2425-MA140 Engineering Calculus

Week 04, Lecture 2
Differentiation Rules

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Calcalas | **Calculus**

Díorthaigh | **Derivatives**

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
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Riail an toraidh	$y = uv$ $\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	**Product rule**
Riail an lín	$y = \frac{u}{v}$ $\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	**Quotient rule**
Cuingriail	$f(x) = u(v(x))$ $\Rightarrow f'(x) = \frac{du}{dv}\frac{dv}{dx}$	**Chain rule**

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Assignments, etc

Assignment 2

- **Assignment 2** is open. See https://universityofgalway.instructure.com/courses/35693/assignments/96620.
  Deadline is 5pm, Friday, 11 October.
- The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2065926

In today's class...

1 Differentiation by rule
2 The Basic Rules
- 1. The Constant Rule
- 2. The Power Rule
- 3. The constant multiple rule
- 4. The Sum and Difference Rules

3 The Product Rule
4 The Quotient Rule
5 Page 16 of the "log tables"
6 Exercises

See also:

- Sections 3.3 of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)
- Section 8.2 of *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Differentiation by rule

Yesterday, we computed derivatives of some functions using the "limit" definition (i.e., **differentiation from first principles**). However, that approach is tedious, and unnecessary in many case.

Instead we can use a set of "**rules**" which makes the process much more efficient. These rules are themselves derived from the "limit" definition – but we don't have to use that every time.

Notation

In today's class we'll make use of various notations for the derivative of a function: e.g., $f'(x)$, $\dfrac{df}{dx}$, $\dfrac{d}{dx}(f)$

The Constant Rule

If f is a constant function, i.e. $f(x) = c$ for all x, then:

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Why:

We've already deduced that

- The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- The derivative of $f(x) = x^{1/2}$ is $f'(x) = \frac{1}{2}x^{-1/2}$

These are particular examples of the **Power Rules**

The Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Examples Calculate the derivatives of the following functions

1. $f(x) = x^6$
2. $f(x) = \sqrt[3]{x}$

The constant multiple rule

Let $f(x)$ be any differentiable function, and let k be constant, then

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big).$$

Example: Find the derivative of $f(x) = 5x^4$.

The Sum and Difference Rules

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}\big(u(x)+v(x)\big)=\frac{d}{dx}\big(u(x)\big)+\frac{d}{dx}\big(v(x)\big).$$

Similarly, $\frac{d}{dx}\big(u(x)-v(x)\big)=\frac{d}{dx}\big(u(x)\big)-\frac{d}{dx}\big(v(x)\big).$

Example: Find the derivative of $f(x)=1+x+x^2$.

Actually, the "**Difference Rule**", which states that

$$\boxed{\frac{d}{dx}\big(u(x) - v(x)\big) = \frac{d}{dx}\big(u(x)\big) - \frac{d}{dx}\big(v(x)\big).}$$

can be combined by combining the **Sum Rule** and the **Constant Multiple Rule**.

Example

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$;

(b) The slope of the tangent line at $x = 2$;

(c) The equation of the tangent at $x = 2$.

(a) $f'(x) = -15x^2 + 6x - 9$

(b) The slope of the tangent line at $x = 2$ is $f'(2)$:

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57.$$

(c) The equation of the line with slope M and passing through a point (x_1, y_1) is

$$y - y_1 = M(x - x_1)$$

The y coordinate at $x = 2$ is

$$\begin{aligned} f(2) &= -5(2)^3 + 3(2)^2 - 9(2) + 7 \\ &= -5(8) + 3(4) - 18 + 7 \\ &= -40 + 12 - 18 + 7 \\ &= -39. \end{aligned}$$

So the tangent line passes through the point $(2, -39)$ and the slope of the line is -57.

Therefore, the equation of this line is $y + 39 = -57(x - 2)$

Ans: The equation of the tangent line is $x = 2$ is $y = 75 - 57x$.

The Product Rule

We now consider some advanced rules, which are a little more complicated and, I think, less obvious.

The first concerns the derivative of the **product** of two functions.

The Product Rule

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

First, let's convince ourselves that the following "product rule" is misinformation:

$$\frac{d}{dx}(uv) \overset{???}{=} \frac{du}{dx}\frac{dv}{dx}.$$

The Product Rule

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example Use the **product rule** to find the derivative of $f(x) = x^3(x^2 + 1)$.

The Product Rule

Example: use the product rule to show that, if $f(x) = x\sin(x)$, then $f'(x) = x\cos(x) + \sin(x)$.

The Quotient Rule

The Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then $f(x) = \dfrac{u(x)}{v(x)}$ is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example: Use this rule to find the derivative of $f(x) = \dfrac{x+1}{x^2}$

The Quotient Rule

Example

We know that

- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$,
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

Use these facts, and the Quotient Rule to show that

$$\frac{d}{dx}(\tan(x)) = \Big(\frac{1}{\cos(x)}\Big)^2.$$

The Quotient Rule

Page 16 of the "log tables"

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Riail an toraidh — **Product rule**

$$y = uv$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Riail an lín — **Quotient rule**

$$y = \frac{u}{v}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Cuingriail — **Chain rule**

$$f(x) = u(v(x))$$

$$\Rightarrow f'(x) = \frac{du}{dv}\frac{dv}{dx}$$

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Exercises

Exercises 4.2.1 (Based on Q2(a), 2023/2024)

Find the derivative of $f(x) = \dfrac{\sin(x)}{\sqrt{x}}$.

Exercise 4.2.2 (Based on Q2(b), 2019/2020

Find the derivative of $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$.