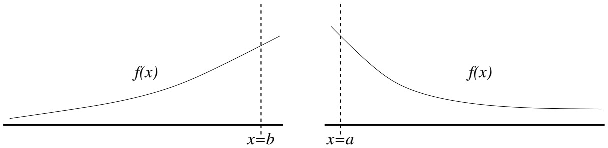
MA211

Lecture 19: Improper Integrals -Type 1

Wed 12$^{\text{th}}$ Nov 2008



Topics of the day...

1. Proper Integrals
2. Improper Integrals
3. Improper Integrals of Type I
   - $\int_a^\infty f(x)dx$
   - $\int_{-\infty}^b f(x)dx$
   - $\int_{-\infty}^\infty f(x)dx$

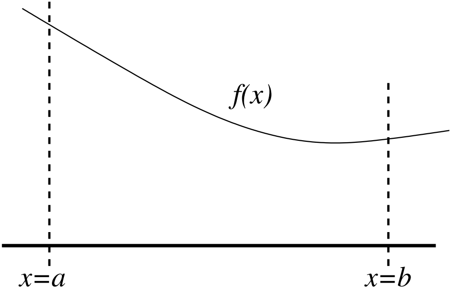
See also Section 7.7 of Stewart.

Proper Integrals

So far, the definite integrals we have considered:

$$\int_a^b f(x)dx,$$

have all been *Proper*: they are integrals of bounded functions on closed, finite intervals.

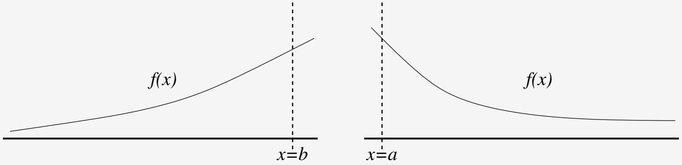


So we when we think of the integral as the area between the graph of the function and the x-axis, it is clear that that is well-defined.

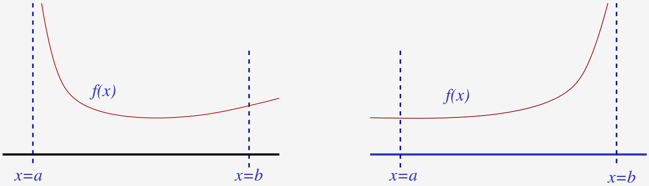
Improper Integrals

A definite integral $\int_a^b f(x)dx$ is *Improper* if:

Type I: if $a = -\infty$ or $b = \infty$



Type II: if $f(x)$ is unbounded (infinite) near a or b.

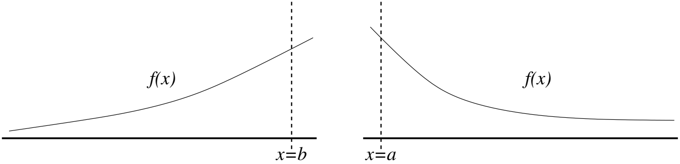


- *Some* improper integrals evaluate as a real, finite number. These are are said to **converge**, or to be *convergent* or **to exist**.
- Those that don't evaluate to a finite number are said to **diverge**, or to be *divergent* or **not to exist**.

Improper Integrals of Type I

Improper Integrals of Type I are of the form

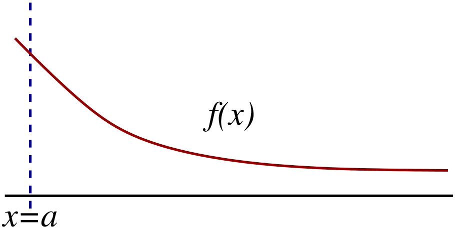
$$\int_a^{\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{b} f(x)\,dx.$$



To evaluate these, note that $\int_a^{\infty} f(x)dx = \lim_{t=\infty} \int_a^t f(x)dx$. So:

- Evaluate $\mathcal{I}(t) = \int_a^t f(x)dx$;
- and then compute $\lim_{t=\infty} \mathcal{I}(t)$.

$$\int_a^\infty f(x)dx$$



1. Evaluate $\mathcal{I}(t) = \displaystyle\int_a^t f(x)dx$;
2. and then compute $\mathcal{I} = \lim\limits_{t=\infty} \mathcal{I}(t)$.
3. If the limit exists, call it L and write $\int_a^\infty f(x)\,dx = L$. We say that $\int_a^\infty f(x)\,dx$ **converges to** L.
4. If no such limit exists, $\int_a^\infty f(x)\,dx$ is said to **diverge**.

Example

Evaluate $\mathcal{I} = \displaystyle\int_1^\infty \frac{1}{x^2} dx$

Example

Evaluate the improper integral $\mathcal{I} = \displaystyle\int_1^\infty \frac{dx}{x}$

Example

Evaluate $\mathcal{I} = \displaystyle\int_1^\infty \frac{1}{\sqrt{x}}dx$

$$\int_1^\infty 1/x^p\,dx \begin{cases} \texttt{converges} & \text{for } p > 1, \\ \texttt{diverges} & \text{for} p \leq 1. \end{cases}$$

Proof: If $p = 1$ then

$$\int_1^t x^{-p} dx = \int_1^t \frac{1}{x} dx = \ln(x)\Big|_1^t = \ln(t) - \ln(1) = \ln(t).$$

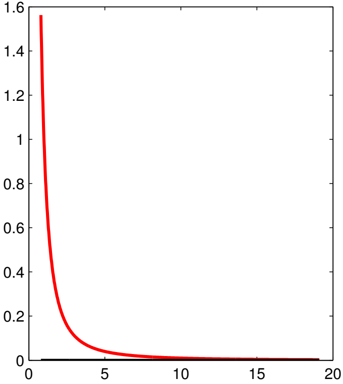
But $\lim_{t\to\infty} \ln(t)$ does not exists, so $\displaystyle\int_1^t \frac{1}{x} dx$ diverges.

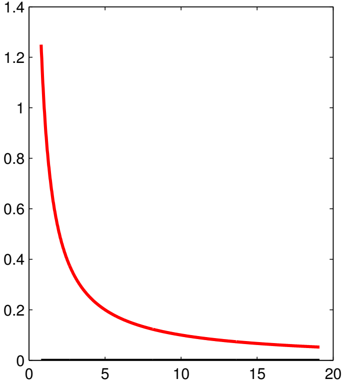
If $p \neq 1$ then $\displaystyle\int_1^t x^{-p} dx = \frac{x^{1-p}}{1-p}\Big|_1^t = \frac{t^{1-p} - 1}{1-p}$.

If $p < 1$ then $1 - p > 0$ so the limit $\displaystyle\lim_{t\to\infty} t^{1-p}$ does not exist, so the integral diverges in that case.

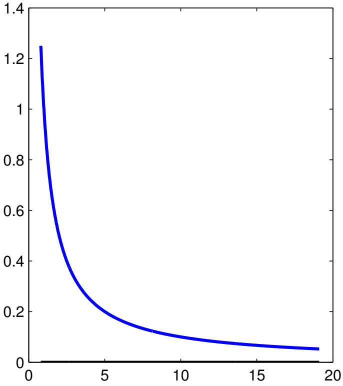
If however $p > 1$ then $1 - p < 0$ and $\displaystyle\lim_{t\to\infty} t^{1-p} = 0$, so the integral converges to $\displaystyle\frac{-1}{1-p}$.

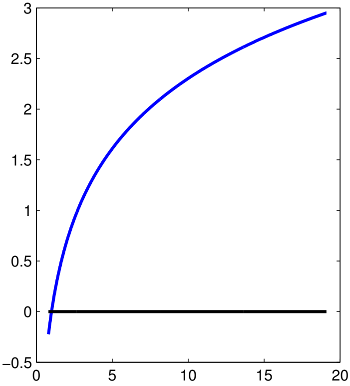
Example: $\int_a^\infty x^{-2} dx$



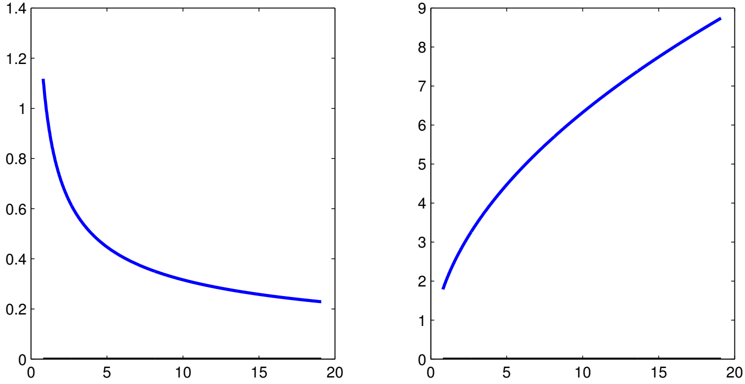


Example: $\int_a^\infty x^{-1} dx$





Example: $\int_a^\infty x^{-1/2} dx$

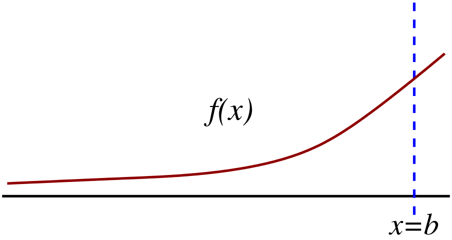


Example

Evaluate the integral $\int_1^\infty \frac{1}{1+x^2}dx$

For problems of the form:

$$\int_{-\infty}^{b} f(x)dx$$



1. Evaluate $\mathcal{I}(t) = \int_{t}^{b} f(x)dx$;
2. and then compute $\mathcal{I} = \lim_{t=-\infty} \mathcal{I}(t)$.
3. If the limit exists, call it $\mathcal{I}$ and write $\int_{-\infty}^{b} f(x)\,dx = L$. We say that the integral **converges to** L.
4. If no such limit exists, it is said to **diverge**.

Example

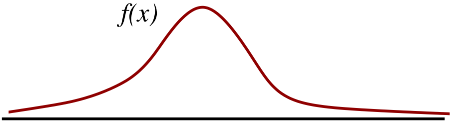
Evaluate $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2}$

Example

Show that $\displaystyle\int_{-\infty}^{0} e^x dx$ converges, but that $\displaystyle\int_{0}^{\infty} e^x dx$ diverges.

We also have to deal with the case where *both* limits of integration are at infinity:

$$\int_{-\infty}^{\infty} f(x)\,dx$$



To do this we recall that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx.$$

So $\int_{-\infty}^{\infty} f(x)dx$ converge if and only if *both* $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ converge.

Example

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$

Exercise (19.1)

Evaluate each of the following improper integrals:

(i) $\displaystyle\int_1^\infty \frac{1}{\ln(e^x)}dx.$

(ii) $\displaystyle\int_0^\infty \frac{x^2}{1+x^2}dx.$

(iii) $\displaystyle\int_3^\infty \frac{dx}{(2x-1)^{2/3}}$

(iv) $\displaystyle\int_0^\infty \frac{x}{1+2x^2}dx$

(v) $\displaystyle\int_0^\infty \frac{1}{1+e^x}dx$