

MA211

Lecture 18: Integration by parts

Mon 10th Nov 2008

$$\int u dv = uv - \int v du$$

Today...

See also Section 7.1 of Stewart.

Integration by parts

Recall that if we are differentiating the product of two functions u and v then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Now, using that $\int \frac{d}{dx}(uv) dx = uv$ we can deduce the following

Integration by parts

The most important technique for integrating is:

Integration by Parts

$$\int u dv = uv - \int v du.$$

Using this formula, we try to replace the integral with one that is easier to solve.

The main “*trick*” is in choosing u and dv .

Usually we are trying to integrate the product of 2 functions (though this is not always obvious!).

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Example (1)

Evaluate $\mathcal{I} = \int x e^x dx.$

Soln: Let $u = x,$ $dv = e^x dx$

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Example (3)

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Example (4)

(From Q1 (c) (i), Aut 06/07)

Evaluate $\mathcal{I} = \int \ln(x) dx.$

Example (5)

(From Q1 (b), Aut 05/06)

Evaluate $\mathcal{I} = \int \tan^{-1}(x) dx$.

Exercise (18.2)

Using *Integration by parts*, evaluate the following integrals

(i) $\int x \cos(x) dx.$

(ii) $\int (\ln(x))^2 dx.$

(iii) $\int x \tan^{-1}(x) dx.$

(iv) $\int x^2 \tan^{-1}(x) dx.$

(v) $\int (x + 3)e^{2x} dx.$

Integration by Parts for Definite Integrals

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

Example (6)

Use that $\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$ to evaluate $\int_1^2 \frac{\ln(x)}{x} dx$

Example (7)

Use that $\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$ to evaluate $\int_1^e x^3 \ln(x) dx$

Exercise (18.3)

Evaluate the following definite integrals

(i) $\int_1^2 \ln(x) dx$

(ii) $\int_1^2 \frac{\ln(x)}{x} dx$

(iii) $\int_{\pi/6}^{\pi/2} \frac{x}{\sin^2(x)} dx.$

Hint: if $f(x) = \frac{\cos(x)}{\sin(x)}$, what is $f'(x)$?

We'll now have a look at how to use integration by parts:

Integration by Parts

$$\int u dv = uv - \int v du,$$

to to replace an integral... with itself! Surprisingly, this turns out to be useful.

Reduction Formulae

Some times we can use integration by parts twice:

Example (1)

Show that $\int e^x \cos(x) dx = \frac{1}{2}(e^x \sin(x) + e^x \cos(x)) + C.$

Example (2)

Show that

$$\int \sin^n(x) dx = -\frac{1}{n} \cos(x) \sin^{n-1}(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx.$$

Exercise (Q18.4)

Using *Integration by parts* to answer the following questions

- (i) Evaluate $\int x^2 e^x dx$.
- (ii) Evaluate $\int x^5 e^{x^2} dx$. (*Hint: first use a substitution, then use the answer to part (i)*).
- (iii) Evaluate $\int e^x \sin(x) dx$.
- (iv) Let $\mathcal{I}_n = \int_0^1 x^n e^x dx$. Show that $\mathcal{I}_n + n\mathcal{I}_{n-1} = e$.
- (v) Evaluate $\int \sin(\ln(x)) dx$.

Partial Fractions

Suppose we want to evaluate the integral of $\frac{x+4}{x^2-5x+6}$.

We know that we can write

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)}$$

and that this can be further simplified using *partial fractions*:

Example (1)

Evaluate $\int \frac{1}{x(x^2 + 1)} dx$.

Exercise (Q18.5)

Evaluate the following:

$$(i) \int \frac{1}{x(x^2 - 1)} dx$$

$$(ii) \int \frac{x^3 + 2}{x^3 - 1} dx$$

$$(iii) \int \frac{2x + 1}{x^2 + 4x + 4} dx$$

$$(iv) \int_2^3 \frac{3x^3 + 1}{x^3 - 2x^2 + x} dx$$