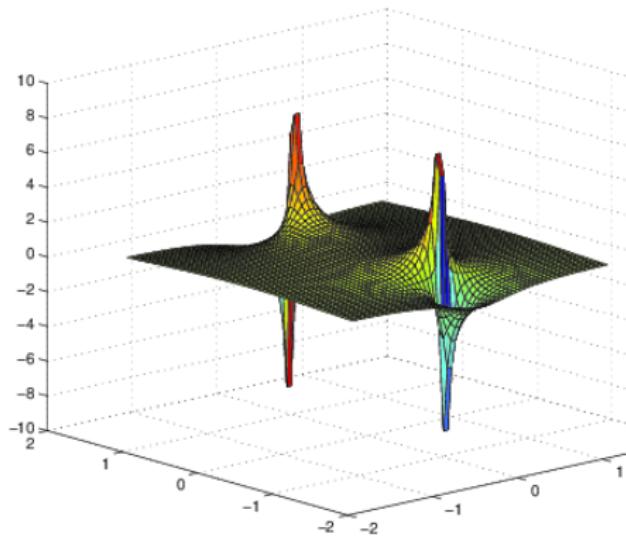


## MA378 Chapter 1: Interpolation

### §1.5 Wrap-up: Convergence & Runge's Example

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Start: 30 January  
2026 (W03.2)



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## 5.1 Convergence

The celebrated Weierstrass approximation theorem states that, given  $f$  and a positive number  $\varepsilon$ , there is a polynomial  $p$  such that

$$\|f - p\|_{\infty} := \max_{x \in [a, b]} |f(x) - p(x)| \leq \varepsilon.$$

That means: “**For any function,  $f$ , you can find a polynomial that approximates it as accurately as you would like**”.

Now suppose that  $f$  is a continuous function on  $[a, b]$  and that  $\{p_n\}_{n=0}^{\infty}$  is a sequence of polynomials that interpolate  $f$  at  $n + 1$  **equally spaced** points. One might be inclined to believe that

$$\lim_{n \rightarrow \infty} \|f - p_n\|_{\infty} = 0.$$

## 5.1 Convergence

Another way of thinking about this is recalling the error bound:

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|$$

we might expect that

$$\lim_{n \rightarrow \infty} \max_{x \in [a,b]} \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)| = 0.$$

In other words, we might think that, in order to find an interpolating polynomial that is as accurate as we would like, we just need to choose large enough  $n$ .

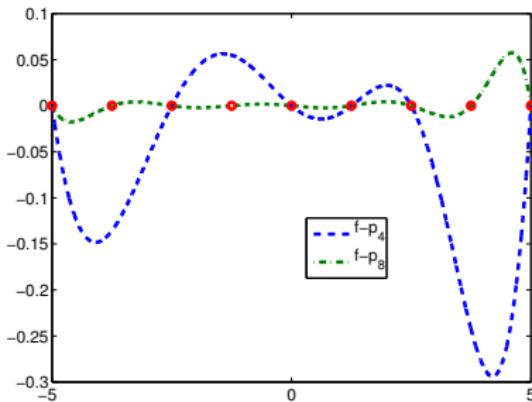
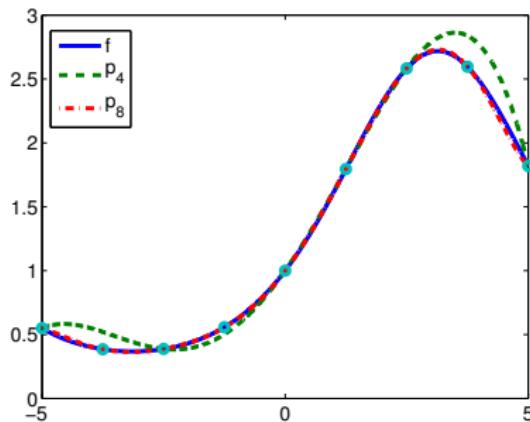
## 5.1 Convergence

And **some times** this is true. For example, suppose that  $a = -5$ ,  $b = 5$ , and  $f(x) = e^{\sin(x/2)}$ . In Table 1 the errors for successive interpolants are shown.

Table: Errors in polynomial interpolants to  $e^{\sin(x/2)}$  on  $[-5, 5]$

$n$	$\ f - p_n\ _\infty$
2	1.27e-00
4	2.94e-01
6	8.39e-02
8	5.75e-02
16	1.07e-03

## 5.1 Convergence



Polynomial interpolants,  $p_4$  and  $p_8$ , to  $e^{\sin(x/2)}$  on  $[-5, 5]$ , and their errors (right)

## 5.1 Convergence

However, there is a famous example of a simple function that cannot be successfully interpolated in this manner.

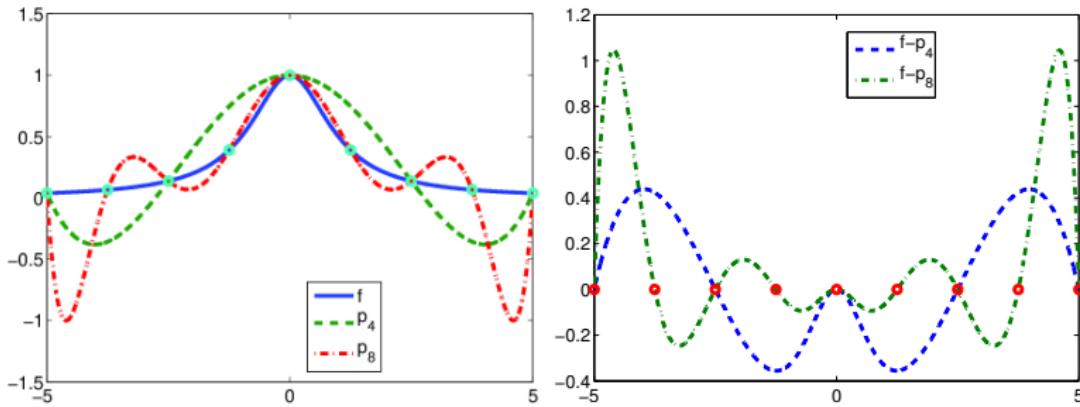
### Runge's Example

$$f(x) = \frac{1}{1+x^2} \quad \text{on } [-5, 5].$$

Errors for some  $n$  are shown below. Notice they *increase* with  $n$ .

$n$	$\ f - p_n\ $
2	0.65
4	0.44
6	0.62
8	1.05
16	14.39
20	59.66
22	122.91
24	257.21

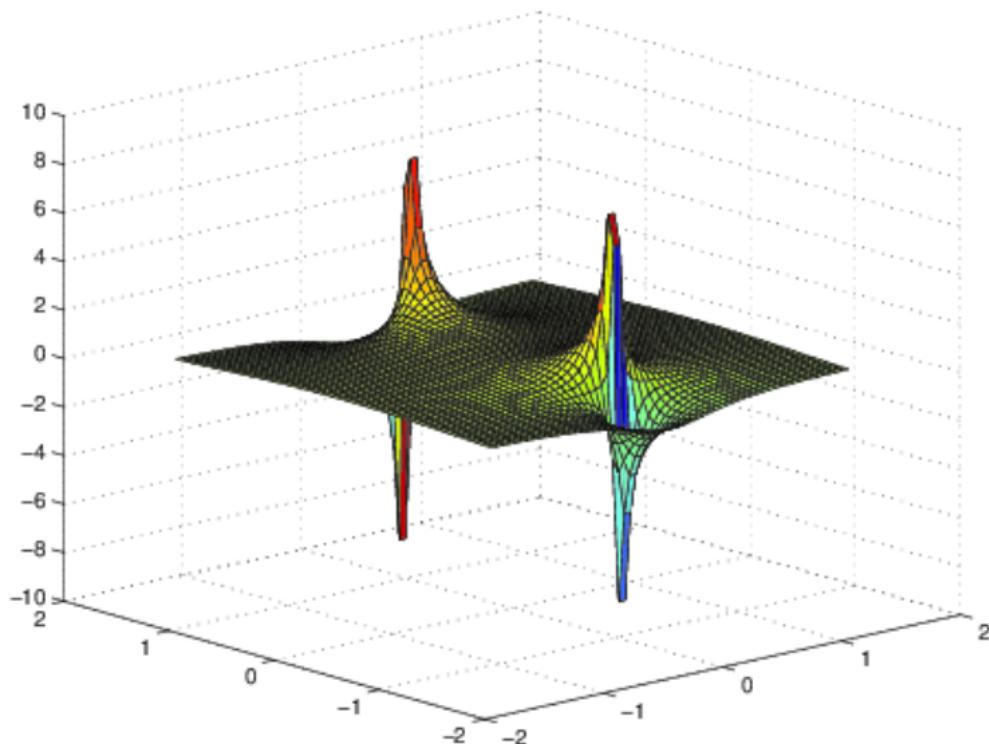
## 5.1 Convergence



*Polynomial interpolants to  $\frac{1}{1+x^2}$  on  $[-5, 5]$*

## 5.1 Convergence

## 5.1 Convergence



## 5.2 Where to from here?

So now it looks like polynomial interpolation is bad, at least on equidistant points.

However, in Lab 1 we'll do an exercise that might lead us to be more optimistic: it is possible to find a set of points that made the approximation as good we wanted (until round-off error dominated).

## 5.2 Where to from here?

Unfortunately, just because we have a good set of points for interpolating one particular function, it does not follow that that set is good for every continuous function: this is **Faber's Theorem**. This has often led numerical analysts to abandon the idea of interpolation by high-order polynomials completely.

However, there is a set of points that are useful, if  $f$  is smooth enough: the **Chebyshev** points of Lab 1. If you are interested, read the essay **Inverse Yogiisms** by Lloyd N. (Nick) Trefethen, Notices AMS, Dec. 2016. To investigate this numerically in MATLAB, try exploring the [Chebfun toolbox](#).

(I really do recommend you read that essay: it is beautifully written, and includes musings on polynomial interpolation, turbulence in fluid flows, and  $P = NP$ )

## 5.2 Where to from here?

The approach we will take is different. We say that if  $p_1$  is the polynomial of degree 1 that interpolates the function  $f$  at the points  $x_0$  and  $x_1$ , with  $h = x_1 - x_0$ , then

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

So, assuming  $M_2$  is bounded (which is reasonable), we can make  $p_1$  as close to  $f$  as we would like by taking a small enough interval  $[x_0, x_1]$ . The next section of this module is devoted to seeing how this can be used in theory and practice.