Symmetric Positive Definite (SPD) Matrices

Definition 1 (SPD)**.** A matrix, $A \in M_{n\times n}(\mathbb{R})$, is *symmetric positive definite* (SPD) if and only if

(a) $A = A^T$ (i.e., it is symmetric)

(b) and $\vec{x}^T A\vec{x} > 0$ for all vectors $\vec{x} \neq 0$.

Often it is easy to check if a matrix is *not* SPD: just find one x for which $x^T Ax \leqslant 0$. Verifying a matrix *is* SPD may take a little more work.

Example 2. Which of these matrices is SPD?

$$(i)\ \ A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad (ii)\ \ A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

SPD matrices have lots of important properties. One of the more obvious ones is that such a matrix is nonsingular. If A were SPD and singular, then there would exist some vector x such that $Ax = 0$, and so $x^T Ax = 0$. This is not possible since $x^T Ax > 0$ for all x.

SPD matrices occur in lots of applications. In particular, if there is some $n \times n$ matrix C for which $A = C^T C$, then A is SPD. Such matrices occur in, for example, the solution of least-squares problems by the normal equations.

Here are some more properties of SPD matrices, which may be a little less obvious, but which can be really useful in various applications.

For Part (b) we need the following definition.

Definition 3 (Principal submatrix)**.** The $k \times k$ matrix B is called *principal submatrix* of A if it can be obtained by removing all but k particular rows and column of A (that is, if we removed row r of A then we also removed column r). B is called a *leading principal submatrix or order* k if (in MATLAB notation) $B = A(1 : k, 1 : k)$.

Theorem 4. (a) Let A and B both be real $n \times n$ matrices. If B^{-1} exists, then A is SPD $\iff$ $B^T AB$ is SPD.

(b) If A is SPD, then any principle submatrix of A is SPD.

(c) A is SPD if and only if $A = A^T$ and all the eigenvalues of A are positive.

(d) If A is SPD, then $a_{ii} > 0$ for all i, and $\max_{ij} |a_{ij}| = \max_i a_{ii}$.

(e) A is SPD of the determinant of every leading principal submatrix is positive.

(f) A is SPD $\iff$ there exists a unique lower triangular matrix L with positive diagonal entries such that $A = LL^T$. *This is called the Cholesky factorisation of* A.

In class we'll step through the proofs most parts of Theorem 4, except for Parts (e) and (f). For Part (e) it is easy to prove that the determinant of any principal submatrix of A is positive (the determinant of a matrix is the product of its eigenvalues). The converse is a little trickier, and usually involves an *interlacing* theorem.

Part (f) is very important in other contexts, but we probably won't get to it.

Definition 5 (Positive semi-definite)**.** A matrix, $A \in M_{n\times n}(\mathbb{R})$, is *symmetric positive semi-definite* if and only if $A = A^T$ (i.e., it is symmetric) and $\vec{x}^T A\vec{x} \geqslant 0$ for all vectors $\vec{x} \neq 0$.

We encounter semi-definite matrices in, for example, areas of graph theory.