(\frac{8}{x+4} - \frac{5}{x+3}\right) dx$$
$$= \underbrace{\int \frac{8}{x+4}\,dx}_{\mathrm{I}_1} - \underbrace{\int \frac{5}{x+3}\,dx}_{\mathrm{I}_2}\,.$$

First, we evaluate I_1.

$$\mathrm{I}_1 = \int \frac{8}{x+4}\,dx = 8\int \frac{1}{x+4}\,dx\,.$$

If we let $u = x+4$, then $du = dx$ and, hence,

$$\mathrm{I}_1 = 8\int \frac{1}{x+4}\,dx = 8\int \frac{1}{u}\,du = 8\ln|u| + C_1 = 8\ln|x+4| + C_1\,.$$

Similarly, we find that: $\mathrm{I}_2 = \int \frac{5}{x+3}\,dx = 5\ln|x+3| + C_2$.

To conclude:

$$\begin{aligned}
\int \frac{3x+4}{x^2+7x+12}\,dx &= \int \left(\frac{8}{x+4} - \frac{5}{x+3}\right) dx \\
&= \int \frac{8}{x+4}\,dx - \int \frac{5}{x+3}\,dx \\
&= \mathrm{I}_1 - \mathrm{I}_2 \\
&= \left(8\ln|x+4| + C_1\right) - \left(5\ln|x+3| + C_2\right) \\
&= 8\ln|x+4| - 5\ln|x+3| + C.
\end{aligned}$$

Integration by Parts

The Product Rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

We can use this rule to develop another integration technique after a little rearrangement. From the above we have

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

and integrating both sides gives

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

This method is called **Integration by Parts**: it is, by some distance, the most important technique for integration, in both theory and practice.

Integration by Parts

Integration by Parts

If u and v are differentiable functions in the variable x, then

$$\int uv' \, dx = uv - \int vu' \, dx \, .$$

Recall that we can write

$$du = u' \, dx \quad \text{and} \quad dv = v' \, dx \, .$$

Therefore, we can rewrite the formula for Integration by Parts as

$$\int u \, dv = uv - \int v \, du \, .$$

Integration by Parts

Example

Evaluate $\mathrm{I} = \int x \cos(x)\, dx$

We'll take $u = x$ and $dv = \cos(x) dx$.

One of the challenges of Integration by Parts is knowing how to choose u and dv. When integrating $\boxed{\int x \cos(x)\, dx}$ we choose $u = x$, because its derivative, $u' = 1$ is simpler.
Suppose we had made the bad choice of

$$u(x) = \cos(x), \qquad dv = x\, dx,$$

then we'd get:

More generally, given choices for u and dv, we proceed as follows:

1. Some functions are easy to differentiate (and maybe not so easy to integrate), and so make a good choice for u. Important examples include **logarithms** and **inverse trigonometric** functions.
2. Some functions (such as polynomials) have simple(r) derivatives, so are also a good choice for u.
3. Trigonometric and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for dv.

Example (of choosing u

Evaluate $\mathrm{I} = \int \frac{\ln(x)}{x^2}\, dx$.

Example

Evaluate $\mathrm{I} = \int \ln(x)\, dx$.

Since $\int \ln(x) dx$ can be written as $\int \ln(x) \cdot 1 dx$, we use integration by parts, with $u = \ln(x)$ and $dv = dx$.

Sometimes, we have to apply Integration by Parts more than once.

Example

Evaluate $\mathrm{I} = \int x^2 e^x \, dx$.

Example of repeated IbP

Evaluate $\mathrm{I} = \int e^x \cos(x)\,dx$.

Definite Integrals

Integration by Parts for Definite Integrals

$$\int_a^b u dv = (uv)\Big|_a^b - \int_a^b v\, du$$

Example: Use Integration By Parts to evaluate $\displaystyle\int_0^1 xe^{-x}\, dx$.

Exercises

Exer 8.1.1

Evaluate the follow integrals

1. $\displaystyle\int \sin(\ln x)\frac{1}{x}\,\mathrm{d}x.$
2. $\displaystyle\int x^2(x^3+5)^9\,\mathrm{d}x$
3. $\displaystyle\int \frac{\sin(x)}{\cos^3(x)}\,dx$

Exer 8.1.2

Evaluate $\displaystyle\int_0^1 x^2(x^3+5)^9\,\mathrm{d}x$

Exercises

Exer 8.1.3

Evaluate the follow integrals

1. $\displaystyle\int xe^{2x}\,dx$.
2. $\displaystyle\int x^2\cos(x)\,dx$.

Exer 8.1.4

Evaluate $\displaystyle\int_1^e \ln(x^2)\,dx$.