

This is a sample paper for 2526-MA385. It is similar to the final Semester 1 exam paper in the following ways:

- It features 5 questions; all to be attempted.
- Questions 1 and 2 are based on material from Section 1 (may have some over-lapping content. E.g., Newton's Method, or FPI could feature on both).
- Questions 3, 4 and 5 are based on Sections 2, 3, and 4 respectively with minimal overlap (and only in so far as Sections 3 and 4 overlap a little)
- Questions feature a mixture of definitions, theory and calculations.
- The questions on the exam will, of course, be different. However, if you can attempt this paper unseen, you are well prepared.

Q1. Suppose we wish to find $\tau \in [a, b]$ such that $f(\tau) = 0$ for some nonlinear function $f(x)$.

- State the **Secant Method** for this problem. Provide a justification for it.
- Suppose that $f(x) = 2x^2 - 5$. Show that $f(x) = 0$ has a solution in $[1, 2]$.
- Taking $x_0 = 1$ and $x_1 = 2$, carry out **three** iterations of the Secant Method to estimate the solution to $2x^3 - 4 = 0$. Show your calculations to 4 decimal places.

- Q2. (a) What does it mean for a function, g , to be a contraction on an interval $[a, b]$?
- (b) Suppose that we have a fixed point iteration (FPI) method $x_{k+1} = g(x_k)$, and that g is known to be a contraction, with a fixed point τ . Show that the sequence generated by the method, $\{x_0, x_1, x_2, \dots\}$ converges *at least linearly* to τ .
- (c) Suppose that we want to solve $2x^2 - 5 = 0$ using FPI, in order to approximate $\tau = \sqrt{10}/2$. That is, we choose a function $g = g(x)$, and initial guess $x_0 \in [1, 2]$, and set $x_{k+1} = g(x_k)$ for $k = 0, 1, 2, \dots$. Consider the following functions:

$$g_1(x) = 2x^2 + x - 5, \quad g_2(x) = x/2 + 5/(4x), \quad g_3(x) = x^2/5 - 1/2.$$

For each of these, determine whether or not it is a suitable choice of g in the FPI.

- (d) Show that Newton's method for solving $f(x) = 0$ can be considered as a FPI method. What FPI method does it yield when we use it to solve $f(x) = 2x^2 - 5 = 0$?

Q3. Consider the general two-stage Runge-Kutta (RK2) method: $y_{i+1} = y_i + f\Phi(t_i, y_i; h)$, where

$$k_1 = f(t_i, y_i), \quad k_2 = f(t_i + \alpha h, y_i + \beta h k_1)$$

and

$$\Phi(t_i, y_i; h) = a k_1 + b k_2.$$

For a specific method we can take

$$\alpha = 2, \text{ and } a = 1/4. \quad (1)$$

- (a) What does it mean for a one-step method to be *consistent*? Determine the value of b_2 for the method to be consistent.
- (b) Suppose y solves the initial value problem

$$y(1) = 1, \quad y'(t) = 2t \quad \text{for } t > 1.$$

Explain why the RK2 method should compute the exact solution. Use this fact to determine the value for α .

- (c) Suppose that we attempt to solve

$$y'(t) = \lambda y(t) \quad y(0) = 1,$$

with a RK2 method. Use the fact that the RK2 solution should agree with the Taylor series for $y(t_{i+1})$ about t_i , up to terms of order h^2 , to find a value of β .

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- Q4. (a) Let $L \in \mathbb{R}^{n \times n}$ be a non-singular lower triangular matrix, and $\mathbf{b} \in \mathbb{R}^n$ be such that $b_i = 0$ for $i = 1, \dots, k \leq n$. If \mathbf{y} solves $L\mathbf{y} = \mathbf{b}$, show that $y_i = 0$ for $i = 1, \dots, k \leq n$. Hence or otherwise, show that the inverse of a nonsingular lower triangular matrix is also lower triangular.
- (b) Define the LU factorization of a matrix. What assumptions must be made on the matrix to ensure that such a factorization exists?
- (c) Find the LU -factorisation of

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix}.$$

Use this factorization to solve $Ax = b$, where $b = (4, -4, 0, 8)^T$.

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- Q5. (a) Recall the definition of the Euclidean norm on \mathbb{R}^n : $\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}^T \mathbf{u}}$. Prove the Cauchy-Schwarz inequality:

$$\left| \sum_{i=1}^n u_i v_i \right| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Hence show that $\|\cdot\|_2$ satisfies the triangle inequality.

- (b) Let A be any matrix in $\mathbb{R}^{n \times n}$. What are the *singular values* of A ? Show that they are real and non-negative. Define the *subordinate matrix norm* on $\mathbb{R}^{n \times n}$ associated with $\|\cdot\|_2$ and show that $\|A\|_2$ is the largest singular value of A .
- (c) State the Gerschgorin First Circle Theorem, and use it to find an upper bound on $\|A\|_2$ when

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix}.$$