MA378: Tutorial Sheet 2

In all the questions below, when needed, the function f is
$$f(x) = \ln(x)$$

1. (a) Write down the Lagrange form for the polynomial, $p_2(x)$, that interpolates f at the points $x_0 = 1$, $x_1 = 2$, and $x_2 = 3$.

   (b) What bound does Equation (1) give for $|f(4/3) - p_2(4/3)|$?

2. (a) Give a formula for the piecewise linear interpolant, $l(x)$, that interpolates f, at the points $x_0 = 1$, $x_1 = 2$, and $x_2 = 3$.

   (b) Use Equation (2) to give an upper bound for $\|f(x) - l(x)\|_\infty$.

   (c) What value of N would you have to choose so that $\|f - l\|_\infty \leq 10^{-6}$?

3. Suppose that S is a natural cubic spline on $[0, 2]$ with
$$S(x) = \begin{cases} -x + 2(1-x) + a(1-x)^3 + \dfrac{2}{3}x^3, & \text{for } 0 \leq x < 1, \\ b(2-x) + c(2-x)^3 + d(x-1)^3, & \text{for } 1 \leq x \leq 2. \end{cases}$$
   Find a, b, c, and d.

Some useful formulae.

- Cauchy's theorem: If p_n is the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that
$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \tag{1}$$
  where $\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ denotes the nodal polynomial.
- $\|g\|_\infty :- \max\limits_{a \leq x \leq b} |g(x)|$.
- If l is the linear spline interpolant to a function f on the equally spaced points $a = x_0 < x_1 \cdots < x_N = b$ with $h = x_i - x_{i-1} = (b-a)/N$, then
$$\|f - l\|_\infty \leq \frac{h^2}{8}\|f''\|_\infty. \tag{2}$$