CS4423: Networks

Week 3, Lecture 1: Matrices and Walks

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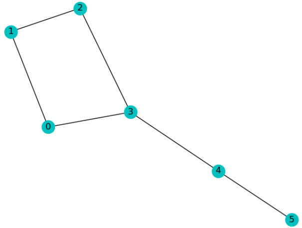


This version of the slides are by Niall Madden. Elements are based on notes by Dr Angela Carevale and "A First Course in Network Theory" by Estrada and Knight

News.

Lab 0 : this week. Next one at 10 am in CA 116a (I think).

Lab 1 : next week.

Part of an assignment due in Week 5.

News.Outline

1. Data collection
2. Adjacency Matrices (again)
   - Graphs from matrices
3. Degree
   - Properties (1)
   - Handshaking
4. Walks
   - Adjacency matrices
   - Shortest Path
5. Exercises

Data collection

(Stealing an idea from Angela Carnevale) I'd like to gather some data for use in the class. So, I'm going to run a little survey on what programmes/shows people watch. To do that, I need some ideas... so I'm going to ask you to suggest some things people watch. I'll start:

Niall: Only Murders in the Building.

Breaking Bad. Penguin.

Succession. Squid Game.

The Bear.

Adjacency Matrices (again)

Recall... Boolean Matrix : $a_{ij} \in \{0,1\}$.

Definition (Adjacency Matrix)

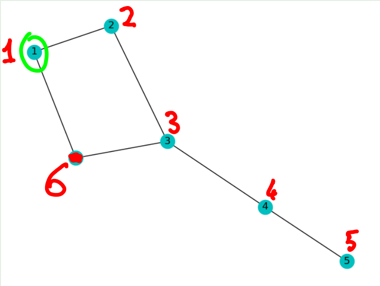
The **adjacency matrix** of a graph, G of order n, is a square $n \times n$ matrix, $A = (a_{ij})$, with rows and columns corresponding to the nodes of the graph. That is, we number the nodes $1, 2, \ldots, n$. Then A is given by

$$a_{ij} = \begin{cases} 1 & \text{if node } i \text{ and } j \text{ are joined by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Put another way: a_{ij} is the number of edges between node i and node j.

Example

Write down the adjacency matrix for the following graph.



A is a 6×6 matrix

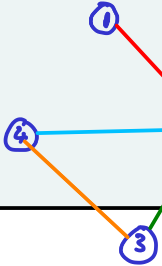
$$A = \begin{array}{cc} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ 4 \\ 5 \\ 6. \end{matrix} \end{array}$$

Any matrix $M = (m_{ij})$, with the properties that all entries are zero or one, and the diagonal entries are zero (i.e., $m_{ii} = 0$), is an adjacency matrix of *some* graph (as long as we don't mind too much about node labels).

Example

Sketch the graph with adjacency matrix : G has order 4

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$



However, in a sense **every** square **matrix** defines a graph if

- we allow loops (and edge between a node and itself)
- every edge has a weight. This is equivalent to the case for our more typical graphs that every potential edge is weighted zero (is not in the edge set) or one (is in the edge set).
- there are two edges between each node (one in each direction) and they can have different weights.

It is a topic we will mention only in passing during the rest of the module.

Degree

As we know the **degree** of a node in a simple graph is the number of nodes it is adjacent to (i.e., its number of *neighbours*)

For a node v we denote this number $\deg(v)$.

The degree can serve as a (simple) measure of the importance of a node in a network.

Basic properties of adjacency matrices

1. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \sum_{u \in X} \deg(u)$, where $\deg(u)$ is the degree of u, and n is the order of the graph.
2. All graphs we've seen so far are *undirected*. For all such graphs, A is symmetric: $A = A^T$; equivalently $a_{ij} = a_{ji}$
3. $a_{ii} = 0$ for all i. – no loops.
4. In real-world examples, A would usually be sparse, which means that $\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \ll n^2$. That is, *the vast majority of the entries are zero.* (This fact will be important in future classes).

So. $\sum_{j=1}^{n} a_{ij} = \deg(i)$

The first of these facts relates to a (crude) measure of how connected a network is: the average degree:

$$\text{Average degree} = \frac{1}{n}\sum_{u \in X} \deg(u) = \frac{1}{n}\sum_{i,j}^{n} a_{ij}$$

However, if the size of the network is m, then is quantity is $2m/n$.

$$\left(\text{Ave degree} = \frac{\text{size}}{\text{order}} = \frac{\text{\# of edges}}{\text{\# of nodes}} \right)$$

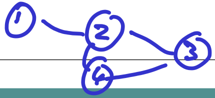
What can we deduce from the fact that the degree sum is twice the size of the graph?

Size of graph = number of edges.

So the degree sum is always even.

$\Rightarrow$ Every graph has an even number of nodes with odd degree

This relates to the famous Königsberg Bridges Problem.



Definition (walk)

A **walk** in a network/graph is a series of edges (perhaps with some repeated)

$$u_1 - v_1, \quad u_2 - v_2, \quad u_3 - v_3, \quad \ldots, \quad u_p - v_p,$$

with the property that $v_i = u_{i+1}$ If $v_p = u_1$ it is a **closed walk**.
The **length** of a walk is the number of edges in it.

Example: These are walks :

$w_1 =$ 1−2, 2−3, 3−4 . $w_3 =$ 3−4, 4−2, 2−1, 1−2, 2−3

$w_2 =$ 2−3, 3−4, 4−2, 2−3 . w_1 has length 3, w_3 has length 5.

These are not walks:

1−2, 2−3, 3−4, 4−1

since 4−1 not an edge

1−2, 2−3, 4−2

not a walk ("Jump" from 3 to 4).

Adjacency matrices can be used to enumerate the number of walks of a given length between a pair of vertices.

Obviously, a_{ij} is the number of walks of length 1 between node i and node j.

We can extract that information for node j by computing the product of A and e_j (column j of the identity matrix).

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{e_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

← 1 walk of length 1 from 2 to 1

Now repeat the process and interpret the results...

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{(Ae_2)} = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

But this is same as $A(e_1 + e_3 + e_4)$

which each represented the nodes reachable from node 2 by a walk of length 1.

So the result represents the number of walks of length $\underline{\underline{2}}$ starting at node 2.

Now repeat the process and interpret the results...

So $A(Ae_j) = A^2 e_j$ has, as entry k, the number of walks of length 2 from node j to node k.

However: Ae_j is column j of A.

So just compute A^2.

Finished here.

We can conclude that, if $B = A^2$, then $b_{i,j}$ is the number of walks of length 2 between nodes i and j.

Note: b_{ii} is the degree of node i.

And more is true: $B = A^k$, then $b_{i,j}$ is the number of walks of length k between nodes i and j.

Example: (K_{22})

Definition (Trails and Paths)

A **trail** is a walk with no repeated edges.
A **path** is walk with no nodes (and so no edges) repeated. (The idea of a **path** is hugely important in network theory. We'll return to it often)

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Path length and shortest path

The **length** of a path is the number of edges in it.
A path from node u to node v is a **shortest** path, if there is no path between them that is shorter (though there could be other paths of the same length)

Finding shortest paths in a network is a major topic...

Exercises

1. Let G be any graph of order n. Let $\bar{G}$ be its compliment. Call their adjacency matrices A_G and $A_{\bar{G}}$, respectively. Let H be the graph with adjacency matrix $A_G + A_{\bar{G}}$. By what name is H more commonly known?