MA284 : Discrete Mathematics

Week 4: Algebraic and Combinatorial Proofs

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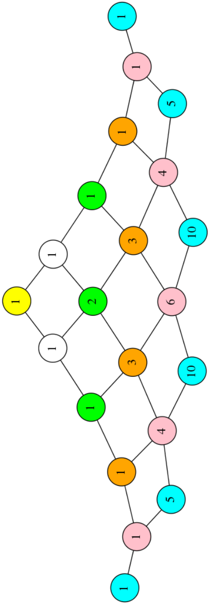
29 September and & 1 October, 2021

1. Part 1: A short summary
   - Binomial coefficients
2. Part 2: Pascal's Triangle (again)
3. Part 3: Algebraic and Combinatorial Proofs
4. Part 4: How combinatorial proofs work
   - Which is better?
5. Exercises

These slides are based on §1.3 and §1.4 of Oscar Levin's *Discrete Mathematics: an open introduction*.



Tutorials

Tutorials started last week. You should attend one of the sessions listed below. The venues for Wednesday at 11 has changed from that originally advertised, and the Thursday at 4 one is new.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			CD: **MRA201**		
12 – 1		EM: CA117			
1 – 2					
2 – 3			AH Online		
3 – 4		AH: Online		CD: Online	
4 – 5				EM: **AMB-G008**	

Online class will be held on the course room in the Blackboard Virtual Classroom: eu.bbcollab.com/guest/768da44b88344e86bf5eae54357e2be9

ASSIGNMENT 1 is now open!

To access the assignment, go to the 2122-MA284 Blackboard page, select `Assignments` ... `Assignment 1`.

There are 10 questions.
You may attempt each one up to 10 times.
This assignment contributes approximately 8% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday 1 October 2021.

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Start of ...

PART 1: A short summary

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as "} n \text{ choose } k \text{"}$$

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ **is the number of ways to select k objects from a total of n objects.**

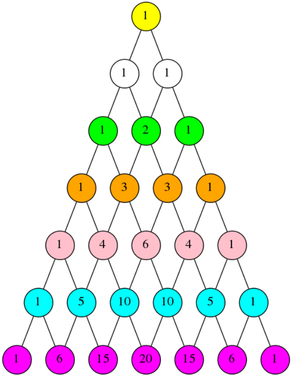
There is a formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can also calculate binomial coefficients using Pascal's identity.

Pascal's Identity: a recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

Permutations of k objects from n

The number of permutations of k objects out of n, $P(n, k)$, is

$$P(n, k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

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END OF PART 1

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PART 2: Pascal's Triangle (again)

At the end of Week 4, we "proved" that

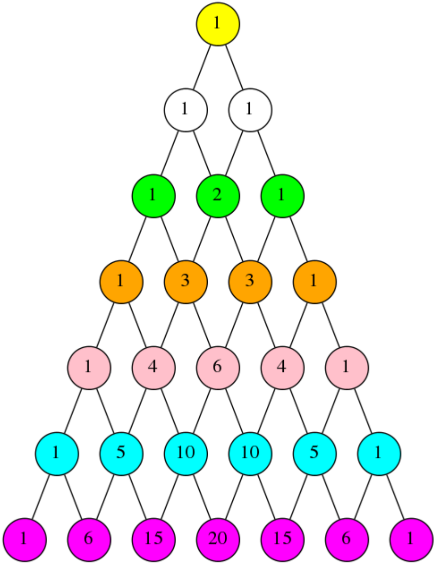
$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We did this by counting $P(n, k)$ in two different ways.

This is a classic example of a *Combinatorial Proof*, where we establish a formula by counting something in 2 different ways.

For much of this week, we will study this style of proof. See also Section 1.4 of the text-book.

But first, we will form some conjectures, using **Pascal's Triangle**.



Binomial coefficients have many important properties.
Looking at their arrangement in Pascal's Triangle, we can spot some:

(i) For all n, $\binom{n}{0} = \binom{n}{n} = 1$

(ii) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

(iii) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

(iv) $\binom{n}{k} = \binom{n}{n-k}$

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END OF PART 2

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PART 3: Algebraic and Combinatorial Proofs

Proofs

Proofs of identities involving Binomial coefficients can be classified as

- **Algebraic:** if they rely mainly on the formula for binomial coefficients.
- **Combinatorical:** if the involve counting a set in two different ways.

For our first example, we will give two proofs of the following fact:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Algebraic proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Combinatorial Proofs

Proofs of identities involving **binomial coefficients** can be classified as either

- **Algebraic:** if they rely mainly on the formula for binomial coefficients; or
- **Combinatorical:** if the involve counting a set in two different ways.

Example

Give two proofs of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

First, we check:

Algebraic proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Combinatorial proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

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END OF PART 3

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PART 4: How combinatorial proofs work

Which are better: algebraic or combinatorial proofs?

When we first study discrete mathematics, *algebraic* proofs make seem easiest: they reply only on using some standard formulae, and don't require any deeper insight. Also, they are more "familiar".

However,

- Often algebraic proofs are quite tricky;
- Usually, algebraic proofs give no insight as to why a fact is true.

Example (MA284 - Semester 1 exam, 2016/2017)

Give a combinatorial proof of the following fact

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

We wish to show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

What is a "combinatorial proof" really?

1. These proofs involve finding two different ways to answer the same counting question.
2. Then we explain why the answer to the problem posed one way is A
3. Next we explain why the answer to the problem posed the other way is B.
4. Since A and B are answers to the same question, we have shown it must be that $A = B$.

Example

Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 1:

Example

Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 2:

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END OF PART 4

Unless indicated otherwise, these questions identical to, or variants on, Sections 1.4, 1.5 and 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. Put the following numbers in increasing order.

(a) The number of subsets of the set $\{a, b, c, d, e, g, h, i\}$.
(b) $\binom{10}{5}$
(c) $\binom{12}{3}$.
(d) $\binom{12}{3}$.
(e) $5!$
(f) $P(7, 4)$
(g) $P(8, 5)$

Q2. Compute $\binom{7}{3}$ using Pascal's Identity. Check you got the right answer by also doing this using the factorial formula.

Q3. Write out **all** permutations of the letters `A`, `B`, `C`, and `D` that use all four letters. Verify you get 24.
Now write out **all** permutations of the 4 letters `A`, `B`, `C`, and `C` (i.e., C is repeated). How many do you get?

Q4. Give a combinatorial proof for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

Q5. Give an algebraic proof, using induction, for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

Q6. Give a combinatorial proof of the fact that $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy$

Q7. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

Q8. Consider the bit strings in $\mathbf{B}_2^6$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 01?
(b) How many of those bit strings start with 001?
(c) Are there any other strings we have not counted yet? Which ones, and how many are there?
(d) How many bit strings are there total in $\mathbf{B}_2^6$?
(e) What binomial identity have you just given a combinatorial proof for?

Q9. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}.$$

Q10. (MA284 – Semester 1 exam, 2017/2018) combinatorial argument, or otherwise, prove the following statement.

$$\binom{n}{5} = \binom{2}{2}\binom{n-3}{2} + \binom{3}{2}\binom{n-4}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n-3}{2}\binom{2}{2}.$$