Linear Algebra I - Tutorial Sheet

The questions on this sheet are designed to help you prepare for Assignment 4.

Q1 Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n, in the variable $t \in \mathbb{R}$. Which of the following are **subspaces** of $\mathbb{P}_3$? Explain your answers.

(a) $H_0 := \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{P}_3$.

(b) $H_1 := \operatorname{span}\{t^{-1}, t^2\}$

(c) $H_2 := \{0, 2, -3t, t^3\}$

(d) $H_3 := \operatorname{span}\{t, t^3\}$

(e) $H_4 := \{p(t) \in \mathbb{P}_2\}$.

(f) $H_5 := \{p(t) \in \mathbb{P}_3\}$.

(g) $H_6 := \{p(t) \in \mathbb{P}_2 : p(0) = 0\}$.

(h) $H_7 := \{p(t) \in \mathbb{P}_2 : p(0) = 1\}$.

Tip: in Week 2 we saw that, in order to verify that H is a subspace of a real vector space V, we have to check that every element of H is also an element of V, and further that

- That the zero vector in V is also in H;
- If $u, v \in H$ then $u + v \in H$.
- If $u \in H$ then $cu \in H$ for any scalar $c \in \mathbb{R}$.

Q2 (This question is similar to Q1(b) from the 2020/2021 exam paper). Let

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 1 & 9 & -1 \\ 0 & 1 & 3 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -4 \end{bmatrix}, \text{ and } y = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix},$$

(a) Decide (with justification) if $x \in \operatorname{Nul} A$, and if $x \in \operatorname{Col} A$.

(b) Decide (with justification) if $y \in \operatorname{Nul} A$, and if $y \in \operatorname{Col} A$.

Q3 (Q2b from 2021/2022 exam) Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} -3p - 3q \\ 3p - q + 8r \\ q - 2r \\ 3p + 6r \end{bmatrix} : p, q, r \in \mathbb{R} \right\},$$

of $\mathbb{R}^4$ and give a basis for it.

Q4 (Parts (a)–(c) of this question are based on Q3(b) from the 2020/2021 exam paper).

(a) What is the largest possible rank of an 5×9 matrix?

(b) If the null space of a 7×5 matrix 1-dimensional, what is the dimension of its column space?

(c) Give an example of a 3×4 that has null space of dimension 2.

(d) Give an example of a 3×3 matrix that has $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$ in its column space, and $z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in its null space.

Q5 (This is similar to Q2(a) on the 2018/2019 exam paper)

(a) Decide, with justification, if t is a linear combination of the polynomials $2t^2 + 2t$ and $t - 5$ in $\mathbb{P}_2$.

(b) Decide, with justification, if the polynomial $t - 5$ is a linear combination of the polynomials $2t^2 + 2t$ and $t^2 - 5$ in $\mathbb{P}_2$.