MA378 Chapter 3: Numerical Integration

§3.5 Orthogonal Polynomials

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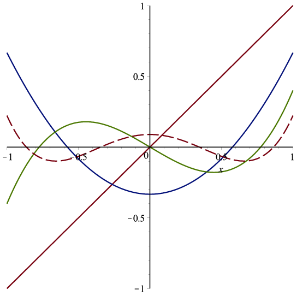


5.1 Orthogonal Polynomials

High order Newton-Cotes methods are of little use because of the problems associated with interpolation be high degree polynomials at equally spaced points. However, high-order Gaussian methods are very useful.

Driving such methods by undetermined coefficients is not practical, however. There is a simpler way, but some mathematical preliminaries are required, including the ideas of **vector spaces** and **inner products**.

Definition 5.1 (Vector space: informal)

A **vector** space is a collection of objects, called **vectors**, where it makes sense to

- add two vectors to get another one;
- multiply a vector by a scalar to get another one.

Definition 5.2 (Vector Space: formal)

V is a *vector space* (a.k.a., a *linear space*) over a field F (e.g, the real or complex numbers) if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

(i) $\mathbf{u} + \mathbf{v} \in V$ (closed under addition)

(ii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)

(iii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)

(iv) V has a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(v) $-\mathbf{u} \in V$

(vi) $a\mathbf{u} \in V$

(vii) $a(b\mathbf{u}) = (ab)\mathbf{u}$

(viii) F contains 0 and 1 such that $1\mathbf{u} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$.

(ix) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Examples: (of vector spaces)

$\mathbb{R}^n$ – vectors with n entries.

$\mathbb{P}^n$ – polynomials of degree at most n.

Eg, if $p, q \in \mathbb{P}^2$

$$p(x) = p_0 + p_1 x + p_2 x^2$$

$$q(x) = q_0 + q_1 x + q_2 x^2$$

Then $p + q \in \mathbb{P}^2$ and $\alpha p \in \mathbb{P}^2$

for any $\alpha \in \mathbb{R}$.

Definition 5.3 (Inner Product)

Let V is a real vector space. An **Inner Product** (IP) is a real-valued function $(\cdot,\cdot)$ on $V \times V$ such that, for all $f, g, h \in V$,

(i) $(f+g,h) = (f,h) + (g,h)$,
(ii) $(\lambda f, g) = \lambda(f,g)$, for $\lambda \in \mathbb{R}$. } linear
(iii) $(f,g) = (g,f)$, – symmetry.
(iv) $(f,f) \geq 0$. $\quad (f,f) = 0 \Leftrightarrow f \equiv 0$. – "positivity" + "non-degenerate"

An IP maps a pair of vectors to a Real number.

5.2 Inner products

Example 5.4

Let $\mathbb{R}^n$ be our vector space, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$. Then the following is an inner product:

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i y_i,$$

Eg $x = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ $\quad (x, x) = (1)(1) + (2)(2) + (-3)(-3)$

$= 14.$

$y = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$ $\quad (x, y) = (1)(-2) + (2)(1) + (-3)(0) = 0.$

5.2 Inner products

Example 5.5

The set of real-valued functions that are continuous and defined on the ionterval $[a, b]$, denoted $C[a, b]$, is a vector space. And

$$(f, g) := \int_a^b f(x)g(x)dx, \tag{1}$$

is an inner product.

This is so typical for vector spaces of functions we call it the "usual" inner product.

[See board to see why it is an IP].

5.3 Sequence of Orthogonal Monic Polynomials

(See Lecture 23 of Stewart's "Afternotes" for more details).

Definition 5.6 (Monic Polynomial)

A polynomial is *monic* if the coefficient of its leading term is 1.

Examples: $p(x) \equiv 1 = (1x^0)$

$p(x) = x^2 + 3x - 72.$ $\quad p(x) = 4x^4 + x^5 - 12.$

are all monic. So too are $\{1, x, x^2, x^3, \dots\}$

These are not:

$p(x) = 2,$ $\quad p(x) = -x^2 + 3x - 72$

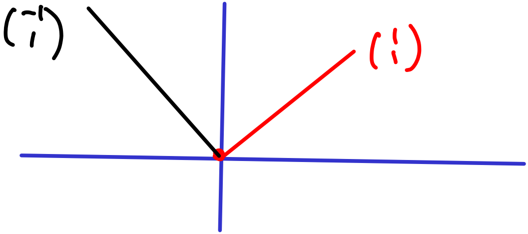
$p(x) = x^4 + 5x^5 - 12.$

Definition 5.7

Two elements a, b, of a vector space are *orthogonal* with respect to a given inner product $(\cdot,\cdot)$ if $(a, b) = 0$.

Example: $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $y = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

then $(x, y) = x_1 y_1 + x_2 y_2 = (1)(1) + (-1)(1) = 0$



Example 5.8

Take the space of polynomials of degree 2 or less and the IP

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$

Let $p(x) \equiv 1$, $q(x) \equiv x$, $r(x) \equiv \underline{x^2 - 1/3}$, and $f(x) = 3x - 4$

We can check that $(r, p) = 0$, and $(r, q) = 0$. We can the $\boxed{\text{verify}}$ that $(r, f) = 0$. **Details:**

$$(r,p) = \int_{-1}^{1} (x^2 - 1/3)\,dx = \frac{1}{3}x^3 - \frac{x}{3}\Big|_{-1}^{1} = 1/3 - 1/3 - (-1/3 + 1/3) = 0$$

$$(r,q) = \int_{-1}^{1} x^3 - x/3\,dx = 1/4\,x^4 - \frac{x^2}{6}\Big|_{-1}^{1} = 1/4 - 1/6 - (1/4 - 1/6) = 0$$

Note $f(x) = 3q(x) - 4p(x)$. So $(r, f) = (r, 3q - 4p)$

$$= (r, 3q) - (r, 4p) = 3(r,q) - 4(r,p) = 0$$

5.3 Sequence of Orthogonal Monic Polynomials

As given above, a polynomial is **monic** if the coefficient of the leading term is 1:

$$p_n = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-1} + \cdots + c_1 x + c_0.$$

We'll now look at a sequence of such polynomials

$$\{\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2, \dots, \widetilde{p}_n, \dots\}$$

that have the property they are orthogonal to each other:

$$(\widetilde{p}_i, \widetilde{p}_j) := \int_a^b \widetilde{p}_i(x)\widetilde{p}_j(x)dx = 0 \quad \text{if } i \neq j.$$

We want to establish some important facts about monic polys:

- A set of monic polys of degrees $1, \dots, n$, forms a basis for $\mathcal{P}_n$.
- If the members of that set are orthogonal to each other, then they are orthogonal to *all* polynomials of lower degree.
- We can construct such as set.

Theorem 5.9

Let $\{\widetilde{p}_i\}_{i=0}^{n}$ be a sequence of polynomials where each $\widetilde{p}_i$ is monic an exactly of degree i. This sequence forms as basis for $\mathcal{P}_n$.

Proof: Recall that $\mathbb{P}_n$ is the space of all polynomials of degree n or less. First take $n=1$. So $\tilde{p}_0 = 1$. and any element of $\mathbb{P}_0$ is a multiple of $\hat{p}_0$. So $\{\hat{p}_0\}$ is a basis for $\mathbb{P}_0$. We proceed by induction. Suppose that $\{\hat{p}_0, \hat{p}_1, \dots, \hat{p}_n\}$ is a basis for $\mathbb{P}_n$.

Theorem 5.9

Let $\{\widetilde{p}_i\}_{i=0}^{n}$ be a sequence of polynomials where each $\widetilde{p}_i$ is monic an exactly of degree i. This sequence forms as basis for $\mathcal{P}_n$.

Proof: Let $q \in \mathbb{P}_{n+1}$. So we can write

$$q(x) = a_{n+1} x^{n+1} + a_n x^n + a_{n-1} x^{n-1} + \dots + a_0.$$

Then

$$q(x) = a_{n+1} \bar{p}_{n+1}(x) + \boxed{b_n x^n + b_{n-1} x^{n-1} + \dots + b_n}$$

poly of degree. n

By inductive hypothesis this is a linear combination of $\tilde{p}_n, \tilde{p}_{n-1}, \dots, \tilde{p}_0$.

5.3 Sequence of Orthogonal Monic Polynomials

Theorem 5.9 means that if q is a polynomial of degree n then it can be written uniquely as a linear combination of the $\widetilde{p}_i$:

$$q(x) = \sum_{i=0}^{n} a_i \widetilde{p}_i(x),$$

for some unique choice of the real coefficients a_i.

5.3 Sequence of Orthogonal Monic Polynomials

Definition 5.10

The sequence $\{\widetilde{p}_i\}_{i=0}^n$ is a sequence of *monic*, *orthogonal* polynomials if each $\widetilde{p}_i$ is monic and *exactly* of degree i and

$$(\widetilde{p}_i, \widetilde{p}_j) = 0 \quad \text{if } i \neq j.$$

Theorem 5.11

If $\widetilde{p}_j \in \{\tilde{p}_i\}_{i=0}^{\infty}$ then $\widetilde{p}_j$ is orthogonal to all polynomials of degree less than j.

Proof:

Let p be any polynomial of degree n

From Thm 5.9 we can write

$$p(x) = \sum_{j=0}^{n} a_j \hat{p}_j(x)$$

The $(p, \hat{p}_{n+1}) = \left(\sum_{j=0}^{n} a_j \hat{p}_j, \hat{p}_{n+1} \right) =$

$$= \sum_{j=0}^{n} a_j (\hat{p}_j, \hat{p}_{n+1}) = 0.$$

5.4 Constructing the Sequence

Theorem 5.12

The sequence $\{\tilde{p}_i\}_{i=0}^{\infty}$ exists and can be constructed as follows: Let α and β be defined as

$$\alpha_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_n)}{(\widetilde{p}_n, \widetilde{p}_n)}, \quad \text{and} \quad \beta_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_{n-1})}{(\widetilde{p}_{n-1}, \widetilde{p}_{n-1})},$$

then the sequence is given by

$$\widetilde{p}_0(x) \equiv 1, \qquad \widetilde{p}_1(x) = x - \alpha_1$$

and

$$\underline{\widetilde{p}_{n+1}(x)} = (x - \alpha_{n+1})\boxed{\widetilde{p}_n(x)} - \beta_{n+1}\boxed{\widetilde{p}_{n-1}(x)},$$

for $n \geq 1$.

The proof uses *Gram-Schmidt Orthogonalization*.

 (FInished here 11am, Fri 8 March, 2024)

Idea: we want, e.g,

$$(\hat{p}_{n+1}, \hat{p}_n) = 0.$$

So $\big((x - \alpha_{n+1})\hat{p}_n - \beta_{n+1}\hat{p}_{n-1}, \hat{p}_n \big) = 0$

$$\Rightarrow (x\hat{p}_n, \hat{p}_n) - \alpha_{n+1}(\hat{p}_n, \hat{p}_n) - \beta_{n+1}\underbrace{(\hat{p}_{n-1}, \hat{p}_n)}_{=0} = 0$$

$$\Rightarrow \quad (x\hat{p}_n, \hat{p}_n) = \alpha_{n+1}(\hat{p}_n, \hat{p}_n).$$

which gives $\alpha_{n+1} = \dfrac{(x\hat{p}_n, \hat{p}_n)}{(\hat{p}_n, \hat{p}_n)}$.

For β_{n+1}, use $(\hat{p}_{n+1}, \hat{p}_{n-1}) = 0$.

Example 5.13

If we use the inner product $(f,g) := \int_{-1}^{1} f(x)g(x)$ then the first 3 polynomials in the sequence are:

$$\widetilde{p}_0 = 1, \quad \widetilde{p}_1 = x, \quad \text{and} \quad \widetilde{p}_2 = x^2 - 1/3.$$

Note, eg $(\hat{p}_1, \hat{p}_2) = \int_{-1}^{1} x(x^2 - 1/3)\,dx = \left(\frac{1}{4}x^4 - \frac{1}{6}x^2\right)\Big|_{-1}^{1}$

$= 0$

Example 5.14

The zeros of $\widetilde{p}_2$ are ...

If $\hat{p}_2(x) = 0 \Rightarrow x^2 - 1/3 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

5.5 Properties of the sequence

One of the ways of constructing Gaussian Quadrature rule $G_n(\cdot)$ on $n+1$ is to take the quadrature points as the roots of $\widetilde{p}_{n+1}$. We know (from the fundamental theorem of algebra) a polynomial of degree $n+1$ has exactly $n+1$ roots in $\mathbb{C}$ up to multiplicity.

However, the polynomials $\widetilde{p}$ have the special properties, established in the following lemma. (*A slightly different proof of these facts is given in Thm 9.4 of Suli and Mayers.*).

Theorem 5.15

Let $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\widetilde{p}_0, \widetilde{p}_1, \dots\}$ *be the set of monic polynomials that are orthogonal with respect to the inner product*
$(u, v) := \displaystyle\int_a^b u(x)v(x)dx.$ *Then:*

(i) *The zeros of each* $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ *are simple (not repeated).*

(ii) *All the zeros of* $\widetilde{p}_i$ *are real numbers in the interval* $[a, b]$.

(i) Suppose that $\tilde{p}_i$ has a repeated root, at $x = c$.

Then we can write it as

$$\tilde{p}_i(x) = (x-c)^2 q(x) \quad \text{where } \deg(q) = i-2$$

Since $\deg(q) < \deg(\hat{p}_i)$ we should have that

$$(\hat{p}_i, q) = 0$$

But $$(\hat{p}_i, q) = \int_a^b \underbrace{(x-c)^2 q(x)}_{p_i} q(x)\,dx.$$

$$= \int_a^b \left[(x-c)q(x)\right]^2 dx > 0$$

since $\left[(x-c)q(x)\right]^2 \geq 0$ for all x, and not 0 for all x.

But $(\hat{p}_i, q) > 0$ is not possible: Contradiction!!

5.5 Properties of the sequence

(ii) Suppose that $\hat{p}_i$ has k zeros in $[a,b]$ and $k < i$.

If so we can write $\hat{p}_i$ as

$$\hat{p}_i(x) = q(x)\, r(x) \quad \text{where}$$

r has degree $k < i$ and has k zeros in $[a,b]$

q has no zeros in $[a,b]$, and so does not change sign on $[a,b]$.

Then $(\hat{p}_i, r) = 0$ since $\deg(r) = k < \deg(\hat{p}_i)$

But $\int_a^b \hat{p}_i(x) r(x)\, dx = \int_a^b q(x) [r(x)]^2 \, dx \neq 0$

since $q(x)$ does not change sign and $[r(x)]^2 \geq 0$ $\boxed{\text{NU}}$

5.5 Properties of the sequence

5.6 Exercises

Exercise 5.1

$\mathcal{P}_n$, the space of polynomials of degree (at most) n forms a vector space. Is it true that the space of *monic* polynomials of degree n forms a vector space?

Exercise 5.2 ($\star$)

(i) Using the Inner Product

$$(f, g) := \int_0^1 f(x)g(x)dx,$$

find $\widetilde{p}_0(x)$, $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$.

(ii) Find the zeros of $\widetilde{p}_2(x)$ and call them x_0 and x_1. Construct a quadrature rule for $\int_{-1}^{1} f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials.