Annotated slides

2425-MA140 Engineering Calculus

Week 06, Lecture 3
Limits at infinity

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Assignments, etc

- **Assignment 4** is open, due Tuesday 28 Oct at 17:00.
- **Assignment 5** just opened, due Monday, 3 Nov at 17:00.

In today's class...

See also: 4.6 (Limits at Infinity and Asymptotes) in **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Limits at infinity

We now know how to use the first and second derivatives of a function to describe the shape of a graph on a domain (a, b).

However, sometimes we'll wish to graph a function, f, defined on an unbounded domain. So we'll need to know f behaves as $x \to -\infty$ and/or $x \to \infty$.

To that end, we'll learn about **limits at infinity**, and how these limits affect the graph of a function.

"unbounded domain" would be something like "all $x > 0$" or (equivalently) $x \in (0, \infty)$.

Limits at infinity

Recall...

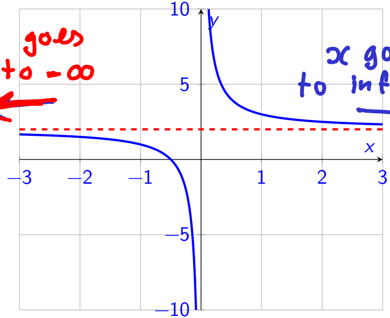
We learned in Week 2, that if we write $\lim_{x\to a} f(x) = L$, then the value of $f(x)$ approaches L as x approaches a (regardless of what actually happens at a).
Now we consider what happens as $x \to \pm\infty$.

Limits at infinity

Here we show the graph of $f(x) = 2 + \dfrac{1}{x}$. Observe that

- As $x \to \infty$, $f(x) \to 2$. This is because, as x gets very large, so $1/x$ gets very small.
- Similarly, as $x \to -\infty$ we see that, again $f(x) \to 2$.

x goes to $-\infty$



x goes to infinity

$\lim\limits_{x\to\infty} f(x) = 2$

So we write

$\lim\limits_{x\to-\infty} f(x) = 2$

$$\lim_{x\to-\infty} f(x) = 2, \qquad \text{and} \qquad \boxed{\lim_{x\to\infty} f(x) = 2.}$$

Limit at infinity: Informal definition

We write $\boxed{\lim_{x\to\infty} f(x) = L}$ if the value of $f(x)$ can be made as close to L as we like, by taking x as large as needed. (And $f(x)$ is closer still to L for any larger x).

We write $\lim_{x\to-\infty} f(x) = L$ if, for $x < 0$, the value of $f(x)$ can be made as close to L as we like, by taking $\boxed{-x \text{ as large}}$ as needed. (And $f(x)$ is closer still to L for any larger $-x$).

Horizontal Asymptote

If $\lim_{x\to\infty} f(x) = L$, or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

With the example from earlier,

$$f(x) = 2 + \frac{1}{x}$$

we had $y = 2$ as a Horizontal Asymptote.

Computing limits at infinity

The key facts to know are:

- $\boxed{\lim_{x\to\infty} \frac{1}{x} = 0;}$
- The properties of limits from Week 2, Lecture 2 still hold. In particular (assuming the limits exist)
  - $\lim_{x\to\infty} \big(f(x)+g(x)\big) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$.
  - $\lim_{x\to\infty} \big(f(x)g(x)\big) = \big(\lim_{x\to\infty} f(x)\big)\big(\lim_{x\to\infty} g(x)\big)$.
  - The Squeeze Theorem

If x is really really large, then $1/x$ is really really small.

Similarly $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \left(\frac{1}{x}\right) \lim_{x\to 0} \left(\frac{1}{x}\right) = (0)(0) = 0.$

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Eg $\lim_{x\to\infty}\left(x+\frac{1}{x}\right) = \underbrace{\lim_{x\to\infty}(x) + \lim_{x\to\infty}\left(\frac{1}{x}\right)}_{\text{"sum rule"}}$

$= \infty + 0 = \infty.$

Example: Find the limit of $f(x) = \dfrac{\sin(x)}{x}$ as $x \to \infty$.

First note: $\lim\limits_{x \to \infty} \sin(x)$ does not exist,

since $\sin(x)$ is always oscillating between -1 & $+1$.

That is $-1 \le \sin(x) \le 1$

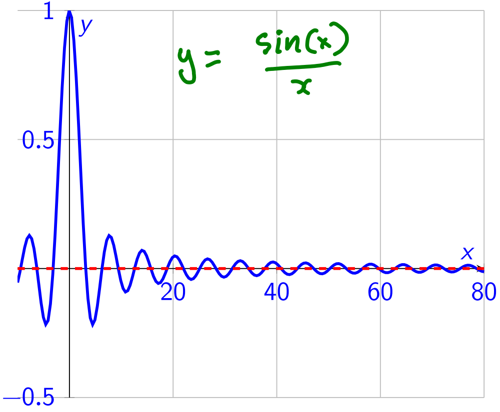
So, let $g(x) = \dfrac{-1}{x}$ and $h(x) = \dfrac{1}{x}$.

So $g(x) \le f(x) \le h(x)$.

Also $\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to \infty} -\dfrac{1}{x} = 0$

& $\lim\limits_{x \to \infty} h(x) = 0$. So $\lim\limits_{x \to \infty} f(x) = 0$, by Squeeze thm.

Computing limits at infinity

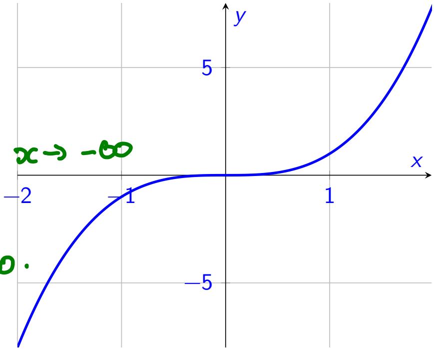


This has a horizontal asymptote at at $y=0$, but $f(x)$ crosses $y=0$ an infinite number of times.

Computing limits at infinity

Of course, many functions do not have a finite limit at infinity. For example,

$$\lim_{x\to-\infty} x^3 = -\infty, \qquad \text{and} \qquad \lim_{x\to-\infty} x^3 = \infty.$$



↑ y → ∞

→ ∞

y → −∞.

When computing the limit at infinity of a **rational function**,

- Divide the numerator and denominator by the highest power of x in the denominator
- Apply the limit laws.

Example: Evaluate $\lim_{x\to\infty} \frac{3x^2-1}{2x^2+4}$.

f(x) is a rational function if it can be written as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$, $q(x)$ are polynomials.

Recall: the degree of a poly is it highest power of x. Eg $\deg(5x^3+4x^2) = 3$

When computing the limit at infinity of a **rational function**,

- Divide the numerator and denominator by the highest power of x in the denominator
- Apply the limit laws.

~~**Example:** Evaluate~~ $\cancel{\lim_{x\to\infty} \frac{3x^2-1}{2x^2+4}}$.

Let $f(x) = \dfrac{p(x)}{q(x)}$:

- If $\deg(p) > \deg(q)$ then $\lim_{x\to\infty} f(x) = \pm\infty$
- If $\deg(p) = \deg(q)$ then $\lim_{x\to\infty} f(x)$ is finite.
- If $\deg(p) < \deg(q)$ then $\lim_{x\to\infty} f(x) = 0$

When computing the limit at infinity of a **rational function**,

- Divide the numerator and denominator by the highest power of x in the denominator
- Apply the limit laws.

Example: Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2-1}{2x^2+4}$.

$$\lim_{x\to\infty} \frac{3x^2-1}{2x^2+4} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{4}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 - 1/x^2}{2 + 4/x^2} = \frac{\lim_{x\to\infty} (3 - 1/x^2)}{\lim_{x\to\infty} (2 + 4/x^2)} = \frac{3}{2}.$$

Examples

Evaluate the following limits

(i) $\displaystyle\lim_{x\to\infty} \frac{x+123}{x^2+1}$ (ii) $\displaystyle\lim_{x\to\infty} \frac{x^2-9}{x+3}$.

(i) $$\lim_{x\to\infty} \frac{x+123}{x^2+1} = \lim_{x\to\infty} \frac{x/x^2 + \frac{123}{x^2}}{x^2/x^2 + 1/x^2}$$

$$= \frac{\lim_{x\to\infty} \left(1/x + 123/x^2\right)}{\lim_{x\to\infty} \left(1 + 1/x^2\right)} = \frac{0}{1} = 0.$$

Examples

Evaluate the following limits

(i) $\cancel{\lim_{x\to\infty} \frac{x+123}{x^2+1}}$ (ii) $\lim_{x\to\infty} \frac{x^2-9}{x+3}$.

(ii) $\lim_{x\to\infty} \frac{x^2-9}{x+3}$

"dividing by highest power of x in denom."

$$= \lim_{x\to\infty} \frac{\left(\frac{x^2}{x} - \frac{9}{x}\right)}{\left(1 + \frac{3}{x}\right)} = \lim_{x\to\infty} \frac{\left(x - \frac{9}{x}\right)}{1 + \frac{3}{x}}$$

$$= \frac{\infty}{1} = \infty$$

Curve Sketching (over large domains)

In order to roughly **sketch the graph** of a function, f, over a large domain, the approach is similar to yesterday, but we also calculate the limits at infinity:

ie $f'(x) = 0$
or $f'(x)$ does not exist.

1. Compute $f'(x)$ and $f''(x)$.
2. Find the critical points. Determine if they correspond to maxmima, minima or neither (using the 2nd Derivative test as needed).
3. Find points of inflection. eg $f''(x) = 0$ + tests.
4. Evaluate the limits at $\pm\infty$, and add any horizontal asymptotes.
5. Compute some specific points, e.g. at the critical and inflection points, y-intercept and, if possible, and x-intercept.
6. Plot the points from the previous step, and fill in the graph using information on the local max/min and inflection points.

Example

Sketch the graph of

$$f(x) = \frac{(1+x)^2}{1+x^2}$$

Note: $f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$ and $f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}$.

1. Find & classify critical points. that is solve $2\frac{(1-x^2)}{(1+x^2)^2} = 0$. since $(1+x^2)^2 > 0$ for all x, just solve $1-x^2=0$

So critical points at $x=1$ & $x=-1$.

Check $f''(-1) = \frac{4(-1)(1-3)}{8} > 0$ so min at $x=-1$.

Curve Sketching (over large domains)

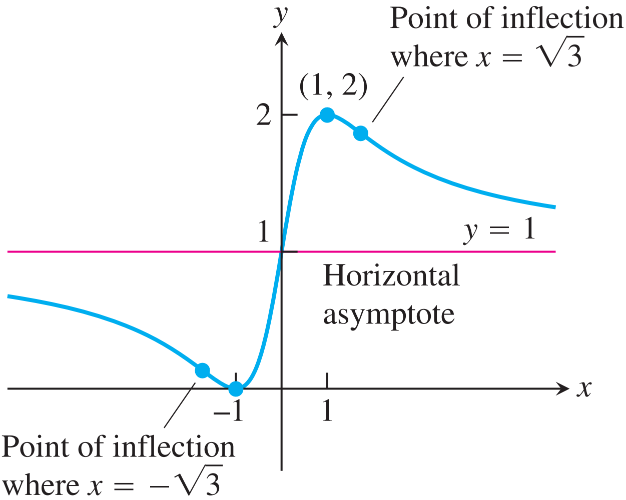
Niall will add the details later.

If you are reading this, he has probably forgotten, and you should probably send him a gentle reminder (and a math joke).

FINISHED HERE THURSDAY

Curve Sketching (over large domains)

Curve Sketching (over large domains)



Now that we know how to find maxima and minima of functions, we can solve **optimization** problems. Here is a classic example:

Example

What is the largest rectangular field we could enclose with 40m of fencing?

We can "solve" this problem by checking a few cases.

Now use calculus:

Here is a more general approach:

1. Write down a function, f, describing the quantity to be minimized/maximized.
2. If f is in more than one variable, use other information, linking the variables, to reduce it to a function of one variable.
3. Differentiate f, and find its critical points. Determine which correspond to maxima and minima.

Example:

A stretch of land is bordered by a (remarkably straight) river. What is the largest field we could enclose with 40m of fencing, if we don't have fence along by the river?

Sometimes, we are given the formula of the quantity to be optimised explicitly.

Example

Suppose that if a particular vehicle is been driven at a speed of x km/hr then its fuel usage, measured, in L/100km is given by

$$y = \frac{x^2}{1000} - \frac{1}{10}x + 10,$$

1. What speed should you drive at in order to minimise your fuel usage?
2. What is the fuel usage (in $L/100$km) at that speed?

Exercises

Exer 6.3.1 (Example 4.6.9 from the textbook)

Sketch the graph of $f(x) = \dfrac{x^2}{1 - x^2}$.