MA313 : Linear Algebra 1 ("Linear Algebra for Data Science")

Week 1: Introduction to MA313 and to Vector Spaces

Dr Niall Madden

6 and 9 September, 2022

Image taken from the Burren College of Art Logo

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Outline

MA313
Week 1: Introduction to MA313 and to Vector Spaces

Start of ...

PART 1: All about MA313

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The best way to contact me is by email.

https://commons.wikimedia.org/wiki/File:%C3%81ras_de_Br%C3%BAn.jpg

This module is taken by about 20 students in

- 3rd Arts: 3BA1 and 3CMS1
- 4th Arts: 4BCS1, 4BDA1, 4BMU1
- 3rd Science: 3BS9
- Visiting student (perhaps).
- Anyone else?

This group has different backgrounds. So please complete this form to help me understand: https://forms.office.com/r/Me6nmgBk5R While I'll try to take that into account, please let me know if I am incorrectly assuming your prior knowledge.

Updates:

- Almost everyone knows R! So I'll sprinkle a little through the semester.
- People want videos. So people get (old) videos.

This is *Linear Algebra 1*: a mathematics module focused on topics of

- *vector spaces*
- *linear transformations*
- *orthogonality*

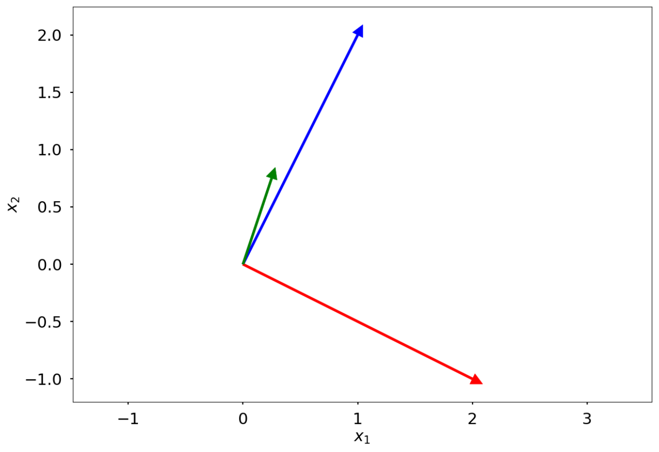
I've subtitled it "Linear Algebra for Data Science", because most of the applications we'll look come from data science. These include

- determining how much *independent information* is in a data set, and explaining what that means.
- finding concise ways of expressing data sets, which reveal some intrinsic information.
- ***fitting*** mathematical functions to data.

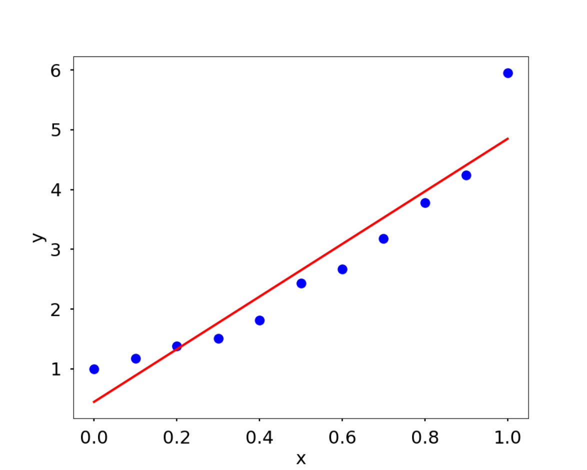
Along the way, we'll learn about the following ideas:

1. **Vectors**, and collections of vectors: *vector spaces*
2. **Subspaces** and how to identify them.
3. Combining vectors: **Linear combinations**
4. **Spans**, Spanning sets, **Linear Independence** and **Bases**.
5. **Dimension** of vector spaces; **Rank** and Nullity of matrices.
6. The so-called "**fundamental subspaces** associated with a matrix, A: the **Column space** and **row space** of A, and the **Null space** of a A and of A^T.
7. Reduced **Row Echelon Form**
8. **Linear Transformations**, and their link to matrices.
9. **Coordinate vectors** and coordinate mappings
10. **Inner products**, and angles between vectors.
11. **Orthogonality**, the theorem of Pythogoras.
12. **Length** (norm) of a vector, and the distance between two vectors.
13. **Orthogonality** and Least squares.

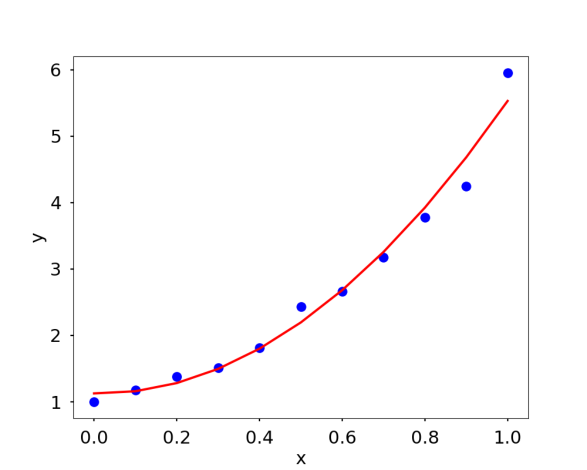
Example: which pair of vectors are most "similar"?



What line best fits a data set?



What "curve" best fits a data set?



Lectures: Tuesday, 13.00-13.50 in AC202
Friday, 12.00-12.50 in AC214.

Tutorials: Need to find a time! More of this presently.

Format: All classes are **in-person**. If live-streaming or recording is requested, we can discuss. Should it happen that we need to go online (e.g., if I get COVID), we'll use Blackboard Collaborate.

Blackboard: At `http://NUIGalway.BlackBoard.com`

- Slides from lectures, including annotated ones.
- Announcements;
- Grade centre;
- Access to assignments;
- *Videos from 2021/2022* (Warning: some things will have changed)
- etc...

Work load: 5 ECTS (60 is the typical yearly total for a full-time programme)

24 lectures, all in Semester 1

Roughly 120 hours of student effort time.

Lecture materials: Slides for each week's classes will be available for download in advance of the Tuesday lecture.

These contain the main definitions, ideas, and examples, as well as exercises that are of a similar style and standard as those on the final exam.

When slides are annotated in class, the annotated version will be posted at the end of the week. (If I forget, a gentle reminder is welcome!).

SUMS: The School of Maths provides a free drop-in centre called **SUMS: Support for Undergraduate Maths Students**.

SUMS opens from **2pm to 5pm, Monday to Friday**, from Monday of Week 3. For more information, see http://www.maths.nuigalway.ie/sums/

Devices: The use of portable electronic devices during class is *encouraged*. For example, you might want to use it to check Wikipedia, or access the textbook.
Be aware that these can be distracting to other students. Please be considerate.

Your achievement in MA313 will be assessed as follows:

Final exam: 50%. 2 hour written exam.

Online assignments: 20%. Four WeBWorK assignments. They will be open for at least 5 working days. Multiple attempts can be made. Scoring (right/wrong) is provided immediately for most questions.

Written assignment: 10%. There will be one written homework assignment. Questions on this will more closely resemble "exam questions" that the WeBWorK assignments can.

Communication skills: 20%. Completed jointly with MA335 (Algebraic Structures). You'll write an essay on a agreed topic. If taking MA335, you'll also give presentation. (If not, presentation is optional).

Tentative schedule of deadlines:

Assignment 1 (WeBWorK) - end of Week 2 (16 September)

Assignment 2 (WeBWorK) - end of Week 4 (30 September)

Assignment 3 (Written) - end of Week 6 (14 October)

Assignment 4 (WeBWorK) - end of Week 8(?) (28 October)

Assignment 5 (WeBWorK) - end of Week 11(?) (18 September)

Essay : details to be confirmed.

These dates can be flexible, if discussed in advance.

Homework!

Verify that you can access the homework system by trying the "Demo" assignment. Link is on `Blackboard/Assignments`.

Tutorials will start in Week 3. When:
https://forms.office.com/r/0ya9Bp8qBU

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12					
12 – 1					Lecture
1 – 2		Lecture			
2 – 3					
3 – 4					
4 – 5					

So far, least worst times are:

- Monday at 13.00 (7/11)
- Tuesday at 15.00 (7/11)
- Wednesday at 11.00 (8/11) or 14.00 (9/11)
- Thursday at 12.00 (8/11)
- Friday at 11.00 (9/11)

There is no required textbook for MA313.

However, several are recommended:

- David Lay, Steven Lay, and Judith McDonald: *Linear algebra and its applications*. Free online access available: https://search.library.nuigalway.ie/permalink/f/1pmb9lf/353GAL_ALMA_DS5192067630003626. *Quite extensive, and has lots of examples and exercises. But aimed at an electronic engineering audience. You are strongly recommended to review Chapters 1 and 2. We'll start at Chapter 4.*
- Shaina Race Bennet: Linear Algebra for Data Science with examples in R. *Free and open, and with the right emphasis. Though we won't use R.*

Lecture notes will have most of the material needed, but the text is great for providing more examples, different explanations, and exercises.

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PART 2: Mathematical Preliminaries

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In spite of the fact that this module is called "**Linear Algebra 1**", everyone here has taken at least one previous module on linear algebra: MA203 or MA283. And linear algebra features in all fist year mathematics modules. So you all know the basic idea of

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Examples:

- Matrices: rectangular tables of numbers.
  Examples:

Part 2: Mathematical Preliminaries

- How matrix-vector multiplication works: Examples (including identifying rows and columns):

Exercises

1. Read Sections 2.1, 2.2 and 2.3 of the text-book.
2. Verify that you can access the homework system by trying the "Demo" assignment, which also checking your matrix algebra skills. Link is on `Blackboard/Assignments`.
3. Complete the survey at https://forms.office.com/r/Me6nmgBk5R by 5pm Thursday, 8 September 2022.

Part 2: Mathematical Preliminaries

There are some other concepts relating to vectors and matrices that you should know from previous courses:

- determinants of (square) matrices
- eigenvalues of matrices, and corresponding eigenvectors;
- transpose of a vector or matrix; symmetric matrix.
- solving linear systems of equations with Gaussian Elimination; row reduction (to row reduced echelon form).

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PART 3: The big idea

When we first study *vectors*, we are taught to think of them as points in space, or as lines of a particular length and direction.

Then we can think of operations on vectors like

- adding vectors
- changing the length of a vector
- rotating a vector
- calculating the angle between two vectors.

The intuition that we gain from this is very valuable, but also limiting.

This entire module is based around the idea of the **abstract** definition of a vector space.

This concept of **abstraction** is central to modern mathematics. The idea is to strip away parts of the concept that just come from our intuition, so that we can see the real essence of the object.

Image taken from the Burren College of Art Logo

We'll do this with the idea of a vector: try to distil what makes a vector a vector in Euclidean space, so that we can look at lots of other examples.

Suppose we have three vectors in $\mathbb{R}^2$, called u, v and w.

- We can add any pair of them (in any order):

- Adding them has a geometric meaning (but don't give this too much importance):

- Add all three of them (in any order):

- Multiply any one of them by a scalar (i.e., something that is "just a number", and not a vector).

- Multiplying by zero is particularly important.

Apart from the geometry bit, everything we've said about vectors in $\mathbb{R}^2$ is true in $\mathbb{R}^3$ or $\mathbb{R}^4$ or $\mathbb{R}^{2021}$, or $\mathbb{R}^n$.

Example:

You will also have met vectors of complex numbers. E.g., $\mathbb{C}^2$ and $\mathbb{C}^3$.

But there are lots of other collection of things that seem have obey the same rules, about how we can add them and multiply them by scalars, etc. Matrices are an obvious example, but my favourite is a collection of polynomials of degree at most n, when we'll call $\mathbb{P}_n$.

Example

The set $\mathbb{P}^3$

THE BIG IDEA

So, our plan is to identify first the properties that are common to all examples so far, and then the properties that are common to **collections** of these vectors.

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PART 4: Definition of a Vector Space

See Section 4.1 of the text-book:
https://search.library.nuigalway.ie/permalink/f/1pmb9lf/353GAL_ALMA_DS51920676300003626

Definition of a vector space (1/2)

A **vector space** consists of

- a (non-empty!) set V, whose elements we call **vectors**,
- an operation called **addition** which assigns a vector

$$u + v \in V$$

  to any two vectors $u, v \in V$, and
- an operation called **scalar multiplication** which assigns a vector

$$cu \in V$$

  to each scalar $c \in \mathbb{R}$ and vector $u \in V$

such that the axioms on the following slides are satisfied.

Definition of a vector space (2/2)

We require that the following conditions **V1–V8** are satisfied for all vectors $u, v, w \in V$ and scalars $c, d \in \mathbb{R}$:

V1. $u + v = v + u$ (**commutativity** of addition)

V2. $(u + v) + w = u + (v + w)$ (**associativity** of addition)

V3. There exists $\mathbf{0} \in V$, called the **zero vector** such that $u + \mathbf{0} = u$ for all $u \in V$,

V4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$

V5. $c(u + v) = cu + cv$ (**distributivity I**)

V6. $(c + d)u = cu + du$ (**distributivity II**)

V7. $c(du) = (cd)u$

V8. $1u = u$

Example

$\mathbb{R}^n$ is a vector space We define

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$$

with addition defined as $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$ and

scalar multiplication defined as $c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$.

Then $\mathbb{R}^n$ is a vector space. The proof is a quite tedious, but quite easy.

It would take too long to show that $\mathbb{R}^n$ satisfies each of the 8 axioms. So we'll just verify the three of them.

V1. $u + v = v + u$ (**commutativity** of addition)

V3. There exists $\mathbf{0}$, called the **zero vector**, such that $u + \mathbf{0} = u$ for all $u \in V$.

V4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$.

For an integer $n \geqslant 0$, let $\mathbb{P}_n$ consist of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

of degree at most n, where $a_0, \ldots, a_n \in \mathbb{R}$.

We can add polynomials in $\mathbb{P}_n$ in the usual way:

$$\begin{aligned}(a_0 + a_1 t + \cdots + a_n t^n)& \\ &+ (b_0 + b_1 t + \cdots + b_n t^n) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n.\end{aligned}$$

Also,

$$cp(t) = ca_0 + ca_1 t + \cdots + ca_n t^n,$$

where $c \in \mathbb{R}$.

Claim: These operations turn $\mathbb{P}_n$ into a vector space.

The reasoning again just boils down to to properties of real numbers.

Example

Function spaces Let $\mathbb{D}$ be an arbitrary set.
Let V be the set of **all** functions $f : \mathbb{D} \to \mathbb{R}$.
Given $f, g \in V$ and $c \in \mathbb{R}$, we define $f + g \in V$ and $cf \in V$ via

$$(f + g)(x) := f(x) + g(x)$$

and

$$(cf)(x) := cf(x)$$

for $x \in \mathbb{D}$.
Claim: These operations turn V into a vector space.

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PART 5: Not everything is a vector space

Part 5: Not everything is a vector space

So far, all of the examples we have looked at correspond to vector spaces. But not every set equipped with addition and scalar multiplication is a vector space.

Here are a few examples of things that are not vector spaces.

1. The set of vectors in $\mathbb{R}^2$ with strictly positive entries.

2. The set of vectors in $\mathbb{R}^2$ with non-negative entries.

3. The set of polynomials of degree **exactly** 3.

Part 6: Exercises

Here are a set of exercises to help you work through the material presented during this week's classes.

All but the last are taken either directly from the textbook, or with minor edits.

These are not homework assignments, and you don't have to submit them.

Part 6: Exercises

Q1. Let $M_{m\times n}$ be the set of $m \times n$ matrices with real entries. Then $M_{m\times n}$ is a vector space with respect to the following operations:

$$\begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \ddots & \ddots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & \dots & b_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ b_{m1} & \ddots & \ddots & b_{mn} \end{bmatrix}$$

$$:= \begin{bmatrix} a_{11}+b_{11} & \dots & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1}+b_{m1} & \ddots & \ddots & a_{mn}+b_{mn} \end{bmatrix}$$

$$c \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \ddots & \ddots & a_{mn} \end{bmatrix} := \begin{bmatrix} ca_{11} & \dots & \dots & ca_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ ca_{m1} & \ddots & \ddots & ca_{mn} \end{bmatrix}$$

Part 6: Exercises

These are just the usual operations of adding matrices and multiplying a matrix by a scalar. *Verify that the axioms **V1–V8** are really satisfied.*