MA385: Assignment

This assignment contributes 10% to your final grade for MA385. Your solutions must be clearly written (ideally, by hand), and neatly presented. For numerical calculations (such as in Q3), you make use a calculator or computer; details of the calculations are not required. Write your name and ID number on the first page. Number every page. Upload a scanned version to canvas at universityofgalway.instructure.com/courses/46945/assignments/130676.
If you really want, you can submit a hard copy at the lecture on the 20th or 23rd. If doing so, staple the pages together, and write your name and ID number at the top of each page.
The assignment is collaborative*: you are allowed work with classmates. However, each of you needs to submit your own hand-written version of your solutions, and you should include a clear statement on who you collaborated with, and on what aspects.*

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Q1. Suppose that we have a fixed point iteration (FPI) method $x_{k+1} = g(x_k)$ which we know to be converges to fixed point of g, denoted τ. Show that the method converges with at least order p if

$$g'(\tau) = g''(\tau) = \dots g^{(p-1)} = 0$$

Q2. Suppose we want to estimate $\sqrt{3}$, via FPI using a method of the form

$$x_{k+1} = ax_k + b/x_k, \qquad \text{for } k = 0, 1, 2, \dots. \tag{1}$$

Show that one needs to choose $b = 3 - 3a$.

Q3. Take $a = 3/4$ in (**??**).

(a) Show that the resulting g is a contraction on $[1, 3]$.
(b) Take $x_0 = 1.5$, and compute the corresponding values of x_1, x_2 and x_3. Compute the errors $\mathcal{E}_k = |\tau - x_k|$ for $k = 0, 1, 2, 3$.
(c) Show that $\mathcal{E}_{k+1}/\mathcal{E}_k$ is roughly constant. What can we infer from that?

Q4. Determine the values of a and b in (**??**) that would correspond to Newton's method applied to solving $x^2 - 3 = 0$. Show that this method converges with at least order 2.

Q5. Is it possible to determine values for a and b in (**??**) for which the corresponding method converges with at least order 3 (using the result in Q**??**)? Explain your answer.

Q6. When preparing this assignment, I asked a generative AI model to propose some fixed point methods for estimating $\sqrt{3}$. Its suggestions included taking

$$x_{k+1} = \frac{3}{x_k + 1} \qquad \text{and} \qquad x_{k+1} = x_k + \frac{3 - x_k^2}{2\sqrt{3}}.$$

Both of these are bad. Explain in one line why this is so.

Q7. In his seminal paper of 1901, Carl Runge gave the following example of what we now call a *Runge-Kutta 2 (RK2) method*:

$$\Phi(t_i, y_i; h) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\big).$$

(i) Show that it is consistent.
(ii) Show how this method fits into the general framework of RK2 methods. That is, what are a, b, α, and β? Do they satisfy the conditions

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b?$$