2526-MA140 Engineering Calculus

Derivatives of

Week 05, Lecture 3
Exponentials and Logarithms; Higher-order derivatives

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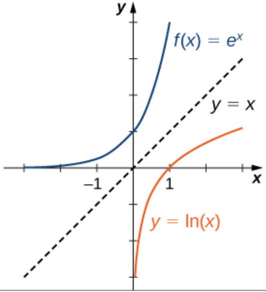


Today's topics:

See also: 3.9 (Derivatives of Exponential and Logarithmic Functions) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

The number e

The number e is a mathematical constant (similar in a sense to the way that $\pi \approx 3.14159$ is one too).

The value of e is roughly 2.7182818284.

It is often called **Euler's Number** after Leonard Euler, who did not discover it: that was (probably) Jacob Bernoulli in 1683 while studying compound interest. Or maybe 100 years earlier by John Napier.

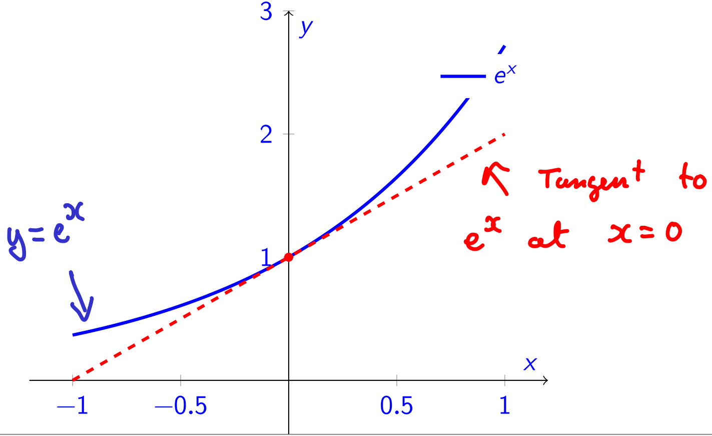
It has some very interesting and important properties...

$$e^{i\pi} = -1$$

Natural Exponential Function

The Natural Exponential Function

The Natural Exponential Function is $f(x) = e^x$. It is special for many reasons, including the its tangent at $x = 0$ has slope 1.



Let's assume that e is the number for which, if $f(x) = e^x$, then $f'(0) = 1$. Using the limit definition of the derivative, this means

$$1 = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

~~From this can deduce that~~ ...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = e^x \Rightarrow f'(0) = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

Let's assume that e is the number for which, if $f(x) = e^x$, then $f'(0) = 1$. Using the limit definition of the derivative, this means

$$1 = \lim_{h\to 0} \frac{e^{0+h} - e^0}{h} = \boxed{\lim_{h\to 0} \frac{e^h - 1}{h}.} = 1 .$$

From this can deduce that...

$$\frac{d}{dx}[e^x] = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{\boxed{e^x}(e^h - 1)}{h} = e^x \lim_{h\to 0} \boxed{\frac{e^h - 1}{h}}$$

$$= e^x(1) = e^x$$

That is $\frac{d}{dx}[e^x] = e^x$

So now we know that

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

That is e^x is the function that is its own derivative!!!

Example

Compute the derivative of $f(x) = e^{\sin(x)}$

$f(x) = u(v(x))$ with $u(v) = e^v$ $v(x) = \sin(x)$

Then $\frac{du}{dv} = e^v$ $\frac{dv}{dx} = \cos(x)$

By the Chain Rule:

$$\frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = e^v \cdot \cos(x) = e^{\sin(x)} \cdot \cos(x)$$

Logarithms

Suppose that $y = f(x)$ is an **exponential** function; that is: $y = b^x$ for some $b > 0$ (and excluding $x = 1$).

Its **inverse** is called a **logarithmic function**, denoted $\log_b$

$$\text{If } y = b^x \quad \text{then} \quad \log_b(y) = x.$$

Examples

- $log_2(8) = 3$ — since $2^3 = 8$
- $\log_{10}(100) = 2$ — since $10^2 = 100$
- $\log_e(e^x) = x$ — since e^x is e^x !

$\log_b a$ = " what power do I need to raise b (base) to, to get a? "

Properties of Logarithms

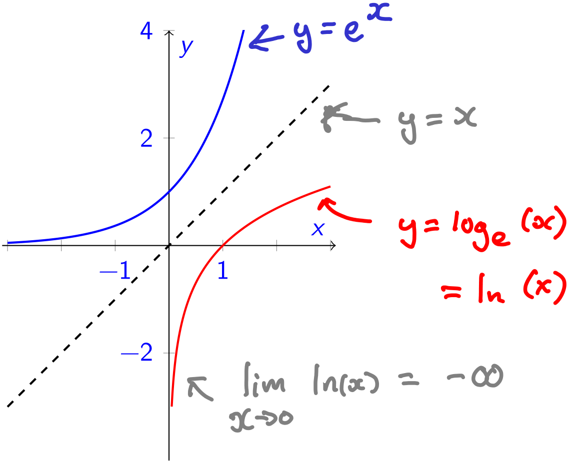
If $a, b, c > 0$ and $b \neq 1$, then

- $\log_b(ac) = \log_b(a) + \log_b(c)$
- $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$
- $\log_b(a^r) = r\log_b(a)$

Check textbook for more details...

Section 1.5

We denote $\boxed{\log_e(x)}$ as $\boxed{\ln(x)}$



$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}} = x^{-1}$$

Why?

Let $y = \ln(x)$ so $\boxed{x = e^y.}$

Differentiate $x = e^y$ with respect to x:

$$\frac{d}{dx}[x] = \frac{d}{dx}[e^y]$$

$$\Rightarrow \quad 1 = \frac{d}{dy}[e^y]\cdot\frac{dy}{dx} \qquad \text{[Chain Rule]}$$

$$\Rightarrow \quad 1 = e^y\cdot\frac{dy}{dx} \quad \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{\boxed{e^y}} = \frac{1}{\boxed{x}}}$$

Example:

Find the derivative of $f(x) = \ln(x^2 + 2x + 3)$.

$f(x) = u(v(x))$ $\quad u(v) = \ln(v)$ $\quad v(x) = x^2 + 2x + 3$

$\Rightarrow \frac{du}{dv} = \frac{1}{v}$ $\quad \frac{dv}{dx} = 2x + 2$

So $\frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{1}{v} \cdot (2x+2) = \frac{2x+2}{x^2+2x+3}$

Logarithmic differentiation

Next: the idea of **logarithmic differentiation**, which helps us differentiate functions with x, or a function of x in the exponent, such as $y = (2x)^{\sin(x)}$ or $y = x^x$.

Strategy:

- Take ln of both sides
- Simplify, using properties of logarithms.
- Differentiate.
- Solve for $\frac{dy}{dx}$

Logarithmic differentiation

Example

Differentiate $f(x) = x^x$.

Set $y = x^x$

Take the Natural log of this equation:

$$\ln(y) = \ln(x^x)$$

$\Rightarrow \ln(y) = x \ln(x)$ since $\ln(a^b) = b\ln(a)$

Left side: $\frac{d}{dx}[\ln(y)] = \frac{d}{dy}(\ln(y)) \cdot \frac{dy}{dx} = \frac{1}{y} \cdot \frac{dy}{dx}$.

Right side: $\frac{d}{dx}[x \ln(x)]$

$u(x) = x$ $\quad \frac{du}{dx} = 1$ $\qquad v(x) = \ln(x)$ $\quad \frac{dv}{dx} = \frac{1}{x}$

Product Rule: $\frac{d}{dx}[x \ln(x)] = x \cdot \frac{1}{x} + \ln(x)(1) = 1 + \ln(x)$

Logarithmic differentiation

Example

Differentiate $f(x) = x^x$.

(continued)

This gives

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln(x)$$

so $$\frac{dy}{dx} = y\left(1 + \ln(x)\right)$$

Ans $$\frac{dy}{dx} = x^x\left(1 + \ln(x)\right).$$

Higher-order Derivatives

We learned last week that the derivative of $f(x)$, denoted $f'(x)$, is itself a function.

That implies that $f'(x)$ can itself be differentiated, which is called the **second derivative** of f. It is denoted as

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x) \quad \text{or} \quad f^{(2)}(x).$$

We can continue this process to get higher-order derivatives as long as the preceding derivative is again differentiable.

The first and second derivatives f' and f'' (if they exist) provide valuable information about the function and its graph, particularly concerning local or global maxima, local/global minima and points of inflection.

$$\frac{d}{dx}\left(\frac{d}{dx}(y)\right) = \frac{d^2}{(dx)^2}y$$

 Finished here

Example

Find the **second** derivative of the functions

(i) $f_1(x) = 3x^2 + 2x + 1$ (iii) $f_3(x) = \ln x$

(ii) $f_2(x) = e^x$ (iv) $f_4(x) = \sin(x)$

Eg $f(x) = x^2$

So $f'(x) = 2x$ & $f''(x) = 2$

Eg $f(x) = a + bx$ (any linear poly)

$f'(x) = b$ $f''(x) \equiv 0$

So 2^{nd} deriv of a line is zero

This section of MA140 is concerned with using techniques of differentiation to finding where a function is

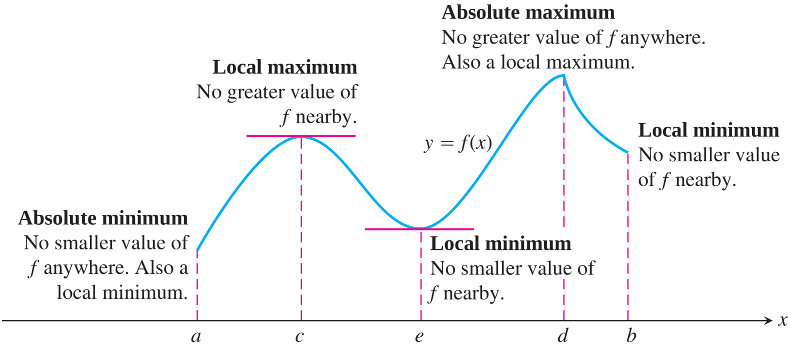
- Increasing
- Decreasing
- Has its maximum value
- Has its minimum value

Along the way we'll learn about **critical values** and the **first derivative test**.

Mathematical English

- The plural of **maximum** is **maxima**;
- The plural of **minimum** is **minima**;
- An **extremum** a maximum or a minimum.
- The plural of **extremum** is **extrema**.

Given an interval $[a,b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$, consider the function $f : [a,b] \to \mathbb{R}$ whose graph is given below. It illustrates local and absolute (="global") maxima and minima. Collectively, these are called **extrema**.



Definition: critical points

Let c in an point in the domain of a function f. We say that $x = c$ is a **critical point** of $f(x)$ if either

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist.}$$

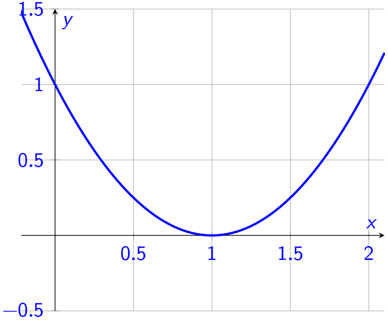
Important: If f has am extremum at $x = c$, then c must be a critical point of f (This is called “Fermat's Theorem”).

So, to find a maximum or minimum of f, it is enough to check at the critical points.

Warning: All extrema are at critical points, but not all critical points correspond to a extrema.

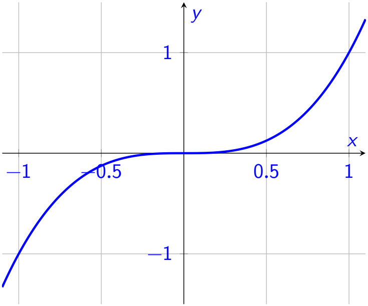
Example

$f(x) = x^2 - 2x + 1$ has one critical point. Find it. Does it correspond to an extremum?



Example

Find all critical points of $f(x) = x^3$. Do they correspond to extrema?



Exercises

Exercise 5.3.1 [2019 exam, Q2(b)(i)]

Differentiate $f(x) = e^{\sin(x)} \cos x$.

Exercise 5.3.2 [2023 exam, Q2(a)(i)]

Differentiate $f(x) = xe^{\sin(x)}$.

Exercise 5.3.3

Let $f(x) = x^2 e^x$. Find $f'(x)$, $f''(x)$ and $f'''(x)$.