§1.3: The secant method

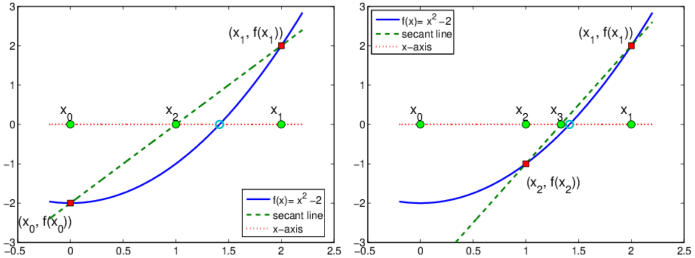
Solving nonlinear equations

MA385 – Numerical Analysis

September 2025

Idea:

- Choose two points, x_0 and x_1.
- Take x_2 to be the zero of the line joining $(x_0, f(x_0))$ to $(x_1, f(x_1))$.
- Take x_3 to be the zero of the line joining $(x_1, f(x_1))$ to $(x_2, f(x_2))$.
- Etc.



6. §1.3 Secant Method (DRAFT!)

The Secant Method

Choose x_0 *and* x_1 *so that there is a solution in* $[x_0, x_1]$. *Then define*

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}. \tag{1}$$

Example 1.4

Use the Secant Method to solve $x^2 - 2 = 0$ *in* $[0, 2]$. Results are shown below. We see that, not only does the method appear to converge to the true solution, it seem to do so *much* more efficiently than Bisection. We'll return to why this is later.

	Secant		Bisection	
k	x_k	$\lvert x_k - \tau \rvert$	x_k	$\lvert x_k - \tau \rvert$
0	0.000000	1.41	0.000000	1.41
1	2.000000	5.86e-01	2.000000	5.86e-01
2	1.000000	4.14e-01	1.000000	4.14e-01
3	1.333333	8.09e-02	1.500000	8.58e-02
4	1.428571	1.44e-02	1.250000	1.64e-01
5	1.413793	4.20e-04	1.375000	3.92e-02
6	1.414211	2.12e-06	1.437500	2.33e-02
7	1.414214	3.16e-10	1.406250	7.96e-03
8	1.414214	4.44e-16	1.421875	7.66e-03

To compare different methods, we need the following concept.

Definition 1.5 (Linear Convergence)

Suppose that $\tau = \lim_{k\to\infty} x_k$*. Then we say that the sequence* $\{x_k\}_{k=0}^{\infty}$ *converges to* τ **at least linearly** *if there is a sequence of positive numbers* $\{\varepsilon_k\}_{k=0}^{\infty}$*, and* $\mu \in (0,1)$*, such that*

$$\lim_{k\to\infty} \varepsilon_k = 0, \tag{2a}$$

and

$$|\tau - x_k| \le \varepsilon_k \quad \text{for} \quad k = 0, 1, 2, \dots. \tag{2b}$$

and

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu. \tag{2c}$$

For Example 1.4, the bisection method converges at least linearly

As we have seen, there are methods that converge more quickly than bisection. Now we'll give a more precise description of what "more quickly" means.

Definition 1.6 (Order of Convergence)

Let $\tau = \lim_{k\to\infty} x_k$. Suppose there exists $\mu > 0$ and a sequence of positive numbers $\{\varepsilon_k\}_{k=0}^{\infty}$ such that (2a) and and (2b) both hold. Then we say that the sequence $\{x_k\}_{k=0}^{\infty}$ converges with at least order q if

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu.$$

Two particular values of q are important to us:

(i) If $q = 1$, and we have that $0 < \mu < 1$, then the rate is **linear**.
(ii) If $q = 2$, the rate is **quadratic** for any $\mu > 0$.

Theorem 1.7

Suppose that f and f' are real-valued functions, continuous and defined in an interval $I = [\tau - h, \tau + h]$ for some $h > 0$. If $f(\tau) = 0$ and $f'(\tau) \neq 0$, then the sequence (1) converges at least linearly to τ.

8. Analysis of the Secant Method

- We wish to show that $|\tau - x_{k+1}| < |\tau - x_k|$.
- From the (MVT), there is a point $w_k \in [x_{k-1}, x_k]$ s.t.

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(w_k). \tag{3}$$

- Also by the MVT, there is a point $z_k \in [x_k, \tau]$ such that

$$\frac{f(x_k) - f(\tau)}{x_k - \tau} = \frac{f(x_k)}{x_k - \tau} = f'(z_k). \tag{4}$$

Therefore $\quad f(x_k) = (x_k - \tau) f'(z_k).$

8. Analysis of the Secant Method

- Using (3) and (4), we can show that

$$\tau - x_{k+1} = (\tau - x_k)\Big(1 - f'(z_k)/f'(w_k)\Big).$$

Therefore
$$\frac{|\tau - x_{k+1}|}{|\tau - x_k|} = \Big|1 - \frac{f'(z_k)}{f'(w_k)}\Big|.$$

- Suppose that $f'(\tau) > 0$. (If $f'(\tau) < 0$ just tweak the arguments accordingly). Saying that f' *is continuous in the region* $[\tau - h, \tau + h]$ means that, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

  $$|f'(x) - f'(\tau)| < \varepsilon \text{ for any } x \in [\tau - \delta, \tau + \delta].$$

  Take $\varepsilon = f'(\tau)/4$. Then $|f'(x) - f'(\tau)| < f'(\tau)/4$. Thus

  $$\frac{3}{4} f'(\tau) \leq f'(x) \leq \frac{5}{4} f'(\tau) \quad \text{for any } x \in [\tau - \delta, \tau + \delta].$$

  Then, so long as w_k and z_k are both in $[\tau - \delta, \tau + \delta]$

  $$\frac{f'(z_k)}{f'(w_k)} \leq \frac{5}{3}.$$

8. Analysis of the Secant Method

Given enough time and effort we *could* show that the Secant Method converges faster that linearly. In particular, that the order of convergence is

$$q = (1 + \sqrt{5})/2 \approx 1.618.$$

This number arises as the only positive root of $q^2 - q - 1$. It is called the **Golden Mean**, and arises in many areas of Mathematics, including finding an explicit expression for the Fibonacci Sequence:

$$\begin{aligned} f_0 &= 1, \\ f_1 &= 1, \\ f_{k+1} &= f_k + f_{k-1} \text{ for } k = 2, 3, \dots. \end{aligned}$$

That gives, $f_0 = 1$, $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, $\dots$.

The connection here is that it turns out that $\varepsilon_{k+1} \leq C\varepsilon_k\varepsilon_{k-1}$. Repeatedly using this we get:

- Let $r = |x_1 - x_0|$ so that $\varepsilon_0 \leq r$ and $\varepsilon_1 \leq r$,
- Then $\varepsilon_2 \leq C\varepsilon_1\varepsilon_0 \leq Cr^2$
- Then $\varepsilon_3 \leq C\varepsilon_2\varepsilon_1 \leq C(Cr^2)r = C^2r^3$.
- Then $\varepsilon_4 \leq C\varepsilon_3\varepsilon_2 \leq C(C^2r^3)(Cr^2) = C^4r^5$.
- Then $\varepsilon_5 \leq C\varepsilon_4\varepsilon_3 \leq C(C^4r^5)(C^2r^3) = C^7r^8$.
- And in general, $\varepsilon_k = C^{f_k - 1}r^{f_k}$.

Exercise 1.6

Suppose we define the Secant Method as follows.

> *Choose any two points x_0 and x_1.*
>
> *For $k = 1, 2, \dots,$ set x_{k+1} to be the point where the line through $\big(x_{k-1}, f(x_{k-1})\big)$ and $\big(x_k, f(x_k)\big)$ that intersects the x-axis.*

Show how to derive the formula for the secant method.

Exercise 1.7

(i) Is it possible to construct a problem for which the bisection method will work, but the secant method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which the secant method will work, but bisection will fail? If so, give an example.