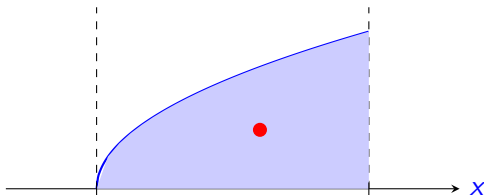


Week 11, Lecture 1  
**Centres of Mass: two-dimensions**

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# The centre of attention today:


- 1 News!
- 2 Centre of Mass: recall
  - Point Masses
  - Variable Density
  - A note on terminology
- 3 Centre of Mass: 2D
  - A lamina
  - Moments
  - Centre of Mass
  - Complex regions
- 4 Solids of Revolution
- 5 Exercises

For more, read Section 6.6 (Moments and Centres of Mass) of **Calculus** by Strang & Herman:

[math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)).

## Tutorials

There is a change to the tutorial plan for this week:

- ▶ Tuesday at 3pm: Teams 1 and 2 will have their in ENG-2003 (JM). Teams 3, 4, 5, and 6 will have a tutorial with MY243 (ST).
- ▶ Thursday at 11am: Teams 9 and 10 attend a tutorial in ENG-3035. Teams 7, 8, 11, and 12's tutorial is Aras Moyola MY129.
- ▶ Tutorials will cover the **Practice Exam Paper**  (link)
- ▶ Friday: no MA140 tutorials!
- ▶ No change to the Irish tutorial.

For more on the final exam, and practice paper: come to class tomorrow and Thursday!

Last week, we learned that if we have a mass-less rod, onto which are attached point masses  $m_1, m_2, \dots, m_n$ , at points  $x_1, x_2, \dots, x_n$ , then the

- ▶ The **moment** of the system is  $M = x_1 m_1 + x_2 m_2 + \dots + x_n m_n$ .
- ▶ The **total mass** is  $m = m_1 + m_2 + \dots + m_n$ .
- ▶ The **centre of mass** is  $\bar{x} = \frac{M}{m}$ .

If we have a rod (which has mass) placed on the  $x$ -axis, with endpoints  $x = a$  and  $x = b$ , with  $a < b$ , and the density of the rod is  $\rho(x)$ , then

- ▶ The moment of the system is  $M = \int_a^b x\rho(x) dx$ .
- ▶ The total mass is  $m = \int_a^b \rho(x) dx$ .
- ▶ The **centre of mass** of the rod is  $\bar{x} = \frac{M}{m} = \frac{\int_a^b x\rho(x) dx}{\int_a^b \rho(x) dx}$ .

If you read up on these topics, you'll often find the terms “**Centroid**” and “**Centre of Mass**” used interchangeably, as though they mean the same thing.

The don't, but are very closely related.

- ▶ A region of space has a **centroid**, also called the **geometric centre**.
- ▶ A lamina is a thin plate: it has a **centre of mass** (point at which it could be balanced on the head of a pin).
- ▶ If the lamina has constant density, and its shape is a region in space, then **centroid** and **centre of mass** are the same.

A **lamina** is a very, very thin plate whose shape is the region in space bounded above by  $y = f(x) > 0$ , below by  $y = 0$ , and left by  $x = a$ , and right by  $y = b$ . For us, it will always have uniform density.

We want to find its **centre of mass** (also called a “centroid”, in the case where we have uniform density), which we denote  $(\bar{x}, \bar{y})$ .

- ▶ The total mass is  $m = \int_a^b f(x) dx$ .
- ▶  $M_x$ , the moment about the  $x$ -axis, is  $M_x = \frac{1}{2} \int_a^b (f(x))^2 dx$ .
- ▶  $M_y$ , the moment about the  $y$ -axis is  $M_y = \int_a^b xf(x) dx$ .

The explanation for the formula for  $M_x$  and  $M_y$  follows...

As in 1D, the key idea we need is that of a **moment**. In a realistic setting, this is the **mass** of the lamina, times its distance from a reference point: usually  $(0, 0)$ .

To start with, it is helpful to think of the moments (in  $x$  and  $y$ ) of a thin rectangle:



Now let's get  $M_x$ , which is the moment about the  $x$ -axis, by summing the moments of all the rectangles, and taking the limit of the resulting Riemann sum:

$$M_x = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{2} (f(x_i^*))^2 \Delta x = \frac{1}{2} \int_a^b (f(x)) dx.$$

Similarly, we get  $M_y$ , which is the moment about the  $y$ -axis as

$$M_y = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_i^* f(x_i^*) \Delta x = \int_a^b x f(x) dx.$$

If the centre of mass is the point  $(\bar{x}, \bar{y})$ , then we could think of the entire “area” as being centred there, but having the same moments.

That is

$$\bar{x}A = M_y, \quad \text{and} \quad \bar{y}A = M_x.$$

giving...

### Centroid of a planar region

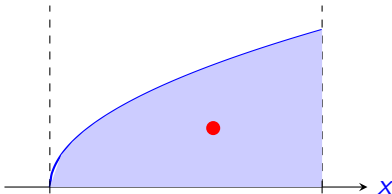
If  $f(x)$  is defined on  $[a, b]$ , then the **centroid**  $(\bar{x}, \bar{y})$  of the region enclosed by the curves  $y = f(x)$ ,  $y = 0$  and the lines  $x = a$  and  $x = b$  is given by

$$\bar{x} = \frac{\int_a^b xf(x) dx}{\int_a^b f(x) dx} \quad \text{and} \quad \bar{y} = \frac{1}{2} \frac{\int_a^b [f(x)]^2 dx}{\int_a^b f(x) dx}$$

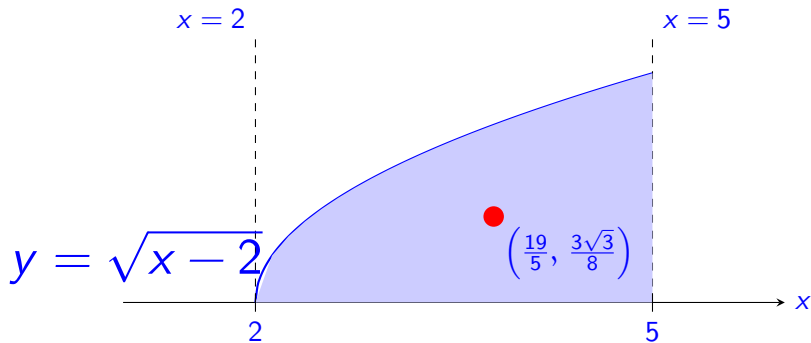
**Example**

Consider the plane region enclosed by the curve  $y = \sqrt{x-2}$ , the  $x$ -axis and the lines  $x = 2$  and  $x = 5$ . Find

- (1) the area of the region;
- (2) the centroid of the region.







The idea can be extended to more complex regions in space, such as the region bounded by two curves,  $f(x)$  and  $g(x)$ . We don't do the derivation here (it is in the textbook).

### Centroid of a planar region bounded by two functions

Take functions  $f(x)$  and  $g(x)$  defined on  $[a, b]$ , when  $f(x) \geq g(x)$ . Consider the region between  $f(x)$  and  $g(x)$ , and between  $x = a$  and  $x = b$ . Its **centroid**,  $(\bar{x}, \bar{y})$ , is given by

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b f(x) - g(x) dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2} \int_a^b f(x)^2 - g(x)^2 dx}{\int_a^b f(x) - g(x) dx}$$

**Example**

Find the centroid of the region between  $f(x) = x$ ,  $g(x) = -x$ ,  $a = 0$  and  $b = 1$ .



## Solids of Revolution

We'll finish this section by considering how to find the centre of mass of a solid of revolution.

Suppose that  $f(x) \geq 0$  on  $[a, b]$  and consider the region enclosed by the curves  $y = f(x)$ ,  $y = 0$  and the lines  $x = a$  and  $x = b$ .

Recall that we can rotate this region about the  $x$ -axis to obtain a solid of revolution.

Intuitively, it is clear that the centroid of such a solid should lie on the  $x$ -axis because of symmetry, so  $\bar{y} = 0$ . So, we only need find  $\bar{x}$ .

# Solids of Revolution

If the solid has uniform density,  $\rho(x, y) \equiv 1$ , then the total mass is the same as the volume.

We know already (Disk Method) that the volume of this region is

$$V = \pi \int_a^b f(x)^2 dx.$$

# Solids of Revolution

To get the moment about the  $y$ -axis, we consider the moment of an individual disk of volume  $\Delta V_r$ , at the point  $x = x_r$ , which is  $x_r \Delta V_r$ . If the solid is divided into  $N$  such rings:

$$M_y \approx \sum_{r=1}^n x_r \Delta V_r = \sum_{r=1}^n x_r (\pi f(x_r)^2 \Delta x)$$

Then, as we have seen repeatedly:

$$M_y = \lim_{n \rightarrow \infty} \sum_{r=1}^n x_r (\pi f(x_r)^2 \Delta x) = \pi \int_a^b x f(x)^2 dx$$

# Solids of Revolution

Putting all this together, and using that  $M_y = V\bar{x}$ , we get...

## Centroid of a solid of revolution

If  $f(x) \geq 0$  on  $[a, b]$ , then the **centroid**,  $(\bar{x}, \bar{y})$  of the solid of revolution obtained by rotating the region enclosed by the curves  $y = f(x)$ ,  $y = 0$  and the lines  $x = a$  and  $x = b$  about the  $x$ -axis is

$$\bar{x} = \frac{M_y}{V} \quad \text{and} \quad \bar{y} = 0.$$

where

$$M_y = \pi \int_a^b x f(x)^2 dx \quad \text{and} \quad V = \pi \int_a^b f(x)^2 dx.$$

# Solids of Revolution

## Example

Consider the plane region enclosed by the curve  $y = \sqrt{x-2}$ , the  $x$ -axis and the lines  $x = 2$  and  $x = 5$ . Find the centroid of the solid of revolution obtained by rotating this region about the  $x$ -axis.

# Solids of Revolution

## Exercises

### Exer 11.1.1

Find the centroid of the region between  $y = 1/x$ ,  $y = 0$ ,  $x = 1$  and  $x = 2$ .

### Exer 11.1.2

Find the centroid of the region between  $f(x) = x^2$ ,  $g(x) = -x$ ,  $x = 0$ , and  $x = 1$ .