2425-MA140 Engineering Calculus

Week 11, Lecture 2 (L32)

Numerical Integration; Exam Preview

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News!

Tutorials

There is a change to the tutorial plan for this week:

- Thursday at 11am: Teams 9 and 10 attend a tutorial in ENG-3035. Teams 7, 8, 11, and 12's tutorial is Aras Moyola MY129.
- Friday: no MA140 tutorials!
- Irish tutorial: no change

Assignments

- **Assignment 6-Q1** has finally been graded. Combined results will be posted later today (I hope!).
- **Assignment 7**: Final grad is now on Canvas.
- **Assignment 8**: WIl be graded very soon.

Today's, we'll run the rule over...

1. Numerical Integration
2. The Midpoint Rule
   - Python
   - MATLAB
   - Error Estimates
   - Other Methods
3. Exam Preview
   - Warning!
   - Organisation
   - Topics
   - Advice
   - Final Grade

For more, read Section **7.6** (Numerical Integration) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax).

Numerical Integration

Regrettably, I may have given you the impression that the best/only way to evaluate $I = \int_a^b f(x)\,dx$ is to find $F(x)$, the antiderivative of f, and then compute $I = F(b) - F(a)$.
That is rather unfortunate, the antiderivative of $f(x)$ maybe hard, or, indeed, **impossible**, to find!

This might be a little surprising, given all the examples we've studied. And also because, one can always find the derivative of $f(x)$, but matter how complicated $f(x)$ is. (Programming this is a major topic in computer science, and very important for modern AI).

Numerical Integration

However, there is good news!

- This is rather liberating, since it means we don't have to try (except for end-of-semester exams).
- There are **algorithms** which can compute $\int_a^b f(x)\,dx$ for you as accurately as you would like! And they are easy to code.
- The **motivation** for these algorithms is easy to understand: they try to estimate areas under curves.
- It is a fascinating area of mathematics.

The Midpoint Rule

Midpoint Rule

To estimate $\int_a^b f(x)\,dx$,

- Choose an integer n, and set $h = (b-a)/N$.
- Split $[a, b]$ in the intervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{N-1}, x_N]$, where $x_k = a + kh$.
- Let m_k be the midpoint of $[x_k, x_{k+1}]$; i.e., $m_k = x_k + h/2$.
- Compute $M_N = h\big(f(m_0) + f(m_1) + \cdots + f(m_{N-1})\big)$.

This can be easily implemented in a few lines of code.

MidPoint.py

```
import numpy as np
f = lambda x : np.sqrt(x) # function to be integrated
a,b=0,1     # limits of integration
N = 10; h=1/N;
x = np.linspace(0,1,N+1)
m = (x[0:-1] +x[1:])/2  # midpoints
M_N = h*np.sum(f(m))
print(f"N={N : 2d}, M_N={M_N : 8.4f}, Error={np.abs(
    M_N-2/3) : .3e}")
```

MidPoint.m

```
f = @(x)sqrt(x); % function to be integrated
a=0; b=1;    % Limits of inetegration
N = 10; h=(b-a)/N;
x = linspace(0,1,N+1);
m = (x(1:end-1) + x(2:end))/2; % midpoints
M_N = h*sum(f(m));
fprintf("N=%2d, M_N=%8.4f, Error=%8.3e\n", N, M_N,
    abs(M_N-2/3));
```

We won't implement the method by hand, but from a mathematical perspective, it is important to note that it can be proved that, error, $\mathcal{E}_N$ is bounded like

$$\mathcal{E}_N := |\int_a^b f(x)\,dx - M_N| \leqslant h^2 \frac{b-a}{24} \max_{a \leqslant x \leqslant b} f''(x).$$

This has lots of consequences:

- If f is a constant or linear polynomial, then the error is zero.
- For most reasonable f, it is clear that, as $N \to \infty$ (and so $h \to 0$) we get that $\mathcal{E}_N \to 0$.
- More practically, it is possible to choose N so that $\mathcal{E}_N$ is as small as you would like.

Example:

Suppose $f(x) = x^3$, and $a = 0$, $b = 1$.

(i) What is the maximum error expected with $N = 10$?

(ii) What N should you choose to ensure the error is no more than 10^{-4}

With more time, we'd investigate methods that are more accurate, but require the same amount of effort, such as the Trapezoidal Rule, Simpson's Rule, Gaussian Quadrature, etc, etc. And their extension to higher dimensions.

If you are interested, read Section 7.6 of the textbook.

DISCLAIMER

The following information only pertains to the **2024/2025 Semester 1** exam for MA140. The Autumn Exam will have similar structure, but different topics.
In particular, anything designated as "`tip`" relates only to the Semester 1 Exam. The Autumn Exam will be different.

The Exam will mostly follow the organisation of the exam from 2023/2024:

- There are 5 questions on the paper.
- All questions are worth 25 marks. Each question is made up of several parts. The number of marks for each part is shown on the paper. These reflect the difficulty of the part, or the time it might take to answer it.
- We don't use "negative" marking, nor to we give "attempt marks".
- Your score will be based on your best 4 questions.

- Topics appear in roughly the order in which they were covered in class.

Q1 is mainly about functions, partial fractions, **limits**, and continuity. `Tips:` "Domain and Range" questions will not be on the exam. Revise "Case 1" of partial fractions. Practice factorizing quadratics.

Q2 is mainly about **differentiation**, including Product Rule, Quotient Rule and Chain Rule. `Tips:` In this exam, there is no "differentiation from first principles" question. Revise the Squeeze Theorem.

Q3 is mainly **applications of differentiation** including to curve sketching and optimisation. `Tip:` revise the 2nd Derivative Test.

- Q4 is about **integration and techniques of integration**. This includes FTC, substitution, integration by parts. `Tip` Revise improper integrals.
- Q5 is based on **applications of integration**, including areas of regions in the plane, arc lengths, volumes and areas of revolution, centres of mass. `Tip:` centres of mass/centroids will **not** feature on the Semester 1 exam. Neither will infinite solids. Revise formulae for arc length, volumes of revolution, and surface areas of revolution. However, you will not be required to explain their derivations.

- Unless you are expressly asked to give an answer to a certain number of decimal places, it is OK to express a solution in terms of quantities like π, e and $\sqrt{2}$.
- Unless you are expressly asked to use a particular method, use any one you like. For example, in some cases l'Hopital's rule might be a good approach.
- If you go give a numeric answer, to do to at least 3 decimal places. Moreover, if you do any intermediate step numerically, do that to at least 4 decimal places.
- No need to write long-winded answers, but do try to write clearly. An illegible answer will be assumed to be wrong. Where you use a particular method (such as L'Hopital's Rule, Chain Rule, Integration by Parts), mention it.

Exam Preview

- Try practising for the exam by working on the 2023/2024 exam paper (except Q2(b)). Older papers are less relevant, especially Q5.
- The Practice Paper is also a good guide...

Unless you have an exemption for some assignments, your final grade will be calculated as

- 67% from the Semester 1 exam in December.
- 33% from CA
- Your CA score will be based on your best 7 (of 8) assignments;

$\boxed{\text{ANY QUESTIONS??}}$

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