MA378 Chapter 3: Numerical Integration

# §3.1 Introduction / Newton-Cotes / The Trapezium Rule

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# 1.1 Introduction

**Problem**

*Given a real-valued function $f$ that is continuous on $[a, b]$, can we find an estimate for*

$$I(f) := \int_a^b f(x)dx?$$

*And if we can, can we say how accurate it is?*

$I(\cdot)$ is the definite integral operator.

$\int_a^b f(x)dx$ is the area between $x=a$, $x=b$, $y=0$ and $y=f(x)$. "Area under the curve".

It is a Real number.

# 1.1 Introduction

Why is this an interesting problem?

- Many problems in applicable mathematics require definite integrals to be evaluated. These methods were originally motivated by problems in astronomy, and brewing. They are now ubiquitous.
- Evaluating them by finding the anti-derivative can be hard, and very hard to automate.
- Some times, although the function is integrable, its anti-derivative doesn't exist in a closed form.

# 1.1 Introduction

The process of numerically estimating a definite integral is called **Numerical Integration** or **Quadrature**.

The formulae we'll derive all look like

$$Q_N(f) := q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + \cdots + q_N f(x_N).$$

Here the points $x_i$ are called **quadrature points** and the $q_i$ are **quadrature weights**.

We need a way of choosing these.

The simplest approach for choosing the quadrature points to take them to be equally spaced, i.e., $x_i = a + hi$ where $h = (b-a)/N$.

## How to choose the weights?

We've spent a lot of time on approximating functions with polynomials. So it is natural to compute a polynomial interpolant to $f$, and take its integral. The appeal of this approach is due to the fact that

- We know how to compute polynomial interpolants.
- Integrating polynomials is easy.
- We can estimate the error easily (yet again, we'll make use of Cauchy's Theorem).

This leads to the **Newton-Cotes** methods, which are the subject of this section, and the next one. Later again, we'll look at more sophisticated methods, called **Gaussian Methods** which use non-uniformly spaced points.

# 1.2 Newton-Cotes methods

## Definition 1.1 (Newton-Cotes quadrature)

The **Newton-Cotes** quadrature rule for $\int_a^b f(x)dx$ with $N+1$ points is derived by integrating exactly the polynomial of degree $\cancel{n}$ $N$ that interpolates $f$ at the $N+1$ equally spaced points $a = x_0 < x_1 < \cdots < x_N = b$. The method is written as

$$Q_N(f) := q_0 f_0 + q_1 f_1 + q_2 f_2 + \cdots + q_N f_N,$$

where we use the notation $f_k := f(x_k)$.

That is, the quadrature weights are chosen so that

$$Q_N(f) = \int_a^b p_N(x)dx,$$

where $p_N$ is the polynomial of degree $n$ that interpolates $f$ at the $N+1$ quadrature points...

# 1.2 Newton-Cotes methods

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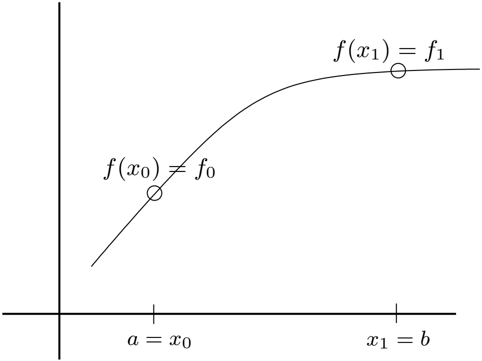
However, it turns out that we can compute the weights $q_0$, $q_1$, ..., $q_N$, **without** knowing $p_N$.

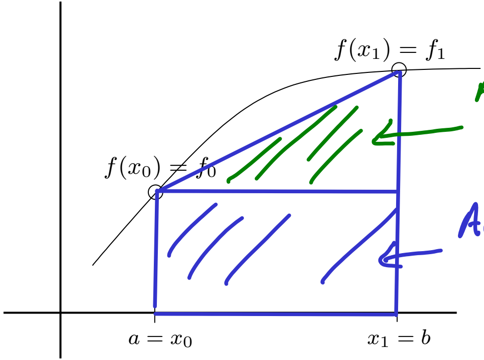
We'll do this for $N=1$ in the next section, and $N=2$ (the most interesting case) in Section 3.2.

# 1.3 The Trapezium rule

Suppose we wanted to estimate the integral of a function, $f$, shown below, on the interval $[a, b]$.



**Method 1:** We could try to estimate the area of the trapezium that fits under the graph:




Area is $\frac{1}{2}(x_1 - x_0)(f_1 - f_0)$

Area is $(x_1 - x_0) f_0$

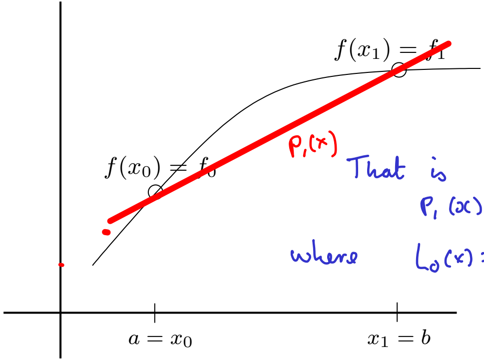
Area of Trapezium is

$$(x_1 - x_0) f_0 + (x_1 - x_0)\left(\frac{f_1}{2} - \frac{f_0}{2}\right) = \frac{1}{2}(f_0 + f_1)(x_1 - x_0)$$

**Method 2:** We could find $p_1$, the polynomial of degree 1 that interpolates $f$ at $x = a$ and $x = b$:

The Lagrange Form of the interpolant is

$$p_1(x) = f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$$




That is

$$p_1(x) = L_0(x) f_0 + L_1(x) f_1$$

where $L_0(x) = \dfrac{x - x_1}{x_0 - x_1}$ $\quad L_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

[See board]

Note that this shows that $q_i = \displaystyle\int_a^b L_i(x)dx$, where, as usual, the $L_i$ are the Lagrange Polynomials.

**Method 3:** The third approach for generating the Trapezium Rule is called the *Method of Undetermined Coefficients.* Because the method is based on integrating a linear function we expect it to yield an exact solution for any constant or linear function (i.e., there should be no error). To keep the algebra simple, we'll take $a = 0$ and $b = 1$. So,

$$Q_1(f) = q_0 f(0) + q_1 f(1),$$

and, setting $f(x) \equiv 1$, and then $f(x) = x$ we get

1. Take $f(x) = 1$, then $\int_0^1 f(x)\,dx = \underline{1}$. And $Q(f) = q_0 + q_1$
So $q_0 + q_1 = \underline{1}$
2. $f(x) = x$, then $\int_0^1 f(x)\,dx = 1/2$. And $Q(f) = q_1$
So $q_1 = 1/2$

Combine to get $q_0 = 1/2$, $q_1 = 1/2$.

## 1.3 The Trapezium rule $\int_a^b g(t)\,dt$ Method 3

Now we need to extend this to estimating $\cancel{\int_a^b g(x)\,dx}$ as follows:

Define a mapping from $[0,1]$ to $[a,b]$ as

$t(x) = a + (b-a)x$. Note that $\boxed{\dfrac{dt}{dx} = (b-a)}$.

Now let $f(x) = g(\underbrace{a + (b-a)x}_{t})$

$$\int_a^b g(t)\,dt = \int_0^1 f(x)\cdot(b-a)\,dx = (b-a)\int_0^1 f(x)\,dx$$

Also $f(0) = g(a)$ and $f(1) = g(b)$

So $\frac{1}{2}\big(f(0) + f(1)\big) = \frac{1}{2}\big(g(a) + g(b)\big)(b-a)$.

So $\displaystyle\int_a^b g(t)\,dt \approx \frac{b-a}{2}\big(g(a) + g(b)\big)$

**Example 1.2**

Use the trapezoid Rule to estimate n

$$\int_0^{\pi/4} \cos(x)dx.$$

Calculate the (exact) error $|\int_a^b f(x)dx - Q_1(f)|$.

$f(x) = \cos(x)$ , $a = 0$, $b = \pi/4$.

$\int_a^b f(x)\,dx = \int_0^{\pi/4} \cos(x)\,dx = \sin(x)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}} = 0.7071$

And $Q_1(f) = \frac{b-a}{2}\left(f(a)+f(b)\right) = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)\left(\cos(0) + \cos(\pi/4)\right)$

$= 0.67038.$ Error is $|0.7071 - 0.67038| = 0.0367.$

# 1.4 Exercises

## Exercise 1.1

(*For simplicity, you may assume that the quadrature rule is integrating $f$ on the interval* $[-1, 1]$.) Let $q_0$, $q_1$, ..., $q_n$ be the quadrature weights for the Newton-Cotes rule $Q_n(f)$. Show that $q_i = q_{n-i}$ for $i = 0, \dots n$.

## Exercise 1.2

Show that $\sum_{i=0}^{n} q_i = b - a$.