MA378 Chapter 1: Polynomial Interpolation

Submit carefully written solutions to Exercises 1.4⋆, 2.3⋆, 2.5⋆, and 4.4⋆.

Deadline: 5pm, Friday 9 February.

Your solutions must be clearly written, and neatly presented. You can submit an electronic copy, through Canvas, or a hard copy (ideally at the lecture on the 9th). Make sure pages of the hard copy are stapled together. Marks will be given for quality and clarity of exposition. Collaboration is encouraged; policy will be discussed in class.

Exercise 1.1. Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

Exercise 1.2. Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

Exercise 1.3. (a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

Exercise 1.4 (⋆). For each of the following interpolation problems, determine (with explanation) if there is no solution, exactly one solution, or more than one solution. In all cases p_n denotes a polynomial of degree (at most) n. You are not required to determine p_n where it exists.

(a) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(b) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = -2$.

(c) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(d) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(e) Find $p_2(x)$ that interpolates (x_0, y_0) and (x_1, y_1) where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$.

Exercise 2.1. The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{0 \leqslant i < j \leqslant n} (x_j - x_i). \qquad (2.0.1)$$

Verify (2.0.1) for the 2×2 and 3×3 cases.

(Note that from Formula (2.0.1) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \ldots, x_n$ are all distinct.)

Exercise 2.2. Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute

$$\max_{0 \leqslant x \leqslant a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayers.

Exercise 2.3 (⋆). Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \qquad \text{for all } x.$$

Exercise 2.4. Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0, 3)$, $(1, 2)$ and $(2, 4)$.

Exercise 2.5 ($\star$)**.** Show that all the following represent the polynomial $T_3(x) = 4x^3 - 3x$ (often called the "Chebyshev Polynomial of Degree 3"),

(a) Horner form: $H_3(x) := \big((4x + 0)x - 3\big)x + 0$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3}\Big(\prod_{j=0,j\neq k}^{3}\frac{x - x_j}{x_k - x_j}\Big)(-1)^{k+1}$, where $x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$.

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and $T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

Exercise 3.1. Read Section 6.2 of An Introduction to Numerical Analysis (Süli and Mayers). Pay particular attention to the proof of Thm 6.2 at `https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=192`.

Exercise 3.2. Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0\leqslant x\leqslant x_2} |f(x) - p_2(x)| \leqslant \frac{1}{6}\frac{2}{3\sqrt{3}}h^3M_3 = \frac{1}{9\sqrt{3}}h^3M_3.$$

Hint: simplify the calculations by taking $t = x - x_1$*, writing* $(x - x_0)(x - x_1)(x - x_2)$ *in terms of* h *and* t.

Exercise 4.1. For *just* the case $n = 1$, state and prove an appropriate version of Theorem 4.2 (i.e., error in the Hermite interpolant). Use this to find a bound for

$$\max_{x_0\leqslant x\leqslant x_1} |f(x) - p_3(x)|$$

in terms of f and $h = x_1 - x_0$.

Exercise 4.2. Let $n = 2$ and $x_0 = -1$, $\mathbf{x_0 = 0}$ and $x_1 = 1$. Write out the formulae for H_i and K_i for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 4.3. Let L_0, L_1, $\ldots$, L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \ldots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

and

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \ldots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0.$$

(a) Show that $K_i(x_k) = 0$, for $k = 0, 1, \ldots n$, and $K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

(b) Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n}\big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$

Exercise 4.4 ($\star$)**.** Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$.

Exercise 4.5. Show that there is a unique solution to the Hermite Polynomial Interpolation Problem.

Exercise 4.6. Take $f(x) = x^3$ and $\{x_0, x_1, x_2\} = \{-1, 0, 1\}$.

(a) Write down the Lagrange form of p_2, the polynomial of degree two that interpolates f at x_0, x_1, and x_2. Simplify the expression for $p_2(x)$ as much as possible.

(b) Use Corollary 3.5 to give an upper bound for

$$\max_{-1\leqslant x\leqslant 1} |f(x) - p_2(x)|.$$

(c) Using calculus, give a sharper bound for $|f(x) - p_2(x)|$ on the interval $[-1, 1]$. That is, find the maxima/minima of the function $g(x) = f(x) - p_2(x)$ on $[-1, 1]$, and thus compute exactly

$$\max_{-1\leqslant x\leqslant 1} |f(x) - p_2(x)|.$$

(d) Suppose we have $\{x_0, x_1, x_2\} = \{-a, 0, a\}$ for some number a, which we can choose. What is the largest value of a that can be permitted if we require that

$$\max_{-a\leqslant x\leqslant a} |f(x) - p_2(x)| \leqslant 10^{-3}?$$

You may use the result in Exercise 3.1 (**without** proof).

(e) Write down the formula for the polynomial that is the Hermite interpolant to $f(x) = x^3$ at $x_0 = -1$ and $x_1 = 1$. (Hint: be lazy; you can do this without figuring out what $H_i(x)$ and $K_i(x)$ are).