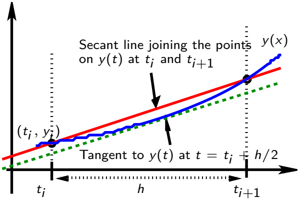
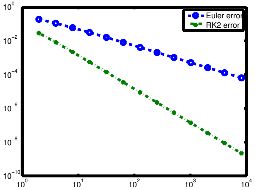
Initial Value Problems

# §2.4 Runge-Kutta 2 (RK2)

MA385/530 – Numerical Analysis 1

October 2019




<< Annotated slides >>>

Recall our original motivation of Euler's method: use the slope of the tangent to $y$ at $t_i$ as an approximation for the slope of the secant line joining the points $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$.

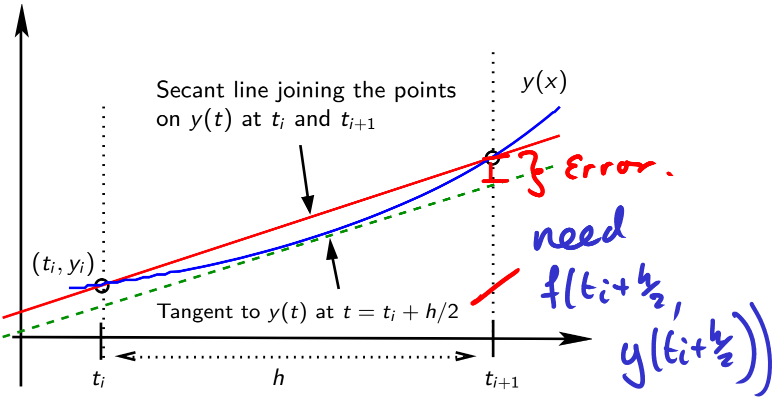
One could argue, given the diagram on the next slide, that the slope of the tangent to $y$ at $t = (t_i + t_{i+1})/2 = t_i + h/2$ would be a better approximation. This would give

$$y(t_{i+1}) \approx y_i + hf(t_i + \frac{h}{2}, y(t_i + \frac{h}{2})). \tag{8}$$

However, we don't know $y(t_i + h/2)$, but can approximate it using Euler's Method: $y(t_i + h/2) \approx y_i + (h/2)f(t_i, y_i)$.

## Modified (Midpoint) Euler's Method

$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right). \tag{9}$$



**Example 2.11**

*Use the Modified Euler Method to approximate $y(1)$ where*

$$y(0) = 1, \qquad y'(t) = y\log(1+t^2).$$

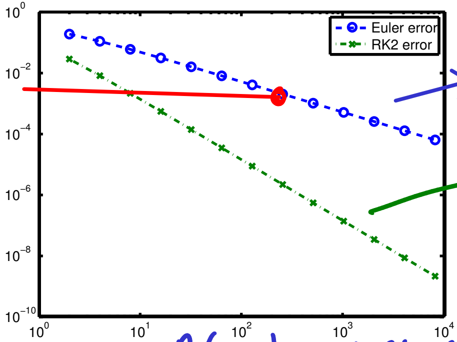
This has the solution $y(t) = (1+t^2)^t \exp(-2t + 2\tan^{-1} t)$.

| | Euler | | Modified | |
|---|---|---|---|---|
| $n$ | $\mathcal{E}_n$ | $\mathcal{E}_n/\mathcal{E}_{n-1}$ | $\mathcal{E}_n$ | $\mathcal{E}_n/\mathcal{E}_{n-1}$ |
| 1 | 3.02e-01 | | 7.89e-02 | |
| 2 | 1.90e-01 | 1.59 | 2.90e-02 | 2.72 |
| 4 | 1.11e-01 | 1.72 | 8.20e-03 | 3.54 |
| 8 | 6.02e-02 | 1.84 | 2.16e-03 | 3.79 |
| 16 | 3.14e-02 | 1.91 | 5.55e-04 | 3.90 |
| 32 | 1.61e-02 | 1.95 | 1.40e-04 | 3.95 |
| 64 | 8.13e-03 | 1.98 | 3.53e-05 | 3.98 |
| 128 | 4.09e-03 | 1.99 | 8.84e-06 | 3.99 |

Clearly we get a much more accurate result using the Modified Euler Method. Even more importantly, we get a higher *order of accuracy*: if $h$ is reduced by a factor of **two**, the error in the Modified method is reduced by a factor of **four**.

We can also make a direct comparison of the two methods by using a log-log plot of the errors. $|\mathcal{E}_n| \leq K h^p \; (= K n^{-p})$.

Error.




Slope is $-1$

Slope is $-2$, ie $p = 2$.

n (number of steps

Figure: Log-log plot of the errors when Euler's and Modified Euler's methods are used to solve Example 2.11

The "*Modified Euler Method*" is an example of one of the (large) family of 2nd-*order Runge-Kutta* (RK2). Recall that that one-step methods are written as $y_{i+1} = y_i + h\Phi(t_i, y_i; h)$

**The general RK2 method is**

$$k_1 = f(t_i, y_i) \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1).$$
$$\Phi(t_i, y_i; h) = (ak_1 + bk_2) \tag{10}$$

**Example:** take $a = 1, b = 0$.

Then $\Phi(t_i, y_i; h) = k_1 = f(t_i, y_i)$

This is just Euler's Method (boring!)

so exclude this case.

**The general RK2 method is**

$$k_1 = f(t_i, y_i) \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1).$$
$$y_{i+1} = y_i + h(a k_1 + b k_2)$$

**Example 2:** take $\alpha = \beta = 1/2$, $a = 0$, $b = 1$.

So

$$y_{i+1} = y_i + h(k_2)$$
$$= y_i + h f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right),$$

ie Modified (Midpoint) Euler Method.

Our aim now is to deduce general rules for choosing $a$, $b$, $\alpha$ and $\beta$. We'll see that if we pick any one of these four parameters, then the requirement that the method be consistent and second-order determines the other three.

By demanding that RK2 be $\boxed{\textbf{\textit{consistent}}}$ we get that $\boxed{a + b = 1.}$

Recall: a one-step method is consistent if

$$\Phi(t_i, y_i; 0) = f(t_i, y_i).$$

For an RK2 method, if $h=0$, then

$$K_1 = f(t_i, y_i) \quad \text{and} \quad K_2 = f(t_i + \cancel{\alpha h}^{\to 0}, y_i + \cancel{\beta h}^{\to 0} K_1).$$

So $\Phi(t_i, y_i; 0) = a f(t_i, y_i) + b f(t_i, y_i)$. This gives that $a+b=1$ for the method to be consistent

Next we need to know how to choose $\alpha$ and $\beta$. The formal way is to use a two-dimensional Taylor series expansion. It is quite technical. (FYI, detailed will be posted as an appendix to these notes). Instead we'll take a less rigorous, ***heuristic*** approach.

(From Wikipedia: "A heuristic technique (Ancient Greek: 'find' or 'discover'), often called simply a heuristic, is any approach to problem solving, learning, or discovery that employs a practical method, not guaranteed to be optimal, perfect, logical, or rational, but instead sufficient for reaching an immediate goal.")

Because we expect that, for a second order accurate method, $|\mathcal{E}_n| \leq Kh^2$ where $K$ depends on $y'''(t)$, if we choose a problem for which $y'''(t) \equiv 0$, we expect no error...

Let's take $y(t) = t^2$. So $y'(t) = 2t$, $y''(t) = 2$

and $y'''(t) = 0$. So $y(t)$ solve

$y(1) = 1$ and $y'(t) = f(t, y) = 2t$ for $t > 1$

Since our method should be exact (ie, no Error) for any $h$, so, for simplicity, take $h = 1$, and solve for $y(2) = 2^2$ with RK2 (ie $t_1 = 2$).

$K_1 = f(t_0, y_0) = 2(t_0) = 2$.

$K_2 = f(t_0 + \alpha h, y_0 + \beta h K_1) = 2(1 + \alpha)$

Because we expect that, for a second order accurate method, $|\mathcal{E}_n| \leq Kh^2$ where $K$ depends on $y'''(t)$, if we choose a problem for which $y'''(t) \equiv 0$, we expect no error...

$$K_1 = f(t_0, y_0) = 2(t_0) = 2.$$

$$K_2 = f(t_0 + \alpha h, y_0 + \beta h K_1) = 2(1 + \alpha)$$

This gives

$$y_1 = y_0 + h(aK_1 + bK_2)$$

$$= 1 + 1(a(2) + b(2 + 2\alpha)).$$

Then, since $y_1 = 2^2 = 4.$ and $a + b = 1$.

So

$$4 = 1 + 2(1 - b) + 2b + 2b\alpha.$$

Rearrange to get $\boxed{\alpha = \frac{1}{2b}.}$ (note, this is defined for any $b \neq 0$)

In the above example, the right-hand side of the differential equation, $f(t, y)$, depended only on $t$. Now we'll try the same trick: using a problem with a simple known solution (and zero error), but for which $f$ depends explicitly on $y$. $y_0 = 1, \; t_0 = 1.$

Consider the DE $\boxed{y(1) = 1}$, $y'(t) = y(t)/t$. It has a simple solution: $y(t) = t$. We now use that any RK2 method should be exact for this problem to deduce that $\alpha = \beta$.

Here, $y(t) = t$, and $y'(t) = \dfrac{y(t)}{t} = \dfrac{t}{t} = 1.$

Again, take $h = 1$, and solve for $y(2) = 2$.

we have

$$K_1(t_0, y_0) = \frac{y_0}{t_0} = 1$$

$$K_2(t_0 + \alpha h, y_0 + \beta h K_1) = \frac{1+\beta}{1+\alpha}.$$

$$K_1(t_0, y_0) = \frac{y_0}{t_0} = 1$$

$$K_2(t_0 + \alpha h, y_0 + \beta h K_1) = \frac{1+\beta}{1+\alpha}.$$

use $a+b=1$, ie $a = 1-b$.

Then, using that $y(2)=2$, we get

$$y_1 = y_0 + aK_1 + bK_2$$

$$\Rightarrow \quad 2 = 1 + (1-b) + b\,\frac{1+\beta}{1+\alpha}.$$

$$\Rightarrow \quad 0 = -b + b\,\frac{1+\beta}{1+\alpha}$$

Then, since $b \neq 0$, we get $\frac{1+\beta}{1+\alpha} = 1$.

Thus $\alpha = \beta$.

Now we collect the above results all together and show that the second-order Runge-Kutta (RK2) methods are:

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(t_i, y_i), \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1),$$

where we choose any $b \neq 0$ and then set

$$a = 1 - b, \qquad \alpha = \frac{1}{2b}, \qquad \beta = \alpha.$$

It is easy to verify that the Modified method satisfies these criteria.

## Exercise 2.6

A popular RK2 method, called the *Improved Euler Method*, is obtained by choosing $\alpha = 1$.

(i) Use the Improved Euler Method to find an approximation for $y(4)$ when

$$y(0) = 1, \qquad y' = y/(1 + t^2),$$

taking $n = 2$. (If you wish, use MATLAB.)

(ii) Using a diagram similar to the one in Figure 1 for the Modified Euler Method, justify the assertion that the Improved Euler Method is more accurate than the basic Euler Method.

(iii) Show that the method is consistent.

(iv) Write out what this method would be for the problem: $y'(t) = \lambda y$ for a constant $\lambda$. How does this relate to the Taylor series expansion for $y(t_{i+1})$ about the point $t_i$?

## Exercise 2.7

In his seminal paper of 1901, Carl Runge gave an example of what we now call a *Runge-Kutta 2 method*, where

$$\Phi(t_i, y_i; h) = \frac{1}{4} f(t_i, y_i) + \frac{3}{4} f\Big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\Big).$$

(i) Show that it is consistent.

(ii) Show how this method fits into the general framework of RK2 methods. That is, what are $a$, $b$, $\alpha$, and $\beta$? Do they satisfy the following conditions?

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b. \tag{11}$$

(iii) Use it to estimate the solution at the point $t = 2$ to $y(1) = 1$, $y' = 1 + t + y/t$ taking $n = 2$ time steps.