2323-MA378: Class Test in Week 9 (Wed, 28 Feb) ANS (with solutions)

Some useful formulae.

- Cauchy's theorem: If p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that
$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \tag{1}$$
where $\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ denotes the nodal polynomial.
- $\|g\|_\infty$ denotes $\max\limits_{a \le x \le b} |g(x)|$.)
- If l be the linear spline interpolant to a function f on the equally spaced points $a = x_0 < x_1 \cdots < x_N = b$ with $h = x_i - x_{i-1} = (b-a)/N$, then
$$\|f - l\|_\infty \le \frac{h^2}{8}\|f''\|_\infty, \tag{2}$$
- If S is the Piecewise Cubic Hermite Interpolating Polynomial that interpolates the function f at the equally spaced points $\{a = x_0 < x_1 < \cdots < x_N = b\}$ with $x_i - x_{i-1} = (b-a)/N =: h$, then
$$\|f - S\|_\infty := \max_{a \le x \le b} |f(x) - S(x)| \le \frac{h^4}{384} \max_{a \le x \le b} |f^{(iv)}(x)|. \tag{3}$$

In all the questions below, the function f is
$$\boxed{f(x) = e^{x/2}}$$

Q1. (50 marks)

(a) Write down the Lagrange form for the polynomial, $p_2(x)$, that interpolates f at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

Answer: [25 Marks] The Lagrange for an interpolant of degree 2 to f is
$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$
For this problem
$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-1)}{(-1)(-2)} = \frac{1}{2}x(x-1),$$
$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(1)(-1)} = -(x+1)(x-1),$$
$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{(2)(1)} = \frac{1}{2}(x+1)x.$$
Using that $f(x_0) = f(-1) = e^{-1/2}$, $f(x_1) = f(0) = 1$ and $f(x_2) = f(1) = e^{1/2}$, we get that the Lagrange form is
$$p_2(x) = e^{-1/2}\frac{x(x-1)}{2} - (x+1)(x-1) + e^{1/2}\frac{(x+1)x}{2}$$
Notes: It is not required to simplify p_2 any further. In fact, you would get full marks for writing the solution as $p_2(x) = e^{-1/2}L_0(x) + L_1(x) + e^{1/2}L_2(x)$, providing the formulae for the L_i are given.

(b) Evaluate $p_2(1/2)$. ANS [5 Marks] $p_2(1/2) = -(1/8)e^{-1/2} + 3/4 + (3/8)e^{1/2} \approx 1.29245.$

(c) What bound does (1) give for $|f(1/2) - p_2(1/2)|$?

Answer: [20 MARKS] Equation (1) gives $f(1/2) - p_2(1/2) = \frac{f'''(\tau)}{3!}\pi_3(1/2)$, for some (unknown) $\tau \in (-1, 1)$. So the bound on $|f(1/2) - p_2(1/2)|$ is

$$|f(1/2) - p_2(1/2)| \leq \frac{\max_{-1 \leq x \leq 1} |f'''(x)|}{3!} |\pi_3(1/2)|. \qquad (*)$$

Differentiating f, we get $f'''(x) = (1/8)e^{x/2}$. Since this is a positive function that is increasing on $[-1, 1]$, we get

$$\max_{-1 \leq x \leq 1} |f'''(x)| = f'''(1) = e^{1/2}/8 \approx 0.206.$$

Also, $1/(3!) = 1/6$. And $\pi_3(x) = (x+1)x(x-1) = x^3 - x$, so $\pi_3(1/2) = 1/8 - 1/2 = -3/8$. Substituting into (*), gives

$$|f(1/2) - p_2(1/2)| \leq \frac{0.2061}{6}\frac{3}{8} \approx 0.01288.$$

Although not asked, if you computed the actual value of $|f(1/2) - p_2(1/2)|$, you'd get $|f(1/2) - p_2(1/2)| \approx |1.2840 - 1.2925| = 0.0084$.

Q2. (30 marks)

(a) Give a formula for the piecewise linear interpolant, $l(x)$, that interpolates f, at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

Answer: [10 MARKS]

$$l(x) = \begin{cases} f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0} & x_0 \leq x \leq x_1 \\ f(x_1)\frac{x - x_2}{x_1 - x_2} + f(x_2)\frac{x - x_1}{x_2 - x_1} & x_1 < x \leq x_2 \\ 0 & \text{otherwise} \end{cases}$$

Using that $f(x_0) = e^{-1/2}$, $f(x_1) = 1$ and $f(x_2) = e^{1/2}$, we get that this simplifies as

$$l(x) = \begin{cases} e^{-1/2}(-x) + (x + 1) & -1 \leq x \leq 0 \\ (1 - x) + e^{1/2}x & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Evaluate $l(1/2)$. ANS [5 MARKS] $l(1/2) = (1 - 1/2) + e^{1/2}(1/2) \approx 1.32436.$

(c) Use (2) to give an upper bound for $\|f(x) - l(x)\|_\infty$.

Answer: [10 MARKS] (2) is

$$\|f - l\|_\infty \leq \frac{h^2}{8}\|f''\|_\infty.$$

In this instance $h = 1$. Also, $f''(x) = e^{x/2}/4$, so $\|f''\|_\infty = f''(1) = e^{1/2}/4 =\approx 0.41218$. So

$$\|f - l\|_\infty \leq \frac{1}{8}0.41218 = 0.05152.$$

(d) What value of N would you have to choose so that $\|f - l\|_\infty \leq 10^{-6}$?

Answer: [5 MARKS] We need h such that $\frac{h^2}{8}\|f''\|_\infty \leq 10^{-6}$. That is $h \leq \sqrt{8 \times 10^{-6}/0.41218} = \sqrt{1.940898 \times 10{-5}} = 4.4056 \times 10^{-3}$. Also, $h = (x_2 - x_1)/N = 2/N$, so $N \geq 2/h = 453.971$. **Take** $N = 454$.

Q3. ($\texttt{18 marks}$) Suppose that S is the **PCHIP** interpolant to the function f at the $N+1$ equally spaced points $\{x_0 = -1 < x_1 < \cdots < x_N = 1\}$. What value of N should one take to ensure that $\|f - S\|_\infty$ is no more than 10^{-6}?

Answer: We'll use (3):

$$\|f - S\|_\infty := \max_{a \le x \le b} |f(x) - S(x)| \le \frac{h^4}{384} \max_{a \le x \le b} |f^{(iv)}(x)|.$$

We wish to choose h so that $\frac{h^4}{384} M_4 \le 10^{-6}$, where $M_4 := \max_{-1 \le x \le 1} |f^{(4)}(x)|$. That is, we need h to satisfy $h^4 \le \frac{384}{M_4} 10^{-6}$. To compute M_4, calculate the 4th derivative of f, finding that $f^{(iv)}(x) = e^{x/2}/16$. Since this is a positive, increasing function, we get that $M_4 = f^{(iv)}(1) = e^{1/2}/16 \approx 0.1030$. So now we know that we need h to satisfy $h \le (\frac{384}{0.1030} 10^{-6})^{1/4} \approx (3.7265 \times 10^{-3})^{1/4} \approx 0.247$. Then, since $N = 2/h$, we get than N must be at least 8.095. However, since N is an integer, we should take $N \ge 9$.

Q4. ($\texttt{2 marks}$) Could there ever be a situation where, if we use the same values of f and N in (2) and (3), the error bound for the linear spline interpolant could be *less* than that PCHIP interpolant? If so, suggest an example.

Answer: Yes. We could, say, choose an f such that $\|f^{(iv)}(x)\|_\infty$ is much larger than $\|f''\|_\infty$. Suppose we set $f(x) = e^{\alpha x}$, and keep $x_0 = -1$, and $x_N = 1$. Then $\|f^{(iv)}(x)\|_\infty = \alpha^2 \|f''\|_\infty$. Then, given an h we can choose α such that $\dfrac{h^2}{8} \le \dfrac{h^4}{384}\alpha^2$. That is, take $\alpha > \dfrac{384}{8h^2}$.