MA385 Part 3: Linear Algebra 1

3.2: Gaussian Elimination + Triangular Matrices

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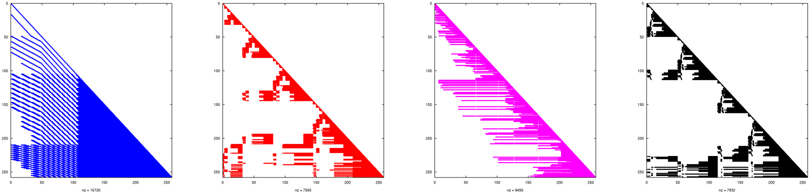

These notes are from the lectures on 23 Oct (Week 7) and 03 Nov (Week 9). (There were not regular lectures in Week 8 due to a public holiday and class test).

1. Outline of Section 3.2

1. Gaussian Elimination
2. Row operations are matrix multiplication
3. Triangular Matrices
4. Exercises

For more, see Section 2.2 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG

2

2. Gaussian Elimination

Gaussian Elimination is an **exact** method for solving linear systems (we replace the problem with one that is easier to solve *and* has the same solution.)
This is in contrast to **approximate** methods studied earlier in the module.
There are approximate methods for solving linear systems, but they are not part of this module.
a major

Carl Freidrich Gauß, Germany, 1777-1855.
Although he produced many very important original ideas, this wasn't one of them. The Chinese knew of "Gaussian Elimination" about 2000 years ago. His actual contributions included major discoveries in the areas of number theory, geometry, and astronomy.

Example 3.2.1

Consider the problem:

$$\begin{pmatrix} -1 & 3 & -1 \\ 3 & 1 & -2 \\ 2 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}$$

We can perform a sequence of elementary row operations to yield the system:

$$\begin{pmatrix} -1 & 3 & -1 \\ 0 & 10 & -5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ -5 \end{pmatrix}.$$

$5x_3 = -5 \Rightarrow x_3 = -1$

$10x_2 = 15 + 5x_3 = 10 \Rightarrow x_2 = 1, \Rightarrow x_1 = -1$

The first steps in detail:

$$\begin{pmatrix} -1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}$$

$$\xrightarrow{r_2 + 3r_1} \begin{pmatrix} -1 & 3 & -1 \\ 0 & 10 & -1 \\ 2 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ -9 \end{pmatrix}$$

That row reduction is the same as

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}$$

3. Row operations are matrix multiplication

Gaussian Elimination: perform elementary row operations such as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

being replaced by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \mu_{21} a_{11} & a_{22} + \mu_{21} a_{12} & a_{23} + \mu_{21} a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A + \mu_{21} \begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix}$$

where $\mu_{21} = -a_{21}/a_{11}$, so that $a_{21} + \mu_{21} a_{11} = 0$.

3. Row operations are matrix multiplication

Note that

$$\begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

so we can write the row operation as $\big(I + \mu_{21} E^{(21)}\big)A$, where $E^{(pq)}$ is the matrix of all zeros, except for $e_{pq} = 1$.

In general each of the row operations in Gaussian Elimination can be written as

$$\big(I + \mu_{pq} E^{(pq)}\big)A \quad \text{where } 1 \le q < p \le n, \tag{1}$$

and $\big(I + \mu_{pq} E^{(pq)}\big)$ is an example of a **Unit Lower Triangular Matrix**.

3. Row operations are matrix multiplication

We can conclude that each step of the process will involve multiplying A by a unit lower triangular matrix, resulting in an upper triangular matrix.

4. Triangular Matrices

Definition 3.2.1 (Lower Triangular)

$L \in \mathbb{R}^{n\times n}$ is a *Lower Triangular (LT) Matrix* if the only non-zero entries are on or below the main diagonal, i.e., if $l_{ij} = 0$ for $1 \le i < j \le n$.
It is a *unit Lower Triangular matrix* if, in addition, $l_{ii} = 1$.

Examples:

$L_2 = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is Lower Triangular.

So too is $L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

And any Identity Matrix.

$\tilde{L}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5 & 1 \end{pmatrix}$

(Finished here)

Definition 3.2.2 (Upper Triangular)

$U \in \mathbb{R}^{n\times n}$ is an *Upper Triangular (UT) matrix* if $u_{ij} = 0$ for $1 \le j < i \le n$. It is a *Unit Upper Triangular Matrix* if $u_{ii} = 1$.

Examples: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a UT matrix.

It is Unit UT if $a = d = 1$.

Any identity matrix is Unit UT.

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ is Unit UT.

If A is LT then A^T is UT.

Triangular matrices have many important properties. A very important one is: **the determinant of a triangular matrix is the product of the diagonal entries**:

Eg: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then

$$\det(A) = a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

If, eg, A is lower Triangular:

$A = \begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{pmatrix}$, so $b = c = 0$. Then

$$\det(A) = a \cdot \det\begin{pmatrix} e & \overset{=0}{f} \\ h & i \end{pmatrix} = a e i.$$

There are other important properties of triangular matrices, but first we need the idea of **matrix partitioning**.

Definition 3.2.3 (Submatrix)

X is a *submatrix* of A if it can be obtained by deleting some rows and columns of A.

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then the following are submatrices:

$X_1 = (1)$ $\quad X_2 = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ $\quad X_3 = \begin{pmatrix} 1 & 3 \\ 4 & 6 \\ 7 & 9 \end{pmatrix}$

But not $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 5 & 9 \end{pmatrix}$

Definition 3.2.4 (Leading Principal Submatrix)

The **Leading Principal Submatrix** of order k of $A \in \mathbb{R}^{n \times n}$ is $A^{(k)} \in \mathbb{R}^{k \times k}$ obtained by deleting all but the first k rows and columns of A. (Simply put, it's the $k \times k$ matrix in the top left-hand corner of A).

Example: Again $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Its leading Principal Submatrices are

$A^{(0)} = (\,)$, $A^{(1)} = (1)$, $A^{2} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$, $A^{(3)} = A$

Matrix partitioning

To **partition a matrix** means to divide it into contiguous blocks that are submatrices.

Example:

$$A = \left(\begin{array}{cc|c} a & b & c \\ d & e & f \\ \hline g & h & i \end{array}\right) \quad \text{is partitioned as}$$

$$A^{(2)} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \qquad B = \begin{pmatrix} c \\ f \end{pmatrix} \qquad A = \left(\begin{array}{c|c} A^{(2)} & B \\ \hline C & D \end{array}\right)$$

$$C = \begin{pmatrix} g & h \end{pmatrix} \qquad D = \begin{pmatrix} i \end{pmatrix}$$

4. Triangular Matrices

Next recall that if A and V are matrices of the same size, and each are partitioned

$$A = \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right), \qquad \text{and} \qquad V = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right),$$

where B is the same size as W, C is the same size as X, etc. Then

$$AV = \left(\begin{array}{c|c} BW + CY & BX + CZ \\ \hline DW + EY & DX + EZ \end{array}\right).$$

Theorem 3.2.1 (Properties of Lower Triangular Matrices)

For any integer $n \geq 2$:

(i) If L_1 and L_2 are $n \times n$ *Lower Triangular* (LT) Matrices that so too is their product $L_1 L_2$.

(ii) If L_1 and L_2 are $n \times n$ *Unit Lower Triangular* matrices, then so too is their product $L_1 L_2$.

(iii) L_1 is nonsingular if and only if all the $l_{ii} \neq 0$. In particular all Unit LT matrices are nonsingular.

(iv) The inverse of a LT matrix is an LT matrix. The inverse of a unit LT matrix is a unit LT matrix.

We restate Part (iv) as follows:

> *Suppose that $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix with $n \geq 2$, and that there is a matrix $L^{-1} \in \mathbb{R}^{n\times n}$ such that $L^{-1}L = I_n$. Then L^{-1} is also a lower triangular matrix.*

Proof is by Induction. First suppose that $n=2$.
Write L as

$$L = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

It is assumed L^{-1} exists so $\det(L) = ad \neq 0$. So $a \neq 0$, $d \neq 0$. Also

$$L^{-1} = \frac{1}{ad}\begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$$

which is an LT matrix.

Next, assume the theorem holds for any Matrix of size up to $(n-1)\times(n-1)$.

Let L be an $n\times n$ LT matrix. Partition it by the last row and column.

$$L = \left(\begin{array}{c|c} L^{(n-1)} & \vec{0} \\ \hline c^T & d \end{array}\right)$$

where $L^{(n-1)}$ is an $(n-1)\times(n-1)$ LT matrix.
$\vec{0}$ is the zero $(n-1)$-vector,
c is an $(n-1)$-vector, and d is a scalar.

Now Portition L^{-1} as

$$L^{-1} = \left(\begin{array}{c|c} W & x \\ \hline y^T & z \end{array}\right).$$

Then, since

$I_n = L L^{-1}$, we have

$$\left(\begin{array}{c|c} I_{n-1} & \vec{0} \\ \hline \vec{0}^T & 1 \end{array}\right) = \left(\begin{array}{c|c} L^{(n-1)} & \vec{0} \\ \hline c^T & d \end{array}\right) \left(\begin{array}{c|c} W & x \\ \hline y^T & z \end{array}\right)$$

$$= \left(\begin{array}{c|c} L^{(n-1)} W + \vec{0} y & L^{(n-1)} x + \vec{0} z \\ \hline c^T W + d y^T & c^T x + d z \end{array}\right)$$

This is, essentially, 4 equations

① : $L^{(n-1)} W = I_{n-1}$. So $W = (L^{(n-1)})^{-1}$

which, by the inductive hypothesis,

• Exists • is an LT matrix

② $L^{(n-1)} x = \vec{0}$ so $x = \vec{0}$.

③ $c^T W + d y^T = \vec{0}^T$ so $y^T = -\frac{1}{d} c^T W$

④ $c^T x + d z = 1$ $\Rightarrow$ $z = 1/d$.

So L^{-1} exist, and from ① & ②

is an LT Matrix. ∎

Theorem 3.2.2 (Properties of Upper Triangular Matrices)

Statements that are analogous to those concerning the properties of lower triangular matrices hold for upper triangular and unit lower triangular matrices. (For proof, see the exercises at the end of this section).

Exercise 3.2.1

Every step of Gaussian Elimination can be thought of as a left multiplication by a unit lower triangular matrix. That is, we obtain an upper triangular matrix U by multiplying A by k unit lower triangular matrices: $L_k L_{k-1} L_{k-2} \dots L_2 L_1 A = U$, where each $L_i = I + \mu_{pq} E^{(pq)}$, and $E^{(pq)}$ is the matrix whose only non-zero entry is $e_{pq} = 1$. Give an expression for k in terms of n.

Exercise 3.2.2

Let L be a lower triangular $n \times n$ matrix. Show that $\det(L) = \prod_{j=1}^{n} l_{jj}$. Hence give a necessary and sufficient condition for L to be invertible. What does that tell us about *Unit* Lower Triangular Matrices?

Exercise 3.2.3

Let L be a lower triangular matrix. Show that each diagonal entry of L, l_{jj} is an eigenvalue of L.

5. Exercises

Exercise 3.2.4

Prove Parts (i)–(iii) of Theorem 3.2.1 (Properties of Triangular Matrices).

Exercise 3.2.5

Suppose the $n \times n$ matrices A and C are both lower triangular matrices, and that there is a $n \times n$ matrix B such that $AB = C$. Must B be a lower triangular matrix?
Suppose A and C are *unit* lower triangular matrices, and $AB = C$. Must B be a unit lower triangular matrix? Why?

Exercise 3.2.6

Construct an alternative proof of the first part of Theorem 3.7 (iv) as follows: Suppose that L is a non-singular lower triangular matrix. If $\mathrm{b} \in \mathbb{R}^n$ is such that $b_i = 0$ for $i = 1, \dots, k \leq n$, and y solves $L\mathrm{y} = \mathrm{b}$, then $y_i = 0$ for $i = 1, \dots, k \leq n$. (Hint: partition L by the first k rows and columns.)
Now use this to give a alternative proof of the fact that the inverse of a lower triangular matrix is itself lower triangular.