Annotated slides from Thursday

CS4423: Networks

Lecture 6: Connectivity and ~~Permutations~~

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These slides are by Niall Madden. Elements are based on "A First Course in Network Theory" by Estrada and Knight

Note: some Examples were done on the white board. Refer to your own notes for those.

Outline

Slides are at:
https://www.niallmadden.ie/2425-CS4423

Data collection

(Stealing an idea from Angela Carnevale) I'd like to gather some data for use in the class. So, I'm going to run a little survey on what programmes/shows people watch. To do that, I need some ideas... So far we have

1. Only Murders in the Building
2. Breaking Bad
3. The Penguin
4. Succession
5. Squid Game
6. The Bear

Any more?

7 The Boys
8 Better Call Saul.
9 Night. Agent.
10. Dr Who.

Notation

$(I + A + A^2 + A^3)_{ij}$

- If we write $A = (a_{ij})$ we mean A is a matrix, and a_{ij} is its entry row i, column j.
- We also write such entries as $(A)_{ij}$. The reason for this slightly different notation is that, for example $(A^2)_{ij}$ is the entry in i, column j of $B = A^2$.
- (Very standard) The **trace** of a matrix is the sum of its diagonal entries. That is $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.
- When we write $A > 0$, we mean all entries of A are positive.

Counting Walks

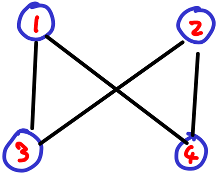
Recall... the **adjacency matrix** of a graph, G of order n, is a square $n \times n$ matrix, $A = (a_{ij})$, with rows and columns corresponding to the nodes of the graph. Set a_{ij} to be the number of edges between nodes i and j.

We learned yesterday that,

- If e_j is the Jth column of the I_n, then $(Ae_j)_i$ is the number of walks of length 1 from node i to node j. (Also, it is just a_{ij}...)
- Moreover, $\big(A(Ae_j)\big)_i = \big(A^2 e_j\big)_i$ is the number of walks of length 2 from node i to node j. We can conclude that, if $B = A^2$, then $b_{i,j}$ is the number of walks of length 2 between nodes i and j.
  Note: b_{ii} is the degree of node i.
- **IN FACT** if $B = A^k$, then $b_{i,j}$ is the number of walks of length k between nodes i and j.

Counting Walks

Example: K_{22}



Eg there are 2 walks of length 2 from 1 to 2:

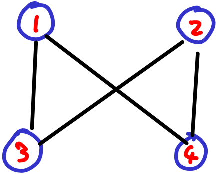
1-3, 3-2 and 1-4, 4-2.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Counting Walks

Example: K_{22}



$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \\ 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \end{pmatrix}$$

4 walks of length $\underline{\underline{3}}$ from 1 to 4

1-4, 4-1, 1-4,
1-3, 3-2, 2-4, etc.

Paths

Definition (Trail)

A **trail** is a walk with no repeated edges.

Definition (Cycle and triangle)

A **cycle** is a trail in which the first and last nodes are the same, but no other node is repeated. A **triangle** is a cycle of length 3.

Definition (Path)

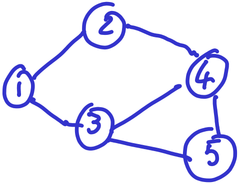
A **path** is walk with no nodes (and so no edges) repeated. (The idea of a **path** is hugely important in network theory. We'll return to it often)

Path length and shortest path

The **length** of a path is the number of edges in it.
A path from node u to node v is a **shortest** path, if there is no path between them that is shorter (though there could be other paths of the same length)

Finding shortest paths in a network is a major topic in networks, and one we'll return to at another time. But, for now, we'll see how to use powers of the adjacency matrix to find the length of such a part (without finding the path itself).

Eg



Note 1-3, 3-4, 4-5 is a path from 1 to 5.
But 1-3, 3-5 is the shortest path. from 1 to 5.

Some facts about **walks** and **paths**

- Every path is also a walk.
- If a particular walk is the shortest walk between two nodes, then it is also the shortest path between those two nodes.
- If k is the smallest natural number for which $(A^k)_{ij} \neq 0$, then the shortest walk from node i to node j is of length k.
- It follows that k is also the length of the shortest path from Node i to node j.

Example: Consider the graph (see board) with adjacency matrix, and its powers:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 4 & 1 & 1 \\ 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 4 & 2 & 3 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$$

$(A^2)_{ii} = \deg(i)$.

Two triangls starting & ending at 4.

Walks of length 3 from 1 to 2: $1-2, 2-1, 1-2$ or $1-2, 2-3, 3-2$.

In the previous example, we observed that, where A is the adjacency matrix of the graph G,

- $(A^2)_{ii}$ is the degree of node i.
- $\text{tr}(A^2)$ is the degree sum of the nodes in G.
- $(A^3)_{ii} \neq 0$ if node i is in a triangle.
- $\text{tr}(A^3)/6$ is the number of triangles in G.
- If G is bipartite, then $(A^3)_{ij} = 0$ for all i, j

That is: a bipartite graph has no triangles. (= 3-cycles)

Connectivity

Let G be a graph, and A its adjacency matrix.

Definition (Reach)

In G, Node i can be **reached** from Node j is there is a path between them.

Fact

If Node i is reachable from Node j, then $(A^k)_{i,j} \neq 0$ for some k.
Also, note that $k \leq n$.
Equivalently, since each power of A is nonnegative, we can say that $(I + A + A^2 + A^3 + \cdots + A^{n})_{i,j} > 0$.

Connectivity

Definition (Connected Graph)

A graph/network is **connected** if there is an path between every pair of nodes. That is, every node is reachable from every other. If the graph is *not* connected, we say it is **disconnected**.

Determining if a graph is connected is important. (We'll see later, this is especially important/interesting with *directed graphs*).

Fact

A graph, G of order n is connected if, and only if, for each i, j, there is some $k \leq n$ for which $(A^k)_{i,j} \neq 0$.

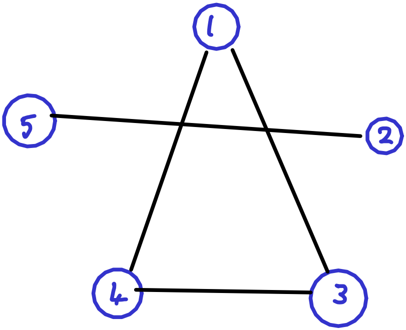
Equivalently : $\left(\sum_{k=0}^{n} A^k\right) > 0$

where $A^0 = I$, the identity matrix.

Finished here

Example

Sketch the graph, G, on the nodes $\{1, 2, 3, 4, 5\}$ with edges $1-3$, $1-4$, $2-5$, $3-4$. Write down its adjacency matrix. Is G connected?



$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

No, G is not connected: there is no path from, eg 1 to 2.

Exercise(s)

1. Write down A, the adjacency matrix of C_5. Try to write down A^2 and A^3 simply by looking at the network it represents.
2. Let u be a vector with n entries. Let $D = \text{diag}(u)$. That is, $D = (d_{ij})$ is the diagonal matrix with entries

   $$d_{ij} = \begin{cases} u_i & i = j \\ 0 & i \neq j. \end{cases}$$

   Verify that $PDP^T = \text{diag}(Pu)$.
3. In all the examples we looked at, we had a symmetric P. Is every permutation matrix symmetric? If so, explain why. If not, give an example.