Annotated slides

2425-MA140 Engineering Calculus

Week 06, Lecture 2
Curve sketching

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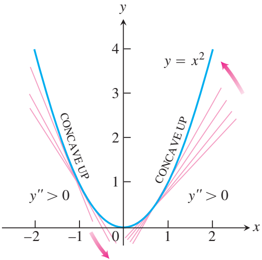
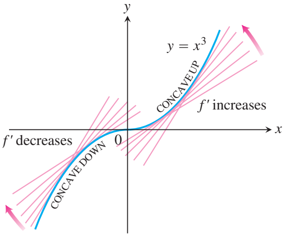
Wednesday, 22 October, 2025





A sketch of today's class...

See also: Section 4.5 (Derivatives and the Shape of a Graph) of **Calculus** by Strang & Herman: Section 4.3 (Maxima and Minima) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

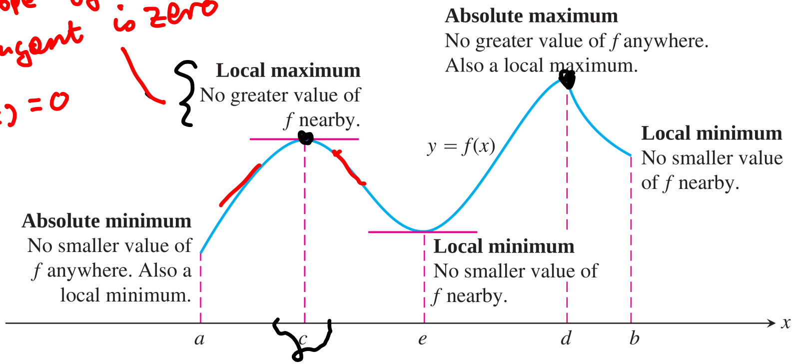
Yesterday, we started studying the application of differentiation in locating (local) maxima and minima in functions.

There are the key points to recall:

- **maximum** and **minimum** points are collectively called **extreme** points.

Slope of tangent is zero

$f'(c) = 0$



The First Derivative Test (again) Review

- $x = c$ is a **critical point** of $f(x)$ if either $f'(c) = 0$ or $f'(c)$ does not exist.
- All extreme points occur at critical points. (but not all critical points correspond to extreme points).
- To find a maximum or minimum of f, we first find the critical points.
- If $f'(x) > 0$ at each point $x \in [a, b]$, then f is increasing on $[a, b]$.
- If $f'(x) < 0$ at each point $x \in [a, b]$, then f is decreasing on $[a, b]$.

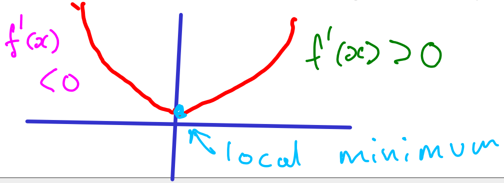
The First Derivative Test (again)

First Derivative Test for local maxima and minima

Suppose that c is a critical point of a differentiable function f.

1. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a **local maximum** of f.
2. If f' changes sign from negative when $x < c$ to positive when $x > c$ then $f(c)$ is a **local minimum** of f.
3. If f' has the same sign for $x < c$ and $x > c$ then $f(c)$ is neither a local maximum nor a local minimum of f.

We finished yesterday with an example that was overly complicated, and wrong! Let's try a better one.



The First Derivative Test (again)

Example

Example (Example 4.5.1 from textbook)

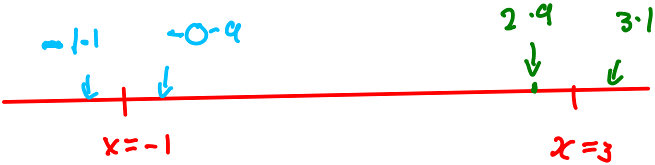
Use the first derivative test to find the location of all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$, and characterize them as maxima or minima.

1. Differentiate $f(x)$ to get $f'(x) = 3x^2 - 6x - 9$.
2. Solve for the critical points. Since $f'(x)$ is defined everywhere, we just need to solve $3x^2 - 6x - 9 = 0$. Simplifying, this is $x^2 - 2\cancel{x} - 3 = 0$. That factorizes as $f'(x) = (x+1)(x-3)$, which has two zeros: at $x = -1$, and $x = 3$.
3. Now we need to know how f' is changing sign at these points. Check the text-book for a technical approach, we'll use a simple one.

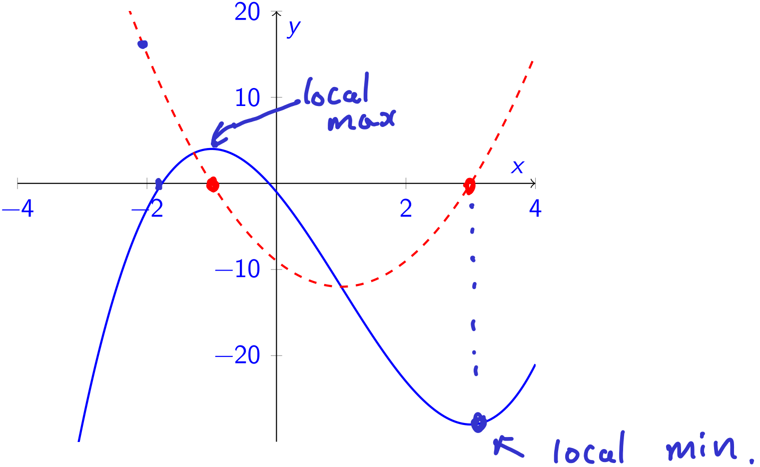
"Characterize an Extremum" = "Say if it is a max or min".

4. By calculation (e.g., with a calculator), we'll check $x = -1$. We see $f'(-1.1) = 1.23$ and $f'(-0.9) = -1.17$. So f' changes from **positive** to **negative** at $x = -1$, so we have a **local maximum**.

5. Similarly, we'll check $x = 3$. We see $f'(2.9) = -1.17$ and $f'(3.1) = 1.23$. So f' changes from **negative** to **positive** at $x = 3$, so we have a **local minimum** there.



A plot of $f(x)$ and $f'(x)$ (but which is which??)



Concave up and down functions

Definition

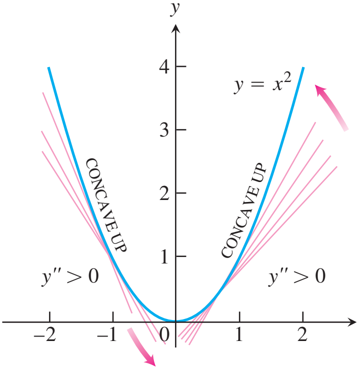
The graph of a differentiable function $y = f(x)$ is:

- **concave up** on an open interval (a, b) if f' is increasing on (a, b);
- **concave down** on an open interval (a, b) if f' is decreasing on (a, b)

Note:

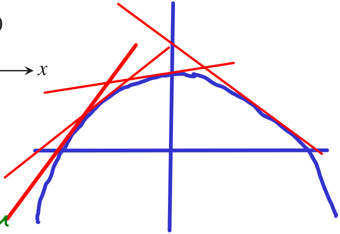
- If the graph of f is **concave up** ("cup"), it is **above** its tangents.
- If the graph of f is **concave down**, it is **below** its tangents.

Concave up and down functions

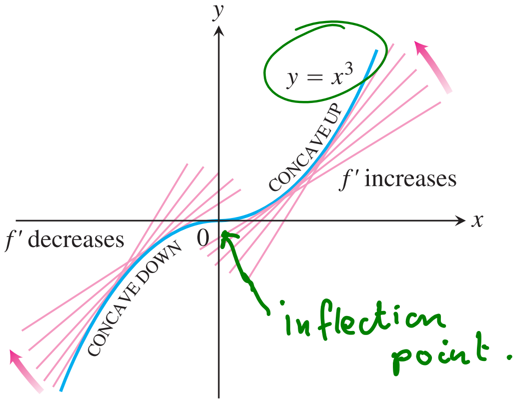


A "Cup" function.

Concave down function



Concave up and down functions



Concave up and down functions

Relating concavity to f''

Let $y = f(x)$ be twice-differentiable on an open interval (a, b).

- If $f'' > 0$ on (a, b), the graph of f is **concave up**
- If $f'' < 0$ on (a, b), the graph of f is **concave down**

Example: $f(x) = x^2$ is concave up (for all x) and $g(x) = -x^2$ is concave down.

$f'(x) = 2x$ | $g'(x) = -2x$

and $f''(x) = 2$ | $g''(x) - 2 < 0$

And $2 > 0.$

Inflection points

Definition: infection point

A **point of inflection** is a point at which the concavity of a function changes.
At such a point, either f'' is zero or does not exist.

Example

Find a point of inflection of the graph of $f(x) = x^3$.

"Concavity changes" means "switch from concave up to concave down" or vice versa.

$$f(x)=x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x \begin{cases} \text{negative} & \text{if } x<0 \\ \text{positive} & \text{if } x>0. \end{cases}$$

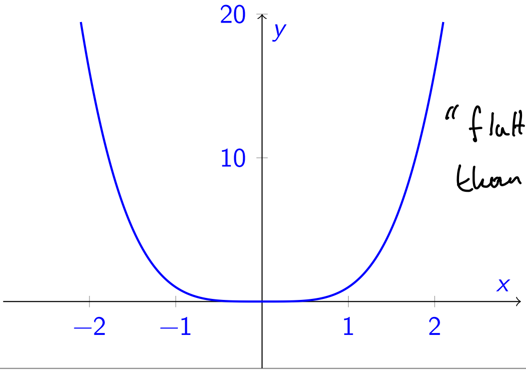
Note $f''(0)=0 \ldots$

Inflection points

Warning: Having $f''(c) = 0$ does not necessarily mean that f has an inflection point at $x = c$.

Example

The curve $y = x^4$ has no inflection point at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.



"flatter at $x=0$ than $y=x^2$".

Second derivative test

c is a critical point.

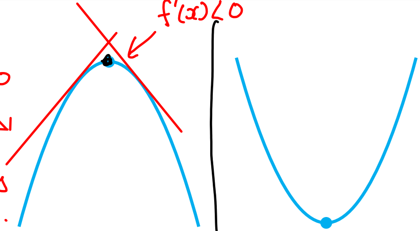
Second Derivative Test

Suppose that f'' is continuous on an interval that contains c.

- If $f'(c) = 0$ and $f''(c) < 0$, then f has a **local max** at $x = c$.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a **local min** at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive. The function f may have a local max, a local min, or neither.

$f'(x) < 0$

$f'(x) > 0$



So $f'(x)$ is decreasing

So $f''(x)$ is negative.

$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Second derivative test

Example

Find and classify the critical and infection points of

$$f(x) = 4x^3 - 21x^2 + 18x + 6.$$

We have $f'(x) = 12x^2 - 42x + 18$.

When $f'(x) = 0$, we have

divide by 6

$$12x^2 - 42x + 18 = 0 \Leftrightarrow 2x^2 - 7x + 3 = 0$$
$$\Leftrightarrow (2x-1)(x-3) = 0.$$

Solve $2x-1=0 \Rightarrow x = \frac{1}{2}$,

Solve $x-3=0 \Rightarrow x=3$.

So the critical points are at $x = \frac{1}{2}$ and $x = 3$.
Next $f''(x) = 24x - 42$ so

$$f''\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right) - 42 = 12 - 42 = -30 < 0,$$

which means there is a local **maximum** at $x = \frac{1}{2}$.

Second derivative test

Also,

$$f''(3) = 24(3) - 42 = 72 - 42 = 30 > 0.$$

so it has a local **minimum** at $x = 3$.

Now solve for $f''(x) = 0$ for any inflection point.

Now, recall that $f''(x) = 24x - 42$. Thus,

$$f''(x) = 0 \Leftrightarrow x = \frac{42}{24} = \frac{7}{4}.$$

Note that

$$x < \frac{7}{4} \implies f''(x) < 0$$

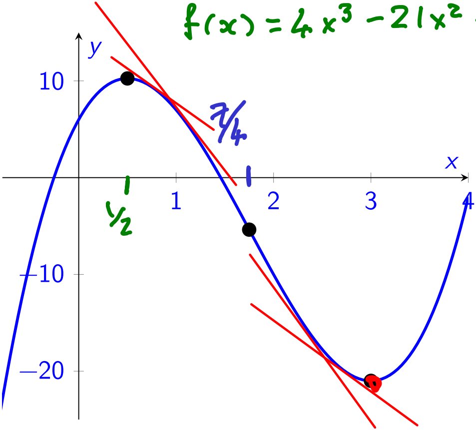
$$x > \frac{7}{4} \implies f''(x) > 0.$$

because $f''(x)$ changes sign.

Therefore, $f(x)$ has a point of inflection at $x = \frac{7}{4}$.

Second derivative test

$f(x) = 4x^3 - 21x^2 + 18x + 6$.



Local max at $x = 1/2$ | Local minimum at $x = 3$ | Inflection point at $x = 7/4$

Second derivative test

Review

If a function f is differentiable on an interval (a, b), then

- $f'(x) > 0$ for $a < x < b$, then it is increasing on (a, b).
- $f'(x) < 0$ for $a < x < b$, then it is decreasing on (a, b).
- $f''(x) > 0$ for $a < x < b$, then it is concave up on (a, b).
- $f''(x) < 0$ for $a < x < b$, then it is concave down on (a, b).

Second derivative test

Review (continued)

1st Derivative Test:
If f' changes sign at a critical point, c, it is a local maximum or minimum.

2nd Derivative Test:

- If $f''(c) < 0$, then there is a local maximum at $x = c$.
- If $f''(c) > 0$, then there is a local mimimum at $x = c$.
- If $f''(c) = 0$ at a critical point c, then the test is inconclusive.

Curve Sketching

In order to roughly **sketch the graph** of a function, f, we can use the following steps:

1. Compute $f'(x)$ and find the **critical points** and inflection points of f. Find the corresponding y-value of these points.
2. Compute $f''(x)$, and use the second derivative test.
3. Make a table showing the intervals on which f is increasing and/or decreasing, and where f is concave up and/or concave down.
4. Plot some specific points (e.g. local max/min, points of inflection, intercepts) and sketch the general shape of the graph of f.

i.e where it cuts the axes.

Example

Sketch the graph of the function $f(x) = x^4 - 4x^3 + 10$

(1) Find $f'(x)$. $f'(x) = 4x^3 - 12x^2$

Solve $f'(x) = 0$ ie $4x^3 - 12x^2 = 0$

$\Rightarrow x^2(4x - 12) = 0$

So $f'(x) = 0$ if $x = 0$ or $4x - 12 = 0$

$\Rightarrow x = 0$ or $x = 3$

So f has critical points at $x = 0$, $x = 3$.

(2) $f''(x) = 12x^2 - 24x$.

Check $f''(0) = 0 \Rightarrow$ test is inconclusive

Check $f''(3) = 12(9) - 24(3) = 108 - 72 = 36 > 0$

so local min.

Curve Sketching

Step 3: Make table to find intervals on which f is increasing/decreasing and on which f is concave up and concave down

		0		2		3	
$4x^2$	$+$	•	$+$		$+$		$+$
$x-3$	$-$		$-$		$-$	•	$+$
$f'(x)$	$-$	•	$-$		$-$	•	$+$
$12x$	$-$	•	$+$		$+$		$+$
$x-2$	$-$		$-$	•	$+$		$+$
$f''(x)$	$+$	•	$-$	•	$+$		$+$

Curve Sketching

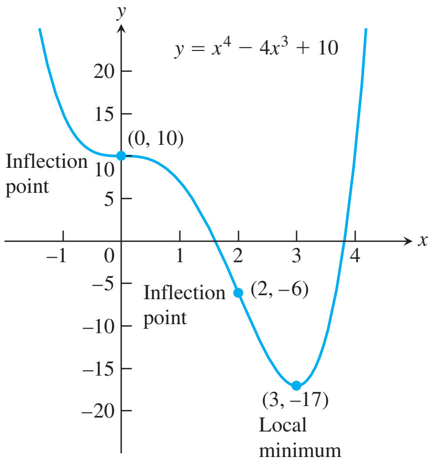
Step 4: Plot specific points (such as local max/min, points of inflection, intercepts) - and sketch general shape of graph of f

This was added after class:
(a) The critical points are x=0, and x=3. The corresponding values y=f(x) are, respectively, y=10 and y=-17.

So there is an inflection point at (0,10) and a local min at (3,-17).

(b) Find intercepts (where convenient)
 * Note that (0,10) is also the y-intercept.
 * In this case, since f(x) is a polynomial of degree 4, so finding the x-intercepts, which is where f(x)=0, is not so eas
 therefore, we'll skip that step.

Step 5: Plot specific points (such as local max/min, points of inflection, intercepts) - and sketch general shape of graph of f



Exercise 6.2.1 : 23/24 Exam, Q3(a)

Let $f(x) = \ln(x^2 + 1)$.

(i) Find all critical point(s) of f and determine whether f has a local minimum, local maximum or neither.

(ii) Determine the interval on which f is increasing.

(iii) Determine the interval on which f is decreasing.

(iv) Find all point(s) of inflection of f, justifying your answer.

Exercises

Exer 6.2.2 (Based on 2019/20 Exam, Q3(a))

Let $f(x) = x^3 - 3x^2$.

1. Find all asymptotes of the graph $f(x)$
2. Determine the interval(s) on which $f(x)$ is increasing and decreasing.
3. Determine the interval(s) on which $f(x)$ is concave up (convex) and concave down (or concave).
4. Find all point(s) of inflection for the graph of $f(x)$.
5. Give a rough sketch the graph of $f(x)$ (your axes need not necessarily have the same scale).