# MA378 Chapter 4: Finite Elements

**Exercise 1.1.** In class we considered the differential operator

$$L(u) := -u''(x) + r(x)u(x).$$

where $r(x) > 0$ for all $x$. Suppose, instead have the more general operator

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x)u(x),$$

where again $r$ is a positive function. Does this $L_q$ also satisfy a maximum principle? If so, provide a proof. If not, give a counter example.

**Exercise 1.2.** Verify that $u(x) = \dfrac{x}{4} + \dfrac{3e^6(e^{-2x} - e^{2x})}{4(e^{12} - 1)}$ is the exact solution to the differential equation $-u''(x) + 4u(x) = x$ for $x \in (0, 3)$, with the boundary conditions $u(0) = 0$, $u(3) = 0$,

**Exercise 1.3.** In this section of the course, we'll always assume homogeneous boundary conditions. That is, that $u(x) = 0$ at the boundaries. Suppose the problem we wish to solve is

$$-u''(x) + r(x)u(x) = f(x) \quad u(0) = \alpha, u(1) = \beta.$$

Show how to find a problem which has the same left-hand side as this one, homogeneous boundary conditions, and with a solution that differs from this one only by a known linear function.

**Exercise 1.4.** Suppose that $u$ solves $-u''(x) + r(x)u(x) = f(x) \quad$ on $(0, 1)$, and $u(0) = u(1) = 0$. Let $\rho$ be such $r(x) \geqslant \rho > 0$, and define

$$C = \max_{0 \leqslant x \leqslant 1} |f(x)|/\rho.$$

Prove that $u(x) \leqslant C$. (Hint: Consider $L(C - u)$).

**Exercise 1.5.** Consider the differential equation:

$$-u''(x) = \exp(x + 1), \text{ on } (0, 2), \text{ and } u(0) = u(2) = 0.$$

(i) State the variational formulation of this differential equation.

(ii) Show that the solution to the variational problem is unique.

**Exercise 2.1.** Show that solving

*Find $u_h \in S$ such that*

$$\mathcal{A}(u_h, v_h) = (f, v_h) \qquad \text{for all} \qquad v_h \in S.$$

is equivalent to solving

*Find $u_h \in S$ such that*

$$\mathcal{A}(u_h, \phi_i) = (f, \phi_i) \qquad \text{for } i = 1, 2, \ldots, N - 1.$$

where the $\phi_i$ form a basis for $S$.

**Exercise 2.2.** Consider the problem:

$$-u''(x) = 9x \qquad u(0) = 0, u(1) = 0.$$

Use the FEM to find an approximate solution on the mesh $\{0, 1/3, 2/3, 1\}$.

Also write down the true solution to this problem.

**Exercise 2.3.** Suppose we want to use a finite element method to solve

$$-u''(x) + u(x) = 1 \text{ on } (0, 1),$$

with $u(0) = u(1) = 0$, using the usual piecewise linear basis functions on the uniform mesh $\{x_0, x_1, \ldots, x_n\}$. Let the resulting linear system is written as the matrix-vector equation $Au_h = F$.

(i) Show that the matrix $A$ is symmetric (i.e. $a_{ij} = a_{ji}$).

(ii) Show that $A$ is tridiagonal (i.e., if $|i - j| > 1$ then $a_{ij} = 0$).

(iii) Derive the formula for the entries of $A$ in terms of $h = x_i - x_{i-1}$. That is, give an expression for $a_{i,i-1}$, $a_{i,i}$ and $a_{i,i+1}$.

*Since this exercise is not covered in class, or in tutorials, solutions are given on the following slides.*

### (i) Show $A$ is symmetric

The entries in $A$ are $a_{ij} = \mathcal{A}(\psi_j, \psi_i)$, where $\mathcal{A}$ is the bilinear form:

$$\mathcal{A}(u, v) = \int_0^1 u'(x)v'(x) + u(x)v(x)\,dx.$$

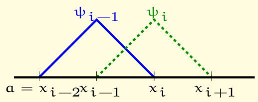
But $\mathcal{A}(v, u) = \int_0^1 v'(x)u'(x) + v(x)u(x)\,dx = \mathcal{A}(u, v)$. So $a_{ij} = \mathcal{A}(\psi_j, \psi_i) = \mathcal{A}(\psi_i, \psi_j) = a_{ji}$.

### (ii) Show that $A$ is tridiagonal

The basis functions for the method, $\{\psi_1, \ldots, \psi_{n-1}\}$, have the formulae

$$\psi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x_{i-1} \leqslant x < x_i \\ \frac{x_{i+1} - x}{h} & x_i \leqslant x \leqslant x_{i+1} \\ 0 & \text{otherwise,} \end{cases} \tag{2.0.1}$$

where $h = x_i - x_{i-1}$. They are pictured below.




Note that $\psi_i(x)$ is only non-zero on $[x_{i-1}, x_{i+1}$. Therefore, if $i > j + 1$ or $j > i + 1$, then $\psi_i(x)\psi_j(x) = 0$ for all $x$. As mentioned above,

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j) = \int_0^1 \psi_i'(x)\psi_j'(x) + \psi_i(x)\psi)j(x)\,dx$$

So, if $|i - j| > 1$, then $a_{ij} = 0$.

### (iii) Derive formulae for the $a_{i,j}$ in terms of $h$.

First we'll compute $a_{i,i-1} = \mathcal{A}(\psi_i, \psi_{i-1})$.

$$\mathcal{A}(\psi_i, \psi_{i-1}) = \int_0^1 \psi_i'(x)\psi_{i-1}'(x) + \psi_i(x)\psi_{i-1}(x)\,dx$$

$$= \int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)\,dx + \int_{x_{i-1}}^{x_i} \psi_i(x)\psi_{i-1}(x)\,dx,$$

because, as illustrated in Part (ii), the only interval where $\psi_{i-1}$ and $\psi_i$ are both non-zero is $[x_{i-1}, x_i]$.
From (2.0.1) in Part (2),

$$\psi_i'(x) = \begin{cases} 1/h & x_{i-1} \leqslant x < x_i \\ -1/h & x_i \leqslant x \leqslant x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)\,dx = \int_{x_{i-1}}^{x_i} \frac{1}{h}\frac{-1}{h}\,dx = -\frac{x}{h^2}\bigg|_{x_{i-1}}^{x_i} = -\frac{x_i - x_{i-1}}{h^2} = -\frac{1}{h}.$$

**(iii) continued**

For $\int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)\,dx$, we can simplify a little by setting $s = x - x_{i-1}$. Then $x_i - x = x_{i-1} + h - x = h - s$.

$$\int_{x_{i-1}}^{x_i} \psi_i(x)\psi_{i-1}(x)\,dx = \int_{x_{i-1}}^{x_i} \frac{x - x_i}{h}\frac{x_{i-1} - x}{h}\,dx = \int_0^h \frac{s}{h}\frac{h-s}{h}\,ds$$

$$= \frac{1}{h^2}\left(-\frac{1}{3}s^3 + \frac{1}{2}hs^2\right)\Big|_0^h = \frac{h}{6}.$$

So $\mathbf{a_{i,i-1} = -1/h + h/6 = a_{i,i+1}}$.
Next we'll calculate $a_{i,i}$:

$$a_{ii} = \mathcal{A}(\psi_i, \psi_i) = \int_0^1 \psi_i'(x)\psi_i'(x) + \psi_i(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} (\psi_i'(x))^2 + (\psi_i(x))^2\,dx.$$

First,

$$\int_{x_{i-1}}^{x_{i+1}} (\psi_i'(x))^2\,dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)^2 dx = \frac{2}{h}.$$

For the $\int_{x_{i-1}}^{x_{i+1}} (\psi_i'(x))^2\,dx$ term, we'll again simplify by setting $s = x - x_{i-1}$.

**(iii) continued**

This gives

$$\int_{x_{i-1}}^{x_{i+1}} (\psi_i(x))^2\,dx = \int_0^h \left(\frac{s}{h}\right)^2 ds + \int_h^{2h} \left(\frac{h-s}{h}\right)^2 ds = \frac{s^3}{3h^2}\Big|_0^h + \frac{-(h-s)^3}{3h^2}\Big|_h^{2h} = \frac{h}{3} + \frac{h}{3}$$

To finish: $\mathbf{a_{i,i} = \int_{x_{i-1}}^{x_{i+1}} (\psi_i'(x))^2 dx + \int_{x_{i-1}}^{x_{i+1}} (\psi_i(x))^2 dx = \frac{2}{h} + \frac{4h}{6}}$.

**Exercise 3.1.** Suppose that we want to solve

$$-u''(x) + u'(x) = 1 \text{ on } (a, b),$$

(a) Write down the system of linear equations that we would have to solve in terms of $h$.

(b) Explain why the analysis of Lemma **??** does not apply directly to this problem.

**Exercise 3.2.** Show that, for any function $f \in C^2[a, b]$,

$$\|f\|_2 \leqslant \sqrt{b-a}\,\|f\|_\infty,$$

where $\|f\|_2 := \left(\int_a^b (f(x))^2 dx\right)^{1/2} = \sqrt{(f, f)}$, and $\|f\|_\infty := \max_{a \leqslant x \leqslant b} |f(x)|$.

*Exercise 3.2 shows that if we have a bound for $\|f\|_\infty$, we can get one for $\|f\|_2$. However, as the next exercise shows, the converse is not true.*

**Exercise 3.3.** Show that, given any $\epsilon > 0$, no matter how small, it is possible to construct a function $f \in C^2[a, b]$, for which

$$\|f\|_2 \leqslant \epsilon$$

but

$$\|f\|_\infty = 1.$$