2425-MA140 Engineering Calculus

Week 10, Lecture 1 (L28) Infinite Areas and Volumes; Cylindrical Shells

Dr Niall Madden

School of Maths, University of Galway

Tuesday, 19 November, 2024

Detail from the Bass Performance Hall, Fort Worth, Texas, USA

Assignments,etc

1. **Assignments 6 and 7:** Deadline was moved to 5pm 19 November (today!) because the server was off-line for about 16 horus on Thursday night and Friday morning.
2. **Assignment 6:** On Canvas, `Assignment-6-Q1-Q7` records your score for Assignment 6 as it was before Sunday 10, November. A score of 89% means you had a perfect score for Questions 1-7; so you only need to complete the ungraded Q8. Do that as part of `Assignment-6` or `Assignment-6-Q8`. If you have a score of less than 89%, you can choose to keep it, or redo **Assignment-6**. Tedious, but you have a whole extra week to improve your grade!
3. **Assignment 8** is open, and the tutorial sheet is available.

Today, we'll hear about...

1. Areas of Rotation (again)
   - Example
2. Infinite solids
   - Volume of Rotation
   - Area of Revolution
3. Assignment 8, Q4
   - The "washer" method
4. Cylindrical Shells
   - Assignment 8, Q4 again
5. Exercises

See also: Sections **6.3** (Volumes of Revolution - Cylindrical Shells) and **6.4** (Arc Length of a Curve and Surface Area) in **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Areas of Rotation (again)

On Thursday (Week 9, Lecture 3) we introduced the idea of a "surface or rotation". That is we wanted to calculate the **surface area** of a solid generated by rotating a curve $y = f(x)$ about the x-axis. We deduced the following:

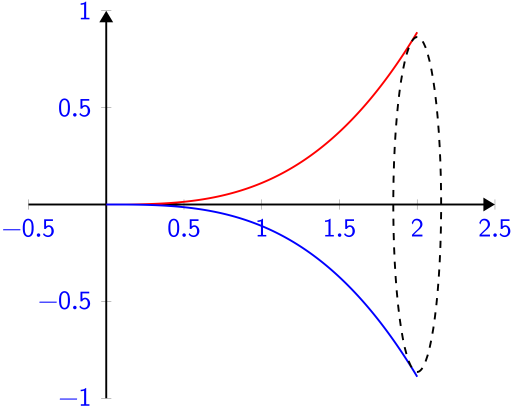
Surface Area

The surface area of the solid obtained by rotating the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

$$S = 2\pi \int_a^b y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx\,.$$

Example

Find the area of the surface generated by revolving the curve $y = \dfrac{x^3}{9}$ between $x = 0$ and $x = 2$ about the x-axis.

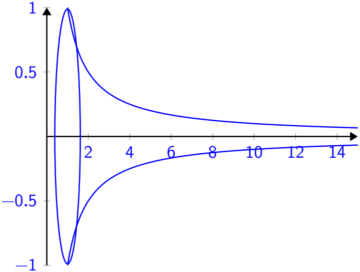


Infinite solids

On Thursday of Week 8, we learned to to compute integrals over infinite domains. And since volumes and areas of solids of revolution are expressed as integrals, there is nothing stopping us from (trying to) compute the volume or area of an infinite solid.

There is one very famous example to consider, sometimes called **Gabriel's Horn** or **Torricelli's Trumpet**.

It is constructed by rotating the graph of $f(x) = 1/x$, on the domain $[1, \infty)$, about the x-axis.



Infinite solids

First, let's compute the Volume of Rotation of $f(x) = 1/x$, for $x \geqslant 1$, when rotated about the x-axis.

We know that the general formula is $\boxed{V = \pi \int_a^b \big(f(x)\big)^2\, dx}$ For us, this will be

$$V = \pi \int_1^\infty x^{-2}\, dx = \lim_{t\to\infty} \pi \int_1^t x^{-2}\, dx$$

$$= \pi \lim_{t\to\infty} (-x^{-1})\Big|_1^t = \pi \lim_{t\to\infty} \Big(\frac{-1}{t} - \frac{1}{-1}\Big) = \pi\Big(1 - \lim_{t\to\infty} \frac{1}{t}\Big) = \pi$$

We know that the general form for an Area of Revolution is

$$\boxed{S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx}$$

For our problem, this becomes

$$S = 2\pi \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\, dx$$

It is a little tricky to compute the antiderivative of $x^{-1}\sqrt{1+x^{-4}}$. However, note that for $x \geqslant 1$, $\sqrt{1+x^{-4}} \geqslant 1$ So,

$$S = 2\pi \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\, dx \geqslant 2\pi \int_1^\infty x^{-1}\, dx = 2\pi \lim_{t\to\infty} (\ln(x))\Big|_1^t = \infty$$

(For the last part, see Slide 11 of Week 8, Lecture 3).

So, we have reached an apparent paradox – the Volume of "Gabriel's Horn" is π (and so finite), but it has an infinite surface area!

This is sometimes expressed as "the horn can hold only a finite amount of paint, but it would take an infinite amount of paint to paint the inside!". This is known as the "Painter's Paradox".

Assignment 8, Q4

In the tutorial sheet of Q4 of Assignment 8, we are asked:

Assignment 8, Q4

Find the volume of the solid formed by rotating the region enclosed by $x = 0$, $x = 1$, $y = 0$, and $y = 5 + x^9$, about the y-axis.

There are two ways (at least) of solving this:

- The "washer method" from last Wednesday;
- The Method of **Cylindirical Shells**, which we have yet to study.

We'll now do each of these.

First we'll sketch the object in question: We should be able to convince ourselves that $V = V_1 - V_2$, where

- V_1 is the volume of $f(y) = 1$, from $y = 0$ to $y = 6$, rotated about the y-axis;
- V_2 is the volume of $f(y) = (y-5)^{1/9}$, from $y = 5$ to $y = 6$, rotated about the y-axis.

It should be easy to check that $V_1 = 6\pi$, and almost as easy to check that $V_2 = \dfrac{9}{11}\pi$. This gives that $V = \dfrac{57}{11}\pi$.

Cylindrical Shells

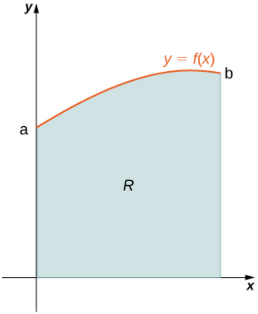
There is another approach, that is arguably easier.

So far, we've used the "disk method" for volumes of rotation. This came from the idea that every "slice" is a disk, whose area we can compute. Then integrating over the domain, we get the "sum" of all the disks: which is the volume.

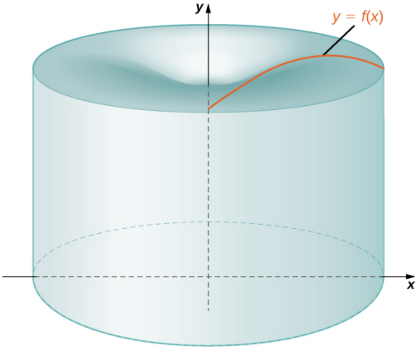
An alternative approach, is to construct think of the solid as an infinite sum of cylinders...

Cylindrical Shells

We state the problem as follows: *Find the volume of the solid obtained by rotating the region between* $y = f(x)$ *and* $y = 0$, *and* $x = 0$ *and* $x = b$, *around the* y*-axis*



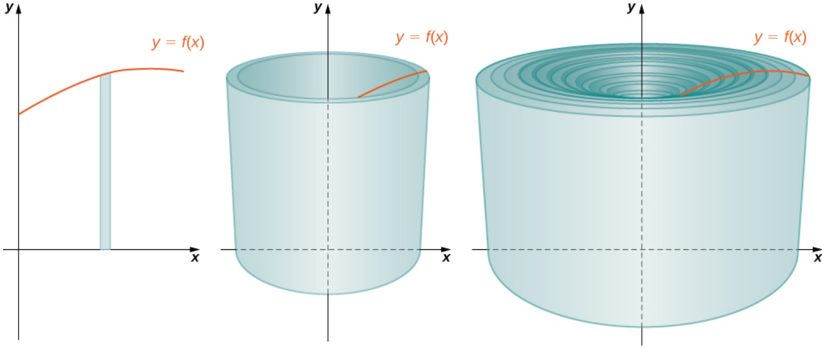
(a)



(b)

Cylindrical Shells

We can think of the region as made up of many small rectangles. When rotated about the y-axis, these become cylinders.



(a) (b) (c)

Cylindrical Shells

We can visualise this for a problem related to "Gabriel's Horn" from earlier, but over a finite region.

> *Let $f(x) = 1/x$. Construct a solid of revolution by rotating the region between $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$ about the y-axis.*

The visualisation from the textbook may be found at this link.

Cylindrical Shells

Very roughly, a cylinder with height $f(x)$, and thickness $\Delta x_i = x_i - x_{i-1}$ has volume

$$V_i = \pi(x_i^2 - x_{i-1}^2)f(x_i) = \pi f(x_i)(x_i + x_{i-1})(x_i - x_{i-1})$$
$$= 2\pi f(x_i)\frac{x_i + x_{i-1}}{2}\Delta x_i \approx 2\pi f(x_i)x_i\Delta x_i$$

Then

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i)x_i\Delta x_i = 2\pi\int_a^b xf(x)\,dx.$$

See text-book for more details!

The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Rotate about the y-axis, the region bounded above by $y = f(x)$, below by $y = 0$, on the left by $x = a$, and on the right by $x = b$.
Then the volume of the resulting solid of revolution is

$$V = 2\pi \int_a^b x f(x)\, dx.$$

Assignment 8, Q4

Use the Method of Cylindrical Shells to find the volume of the solid formed by rotating the region enclosed by $x = 0$, $x = 1$, $y = 0$, and $y = 5 + x^9$, about the y-axis.

Exercises

Exer 10.1.1 (from textbook)

Let $f(x) = 1/x$. Find the volume of the solid of revolution by rotating the region between $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$ about the y-axis.