4.0 Annotated slides from 3rd and 8th of March

MA378 Chapter 3: Numerical Integration

§3.4 Gaussian Quadrature

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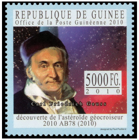

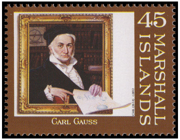

Johann Carl Friedrich Gauß, born 1777 in Braunschweig, died 1855 in Göttingen

4.1 Gaussian Quadrature

This section is perhaps the most mathematically rich in the course. I'd encourage you to read further: Chapter 10 of Süli and Mayers is devoted to this. However, much of the basic theory is developed in Section 9.4 on Orthogonal Polynomials. See also Lectures 22 and 23 of Stewart's "Afternotes goes to Grad School".

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To date we have found some numerical schemes that approximate $\int_a^b f(x)dx$ as the weighted average of values of f at $n+1$ equally spaced points. These methods have precision n (or $n+1$ in special cases).

With Gaussian Quadrature we choose both the quadrature weights and points in such as way as to maximize the precision of the method.

4.1 Gaussian Quadrature

There are three equivalent approaches to finding these points and weights that maximize the precision.

(i) **Undetermined Coefficients**: The obvious way for, say, $n = 2$. But, unlike Newton-Cotes, we have to solve a system of *nonlinear* equations. For larger n this can become difficult.

(ii) Base the method on integrating the **Hermite Interpolant** of the integrand, f, and choose the points so that the coefficients of $f'(x_i)$ are zero. This approach is the easiest to analyse, but less useful for construction.

(iii) **Finding the zeros of the members of a sequence of orthogonal monic polynomials**. This is the approach we will emphasise most, as it gives us an easy way of proving the precision of the methods.

 $\int_{-1}^{1} x^2 = \frac{1}{3}x^3 \Big|_{-1}^{1} \Rightarrow \frac{2}{3}$

Example 4.1

Find a two point rule

$$\int_{-1}^{1} f(x)dx \approx G_1(f) := w_0 f(x_0) + w_1 f(x_1),$$

that is exact for all polynomials of degree 3 or less.

This leads a the non-linear system which we must solve:

$\int_{-1}^{1} 1\,dx = G_1(1) \Rightarrow \qquad \omega_0 + \omega_1 = 2$

$\int_{-1}^{1} x\,dx = G_1(x) \Rightarrow \qquad \omega_0 x_0 + \omega_1 x_1 = 0$

$\int_{-1}^{1} x^2 dx = G_1(x^2) \Rightarrow \qquad \omega_0 x_0^2 + \omega_1 x_1^2 = 2/3$

$\int_{-1}^{1} x^3 dx = G_1(x^3) \Rightarrow \qquad \omega_0 x_0^3 + \omega_1 x_1^3 = 0$

4.2 Undetermined Coefficients [Finished here 3/3/23

(i) $w_0 + w_1 = 2$

(ii) $w_0 x_0 + w_1 x_1 = 0$

(iii) $w_0 x_0^2 + w_1 x_1^2 = 2/3$

(iv) $w_0 x_0^3 + w_1 x_1^3 = 0$

(ii) $\Rightarrow$ $w_0 x_0 = -w_1 x_1$

Then (iv) $\Rightarrow$ $w_0 x_0^3 = -w_1 x_1^3 \Rightarrow w_0 x_0^3 = w_0 x_0 x_1^2$

So, if $x_0 \neq 0$, $x_0^2 = x_1^2$. Since $x_0 \neq x_1$, we get $x_0 = -x_1$

Then (iii) $\Rightarrow$ $w_0 x_0^2 + w_1 x_0^2 = 2/3$

And (i) $\Rightarrow$ $2 x_0^2 = 2/3 \Rightarrow x_0 = -\frac{1}{\sqrt{3}}$, $x_1 = \frac{1}{\sqrt{3}}$

That is

$$\int_{-1}^{1} f(x)dx \approx G_1(f) := f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right). \tag{1}$$

4.2 Undetermined Coefficients

Example 4.2

Let $f(x) = \exp(-x)$.
If we estimate $\int_{-1}^{1} f(x)dx$ using each of

(a) **Trapezium Rule**, we get error of 0.735

(b) **Simpson's Rule**: error is 0.01165

(c) the 2-point **Gaussian Rule**: error is 0.00771.

Using the **composite Trapezium rule**, we find we would have to take $N = 11$ to obtain an estimate that is more accurate than the two-point Gaussian Rule.

4.2 Undetermined Coefficients

Example 4.3

If you use the $G_1(\cdot)$ rule to estimate $\int_0^{\pi/4} \cos(x)dx$ you'll get

$$G_1(\cos) = 0.07070432596.$$

Computing the exact error, we find that

$$\Big| \int_0^{\pi/4} \cos(x)dx - G_1(\cos)\Big| = \Big|\frac{1}{\sqrt{2}} - 0.07070432596\Big| \approx 6.35 \times 10^{-5}.$$

Compare with results for the same problem when

- The Trapezium rule is used.
- Simpson's rule is used.

Example 4.4 (3-point Gauss-Lobatto method)

The **Gauss-Lobatto method** is a variation on Gaussian quadrature.
Rather than allowing all of the quadrature points to vary in order to maximize the precision of the method, we fix some of them — usually the end points.
Use undetermined coefficients to derive the 3-point rule:

$$\int_{-1}^{1} f(x)dx \approx w_0 f(-1) + w_1 f(x_1) + w_2 f(1).$$

There are 4 parameters to determine, so we can expect this has precision 3.
So, choose w_0, w_1, w_2, x_1 so it is exact for
$f(x) = 1 \quad f(x) = x \quad f(x) = x^2 \quad f(x) = x^3$

4.2 Undetermined Coefficients

We get 4 equations:

(i) $w_0 + w_1 + w_2 = 2$

(ii) $-w_0 + x_1 w_1 + w_2 = 0$

(iii) $w_0 + x_1^2 w_1 + w_2 = 2/3$

(iv) $-w_0 + x_1^3 w_1 + w_2 = 0.$

(iv) − (ii) $\Rightarrow$ $(x_1^3 - x_1) w_1 = 0$. Since $w_1 \neq 0$ we have $x_1(1 - x_1^2) = 0$. This has 3 solution: $x_1 = -1$, $x_1 = 0$, & $x_1 = 1$. But we can't have $x_1 = x_0$ or $x_1 = x_2$. So the only possibility is $x_1 = 0$.

We can now find w_0, w_1, w_2 giving the method

$$\frac{1}{3}\left(f(-1) + 4f(0) + f(1) \right)$$

Simpson's Rule again !!

4.2 Undetermined Coefficients

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