MA385 Part 1: Solving nonlinear equations

1.5: Newton's Method

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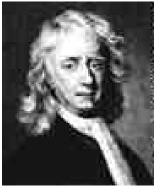
(Started and finished Monday 22 Sep = W03.1)



Sir Isaac Newton, 1643 - 1727, England. Easily one of the greatest scientist of all time. The method we are studying appeared in his celebrated *Principia Mathematica* in 1687, but it is believed he had used it as early as 1669.

0. Outline

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For more details, see Section 1.4 (Relaxation and Newton's method) of Süli and Mayers, *An Introduction to Numerical Analysis*

Also, Chapter 3 of Epperson:
https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9781118730966

1. Motivation

Secant method can be written as

$$x_{k+1} = x_k - f(x_k)\phi(x_k, x_{k-1}),$$

where the function ϕ is chosen so that x_{k+1} is the root of the secant line joining the points $\big(x_{k-1}, f(x_{k-1})\big)$ and $\big(x_k, f(x_k)\big)$.

A closely related idea leads to **Newton's Method**: set $x_{k+1} = x_k - f(x_k)\lambda(x_k)$, where we choose λ so that x_{k+1} is the zero of the tangent to f at $\big(x_k, f(x_k)\big)$.

$x=0$

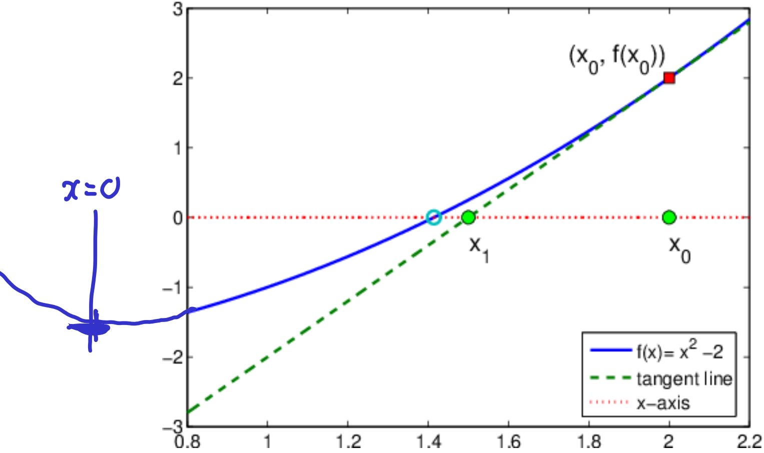

Figure 1: Estimating $\sqrt{2}$ by solving $x^2 - 2 = 0$ using Newton's Method

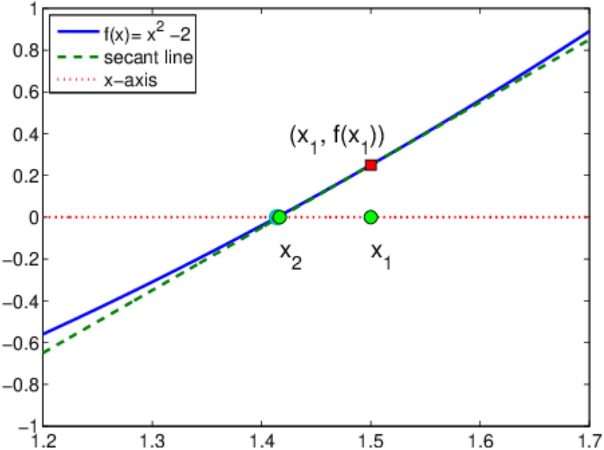

Figure 1: Estimating $\sqrt{2}$ by solving $x^2 - 2 = 0$ using Newton's Method

2. Newton's Method

The formula for Newton's method may be deduced writing down the equation for the line at $\big(x_k, f(x_k)\big)$ with slope $f'(x_k)$, and setting x_{k+1} to be its zero; see Exercise 1.5.1.

Newton's Method

1. Choose any x_0 in $[a, b]$,
2. For $k = 0, 1, \dots$, set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{1}$$

Eg 1.5.1

Use bisection, secant, and Newton's Method to solve $x^2 - 2 = 0$ *in* $[0, 2]$.

So $f(x) = x^2 - 2$

$\Rightarrow \dfrac{f(x)}{f'(x)} = \dfrac{x^2-2}{2x}$.

For this case, Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k},$$

which simplifies as

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

Taking $x_0 = 2$, we get $x_1 = 3/2$.
Then $x_2 = 2x_1 + 1/x_1 = 17/12 = 1.46667$.
Then $x_3 = 1.4142$, etc.

Iter	Bisection	Secant	Newton
k	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$
0	1.41	1.41	5.86e-01
1	5.86e-01	5.86e-01	8.58e-02
2	4.14e-01	4.14e-01	2.45e-03
3	8.58e-02	8.09e-02	2.12e-06
4	1.64e-01	1.44e-02	1.59e-12
5	3.92e-02	4.20e-04	2.34e-16
6	2.33e-02	2.12e-06	—
7	7.96e-03	3.16e-10	—
8	7.66e-03	4.44e-16	—
9	1.51e-04	—	—
10	3.76e-03	—	—
11	1.80e-03	—	—
⋮	⋮	⋮	⋮
22	5.72e-07	—	—

Deriving Newton's method geometrically certainly has an intuitive appeal. However, to analyse the method, we need a more abstract derivation based on a **Truncated Taylor Series**.

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x-x_k)^2}{2!}f''(x_k) + \dots \\ + \frac{(x-x_k)^n}{n!}f^{(n)}(x_k) + \frac{(x-x_k)^{n+1}}{(n+1)!}f^{(n+1)}(\eta_k)$$

where $\eta_k \in (x, x_k)$. Truncate at the second term (i.e., take $n = 1$)...

2. Newton's Method

Deriving Newton's Method

Take $n = 1$, and then

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(\eta_2)$$

Taking $x = \tau$ and using that $f(\tau) = 0$

$$0 = f(x_k) + (\tau - x_k)f'(x_k) + \frac{(\tau - x_k)^2}{2}f''(\eta_2).$$

Rearranging:

$$-(\tau - x_k)f'(x_k) = f(x_k) + \frac{(\tau - x_k)^2}{2}f''(\eta_2)$$

$$\Rightarrow -\tau + x_k = \frac{f(x_k)}{f'(x_k)} + \frac{1}{2}(\tau - x_k)^2\frac{f''(\eta_2)}{f'(x_k)}.$$

Set $\tau := x_{k+1}$

$$\Rightarrow x_{k+1} \approx x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2}(\tau - x_k)^2\frac{f''(\eta_2)}{f'(x_k)}.$$

3. Newton Error Formula

We now want to show that Newton's converges **quadratically**, that is, with (at least) **order** $q = 2$. To do this, we need to

1. Write down a recursive formula for the error.
2. Show that it converges.
3. Then find the limit of $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^2}$.

Step 2 is usually the crucial part.

There are two parts to the proof. The first involves deriving the so-called "**Newton Error formula**".

We'll assume that the functions f, f' and f'' are defined and continuous on the an interval $I_\delta = [\tau - \delta, \tau + \delta]$ around the root τ.

The following proof is essentially the same as the above derivation (see also Section 3.6 of Epperson:
https://search.library.nuigalway.ie/permalink/f/

Theorem 1.5.1 (Newton Error Formula)

If $f(\tau) = 0$ and $f'(x_k) \neq 0 \ldots$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point η_k between τ and x_k such that

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2}\frac{f''(\eta_k)}{f'(x_k)}, \qquad (2)$$

Proof: From Taylor's Thm

$$f(\tau) = f(x_k) + (\tau - x_k) f'(x_k) + \tfrac{1}{2}(\tau - x_k)^2 \cdot f''(\eta_k)$$

for some $\eta_k \in [x_k, \tau]$. Since $f(\tau) = 0$, we rearrange to get

$$f(x_k) + (\tau - x_k) f'(x_k) = -\tfrac{1}{2}(\tau - x_k)^2 \cdot f''(\eta_k).$$

If $f'(x_k) \neq 0 \ldots$ *then...*

Theorem 1.5.1 (Newton Error Formula)

If $f(\tau) = 0$ and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point η_k between τ and x_k such that

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2}\frac{f''(\eta_k)}{f'(x_k)}, \tag{2}$$

$$\tau - \underbrace{\left(x_k - \frac{f(x_k)}{f'(x_k)}\right)} = -\frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}$$

$$\Rightarrow \quad \tau - x_{k+1} = -\frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}.$$

4. Applying the Newton Error Formula

The Newton Error Formula is important in theory (for proving that Newton's Method converges) and practice (to estimate the error when it is applied to specific problems).

In practical applications, we can use the (2) as follows. So suppose we are applying Newton's Method to solving $f(x) = 0$ on $[a, b]$. Denote the error at Step k by $\varepsilon_k = |\tau - x_k|$. Then we can deduce that

$$\varepsilon_{k+1} \leq \varepsilon_k^2 \left(\frac{\max_{a \leq x \leq b} |f''(x)|}{2|f'(x_k)|} \right). \tag{3}$$

Then, using that $\varepsilon_0 \leq |b - a|$, (3) can be used repeatedly to bound ε_1, ε_2, etc.

5. Convergence of Newton's Method

We'll now complete our analysis of this section by proving the convergence of Newton's method.

Theorem 1.5.2

Let us suppose that f is a function such that

- f is continuous and real-valued, with continuous f'', defined on some closed interval $I_\delta = [\tau - \delta, \tau + \delta]$,
- $f(\tau) = 0$ and $f''(\tau) \neq 0$,
- there is some positive constant A such that

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all} \quad x, y \in I_\delta.$$

Let $h = \min\{\delta, 1/A\}$. If $|\tau - x_0| \leq h$ then Newton's Method converges quadratically.

The Newton Error formula is

$$\tau - x_{k+1} = -\frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)} \qquad \eta_k \in [\tau, x_k].$$

So

$$|\tau - x_{k+1}| = \frac{1}{2}|\tau - x_k|^2 \cdot \left|\frac{f''(\eta_k)}{f'(x_k)}\right|$$

In particular

$$|\tau - x_1| = \frac{1}{2}|\tau - x_0| \cdot \underbrace{|\tau - x_0|}_{\le 1/A} \underbrace{\left|\frac{f''(\eta_0)}{f'(x_0)}\right|}_{\le A}$$

$$\Rightarrow |\tau - x_1| \le \frac{1}{2}|\tau - x_0| \quad (1).$$

Repeating this argument will give

$$|\tau - x_{k+1}| \le \left(\frac{1}{2}\right)^k |\tau - x_0|.$$

So

$$\lim_{k\to\infty} |\tau - x_{k+1}| = 0.$$

5. Convergence of Newton's Method

Now recall that

$$|\tau - x_{k+1}| = \frac{1}{2}|\tau - x_k|^2 \cdot \left|\frac{f''(\eta_k)}{f'(x_k)}\right|$$

But $\eta_k \in [\tau, x_k]$ & $x_k \to \tau$.

So
$$\lim_{k\to\infty} \frac{|\tau - x_{k+1}|}{|\tau - x_k|^2} = \underbrace{\frac{1}{2}\frac{|f''(\tau)|}{|f'(\tau)|}}_{\mu}.$$

Exercise 1.5.1

Write down the equation of the line that is tangential to the function f at the point x_k. Give an expression for its zero. Hence show how to derive Newton's method.

Exercise 1.5.2

(i) Is it possible to construct a problem for which the bisection method will work, but Newton's method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which Newton's method will work, but bisection will fail? If so, give an example.

Exercise 1.5.3

(i) Let q be your student ID number. Find k and m where $k-2$ is the remainder on dividing q by 4, and $m-2$ is the remainder on dividing q by 6.

(ii) Show how Newton's method can be applied to estimate the postive real number $m^{1/k}$. That is, state the nonlinear equation you would solve, and give the formula for Newton's method, simplified as much as possible.

(iii) Do three iterations by hand of Newton's Method for this problem, taking $x_0 = 1$.

Exercise 1.5.4

Suppose we want apply to Newton's method to solving $f(x) = 0$ where f is such that $|f''(x)| \leq 10$ and $|f'(x)| \geq 2$ for all x. How close must x_0 be to τ for the method to converge?

See version at
https://www.niallmadden.ie/2526-MA385/1-5-Newton.pdf
for the full list of exercises.