MA378 Chapter 4: Finite Element Methods

§4.3 Analysis: Cea's Lemma

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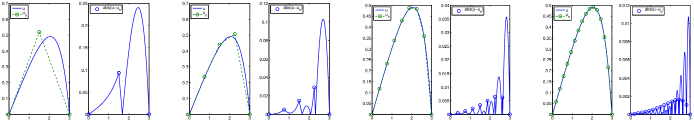

3.1 Error analysis

Recall that we wrote the differential equation

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (a,b), \qquad u(a) = u(b) = 0,$$

in a variational form:

Define $A(u,v) := (u',v') + (ru,v)$. *Find* $u \in H_0^1(a,b)$ *such that*

$$A(u,v) = (f,v) \quad \text{for all} \quad v \in H_0^1(a,b). \tag{1}$$

$$(u,v) = \int_a^b u(x)v(x)\,dx.$$

We then defined the **FEM**:

Definition 3.1 (The Finite Element Method)

Let S be a finite dimensional subspace of $H_0^1(a,b)$. The *Galerkin Finite Element method* is: *find* $u_h \in S$ *such that*

$$\mathcal{A}(u_h, v_h) = (f, v_h) \qquad \text{for all} \qquad v_h(x) \in S. \tag{2}$$

We now show that the member of S found by the FEM is the "closest" to the true solution.

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Lemma 3.2 (Cea's Lemma; Thm 14.6 of Süli and Mayers)

Let u be the solution to (1), i.e., the true solution, and let u_h be the solution to (2), i.e, the FE approximation.

(i) *The difference between the true and approximate solutions is orthogonal to S, i.e.,*

$$\mathcal{A}(u-u_h, v_h) = 0 \text{ for all } v_h \in S,$$

and

(ii) *There is no element of S that is closer to u than u_h:*

$$\mathcal{A}(u-u_h, u-u_h) = \min_{v_h \in S} \mathcal{A}(u-v_h, u-v_h),$$

3.2 Cea's Lemma

First we will prove that

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

(which is a property known as **Galerkin Orthogonality**).

we know that u solves

$$A(u, v) = (f, v) \quad \text{for all } v \in H_0^1(a,b).$$

Since $S \subseteq H_0^1(a,b)$, we also get that

$$A(u, v_h) = (f, v_h) \quad \text{for all } v_h \in S.$$

Also $$A(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in S.$$

Subtracting, we get $A(u, v_h) - A(u_h, v_h) = 0$

$$\Rightarrow \quad A(u - u_h, v_h) = 0.$$

Next we will prove that

$$\mathcal{A}(u-u_h, u-u_h) \leq \mathcal{A}(u-v_h, u-v_h) \text{ for any } v_h \in S.$$

Proof: For any $v_h \in S$

$$A(u-v_h, u-v_h) = A(\underbrace{u-u_h}_{1} + \overbrace{u_h-v_h}^{2}, \underbrace{u-u_h}_{3} + \overbrace{u_h-v_h}^{4}).$$

$$= A(\overbrace{u-u_h}^{1}, \overbrace{u-u_h}^{3}) + A(\underbrace{u-u_h}_{1}, \underbrace{u_h-v_h}_{4})$$

$$+ A(\underbrace{u_h-v_h}_{2}, \underbrace{u-u_h}_{3}) + A(\underbrace{u_h-v_h}_{2}, \underbrace{u_h-v_h}_{4}).$$

Since both $u_h \in S$ and $v_h \in S$, so $(u_h-v_h) \in S$.

Then $A(u-u_h, u_h-v_h) = 0$ by Part (i).

Next we will prove that (continued).

$$\mathcal{A}(u-u_h, u-u_h) \leq \mathcal{A}(u-v_h, u-v_h) \text{ for any } v_h \in S.$$

Similarly $A(u_h - v_h, u - u_h) = 0$ since $A(\cdot,\cdot)$ is symmetric.

Finally, recall that $A(u,v) = (u', v') + (ru, v)$.
And $(u,u) \geq 0$ for any u.
Similarly $(u', u') \geq 0$ for any u.
Then $A(u_h - v_h, u_h - v_h) \geq 0$. Therefore

$$A(u-v_h, u-v_h) = A(u-u_h, u-u_h) + A(u_h - v_h, u_h - v_h) \geq A(u-u_h, u-u_h) \quad \boxed{\text{W}}$$

3.2 Cea's Lemma

Since $\mathcal{A}(\cdot,\cdot)$ is an inner product (see Definition 3.6.1) it induces a **norm:**

$$|||u||| := \sqrt{\mathcal{A}(u,u)}.$$

So we can write (ii) of Cea's Lemma as

$$|||u - u_h||| \leq |||u - v_h||| \qquad \text{for all } v_h \in S.$$

3.3 An example

This is as far as we will take the analysis. With a bit more work (and a little Fourier analysis) we could show that

$$\|u - u_h\|_2 \leq Ch^2 \|u''\|_2.$$

That is, the error is proportional to h^2. We can then further deduce that the method converges:

$$\lim_{h \to 0} \|u - u_h\|_2 = 0.$$

3.3 An example

In place of a rigorous analysis, let us reason as follows. Let l be the piecewise linear interpolant to u as described in Section 2.1. Note that l belongs to S. So, u_h as at least as good an approximation to u as l. That is

$$\|u - u_h\|_2 \leq \|u - l\|_2$$

And Theorem 2.1.3 told us that

$$\|u - l\|_\infty \leq \frac{h^2}{8}\|u''\|_\infty.$$

So, if you believe that

$$\|u - l\|_2 \approx \|u - l\|_\infty,$$

Then it will follow that

$$\|u - u_h\|_2 \lessapprox Ch^2,$$

for some constant C. One can also show that

$$|||u - u_h||| \lessapprox Ch.$$

3.3 An example

Note, however that we have used three different norms here. Therefore much more work would be required to prove a rigorous result. However, we can **demonstrate numerically** that the method converges...

3.3 An example

The table opposite shows the maximum error, over all mesh points, in the finite element solution to

$$-u'' + 3u = x \text{ on } (0,3),$$

$$u(0) = u(3) = 0.$$

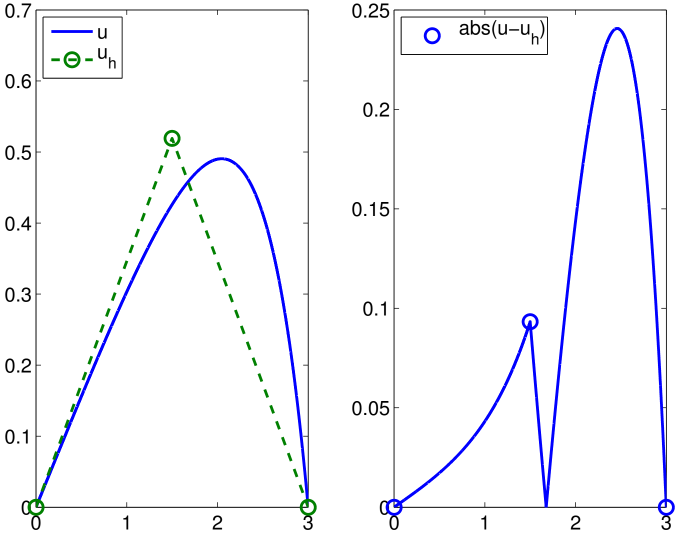
One can see that the error is proportional to n^{-2} (and thus to h^2).

n	$\|u - u_h\|_\infty$
8	6.446e-03
16	1.629e-03
32	4.043e-04
64	1.009e-04
128	2.522e-05
256	6.304e-06
512	1.576e-06
1024	3.940e-07

These results were generated by a Jupyter Notebook (using the Octave kernel), that can be downloaded from `https://www.niallmadden.ie/2324-MA378`

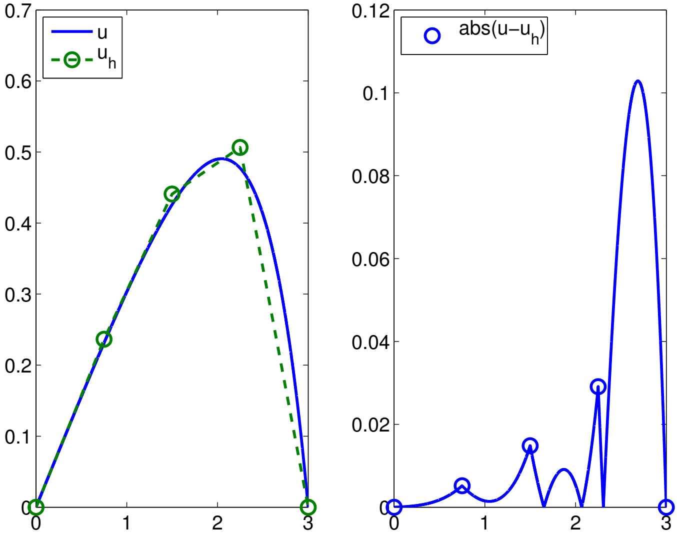
3.3 An example

Solution and error with $n = 2$

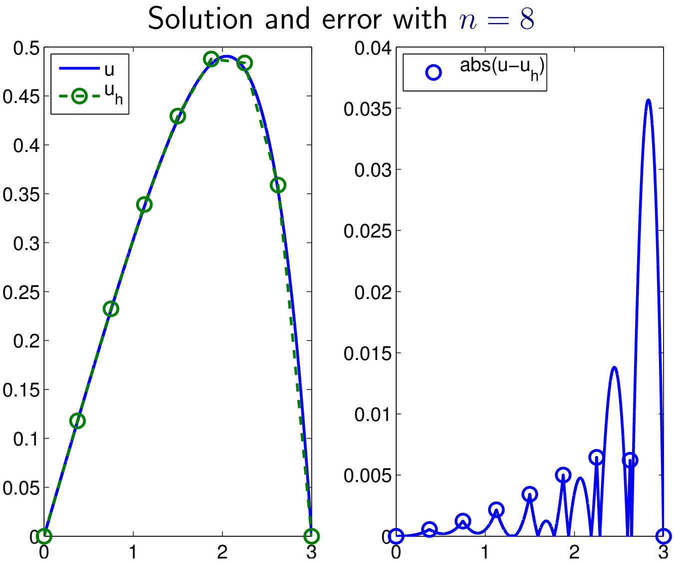


3.3 An example

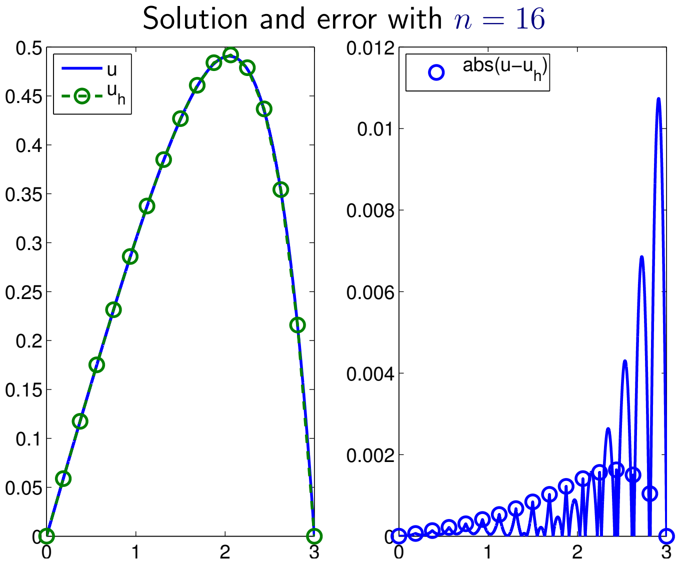
Solution and error with $n = 4$



3.3 An example



3.3 An example



3.4 FE Wrap-Up

There are many aspects of finite element methods that we did not cover, including

1. There are many **other** choices of basis functions given here. One could used cubic splines, or, indeed, higher-order polynomials.
2. When we try to improve the accuracy of the method by reducing h where the error seems large. This is called a h-FEM (and is the most common type of adaptive method).
3. We can also try to improve the accuracy of the method by increasing the order of the polynomials. This is called a p-FEM.
4. The ideas presented here extend to far more general problems. In particular, they work very well for problems in higher dimensions, and on weird-shaped domains.

3.5 Exercises

Exercise 3.1

Suppose that we want to solve

$$-u''(x) + u'(x) = 1 \text{ on } (a, b),$$

(a) Write down the system of linear equations that we would have to solve in terms of h.

(b) Explain why the analysis of Lemma 3.2 does not apply directly to this problem.

Exercise 3.2

Show that, for any function $f \in C^2[a, b]$,

$$\|f\|_2 \leq \sqrt{b-a}\|f\|_\infty,$$

where $\|f\|_2 := \left(\int_a^b (f(x))^2 dx \right)^{1/2} = \sqrt{(f, f)}$, and $\|f\|_\infty := \max_{a \leq x \leq b} |f(x)|$.

3.5 Exercises

Exercise 3.2 shows that if we have a bound for $\|f\|_\infty$, we can get one for $\|f\|_2$. However, as the next exercise shows, the converse is not true.

Exercise 3.3

Show that, given any $\epsilon > 0$, no matter how small, it is possible to construct a function $f \in C^2[a,b]$, for which

$$\|f\|_2 \leq \epsilon$$

but

$$\|f\|_\infty = 1.$$