2425-MA140 Engineering Calculus

Week 06, Lecture 2
Limits at infinity

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Assignments, etc

- **Assignment 3**: The saga continues...
  1. If you think your correct grade is not showing, send me an email with your **results summary**.
  2. If you like, I'll let you reattempt. (Deadline Friday).
- **Assignment 4**: this "optional" assignment has started. Deadline 5pm, Tuesday 29th October.

In today's class...

1. Limits at infinity
   - Definition
2. Computing limits at infinity
   - Rational functions
3. Curve Sketching (over large domains)
4. Exercises

See also: 4.6 (Limits at Infinity and Asymptotes) in **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Limits at infinity

We now know how to use the first and second derivatives of a function to describe the shape of a graph on a domain (a, b).

However, sometimes we'll wish to graph a function, f, defined on an unbounded domain. So we'll need to know f behaves as $x \to -\infty$ and/or $x \to \infty$.

So now we'll learn about **limits at infinity**, and how these limits affect the graph of a function.

Limits at infinity

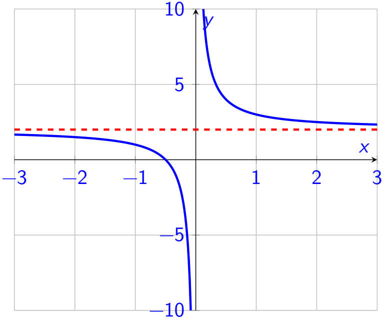
Recall...

We learned in Week 2, that if we write $\lim_{x \to a} f(x) = L$, then the value of $f(x)$ approaches L as x approaches a (regardless of what actually happens at a).
Now we consider what happens as $x \to \pm\infty$.

Limits at infinity

Here we show the graph of $f(x) = 2 + \dfrac{1}{x}$. Observe that

- As $x \to \infty$, $f(x) \to 2$. This is because, as x gets very large, so $1/x$ gets very small.
- Similarly, as $x \to -\infty$ we see that, again $f(x) \to 2$.



So we write

$$\lim_{x\to-\infty} = 2, \qquad \text{and} \qquad \lim_{x\to\infty} = 2.$$

Limit at infinity: Informal definition

We write $\lim_{x \to \infty} f(x) = L$ if the value of $f(x)$ can be made as close to L as we like, by taking x as large as needed. (And $f(x)$ is closer still L for any larger x).

We write $\lim_{x \to -\infty} f(x) = L$ if, for $x < 0$, the value of $f(x)$ can be made as close to L as we like, by taking $|x|$ as large as needed. (And $f(x)$ is closer still L for any larger $|x|$).

Horizontal Asymptote

If $\lim_{x \to \infty} f(x) = L$, or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

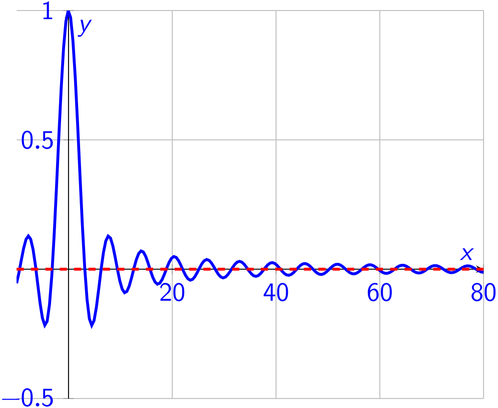
Computing limits at infinity

The key facts to know are:

- $\lim_{x\to\infty} \frac{1}{x} = 0$;
- The properties of limits from Week 2, Lecture 2 still hold. In particular (assuming the limits exist)
  - $\lim_{x\to\infty} \big(f(x) + g(x)\big) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$.
  - $\lim_{x\to\infty} \big(f(x)g(x)\big) = \big(\lim_{x\to\infty} f(x)\big)\big(\lim_{x\to\infty} g(x)\big)$.
  - The Squeeze Theorem

Example: Find the limit of $f(x) = \dfrac{\sin(x)}{x}$ as $x \to \infty$.

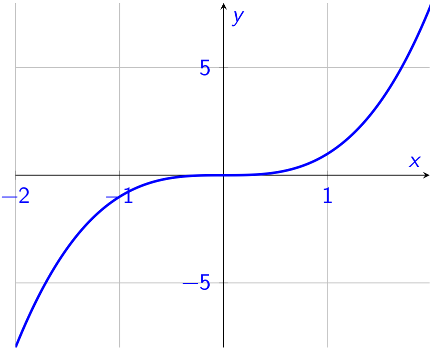
Computing limits at infinity



Computing limits at infinity

Of course, many functions do not have a finite limit at infinity. For example,

$$\lim_{x\to-\infty} x^3 = -\infty, \qquad \text{and} \qquad \lim_{x\to-\infty} x^3 = \infty.$$



When computing the limit at infinity of a rational function,

- Divide the numerator and denominator by the highest power of x in the denominator
- Apply the limit laws.

Example: Evaluate $\lim_{x\to\infty} \dfrac{3x^2-1}{2x^2+4}$.

Examples

Evaluate the following limits

(i) $\displaystyle\lim_{x\to\infty} \frac{x+123}{x^2+1}$ (ii) $\displaystyle\lim_{x\to\infty} \frac{x^2-9}{x+3}$.

Curve Sketching (over large domains)

In order to roughly **sketch the graph** of a function, f, over a large domain, the approach is similar to yesterday, but we also calculate the limits at infinity:

1. Compute $f'(x)$ and find the critical points. Do they correspond to maxmima, minima or neither (note: may need the 2nd Derivative test).
2. Compute $f''(x)$, and find points of inflection.
3. Evaluate the limits at $\pm\infty$, and add any horizontal asymptotes.
4. Compute some specific points, e.g. at the critical and inflection points, y-intercept and, if possible, and x-intercept.
5. Plot the points from the previous step, and fill in the graph using information on the local max/min and inflection points.

Curve Sketching (over large domains)

Example

Sketch the graph of
$$f(x) = \frac{(1+x)^2}{1+x^2}$$

Note: $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2}$ and $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$.

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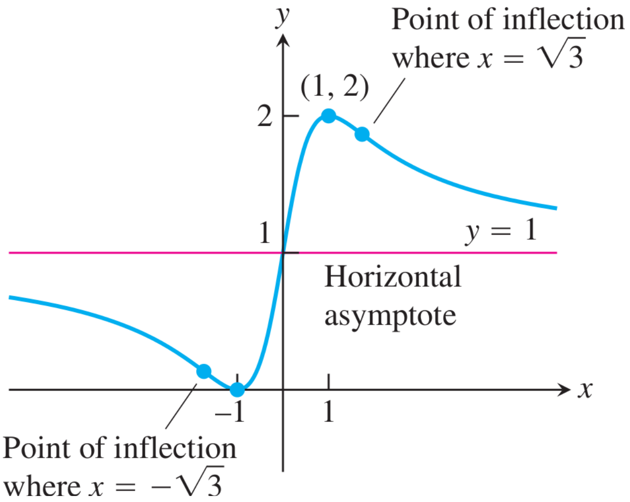

Exercises

Exer 6.2.1 (Example 4.6.9 from the textbook)

Sketch the graph of $f(x) = \dfrac{x^2}{1-x^2}$.