2526-MA140 Engineering Calculus

Week 04, Lecture 3
The Chain Rule and Inverse Functions

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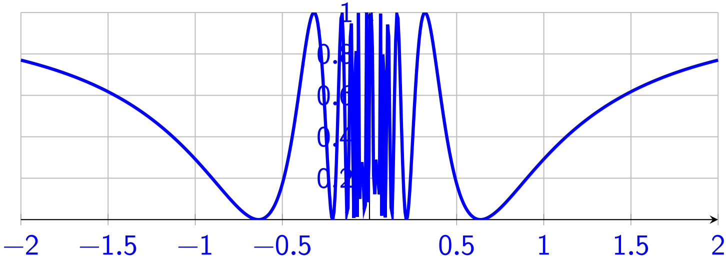
University of Galway

Thursday, 09 October, 2025



Assignments, etc

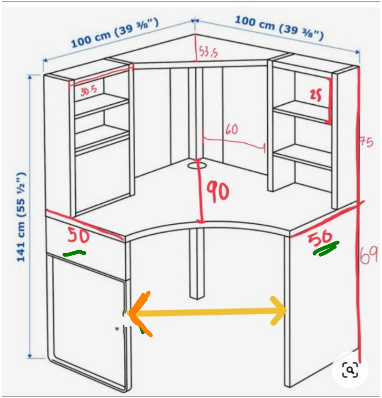
If emailing: include MA140 in the subject.

Assignments

- **Assignment 2** is open. See https://universityofgalway.instructure.com/courses/35693/assignments/96620.
  Due by 17:00, Monday 13 October.
- The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2065926
- **Assignment 3** is also open. Access through Canvas, or at https://universityofgalway.instructure.com/courses/46734/assignments/130491 Due by 17:00. Monday 20 October.

D Class Test Tuesday!

Warm-up



"Olive" is thinking of buying this desk unit in IKEA. Her (wheel)chair is 55cm. Is the sitting region of the desk indicated by the yellow line, wide enough?

IKEA MIKNE ?

What we'll do today:

See also: Sections 3.6 (The Chain Rule) and 3.8 (Implicit Differentiation) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Chain Rule (again)

Yesterday, we first learned about the *most important* differentiation rule: **chain rule**. It applies to a "function of a function"

The Chain Rule

If $u(x)$ and $v(x)$ are differentiable, and f is the composite function $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

"f of u of v of x"

Example (Ex 3.6.1 in text-book)

Find the derivative of $f(x) = \dfrac{1}{(3x^2+1)^2}$. $= [3x^2 - 1]^{-2}$

$u(v) = v^{-2}$

$v(x) = 3x^2+1$

$\dfrac{du}{dv} = -2v^{-3}$

$\dfrac{dv}{dx} = 6x$

$\dfrac{df}{dx} = \dfrac{du}{dv} \cdot \dfrac{dv}{dx} = \left(-2v^{-3}\right) \cdot 6x$

$= \dfrac{-12x}{v^3} = \dfrac{-12x}{(3x^2+1)^3}.$

Composites of 3 or more functions

One can apply the **Chain Rule** to "functions of functions of functions": if $y(x) = t\Big(u\big(v(x)\big)\Big)$, then

$$\frac{dy}{dx} = \frac{dt}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$ when $y = \sin^4(x^5 + 7)$. $= \left[\sin(x^5+7)\right]^4$

$t(u) = u^4$

$\frac{dt}{du} = 4u^3$
$= 4(\sin(v))^3$
$= 4\sin^3(x^5+7)$

$u(v) = \sin(v)$

$\frac{du}{dv} = \cos(v)$
$= \cos(x^5+7)$

$v(x) = x^5+7$

$\frac{dv}{dx} = 5x^4$

$\frac{dy}{dx} = 4\sin^3(x^5+7)\cdot\cos(x^5+7)\cdot 5x^4.$

Composites of 3 or more functions

Example

Show that the derivative of $y = \cos^2(1/x)$ is

$$\frac{dy}{dx} = 2\frac{\sin(1/x)\cos(1/x)}{x^2}.$$

$\left[\cos\left(\frac{1}{x}\right)\right]^2$

$f(x) = t(u(v(x)))$

$t(u) = u^2$

$\frac{dt}{du} = 2u$

$= 2\cos(v) = 2\cos\left(\frac{1}{x}\right)$

$u(v) = \cos(v)$

$\frac{du}{dv} = -\sin(v)$

$= -\sin\left(\frac{1}{x}\right)$

$v(x) = \frac{1}{x} = x^{-1}$

$\frac{dv}{dx} = -x^{-2}$

$= -1/x^2$

$$\frac{dy}{dx} = \frac{dt}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} = 2\cos(1/x)(-\sin(1/x))\cdot\left(-\frac{1}{x^2}\right)$$

$$= \frac{2}{x^2}\sin(1/x)\cos(1/x).$$

Inverse functions

Suppose that $y = f(x)$. That is, f maps x to y.

Then the **inverse** of f is the function, f^{-1}, that maps y back to x.

Example

- The inverse of $f(x) = \frac{1}{2}x$ is $f^{-1}(x) = 2x$.
- The inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$. ✓

Warning: $f^{-1}(x)$ is not the same as $\dfrac{1}{f(x)}$.

Check that $f^{-1}(f(x)) = x = f(f^{-1}(x))$

(f, eg, $f(x) = \frac{x}{2}$ & $f^{-1}(x) = 2x$

then $f^{-1}(f(x)) = f^{-1}\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) = x$ ✓

It is often useful to be able to express the derivative (assuming there is one) of an inverse function $f^{-1}(x)$ in terms of the derivative of $f(x)$.

To do this, we use the following rule:

Inverse-Function Rule

If $\underline{y = f^{-1}(x)}$, then $x = f(y)$ and also

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(\underline{y})} = \frac{1}{f'(\underline{f^{-1}(x)})}.$$

Example

If $y = x^{1/3}$, use the **Inverse Rule** to find $\frac{dy}{dx}$.

Note: we can solve this just using the **Power Rule:** $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$.

But we'll also do this with the **Inverse Rule** for purposes of *exposition*.

If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2.$$

By the inverse rule, $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}.$$

As $y = x^{\frac{1}{3}}$ we have

$$\frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}}.$$

Example

Find the derivative of $\sin^{-1}(x)$. Note: this is not $\frac{1}{\sin(x)}$!!

Let $y = \sin^{-1}(x)$, then $x = \sin(y)$ $(\star)$, so

$$\frac{dx}{dy} = \cos(y)\,. \qquad (\star\star)$$

From $\sin^2(y) + \cos^2(y) = 1$ we find $\cos(y) = \sqrt{1 - \sin^2(y)}$ $= \sqrt{1+(\sin(y))^2}$
(choosing the positive square root as $\cos(y)$ is positive for y here).
Using $(\star)$:

$$\cos y = \sqrt{1 - x^2}\,.$$ because $\sin(y) = x$

Now using the inverse rule and $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}}\,.$$

Implicit differentiation

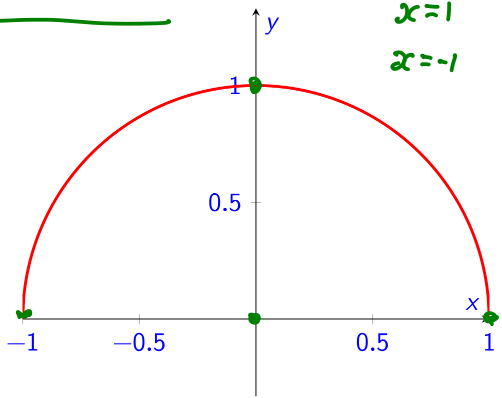
To date, most functions we have studied have been **explicitly** defined. Such functions and be written as $y = f(x)$: given a value of x we can substitute it into $f(x)$ to get the corresponding value of y

Example: $y = \sqrt{1 - x^2}$.

Note $x=0$ $y=1$

$x=1$ $y=0$

$x=-1$ $y=0$



Implicit differentiation

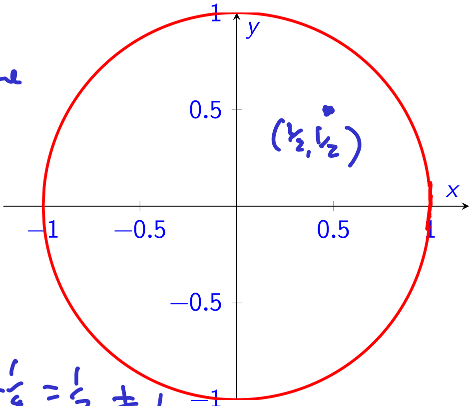
However, sometimes we are given an equation involving x and y where these two terms are not "separated" entirely; e.g, $x^2 + y^2 = 1$. Here y is **implicitly** defined: for any pair (x, y) we can check if it is on the curve described by the equation.

Eg: (x, y)

is $(1,0)$ on the curve? yes: $1^2 + 0^2 \stackrel{\checkmark}{=} 1$.

is $(\frac{1}{2}, \frac{1}{2})$ on the curve? No!

$\frac{1}{2}^2 + \frac{1}{2}^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1$.



Finished here

Implicit differentiation

Since **implicit equations** define curves, we can use **implicit differentiation**, for example, finding tangents to these curves.

Method:

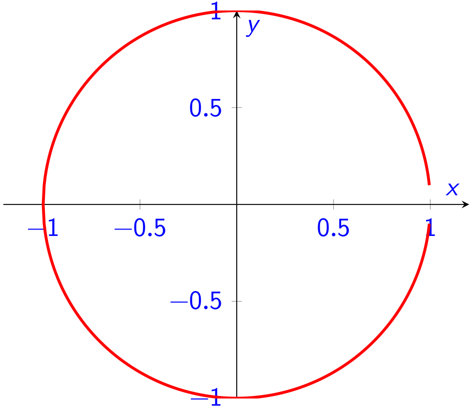
1. Differentiate both size of the equation, with respect to x, keeping in mind that y is a function of x, using the Chain Rule where needed.
2. Solve for dy/dx.

Implicit differentiation

If y is defined by $x^2 + y^2 = 1$, find $\dfrac{dy}{dx}$.

Implicit differentiation

Now we know that if $x^2 + y^2 = 1$, then $\dfrac{dy}{dx} = -\dfrac{x}{y}$. We can check that this relates to the slope of the tangents to this curve at various places:



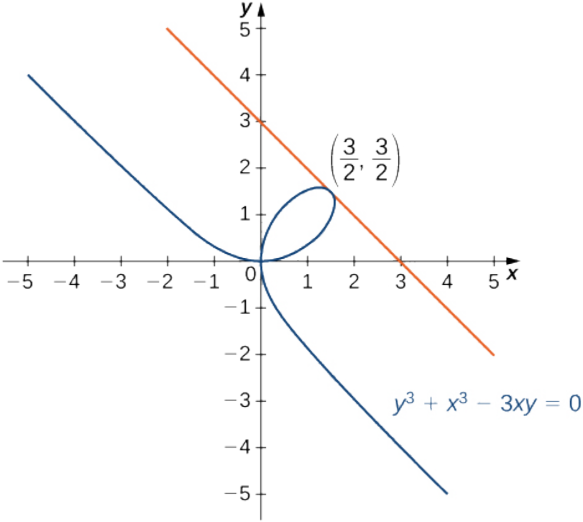
Implicit differentiation

Find the tangent to the curve $x^2+y^2=25$, at the point $(3,-4)$.

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$.

Implicit differentiation



Exponential functions

Earlier in this course we met functions such as $y = x^2$; this is a **power** function.

Now we consider **exponential functions**, such as $\boxed{y = 2^x}$.

Such functions occur in many applications. For example: if I invest €100 with an annual interest rate of 20%, then after x years, I will have €$100 \times (1.2)^x$. **Why?**

Exponential functions

Exponential functions grow quite fast: if my investment is indeed worth $f(x) = 100 \times (1.2)^x$ euros after x years, then...

- After 1 year, I have €120
- After 10 years, I have €619.17
- After 20 years, I have €3,833.80
- After 25 years, I have €9,539.60
- After 50 years, and 190 days, I'll be a millionaire!

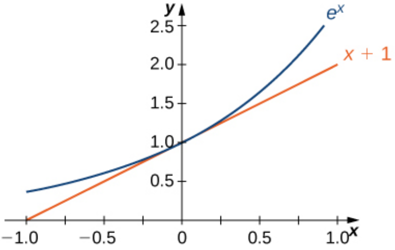
Here I remind you of some properties of exponents that you should already know: for any positive numbers a and b,

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x a^y$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{a^y}$

The number $e \approx 2.7182818284$. It is often called **Euler's Number** after Leonard Euler, who did not discover it: that was (probably) Jacob Bernoulli in 1683 while studying compound interest. Or maybe 100 years earlier by John Napier.

The Natural Exponential Function

The Natural Exponential Function is $f(x) = e^x$. It is special for many reasons, including the its tangent at $x = 0$ has slope 1.



Let's assume that e is the number for which, if $f(x) = e^x$, then $f'(0) = 1$. Using the limit definition of the derivative, this means

$$1 = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

From this can deduce that...

So now we know that

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

That is e^x is the function that is its own derivative!!!

Example

Compute the derivative of $f(x) = e^{\sin(x)}$

Exercises

Exercise 4.3.1

Find the derivative of

1. $f(x) = x^3 \cos(x^2)$
2. $f(x) = \tan^3\left(\sin^2(x^4)\right)$

Exercise 4.3.2

Show that $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$.

Exercise 4.3.3

Find the equation of the tangent to the curve defined by $x^2 - y^2 = 16$ at the point $(5, 3)$.