MA378: Assignment 2 (Version 2.0) ANS with solutions

Deadline: 13:00, Wednesday 20 March.

Your solutions must be clearly written, and neatly presented. You can submit an electronic copy, through blackboard, or a hard copy. If submitting a hard copy, do so at the 1pm lecture in the 20th. Staple pages of the hard-copy, and write your name/ID at the topi of each page. Marks will be given for quality and clarity of exposition ([15 Marks]). Usual collaboration policy applies.

Chapter 2: Piecewise Polynomial Interpolation

Exer 2.4 [20 Marks] Take $f(x) = \ln(x)$, $x_0 = 1$, $x_N = 2$. What value of N would you have to take to ensure that $|\ln(x) - S(x)| \leqslant 10^{-4}$ for all $x \in [1, 2]$, where S is the natural cubic spline interpolant to f.

Answer: In Theorem 2.3 we learned that $\|f - S\|_\infty \leqslant \dfrac{5}{384} M_4 h^4$, where $M_4 = \max\limits_{1 \leqslant x \leqslant 2} |f^{(iv)}(x)|$. So we need to ensure that $\frac{5}{384} M_4 h^4 \leqslant 10^{-4}$. First calculate that $M_4 = \max\limits_{1 \leqslant x \leqslant 2} |-6/x^4| = 6$. With this, we see we need h such that $h^4 \leqslant 10^{-4} \times \frac{384}{30} = 1.28 \times 10^{-3}$. That gives $h \leqslant 0.1891$ Using that, in this case $N = 1/h$, we get the requirement that $N \geqslant 5.2869$. Since N must be an integer, the answer is **we must take** $N = 6$.

Exer 2.6 [20 Marks] Suppose that S is a natural cubic spline on $[0, 2]$ with

$$S(x) = \begin{cases} 3x + a(1-x)^3 + bx^3, & \text{for } 0 \leqslant x < 1, \\ c(2-x) - (2-x)^3 + d(x-1)^3, & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

Find a, b, c, and d.

Answer: First note that

$$S'(x) = \begin{cases} 3 - 3a(1-x)^2 + 3bx^2, & \text{for } 0 \leqslant x < 1, \\ -c + 3(2-x)^2 + 3d(x-1)^2, & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

and

$$S''(x) = \begin{cases} 6a(1-x) + 6bx, & \text{for } 0 \leqslant x < 1, \\ -6(2-x) + 6d(x-1), & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

A natural spline has $S''(0) = 0$, so that gives $a = 0$. Similarly, requiring that $S''(2) = 0$ gives that $d = 0$.
Next use that S must be continuous at $x = 1$, to get that $3 + b = c - 1$, and
S' must be continuous at $x = 1$, which gives $3 + 3b = -c + 3$
Solving these equations gives $a = 0, b = -1, c = 3$ and $d = 0$.

Chapter 3: Numerical Integration

Exer 1.1 [10 Marks] (*For simplicity, you may assume that the quadrature rule is integrating f on the interval $[-1, 1]$.*) Let q_0, q_1, ..., q_N be the quadrature weights for the Newton-Cotes rule $Q_N(f)$. Show that $q_i = q_{N-i}$ for $i = 0, \dots N$.

Answer: There are a few possible ways of answering this one. Here is one. Recall that $q_i = \int_{-1}^{1} L_i(x)dx$, where L_i is the ith Lagrange polynomial associated with the points $-1 = x_0 < x_1 < \cdots < x_n = 1$. That is, $L_i(x)$ and $L_{n-i}(x)$ are the unique polynomials of degree n with the properties that

$$L_i(x_j) = \begin{cases} 1 & x_j = x_i \\ 0 & x_j \neq x_i, \end{cases} \qquad \text{and} \qquad L_{n-i}(x_j) = \begin{cases} 1 & x_j = x_{n-i} \\ 0 & x_j \neq x_{n-i}. \end{cases}$$

Since the x_i are uniformly spaced on $[-1, 1]$ we can see that $x_i = -x_{n-i}$. Therefore,
$L_{n-i}(-x_j) = \begin{cases} 1 & x_j = -x_{n-i} = x_i \\ 0 & x_j \neq -x_{n-i} = x_i. \end{cases}$ Thus $L_{n-i}(x) = L_i(-x)$. With the substitution $y = -x$, we can see that $q_{n-i} = \int_{-1}^{1} L_{n-i}(x)dx = \int_{-1}^{1} L_i(-x)dx = -\int_{1}^{-1} L_i(y)dy = \int_{-1}^{1} L_i(y)dy = q_i$ (note the change in the limits of integration). So $q_i = q_{n-i}$.

Exer 3.5 [20 Marks] Consider the rule (which is not, strictly speaking, a Newton-Cotes rule):

$$R(f) = q_0 f(\frac{1}{3}) - f(\frac{1}{2}) + q_2 f(\frac{3}{4})$$

for approximating $\int_0^1 f(x)dx$.

(a) Determine values of q_0 and q_2 that ensure this rule has precision 2.

(b) What is the maximum precision of $R(\cdot)$ with the values of q_1 and q_2 that you have determined?

(c) Why is this not, strictly speaking, a Newton-Cotes rule?

Answer: (a) We need to find q_0 and q_2 so that $R(f) = \int_0^1 p_2(x)dx$ where p_2 is any polynomial of degree 2. Since that space of polynomials is spanned by the set $\{1, x, x^2\}$, we take q_0 and q_2 to satisfy the equations $q_0 - 1 + q_2 = 1$, $q_0/2 - 1/2 + q_2(3/4) = 1/2$, and $q_0/9 - 1/4 + q_2(9/16) = 1/3$. These equations are not linearly independent (since there are only two unknowns. Solving any pair of them should give $q_0 = 6/5$ and $q_2 = 4/5$. So $R(f) = \frac{6}{5}f(\frac{1}{3}) - f(\frac{1}{2}) + \frac{4}{5}f(\frac{3}{4})$.

Answer: (b) Could this method be exact for some higher degree polynomials? Checking with $f(x) = x^3$, we should find that $R(x^3) = 37/144 \neq \int_0^1 x^4 dx$. So the precision is at most 2.

Answer: (c) Either one of the following reasons would suffice: the limits of integration are not included as quadrature points, and the points are not equally spaced.

Exer 5.2 [15 Marks]

(i) Using the Inner Product $(f, g) := \displaystyle\int_0^1 f(x)g(x)dx$, find $\widetilde{p}_0(x)$, $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$.

Answer: We'll use Thm 5.12. Define

$$\alpha_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_n)}{(\widetilde{p}_n, \widetilde{p}_n)}, \quad \text{and} \quad \beta_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_{n-1})}{(\widetilde{p}_{n-1}, \widetilde{p}_{n-1})},$$

and

$$\widetilde{p}_0(x) \equiv 1, \widetilde{p}_1(x) = x - \alpha_1, \quad \text{and} \quad \widetilde{p}_{n+1}(x) = (x - \alpha_{n+1})\widetilde{p}_n(x) - \beta_{n+1}\widetilde{p}_{n-1}(x), \text{ for } n \geqslant 1.$$

- $n = 0$: $\alpha_1 = (x, 1)/(1, 1) = 1/2$ which gives that $\mathbf{\widetilde{p}_1 = x - 1/2}$;
- $n = 1$: $\alpha_2 = (1/24)/(1/12) = 1/2$ and $\beta_2 = (1/12)/1 = 1/12$, which gives that $\widetilde{p}_2 = (x - 1/2)^2 - 12$. Can simplify as $\mathbf{\widetilde{p}_2(x) = x^2 - x + 1/6}$.
- $n = 2$: $\alpha_3 = (1/360)/(1/180) = 1/2$ and $\beta_3 = (1/180)/(1/12) = 1/15$, which gives that $\widetilde{p}_3 = (x-1/2)\big((x-1/2)^2 - 1/12\big) - x/15 + 1/30$. Can simplify this as $\mathbf{\widetilde{p}_3(x) = x^3 - (3/2)x^2 + (3/5)x - 1/20}$.

(ii) Find the zeros of $\widetilde{p}_2(x)$ and call them x_0 and x_1. Construct a quadrature rule for $\displaystyle\int_0^1 f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials.

Answer: The zeros of $\widetilde{p}_2(x) = x^2 - x + 1/6$ are $x_0 = 1/2 - \sqrt{3}/6$ and $x_1 = 1/2 + \sqrt{3}/6$.
The associated Lagrange Polynomials are

- $L_0 = \frac{x - x_1}{x_0 - x_1} = \frac{x - 1/2 - \sqrt{3}/6}{-\sqrt{3}/3} = -\sqrt{3}x + (1 + \sqrt{3})/2$
- $L_1 = \frac{x - x_0}{x_1 - x_0} = \frac{x - 1/2 + \sqrt{3}/6}{\sqrt{3}/3} = \sqrt{3}x + (1 - \sqrt{3})/2$

With a little calculus we can see that $w_0 = \int_0^1 L_0(x)dx = \frac{1}{2}$ and $w_1 = \int_0^1 L_1(x)dx = \frac{1}{2}$. However, it is OK to derive the values of w_0 and w_1 using, e.g., undetermined coefficients.