2425-MA140 Engineering Calculus

Week 05, Lecture 2
Logarithmic Functions; Higher-order derivatives

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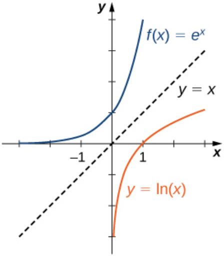


Assignments, etc

Assignment 3

- If you take a break while doing the assignment, click on Pause. You can then `Resume` it later.
- Make sure any completed question shows Answer saved
- When you've finished, click End Exam
- Keep a record of your attempt: click Print this results summary and save the PDF.
- Don't re-attempt the assignment, unless you don't mind losing your earlier grade.

There seem to be some issues with save results. We are working to understand that. If you have concerns, get in touch with me closer to the deadline. Include a copy of your summary results.

And now for something quite interesting...

See also: Sections 3.8 (Implicit Differentiation) and 3.9 (Derivatives of Exponential and Logarithmic Functions) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

We'll also recap some basic information about exponential and logarithmic functions from Section 1.5 of that text.

The Natural Exponential

Yesterday, we met Euler's Number: $e \approx 2.7182818284$.

The associated exponential function is one of the most important in mathematics:

The Natural Exponential Function

The Natural Exponential Function is $f(x) = e^x$. Its properties include that its tangent at $x = 0$ has slope 1.

Using only that $f'(0) = 1$, we deduced that

$$\boxed{\frac{d}{dx} e^x = e^x.}$$

That is e^x is the function that is its own derivative!!!

The Natural Exponential

Example

Compute the derivative of $f(x) = e^{\sin(x)}$

Logarithms

Suppose that $y = f(x)$ is an **exponential** function; that is: $y = b^x$ for some $b > 0$ (and excluding $x = 1$).

Its **inverse** is called a **logarithmic function**, denoted $\log_b$

$$\text{If } y = b^x \quad \text{then} \quad \log_b(y) = x.$$

Examples

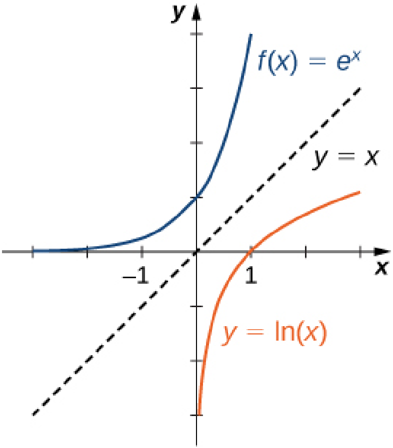
- $\log_2(8) = 3$
- $\log_{10}(100) = 2$
- $\log_e(e^x) = x$

Properties of Logarithms

If $a, b, c > 0$ and $b \neq 1$, then

- $\log_b(ac) = \log_b(a) + \log_b(c)$
- $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$
- $\log_b(a^r) = r\log_b(a)$

We denote $\boxed{\log_e(x)}$ as $\boxed{\ln(x)}$



$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Why?

Example:

Find the derivative of $f(x) = \ln(x^2 + 2x + 3)$.

To finish we introduce the idea of **logarithmic differentiation**, which helps us differentiate functions with x, or a function of x in the exponent, such as $y = (2x)^{\sin(x)}$ or $y = x^x$.

Strategy:

- Take ln of both sides
- Simplify, using properties of logarithms.
- Differentiate.
- Solve for $\frac{dy}{dx}$

Example [2019 exam, Q2(b)(iii)]

Differentiate $f(x) = x^x$.

Higher-order Derivatives

We learned last week that the derivative of $f(x)$, denoted $f'(x)$, is itself a function.

That implies that $f'(x)$ can itself be differentiated, which is called the **second derivative** of f. It is denoted as

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x) \quad \text{or} \quad f^{(2)}(x).$$

We can continue this process to get higher-order derivatives as long as the preceding derivative is again differentiable.

The first and second derivatives f' and f'' (if they exist) provide valuable information about the function and its graph, particularly concerning local or global maxima, local/global minima and points of inflection.

Higher-order Derivatives

Example

Find the **second** derivative of the functions

(i) $f_1(x) = 3x^2 + 2x + 1$

(ii) $f_2(x) = e^x$

(iii) $f_3(x) = \ln x$

(iv) $f_4(x) = \sin(x)$

Exercises

Exercise 5.2.1 [2019 exam, Q2(b)(i)]

Differentiate $f(x) = e^{\sin(x)} \cos x$.

Exercise 5.2.2 [2023 exam, Q2(a)(i)]

Differentiate $f(x) = xe^{\sin(x)}$.

Exercise 5.2.3

Let $f(x) = x^2 e^x$. Find $f'(x)$, $f''(x)$ and $f'''(x)$.