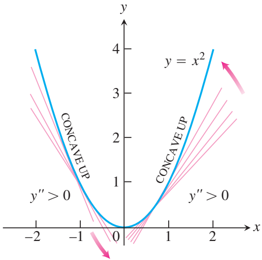
2425-MA140 Engineering Calculus

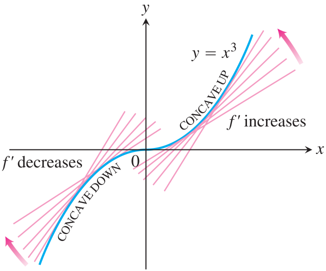
Week 06, Lecture 1
Curve Sketching

Dr Niall Madden

School of Maths, University of Galway

Tuesday, 22 October, 2024





Survey, Assignments, etc

- The module survey for MA140 has started. Please take a few minutes to complete it. See https://universityofgalway.instructure.com/courses/35693/discussion_topics/127822
- **Assignment 3**: if you think your correct grade is not showing, send me an email with your **results summary**.
- **Assignment 4**: this "optional" assignment has started. Deadline 5pm, Tuesday 29th October.

In today's class...

1. Review
   - The 1st Derivative Test
2. Concave up and down functions
3. Inflection points
4. Second derivative test
5. Curve Sketching
6. Exercise

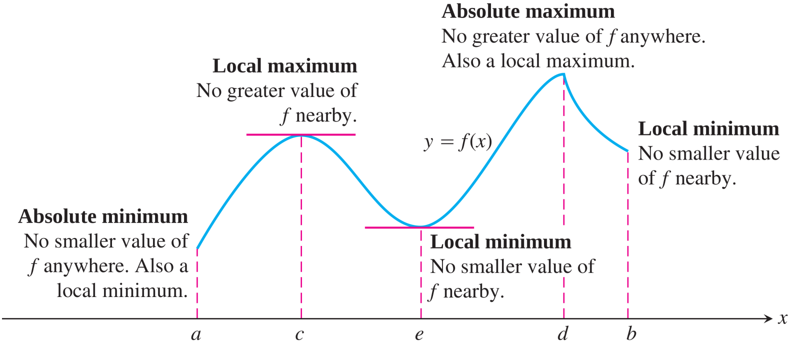
See also: Section 4.5 (Derivatives and the Shape of a Graph) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Review

In the last lecture, (Week 5, Lecture 3), we started studying the application of differentiation in locating (local) maxima and minima in functions.

There are the key points to recall:

- **maximum** and **minimum** points are collectively called **extreme** points.



Review

- $x = c$ is a **critical point** of $f(x)$ if either $f'(c) = 0$ or $f'(c)$ does not exist.
- All extreme points occur at critical points. (but not all critical points correspond to extreme points).
- To find a maximum or minimum of f, we first find the critical points.
- If $f'(x) > 0$ at each point $x \in [a, b]$, then f is increasing on $[a, b]$.
- If $f'(x) < 0$ at each point $x \in [a, b]$, then f is decreasing on $[a, b]$.

First Derivative Test for local maxima and minima

Suppose that c is a critical point of a differentiable function f.

1. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a **local maximum** of f.
2. If f' changes sign from negative when $x < c$ to positive when $x > c$ then $f(c)$ is a **local minimum** of f.
3. If f' has the same sign for $x < c$ and $x > c$ then $f(c)$ is neither a local maximum nor a local minimum of f.

Concave up and down functions

Definition

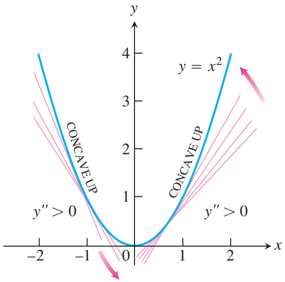
The graph of a differentiable function $y = f(x)$ is:

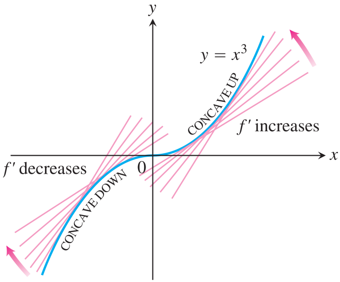
- **concave up** on an open interval (a, b) if f' is increasing on (a, b);
- **concave down** on an open interval (a, b) if f' is decreasing on (a, b)

Note:

- If the graph of f is **concave up** ("cup"), it is **above** its tangents.
- If the graph of f is **concave down**, it is **below** its tangents.

Concave up and down functions





Concave up and down functions

Relating concavity to f''

Let $y = f(x)$ be twice-differentiable on an open interval (a, b).

- If $f'' > 0$ on (a, b), the graph of f is **concave up**
- If $f'' < 0$ on (a, b), the graph of f is **concave down**

Example: $f(x) = x^2$ is concave up (for all x) and $g(x) = -x^2$ is concave down.

Inflection points

Definition: infection point

A **point of inflection** is a point at which the concavity of a function changes.
At such a point, either f'' is zero or does not exist.

Example

Find a point of inflection of the graph of $f(x) = x^3$.

Inflection points

Warning: Having $f''(c) = 0$ does not necessarily mean that f has an inflection point at $x = c$.

Example

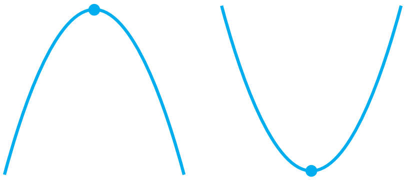
The curve $y = x^4$ has no inflection point at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.

Second derivative test

Second Derivative Test

Suppose that f'' is continuous on an interval that contains c.

- If $f'(c) = 0$ and $f''(c) < 0$, then f has a **local max** at $x = c$.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a **local min** at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive. The function f may have a local max, a local min, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Second derivative test

Example

Find and classify the critical and infection points of

$$f(x) = 4x^3 - 21x^2 + 18x + 6.$$

We have $f'(x) = 12x^2 - 42x + 18$.

When $f'(x) = 0$, we have

$$\begin{aligned} 12x^2 - 42x + 18 = 0 &\Leftrightarrow 2x^2 - 7x + 3 = 0 \\ &\Leftrightarrow (2x-1)(x-3) = 0. \end{aligned}$$

So the critical points are at $x = \frac{1}{2}$ and $x = 3$.
Next $f''(x) = 24x - 42$ so

$$f''(\frac{1}{2}) = 24(\frac{1}{2}) - 42 = 12 - 42 < 0,$$

Second derivative test

which means we have a local max. at $x = \frac{1}{2}$. Also, we have a local min. at $x = 3$ because

$$f''(3) = 24(3) - 42 = 72 - 42 > 0.$$

Now, recall that $f''(x) = 24x - 42$. Thus,

$$f''(x) = 0 \Leftrightarrow x = \frac{42}{24} = \frac{7}{4}.$$

Note that

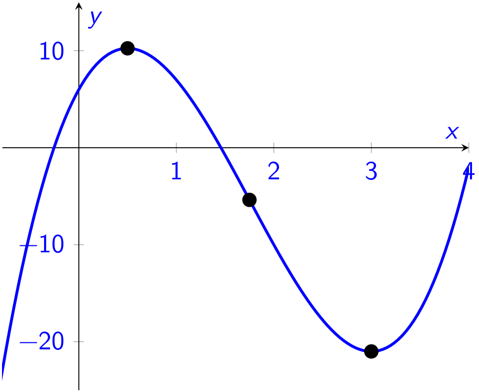
$$x < \frac{7}{4} \implies f''(x) < 0$$

$$x > \frac{7}{4} \implies f''(x) > 0.$$

Therefore, $f(x)$ has a point of inflection at $x = \frac{7}{4}$.

Second derivative test

(This figure was added after class).



Second derivative test

Review

If a function g is differentiable on an interval (a, b), then

- $g'(x) > 0$ for all $x \in I \Leftrightarrow g$ increasing on (a, b).
- $g'(x) < 0$ for all $x \in I \Leftrightarrow g$ decreasing on (a, b).
- $g''(x) > 0$ for all $x \in I \Leftrightarrow g$ concave up on (a, b).
- $g''(x) < 0$ for all $x \in I \Leftrightarrow g$ concave down on (a, b).

First Derivative Test. If the sign of g' changes at a critical point, then we have a local max/min.

Second Derivative Test. If g concave down/up at a critical point, then we have a local max/min.

If $g''(c) = 0$ at a critical point c, then the test is inconclusive.

Curve Sketching

In order to roughly **sketch the graph** of a function, f, we can use the following steps:

1. Compute $f'(x)$ and find the critical (stationary) points and inflection points of f. Find the corresponding y-value of these points.
2. If necessary, compute $f''(x)$, and use the second derivative test (optional).
3. Make a table showing the intervals on which f is increasing and/or decreasing, and where f is concave up and/or concave down.
4. Plot some specific points (e.g. local max/ min, points of inflection, intercepts) and sketch the general shape of the graph of f.

Curve Sketching

Example

Sketch the graph of the function $f(x) = x^4 - 4x^3 + 10$

Step 3: Make table to find intervals on which f is increasing/decreasing and on which f is concave up and concave down

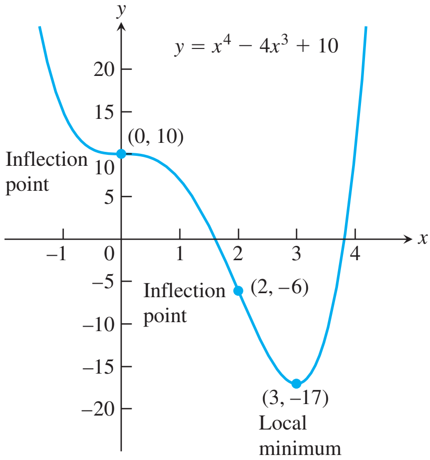
		0		2		3	
$4x^2$	$+$	$\bullet$	$+$		$+$		$+$
$x-3$	$-$		$-$		$-$	$\bullet$	$+$
$f'(x)$	$-$	$\bullet$	$-$		$-$	$\bullet$	$+$
$12x$	$-$	$\bullet$	$+$		$+$		$+$
$x-2$	$-$		$-$	$\bullet$	$+$		$+$
$f''(x)$	$+$	$\bullet$	$-$	$\bullet$	$+$		$+$

Curve Sketching

Step 4: Plot specific points (such as local max/min, points of inflection, intercepts) - and sketch general shape of graph of f

Curve Sketching

Step 5: Plot specific points (such as local max/min, points of inflection, intercepts) - and sketch general shape of graph of f



Exercise

Exer 6.1.1 (Based on 2019/20 Exam, Q3(a))

Let $f(x) = x^3 - 3x^2$.

1. Find all asymptotes of the graph $f(x)$
2. Determine the interval(s) on which $f(x)$ is increasing and decreasing.
3. Determine the interval(s) on which $f(x)$ is concave up (convex) and concave down (or concave).
4. Find all point(s) of inflection for the graph of $f(x)$.
5. Give a rough sketch the graph of $f(x)$ (your axes need not necessarily have the same scale).