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CS4423-Networks: Lecture 9 [$\color{red}{\text{Draft}}$]

Week 5, Lecture 2: Trees and Algorithms

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This Jupyter notebook, and PDF and HTML versions, can be found at
https://www.niallmadden.ie/2425-CS4423/#Week05

This notebook was written by Niall Madden, adapted from notebooks by Angela Carnevale.

Modules for this notebook

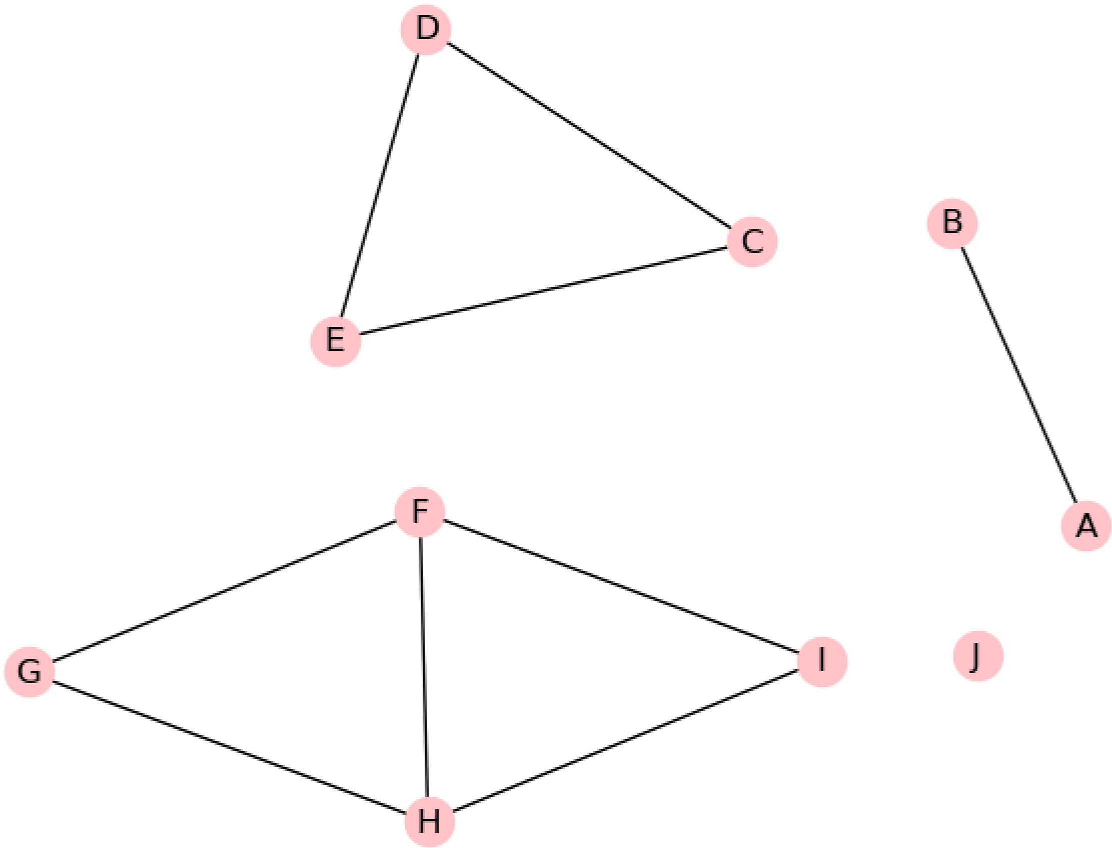
Today, we'll default to light rose-coloured nodes, with has an RGB code of `#ffc5cb`. For more options, see https://matplotlib.org/stable/users/explain/colors/colors.html

In [1]:
```
import networkx as nx
import numpy as np
opts = { "with_labels": True,  "node_color": '#ffc5cb' } # show labels; rose noodes
```

Example

Short discussion (again) of paths and cycles, and connected componets

In [2]:
```
nodes = 'ABCDEFGHIJ'
edges = ['AB', 'CD', 'DE', 'CE', 'FG', 'FH', 'FI', 'GH', 'HI']
G2 = nx.Graph()
G2.add_nodes_from(nodes)
G2.add_edges_from(edges)
nx.draw_kamada_kawai(G2, **opts)
```



- A cycle in a simple graph provides, for any two nodes on that cycle, (at least) two different paths from one to the other.
- It can be useful to provide alternative routes for connectivity in case one of the edges should fail (e.g. in a electricity networks).
- (C, D, E, C) is a 3-cycle; there are others too.
- The graph is not connected: there are 4 connected components.

Trees

- A graph is called **acyclic** if it does not contain any cycles.
- A **tree** is a (simple) graph that is **connected** and **acyclic**.

In other words, between any two vertices in a tree there is **exactly one simple path**.

Trees can be characterized in many different ways.

Theorem. Let $G = (X, E)$ be a (simple) graph of order $n = |X|$ and size $m = |E|$. Then the following are equivalent:

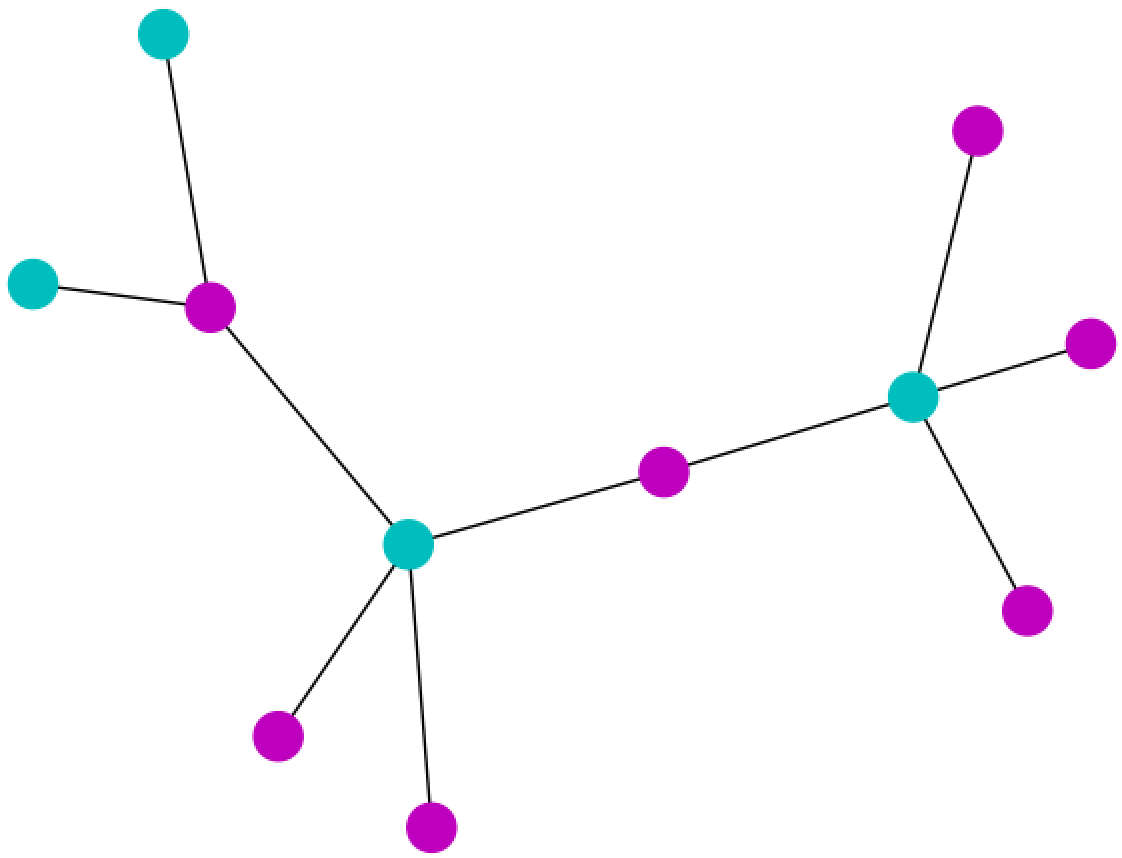
- G is a tree (i.e. acyclic and connected);
- G is connected and $m = n - 1$;
- G is a minimally connected graph (i.e., removing any edge will disconnect G);
- G is acyclic and $m = n - 1$;

- G is a maximally acyclic graph (i.e., adding any edge will introduce a cycle in G).
- There is a unique path between each pair of nodes in G.

Another fact about trees

All trees are bipartite. There are a few ways of thinking about this. One is the a graph is bipartite if has no cycles of odd-length. Since a tree has no cycles - it must be bipartite!

In [3]:
```
G3 = nx.Graph(["ac","bc","cd","de", "df", "dg","gh", "hi", "hj", "hk"])
top,bottom = nx.bipartite.sets(G3)
G3_colours = ['c' if node in top else 'm' for node in G3.nodes()]
nx.draw(G3, node_color=G3_colours)
```



How many trees are there?

1. There is one tree with a single node.
2. There is also just one tree with two nodes.
3. We can easily see that there are 3 trees with 3 nodes (see notes on the board).
4. After that, it gets a little harder to count!

Cayley's Formula

Theorem (Cayley's Formula). There are exactly n^{n-2} distinct (labelled) trees on the n-element vertex set $X = \{0, 1, 2, \ldots, n-1\}$, if $n > 1$.

We'll later see why this is true. But let's see what the numbers look like:

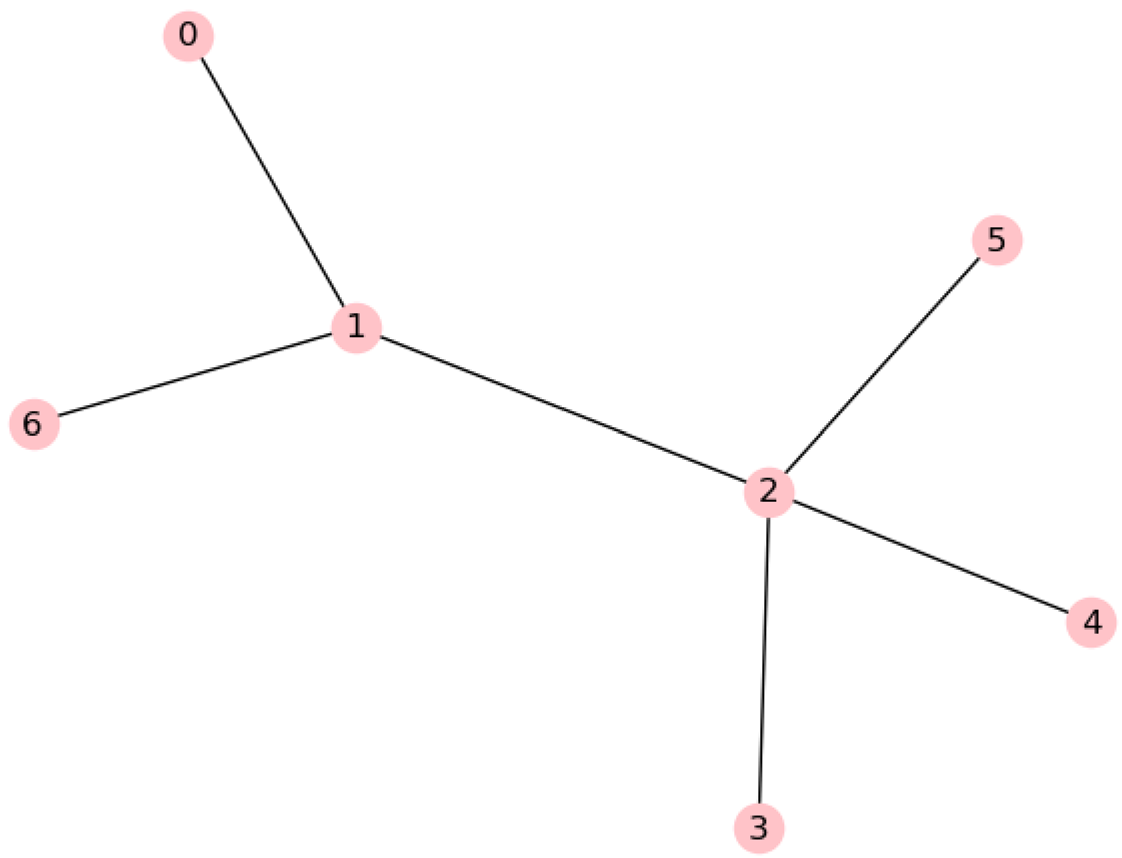
In [4]:
```
domain = range(2, 10)
print(np.array([domain, [n**(n-2) for n in domain]]))
```

```
[[       2       3       4       5       6       7       8       9]
 [       1       3      16     125    1296   16807  262144 4782969]]
```

To see why this is true, we'll learn about **Prüfer Codes**.

Let's look at an example: a tree of order $n = 7$

In [5]:
```
T4 = nx.Graph()
T4.add_nodes_from(range(0,7))
T4.add_edges_from([(0,1),(1,2),(2,3),(2,4),(2,5),(1,6)])
nx.draw(T4, **opts)
```



Computing the Prüfer code

How to determine the Prüfer code of a tree T (destructively):

- Start with a tree, T with nodes labeled $0, 1, \ldots, n-1$, and empty list $\mathbf{a}$.

1. Find the **leaf** x with the smallest label (a "*leaf*" is a node of degree 1. Every tree must have at least 2).
2. Append the label of its unique neighbour, y to the list $\mathbf{a}$
3. Remove x (and the edge $x - y$) from T.
4. Repeat Steps 1-3 until T has only 2 nodes left. We now have the code as a list of length $n-2$.

So the graph above has Pruefer code $\{1, 2, 2, 2, 1\}$

We'll write some code to compute the Prufer code of a tree.

Since the algorithm is recursive, we first write a function that does Steps 1-3:

- Find the **leaf** `x` with the smallest label
- Set `y` to be its neighbour.
- Delete `x` from `T`
- Return `y`

One of the steps involves finding the neighbour of x. A minor technical issue is that the method `T.neighbours(x)` returns a iterable. To get its one and only item, we'll use the `next()` function (there are a few other ways to do this, including converting it to a list)`.

In [6]:
```
def pruefer_node(tree):
    for x in tree:   # go through nodes in order
        if tree.degree(x) == 1:  # first one of degree 1
            y = next(tree.neighbors(x))  # y is its only neighbour
            tree.remove_node(x)
            return(y)
```

Since our function destroys the list, we'll make a copy before we start. Also, since we know the list has length $n-2$, we just call this function $n-2$ times, adding the value returned to the list:

In [7]:
```
n = T4.order()
T = T4.copy()
a = [] # empty list
for k in range(n-2):
    y = pruefer_node(T)
    a+=[y]
print(a)
```

```
[1, 2, 2, 2, 1]
```

If you prefer list comprehension:

In [8]:
```
T = T4.copy()
a = [pruefer_node(T) for k in range(n-2)]
print(a)
```

```
[1, 2, 2, 2, 1]
```

Let's wrap this up as a `python` function

In [9]:
```
def pruefer_code(tree):
    return [pruefer_node(tree) for k in range(tree.order() - 2)]
```

Test it:

In [10]:
```
T = T4.copy()
code = pruefer_code(T)
code
```

Out[10]:
```
[1, 2, 2, 2, 1]
```

Making a tree from a Prüfer code

Maybe surprisingly, the tree can be reconstructed from its Prüfer code. This is based on the following fact and shows that the map from trees to codes is a bijection!

Fact: The degree of node x is 1 plus the number of entries x in the Prüfer code of T.

Example

In [11]:
```
d = n*[1] # list of n 1's/
for k in code:
    d[k] += 1
print(f"degree list: {d}")
```

```
degree list: [1, 3, 4, 1, 1, 1, 1]
```

In [12]:
```
print(f'Check: {[T4.degree[x] for x in T4]}')
```

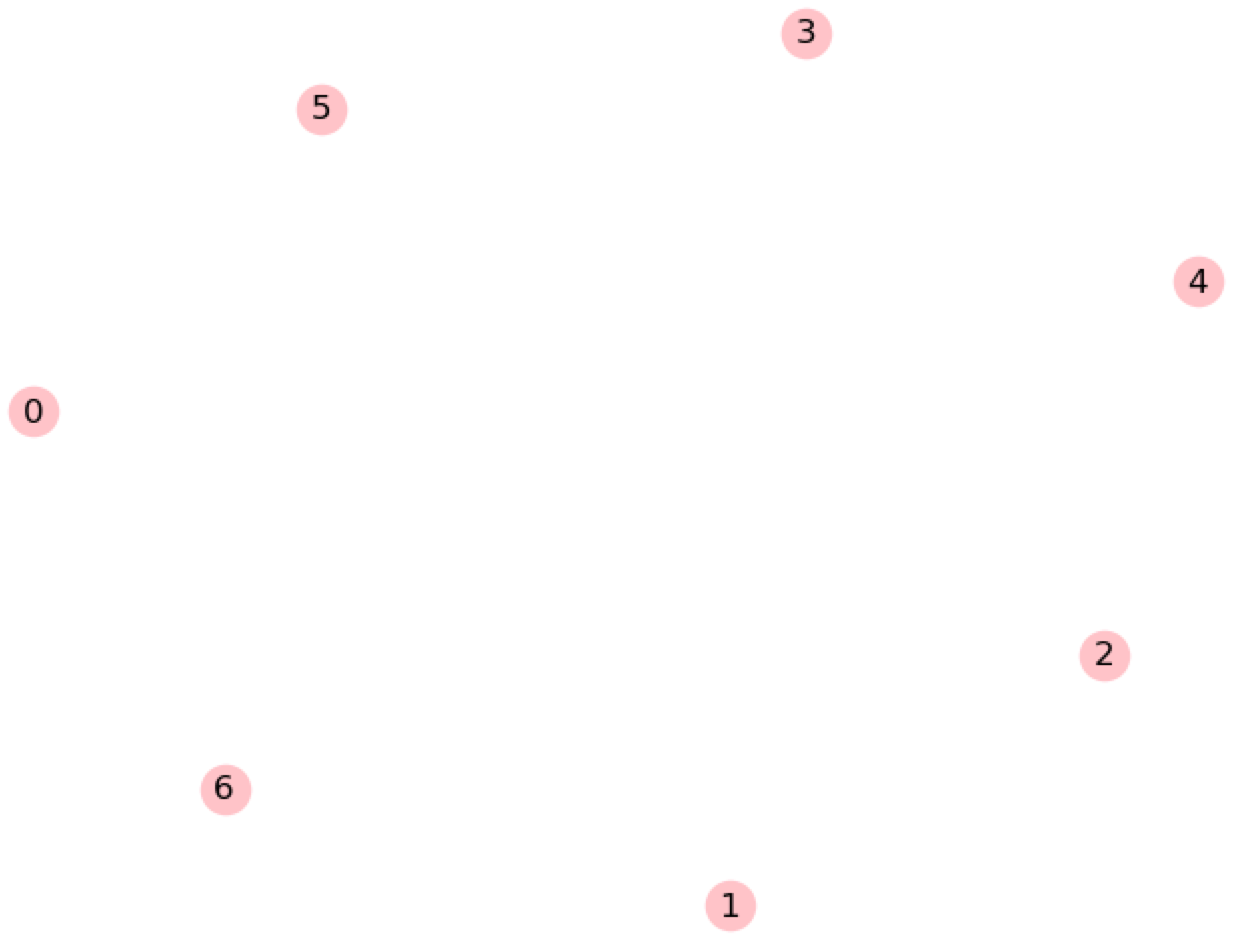
```
Check: [1, 3, 4, 1, 1, 1, 1]
```

How to compute a tree from a Prüfer code `a` (Note that `a` is a list of length $n-2$, with all entries numbers 0 to $n-1$).

1. Set `G` to be a graph with node list `[0, 1, 2, ..., n-1]` (and no edges yet).
2. Compute the list of node degrees `d` from the code.
3. For $k = 0, 1, \ldots n-2$
   - Set `y=a[k]`
   - Set `x` to be the node with smallest degree in `d`
   - Add the edge `(x,y)` to `G`
   - Set `d[x]=d[x]-1` and `d[y]=d[y]-1` (i.e., decrease the degrees of both `x` and `y` by 1).
4. Finally, connect the remaining 2 nodes of degrees 1 by an edge.

Tip: if `d` is a list, `d.index(1)` returns the index of the first entry of `d` that has the value `1`.

In [13]:
```
T4a = nx.empty_graph( T4.order() )
nx.draw(T4a, **opts)
```



In [14]:
```
code
```

Out[14]:
```
[1, 2, 2, 2, 1]
```

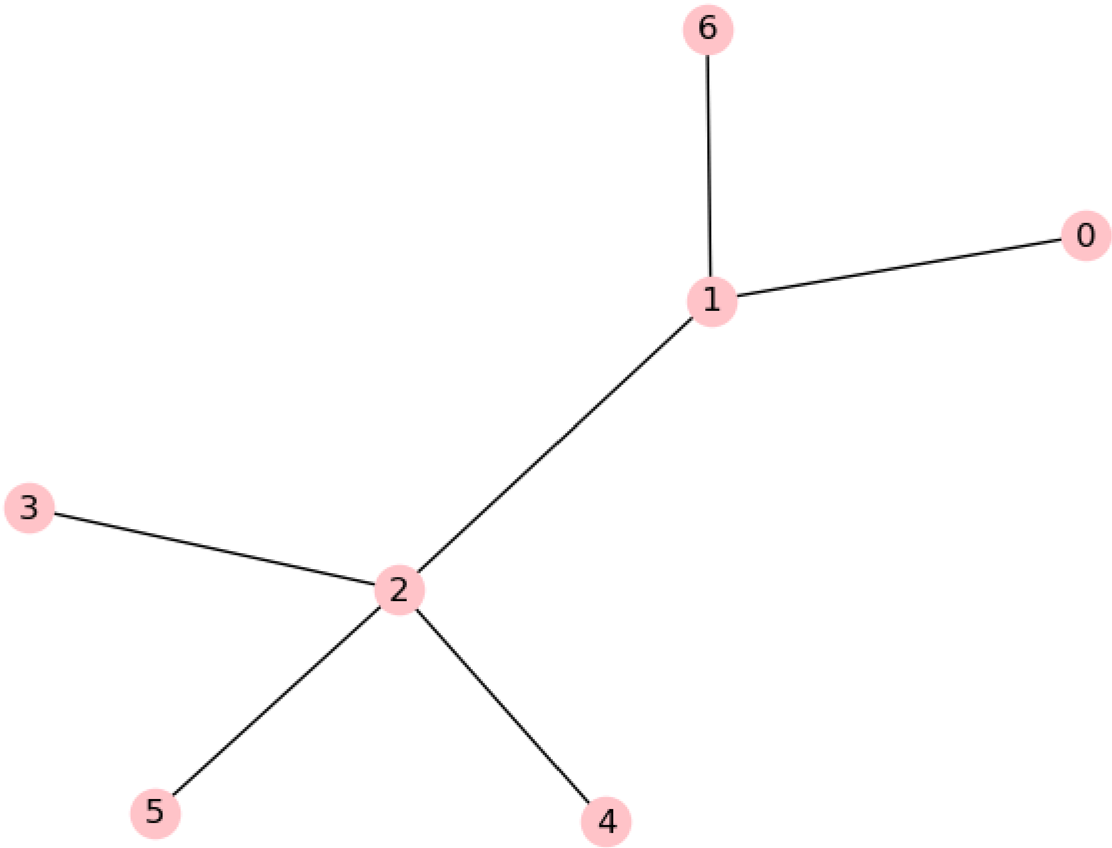
In [15]:
```
d = n*[1] # list of n 1's/
for k in code:
    d[k] += 1
# repeat n-2 times:
for k in range(n-2):
    y = a[k]
    x = d.index(1) # firsty
    T4a.add_edge(x, y)
    d[x] -= 1;  d[y] -= 1
    print(f'Degrees = {d} : adding edge {x}-{y}')
```

```
Degrees = [0, 2, 4, 1, 1, 1, 1] : adding edge 0-1
Degrees = [0, 2, 3, 0, 1, 1, 1] : adding edge 3-2
Degrees = [0, 2, 2, 0, 0, 1, 1] : adding edge 4-2
Degrees = [0, 2, 1, 0, 0, 0, 1] : adding edge 5-2
Degrees = [0, 1, 0, 0, 0, 0, 1] : adding edge 2-1
```

Add the final edge, by find the index to the remaining two 1's. We can find the first with `x=d.index(1)`, and the second with `y=d.index(1, x+1)` (could also use list comprehension, of course: see below).

In [16]:
```
x = d.index(1)
y = d.index(1, x+1)
T4a.add_edge(x,y)
```

In [17]:
```
nx.draw(T4a, **opts)
```



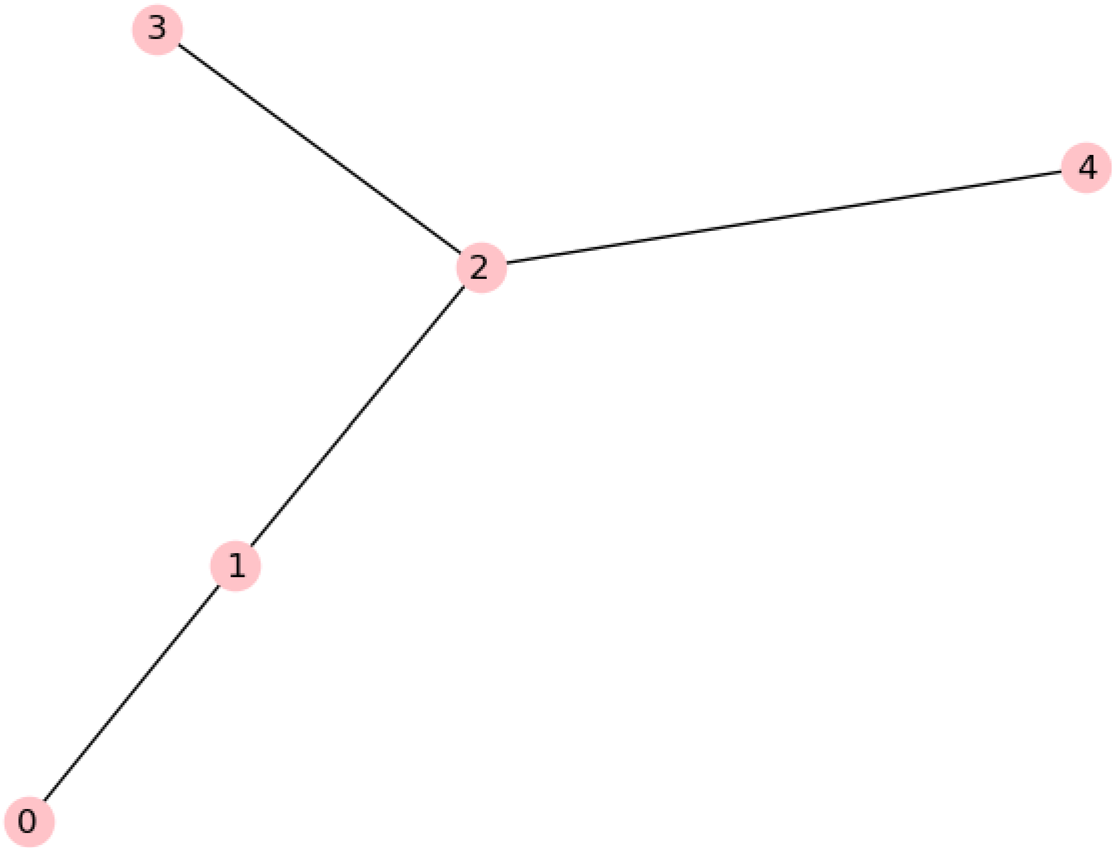
Turn the entire procedure into a `python` function:

In [18]:
```
def pruefer_to_tree(code):
    # initialize graph and defects
    n = len(code) + 2
    tree = nx.empty_graph(n)
    d = n*[1]
    for y in code:
        d[y] += 1

    # add edges
    for y in code:
        x = d.index(1)
        tree.add_edge(x, y)
        d[x]-=1; d[y]-=1;
    # final edge
    e = [x for x in tree if d[x] == 1]
    tree.add_edge(*e)
    return tree
```

Let's check it works:

In [19]:
```
T4b = pruefer_to_tree([1,2,2])
nx.draw(T4b, **opts)
```



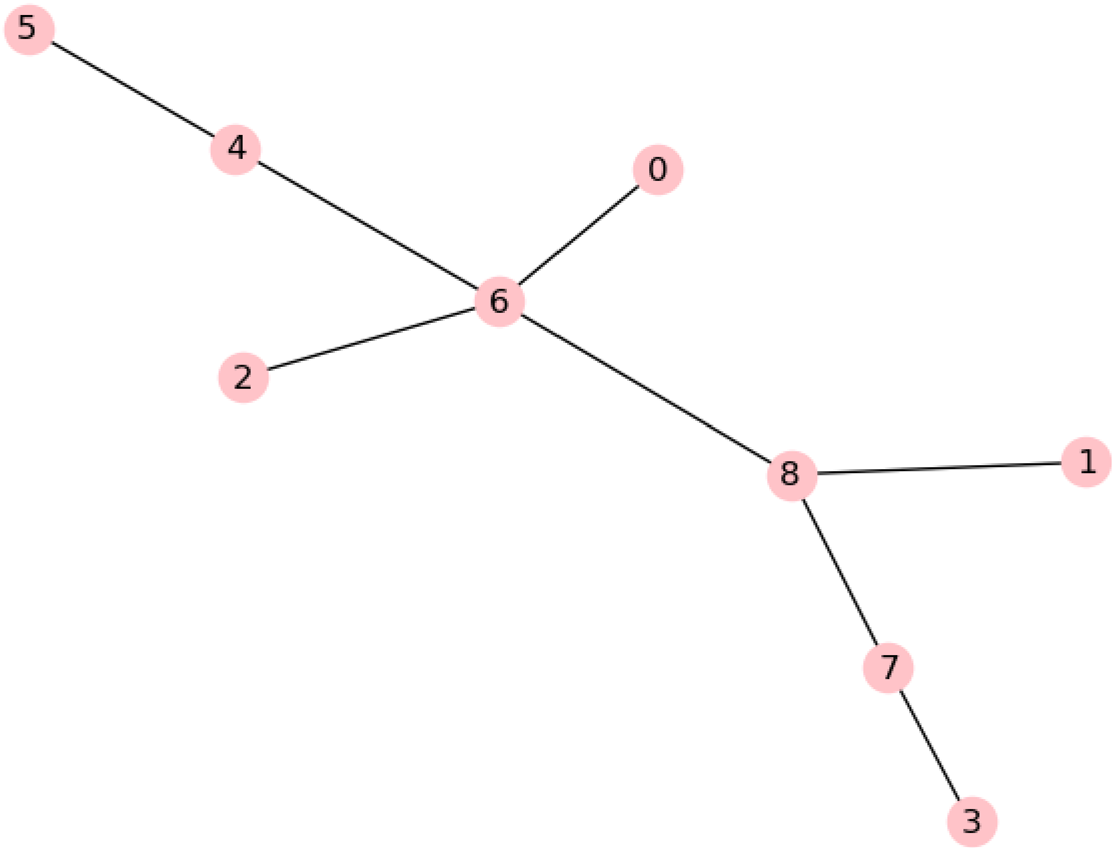
Since we have now sown that there is a bijection between labeled trees and Prüfer codes, we can prove Cayley's Theorem easily:

- A tree with n nodes has a Prüfer code of length $n-2$.
- There are n choices for each entry in the code.
- So there are n^{n-2} possible codes for a tree with n nodes
- So there are n^{n-2} possible trees with n nodes.

Random Trees

We can ask `networkx` to produce a **random tree** with a given number of nodes:

In [20]:
```
n = 8
T5 = nx.random_tree(9)
nx.draw(T5, **opts)
```

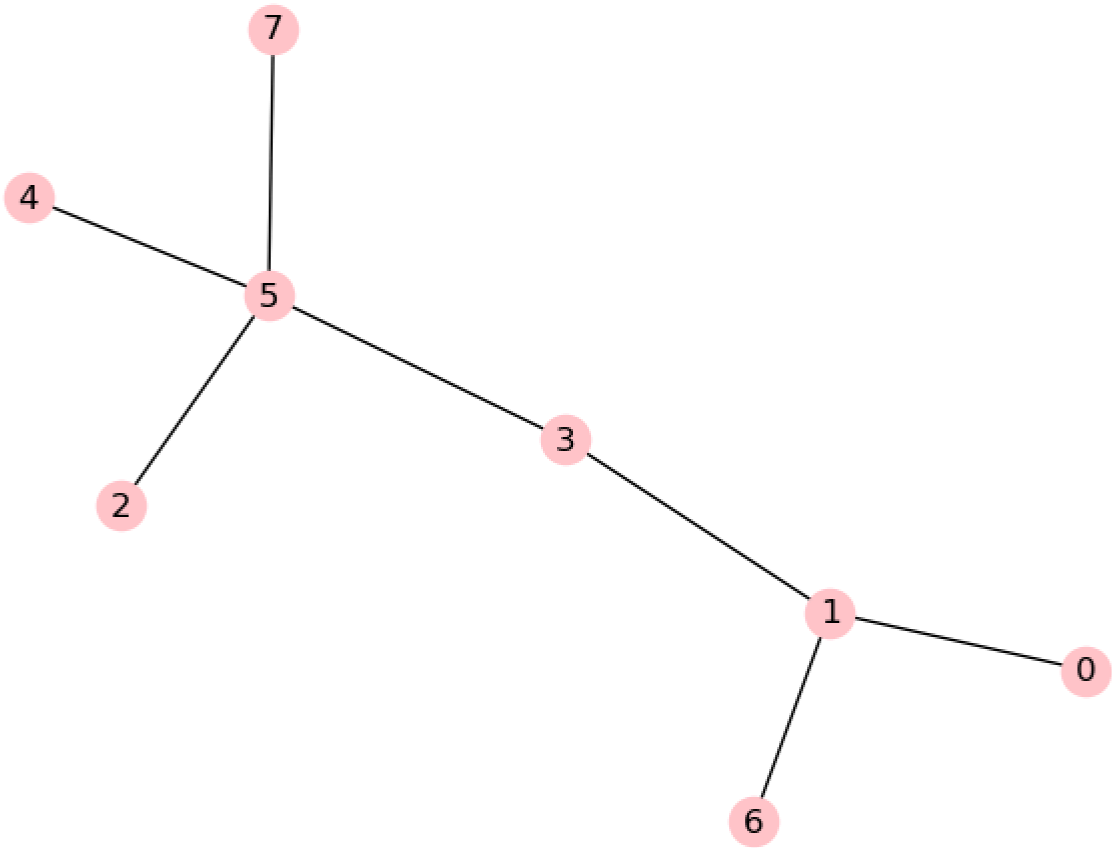


However, are can also construct a random tree on n nodes from a random Prüfer code of length $n-2$
.

In [21]:
```
code = np.random.randint(n, size=n-2)
print(f"code={code}")
```

```
code=[1 5 5 1 3 5]
```

In [22]:
```
T5a = pruefer_to_tree(code)
nx.draw(T5a, **opts)
```



Graph and Tree Traversal

Often one has to search through a network to check properties of nodes (e.g., finding the node with largest degree). For large unstructured networks, this could be challenging. Fortunately, there are simple and efficient algorithms:

- DFS
- BFS

Depth First Search (DFS)

DFS works by starting at a root node, and travelling as far along one of its branches as it can, then returning the the last unexplored branch.

The main data structure we'll need is a **stack**), also called a "*Last In First Out (LIFO) queue*". It has two operations:

- `S.push(x)` : pushes `x` onto the top of the stack (We'll use the `extend()` method)
- `y=S.pop()` : pops/removes the item from the top of the stack and stores it in `y

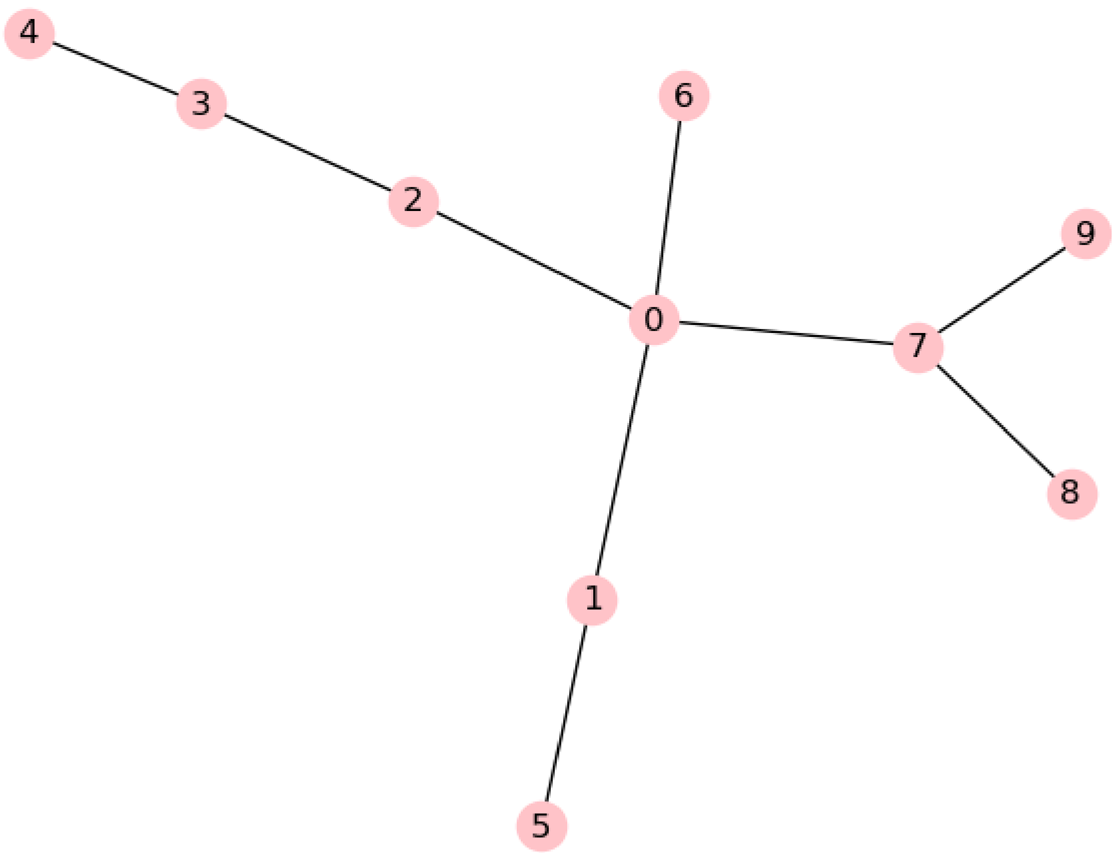
DFS: Given a rooted tree T with root x, visit all nodes in the tree. Start with an empty stack, `S` :

- `S.push(x)`
- while $S \neq \emptyset$:
  - `y = S.pop()`
  - visit(`y`)
  - `S.push(y.children)`

Let's create a tree to try this:

In [23]:
```
T6 = nx.Graph()
T6.add_nodes_from(range(10))
T6.add_edges_from([(0,1), (0,2), (2,3), (3,4), (1,5), (0,6),(0,7),(7,8),(7,9)])
nx.draw(T6, **opts)
print(f"Edges of T6 are {T6.edges()}")
```

```
Edges of T6 are [(0, 1), (0, 2), (0, 6), (0, 7), (1, 5), (2, 3), (3, 4), (7, 8), (7,
9)]
```



Now try the algorithm

In [24]:
```
T = T6.copy()
x = 0
S = [x]
while len(S) > 0:
    y = S.pop()
    S.extend(T[y])
    T.remove_node(y)
    print(y, S)
```

```
0 [1, 2, 6, 7]
7 [1, 2, 6, 8, 9]
9 [1, 2, 6, 8]
8 [1, 2, 6]
6 [1, 2]
2 [1, 3]
3 [1, 4]
4 [1]
1 [5]
5 []
```

Breadth First Search (BFS)

BFS works by starting at a root node, and explores all the neighbouring nodes ("Level 1") first. Next it searches their neighbours ("Level 2"), etc.

The main data structure we'll need is a **queue**, also called a "*First In First Out (FIFO) queue*". It has two operations:

- `Q.extend(l)` : adds the items in the list `l` to the end of `Q`
- `y=S.pop(0)` : pops/removes the *first* item from queue, and stores it in `y

BFS: Given a rooted tree T with root x, visit all nodes in the tree. Start with an empty list/queue, `Q` :

- `Q.push(x)`
- while $Q \neq \emptyset$:
  - `y = Q.pop(0)`
  - visit(y)
  - `Q.push(y.children)`

Let's test it:

In [25]:
```
T = T6.copy()
x = 0
Q = [x]
while len(Q) > 0:
    y = Q.pop(0)
    Q.extend(T[y])
    T.remove_node(y)
    print(y, Q)
```

```
0 [1, 2, 6, 7]
1 [2, 6, 7, 5]
2 [6, 7, 5, 3]
6 [7, 5, 3]
7 [5, 3, 8, 9]
5 [3, 8, 9]
3 [8, 9, 4]
8 [9, 4]
9 [4]
4 []
```

Many questions on networks concerning distance and connectivity can be answered by a versatile strategy involving a subgraph which is a tree, and then searching that. Such a tree is called a **spanning tree** of the underlying graph.

Alternative Implementations (Extra: won't do in class)

Both DFS and BFS are more like strategies, rather than specific algorithms. Different problems might require different implementations. Sometimes, the stack, or the queue don't have to be made explicit:

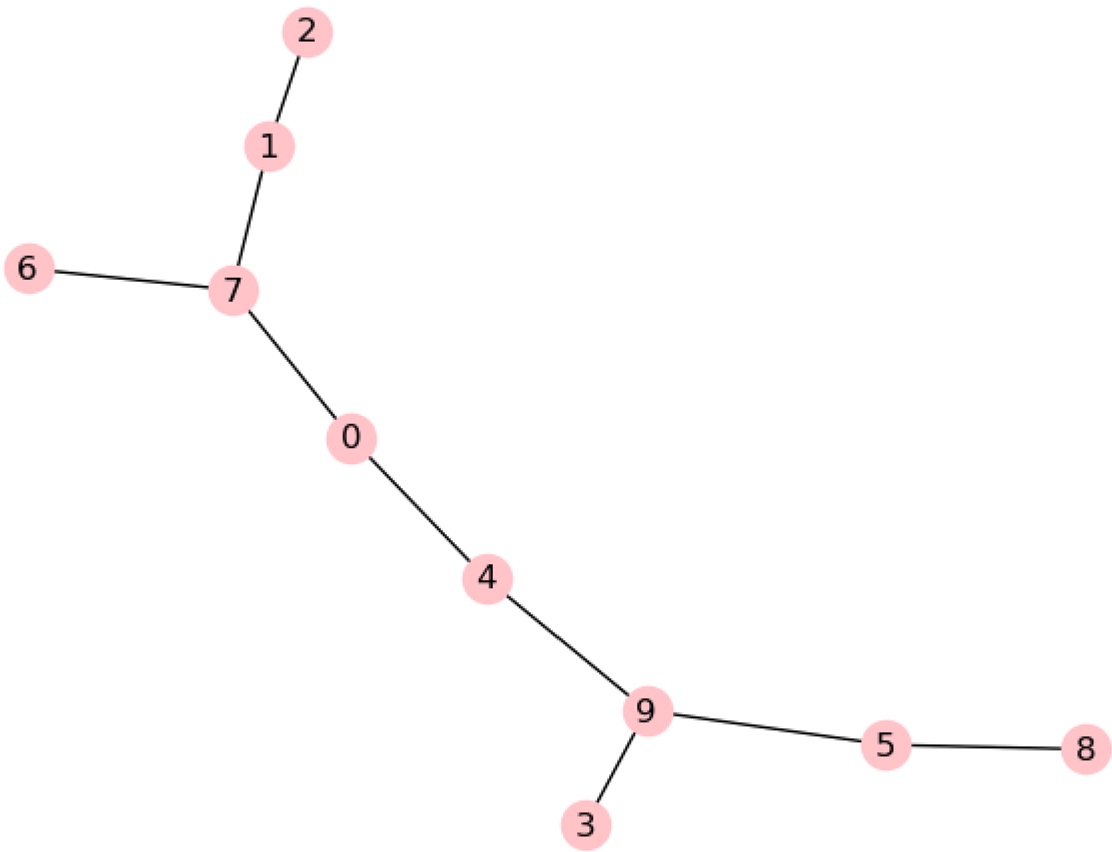
- In a recursive implementation, DFS can make use of the (`python`) interpreter's **function call stack**.

- BFS can take advantage of the fact that some types of lists in a (`python`) `for` loops are largely organized as **queues**.

In order to keep track of which nodes have already been visited, we maintain for each node an attribute `"seen"` that is initially `False` , and becomes `True` when the DFS/BFS visits the node.

In `networkx` , the attributes of a node `x` in a graph `G` are kept in a dictionary `G.nodes[x]` .

In [26]:
```python
n = 10
T6a = nx.random_tree(n)
nx.draw(T6a, **opts)
```



In [27]:
```python
TT = T6a.copy()
for x in TT:
    TT.nodes[x]['seen'] = False
TT.nodes('seen')
```

Out[27]:
```
NodeDataView({0: False, 1: False, 2: False, 3: False, 4: False, 5: False, 6: False,
7: False, 8: False, 9: False}, data='seen')
```

- DFS on a tree:

In [28]:
```python
def dfs(tree, x):
    print(x, end=', ')
    tree.nodes[x]['seen'] = True
    for z in tree[x]:
        if not tree.nodes[z]['seen']:
            dfs(tree, z)
```

In [29]:
```python
dfs(TT, 3)
```

```
3, 9, 4, 0, 7, 1, 2, 6, 5, 8,
```

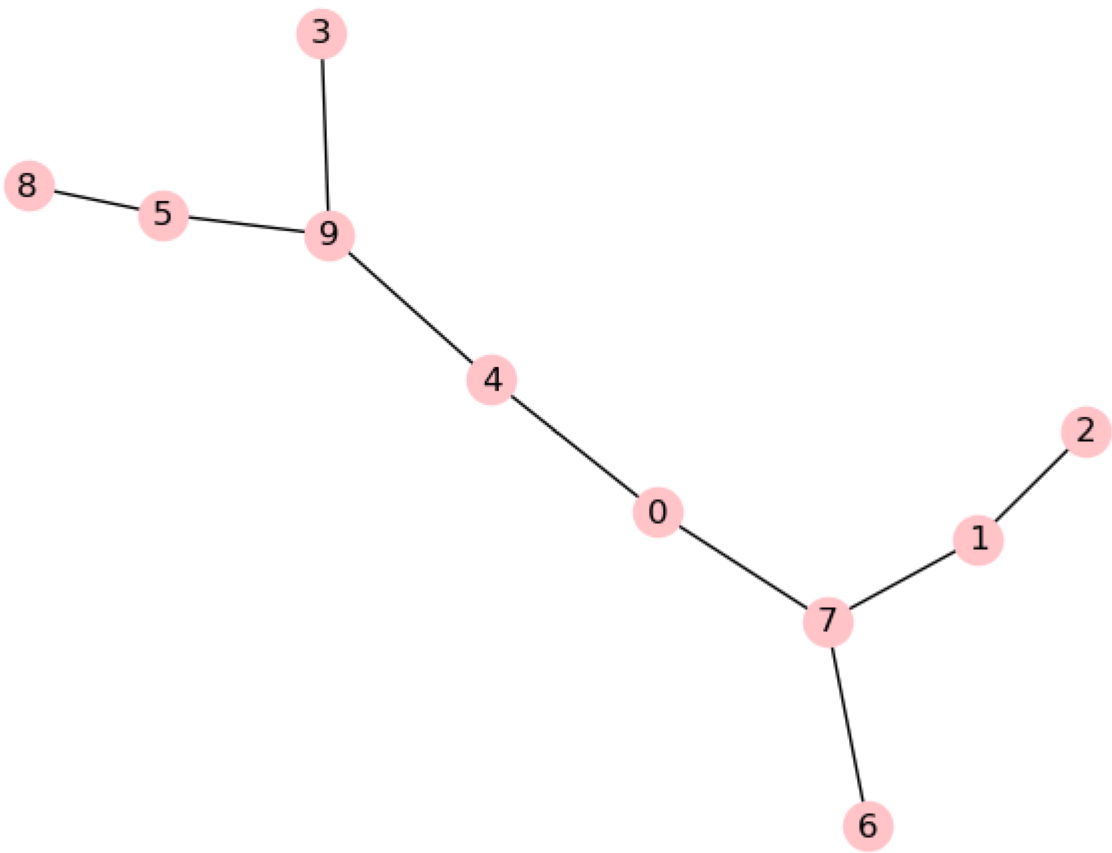
- BFS on a tree:

In [30]:
```python
TT = T6a.copy()
for x in TT:
    TT.nodes[x]['seen'] = False
```

In [31]:
```python
Q = [3]
TT.nodes[3]['seen'] = True
for y in Q:
    print(y, end=', ')
    for z in TT[y]:
        if not TT.nodes[z]['seen']:
            Q.append(z)
            TT.nodes[z]['seen'] = True
```

```
3, 9, 4, 5, 0, 8, 7, 1, 6, 2,
```

In [32]:
```python
nx.draw(TT, **opts)
```



Exercises

1. A tree T uniquely determines its Prüfer code, and hence the two nodes that remain after (destructively) computing the code. What are those two nodes, in terms of properties of T, or its Prüfer code?

2. A. What tree has Prüfer code $[0, 1, 2, \ldots, n-3]$?
   B. What tree has Prüfer code $[\underbrace{0, 0, 0, \ldots, 0}_{n-2 \text{ zeros}}]$?
   C. What tree has Prüfer code $[0, 1, 2, \ldots, n-3]$?
3. Give the Prüfer for the tree with nodes $\{0, 1, 2, 3, 4, 5\}$ and edges $0-1$, $0-2$, $1-3$, $1-4$, $2-5$