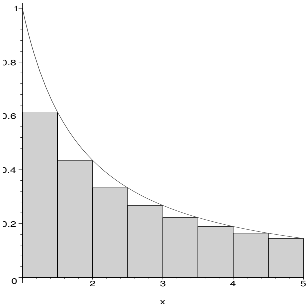
MA211

Lecture 16: Series Solutions. Integration

Mon 3^{rd} Nov 2008



Today...

1. Power Series
   - Initial Value problems
2. Integration
   - Preliminaries
3. Area Under a Curve
4. Definite Integrals
   - The Fundamental Theorem of Calculus
   - Examples
   - The Mathematical Tables

For more on *Series Solutions*, see Section 17.4 of Stewart *Calculus: early transcendentals*.

For further examples on *Integration*, have a look at Chapter 5, but especially Sections 5.5,

Power Series

Toward the end of last Wednesday's class, we started a section on **Series Solutions**.

This is a technique that allows use to write down approximate solutions to problems with *nonconstant coefficients*.

For example

$$y'' - xy' + y = 0.$$

Power Series

Power Series

The key idea is that we suppose that we can write y as

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

The general solution will always have arbitrary constants, so we let these be c_0 and c_1.

Then we substitute the power series is into the differential equation, and get equations for c_2, c_3, c_4, ...

The more terms we take, the more accurate the solution is.

Power Series

Example

Use a power series to solve the DE

$$y'' - xy = 0.$$

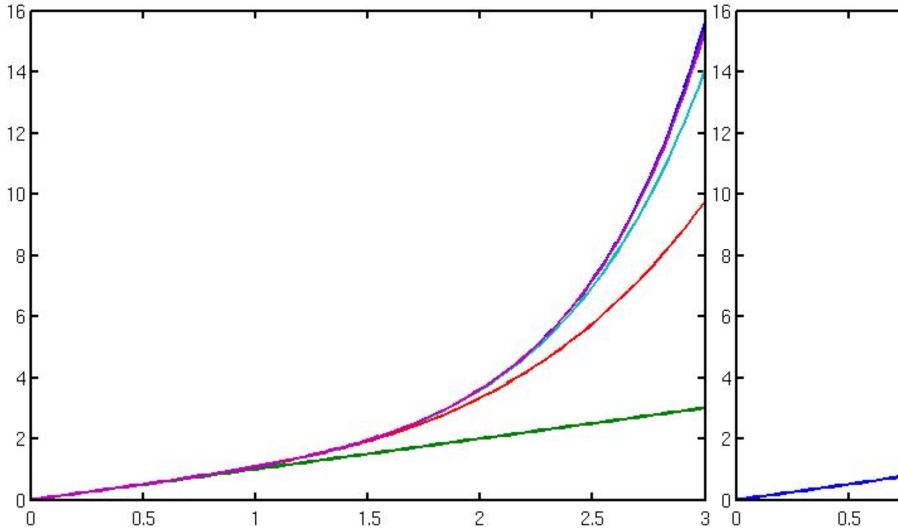
Power series methods are particularly useful for getting solutions to *initial value problems* where we are given, not only the differential equation, but also the value of y and y' at some initial point.

These allow us to solve for c_0 and c_1.

Example

Use a power series to solve the initial value problem

$$y'' - xy = 0, \qquad y(0) = 0, y'(0) = 1.$$


16
14
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3
16
14
12
10
8
6
4
2
0
0
0.5

Exercise (Q16.1)

For each of the following differential equations, find a recurrence relation for the coefficients of the power series solution, and write out the solution up to the x^5 term.

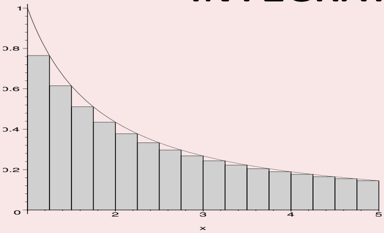
1. $y'' + xy = 0.$
2. $y'' + x^2y = 0.$
3. $y'' - 2xy' + y = 0.$
4. $y'' - 2xy' + y = 0, \quad y(0) = 1, y'(0) = -1$
5. $y'' - xy' = 0, \qquad y(0) = 0, y'(0) = 2$

We've now finished the section on solving 2nd order problem.

For the next few lectures we will study

New Section

INTEGRATION



$$\approx \int_a^b f(x)dx$$

Later we'll return to the topic of solving 1st order problems.

Integration

In this section of the course we return to the problem of:

Given a function F, find a function f such that

$$f'(x) = F(x).$$

We call f an *anti-derivative* if F. (See Lecture 6)

More often we write this as an *Integral* problem:

Given a function F, find

$$f(x) = \int F(x)dx.$$

Sigma Notation

We can write a sum $f_0, f_1, f_2, \ldots, f_n$ as $\displaystyle\sum_{k=0}^{n} f_k$.

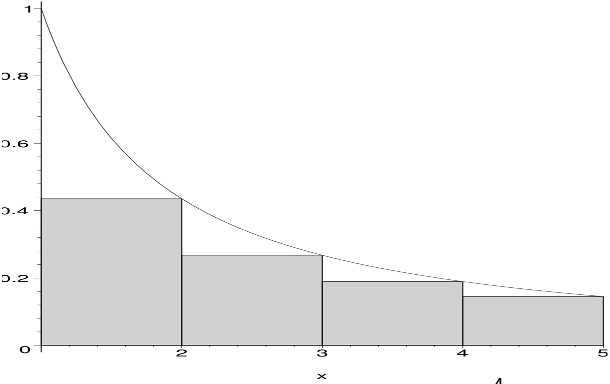
Example

$$1 + 2 + 3 + 4 + \cdots + 10 = \sum_{k=1}^{10} k.$$

$$1 + 3 + 5 + 7 + \cdots + 13 = \sum_{k=1}^{7} (2k - 1).$$

Area Under a Curve

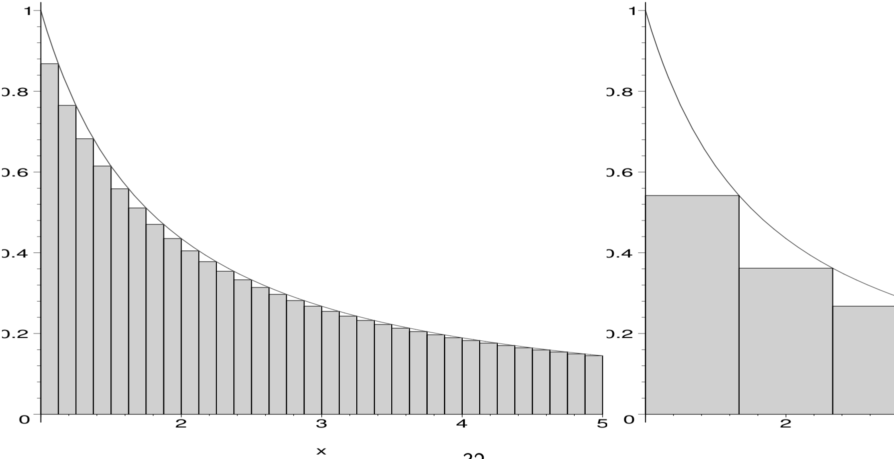
One can approximate the area from a to b bounded above by a given function, below by the x-axis by the area of boxes under the curve:



$$Area \approx A_1 + A_2 + A_3 + A_4 = S_4 = \sum_{k=1}^{4} A_k.$$

Area Under a Curve

As we increase the number of boxes, the approximation improves...



$$Area \approx S_{32} = \sum^{32} A_k.$$

Area Under a Curve

First we divide the interval $[a, b]$:

$$a = x_0 < x_1 < x_2 \cdots < x_n = b \quad \text{and } \delta x = x_i - x_{i-1} = \frac{b-a}{n}.$$

Then the area of each box is:

$$A_k = f(x_k)\delta x.$$

Define the sums of the areas of n boxes as

$$S_n = A_1 + A_2 + \cdots + A_n,$$

And now:

$$Area = \lim_{n\to\infty} S_n = \int_a^b f(x)dx.$$

Definite Integrals

The integral of a function f from a to b

$$\int_a^b f(x)dx,$$

where

- $\int$ is the integration symbol
- a and b are the lower and upper *limits of integration*
- dx means we are integrating with respect to x.

Definite Integrals

You should know the following properties:

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
- $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$ for any constant C.
- $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Definite Integrals

Exercise (Q16.2)

Which of the following statements is true? Why?

$$\int_a^b |f(x)| dx \leq \left| \int_a^b f(x) dx \right|,$$

or

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Theorem (The Fundamental Theorem of Calculus)

Let $F(x)$ be defined as

$$F(x) = \int_a^x f(x)dx.$$

Then $F'(x) = f(x)$. That is

$$\frac{d}{dx}\int_a^x f(x)dx = f(x).$$

Furthermore, let $g(x)$ be any antiderivative of f. That is: $G'(x) = f(x)$. Then

$$\int_a^b f(x)dx = G(b) - G(a) := G(x)\Big|_a^b.$$

To evaluate a definite integral of the form

$$\int_a^b f(x)dx,$$

find an anti-derivative G of f so that $G'(x) = f(x)$. Then

$$\int_a^b f(x)dx = G(b) - G(a).$$

Example

Evaluate each of the following

$$(a) \int_0^2 x^2 dx; \qquad (b) \int_1^2 \frac{(x+2)^2}{x} dx; \qquad (c) \int_0^{\pi} \sin(\frac{x}{3}) dx.$$

In these examples we relied on knowing the anti-derivative of some elementary functions.

If you don't remember these, or others, look them up on pages 41 and 42 of the Mathematical Tables.

DIFREÁIL (DIFFERENTIATION)

$f(x)$	$f'(x) \equiv \frac{d}{dx}[f(x)]$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$

SUIMEÁIL (INTEGRATION)

Glactar $a>0$ agus fágtar tairisigh na suimeála ar lár.

We take $a>0$ and omit constants of integration.

$f(x)$	$\int f(x)\, dx$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln \lvert \sec x \rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$
$\operatorname{cosec} x$	$\ln \lvert \tan \frac{x}{2} \rvert$
$\cot x$	$\ln \lvert \sin x \rvert$
e^x	e^x

$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$
$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\operatorname{cosec}^{-1}\frac{x}{a}$	$-\frac{a}{x\sqrt{x^2-a^2}}$
$\cot^{-1}\frac{x}{a}$	$-\frac{a}{a^2+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ln a}$
$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$

$\coth^{-1} x$	$-\dfrac{1}{x^2-1}$
$\operatorname{sech}^{-1} x$	$-\dfrac{1}{x\sqrt{1-x^2}}$
$\operatorname{cosech}^{-1} x$	$-\dfrac{1}{x\sqrt{x^2+1}}$

Torthaí agus Líonta:
Products and Quotients:

$$y = uv\;;\;\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = \frac{u}{v}\;;\;\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\coth x$	$\ln \lvert \sinh x \rvert$
$\operatorname{sech} x$	$\tan^{-1}(\sinh x)$
$\operatorname{cosech} x$	$\ln \left\lvert \tanh \dfrac{x}{2} \right\rvert$
$\cos^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sin 2x]$
$\sin^2 x$	$\frac{1}{2}[x - \frac{1}{2}\sin 2x]$
$\cosh^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sinh 2x]$
$\sinh^2 x$	$\frac{1}{2}[-x + \frac{1}{2}\sinh 2x]$
$\dfrac{1}{x\sqrt{a^2-x^2}}$	$-\dfrac{1}{a}\operatorname{sech}^{-1}\dfrac{x}{a}$
$\dfrac{1}{x\sqrt{x^2+a^2}}$	$-\dfrac{1}{a}\operatorname{cosech}^{-1}\dfrac{x}{a}$