Vector norms (again). In Lecture 4 we learned about the p-norms of vectors in $\mathbb{C}^n$:

$$\|u\|_p = \begin{cases} \left(\sum_{i=1}^n |u_i|^p\right)^{1/p} & 1 \leqslant p < \infty \\ \max_i |u_i| & p = \infty. \end{cases}$$

Of these, the most important is the the "Euclidean" or "2"-norm: $\|u\|_2 := \sqrt{u^\star u} = \sqrt{(u,u)}$.

- If $\|u\| = 0$ then u is the zero vector (this is true for any norm).
- If $\|u\| = 1$ we say that u is *normalised*.

Unitary matrices (again).

Definition 1. *A matrix* $U \in M_n(\mathbb{C})$ *is* unitary *if its Hermitian transpose is equal to its inverse, i.e. if* $U^* U = I_n$. *If* $U \in M_n(\mathbb{R})$ *has this property,* U *is called an* orthogonal *matrix. This means that the (ordinary) transpose of* U *is equal to the inverse of* U*, i.e.* $U^T U = I_n$.

If $U = (u_1|u_2|\cdots|u_n)$ is unitary then $u_i^\star u_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.
Important properties of unitary matrices include that:

- If u_i is column i of a unitary matrix, then $\|u_i\| = 1$.
- For the 2-norm:
$$\|Ux\| = \sqrt{(Ux)^\star Ux} = \sqrt{x^\star U^\star Ux} = \sqrt{x^\star x} = \|x\|.$$

Matrix norms. We will often need to quantify how close (or otherwise) one matrix is to another. A specific example of this is: "*find the best rank 1 approximation to* A". To answer this, we need to be able to quantify "best". So we need matrix norms.
There are two ways we can build matrix norms from vector norms.

(i) Treat the matrix as a vector, e.g., by stacking the columns of the matrix to form a vector. Now apply a vector norm. If the vector norm is the 2-norm, the corresponding matrix norm is called the *Frobenius* norm:
$$\|A\|_F := \Big(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\Big)^{1/2}.$$
Here are two other ways of thinking of the $\|\cdot\|_F$-norm:
$$\|A\|_F = \Big(\sum_{j=1}^n \|a_j\|_2^2\Big)^{1/2} = \sqrt{\mathrm{Tr}(A^\star A)},$$
where $\mathrm{Tr}(A)$ is the *trace* of A, i.e., the sum of its diagonal entries.

(ii) **Operator norms:** Any vector norm "induces" a matrix norm: given $A \in M_{m,n}(\mathbb{C})$, we can define
$$\|A\|_p := \sup_{x \in \mathbb{C}^n} \frac{\|Ax\|_p}{\|x\|_p}, \tag{1}$$
where on the right $\|\cdot\|$ denotes any vector p-norm.

Example (not done in class): If A is a diagonal matrix, then, for any p-norm, $\|A\|_p = \max_i |a_{ii}|$.
In general, (1) is not very practical. However, if $p = 1$, then it is equivalent to the maximum column sum of the matrix. To see this, consider the following.

- We can write a matrix-vector product as $Ax = \sum_{j=1}^{n} a_j x_j$, where a_j represents column j of A, and x_j is entry j of the vector x.
- So, for any x, $\|Ax\|_1 = \|\sum_{j=1}^{n} a_j x_j\|_1 \leqslant \sum_{j=1}^{n} \|a_j\|_1 |x_j|$.
- If we require that $\|x\|_1 = 1$, then $\sum_{j=1}^{n} \|a_j\|_1 |x_j| \leqslant \max_j \|a_j\|_1$.
- This gives that $\|A\|_1 \leqslant \max_j \|a_j\|_1$.
- To get equality, suppose that $\|a_k\| = \max_j \|a_j\|_1$, and take x to be the vector whose only non-zero entry is $x_k = 1$.

If $p = \infty$, then it is equivalent to the maximum row sum (see exercises). And when $p = 2$ it is, we'll learn, the largest singular value of A.

Good and bad norms. We shall learn that some norms have important properties that we need later, and some lack those. The properties we need are:

1. $\|AB\|_\star \leqslant \|A\|_\star \|B\|_\star$. This is the "submultiplicity" property, sometimes also called the "consistency" property. We proved this in class for operator norms. It also holds for the $\|\cdot\|_F$ norm (see exercises). Generally, it is not true for other norms.
2. $\|UA\|_\star = \|A\|_\star$ when U is a unitary matrix. This holds true for the $\|\cdot\|_2$ and $\|\cdot\|_F$ norms.