MA385 Part 4: Linear Algebra 2
## 4.3: Condition Numbers
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For more, see Section 2.7 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.
action?docID=221072&ppg=51&c=UERG

# 2. Motivation

Numerical solutions to some linear systems are adversely affected by round-off errors.

This phenomenon is due to the *matrices* in the linear systems. Those matrices for which the issue is particularly prevalent are referred to as being **ill-conditioned**.

For any matrix, we can assign a numerical *score* that gives an indication of whether it is ill-conditioned. That score, called the **condition number**, is defined in terms of matrix norms, and is the subject of these section.

# 3. Consistency of matrix norms

Suppose we have a vector norm, $\| \cdot \|$ and associated subordinate matrix norm. It is not hard to see that

$$\|A\mathbf{u}\| \le \|A\|\|\mathbf{u}\| \quad \text{for } \textit{any } \mathbf{u} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}.$$

Here is why:

# 3. Consistency of matrix norms

There is an analogous statement for the product of two matrices:

### Definition 4.3.1 (Consistent matrix norm)

A matrix norm $\|\cdot\|$ is **consistent** (or "*sub-multiplicative*") if

$$\|AB\| \leq \|A\|\|B\|, \qquad \text{for all } A, B \in \mathbb{R}^{n \times n}.$$

### Theorem 4.3.1

Any subordinate matrix norm is consistent.

The proof is left to an exercise. That exercises also demonstrates that there are matrix norms which are *not* consistent.

# 4. Computer representation of numbers

**[Please read this slide in your own time!]**

Modern computers don't store numbers in decimal (base 10), but in binary (base 2) "floating point numbers" of the form :

$$x = \pm a \times 2^{b-M}.$$

Most use *double precision*, where 8 bytes (64 bits or *binary digits*) are used to store

- ▶ the sign (1 bit),
- ▶ $a$, called the "significand" or "mantissa" (52 bits)
- ▶ and the exponent, $b - 1023$ (11 bits)

Note that $a$ has roughly 16 decimal digits.

(Some older computer systems sometimes use *single precision* where $a$ has 23 bits — giving 8 decimal digits — and $b$ has 7; so too do many new GPU-based systems).

# 4. Computer representation of numbers

When we try to store a real number $x$ on a computer, we actually store the nearest floating-point number. That is, we end up storing $x + \delta x$, where $\delta x$ is the "round-off" error.

But the quantity we are mainly interested in is the **relative error:** $|\delta x|/|x|$.

Since this is not a course on computer architecture, we'll simplify a little and just take it that single and double precision systems lead to a relative error of $10^{-8}$ and $10^{-16}$ respectively.

# 5. Condition Numbers

(*See p68–70 of Süli and Mayers for a full development of the concept of a condition number*).

Suppose we use, say, $LU$-factorization and back-substitution on a computer to solve

$$A\mathbf{x} = \mathbf{b}.$$

Because of the "round-off error" we actually solve

$$A(\mathbf{x} + \delta\mathbf{x}) = (\mathbf{b} + \delta\mathbf{b}).$$

Our problem now is, for a given $A$, if we know the (relative) error in $\mathbf{b}$, can we find an upper-bound on the relative error in $\mathbf{x}$?

# 5. Condition Numbers

> **Definition 4.3.2**
>
> The *condition number* of a matrix, with respect to a particular matrix norm $\| \cdot \|_\star$ is
>
> $$\kappa_\star(A) = \|A\|_\star \|A^{-1}\|_\star.$$
>
> If $\kappa_\star(A) \gg 1$ then we say $A$ is *ill-conditioned*.

**Example:** Find the condition number $\kappa_\infty$ of

$$A = \begin{pmatrix} 10 & 12 \\ 0.08 & 0.1 \end{pmatrix}.$$

# 5. Condition Numbers

> ### Theorem 4.3.2
>
> Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular and that $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$ are non-zero vectors. If $A\mathbf{x} = \mathbf{b}$ and $A(\mathbf{x} + \delta\mathbf{x}) = (\mathbf{b} + \delta\mathbf{b})$ then
> $$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}.$$

**Example 4.3.1**

Suppose we are using a computer to solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 10 & 12 \\ 0.08 & 0.1 \end{pmatrix} \quad \text{and } \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

But, due to round-off error, right-hand side has a relative error (in the $\infty$-norm) of $10^{-6}$. Give a bound for the relative error in $\mathbf{x}$ in the $\infty$-norm.

For every matrix norm we get a different condition number.

### Example 4.3.2

Let $A$ be the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

What are $\kappa_1(A)$, and $\kappa_\infty(A)$?

First we compute $\|A\|_1$ and $\|A\|_\infty$.

For this very special example, it is easy to write down the inverse of $A$:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ -1 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

To compute $\kappa_1(A)$ and $\kappa_\infty(A)$, we need to know $A^{-1}$, which is usually not practical. However, for $\kappa_2$, we are able to *estimate* the condition number of $A$ without knowing $A^{-1}$.

Recall that $\|A\|_2 = \sqrt{\lambda_n}$ where $\lambda_n$ is the largest eigenvalue of $B = A^T A$.

We can also show that $\|A^{-1}\|_2 = \dfrac{1}{\sqrt{\lambda_1}}$ where $\lambda_1$ is the smallest eigenvalue of $B$

▶ $A^T A$ and $A A^T$ have the same eigenvalues (they are "*similar*"):

▶ For any non-singular matrix $X$, we have that $(X^T)^{-1} = (X^{-1})^T$.

▶ $(A^T A)^{-1} = (A A^T)^{-1}$

So

$$\kappa_2(A) = \left( \lambda_n / \lambda_1 \right)^{1/2}.$$

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Motivated by this, we'll finish MA385, by studying an easy way of estimating the eigenvalues of a matrix.

# 8. Exercises

## Exercise 4.3.1

(i) Prove that, if $\|\cdot\|$ is a subordinate matrix norm, then it is *consistent*, i.e., for any pair of $n \times n$ matrices, $A$ and $B$, we have $\|AB\| \leq \|A\|\|B\|$.

(ii) One might think it intuitive to define the "max" norm of a matrix as follows:
$$\|A\|_{\widetilde{\infty}} = \max_{i,j} |a_{ij}|.$$

Show that this is indeed a norm on $\mathbb{R}^{n \times n}$. Show that, however, it is not consistent.

**Exercise 4.3.2**

Let $A$ be the matrix

$$A = \begin{pmatrix} 0.1 & 0 & 0 \\ 10 & 0.1 & 10 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Compute $\kappa_\infty(A)$. Suppose we wish to solve the system of equations $A\mathbf{x} = \mathbf{b}$ on *single precision* computer system (i.e., the relative error in any stored number is approximately $10^{-8}$). Give an upper bound on the relative error in the computed solution $\mathbf{x}$.