

## MA378 Chapter 1: Interpolation

### §1.2 Lagrange Interpolation

Dr Niall Madden

January 2023



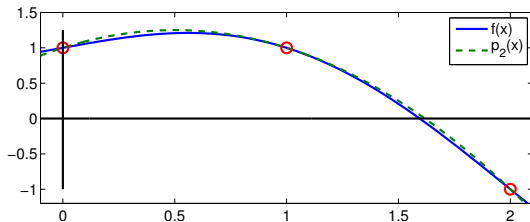
Source: <http://jeff560.tripod.com/stamps.html>

Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics.

## 2.1 Finding the polynomial

### Example 2.1

Show that the polynomial of degree 2 that interpolates  $f(x) = 1 - x + \sin(\pi x/2)$  at the points  $x_0 = 0$ ,  $x_1 = 1$  and  $x_2 = 2$  is  $p_2 = -x^2 + x + 1$ .



How do we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP?*

As a first step, we'll prove the following:

### Theorem 2.2

*If  $p_n \in \mathcal{P}_n$  has  $n + 1$  zeros, then  $p_n \equiv 0$  (i.e.,  $p_n(x) = 0$  for all  $x$ ).*

### Theorem 2.3 (There is a unique solution to the PIP)

*There is at most one polynomial of degree  $\leq n$  that interpolates the  $n + 1$  points  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$  where  $x_0, x_1, \dots, x_n$  are distinct.*

## 2.2 The Vandermonde matrix method

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find  $p_2$  such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \text{and} \quad p_2(x_2) = y_2.$$

Since  $p_2(x)$  is of the form  $a_0 + a_1x + a_2x^2$ , this just amounts to finding the values of the coefficients  $a_0$ ,  $a_1$ , and  $a_2$ . One might be tempted to solve for them using the system of equations

$$a_0 + a_1x_0 + a_2x_0^2 = y_0$$

$$a_0 + a_1x_1 + a_2x_1^2 = y_1$$

$$a_0 + a_1x_2 + a_2x_2^2 = y_2$$

This is known as the *Vandermonde System*.

## 2.2 The Vandermonde matrix method

Writing

$$a_0 + a_1x_0 + a_2x_0^2 = y_0$$

$$a_0 + a_1x_1 + a_2x_1^2 = y_1$$

$$a_0 + a_1x_2 + a_2x_2^2 = y_2$$

in matrix-vector format we get

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \quad \text{or} \quad Va = y. \quad (1)$$

But this may not be a good idea. (*Unfortunately, to see exactly why, you needed to have studied MA385. If you didn't, you can skip the next bit*).

## 2.2 The Vandermonde matrix method

In MA385 we learned about the relationship between the *condition number* of a matrix,  $V$ , and the relative error in the (numerical) solution to a matrix-vector equation with  $V$  as the coefficient matrix. The condition number is  $\kappa(V) = \|V\| \|V^{-1}\|$ , for some subordinate matrix norm  $\|\cdot\|$ .

## 2.2 The Vandermonde matrix method

### Example 2.4 (Stewart's "Afternotes...", Lecture 19)

Suppose  $x_0 = 100$ ,  $x_1 = 101$  and  $x_2 = 102$ . Then it is not hard to check that

$$\|X\|_{\infty} = \max_i \sum_j |X_{ij}| = 10,507.$$

Also,

$$V^{-1} = \frac{1}{2} \begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

so  $\|V^{-1}\|_{\infty} = 20401$ . So  $\kappa(V) = 214,353,307$ .



## 2.3 Lagrange Interpolation

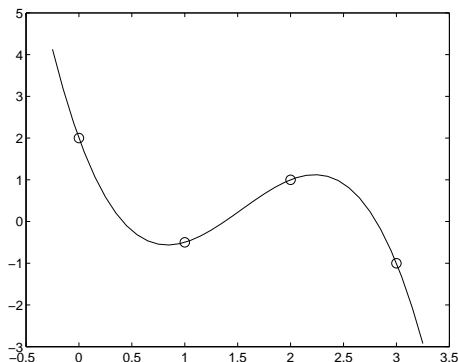
We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (Here “constructive” means that we'll prove it exists by actually computing it).

## 2.3 Lagrange Interpolation

### Example

Consider the problem: *find  $p_3 \in \mathcal{P}_3$  such that*

$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$

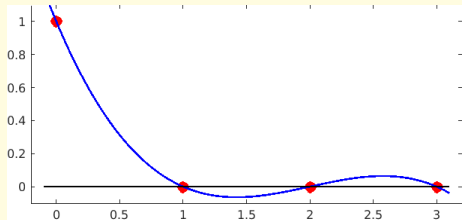


## 2.3 Lagrange Interpolation

Here is an easier problem to solve: Find  $L_0 \in \mathcal{P}_3$  such that

$$L_0(0) = 1, \quad L_0(1) = 0,$$

$$L_0(2) = 0, \quad L_0(3) = 0.$$



Because  $L_0$  is a cubic and has zeros at  $x = 1, 2, 3$  it is of the form  $L_0(x) = C(x - 1)(x - 2)(x - 3)$ .

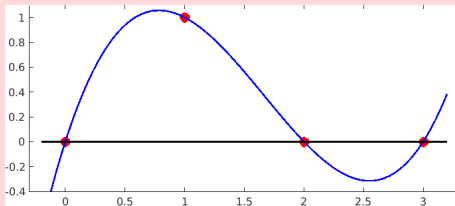
Choosing  $C$  so that  $L_0(0) = 1$ , we get

$$L_0(x) =$$

## 2.3 Lagrange Interpolation

Similarly, find  $L_1 \in \mathcal{P}_3$  such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

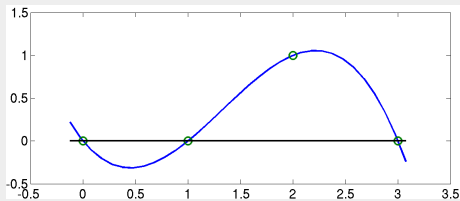


Then

$$L_1(x) =$$

## 2.3 Lagrange Interpolation

In the same style, let  $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



$$L_2(x) =$$

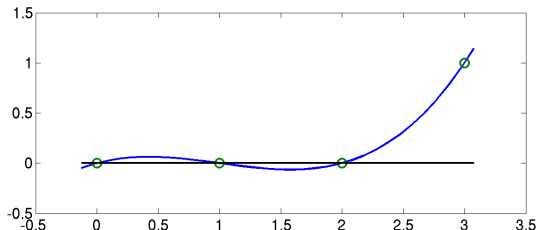
## 2.3 Lagrange Interpolation

Finally, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$$L_3(x) = \frac{(x-0)(x-1)(x-3)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j \neq 3}^n \frac{(x-x_j)}{(x_3-x_j)}.$$



## 2.3 Lagrange Interpolation

Because each of  $L_0$ ,  $L_1$ ,  $L_2$ , and  $L_3$  is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - \left(\frac{1}{2}\right)L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. Furthermore...

## 2.3 Lagrange Interpolation

$$\begin{aligned}p_3(0) &= 2L_0(0) - (1/2)L_1(0) + (1)L_2(0) + (-1)L_3(0) \\&= 2(1) - (1/2)(0) + (1)(0) + (-1)(0) \\&= 2, \\p_3(1) &= 2L_0(1) - (1/2)L_1(1) + (1)L_2(1) + (-1)L_3(1) \\&= 2(0) - (1/2)(1) + (1)(0) + (-1)(0) \\&= -1/2, \\p_3(2) &= 2L_0(2) - (1/2)L_1(2) + (1)L_2(2) + (-1)L_3(2) \\&= 2(0) - (1/2)(0) + (1)(1) + (-1)(0) \\&= 1, \\p_3(3) &= 2L_0(3) - (1/2)L_1(3) + (1)L_2(3) + (-1)L_3(3) \\&= 2(0) - (1/2)(0) + (1)(0) + (-1)(1) \\&= -1.\end{aligned}$$

Thus  $p_3$  solves the problem!



## 2.4 The Lagrange Form

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation.

### Definition 2.5 (Lagrange Polynomials)

The **Lagrange Polynomials** associated with  $x_0 < x_1 < \cdots < x_n$  is the set  $\{L_i\}_{i=0}^n$  of polynomials in  $\mathcal{P}_n$  such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \quad (2a)$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j}. \quad (2b)$$

## 2.4 The Lagrange Form

### Definition 2.6

The **Lagrange form of the Interpolating Polynomial**

$$p_n(x) = \sum_{i=0}^n y_i L_i(x), \quad (3a)$$

or

$$p_n(x) = \sum_{i=0}^n f(x_i) L_i(x). \quad (3b)$$

Take care not to confuse

- ▶ the *Lagrange Polynomials*, which are the  $L_i$  with
- ▶ the *Lagrange Interpolating Polynomial*, which is the  $p_n$  defined in (3).

## 2.4 The Lagrange Form

### Theorem 2.7 (Lagrange)

*There exists a solution to the Polynomial Interpolation Problem and it is given by*

$$p_n(x) = \sum_{i=0}^n y_i L_i(x).$$

## 2.5 Example

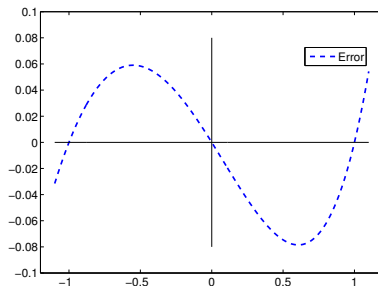
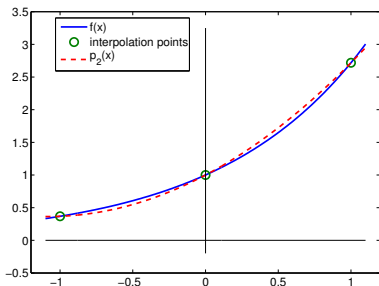
### Example 2.8 (Süli and Meyers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function  $f(x) = e^x$  at interpolation points  $\{-1, 0, 1\}$ .

## 2.5 Example

The figure below shows the solution to Example 2.8 (top) and the difference between the function  $e^x$  and its interpolant (bottom). It would be interesting to see how this error depends on

- (i) the function (and its derivatives)
- (ii) the number of points used.



## 2.6 Exercises

### Exercise 2.1

The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{0 \leq i < j \leq n} (x_j - x_i). \quad (4)$$

Verify this for the  $2 \times 2$  and  $3 \times 3$  cases.

## 2.6 Exercises

(Note that from Formula (4) we can deduce directly that the PIP has a unique solution *if and only if* the points  $x_0, x_1, \dots, x_n$  are all distinct.)

### Exercise 2.2

Find the polynomial  $p_1$  that interpolates the function  $f(x) = x^3$  at the points  $x_0 = 0$  and  $x_1 = a$ . Find the point  $\sigma \in [0, a]$  that maximises  $|f(x) - p_1(x)|$ , and hence compute

$$\max_{0 \leq x \leq a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayer.

## 2.6 Exercises

### Exercise 2.3

Show that

$$\sum_{i=0}^n L_i(x) = 1 \quad \text{for all } x.$$

### Exercise 2.4

Write down the Lagrange Form of  $p_2$ , the polynomial of degree 2 that interpolates the points  $(0, 3)$ ,  $(1, 2)$  and  $(2, 4)$ .

Source: Chapter 2 of Stoer and Bulirsch.



## 2.6 Exercises

### Exercise 2.5

Show that all the following represent the same polynomial (usually called the “Chebyshev Polynomial of Degree 3”),

$$T_3(x) = 4x^3 - 3x.$$

(a) Horner form:  $((4x + 0)x - 3)x + 0$ .

(b) Lagrange form:  $\sum_{k=0}^3 \left( \prod_{j=0, j \neq k}^3 \frac{x - x_j}{x_k - x_j} \right) (-1)^{k+1}$ , where  
 $x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$ .

(c) Recurrence relation:  $T_0 = 1$ ,  $T_1 = x$ , and  
 $T_n = 2xT_{n-1} - T_{n-2}$  for  $n = 2, 3, \dots$

(d) Trigonometric form:  $T_3(x) = \cos(3 \cos^{-1}(x))$ .