Annotated slides

2526-MA140: Week 01, Lecture 3 (L03)

Polynomials and Partial Fractions

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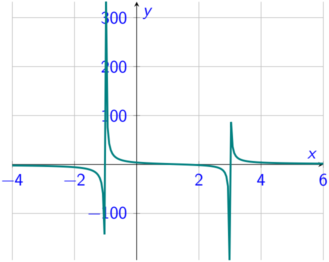
University of Galway

18 September, 2025



Outline

See also Sections 1.2 and 7.4(!) of https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/01%3A_Functions_and_Graphs

Slides are on canvas, and at https://www.niallmadden.ie/2526-MA140/

Tutorials start next week. Here is the schedule:

- Teams 1+2: Tuesday 15:00 ENG-**2003**
- Teams 3+4: Tuesday 15:00 ENG-**2034**
- Teams 11+12: Thursday 11:00 ENG-**2002**
- Teams 9+10: Thursday 11:00 ENG-**3035**
- Teams 5+6: Friday 13:00 Eng-**2002**
- Teams 7+8: Friday 13:00 Eng-**2035**

Note: I think the schedule is correct. If there are any changes, you'll be informed on Canvas.

Would you be interested to taking a tutorial through Irish? (Show of hands?) If so, please fill out this form:
https://forms.office.com/e/13kQHhwG8K

You don't have to complete a graded assignment next week. However, this is a "practice" one available. See https://universityofgalway.instructure.com/courses/46734/assignments/128373

During tutorials, the tutor will solve some similar questions. You can access the **tutorial sheet** at https://universityofgalway.instructure.com/courses/46734/files/2842617?module_item_id=925893. You can also access this through the Canvas page: Modules... Tutorial Sheets.

The Tutorial Sheet has questions that are nearly identical to your own version.

Also: bring to SUMS - opens Monday.

Polynomials (again)

Yesterday, we saw that...

Polynomials

Polynomials are functions of the form

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0, \quad x \in \mathbb{R},$$

where $a_0, a_1, \ldots, a_n$ are real numbers called the **coefficients** of the polynomial. The number n is called the **degree** of the polynomial.

Examples: $y = x^3 - 2x^2 - 1$ is a cubic $(n = 3)$

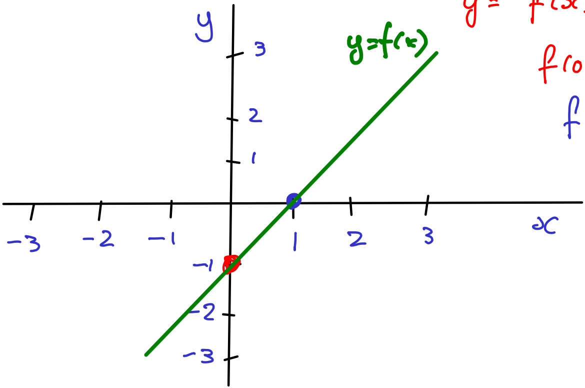
$y = \frac{1}{2} + 54.125\, x^4 - x$ is a poly of degree $n = 4$.

$y = e + \pi x$ is a poly of degree 1 (Linear).

But not $y = x^{-1} = 1/x$ or $y = x^{1/2} = \sqrt{x}$

Example: Linear Polynomial

A polynomial of degree $n = 1$ is called "linear". Its graph is a straight line. E.g. $y = x - 1$ is a **linear** polynomial.



$y = f(x) = x - 1$

$f(0) = -1$

$f(1) = 0$

Example: quadratic

$x^2 - 2x - 3$ is a **quadratic** polynomial: it has degree $n = 2$.

There are many occasions when we want to **factorise** such quadratics, meaning we write them as the product of a pair of linear polynomials.

For example, we can **factorise** $x^2 - 2x - 3$ as

$$x^2 - 2x + 3 = (x - 3)(x + 1)$$

$$(x-3)(x+1) = (x-3)x + (x-3)(1)$$

$$= x^2 - 3x + x - 3 = x^2 - 2x - 3.$$

Note: can see now that $x^2 - 2x - 3$ has zeros at $x = 3$, $x = -1$.

Example: quadratic

$x^2 - 2x - 3$ is a **quadratic** polynomial: it has degree $n = 2$.

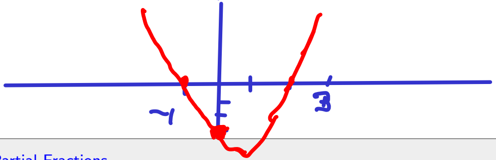
There are many occasions when we want to **factorise** such quadratics, meaning we write them as the product of a pair of linear polynomials.

For example, we can **factorise** $x^2 - 2x - 3$ as

$$x^2 - 4x + 3 = (x - 3)(x + 1)$$

To sketch this: $f(x) = x^2 - 2x - 3$.

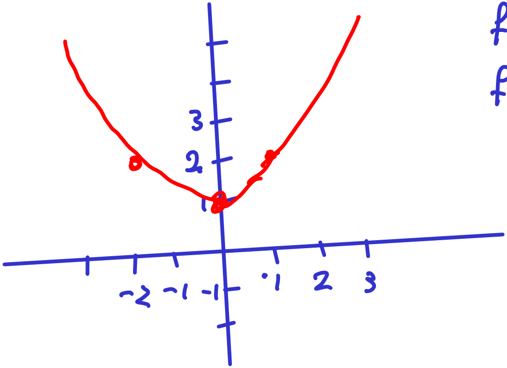
Use that $f(0) = -3$, $f(3) = 0$ $f(-1) = 0$.



It is important to note that not all quadratic polynomials can be factorised as two linear polynomials. Such quadratics are called **irreducible**.

For example, $x^2 + 1$ is irreducible.

$f(x) = x^2 + 1$



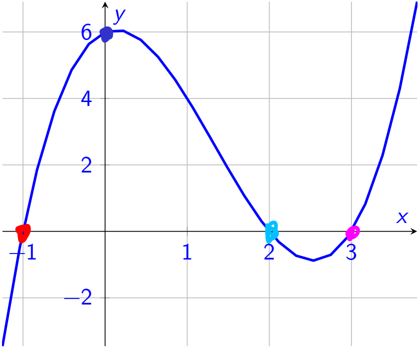
$f(0) = 1$

$f(1) = 2$

$f(-1) = 2$.

Example

$y = x^3 - 4x^2 + x + 6$ is a **cubic** function with degree $n = 3$.



Notes

$f(x) = x^3 - 4x^2 + x + 6$

$f(0) = -6$

$f(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$= -1 - 4 - 1 + 6 = 0$ ✓

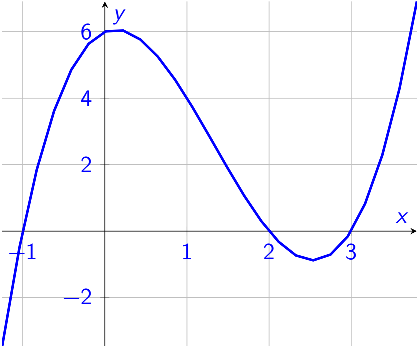
$f(2) = 0$ $f(3) = 0$.

Check!

So $f(x) = (x+1)(x-2)(x-3)$

Example

$y = x^3 - 4x^2 + x + 6$ is a **cubic** function with degree $n = 3$.



Note: every cubic has at least one root, (and so at least one linear factor).

Fact

A polynomial function of ~~grade~~ degree n has **up to** $n-1$ turning points ("bends").

Examples:

A liner poly $(n=1)$ has no bends.
A quadratic $(n=2)$ has 1 bend.
A cubic $(n=3)$ can have up to 2
(as in the previous example).
Locating these "bends" is a major topic in Calculus.

When sketching the graph of a function, we first find the **intercepts**:

- The y-**intercept** is where the graph of the function cuts the y-axis: found by letting $x = 0$.
- The x-**intercepts** are where the function's graph cuts the x-axis. These points are also called the **roots** (or **zeros**). To find them, set y equal to zero and solve for x.

We did this in previous examples, but will do it a little more methodically . . .

Example

Sketch the graph of $y = -x^3 + x^2 + 2x$ $f(x) = -x^3 + x^2 + 2x$

1. Find y-intercept : $f(0) = 0$ ✓

2. To find the x-intercepts.

$$y = x(-x^2 + x + 2).$$

and find the zeros of $-x^2 + x + 2$.

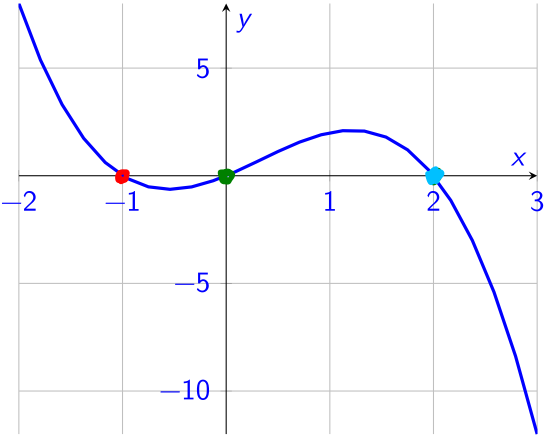
check!

Factorizing we get $-x^2 + x + 2 = (-x+2)(x+1)$.

So $x = 2$ & $x = -1$ are also x-intercepts (= "zeros" or "roots"), Actual plot of $y = \boxed{-x^3} + x^2 + 2x$

Also check that if $x < -1$ then $y > 0$

& $x > 2$ $y < 0$



Rational Functions

Rational Functions have the general form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

$$\frac{x+1}{x^2-1}$$

- If degree of $p(x)$ < degree of $q(x)$,
  $f(x)$ is called a **strictly proper rational function**.

$$\frac{x^2+1}{x^2-1}$$

- If degree of $p(x)$ = degree of $q(x)$,
  $f(x)$ is called a **proper rational function**.
- If degree of $p(x)$ > degree of $q(x)$,
  $f(x)$ is called an **improper rational function**.

$$\frac{x^3+x^2+1}{x^2-1}.$$

Rational Functions

An improper or proper rational function can always be expressed as a polynomial plus a strictly proper rational function, for example by algebraic division.

Example

$$\frac{4x^3+4x^2+4}{x^2-3} = 4x+4+\frac{12x+16}{x^2-3}$$

Check!

Try $4x^3+4x^2+4 = (x^2-3)(4x+4)+12x+16$

For the previous example, we can work this out ourselves using **Long Division** to divide numerator by denominator:

$$
\begin{array}{r|l}
 & 4x + 4 \\
\hline
x^2-3 & 4x^3 + 4x^2 + 4 \\
 & -(4x^3 - 12x) \\
\hline
 & 4x^2 + 12x + 4. \\
 & -(4x^2 - 12) \\
\hline
 & 12x + 16. \quad \text{remainder.}
\end{array}
$$

So $4x^3 + 4x^2 + 4 = (x^2-3)(4x+4) + 12x + 16.$

Example 2.30 from text book

Use long division to show that

$$\frac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3} = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

Finished here on
Thurdsday

Partial Fractions

A (proper) rational function can often be written as a sum of simpler ones: **partial fractions.**

For example

$$\frac{8x - 12}{x^2 - 2x - 3}$$

can be written as

$$\frac{3}{x-3} + \frac{5}{x+1}$$

Check:

Partial Fractions

Note: Any polynomial (with real coefficients) can be factorised fully into the product of

- linear
- and irreducible quadratic factors.

We get different combinations of factors in the denominator. Let's look at **four cases**, and how to find the partial fractions in each case.

Partial Fractions

The four cases

1. Linear factors to the power of 1 in the denominator.
2. Linear factors to the power greater than 1 in the denominator, (i.e repeated linear factors).
3. Irreducible quadratic factors.
4. Irreducible quadratic factors to power greater than 1.

Partial Fractions

(1) Linear factors to the power of 1 in the denominator.

Example

$$\frac{3x}{(x-1)(x+2)}$$

Partial Fractions

We have **two methods** to find A and B.

Method 1: Comparing coefficients

Partial Fractions

Method 2: Substituting specific values for x.

Partial Fractions

Example

Write $\dfrac{8x-12}{x^2-2x-3}$ as sum of partial fractions.

Exercises

Exercise 1.3.1

Sketch the graphs of

(i) $y = 5x^2 - 7$

(ii) $y = x^2 - 4x + 3$

(iii) $y = x^3 - 6x^2 - 11x - 6$

Exer 1.3.2

Find the constants A, B and C, so that

$$\frac{2x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$