Week 04, Lecture 3
The Chain Rule

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Calcalas **Calculus**

Díorthaigh **Derivatives**

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$

Riail an toraidh	$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	**Product rule**
Riail an lín	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	**Quotient rule**
Cuingriail	$f(x) = u(v(x)) \Rightarrow f'(x) = \frac{du}{dv}\frac{dv}{dx}$	**Chain rule**

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Assignments, etc

Assignments

- **Assignment 2** is open. See https://universityofgalway.instructure.com/courses/35693/assignments/96620.
  Deadline is 5pm, Friday, 11 October.
- The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2065926
- **Assignment 3** opens tomorrow morning.

In today's class...

See also:

- Sections 3.6 (The Chain Rule) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)
- Section 8.3 of *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Chain Rule

Of all the differentiation rules, the **chain rule** is the most important: most other rules are actually just special cases of it. It applies to a "function of a function"

The Chain Rule

If $u(x)$ and $v(x)$ are differentiable, and f is the composite function $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

Example: What is the derivative of $f(x) = \cos(x^2)$?

First note that this is a composite function...

Chain Rule

The Chain Rule

If $f(x) = u\big(v(x)\big)$, then

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}.$$

Example: What is the derivative of $f(x) = \cos(x^2)$?

Chain Rule

Example

Find $\frac{dy}{dx}$ if $y = (x^3 + 4x^4 + 7)^{12}$.

Example: Let $u(v) = v^{12}$ and $v(x) = x^3 + 4x^4 + 7$, then y is $y = u\big(v(x)\big)$.

Note that
$$\frac{du}{dv} = 12v^{11} \quad \text{and} \quad \frac{dv}{dx} = 3x^2 + 16x^3.$$

By the Chain Rule we have
$$\frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dx} = 12v^{11}(3x^2 + 16x^3),$$

and therefore
$$\frac{dy}{dx} = 12(x^3 + 4x^4 + 7)^{11}(3x^2 + 16x^3).$$

Chain Rule

Example (Skimmed this in class)

Find $\dfrac{dy}{dx}$ if $y = \dfrac{1}{(x^4 + 2x^2 + 8)^{40}}$.

We have $y = (x^4 + 2x^2 + 8)^{-40}$. We can write y as $y(x) = u(v(x))$ with

- $u(v) = v^{-40}$ and so $\dfrac{du}{dv} = -40u^{-41}$; and
- $v(x) = x^4 + 2x^2 + 8$, so $\dfrac{dv}{dx} = 4x^3 + 4x$.

Applying the Chain Rule: $\dfrac{dy}{dx} = \dfrac{du}{dv}\dfrac{dv}{dx}$. we get

$$\frac{dy}{dx} = -40u^{-41}(4x^3 + 4x) = \frac{-40(4x^3 + 4x)}{(x^4 + 2x^2 + 8)^{41}}$$

Often we apply the **Chain Rule** to "functions of functions of functions": if $y(x) = t\Big(u\big(v(x)\big)\Big)$, then

$$\frac{dy}{dx} = \frac{dt}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$ when $y = \sin^4(x^5 + 7)$.

Example

Find the derivative of $y = x^2\, e^{\sin(x)}$

Inverse functions

Suppose that $y = f(x)$. That is, f maps x to y.

Then the **inverse** of f is the function, f^{-1}, that maps y back to x.

Example

- The inverse of $f(x) = \dfrac{1}{2}x$ is $f^{-1}(x) = 2x$.
- The inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$.

Warning: $f^{-1}(x)$ is not the same as $\dfrac{1}{f(x)}$.

It is often useful to be able to express the derivative (assuming there is one) of an inverse function $f^{-1}(x)$ in terms of the derivative of $f(x)$.

To do this, we use the following rule:

Inverse-Function Rule

If $y = f^{-1}(x)$, then $x = f(y)$ and also

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}.$$

Alternatively: If f and f^{-1} are inverse and differentiable, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example

If $y = x^{1/3}$, use the Inverse Rule to find $\frac{dy}{dx}$.

Note: We can solve this just using the **Power Rule:**
$\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$. So we are just doing this because it is *instructive* If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2 .$$

By the inverse rule, $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2} .$$

As $y = x^{\frac{1}{3}}$ we have

$$\frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}} .$$

Example

Find the derivative of $\sin^{-1}(x)$

Let $y = \sin^{-1}(x)$, then $x = \sin(y) \quad (\star)$, so

$$\frac{dx}{dy} = \cos(y)\,. \qquad (\star\star)$$

From $\sin^2(y) + \cos^2(y) = 1$, we find $\cos(y) = \sqrt{1 - \sin^2(y)}$ (choosing the positive square root as $\cos(y)$ is positive for y here). Using $(\star)$:

$$\cos y = \sqrt{1 - x^2}\,.$$

Now using the inverse rule and $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}}\,.$$

Implicit differentiation

When $y = f(x)$, we say that y **is explicitly defined**. E.g., $y = \sqrt{1 - x^2}$.

Often times, however, we are given an equation involving x and y where these two terms are not "separated" entirely; e.g, $x^2 + y^2 = 1$. Here y is **implicitly** defined.

The tool of **implicit differentiation** allows us to, say, find tangents to these curves.

Method:

1. Differentiate both size of the equation, wrt x. keeping in mind that y is a function of x, using the Chain Rule where needed.
2. Solve for dy/dx.

Implicit differentiation

If y is defined by $x^2 + y^2 = 1$, find $\dfrac{dy}{dx}$.

Implicit differentiation

Find the tangent to the curve $x^2+y^2=25$, at the point $(3,-4)$.

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$.