Week 09, Lecture 3
Arc Lengths and Surface Areas

Dr Niall Madden

University of Galway

Thursday, 13 November, 2025

About the class test...

1. TBC

Today, we'll discuss at length:

See also: Section **6.4** (Arc Length of a Curve and Surface Area) in **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Yesterday, we started on how to calculate the length of curve that is defined by a function, f. We finished by deriving that...

Arc length of a curve

The **arc length**, L, of the curve $y = f(x)$, from $x = a$ to $x = b$, is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,.$$

We finished by discussing **Catenary Arches**. These have the shape taken on by a free hanging chain. When used as an arch, it has an "optimal" shape in the sense that it can support its own weight. It has been known since ancient times. The example below is from Casa Milá in Barcelona.
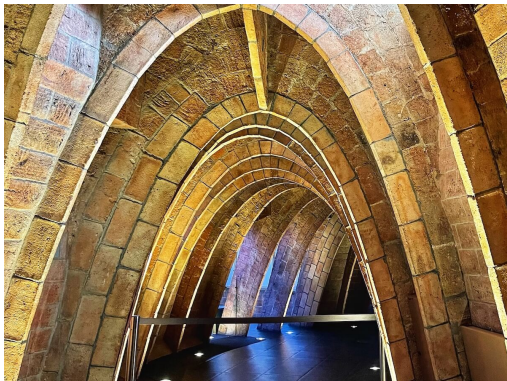
As a hanging chain, a catenary can be described by the function

$$f(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$
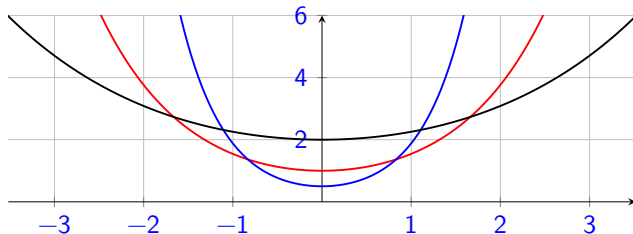
The function

$$f(x) = \frac{e^{x/a} + e^{-x/a}}{2}$$

is also known as the **hyperbolic cosine**, or cosh function. That is

$$\cosh(x) = \frac{e^{x/a} + e^{-x/a}}{2}.$$

You can read about it in the textbook (Section 1.5). But we don't need that for the following example.

Example (Example from Civil Engineering)

Metal posts have been installed $4m$ apart across a gorge. Find the length for rope bridge that follows the curve

$$f(x) = \frac{1}{2}\bigl(e^x + e^{-x}\bigr).$$

A note of caution

Computing the arc length of a function involves evaluating integrals where the integrand is of the form $\sqrt{1 + (g(x))^2}$. In some (rare) cases, this is easy. In others, it is possible to use a method called **trigonometric substitution**, which is not on our syllabus.

In the "real world" we actually use highly accurate numerical approximations. The details are beyond this course.
There is some interesting mathematics involved in determining how to take n large enough to ensure the error is small.

We'll mention this again briefly at the end of the semester.

Surface area of a cylinder

Suppose we have the line $y = r$, for some $r > 0$, and two points $x = a$ and $x = b$. The length of the segment of the line between a and b is denoted $\Delta x = b - a$.

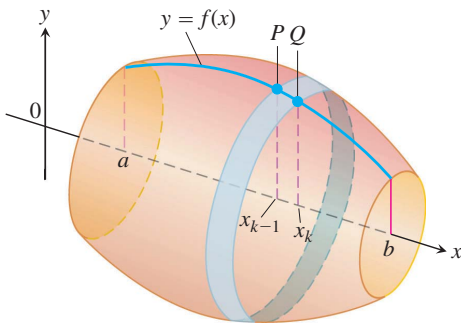Now rotate the line segment about the $x-$axis, to make a cylinder with curved surface area (see p10 of the "Log Tables")

$$A = 2\pi r \Delta x.$$

This area is the same as that of a rectangle with side lengths Δx and $2\pi y$.

NOT TO SCALE
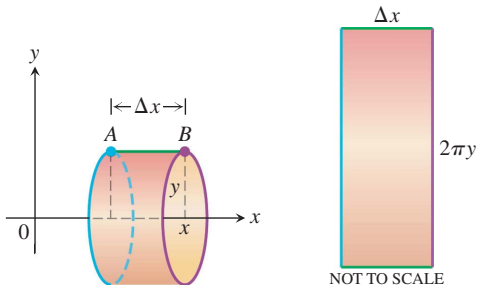
Areas of Revolution

We are interested in the **surface area of a solid** generated by rotating/revolving a curve $y = f(x)$ about the x-axis.

If the arc length from P to Q is L_k, then the surface area of the typical band is approximately

$$2\pi f(x_k) L_k .$$

Areas of Revolution

Summing over all the bands, we get that the surface area of the solid can be approximated as

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k) L_k.$$

We know that the arc length L_k can be computed as

$L_k = \sqrt{1 + \left[\dfrac{\Delta y_k}{\Delta x_k}\right]^2} \cdot \Delta x_k$. Thus, we can approximate the surface area of the solid as follows:

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k) \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x_k}\right]^2} \cdot \Delta x_k.$$

For $n \to \infty$, we get the Riemann sum

$$S = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi f(x_k) \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x_k}\right]^2} \cdot \Delta x_k.$$

Areas of Revolution

We now get the formula for the surface area of a solid obtained by rotating a curve.

Surface Area

If f is continuously differentiable on the interval $[a, b]$, then the surface area of the solid obtained by rotating the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

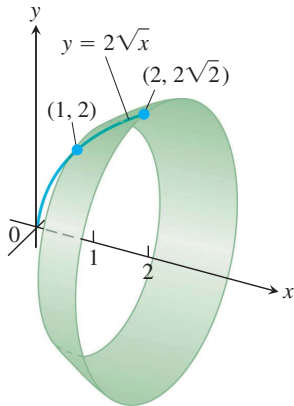$$S = 2\pi \int_a^b y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

You can also write this as

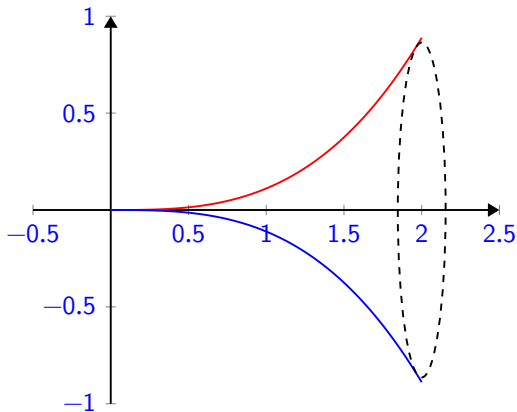$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx \, .$$

Areas of Revolution

Example

Find the area of the surface generated by revolving the curve
$y = 2\sqrt{x}$, $1 \leqslant x \leqslant 2$, about the $x-$axis

Areas of Revolution

Example

Find the area of the surface generated by revolving the curve $y = \dfrac{x^3}{9}$ between $x = 0$ and $x = 2$ about the x-axis.

Infinite solids [Extra: not examinable]

On Thursday of Week 8, we learned to to compute integrals over infinite domains. And since volumes and areas of solids of revolution are expressed as integrals, there is nothing stopping us from (trying to) compute the volume or area of an infinite solid.

There is one very famous example to consider, sometimes called **Gabriel's Horn** or **Torricelli's Trumpet**.

It is constructed by rotating the graph of $f(x) = 1/x$, on the domain $[1, \infty)$, about the x-axis.
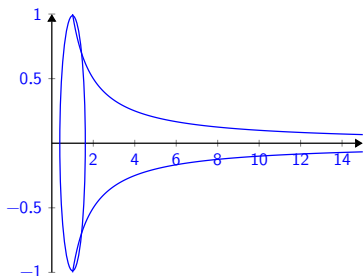
First, let's compute the Volume of Rotation of $f(x) = 1/x$, for $x \geqslant 1$, when rotated about the x-axis.

We know that the formula is $\boxed{V = \pi \int_a^b \left(f(x)\right)^2 dx}$.

For us, this will be

$$V = \pi \int_1^\infty x^{-2}\, dx = \lim_{t \to \infty} \pi \int_1^t x^{-2}\, dx$$
$$= \pi \lim_{t \to \infty} (-x^{-1})\big|_1^t = \pi \lim_{t \to \infty} \left(\frac{-1}{t} - \frac{1}{-1}\right) = \pi\left(1 - \lim_{t \to \infty} \frac{1}{t}\right) = \pi$$

We know formula for Area of Revolution is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

For our problem, this becomes

$$S = 2\pi \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\, dx$$

It is a little tricky to compute the antiderivative of $x^{-1}\sqrt{1 + x^{-4}}$. However, note that for $x \geqslant 1$, $\sqrt{1 + x^{-4}} \geqslant 1$ So,

$$S = 2\pi \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\, dx \geqslant 2\pi \int_1^\infty x^{-1}\, dx = 2\pi \lim_{t\to\infty} \ln(x)\big|_1^t = \infty$$

(For the last part, see Week 8, Lecture 3).

So, we have reached an **apparent paradox**[1]. It is best discussed :

- ▶ the Volume of "Gabriel's Horn" is π (and so finite)
- ▶ but it has an infinite surface area!

This is sometimes called **The Painter's Paradox**: "the horn can hold only a finite amount of paint, but it would take an infinite amount of paint to paint the inside!".

This "paradox" predates calculus, but the deep discussion (late 17th century) involving it helped refine the ideas the led to your favourite subject.

---

[1] We say "apparent" because it is possible to resolve. Try to think of possible resolutions.

Exercises

Exer 9.3.1

What is the arc length of the graphs of $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 1$ to $x = 2$?

Exer 9.3.2 (A little tricky)

Find the length of the Catenary function $f(x) = e^{x/2} - e^{-x/2}$ from $x = -2$ to $x = 2$.

Exer 9.3.3

What is the area of the surface formed by rotating the curve of

$$f(x) = \frac{3\sqrt{x}}{\sqrt{2}}$$

between $x = 0$ and $x = 2$, about the x-axis?

Exercises

Exer 9.3.4

Use calculus to determine the area of the surface formed by rotating the curve of $f(x) = x$, between $x = 1$ and $x = 2$, about the x-axis.
Can you verify this using the formula for the surface area of a cone ($A = \pi r l$), where r is the radius of the base, and l is the length of the (sloping) side?