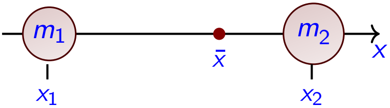
2526-MA140 Engineering Calculus

Week 10, Lecture 3
Root Mean Square, Moments and Centroids

Dr Niall Madden

University of Galway

Thursday, 20 November, 2025



Today's lecture will be centred on...

1. Root-Mean-Square Values
   - Average value
   - RMS
2. Centre of Mass: over view
3. Moments
   - Centre of Mass
4. A 1D rod with variable density
   - Total mass
5. Moments
   - Centre of Mass
6. Two dimensions
   - Moments
   - Centre of Mass
7. Exercises

For more, read Section **6.6** (Moments and Centres of Mass) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax).

Yesterday we learned:

Average value of a function

The constant $\bar{f}$ is the **average** value of $f(x)$ on $[a, b]$, if

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx \Rightarrow \int_a^b \bar{f}\,dx = \int_a^b f(x)\,dx.$$

Example:

1. What is the Average Value of $f(x) = x^2$ on $[-1, 1]$?
2. ~~What is the Average Value of $f(x) = x^3$ on $[-1, 1]$?~~

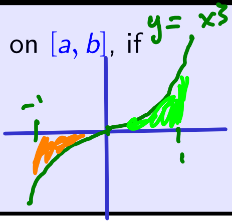
$$\bar{f} = \frac{1}{1-(-1)}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left(\frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{1}{2}\cdot\frac{1}{3}\left(1-(-1)\right) = \frac{1}{3}$$

Yesterday we learned:

Average value of a function

The constant $\bar{f}$ is the **average** value of $f(x)$ on $[a, b]$, if

$$\int_a^b \bar{f}\, dx = \int_a^b f(x)\, dx.$$



Example:

1. ~~What is the Average Value of $f(x) = x^2$ on $[-1, 1]$?~~
2. What is the Average Value of $f(x) = x^3$ on $[-1, 1]$?

$$\bar{f} = \frac{1}{2}\int_{-1}^{1} x^3\, dx = \frac{1}{8} x^4 \Big|_{-1}^{1} = \frac{1}{8}(1 - 1) = 0.$$

Root-Mean-Square Values (RMS).

In some contexts, the **average value** of a function is a useful summary statistic. But it can be misleading too, as the last example showed.

Notable examples of this include

- The average value of an alternating current is zero;
- The average motion of a piston is zero.

Therefore (especially in power electronics) we need another measure to summarise a function

Root Mean Squared (RMS)

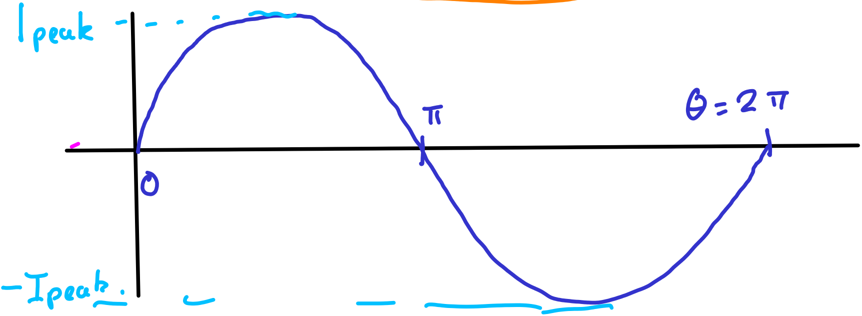
The **root mean square (RMS)** of a function $f(x)$ is

$$f_{\mathrm{RMS}} := \left(\frac{1}{b-a} \int_a^b [f(x)]^2 \, dx \right)^{1/2} = \sqrt{\quad \cdots}$$

Example

An electric current $i(\theta)$ is given by $i(\theta) = I_{\text{peak}} \sin(\theta)$ where I_{peak} is a constant. Find the root mean square of $i(\theta)$ over the interval $[0, 2\pi]$.

(*Hint: use that* $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$).



Example

An electric current $i(\theta)$ is given by $i(\theta) = I_{\text{peak}} \sin(\theta)$ where I_{peak} is a constant. Find the root mean square of $i(\theta)$ over the interval $[0, 2\pi]$.

(*Hint: use that* $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$).

$i(\theta) = I_{peak} \sin(\theta)$. $\quad a = 0 \quad b = 2\pi$.

$$(RMS)^2 = \frac{1}{b-a} \int_a^b (i(\theta))^2 \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (I_{peak})^2 \sin^2(\theta) \, d\theta$$

$$= \frac{(I_{peak})^2}{2\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta$$

Example

An electric current $i(\theta)$ is given by $i(\theta) = I_{\text{peak}} \sin(\theta)$ where I_{peak} is a constant. Find the root mean square of $i(\theta)$ over the interval $[0, 2\pi]$.

(*Hint: use that* $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$).

$$(RMS)^2 = \frac{(I_{peak})^2}{2\pi} \int_0^{2\pi} \frac{1}{2}\left(1 - \cos(2\theta)\right) d\theta$$

$$= \frac{(I_{peak})^2}{4\pi} \left[\theta - \frac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi}$$

$$= \ldots = \left(\frac{I_{peak}}{2}\right)^2 \Rightarrow RMS = \frac{I_{peak}}{\sqrt{2}}.$$

Centre of Mass: over view

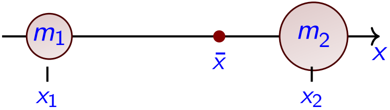
In this section, we what to study the **centre of mass** of an object, such as an irregularly shaped thin plate.

Intuitively, this is the point at which at which a plate could be perfectly balanced on the tip of a pin. (This is related to the concept of the *average value pf a function*).

But first we study two one-dimensional examples, the first of which does not even require calculus. But all have the concept of "balance" at their heart.

Moments

Suppose we have a thin rod with negligible mass. We attach objects with mass m_1 and m_2, at the points x_1 and x_2. We want to find $\bar{x}$: the point at which the rod is balanced (e.g., if suspended from a string at that point).



Suppose $m_1 < m_2$. Then we know that x_1 should be further from $\bar{x}$ than x_2. More precisely, we need

$$m_1|x_1 - \bar{x}| = m_2|x_2 - \bar{x}|.$$

Find $\bar{x}$.

Moments

Starting from

$$m_1|x_1 - \bar{x}| = m_2|x_2 - \bar{x}|.$$

we can solve for $\bar{x}$:

$x_1 < \bar{x} \Rightarrow |x_1 - \bar{x}| = \bar{x} - x_1$

$x_2 > \bar{x} \Rightarrow |x_2 - \bar{x}| = x_2 - \bar{x}$

So we solve $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$ for $\bar{x}$.

$\Rightarrow \bar{x}(m_1 + m_2) = m_1 x_1 + m_2 x_2$

$$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Moments

- For this scenario, we have deduced that $\bar{x} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$.
- The quantity $m_1x_1 + m_2x_2$ is called the *(first) moment of the system (with respect to the origin)*.
- It can also be interpreted as: $\bar{x}(m_1 + m_2) = m_1x_1 + m_2x_2$. This means: "*if all the mass was concentrated at $x = \bar{x}$, the moment would not be changed*".
- If there are three masses, m_1, m_2 and m_3, at the points x_1, x_2 and x_3, the formula extends: $\bar{x} = \dfrac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$.
- And for n masses:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k}$$

Center of Mass of Objects on a Line

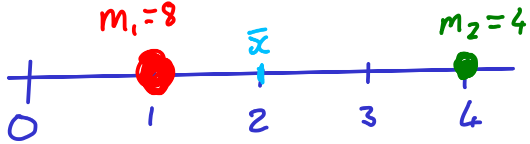
Let m_1, m_2, ..., m_n be point masses placed on a number line at points x_1, x_2, ..., x_n, respectively. The **total mass** of the system is $m = \sum_{k=1}^{n} m_k$.

Then the **moment of the system**, with respect to the origin, is $M = \sum_{k=1}^{n} m_k x_k$.

And the **centre of mass** is $\bar{x} = \dfrac{M}{m}$.

Example:

Find the centre of mass of a system where a mass of $8kg$ is placed on the number line at $x = 1$, and a mass of $4kg$ is placed at $x = 4$,



$$M = x_1 m_1 + x_2 m_2 = (1)(8) + (4)(4) = 8 + 16 = 24.$$

$$m = m_1 + m_2 = 8 + 4 = 12$$

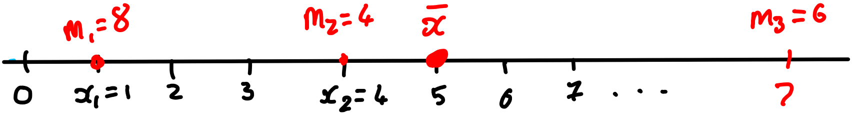
$$\bar{x} = \frac{M}{m} = \frac{24}{12} = 2.$$

Example:

We have a system where

- a mass of $8kg$ is placed on the number line at $x = 1$,
- a mass of $4kg$ is placed at $x = 4$,
- a mass of $6kg$ is placed at $x = x_3$.

If the centre of mass is at $x = 5$, find x_3.



$M = m_1x_1 + m_2x_2 + m_3x_3 = (8)(1) + (4)(4) + (6)(x_3)$

$m = 8 + 4 + 6 = 18.$ $\bar{x} = 5$

So $5 = \dfrac{M}{m} \Rightarrow \dfrac{24 + 6x_3}{18} \Rightarrow 5 = \dfrac{4}{3} + \dfrac{x_3}{3} \Rightarrow .. \Rightarrow x_3 = 11.$

A 1D rod with variable density

In our previous examples, we assumed the rod we were hanging masses from was itself mass-less. That was just to simplify calculations, as was the assumption that the masses were "point masses".

But suppose the rod does have mass, and it varies along the length. How do we find the centre of gravity?

First, it helps to understand that when we say that "the mass can vary", what we really mean is that the **density** (i.e., mass per unit length) can vary.

That is, there is a function $\rho(x)$, which is the density of the rod at x.

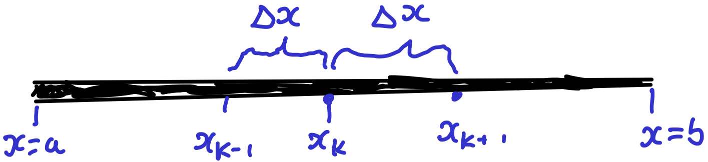
"Rho of x".

A 1D rod with variable density

To get the total mass, we reason as follows.

- The mass of a "slice" from x_k to $x_k + \Delta x$ is $m_k = \rho(x_k)\Delta x$.
- Summing over all slices of such length we get the total mass is $m \approx \sum_{k=1}^{n} \rho(x_k)\Delta x$, where $\Delta x = (b-a)/n$, $x_0 = a$, and $x_k = x_0 + k\Delta x$.
- Doing our usual trick of letting $n \to \infty$, we get i.e. $\Delta x \to 0$

$$\boxed{m = \int_a^b \rho(x)\, dx.}$$



Moments

For a discrete set of points (and masses), we know the moment is

$$M = \sum_{k=1}^{n} x_k m_k.$$

With our $\Delta x = (b-a)/n$ notation, this is

$m_k = \rho(x_k)\Delta x$

$$M \approx \sum_{k=1}^{n} x_k \rho(x_k) \Delta x.$$

Again, we let $n \to \infty$, and we get

$$\boxed{M = \int_a^b x \rho(x)\, dx.}$$

We can now conclude that the **centre of mass** of a rod on the x-axis with end-points at $x = a$ and $x = b$ (with $a < b$), and density $\rho(x)$ is

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.}$$

Example Use that the **centre of mass** of a rod on the x-axis with end-points at $x = a$ and $x = b$ (with $a < b$), and density $\rho(x)$ is

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx.}}$$

to find the centre of mass when $a = 0$, $b = 1$, and $\rho(x) = x^2$.

$$\bar{x} = \frac{\int_0^1 x\,(x^2)\,dx}{\int_0^1 (x^2)\,dx} = \frac{\int_0^1 x^3\,dx}{\int_0^1 x^2\,dx} = \frac{\frac{1}{4}x^4\Big|_0^1}{\frac{1}{3}x^3\Big|_0^1}$$

$$= \frac{1/4}{1/3} = \frac{3}{4}.$$

Two dimensions

Suppose we are given a (positive) function $f(x)$, have a region in the plane bounded above by $y = f(x)$, below by $y = 0$, and left by $x = a$, and right by $y = b$. A thin plate defined by this region is sometimes called a **lamina**. Its area is $A = \int_a^b f(x)\,dx$.

We now want to consider how to find its **centre of mass ("centroid")**, which we denote $(\bar{x}, \bar{y})$.

Intuitively, (again): this is the point at which at which a cutout of the region could be **perfectly balanced** on the tip of a pin.

Again, the key idea we need is that of a **moment**. In a realistic setting, this is the **mass** of the lamina, times its distance from a reference point: usually $(0, 0)$.

In our setting, we'll just use the **area** of a region as a proxy for the mass. (This is physically reasonable if the limina has uniform density, and is very, very thin).

To start with, it is helpful to think of the moments (in x and y) of a thin rectangle:

Now let's get M_x, which is the moment about the x-axis, by summing the moments of all the rectangles, and taking the limit of the resulting Riemann sum:

$$M_x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\big(f(x_i^\star)\big)^2 \Delta x = \int_a^b \frac{\big(f(x)\big)^2}{2}\,dx.$$

Similarly, we get M_y, which is the moment about the y-axis as

$$M_y = \lim_{n\to\infty} \sum_{k=1}^{n} x_i^\star f(x_i^\star)\Delta x = \int_a^b x f(x)\,dx.$$

If the centre of mass is the point $(\bar{x}, \bar{y})$, then we could think of the entire "area" as being centred there, but having the same moments.

That is

$$\bar{x}A = M_y, \quad \text{and } \bar{y}A = M_x.$$

giving...

Centroid of a planar region

If $f(x)$ is defined on $[a, b]$, then the **centroid** $(\bar{x}, \bar{y})$ of the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$ is given by

$$\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} \quad \text{and} \quad \bar{y} = \frac{\int_a^b [f(x)]^2\,dx}{2\int_a^b f(x)\,dx}$$

Example

Consider the plane region enclosed by the curve $y = \sqrt{x-2}$, the x-axis and the lines $x = 2$ and $x = 5$. Find

(1) the area of the region;

(2) the centroid of the region.

Exercises

Exer 10.3.1

Find $b > 0$ such that the average value of $f(x) = x^2 - 2x + 3/4$ on the interval $[0, b]$ is zero.
Compute the root mean squared of $f(x)$ on the same interval.

Exer 10.3.2

Find the centre of mass, $\bar{x}$, of a system with thin rod of negligible mass, placed on the x-axis, with a mass of $m_1 = 1$ placed at $x_1 = -1$, and $m_2 = 3$ placed at $x_2 = 2$.

Exer 10.3.3

A system consists of a thin rod of negligible mass, placed on the x-axis, with a mass of $m_1 = 10$ placed at $x_1 = 0$, and $m_2 = 5$ placed at $x_2 = 2$, and a mass m_3 at $x_3 = 3$. If $\bar{x} = 1$, find m_3.

Exercises

Exer 10.3.4

Find the centre of mass of a rod with density $\rho(x) = \sqrt{x}$, that is on the x-axis, with end points at $x = 0$ and $x = 1$.