2526-MA140: Week 03, Lecture 1 (L04)

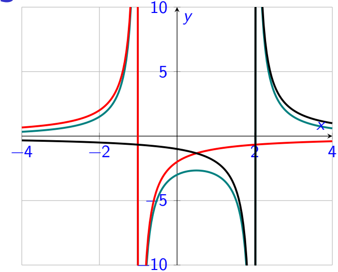
Partial Fractions

Dr Niall Madden

University of Galway

Tuesday, 23 September, 2025

Annotated slides



Outline

Slides are on canvas, and at

https://www.niallmadden.ie/2526-MA140/

Tutorials start **this** week. The schedule is:

Teams	Time	Venue	Leader
1, 2	Tuesday 15:00	ENG-**2003**	ST
3, 4	Tuesday 15:00	ENG-**2034**	JM
9, 10	Thursday 11:00	ENG-**2002**	ST
11, 12	Thursday 11:00	ENG-**3035**	JM
5, 6	Friday 13:00	Eng-**2002**	ST
7, 8	Friday 13:00	Eng-**2035**	JM

Rang teagaisc trí Ghaeilge (Irish tutorial): Dé Máirt (Tuesday) 15:00, Áras na Gaeilge 221.

- There is currently a "practice" assignment open. See https://universityofgalway.instructure.com/courses/46734/assignments/128373
- During tutorials, the tutor will solve some similar questions. You can access the **tutorial sheet** at https://universityofgalway.instructure.com/courses/46734/files/2842617?module_item_id=925893. You can also access this through the Canvas page: `Modules`... `Tutorial Sheets`.
- **Assignment 1** will be due 5pm, Monday 5 October.

. . . .

Also: try the exercises at the end of each set of lecture slides: they are similar in style and standard to exam questions.

News!

There are two class test planned for this module:

- MCQ format;
- both worth 10% of the final grade;
- Test 1: **Tuesday, 14 October** (Week 5)
- Test 2: **Tuesday, 18 November** (Week ~~9~~ 10)
- Contact Niall if you have any concerns, or wish to avail of alternative arrangements, as provided by LENS reports.

Partial Fractions

Rational Functions have the general form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

An (proper) rational function can often be written as a sum of simpler ones: **partial fractions.**

For example

$$\frac{8x-12}{x^2-2x-3}$$

can be written as

$$\frac{3}{x-3} + \frac{5}{x+1}$$

Simpler = denominators have lower degree

Check: (*next slide*)

Partial Fractions

Show that

$$\frac{8x-12}{x^2-2x-3} = \frac{3}{x-3} + \frac{5}{x+1}.$$

First, show

$$(x-3)(x+1) = x^2 - 3x + x - 3$$

$$= x^2 - 2x - 3$$

Next

$$\frac{3}{x-3} + \frac{5}{x+1} = \frac{3(x+1)}{(x-3)(x+1)} + \frac{5(x-3)}{(x+1)(x-3)}$$

$$= \frac{3x+3}{x^2-2x-3} + \frac{5x-15}{x^2-2x-3} = \frac{8x-12}{x^2-2x-3}$$

Partial Fractions

Note: Any polynomial (with real coefficients) can be factorised fully into the product of

- linear
- and irreducible quadratic factors.

Examples: $x^2 - 1 = (x-1)(x+1)$

so it is the product of 2 linear factors

---

x^2+1 is irreducible. (note $x^2 = -1$
(over the reals). & $x = \pm\sqrt{-1}$ not real)

We get different combinations of factors in the denominator. Let's look at **four cases**, and how to find the partial fractions in each case.

The four cases

1. Denominator has **linear factors to the power of 1**
2. Denominator has **factors to the power greater than 1 (i.e repeated linear factors).**
3. Denominator has irreducible quadratic factors.
4. Denominator has irreducible quadratic factors to power greater than 1.

1: $\frac{1}{(x-1)(x+1)} = \frac{1}{x^2-1}$

2. $\frac{1}{(x+1)(x+1)} = \frac{1}{x^2+2x+1}$

3. $\frac{1}{x^2+1}$

4) $\frac{1}{(x^2+1)^2} = \frac{1}{x^4+2x^2+1}$

Case 1: Linear factors to the power of 1 in the denominator.

We have **two methods** to find A and B.

Method 1: Comparing coefficients (ie match powers of x)

Example

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}. \quad \text{Find } A, B.$$

$$\frac{3x}{(x-1)(x+2)} = \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x+2)(x-1)}$$

$$= \frac{Ax+2A+Bx-B}{(x-1)(x+2)} = \frac{(A+B)x+(2A-B)}{(x-1)(x+2)}$$

So $A+B=3$
$2A-B=0$ $\overset{\text{solve}}{\Longrightarrow}$ $\boxed{A=1,\ B=2.}$

Method 2: Substituting specific values for x.

Recall $$\frac{3x}{(x-1)(x+2)} = \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x-1)(x+2)}.$$

So $$3x = A(x+2) + B(x-1) \quad \text{for all } x.$$

Set $x=1$ Then $3(1) = A(3) + B(0) \Rightarrow \boxed{A=1.}$

Set $x=-2$ Then $3(-2) = A(-2+2) + B(-2-1)$

$$\Rightarrow -6 = A(0) - 3B$$

$$\boxed{B=2}$$

Example

Write $\dfrac{8x-12}{x^2-2x-3}$ as sum of partial fractions. / From Earlier

Step 1: factorize $x^2-2x-3 = (x-3)(x+1)$.

So find A, B such that

$$\frac{8x-12}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

So

$$8x-12 = A(x+1) + B(x-3).$$

(a) set $x=3 \Rightarrow 8(3)-12 = A(4) + B(0)$

$\Rightarrow 4A = 12 \Rightarrow A = 3$

(b) set $x=-1 \Rightarrow B = 5$ (Check!)

(2) Linear factors to the power greater than 1 in the denominator, (i.e repeated linear factors).

If $(x-\alpha)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1}{x-\alpha} + \frac{A_2}{\underline{(x-\alpha)^2}} + \dots + \frac{A_k}{(x-\alpha)^k}$$

liner quatrabic.

Example

Find A, B and C such that

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Power 2

(Note: we'll find that $A = 5/9$, $B = 4/3$ and $C = -5/9$).

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)}$$

So $3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$.

Solve for A, B, C.

So
$$3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$
Solve for A, B, C.

(a) $x=1$ $\Rightarrow 3(1)+1 = A(1-1)(x+2) + B(1+2) + C(1-1)^2$

$\Rightarrow 4 = A(0) + 3B + C(0)^2 \Rightarrow \boxed{B = \frac{4}{3}}.$

(b) $x=-2$ $\Rightarrow 3(-2)+1 = A(0) + B(0) + C(-3)^2$

$\Rightarrow -5 = 9C \Rightarrow \boxed{C = -\frac{5}{9}}$

We now have
$$3x+1 = A(x-1)(x+2) + \left(\tfrac{4}{3}\right)(x+3) + \left(-\tfrac{5}{9}\right)(x-1)^2$$
Now match powers of x^2 . . .
$$0 = A + \left(-\tfrac{5}{9}\right) \Rightarrow A = \tfrac{5}{9}.$$

(3) Irreducible quadratic factors.

Irreducible quadratic factors can not be factorised using real numbers, e.g. $x^2 + x + 1$.

An irreducible quadratic factor $ax^2 + bx + c$ gives rise to partial fractions of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example 2.34 from textbook

If one writes

$$\frac{5x}{(x^2 + x + 1)(x - 2)} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 2}$$

then we find $A - 10/7$, $B = 5/7$ and $C = 10/7$.

(4) Irreducible quadratic factors to power greater than 1.

Each repeated irreducible quadratic factor $(ax^2+bx+c)^k$ in the denominator will give rise to

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}.$$

These can be done in a similar way to the previous case. But the calculations are pretty messy, so we won't even try!

Finished here.

Towards Limits

When we were considering the domain of a function, we looked at those x-values for which the function was not defined.

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad\qquad g(x) = \frac{x^2 - 1}{x - 1}$$

Neither f nor g are defined at $x = 1$.

But what happens if x gets very closed to 1?

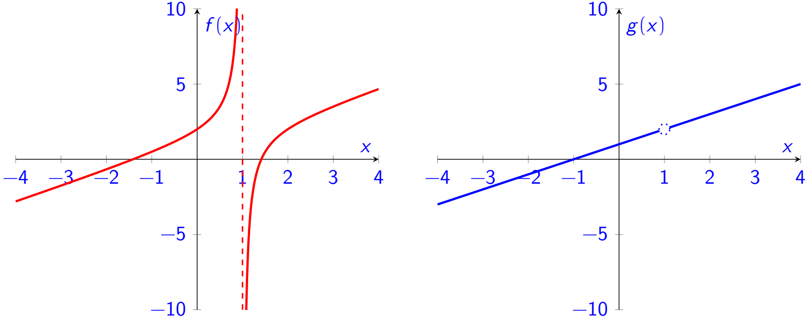
x	0.900	0.990	0.999	1	1.001	1.010	1.100
$f(x)$							
$g(x)$							

Let's look at the graphs of f and g.

Towards Limits

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1}$$



Towards Limits

In the previous example, we saw that, although neither f nor g was defined at $x = 1$, they behaved very differently as x approaches 1.

To discuss this we'll need the concept of a **limit** which, roughly, relates to the value of function as it **approaches** a point (but not actually at that point).

$$\lim_{x \to a} f(x) = L$$

The concept of a limit is a prerequisite for a proper understanding of calculus and numerical methods.

Exercises

Exercise 2.1.1

Find the constants A, B and C, so that

$$\frac{2x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

Exercises

Exercise 2.1.2

Express the following as partial fractions.

1. $\dfrac{6}{x^2 - x - 2}$
2. $\dfrac{2x - 1}{x^2 - x - 2}$
3. $\dfrac{x - 1}{(x+1)(x^2 - x - 2)}$
4. $\dfrac{x}{x^2 + 2x + 1}$
5. $\dfrac{1}{x^3 - 1}$

Exercises

Exercise 2.1.3 (MA140 Exam, 24/25)

Express $\dfrac{3x+1}{x^2-x-2}$ as partial fractions.