MA385 Part 1: Solving nonlinear equations

1.7: Wrap Up

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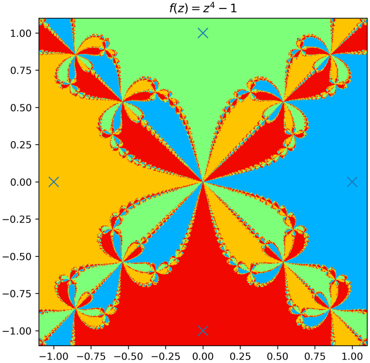

0. Outline

1. Wrapping up

When studying a numerical method (or any piece of Mathematics) you should ask *why* you are doing this. For example, it might be

- because it will help you can understand other topics later;
- because it is interesting/beautiful in its own right; or
- (most commonly) because it is useful.

We'll now give some instances of each of these.

2. Relation to other topics

The analyses we have used in this section allowed us to consider some important ideas in a simple setting.

Examples include

- **Convergence**, including *rates of convergence*
- **Fixed-point theory**, and contractions. We'll be seeing analogous ideas in the next section (Lipschitz conditions).
- **The approximation of functions by polynomials** (Taylor's Theorem). This point will reoccur in the next section, and all through-out next semester.

Applications come from lots of areas of science and engineering. In financial mathematics, the **Black-Scholes equation** for pricing a put option can be written as

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0.$$

- $V(S,t)$ is the current value of the right (but not the obligation) to buy or sell ("put" or "call") an asset at a future time T;
- S is the current value of the underlying asset;
- r is the current interest rate (because the value of the option has to be compared with what we would have gained by investing the money we paid for it)
- σ is the volatility of the asset's price.

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Often one knows S, T and r, but not σ. The method of *implied volatility* is when we take data from the market and then find the value of σ which, if used in the Black-Scholes equation, would match this data. This is a nonlinear problem and so Newton's method can be used. See Chapters 13 and 14 of Higham's "An Introduction to Financial Option Valuation" for more details.

(We will return to the Black-Scholes problem again at the end of the next section).

In the next section of MA385, we'll try to solve differential equations of the form $y'(t) = f(t, y)$.

The methods we'll study are called "Runge-Kutta" methods. We'll mainly focus on *explicit* methods:

$$y_{k+1} = \Phi(t_k, y_k).$$

But we'll also mention **implicit** methods:

$$y_{k+1} = \Phi(t_k, y_{k+1}).$$

for which a nonlinear solver is required.

Suppose we have a function $f(x)$ and want to find some $\tau \in [a, b]$ where (for example) $f(x)$ is **minimized**.

This is equivalent to solving for $f'(x) = 0$, which can be done using any of the methods we presented earlier.

One prominent example is where $f = f(\mathrm{x})$ is a *loss function* for a neural network, and the vector x represents weights in the network. The most popular method for finding x is **stochastic gradient descent**, which is, essentially, Newton's Method in high dimension: choose a dimension (i.e., entry of x) at random, and apply Newton (or Secant) for one iteration.

Finally, we claim that some of these ideas we looked at are interesting and beautiful in their own right.

The **complex n^{th} roots of unity** is the set of numbers $\{z_0, z_1, \dots, z_{n-1}\}$ who's n^{th} roots are 1. For example, the 4th roots of unity are 1, -1, $i = \sqrt{-1}$ and $-i$.

More generally, they can be expressed as

$$z_k = e^{i\theta} \quad \text{where } \theta = \frac{2k\pi}{n} \qquad \text{for } k \in \{0, 1, 2 \dots, n-1\}.$$

Suppose we wanted to estimate these numbers using Newton's method. We could try to solve $f(z) = 0$ with $f(z) = z^n - 1$. The iteration is:

$$z_{k+1} = z_k - \frac{(z_k)^n - 1}{n(z_k)^{n-1}}.$$

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However, there are n possible solutions to

$$z^n - 1 = 0.$$

Given a particular starting point, which root with the method converge to? If we take a number of points in a region of space, iterate on each of them, and then colour the points to indicate the ones that converge to the same root, we get the famous Julia[1] set, an example of a fractal.

[1] Gaston Julia, French mathematician 1893–1978. The famous paper which introduced these ideas was published in 1918. Interest later waned until the 1970s when Mandelbrot's computer experiments reinvigorated interest.

A contour plot of the Julia set for $n = 4$, generated by Python/numpy (see `Julia.py` and `Julia.ipynb` on Canvas).