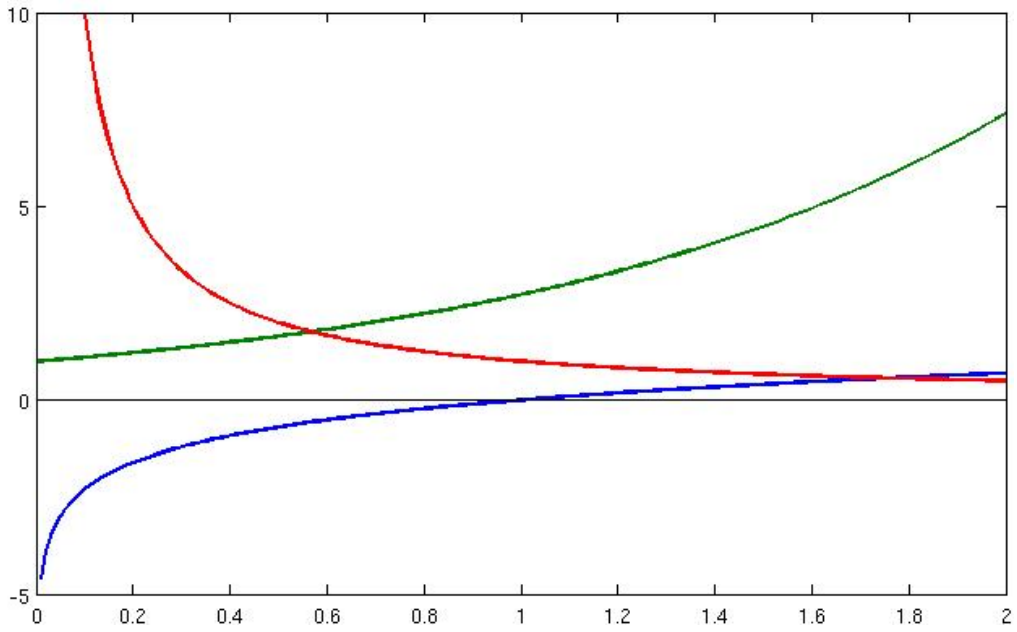
MA211

Lecture 7: Logs and Exponentials

Mon 29 September 2008



Reminder: Problem Set 1

Problem Set 1 is now available from the web-site.

Write out, clearly and carefully, solutions to the selected exercises and submit them at our lecture on Monday, Oct 6th.

Tutorials take place

- Tuesday, 3pm, AC202
- Wednesday, 5pm, QA003 (Physiology lecture room)

In today's lecture

1. Order of a Differential Equation
2. Initial and Boundary Value Problems
   - IVPs
   - BVPs
3. Transcendental Functions
4. Inverse functions
   - Properties
5. Exponents
   - Properties of Exponentials
6. Logarithms
   - Properties of Logarithms
7. The Natural Logarithm

Order of a Differential Equation

There are different types of differential equation that we will study in this course.

The first way of classifying a DE is according to the number of derivatives present.

Definition (Order)

The **Order** of a differential equation is the order of the highest derivative present.

Examples:

In this course we'll only consider 1st- and 2nd-order DEs.

Initial and Boundary Value Problems

There is a further way of classifying DEs to which we seek a *particular solutions*.

1. Initial Value Problems
2. Boundary Value Problems

Examples:

Key idea:

Definition (IVP)

An **Initial Value Problem** is of the form:

Find a function $f(t)$ that satisfies a DE for all $t \geq T$ and one or more of f(T), f'(T), f''(T)... are given.

Examples:

Key idea:

Definition (BVP)

A **Boundary Value Problem** is of the form:

Find a function $f(x)$ that satisfies a DE for all x in the interval (a, b) and $f(a)$ and $f(b)$ are given.

Initial and Boundary Value Problems

BVPs

Mathematically, we will treat these as being the same:

1. Solve the differential Equation to get the *general solution*, including constants of integration.
2. Use the initial or boundary values to solve for the constants and so get the *particular solution*.

Examples:

Exercise (7.1)

Find solutions to the following differential equations

(i) $y'(t) = x - 2$

(ii) $f'(x) = x^{-2} - x^{-3}$ subject to $f(-1) = 0$.

(iii) $y''(x) = x^3 - 1$, with $y'(0) = 0, y(0) = 8$.

(iv) $f''(t) + f(t) = 0$. (Hint: Trig function)

(v) $f''(t) = 9f(t)$ (Hint: Trig function)

Transcendental Functions

This course is mostly concerned with *functions*, and their derivatives and integrals.

However, we've mostly just looked at polynomials and trigonometric functions in any detail.

Most interesting functions that are solutions to differential equations are *Transcendental*: they can't be made of addition and multiplication of the independent variable.

The Trig functions are transcendental, as are the exponential and logarithmic functions, and the hyperbolic functions.

First we'll cover **Inverse functions**.

Inverse functions

Recall...

Definition (One-to-one)

The function $f : A \to B$ is one-to-one if whenever $f(a_1) = f(a_2)$ then $a_1 = a_2$.

Definition (Onto)

The function $f : A \to B$ is onto if for each $y \in B$ there exists $x \in A$ such that y is the image of x.

Inverse functions

When a function is both one-to-one and onto is has an **Inverse**.

Definition

If the function g is the inverse of f then

$$\text{when } f(x) = y, \text{ we get that } g(y) = x.$$

Usually we write $g = f^{-1}$.

Examples:

Properties of Inverse Functions

1. $y = f(x) \iff x = f^{-1}(y)$
2. The domain of f^{-1} is the range of f
3. The range of f^{-1} is the domain of f
4. $f^{-1}\big(f(x)\big) = x$
5. $f\big(f^{-1}(x)\big) = x$
6. $\left(f^{-1}\right)^{-1} = f$.
7. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$.

Some bad notation

Please note that we often use the notation y^{-1} inconsistently:

- When a is a variable, $b = a^{-1}$ means $b = \frac{1}{a}$, i.e., b is the *reciprocal* of a
- When f is a function, $g = f^{-1}$ means $g(f(x)) = x$, i.e., g is the *inverse* of f

Exercise (Q7.2)

For *each* of the following functions, identify the largest possible domain and corresponding range. Is the function one-to-one, onto, or both? Does the function have an inverse? If so, what is it?

(i) $f(x) = 1/(1-x)^3$

(ii) $f(x) = 1/(x+1)^2$.

(iii) $f(x) = \sin^{-1}(x)$

(iv) $f(t) = \log_2(x)$.

(v) $f(x) = a^x$ for $a \in (0,1)$

(vii) $f(x) = \ln(x)$

(viii) $f(t) = \tan^{-1}(x)$.

Exponents

An **exponential** function is a function of the form $f(x) = a^x$.

(Don't confuse this with a **polynomial** such as $f(x) = x^a$.)

If $a > 0$ then, for $n = 1, 2, 3, \ldots$

- $a^0 = 1$
- $a^n = \underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text{ factors.}}$
- $a^{-1} = \dfrac{1}{a}$
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{m/n} = \sqrt[n]{a^m}$

If $a > 0$ and $b > 0$ and $x, y \in \mathbb{R}$, then

(i) $a^0 = 1$

(ii) $a^{x+y} = a^x a^y$

(iii) $a^{-x} = \dfrac{1}{a^x}$

(iv) $a^{x-y} = \dfrac{a^x}{a^y}$

(v) $(a^x)^y = a^{xy}$

(vi) $(ab)^x = a^x b^x$

Limits:

- If $a > 1$, then $\lim_{x \to -\infty} a^x = 0$ and $\lim_{x \to \infty} a^x = \infty$
- If $0 < a < 1$, then $\lim_{x \to -\infty} a^x = \infty$ and $\lim_{x \to \infty} a^x = 0$

Logarithms

When $0 < a < 1$ or $a > 0$, $f : (-\infty, \infty) \to (0, \infty)$ defined by $f(x) = a^x$ is an invertible function.

Definition (Logarithm)

If $a > 0$ and $a \neq 1$, the function $\log_a x$, called the **logarithm of x to the base a** is the inverse of the function a^x:

$$y = a^x \iff x = \log_a y.$$

Exercise

For a given a, what are the domain and range of the function $\log_a(x)$?

If $a > 0$ and $b > 0$ and $x, y \in \mathbb{R}$, then

(i) $\log_a 1 = 0$

(ii) $\log_a(xy) = \log_a(x) + \log_a(y)$

(iii) $\log_a\left(\frac{1}{x}\right) = -\log_a x$

(iv) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

(v) $\log_a\left(x^y\right) = y \log_a x$

(vi) $\log_a x = \frac{\log_b x}{\log_b a}$

These can all be deduced from the properties of exponentials.

Example

Show that $\log_a(xy) = \log_a(x) + \log_a(y)$.

Exercise (7.3)

Show how to deduce the remaining properties of $\log_a$ from the properties of exponentials.

The Natural Logarithm

As we saw last week,

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

So we get that...

$f(x)$	$\int f(x)dx$
$\vdots$	$\vdots$
x^3	$\frac{1}{4}x^4$
x^2	$\frac{1}{3}x^3$
$x^1 = x$	$\frac{1}{2}x^2$
$x^0 = 1$	x
$x^{-1} = \frac{1}{x}$	$\frac{1}{0}x^0$???
x^{-2}	$-x^{-1} = \frac{-1}{x}$
x^{-3}	$\frac{-1}{2}x^{-2}$
$\vdots$	$\vdots$

So what is the antiderivative of $f(x) = 1/x$?

The Natural Logarithm

It turns out that the antiderivative of $f(x) = 1/x$ is a function usually called the "*natural logarithm*" of x, and denoted by $\ln(x)$. However, we'll define it as follows:

Definition (Natural logarithm)

For $x > 0$ let A be the area of the region from $t = 1$ to $t = x$ between the curve $1/t$ and the t-axis. The we define $\ln x$ as

$$\ln(x) = \begin{cases} A, & \text{for } x \geq 1 \\ -A, & \text{for } 0 < x < 1. \end{cases}$$

Theorem

If $x > 0$ then

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

The exponential function

Since the function $\ln(x)$ is one-to-one and onto for the domain $(0, \infty)$, it has an inverse: *The Exponential Function*.

This is usually written as $\exp(x)$ or e^x.

Definition (Exponential Function)

The function $\exp : (-\infty, \infty) \to (0, \infty)$ is the inverse of the natural log function $\ln : (0, \infty) \to (-\infty, \infty)$:

$$y = \ln(x) \quad \Longleftrightarrow \quad x = \exp(y).$$

This means that

$$\ln(\exp(x)) = x \qquad \text{for all } x \in \mathbb{R}$$

and

$$\exp(\ln(x)) = x \qquad \text{for all } x \in \mathbb{R}^+ = (0, \infty)$$

One can easily verify that the natural log function ln has the same properties as $\log_a$, in particular:

(i) $\ln(xy) = \ln(x) + \ln(y)$

(ii) $\ln\left(\frac{1}{x}\right) = -\ln x$

(iii) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

(v) $\ln\left(x^y\right) = y \ln x$

From these properties, one can deduce that:

(i) $\exp(x+y) = \exp(x)\exp(y)$

(ii) $\exp(-x) = \frac{1}{\exp(x)}$

(ii) $\exp(x-y) = \frac{\exp(x)}{\exp(y)}$

(iv) $\exp(x)^y = \exp(xy)$

Definition (e)

The number e is defined as

$$e = \exp(1),$$

and is approximately

$$e \approx 2.718281828459045235360287471352662497757247093699 9$$

Like π, the number e is not rational and is not the solution to any polynomial equation. So it is *transcendental*.
Of particular importance is that

$$exp(x) = e^x.$$

$\frac{d}{dx}e^x$

Perhaps the most important property of the exponential function can be deduced as follows:

$$\frac{d}{dx}e^x = e^x.$$

So the exponential function is its own derivative!

Example

1. Calculate the derivative of $f(x) = 4e^x$
2. Calculate the derivative of $f(x) = e^{x/2}$
3. Calculate the derivative of $f(x) = Ae^{Bx}$ where A and B are constants.

$\int e^x dx$

Because the derivative of e^x is e^x, we also get:

$$\int e^x dx = e^x + C$$

Example

Calculate the integral of $f(x) = Ae^{Bx}$, where A and B are constants.

Solution: $\int Ae^{Bx} dx = A \int e^{Bx} dx = \frac{A}{B} e^{Bx} + C.$

Example

Solve the Initial Value Differential Equation

$$f'(x) - f(x) = 0; f(0) = 2;$$

Solution: