2526-MA140 Engineering Calculus

Week 11, Lecture 1
Centres of Mass: two-dimensions

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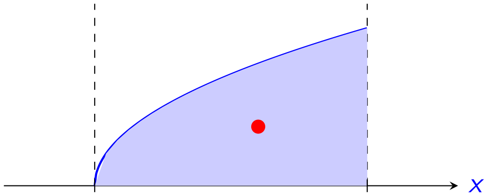


The centre of attention today:

1. News!
2. Centre of Mass: recall
   - Point Masses
   - Variable Density
   - A note on terminology
3. Centre of Mass: 2D
   - A lamina
   - Moments
   - Centre of Mass
   - Complex regions
4. Solids of Revolution
5. Exercises

For more, read Section **6.6** (Moments and Centres of Mass) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax).

News!

Tutorials

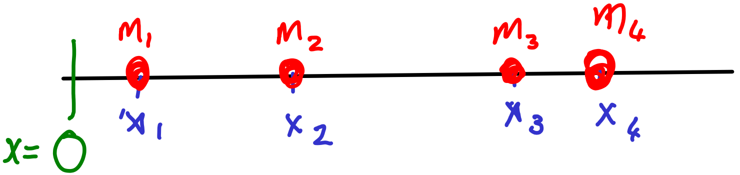
There is a change to the tutorial plan for this week:

- Tuesday at 3pm: Teams 1 and 2 will have their in ENG-2003 (JM). Teams 3, 4, 5, and 6 will have a tutorial with MY243 (ST).
- Thursday at 11am: Teams 9 and 10 attend a tutorial in ENG-3035. Teams 7, 8, 11, and 12's tutorial is Aras Moyola MY129.
- Tutorials will cover the Practice Exam Paper $\longleftarrow$ (link)
- Friday: no MA140 tutorials!
- No change to the Irish tutorial.

For more on the final exam, and practice paper: come to class tomorrow and Thursday!

Last week, we learned that if we have a mass-less rod, onto which are attached point masses $m_1, m_2, \ldots, m_n$, at points $x_1, x_2, \ldots, x_n$, then the

- The **moment** of the system is $M = x_1 m_1 + x_2 m_2 + \cdots + x_n m_n$.
- The **total mass** is $m = m_1 + m_2 + \cdots + m_n$.
- The **centre of mass** is $\bar{x} = \dfrac{M}{m}$.



Moment: "distance from 0" × "mass"

If we have a rod (which has mass) placed on the x-axis, with endpoints $x = a$ and $x = b$, with $a < b$, and the density of the rod is $\rho(x)$, then

- The moment of the system is $M = \displaystyle\int_a^b x\rho(x)\,dx$.
- The total mass is $m = \displaystyle\int_a^b \rho(x)\,dx$.
- The **centre of mass** of the rod is $\bar{x} = \dfrac{M}{m} = \dfrac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$

If you read up on these topics, you'll often find the terms "**Centroid**" and "**Centre of Mass**" used interchangeably, as though they mean the same thing.

The don't, but are very closely related.

- A region of space has a **centroid**, also called the **geometric centre**.
- A lamina is a thin plate: it has a **centre of mass** (point at which it could be balanced on the head of a pin).
- If the lamina has constant density, and its shape is a region in space, then **centroid** and **centre of mass** are the same.

A **lamina** is a very, *very* thin plate whose shape is the region in space bounded above by $y = f(x) > 0$, below by $y = 0$, and left by $x = a$, and right by $y = b$. For us, it will always have uniform density.

We want to find its **centre of mass** (also called a "centroid", in the case where we have uniform density), which we denote $(\bar{x}, \bar{y})$.

- The total mass is $m = \int_a^b f(x)\,dx$. "mass = area".
- M_x, the moment about the x-axis, is $M_x = \dfrac{1}{2}\displaystyle\int_a^b \big(f(x)\big)^2, dx.$
- M_y, the moment about the y-axis is $M_y = \displaystyle\int_a^b xf(x)\,dx.$

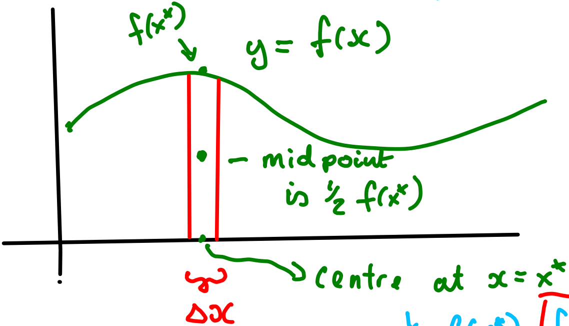
The explanation for the formula for M_x and M_y follows...

As in 1D, the key idea we need is that of a **moment**. In a realistic setting, this is the **mass** of the lamina, times its distance from a reference point: usually $(0, 0)$.

To start with, it is helpful to think of the moments (in x and y) of a thin rectangle:

Now let's get M_x, which is the moment about the x-axis, by summing the moments of all the rectangles, and taking the limit of the resulting Riemann sum:

$$M_x = \lim_{n\to\infty} \sum_{k=1}^{n} \boxed{\frac{1}{2}\big(f(x_i^\star)\big)^2} \Delta x = \frac{1}{2}\int_a^b \big(f(x)\big)^2 dx.$$

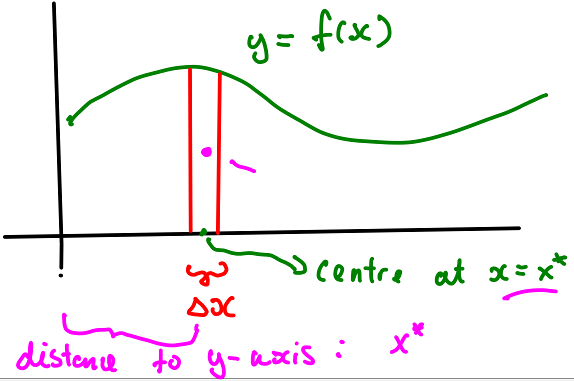


Moment about the x-axis: "distance from midpoint to x-axis" × "mass".

$$M_x = \underbrace{\tfrac{1}{2} f(x^*)}_{\text{Midpoint}} \cdot \underbrace{\boxed{f(x^*)\Delta x}}_{\text{Area}}$$

Similarly, we get M_y, which is the moment about the y-axis as

$$M_y = \lim_{n\to\infty} \sum_{k=1}^{n} x_i^\star f(x_i^\star)\Delta x = \boxed{\int_a^b x f(x)\, dx.}$$



Moment:
"distance from x^* to y-axis" × "Area"
$= x^* f(x^*) \Delta x$

Centre of Mass: 2D

If the centre of mass is the point $(\bar{x}, \bar{y})$, then we could think of the entire "area" as being centred there, but having the same moments.

That is

$$\bar{x}A = M_y, \quad \text{and } \bar{y}A = M_x.$$

giving...

$f(x) > 0$

Centroid of a planar region

If $f(x)$ is defined on $[a, b]$, then the **centroid** $(\bar{x}, \bar{y})$ of the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$ is given by

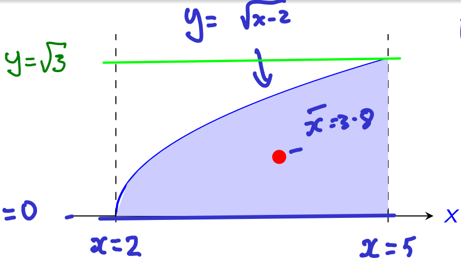
$$\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} \quad \text{and} \quad \bar{y} = \frac{1}{2}\frac{\int_a^b [f(x)]^2\,dx}{\int_a^b f(x)\,dx}$$

Example

Consider the plane region enclosed by the curve $y = \sqrt{x-2}$, the x-axis and the lines $x = 2$ and $x = 5$. Find

(1) the area of the region;

(2) the centroid of the region.



First, we find

$$M = \int_a^b f(x)\,dx$$

$$= \int_2^5 (x-2)^{1/2}\,dx.$$

Use a substitution $u = x-2$.

To get ...

$$M = 2\sqrt{3}.$$

Note, eg $f(x) = \sqrt{x-2} \Rightarrow f(2) = 0$, $f(5) = \sqrt{3}$

Next we find M_y

$$M_y = \int_a^b x\, f(x)\, dx = \int_2^5 \underline{\underline{x}}\, (x-2)^{1/2}\, dx.$$

Let $u = x-2$. So $du = dx$.

Also $x = u+2$. $x=2 \Rightarrow u=0$, $x=5 \Rightarrow u=3$

So $$M_y = \int_{u=0}^{u=3} (u+2)\, u^{1/2}\, du = \int_{u=0}^{u=3} u^{3/2} + 2u^{1/2}\, du$$

$$= \left[\frac{2}{5} u^{5/2} + 2 \cdot \frac{2}{3} u^{3/2}\right]\Bigg|_{u=0}^{3}$$

Thus $M_y = \frac{2}{5}(3)^{5/2} + \frac{4}{3}(3)^{3/2} \overset{\text{check}}{=} \cdots = \sqrt{3}\,\frac{38}{5}$.

Then $\bar{x} = \dfrac{M_y}{m} = \left(\dfrac{\sqrt{3}\cdot 38}{5}\right)\dfrac{1}{2\sqrt{3}} = \dfrac{19}{5} \approx 3.8$

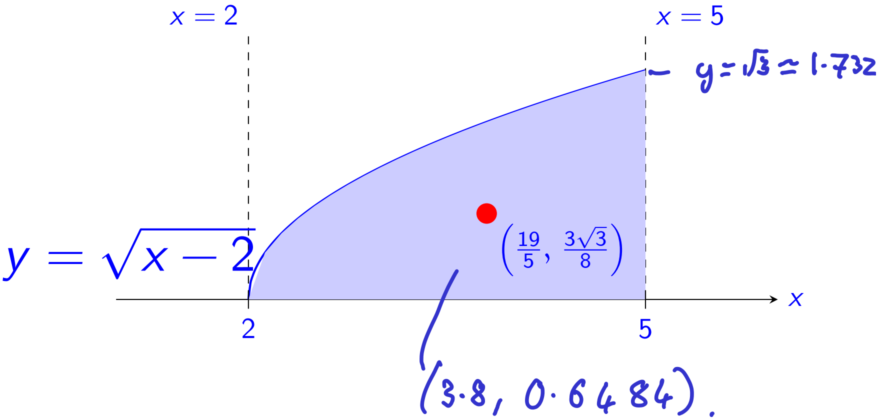
Next we compute M_y

$$M_x = \frac{1}{2}\int_a^b f(x)^2\,dx = \frac{1}{2}\int_2^5 (x-2)\,dx.$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2 - 2x\right]\Big|_2^5$$

$$= \frac{1}{2}\left(\frac{25}{2} - 10 - \frac{4}{2} - 4\right) = \cdots = \frac{9}{4}.$$

Then $\bar{y} = \dfrac{M_x}{M} = \dfrac{9}{4}\cdot\dfrac{1}{2\sqrt{3}} = \dfrac{3\sqrt{3}}{8} \approx 0.6484.$


$x = 2$
$x = 5$
$y = \sqrt{3} \simeq 1.732$
$y = \sqrt{x-2}$
$\left(\frac{19}{5}, \frac{3\sqrt{3}}{8}\right)$
2
5
x
$(3.8, 0.6484)$.

The idea can be extended to more complex regions in space, such as the region bounded by two curves, $f(x)$ and $g(x)$. We don't do the derivation here (it is in the textbook).

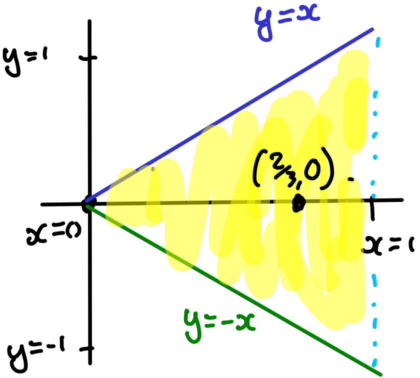
Centroid of a planar region bounded by two functions

Take functions $f(x)$ and $g(x)$ defined on $[a, b]$, when $f(x) \geqslant g(x)$. Consider the region between $f(x)$ and $g(x)$, and between $x = a$ and $x = b$. Its **centroid**, $(\bar{x}, \bar{y})$, is given by

$$\bar{x} = \frac{\int_a^b x\big(f(x) - g(x)\big)\,dx}{\int_a^b f(x) - g(x)\,dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b f(x)^2 - g(x)^2\,dx}{\int_a^b f(x) - g(x)\,dx}$$

Example

Find the centroid of the region between $f(x) = x$, $g(x) = -x$, $a = 0$ and $b = 1$.



First find

$$M = \int_a^b f(x) - g(x)\,dx$$

$$= \int_0^1 x - (-x)\,dx$$

$$= \int_0^1 2x\,dx$$

$$= x^2\Big|_0^1 = 1.$$

Example

Find the centroid of the region between $f(x) = x$, $g(x) = -x$, $a = 0$ and $b = 1$.

Next we need M_x & M_y ...

But $M_x = 0$ since this region is symmetric about the x-axis.

Thus $\bar{y} = \frac{M_x}{m} = \frac{0}{1} = 0$.

$M_y = \int_0^1 x(f(x) - g(x))\,dx = \int_0^1 x(x+x)\,dx$

$= \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = 2/3$. $\Rightarrow \bar{x} = \frac{M_y}{m} = 2/3$

So $(\bar{x}, \bar{y}) = (2/3, 0)$.

Solids of Revolution

We'll finish this section by considering how to find the centre of mass of a solid of revolution.

Suppose that $f(x) \geqslant 0$ on $[a, b]$ and consider the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$.

Recall that we can rotate this region about the x-axis to obtain a solid of revolution.

Intuitively, it is clear that the centroid of such a solid should lie on the x-axis because of symmetry, so $\bar{y} = 0$. So, we only need find $\bar{x}$.

Solids of Revolution

If the solid has uniform density, $\rho(x, y) \equiv 1$, then the total mass is the same as the volume.

We know already (Disk Method) that the volume of this region is

$$V = \pi \int_a^b f(x)^2 \, dx.$$

Solids of Revolution

To get the moment about the y-axis, we consider the moment of an individual disk of volume ΔV_r, at the point $x = x_r$, which is $x_r \Delta V_r$. If the solid is divided into N such rings:

$$M_y \approx \sum_{r=1}^{n} x_r \Delta V_r = \sum_{r=1}^{n} x_r \left(\pi f(x_r)^2 \, \Delta x \right)$$

Then, as we have seen repeatedly:

$$M_y = \lim_{n \to \infty} \sum_{r=1}^{n} x_r \left(\pi f(x_r)^2 \, \Delta x \right) = \pi \int_a^b x f(x)^2 \, dx$$

Solids of Revolution

Putting all this together, and using that $M_y = V\bar{x}$, we get...

Centroid of a solid of revolution

If $f(x) \geqslant 0$ on $[a, b]$, then the **centroid**, $(\bar{x}, \bar{y})$ of the solid of revolution obtained by rotating the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$ about the x-axis is

$$\bar{x} = \frac{M_y}{V} \quad \text{and} \quad \bar{y} = 0\,.$$

where

$$M_y = \pi \int_a^b x f(x)^2\, dx \quad \text{and} \quad V = \pi \int_a^b f(x)^2\, dx.$$

Solids of Revolution

Example

Consider the plane region enclosed by the curve $y = \sqrt{x-2}$, the x-axis and the lines $x = 2$ and $x = 5$. Find the centroid of the solid of revolution obtained by rotating this region about the x-axis.

Solids of Revolution

Exercises

Exer 11.1.1

Find the centroid of the region between $y = 1/x$, $y = 0$, $x = 1$ and $x = 2$.

Exer 11.1.2

Find the centroid of the region between $f(x) = x^2$, $g(x) = -x$, $x = 0$, and $x = 1$.