2425-MA140 Engineering Calculus

Week 2, Lecture 2
Introduction to Limits

Dr Niall Madden

School of Mathematical and Statistical Sciences, University of Galway

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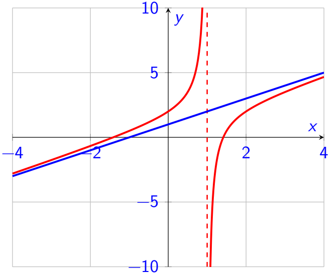


This version of the slides are by Niall Madden, but are adapted from original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Section 7.8.1 (Limit of a function of a real number) in *Modern Engineering Mathematics*:
https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Tutorials started **this** week. And (I'm really, really, sorry) the **correct** schedule is:

- Teams 1+2: Tuesday 15:00 ENG-**2003**
- Teams 3+4: Tuesday 15:00 ENG-**2034** ✓
- Teams 9+10: Thursday 11:00 ENG-**2002**
- Teams 11+12: Thursday 11:00 ENG-**3035**
- Teams 5+6: Friday 13:00 Eng-**2002**
- Teams 7+8: Friday 13:00 Eng-**2035**

- There is currently a "practice" assignment open. See https://universityofgalway.instructure.com/courses/35693/assignments/94873
- A new assignment will open... by tomorrow.

During tutorials, the tutor will solve some similar questions. You can access the tutorial sheet at https://universityofgalway.instructure.com/courses/35693/files/2023552?module_item_id=650912

....

In addition, each week I'll post a set of exercises related to the material covered. You don't have to submit your work for these, but you should try them: they are similar in style and standard to exam questions.

"Not Def" = "not defined".

When we were considering the domain of a function, we looked at those x-values for which the function was not defined.

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1}$$

Neither f nor g are defined at $x = 1$.

But what happens if x gets very closed to 1?

x	0.900	0.990	0.999	1	1.001	1.010	1.100
$f(x)$	11.9	101.99	1001.99	Not Def	−999.99	−99.9	−9.99
$g(x)$	1.9	1.99	1.999	Not Def	2.001	2.01	2.1

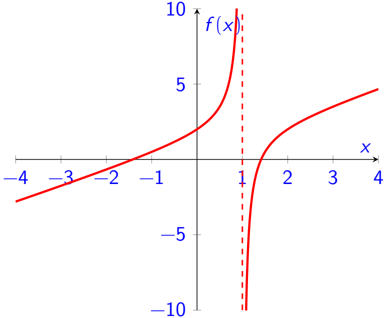
Let's look at the graphs of f and g.

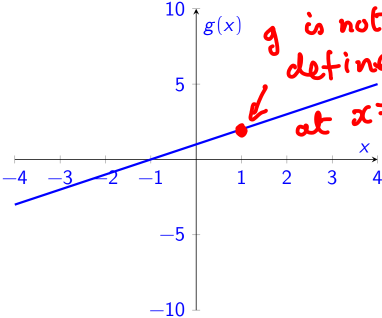
Limits

Except at $x = 1$.

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$





g is not defined at $x = 1$.

Limits

In the previous example, we saw that, although neither f nor g was defined at $x = 1$, they behaved very differently as x approaches 1. To discuss this we need some terminology to help us articulate what it means to be really, really close to value, but not actually at x. We'll also need to be able to discuss what happens for very large or very small x-values.

To do that, we introduce the limit L of a function as x approaches some value $a \in \mathbb{R}$ and denote it by

$$\lim_{x \to a} f(x) = L$$

Note: The concept of a limit is a prerequisite for a proper understanding of calculus and numerical methods.

Definition of a Limit

$x \quad a$

$|x-a| < \delta$

Some conventions and terminology we'll use:

- x is a variable. (ie, a real number)
- a is a fixed number. (some value of x).
- ϵ is a small positive number (that we get to choose).
- δ is another <u>small</u> positive number (determined by ϵ).
- $|x-a| < \delta$ means that the distance between x and a is less than δ, i.e. very small.
- As x approaches a, so $f(x)$ approaches a number L.

When we write

$$\lim_{x \to a} f(x) = L,$$

we read

"*The limit of f, as x goes to a, is L*".

Definition of a Limit

LIMIT: formal definition

$$\lim_{x \to a} f(x) = L,$$

means that, for every number $\epsilon > 0$, it is possible to find a number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

LIMIT: Informal explanation

$$\lim_{x \to a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

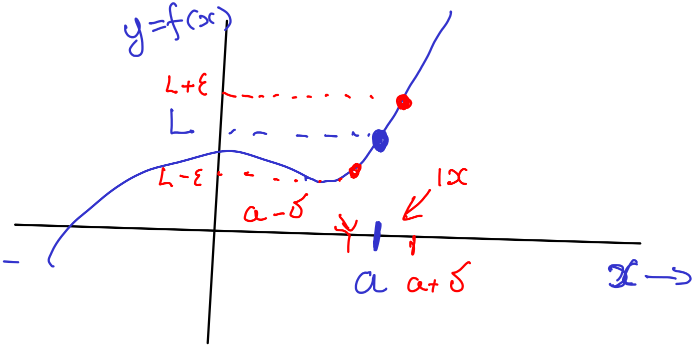
Definition of a Limit

Example

Prove formally that $\lim_{x \to 3} (4x - 5) = 7$.

That is, for arbitrary ϵ, find a δ such that

$$|(4x - 5) - 7| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta.$$



Definition of a Limit

Example

Prove formally that $\lim_{x \to 3} (4x - 5) = 7$.
That is, for arbitrary ϵ, find a δ such that

$$|(4x-5)-7| < \epsilon \quad \text{whenever} \quad |x-3| < \delta.$$

We $|(4x-5)-7| < \varepsilon$

$\Rightarrow |4x - 12| < \varepsilon \quad \Rightarrow 4|x-3| < \varepsilon.$

$\Rightarrow |x-3| < \frac{\varepsilon}{4}$

Since we have $|x-3| < \delta$, take

$\delta < \frac{\varepsilon}{4}$ ✓

Definition of a Limit

The approach we just used is technically correct, but not very practical in many cases.

Fortunately, there are other methods that can be used

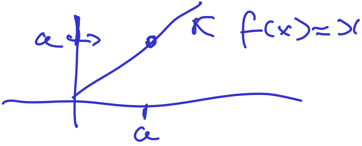
- to show that a limit exists; ✓
- find the limit of a function, $f(x)$ as $x \to a$.

Properties of Limits

Suppose that $\lim_{x \to a} f_1(x) = L_1$, and $\lim_{x \to a} f_2(x) = L_2$ and $c \in \mathbb{R}$ is any constant. Then,

(1) $\lim_{x \to a} c = c$, $c \in \mathbb{R}$.   Eg, if $f(x) = 1$ for all x then $\lim_{x \to a} f(x) = 1$ for any a.

(2) $\lim_{x \to a} x = a$.   That is, if $f(x) = x$ then $\lim_{x \to a} f(x) = a$



(3) $\lim_{x \to a} [c f_1(x)] = cL_1$

Eg $\lim_{x \to a} [5x] = 5 \lim_{x \to a} (x) = 5a$.

Properties of Limits

(4) $\lim_{x \to a} [f_1(x) + f_2(x)] = L_1 + L_2$ and
$\lim_{x \to a} [f_1(x) - f_2(x)] = L_1 - L_2$

Eg $$\lim_{x \to a} (5x + 1) = \underbrace{\lim_{x \to a} (5x)}_{5a} + \underbrace{\lim_{x \to a} (1)}_{1} = 5a + 1$$

(5) $\lim_{x \to a} \left(f_1(x) f_2(x)\right) = L_1 L_2$

Eg $$\lim_{x \to a} \left((x)(x) \right) = \lim_{x \to a} (x) \cdot \lim_{x \to a} (x) = (a)(a) = a^2$$

(6) $\lim_{x \to a} \left((f_1(x))^n \right) = (L_1)^n$

$$\lim_{x \to a} \left((f_1(x))^n \right) = \underbrace{\left(\lim_{x \to a} f_1(x)\right)\left(\lim_{x \to a} f_1(x)\right) \cdots \left(\lim_{x \to a} f_1(x)\right)}_{n \text{ times.}}$$

Properties of Limits

(7) $\lim\limits_{x \to a} \left(\dfrac{f_1(x)}{f_2(x)}\right) = \dfrac{L_1}{L_2}$, providing $L_2 \neq 0$.

Eg $\lim\limits_{x \to 1} \dfrac{x^2+2}{x+3} = \dfrac{\lim\limits_{x \to 1} (x^2+2)}{\lim\limits_{x \to 1} (x+3)} = \dfrac{3}{4}$

(8) $\lim\limits_{x \to a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\boxed{\lim_{x \to a} x^n = a^n} \quad *$$

Example

Evaluate the limit $\lim_{x \to 1}(x^3 + 4x^2 - 3) = L$

$$L = \lim_{x \to 1}(x^3) + \lim_{x \to 1}(4x^2) + \lim_{x \to 1}(-3). \quad [\text{by } (4)]$$

$$= (1)^3 + \underbrace{4 \lim_{x \to 1}(x^2)}_{(3)} + \underbrace{(-3)}_{(1)} = 1 + 4 - 3 = \underline{\underline{2}}$$

Example

Evaluate $\lim_{x\to 1} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = L$

Ans

$$L = \frac{\lim_{x\to 1} (x^4 + x^2 - 1)}{\lim_{x\to 1} (x^2+5)} \quad \text{(by 7)}$$

$$= \frac{(1 + 1 - 1)}{1+5} = \frac{1}{6}.$$

Limits of rational functions

In many cases it's more complicated. In particular, we'll consider numerous examples where we want to evaluate $\lim_{x \to a} f(x)$ where a is not in the domain of f.

A typical example of this is when we evaluate a rational function:

$$\boxed{\lim_{x \to a} \frac{p(x)}{q(x)}}$$

where **both** $p(a) = 0$ and $q(a) = 0$.

Idea: Since we care about the value of p and q **near** $x = a$, but not actually at $x = a$, it is safe to factor out and $(x - a)$ term from both.

Limits of rational functions

Three examples

Evaluate the limits:

$$(a)\ \lim_{x\to 0}\frac{x}{x} \qquad (b)\ \lim_{x\to 0}\frac{x^2}{x} \qquad (c)\ \lim_{x\to 0}\frac{x}{x^2}$$

(a) when $x \neq 0$ $\frac{x}{x} = 1$. So

$$\lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

(b) when $x \neq 0$ $\frac{x^2}{x} = \frac{x}{1} = x$. So $\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0$.

(c) $\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x}$. which is not defined!

Limits of rational functions

Example

Evaluate
$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$$

Limits of rational functions

In that last example, we found that

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x}$$

But these are different functions:

Limits of rational functions

Evaluate the limit

$$\lim_{x \to 2} \left(\frac{\frac{1}{2} - \frac{1}{x}}{x - 2} \right)$$

More limits

Very often, we'll evaluate limits of the form:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

where f and g are not polynomials. Some of the same ideas still apply.

Example

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

Exercises

Exercise 2.2

Evaluate the following limits

(a) $\displaystyle\lim_{x \to \frac{1}{2}} \frac{x - \frac{1}{2}}{x^2 - \frac{1}{4}}$

(b) $\displaystyle\lim_{x \to -4} \frac{x^2 + 3x - 4}{x^2 + x - 12}$

Exercise 2.3

(From 2023/2024 MA140 exam, Q1(a)) Evaluate the limit

$$\lim_{x \to 4} \frac{x - 4}{(\sqrt{x} - 2)(x + 9)}$$