MA211

Lecture 18: Integration by parts

Mon 10^{th} Nov 2008

$$\int u\,dv = uv - \int v\,du$$

Today...

1. Integration by parts
   - $\int u dv = uv - \int v du$.
   - Definite Integrals
2. Reduction Formulae
3. Partial Fractions

See also Section 7.1 of Stewart.

Integration by parts

Recall that if we are differentiating the product of two functions u and v then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{du}$.

Now, using that $\int \frac{d}{dx}(uv)dx = uv$ we can deduce the following

The most important technique for integrating is:

Integration by Parts

$$\int u dv = uv - \int v du.$$

Using this formula, we try to replace the integral with one that is easier to solve.

The main "*trick*" is in choosing u and dv.

Usually we are trying to integrate the product of 2 functions (though this is not always obvious!).

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Evaluate $\mathcal{I} = \int x e^x dx.$

Soln: Let $u = x, \quad dv = e^x dx$

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Example (4)

(*From Q1 (c) (i), Aut 06/07*)

Evaluate $\mathcal{I} = \int \ln(x) dx.$

Example (5)

(*From Q1 (b), Aut 05/06*)

Evaluate $\mathcal{I} = \displaystyle\int \tan^{-1}(x) dx.$

Exercise (18.2)

Using *Integration by parts*, evaluate the following integrals

(i) $\displaystyle\int x\cos(x)dx.$

(ii) $\displaystyle\int \big(\ln(x)\big)^2 dx.$

(iii) $\displaystyle\int x\tan^{-1}(x)dx.$

(iv) $\displaystyle\int x^2\tan^{-1}(x)dx.$

(v) $\displaystyle\int (x+3)e^{2x}dx.$

Integration by Parts for Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Example (6)

Use that $\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$ to evaluate $\int_1^2 \frac{\ln(x)}{x} dx$

Example (7)

Use that $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$ to evaluate $\int_1^e x^3 \ln(x)\,dx$

Exercise (18.3)

Evaluate the following definite integrals

(i) $\displaystyle\int_1^2 \ln(x)dx$

(ii) $\displaystyle\int_1^2 \frac{\ln(x)}{x}dx$

(iii) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin^2(x)}dx.$

Hint: if $f(x) = \dfrac{\cos(x)}{\sin(x)}$, *what is* $f'(x)$?

Reduction Formulae

We'll now have a look at how to use integration by parts:

Integration by Parts

$$\int u dv = uv - \int v du,$$

to to replace an integral... with itself! Surprisingly, this turns out to be useful.

Reduction Formulae

Some times we can use integration by parts twice:

Example (1)

Show that $\int e^x \cos(x)dx = \frac{1}{2}\big(e^x \sin(x) + e^x \cos(x)\big) + C.$

Reduction Formulae

Example (2)

Show that

$$\int sin^n(x)dx = -\frac{1}{n}\cos(x)sin^{n-1}(x) + \frac{n-1}{n}\int sin^{n-2}(x)dx.$$

Reduction Formulae

Exercise (Q18.4)

Using *Integration by parts to answer the following questions*

(i) Evaluate $\int x^2 e^x \, dx$.

(ii) Evaluate $\int x^5 e^{x^2} \, dx$. (*Hint: first use a substitution, then use the answer to part (i)*).

(iii) Evaluate $\int e^x \sin(x) \, dx$.

(iv) Let $\mathcal{I}_n = \displaystyle\int_0^1 x^n e^x \, dx$. Show that $\mathcal{I}_n + n\mathcal{I}_{n-1} = e$.

(v) Evaluate $\displaystyle\int \sin\big(\ln(x)\big) \, dx$.

Partial Fractions

Suppose we want to evaluate the integral of $\dfrac{x+4}{x^2-5x+6}$.

We know that we can write

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)}$$

and that this can be further simplified using *partial fractions:*

Partial Fractions

Example (1)

Evaluate $\int \frac{1}{x(x^2+1)}\,dx$.

Exercise (Q18.5)

Evaluate the following:

(i) $\displaystyle\int \frac{1}{x(x^2-1)}dx$

(ii) $\displaystyle\int \frac{x^3+2}{x^3-1}dx$

(iii) $\displaystyle\int \frac{2x+1}{x^2+4x+4}dx$

(iv) $\displaystyle\int_2^3 \frac{3x^3+1}{x^3-2x^2+x}dx$