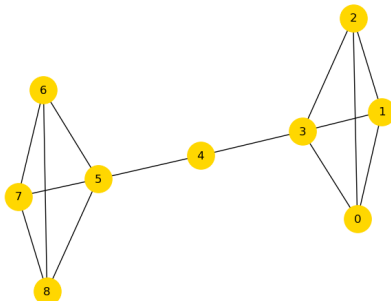


Week 7, Part 1: Closeness and Betweenness Centrality

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Reminders

- ▶ **Assignment 1** Due 5pm Friday, 27th February.
- ▶ **Class Test** 14:00, Thursday 6th March (Week 8)

Outline

Today's notes are split between these slides, and a Jupyter Notebook.

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|---|--------------------------------|--------------------------|
| 1 | Centrality Measures (again) | ■ Distance Matrix |
| 2 | Eigenvector Centrality (again) | 4 Betweenness Centrality |
| 3 | Closeness Centrality | ■ Normalised |
| | ■ Normalised | ■ Examples |

Slides are at:

<https://www.niallmadden.ie/2425-CS4423>



Centrality Measures (again)

Last week we learned about some centrality measures:

Measures of centrality include:

- ▶ The **degree centrality**, c_i^D of Node i in $G = (X, E)$ is the degree of i (i.e., the number of neighbours it has). So $c_i^D = \deg(i)$.
- ▶ The **normalised degree centrality**, C_i^D of Node i is $C_i^D = \deg(i)/(n - 1)$ where n is the order of the network.
- ▶ **Eigenvector Centrality**, which we'll recap now.

Then we'll look at:

- ▶ **Closeness Centrality**, and
- ▶ **Betweenness Centrality**.

Eigenvector Centrality (again)

Eigenvector Centrality

1. Let A be the adjacency matrix of a network. G .
2. We know, thanks to Perron-Frobenius, that A has a positive eigenvalue, λ , which is equal to the spectral radius of A .
3. There is a positive eigenvector, v associated with λ .
4. Choose v so that $v^T v = v_1^2 + v_2^2 + \dots + v_n^2 = 1$.
5. v_i is the **eigenvector centrality** of Node i .

Closeness Centrality

A node x in a network can be regarded as being central, if it is **close** to (many) other nodes, as it can then quickly interact with them.

Recalling the $d(i, j)$ is the distance between Nodes i and j (i.e., the length of the shortest path between them). Then we can use $1/d(i, j)$ as a measure of “closeness”.

Definition (Closeness Centrality)

In a simple, *connected* graph $G = (X, E)$ of order n , the **closeness centrality**, c_i^C , of Node i is defined as

$$c_i^C = \frac{1}{\sum_{j \in X} d(i, j)} = \frac{1}{s(i)},$$

where $s(i)$ is the **distance sum** for node i .

As is usually the case, there is a **normalised** version of this measure.

Normalised closeness centrality

The **normalised closeness centrality** of Node i , defined as

$$C_i^C = (n-1)c_i^C = \frac{n-1}{\sum_{j \in X} d(i,j)} = \frac{n-1}{s(i)}.$$

Note: $0 \leq C_i^C \leq 1$. (Why?)

Example

Compute the normalised closeness centrality of all nodes in the graph on nodes $\{0, 1, 2, 3, 4\}$, with edges $0 - 1$, $0 - 2$, $0 - 3$, $0 - 4$, $1 - 2$, $1 - 3$.

In that example we effectively computed the **distance matrix** of the graph.

Distance Matrix

The **distance matrix** of a graph, G , of order n is the $n \times n$ matrix, $D = (d_{ij})$ such that

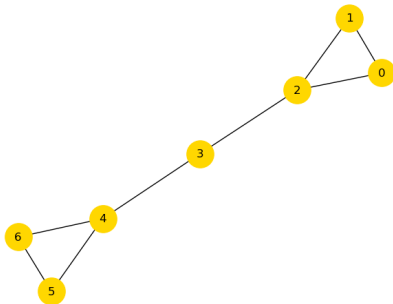
$$d_{ij} = d(i, j).$$

We'll return to how to compute D tomorrow, but for now we note:

- ▶ $s(i)$ is the sum of row i of D ;
- ▶ If \mathbf{s} is the vector of of distance sums, then $\mathbf{s} = D\mathbf{e}$, where $\mathbf{e} = (1, 1, \dots, 1)^T$.

Betweenness Centrality

Consider the following graph (as the 3 – 1 Barbell Graph):



We can, I hope, convince ourselves, that, in a sense:

- ▶ Node 3 is the most central, in the sense that belongs to the most shortest paths.
- ▶ Node 0 (for example), is very much not central in that sense.

Betweenness Centrality

Definition (Betweenness Centrality)

In a simple, connected graph G , the **betweenness centrality** c_i^B of node i is defined as

$$c_i^B = \sum_j \sum_k \frac{n_i(j, k)}{n(j, k)}, \quad j \neq k \neq i$$

where $n(j, k)$ denotes the *number* of shortest paths from node j to node k , and where $n_i(j, k)$ denotes the number of those shortest paths *passing through* node i .

Definition (Normalised Betweenness Centrality)

In a simple, connected graph G , the **normalised betweenness centrality** C_i^B of node i is defined as

$$C_i^B = \frac{c_i^B}{(n-1)(n-2)}$$

