2526-MA140 Engineering Calculus

Week 08, Lecture 2
Integration by Parts; Areas between Curves

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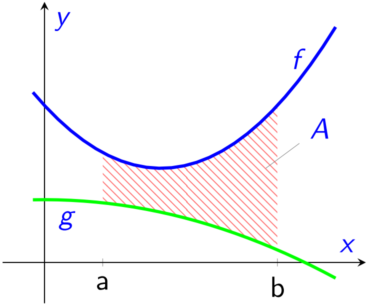
Wednesday, 05 November, 2025



Assignments, etc

- **Problem Set 5** is close, and grades have been posted.
- **Problem Set 6** is open, and will be covered in tutorials this well. Deadline is 5pm next Monday (10 November).
- **Problem Set 7** opened this morning.
- The finally weekly assignment, will open next week.
- Reminder: The second **class test** takes place November 18 (Tuesday of Week 10).
  - If you had extra time and an alternative venue for the 1st test, that should continue.
  - If you had (or wish to) request an low-distraction venue then **contact Niall as soon as possible**, even if you have done so before.

This part is about...

See also Section **7.1** (Integration by Parts) and Section 6.1 (Areas between Curves) in the textbook:

math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Integration by Parts

Yesterday, we learned about integration by parts:

Integration by Parts

Let u and v be differentiable functions. Then

$$\int u\,dv = uv - \int v\,du.$$

One of the challenges of Integration by Parts is knowing how to choose u and dv.
In the last example from yesterday, when integrating
$\boxed{\int x\cos(x)\,dx}$ we choose $u = x$, because its derivative, $u' = 1$ is simpler.
Suppose we had made the bad choice of

$$u(x) = \cos(x), \qquad dv = x\,dx,$$

then we'd get:

To try to get good choices for u and dv, we proceed as follows:

1. Some functions are easier to differentiate than integrate, and so make a good choice for u. Important examples include **logarithms** and **inverse trigonometric** functions.

2. Some functions, such as polynomials, may be good choices for u, since $u'(x)$ may be simpler than $u(x)$.

3. Trigonometric and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for dv.

Example (of choosing u

Evaluate $\mathrm{I} = \int \frac{\ln(x)}{x^2}\, dx$.

Example

Evaluate $\mathrm{I} = \int \ln(x)\,dx$.

Since $\int \ln(x) dx$ can be written as $\int \Big(\ln(x) \Big) (1)\, dx$, we use integration by parts, with $u = \ln(x)$ and $dv = dx$.

Int by Parts: Repeated application

Sometimes, we have to apply Integration by Parts more than once.

Example

Evaluate $\mathrm{I} = \int x^2 e^x \, dx$.

Int by Parts: Repeated application

It is good to check any new rule/method for a simple example we already know the answer to. Now that we know about repeated application, we can do that:

Example

We know that $\mathrm{I} = \int x^2 dx = (1/3)x^3$. We can also use IbP.

Take $u(x) = x$ and $dv = xdx$:

Int by Parts: Repeated application Easy example

Recall: Definite integrals

Last week we introduced the definite integral as follows:

Definition: definite integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral of** f from a to b is given by

$$\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{i=0}^{n-1} f(x_i),$$

where $h = (b-a)/n$ and $x_i = a + ih$, provided the limit exists. Moreover, it is the area of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$ and $x = b$.

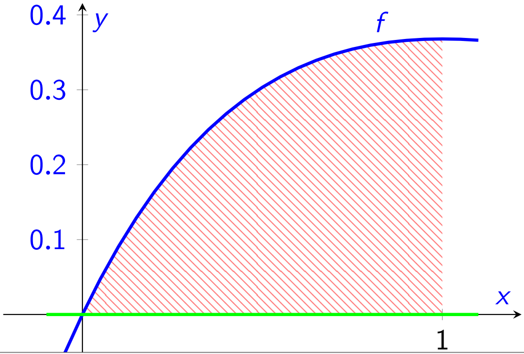
We'll now revisit this idea, and then extend it.

Definite Integrals with IbP

Integration by Parts for Definite Integrals

$$\int_a^b u dv = (uv)\Big|_a^b - \int_a^b v\, du$$

Example: First estimate $\int_0^1 xe^{-1}\, dx$ from the graph of xe^{-x}



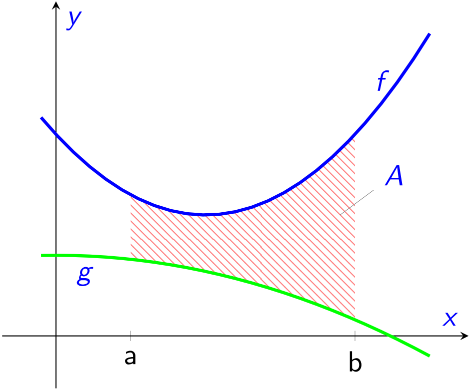
Definite Integrals with IbP

Now use *Integration By Parts* to actually evaluate $\int_0^1 xe^{-x}\,dx$.

Areas Between Curves

We know that $\int_a^b f(x)\,dx$ evaluates as the area of the region between $x = a$ and $x = b$, and between $y = f(x)$ and $y = 0$.

But what if we wanted to evaluate the area between two curves?



Areas Between Curves

Area Between Curves

Let f and g be continuous functions with $f(x) \geqslant g(x)$ throughout the interval $[a, b]$. Then the area A of the region that is

- bounded on the left by $x = a$, and on the right by $x = b$,
- above by the curve $y = f(x)$ and below by $y = g(x)$

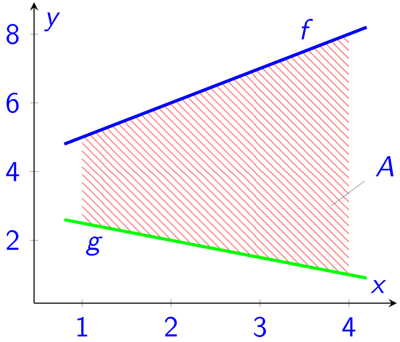
is given by

$$A = \int_a^b (f(x) - g(x))\, dx\,.$$

Areas Between Curves

Example

Find the area of the region bounded above by the graph of $f(x) = x + 4$, and below by the graph of $g(x) = 3 - x/2$ over the interval $[1, 4]$



Areas Between Curves

Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect. That is, we have to find a and b.

Example (from Q5(a) of 2024/2025 Exam paper)

Compute the region bounded by the curves $f(x) = 3x + 4$ and the $g(x) = 2x^2 + 2x + 1$.

First we need to find the points were $f(x)$ and $g(x)$ intersect. That is, we solve $f(x) = g(x)$:

$$\begin{aligned}(3x+4) - (2x^2+2x+1) = 0 \\ \implies -2x^2 + x + 3 = 0 \\ \implies -2(x+1)(x-3/2) = 0 \quad (1)\end{aligned}$$

So they intersect at $x = -1$ and $x = 3/2$.
(Continued)

Areas Between Curves

So the area is given by

$$\int_{-1}^{3/2} f(x) - g(x) dx$$

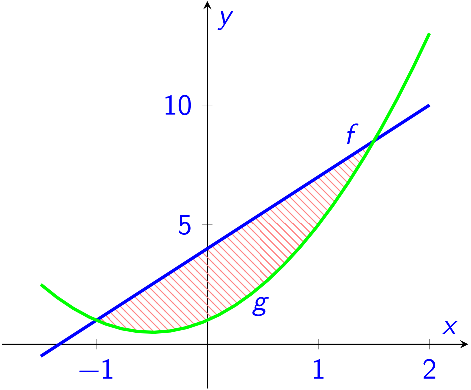
$$= \int_{-1}^{3/2} -2x^2 + x + 3 dx$$

$$= \left(-\frac{2}{3}x^3 + \frac{1}{2}x^2 + 3x\right)\Big|_{-1}^{3/2}$$

$$= \left(-\frac{2}{3}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right)\right) - \left(-\frac{2}{3}(-1) + \frac{1}{2}(1) + 3(-1)\right)$$

$$= 125/24.$$

Areas Between Curves



Compound Regions

In the previous examples, we had $f(x) \geqslant g(x)$ for all $x \in [a, b]$. But what if f and g cross in the domain?

Areas between curves, without $f(x) \geqslant g(x)$

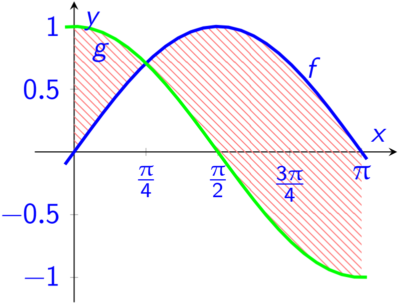
Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ Then A, the area of the region between the graphs of $f(x)$ and $g(x)$, and between $x = a$ and $x = b$, is given by

$$A = \int_a^b |f(x) - g(x)| \, dx.$$

Compound Regions

Example

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$, from $x = 0$ to $x = \pi$.



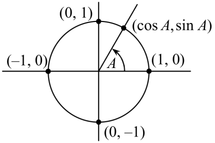
Compound Regions

It will help to consult p13 of the "log" tables.

Triantánacht | **Trigonometry**

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	$0°$	$90°$	$180°$	$270°$	$30°$	$45°$	$60°$	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. $\approx 57.296°$ $\qquad$ $1° \approx 0.01745$ rad.

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Compound Regions

Exercises

Exer 8.2.1 (From 2023/2024 exam)

Evaluate $\int_0^{\pi/2} x\cos(x)\,dx$.

Exer 8.2.2 (From 2019/2020 exam)

The functions $f(x) = 1/x$ and $g(x) = x^2$ intersect at $x = 1$. Calculate the area between their graphs on $[1, 2]$

Exer 8.2.3 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Exercises

Exer 8.2.4 (From 23/24 exam)

Find the area bounded by the curves $f(x) = x^2 - 4x$ and $g(x) = 2x - 5$.