MA378 Solutions to Exercise 4.7 (Assignment 1)

Take $f(x) = x^3$ and $\{x_0, x_1, x_2\} = \{-1, 0, 1\}$.

(a) Write down the Lagrange form of p_2, the polynomial of degree two that interpolates f at x_0, x_1, and x_2. Simplify the expression for $p_2(x)$ as much as possible.

Answer: The Lagrange for an interpolant of degree 2 to f is

$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

For this problem

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-1)}{(-1)(-2)} = \frac{1}{2}x(x-1),$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(1)(-1)} = (x+1)(x-1),$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{(2)(1)} = \frac{1}{2}(x+1)x.$$

Using that $f(x_0) = f(-1) = -1$, $f(x_1) = 0$ and $f(x) = 1$, we get that the Lagrange form is

$$p_2(x) = -\frac{1}{2}x(x-1) + \frac{1}{2}(x+1)x.$$

Simplifying, we should find $p_2(x) = x$.

(b) Use Corollary 3.5 to give an upper bound for

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_2(x)|.$$

Answer: Cor 3.5, for the case $n = 2$ is gives that

$$|f(x) - p_2(x)| \leqslant \frac{1}{3!} \max_{-1 \leqslant \sigma \leqslant 1} |f'''(\sigma)||\pi_3(x)|.$$

Since $f(x) = x^3$, we see that $f'''(x) = 6$. So now we have the bound

$$|f(x) - p_2(x)| \leqslant \frac{6}{6}|(x+1)x(x-1)|.$$

To find the maximum of this quantity over all x in $[-1, 1]$, note that $|(x+1)x(x-1)| = |x(x^2-1)| = |x^3-x|$. It's maximum occurs where $\frac{d}{dx}(x^3 - x) = 0$. That is, solve $3x^2 - 1$. We get that there are two interior extreme points, at $x = \pm 1/\sqrt{3}$. Comparing the values of $\pi_3(x)$ at these points, and at the end points, we can deduce that $|x^3 - x| \leqslant 2\sqrt{3}/9 \approx 0.3849$. In summary,

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_2(x)| \leqslant 0.3849.$$

(c) Using calculus, give a sharper bound for $|f(x) - p_2(x)|$ on the interval $[-1, 1]$. That is, find the maxima/minima of the function $g(x) = f(x) - p_2(x)$ on $[-1, 1]$, and thus compute exactly

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_2(x)|.$$

Answer: Here $g(x) = x^3 - x$, so our goal is (again) to find the max/min of $x^3 - x$. So we get the same answer. (Note: the reason this is the same is largely coincidental. In an earlier version of this problem, I had a more complicated function f to approximate. I decided that was too tedious to work with, and switched to $f(x) = x^3$, without realising that this over-simplified the problem!

(d) Suppose we have $\{x_0, x_1, x_2\} = \{-a, 0, a\}$ for some number a, which we can choose. What is the largest value of a that can be permitted if we require that

$$\max_{-a \leqslant x \leqslant a} |f(x) - p_2(x)| \leqslant 10^{-3}?$$

You may use the result in Exercise 3.1 (**without** proof).

Answer: From Exer 3.1, we know that, if $x_1 - x_0 = x_2 - x_1 = h$, then the expression for the error bound simplifies to

$$\max_{x_0 \leqslant x \leqslant x_2} |f(x) - p_2(x)| \leqslant \frac{h^3}{9\sqrt{3}} M_3.$$

For this problem $f(x) = x^3$. So, as already noted $M_3 = 6$. Also, $h = a$. So we are trying to find a so that

$$\frac{6a^3}{9\sqrt{3}} \leqslant 10^{-3}.$$

With a little calculation we see that $a^3 \leqslant 2.5981 \times 10^{-3}$. That is, we take $a \leqslant 0.3849$.

(e) Write down the formula for the polynomial that is the Hermite interpolant to $f(x) = x^3$ at $x_0 = -1$ and $x_1 = 1$. (Hint: be lazy; you can do this without figuring out what $H_i(x)$ and $K_i(x)$ are).

Answer: The Hermite interpolant at 2 points is a polynomial of degree 3. Also, we proved that there is a unique Hermite interpolant to a given function at a fixed set of points. So, the only such Hermite interpolant to $f(x) = x^3$ is $p_3(x) = x^3$.
Extra: there is only one piecewise linear interpolant to $f(x) = x$ at any set of points, which is $l(x) = x$. However, the (natural) cubic spline interpolant to $f(x) = x^3$ is not x^3 for any set of points. Do you know why?