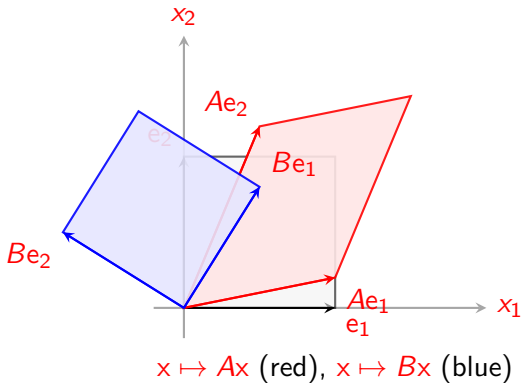


MA385 Part 4: Linear Algebra 2

4.2: Matrix Norms

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1. Outline Section 4.2

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For more, see Section 2.7 of Suli and Mayers:

<https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG>

Vector norms are related to the magnitude of the entries of the vector.

Now we want to generalise to the concept of a **matrix norm**. In a sense, we can just consider the magnitude of the matrix's entries.

However, if we think of a matrix as a linear transformation, or simply as a function that maps (via matrix multiplication) from \mathbb{R}^n to \mathbb{R}^n , we should think about how much it changes a vector.

Definition 4.2.1

Given any (vector) norm $\|\cdot\|$ on \mathbb{R}^n , there is a **subordinate matrix norm** on $\mathbb{R}^{n \times n}$ defined by

$$\|A\| = \max_{v \in \mathbb{R}_*^n} \frac{\|Av\|}{\|v\|}, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ and $\mathbb{R}_*^n = \mathbb{R}^n / \{0\}$.

We define a matrix norm like this because we think of A as an *operator* on \mathbb{R}^n : if $v \in \mathbb{R}^n$ then $Av \in \mathbb{R}^n$. So the norm of A gives us information on how much the matrix can change the size of a vector.

3. Computing Matrix Norms

It is not obvious from the above definition how to calculate the norm of a given matrix. We'll see that

- ▶ The ∞ -norm of a matrix is also the largest absolute-value row sum.
- ▶ The 1-norm of a matrix is also the largest absolute-value column sum.
- ▶ The 2-norm of the matrix A is the square root of the largest eigenvalue of $A^T A$.

4. The max-norm on $\mathbb{R}^{n \times n}$

Theorem 4.2.1

For any $A \in \mathbb{R}^{n \times n}$ the subordinate matrix norm associated with $\|\cdot\|_\infty$ on \mathbb{R}^n can be computed by

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|.$$

4. The max-norm on $\mathbb{R}^{n \times n}$

A similar result holds for the 1-norm, the proof of which is left as an exercise.

Theorem 4.2.2

For any $A \in \mathbb{R}^{n \times n}$ the subordinate matrix norm associated with $\|\cdot\|_\infty$ on \mathbb{R}^n can be computed by

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^n |a_{ij}|. \quad (2)$$

Computing the 2-norm of a matrix is a little harder than computing the 1- or ∞ -norms. However, later we'll need estimates not just for $\|A\|$, but also $\|A^{-1}\|$. And, unlike the 1- and ∞ -norms, we can estimate $\|A^{-1}\|_2$ without explicitly forming A^{-1} .

We begin by recalling some important facts about eigenvalues and eigenvectors.

Definition 4.2.2

Let $A \in \mathbb{R}^{n \times n}$. We call $\lambda \in \mathbb{C}$ an *eigenvalue* of A if there is a non-zero vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x.$$

We call any such x an *eigenvector* of A associated with λ .

- (i) If A is a real symmetric matrix (i.e., $A = A^T$), its eigenvalues and eigenvectors are all real-valued.
- (ii) If λ is an eigenvalue of A , the $1/\lambda$ is an eigenvalue of A^{-1} .
- (iii) If x is an eigenvector associated with the eigenvalue λ then so too is ηx for any non-zero scalar η .
- (iv) An eigenvector may be *normalised* as $\|x\|_2^2 = x^T x = 1$.

- (v) There are n eigenvectors $\lambda_1, \lambda_2, \dots, \lambda_n$ associated with the real symmetric matrix A . Let $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ be the associated normalised eigenvectors. The eigenvectors are linearly independent and so form a basis for \mathbb{R}^n . That is, any vector $v \in \mathbb{R}^n$ can be written as a linear combination:

$$v = \sum_{i=1}^n \alpha_i x^{(i)}.$$

- (vi) Furthermore, these eigenvectors are *orthogonal* and *orthonormal*:

$$(x^{(i)})^T x^{(j)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Here is a useful consequence of (v) and (vi), which we will use repeatedly.

The *singular values* of a matrix A are the square roots of the eigenvalues of $A^T A$. They play a very important role in matrix analysis, applied linear algebra, and statistics (principal component analysis).

Our interest here is in their relationship to $\|A\|_2$.

But first we'll prove a theorem about certain matrices (so called, "normal matrices").

Theorem 4.2.3

For any matrix $A \in \mathbb{R}^{n \times n}$, the eigenvalues of $A^T A$ are real and non-negative.

Part of the above proof involved showing that, if $(A^T A)x = \lambda x$, then

$$\sqrt{\lambda} = \frac{\|Ax\|_2}{\|x\|_2}.$$

This at the very least tells us that

$$\|A\|_2 := \max_{x \in \mathbb{R}_*^n} \frac{\|Ax\|_2}{\|x\|_2} \geq \max_{i=1,\dots,n} \sqrt{\lambda_i}.$$

With a bit more work, we can show that if $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are the the eigenvalues of $B = A^T A$, then

$$\|A\|_2 = \sqrt{\lambda_n}.$$

Theorem 4.2.4

Let $A \in \mathbb{R}^{n \times n}$. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, be the eigenvalues of $B = A^T A$. Then

$$\|A\|_2 = \max_{i=1,\dots,n} \sqrt{\lambda_i} = \sqrt{\lambda_n},$$

Here is the main idea. For full details, see the text-book.

6. Exercises

Exercise 4.2.1

Show that, for *any* subordinate matrix norm on $\mathbb{R}^{n \times n}$, the norm of the identity matrix is 1.

Exercise 4.2.2

Prove that

$$\|A\|_1 = \max_{j=1, \dots, n} \sum_{i=1}^n |a_{i,j}|.$$

Hint: Suppose that $\sum_{i=1}^n |a_{ij}| \leq C$, for $j = 1, 2, \dots, n$. Show that for *any* vector $x \in \mathbb{R}^n$

$$\sum_{i=1}^n |(Ax)_i| \leq C \|x\|_1.$$

Now find a vector x such that $\sum_{i=1}^n |(Ax)_i| = C \|x\|_1$. Now deduce the result.