MA378 Chapter 1: Polynomial Interpolation ANS With solutions!

Submit carefully written solutions to Exercises 1.4⋆, 2.3⋆, 2.5⋆, and 4.4⋆.

Deadline: 5pm, Friday 9 February.

Your solutions must be clearly written, and neatly presented. You can submit an electronic copy, through Canvas, or a hard copy (ideally at the lecture on the 9th). Make sure pages of the hard copy are stapled together. Marks will be given for quality and clarity of exposition. Collaboration is encouraged; policy will be discussed in class.

Exercise 1.1. Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

Exercise 1.2. Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

Exercise 1.3. (a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

Exercise 1.4 (⋆). For each of the following interpolation problems, determine (with explanation) if there is no solution, exactly one solution, or more than one solution. In all cases p_n denotes a polynomial of degree (at most) n. You are not required to determine p_n where it exists.

(a) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

> **Answer:** We want to interpolate the points $(-1, 0)$, $(0, -1)$ and $(1, 1)$ with a polynomial of degree 1. That is, we want to find a single straight line through these points. Since they are not co-linear, that is not possible: no solution exists.

(b) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = -2$.

> **Answer:** We want to interpolate $(-1, 0)$, $(0, -1)$ and $(1, -2)$ with a polynomial of degree 1. These *are* co-linear: $y_i = -1 - x_i$. So exactly one solution exists.

(c) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

> **Answer:** Here $n = 2$ and we have three distinct points: $(-1, 0)$, $(0, -1)$ and $(1, 1)$. So standard theory applies (e.g., Theorems 2.3 and 2.7: there exists exactly one solution.

(d) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

> **Answer:** Among the points we'd have to interpolate are $(x_0, y_0) = (-1, 0)$ and $(x_2, y_2) = (-1, 1)$. Since we can't have a polynomial with two different values at $x = 0$, there is **no** solution to this problem.

(e) Find $p_2(x)$ that interpolates (x_0, y_0) and (x_1, y_1) where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$.

> **Answer:** Since there are only two points, $(x_0, y_0) = (-1, 0)$ and $(x_1, y_1) = (1, -1)$, and $n = 2$, there is an infinite number of solutions. For example, let p_1 be the usual degree 1 interpolant of these points. Then any quadratic of the form $p_1(x) + C(x + 1)(x - 1)$, for any C also interpolates them.

Exercise 2.1. The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{0 \leqslant i < j \leqslant n} (x_j - x_i). \tag{2.0.1}$$

Verify (2.0.1) for the 2×2 and 3×3 cases.

(Note that from Formula (2.0.1) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \ldots, x_n$ are all distinct.)

Exercise 2.2. Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute

$$\max_{0 \leqslant x \leqslant a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayers.

Exercise 2.3 ($\star$). Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \quad \text{for all } x.$$

Answer: One possible solution is: let

$$q(x) = \sum_{i=0}^{n} L_i(x) - 1.$$

Since each of the L_i is a polynomial of degree n, so too is q. Furthermore, for $j = 0, 1, \ldots, n$,

$$\begin{aligned} q(x_j) &= \sum_{i=0}^{n} L_i(x_j) - 1 \\ &= L_j(x_j) - 1 \quad (\text{since } L_i(x_j) = 0 \text{ if } i \neq j) \\ &\quad = 1 - 1 \quad (\text{since } L_j(x_j) = 1) = 0. \end{aligned}$$

That is q is a polynomial of degree n with $n+1$ zeros. By Theorem 2.2, it follows that $q(x) \equiv 0$. Thus

$$\sum_{i=0}^{n} L_i(x_j) - 1 = 0 \iff \sum_{i=0}^{n} L_i(x_j) = 1.$$

Exercise 2.4. Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0, 3)$, $(1, 2)$ and $(2, 4)$.

Exercise 2.5 ($\star$). Show that all the following represent the polynomial $T_3(x) = 4x^3 - 3x$ (often called the "Chebyshev Polynomial of Degree 3"),

(a) Horner form: $H_3(x) := \big((4x + 0)x - 3\big)x + 0$.

Answer: Multiply out the terms on the right of H_3 to get $H_3(x) = (4x)x^2 - 3x = 4x^3 - 3x$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3} \Big(\prod_{j=0, j \neq k}^{3} \frac{x - x_j}{x_k - x_j} \Big) (-1)^{k+1}$, where $x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$.

Answer: This is the Lagrange form of the polynomial of degree 3 that interpolates the four points $(-1, -1)$, $(-1/2, 1)$, $(1/2, -1)$ and $(1, 1)$. Check that $T_3(-1) = 4(-1) - 3(-1) = -1$; $T_3(-1/2) = 4(-1/8) - 3(-1/2) = -1/2 + 3/2 = 1$; $T_3(1/2) = 4(1/8) - 3(1/2) = 1/2 - 3/2 = -1$; and $T_4(1) = 4 - 3 = 1$. Since both these polynomials are of degree $n = 3$, and interpolate the same $n + 1 = 4$ points, they are the same polynomials.

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and $T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

Answer: Note that $T_2 = 2xT_1 - T_0 = 2x^2 - 1$. Then $T_3 = 2xT_2 - T_1 = 2x(2x^2 - 1)^2 - x = 4x^3 - 2x - x = 4x^3 - 3x$.

Exercise 3.1. Read Section 6.2 of An Introduction to Numerical Analysis (Süli and Mayers). Pay particular attention to the proof of Thm 6.2 at `https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=192`.

Exercise 3.2. Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0 \leqslant x \leqslant x_2} |f(x) - p_2(x)| \leqslant \frac{1}{6} \frac{2}{3\sqrt{3}} h^3 M_3 = \frac{1}{9\sqrt{3}} h^3 M_3.$$

Hint: simplify the calculations by taking $t = x - x_1$, *writing* $(x - x_0)(x - x_1)(x - x_2)$ *in terms of* h *and* t.

Exercise 4.1. For *just* the case $n = 1$, state and prove an appropriate version of Theorem 4.2 (i.e., error in the Hermite interpolant). Use this to find a bound for

$$\max_{x_0 \leqslant x \leqslant x_1} |f(x) - p_3(x)|$$

in terms of f and $h = x_1 - x_0$.

Exercise 4.2. Let $n = 2$ and $x_0 = -1$, $\mathbf{x_0 = 0}$ and $x_1 = 1$. Write out the formulae for H_i and K_i for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 4.3. Let L_0, L_1, $\dots$, L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \dots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

and

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0.$$

(a) Show that $K_i(x_k) = 0$, for $k = 0, 1, \dots n$, and $K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

(b) Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$

Exercise 4.4 (⋆). Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$.

Answer: $L_0(x) = 1 - x$ and $L_1(x) = x$. Using the formulae from Exer 4.3, we have that

$$\begin{aligned} H_0 &= (L_0(x))^2(1 - 2L_0'(x))(x - x_0) \\ &= (1 - x)^2(1 + 2x) = 2x^3 - 3x^2 + 1. \end{aligned}$$

$$\begin{aligned} H_1 &= (L_1(x))^2(1 - 2L_1'(x))(x - x_1) \\ &= x^2(1 - 2(x - 1)) = -2x^3 + 3x^2. \end{aligned}$$

Answer: (Continued)

$$K_0 = (L_0(x))^2(x - x_0) = (1 - x)^2 x = x^3 - 2x^2 + x.$$

$$K_1 = (L_1(x))^2(x - x_1) = x^2(x - 1) = x^3 - x^2.$$

Also $f(0) = 0$, $f(1) = \sin(1/2) \approx 0.4794$, $f'(0) = 1/2$ and $f'(1) = \cos(1/2)/2 \approx 0.4388$. So Then

$$q_3(x) = (0.4794)H_1(x) + (1/2)K_0(x) + (0.4388)K_2(x).$$

If one wants to expand this, it can be written as

$$q_3(x) = -0.0201x^3 - 0.0005x^2 + x/2.$$

Answer: (continued) To give an upper bound for $|f(1/2) - q_3(1/2)|$... This is, perhaps, not a very sensible question. There are two valid approaches. First, one can calculate $f(1/2) = 0.2474039593$ and $q_3(x) = 0.2473638592$. Then we calculate $|f(1/2) - q_3(1/2)| \approx 4.01 \times 10^{-5}$.
The second is to use Thm 4.3, from which we can deduce that

$$|f(x) - q(x)| \leqslant= \frac{f^{(iv)}(\tau)}{(4)!}[(x)(x - 1)]^2,$$

where $\tau \in [0, 1]$. In this case $f^{(iv)}(x) = (1/16)\sin(x/2)$, so $|f^{(iv)}(\tau)| \leqslant |f^{(iv)}(1)| \leqslant 0.03$. Then

$$\begin{aligned} |f(\tfrac{1}{2}) - q(\tfrac{1}{2})| &\leqslant \frac{0.03}{24}\left[(\frac{1}{2})(\frac{1}{2} - 1)\right]^2 \\ &= \frac{1}{12800} = 7.8125 \times 10^{-5}. \end{aligned}$$

Exercise 4.5. Show that there is a unique solution to the Hermite Polynomial Interpolation Problem.

Exercise 4.6. Take $f(x) = x^3$ and $\{x_0, x_1, x_2\} = \{-1, 0, 1\}$.

(a) Write down the Lagrange form of p_2, the polynomial of degree two that interpolates f at x_0, x_1, and x_2. Simplify the expression for $p_2(x)$ as much as possible.

Answer: The Lagrange for an interpolant of degree 2 to f is

$$p_2(x) = L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2).$$

For this problem

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-1)}{(-1)(-2)} = \frac{1}{2}x(x-1),$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(1)(-1)} = (x+1)(x-1),$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{(2)(1)} = \frac{1}{2}(x+1)x.$$

Using that $f(x_0) = f(-1) = -1$, $f(x_1) = 0$ and $f(x) = 1$, we get that the Lagrange form is

$$p_2(x) = -\frac{1}{2}x(x-1) + \frac{1}{2}(x+1)x.$$

Simplifying, we should find $p_2(x) = x$.

(b) Use Corollary 3.5 to give an upper bound for

$$\max_{-1\leqslant x\leqslant 1} |f(x) - p_2(x)|.$$

Answer: Cor 3.5, for the case $n = 2$ is gives that

$$|f(x) - p_2(x)| \leqslant \frac{1}{3!} \max_{-1\leqslant\sigma\leqslant 1} |f'''(\sigma)||\pi_3(x)|.$$

Since $f(x) = x^3$, we see that $f'''(x) = 6$. So now we have the bound

$$|f(x) - p_2(x)| \leqslant \frac{6}{6}|(x+1)x(x-1)|.$$

To find the maximum of this quantity over all x in $[-1, 1]$, note that $|(x+1)x(x-1)| = |x(x^2-1)| = |x^3-x|$. It's maximum occurs where $\frac{d}{dx}(x^3-x) = 0$. That is, solve $3x^2-1 = 0$. We get that there are two interior extreme points, at $x = \pm 1/\sqrt{3}$. Comparing the values of $\pi_3(x)$ at these points, and at the end points, we can deduce that $|x^3-x| \leqslant 2\sqrt{3}/9 \approx 0.3849$. In summary,

$$\max_{-1\leqslant x\leqslant 1} |f(x) - p_2(x)| \leqslant 0.3849.$$

(c) Using calculus, give a sharper bound for $|f(x) - p_2(x)|$ on the interval $[-1, 1]$. That is, find the maxima/minima of the function $g(x) = f(x) - p_2(x)$ on $[-1, 1]$, and thus compute exactly

$$\max_{-1\leqslant x\leqslant 1} |f(x) - p_2(x)|.$$

Answer: Here $g(x) = x^3 - x$, so our goal is (again) to find the max/min of $x^3 - x$. So we get the same answer. (Note: the reason this is the same is largely coincidental...

(d) Suppose we have $\{x_0, x_1, x_2\} = \{-a, 0, a\}$ for some number a, which we can choose. What is the largest value of a that can be permitted if we require that

$$\max_{-a\leqslant x\leqslant a} |f(x) - p_2(x)| \leqslant 10^{-3}?$$

You may use the result in Exercise 3.1 (**without** proof).

Answer: From Exer 3.1, we know that, if $x_1 - x_0 = x_2 - x_1 = h$, then the expression for the error bound simplifies to

$$\max_{x_0 \leqslant x \leqslant x_2} |f(x) - p_2(x)| \leqslant \frac{h^3}{9\sqrt{3}} M_3.$$

For this problem $f(x) = x^3$. So, as already noted $M_3 = 6$. Also, $h = a$. So we are trying to find a so that

$$\frac{6a^3}{9\sqrt{3}} \leqslant 10^{-3}.$$

With a little calculation we see that $a^3 \leqslant 2.5981 \times 10^{-3}$. That is, we take $a \leqslant 0.3849$.

(e) Write down the formula for the polynomial that is the Hermite interpolant to $f(x) = x^3$ at $x_0 = -1$ and $x_1 = 1$. (Hint: be lazy; you can do this without figuring out what $H_i(x)$ and $K_i(x)$ are).

Answer: The Hermite interpolant at 2 points is a polynomial of degree 3. Also, we proved that there is a unique Hermite interpolant to a given function at a fixed set of points. So, the only such Hermite interpolant to $f(x) = x^3$ is $p_3(x) = x^3$.
Extra: there is only one piecewise linear interpolant to $f(x) = x$ at any set of points, which is $l(x) = x$. However, the (natural) cubic spline interpolant to $f(x) = x^3$ is not x^3 for any set of points. Do you know why?