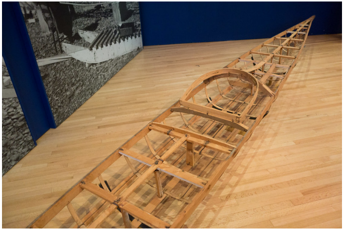
MA378 Chapter 2: Splines

§2.2 (Natural) Cubic Splines

Dr Niall Madden

Start: 6 February 2026 (W04.2)



A detail from an Inuit Kayak frame, Royal Ontario Museum, Toronto, Canada

2.0 In this section:

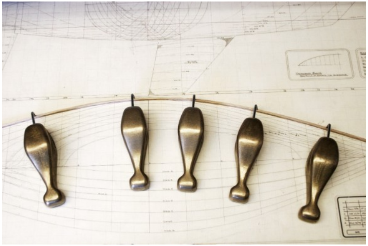
2.1 Introduction

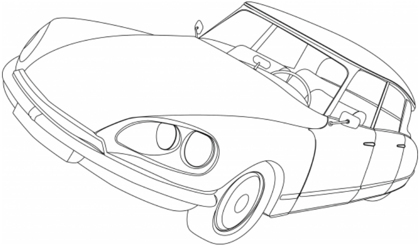
The **Cubic Spline** is perhaps the most useful and popular interpolating function used in numerical analysis, boat building, automotive and aeronautical engineering, computer and film animation, digital photography, financial modelling, and as many other areas as there are human endeavours where continuous processes are modelled by discrete ones.

2.2 Some History

History

It is generally accepted that (mathematical) splines were first described by Isaac Schoenberg (1903-1990) during WW2. However their physical realisation had been used in the ship-building and aircraft industries prior to that. In the 1950s and 1960s major advances were made in the areas of aircraft and car design.





2.2 Some History

For more on cubic splines, see

1. Section 11.4 of “An Introduction to Numerical Analysis”.
2. Lecture 11 of “Afternotes Goes to Graduate School”.

2.2 Some History

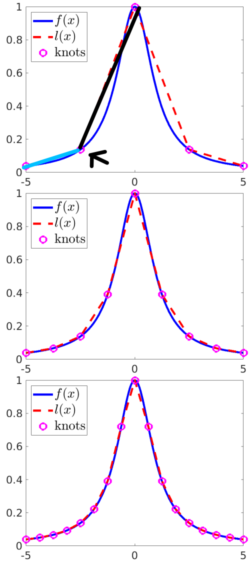
One could argue that the most fundamental short-coming of the linear spline interpolant to f is that they are not "***smooth***": that is, although

$$\underline{l_i(x_i) = l_{i+1}(x_i)},$$

it is generally the case that

$$l_i'(x_i) \neq l_{i+1}'(x_i).$$

Also, the interpolating function cannot capture the "***curvature***" of f. (If you think about this last statement, you'll see that it can be expressed as $l''(x) = 0$ for all $x \in [a, b]$.)



2.3 Definition

Cubic splines try to balance the simplicity of the linear spline approach — by using a low-order polynomial on each interval — with the desire for curvature and more smoothness – by using cubics, and by forcing adjacent ones, and their first and second derivatives, to agree at knot points.

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2.3 Definition

Definition 2.1 (Cubic Spline)

Let f be a function that is continuous on $[a, b]$. The *cubic spline interpolant* to f is the continuous function S such that

(i) for $i = 1, \ldots, N$, on each interval $[x_{i-1}, x_i]$ let $S(x) = s_i(x)$, where each of the s_i is a cubic polynomial.

(ii) $s_i(x_{i-1}) = f(x_{i-1})$ for $i = 1, \ldots, N$,

(iii) $s_i(x_i) = f(x_i)$ for $i = 1, \ldots, N$,

(iv) $s_i'(x_i) = s_{i+1}'(x_i)$ for $i = 1, \ldots, N-1$,

(v) $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, \ldots, N-1$.

$4N-2$ equations

Each $s_i(x)$ is a cubic, so has 4 coef. So there are $4N$ in total.

2.3 Definition

So, we have defined the cubic spline S as a function that...

(a) is a cubic polynomial on each of the $\overset{N}{\cancel{n}}$ intervals $[x_{i-1}, x_i]$. That is, it is ***piecewise cubic***.

(b) interpolates f at $N + 1$ points,

(c) has continuous first derivatives on $[x_0, x_N]$

(d) has continuous second derivatives on $[x_0, x_N]$

2.3 Definition

We know that one can write a cubic as $a_0 + a_1x + a_2x^2 + a_3x^3$. So it takes 4 terms to uniquely define a single cubic. To define the spline we need $4N$ terms. They can be found by solving $4N$ (linearly independent) equations. But the definition only gives $4N - 2$ equations.

So we are short two equations.

2.3 Definition

The “missing”equations can be chosen in a number of ways.

(i) **By setting $S''(x_0) = 0$ and $S''(x_N) = 0$. This is called a *natural* spline, and is the approach we'll take.**

(ii) By setting $S'(x_0) = 0$ and $S'(x_N) = 0$. This is called a *clamped* spline.

(iii) By setting $S'(x_0) = S'(x_N)$ and $S''(x_0) = S''(x_N)$. This is the *periodic* spline and is used for interpolating, say, trigonometric functions.

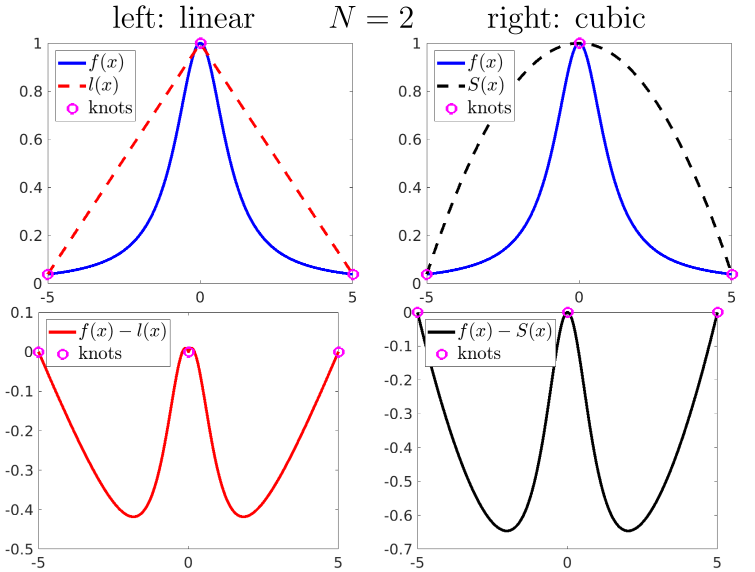
(iv) Only use $N - 2$ components of the spline: $s_2, \ldots, s_{N-1}$. But extend to two end ones so that $s_2(x_0) = f(x_0)$ and $s_{N-1} = f(x_N)$. This is called the ***not-a-knot*** condition.

2.3 Definition

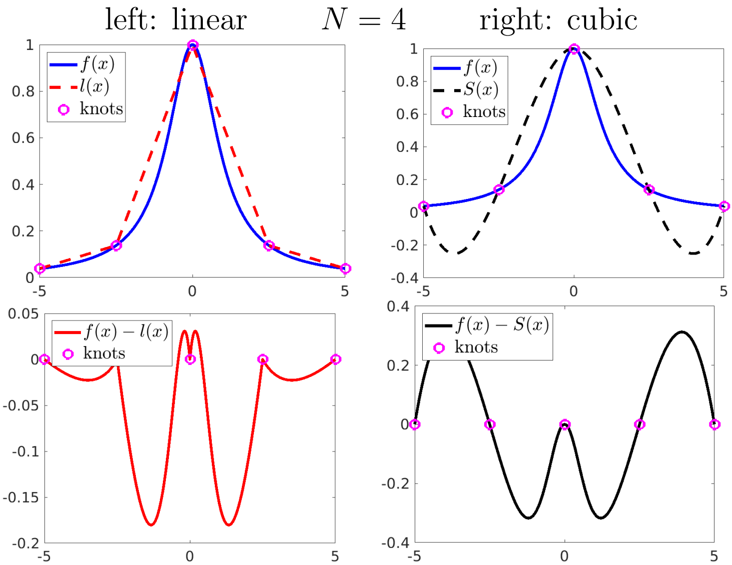
The following figures show the **linear spline** (left) and ***natural cubic spline*** (right), and errors, interpolants to $f(x) = 1/(1 + x^2)$ (with $a = -5$ and $b = 5$ as usual) for various N.

Compared with linear splines, we seem to get significantly better approximation. Moreover, the rate at which the error decreases with respect to h is much faster.

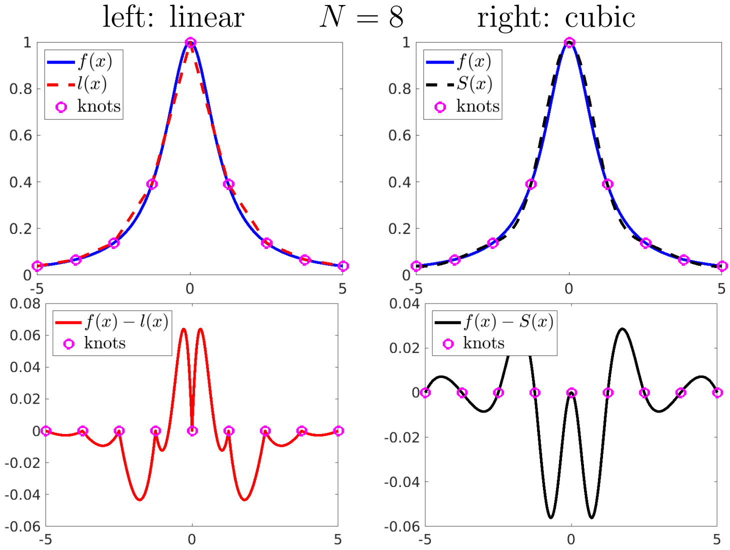
2.3 Definition



2.3 Definition

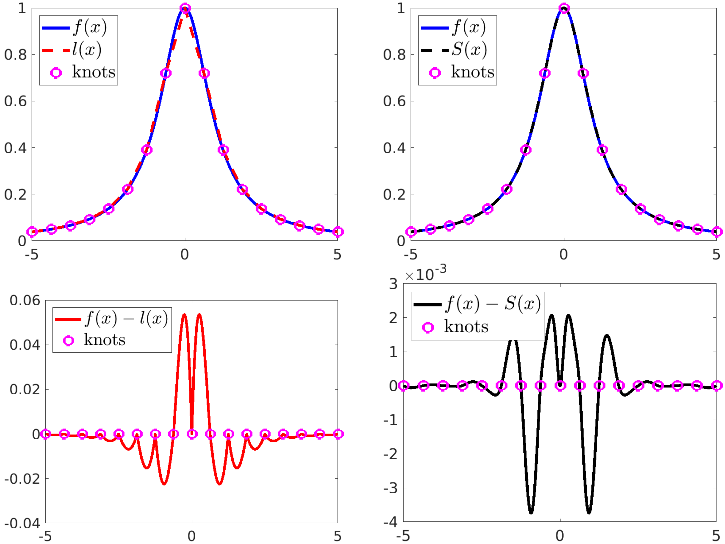


2.3 Definition

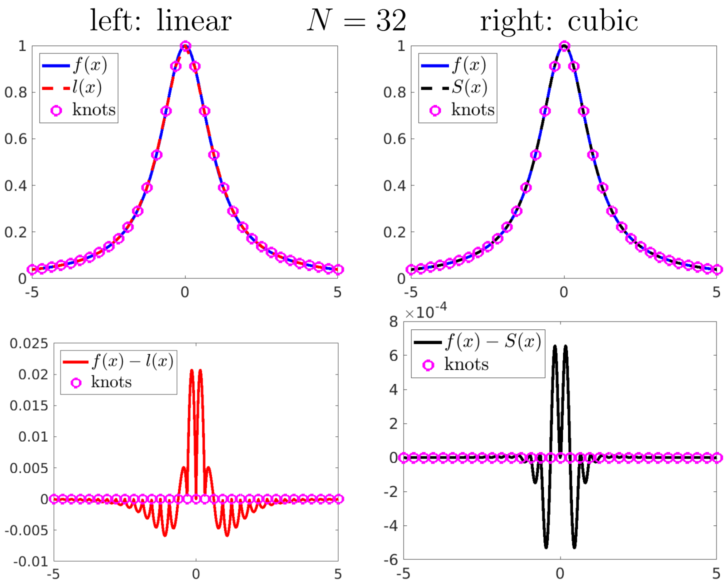


2.3 Definition

left: linear $\quad N = \cancel{8}\ 16 \quad$ right: cubic



2.3 Definition



2.4 Constructing Cubic Splines

Some notation:

- $h = x_i - x_{i-1} = (x_N - x_0)/N$ for all i.
- $f_i := f(x_i)$.

To construct the spline, first observe that if S is piecewise cubic, then S' is piecewise quadratic, and S'' is piecewise linear.

(i) Let $\sigma_i = S''(x_i)$ for $i = 0, \ldots, N$.

Since $S''(x)$ is piecewise linear, we can write each $S_i''(x)$ as

$$S_i''(x) = \sigma_{i-1} \frac{x_i - x}{h} + \sigma_i \frac{x - x_{i-1}}{h}$$

where the σ_i are parameters to be determined. (Note there are $N+1$ σ's).

constant of int.

(ii) Integrate twice:

Since $s_i''(x) = \sigma_{i-1}\frac{(x_i - x)}{h} + \sigma_i\frac{(x - x_{i-1})}{h}$

we have $s_i'(x) = -\sigma_{i-1}\frac{(x_i - x)^2}{2h} + \sigma_i\frac{(x - x_{i-1})^2}{2h} + A_i$

And $s_i(x) = \frac{\sigma_{i-1}(x_i - x)^3}{6h} + \frac{\sigma_i(x - x_{i-1})^3}{3h} + A_i x + B_i$

We rewrite $A_i x + B_i$ as $\alpha_i(x - x_{i-1}) + \beta_i(x_i - x)$

We get

$$s_i(x) = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) + \frac{\sigma_{i-1}}{6h}(x_i - x)^3 + \frac{\sigma_i}{6h}(x - x_{i-1})^3, \quad (1)$$

for $x \in [x_{i-1}, x_i]$.

(iii) The α_i and β_i arose as the constants of integration. To find them:

Lets use that $s_i(x_{i-1}) = f_{i-1}$ & $s_i(x_i) = f(x_i)$

$$s_i(x_{i-1}) = f_{i-1} \Rightarrow h\beta_i + \frac{h^2}{6}\sigma_{i-1} = f_{i-1}$$

$$s_i(x_i) = f_i \Rightarrow h\alpha_i + \frac{h^2}{6}\sigma_i = f_i$$

So, since the f(x_i) terms are given, if we know the simga_i terms, we can calculate the values of alpha_i and beta_i.

and so, for $i = 1, 2, \ldots, N$

$$\alpha_i = \frac{f_i}{h} - \frac{h}{6}\sigma_i, \quad \beta_i = \frac{f_{i-1}}{h} - \frac{h}{6}\sigma_{i-1}. \tag{2}$$

(Note that $\beta_i = \alpha_{i-1}$.)

2.4 Constructing Cubic Splines

(iv) So, now we "just" need the equations for σ_i. Two of these come from the fact that this is a "natural" cubic spline, with $S''(x_0) = 0$ and $S''(x_N) = 0$, therefore $\sigma_0 = 0$ and $\sigma_N = 0$.
The remaining $N - 1$ equations come from the fact that S' is continuous at $x_1, x_2, \ldots, x_{N-1}$. So we get that

$$s_i'(x_i) = s_{i+1}'(x_i).$$

After some work (see exercises...), we can show this means that the system is:

$$\sigma_0 = 0, \tag{3a}$$

$$\frac{1}{6}(\sigma_{i-1} + 4\sigma_i + \sigma_{i+1}) = \frac{1}{h^2}\big(f_{i-1} - 2f_i + f_{i+1}\big) \tag{3b}$$

for $i = 1, \ldots, N-1$,

$$\sigma_N = 0. \tag{3c}$$

2.5 A cubic spline example

Example 2.2

Find the natural cubic spline interpolant to f at the points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ where $f_0 = 0, f_1 = 2, f_2 = 1$ and $f_3 = 0$.

[We won't do all the details in class, so there are included below]

2.5 A cubic spline example

Solution:

From (1) we see we are looking for $S(x)$=

$$S_1(x) = \alpha_1 x + \beta_1(1-x) + \frac{\sigma_0}{6h}(1-x)^3 + \frac{\sigma_1}{6h}x^3,\ x \in [0,1],$$

$$S_2(x) = \alpha_2(x-1) + \beta_2(2-x) + \frac{\sigma_1}{6h}(2-x)^3 + \frac{\sigma_2}{6h}(x-1)^3,\ x \in [1,2],$$

$$S_3(x) = \alpha_3(x-2) + \beta_3(3-x) + \frac{\sigma_2}{6h}(3-x)^3 + \frac{\sigma_3}{6h}(x-2)^3,\ x \in [2,3].$$

2.5 A cubic spline example

We first solve for the σ_i using (3):

$$\frac{1}{6}\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 4 & 1 & 0\\ 0 & 1 & 4 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \begin{pmatrix}0\\ -3\\ 0\\ 0\end{pmatrix}$$

This gives

$$\sigma_0 = 0, \quad \sigma_1 = -\frac{24}{5}, \quad \sigma_2 = \frac{6}{5}, \quad \sigma_3 = 0.$$

Now use (2) to get

$$\alpha_1 = \frac{14}{5}, \quad \alpha_2 = \frac{4}{5}, \quad \alpha_3 = 0.$$

and

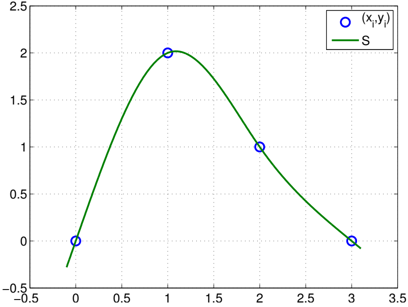
$$\beta_1 = 0 \quad \beta_2 = \frac{14}{5}, \quad \beta_3 = \frac{4}{5}.$$

2.5 A cubic spline example

The answer is

$$S(x) = \begin{cases} \frac{14}{5}x - \frac{4}{5}x^3, & x \in [0,1], \\ \frac{4}{5}(x-1) + \frac{14}{5}(2-x) - \\ \qquad\qquad \frac{4}{5}(2-x)^3 + \frac{1}{5}(x-1)^3, & x \in [1,2], \\ \frac{4}{5}(3-x) + \frac{1}{5}(3-x)^3. & x \in [2,3]. \end{cases}$$

This is shown below.



2.6 Error Estimates

We state the following error estimates without proof. You don't need to know these, or be able to prove them. But you should be able to use them if required.

Theorem 2.3

If $f \in C^4[a,b]$ and S is its cubic spline interpolant on $N+1$ equally spaced points, then

$$\|f - S\|_\infty \leq \frac{5}{384} M_4 h^4,$$
$$\|f' - S'\|_\infty \leq \frac{1}{24} M_4 h^3,$$
$$\|f'' - S''\|_\infty \leq \frac{3}{8} M_4 h^2,$$

where $M_4 = \|f^{(4)}\|_\infty$.

2.6 Error Estimates

Example 2.4

Give an upper bound on the error for the cubic spline interpolant to $f = e^x$ on the interval $[\mathbf{-1}, \mathbf{1}]$ with $N = 10$ mesh points.

2.6 Error Estimates

Example 2.5

What is the smallest value of N that you must take to ensure that, if interpolating $f(x) = e^x$ on N equally sized intervals on $[-1, 1]$, the error is less that 10^{-8}? How does this compare with a linear spline interpolation (see corresponding example for linear splines).

2.6 Error Estimates

Recall the minimum energy property of linear splines. There is an analogous result for natural cubic splines.

Theorem 2.6

Let u be any function that interpolates f at ω^N, and is such that $u \in H^2(x_0, x_N)$. Then

$$\|S''\|_2 \leq \|u''\|_2.$$

The proof is not hard - it is analogous to the proof of Theorem 2.1.9.

2.7 Exercises

Exercise 2.1

[Equation numbers given here refer to those for the slides for Section 2.2.] When deducing the system of equations for the natural cubic spline, we showed how to construct the formulation in (1). and the relationship between σ_i, α_i and β_i in (2). Now carefully show to deduce the system (3).

Exercise 2.2

(For students who did MA385). Write the equations in (3) as a matrix-vector equation $A\sigma = \mathbf{b}$, where A is an $n \times n$ matrix. Show that A is nonsingular, and hence that the system as a unique solution.

Exercise 2.3

Find the natural cubic spline interpolant to $f(x) = \sin(\pi x/2)$ at the nodes $\{x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3\}$. Calculate value of the interpolant at $x = 2.5$. What is the error at this point?

2.7 Exercises

Exercise 2.4

Take $f(x) = \ln(x)$, $x_0 = 1$, $x_N = 2$. What value of N would you have to take to ensure that $|\ln(x) - S(x)| \leq 10^{-4}$ for all $x \in [1, 2]$, where S is the natural cubic spline interpolant to f.

Exercise 2.5

Suppose that S is a natural cubic spline on $[0, 2]$ with

$$S(x) = \begin{cases} -3x + 2(1-x) + a(1-x)^3 + \frac{2}{3}x^3, & x \in [0, 1), \\ b(2-x) + c(2-x)^3 + d(x-1)^3, & x \in [1, 2]. \end{cases}$$

Find a, b, c, and d.

Exercise 2.6

Suppose that S is a natural cubic spline on $[0, 2]$ with

$$S(x) = \begin{cases} 3x + a(1-x)^3 + bx^3, & \text{for } 0 \le x < 1, \\ c(2-x) - (2-x)^3 + d(x-1)^3, & \text{for } 1 \le x \le 2. \end{cases}$$

Find a, b, c, and d.