CS4423: Problem Set 2 ANS with solutions

These exercises should help you prepare for the class test, which will be somewhat similar in structure:

- Q1 will have 10 "true/false" based on material covered up to, and including Week 7.
- Three other questions, again on any material up to and including Week 7.

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Q1. For each of the following, state whether it is **true** or **false**. Explanations are not required. In all cases G represents a graph: $G = (X, E)$ with node set X, and edge set E.

(i) The **order** of G is $|E|$. ANS False

(ii) The **degree** of a node is the number of times it occurs in X ANS False (each node occurs in X exactly ones).

(iii) A bipartite graph is two-colourable. ANS True

(iv) The path graph on n nodes, P_n, is a tree. ANS True

(v) Let G_1 be the graph on the set of nodes $\{0, 1, 2, 3, 4\}$ with edges $0-1$, $0-2$, $0-3$, $1-4$, $2-3$. G_1 is isomorphic to its complement. ANS False

(vi) G_1, the graph in the previous question, has the same order as its line graph. ANS True

(vii) The adjacency matrix of a digraph cannot be symmetric. ANS False

(viii) There exists a 5×5 adjacency matrix with Perron Root $\lambda = 2$, and corresponding eigenvalue $v = (1, -1, 1, -1, 1)$. ANS False

(ix) $a = (4, 3, 2, 1, 4)$ is a valid Prüfer code for a tree with nodes $\{0, 1, 2, 3, 4, 5, 6\}$.

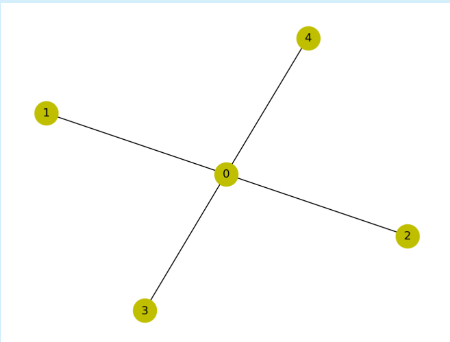
> **Answer:** True (a has length $n - 2$, and all entires correspond to node labels)

(x) The cycle graph on n nodes, C_n, has diameter $\lceil n/2 \rceil$, where $\lceil \cdot \rceil$ is the *ceiling* function. ANS False

Q2. Consider the following matrix:

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

(a) Give a sketch of the graph, G_2, on the nodes $X = \{0, 1, 2, 3, 4\}$ with the that has A_2 as its adjacency matrix.

> **Answer:**
>
> 
>

(b) Is this graph bipartite? If so, indicate a two-colouring in your sketch.

> **Answer:** Yes: it is bipartite. Let Node 0 be red, and all others blue (for example).

(c) Give the *relative degree centrality* of the nodes in G_2.

> **Answer:** They are (in order) $\{1, 1/4, 1/4, 1/4, 1/4\}$

(d) A_2 has as an eigenvector $v = (2, 1, a, b, c)$. Compute a, b and c, as well as the eigenvalue that corresponds to this eigenvector.

> **Answer:** $a = b = c = 1$; the corresponding eigenvalue is $\lambda = 2$.

(e) Compute A_2^2 (Note: this can be done either by matrix multiplication, or just looking at the graph. Either approach is fine). Verify that $A_2 + A_2^2 > 0$. What is the implication of that for the diameter of G_2?

> **Answer:** $A_2^2 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$. We then get $A + A^2 = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. We see that the diameter is 2, since every node is at a distance of at most 2 from every other.

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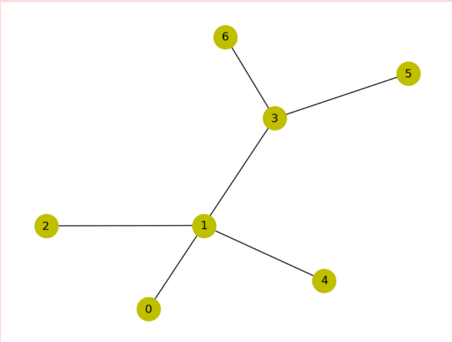
Q3. (a) Sketch the tree, G_3, on the nodes $\{0, 1, 2, 3, 4, 5, 6\}$ with edges $0-1$, $1-2$, $1-3$, $1-4$, $3-5$, $3-6$.

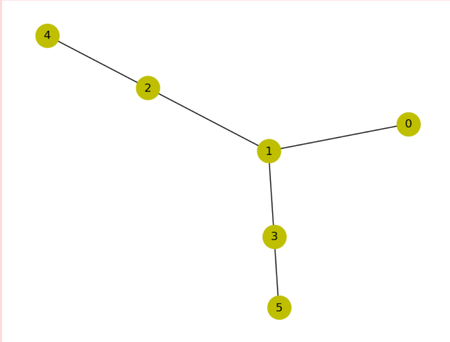
> **Answer:**



(b) Compute the Pruefer code for G_3. ANS (1,1,1,3,3)

(c) Determine the tree on the nodes $\{0, 1, 2, 3, 4, 5\}$ which has Pruefer code $(1, 2, 1, 3)$.

Answer: The tree has edges $0-1$, $1-2$, $1-3$, $2-4$, $3-5$.



Q4. Consider the graph T_4 and G_4 shown in Figure 1a.

(a) List the nodes of T_4 in the order they would be traversed by the **depth-first search** (DFS) algorithm, starting at node A. ANS A, E, D, H, G, J, I, K, C, B, F (corrected 5 Mar)

(b) List the nodes of T_4 in the order they would be traversed by the **breadth-first search** (BFS) algorithm, starting at node A. ANS A, B, C, D, E, F, G, H, I, J, K

(c) For the graph G_4, apply the BFS algorithm to determine the distances from node A to all other nodes in the graph.

Answer: This is probably an overly detailed solution... The algorithm is

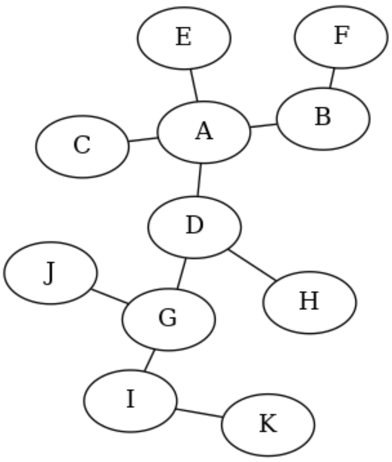
Step 1 [Initialize.] Suppose that $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and that $x = x_j$. Set $d_i \leftarrow \perp$ (undefined) for $i = 0, \ldots, n-1$. Set $d_j \leftarrow 0$ and initialize a queue $Q \leftarrow (x_j)$.

Step 2 [Loop.] While $Q \neq \emptyset$:

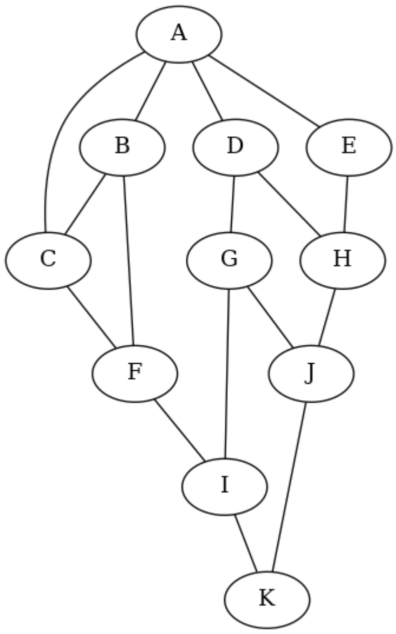
- pop node x_k off Q
- for each neighbor x_l of x_k with $d_l = \perp$: push x_l onto Q and set $d_l \leftarrow d_k + 1$.

Step 3 [Stop.] Return the array $(d_0, \ldots, d_{n-1})$.

x_k	Q	A	B	C	D	E	F	G	H	I	J	K
	[A]	0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
A	[B,C,D,E]	0	1	1	1	1	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
B	[C,D,E,F]	0	1	1	1	1	2	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
C	[D,E,F]	0	1	1	1	1	2	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
D	[E,F,G,H]	0	1	1	1	1	2	2	2	$\perp$	$\perp$	$\perp$
E	[F,G,H]	0	1	1	1	1	2	2	2	$\perp$	$\perp$	$\perp$
F	[G,H,I]	0	1	1	1	1	2	2	2	3	$\perp$	$\perp$
G	[H,I,J]	0	1	1	1	1	2	2	2	3	3	$\perp$
H	[I,J]	0	1	1	1	1	2	2	2	3	3	$\perp$
I	[J,K]	0	1	1	1	1	2	2	2	3	3	4
J	[K]	0	1	1	1	1	2	2	2	3	3	4
K	[]	0	1	1	1	1	2	2	2	3	3	4



(a) T_4: for Q4(a) and (b)



(b) G_4: for Q4(c)

Figure 1: Graphs for Q4