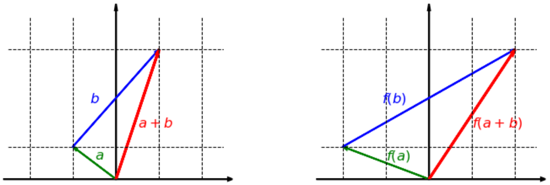
# Annotated slides from Tuesday

MA313 : Linear Algebra I

# Week 4: Spanning sets and column spaces

Dr Niall Madden

## 27 and 30 September, 2022




Adapted from https://commons.wikimedia.org/wiki/File:Streckung_der_Summe_zweier_Vektoren.gif

These slides are adapted (slightly) from ones by Tobias Rossmann.

# Outline



For more details, see Section 4.2 of the text-book:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=5174425

## Assignment 2

- Opened last Monday (19 Sep 2022).
- **Deadline:** 5pm, Friday 30 Sep 2022.
- It contributes 5% to the final grade for MA313.
- Tutorials continue Thursdays at 12 in IT206.

## Communication Skills

1. Topics and Info posted on Blackboard and at https://www.niallmadden.ie/teaching/2223-MA313/22_23_Communication_Skills.pdf
2. Confirm your topic by 5pm, 26 September (Monday of Week 4). To that by first emailing Niall with your choice and, if agreed, entering in on Blackboard.

**MA313**
**Week 4: Spanning sets and column spaces**

***Start of ...***

# PART 1: Recall from last week

# Part 1: Recall from last week

## Linear combinations

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form $c_1 u_1 + \cdots + c_p u_p$ for scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

## Span

The **span** of a set of vectors is the set of all possible liner combinations of them. That is, given vectors $u_1, \ldots, u_p$ in some vector space $V$, their **span** is

$$\mathrm{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

# Part 1: Recall from last week

## Subspaces

Given any set of a vectors in a vector space $V$, their span is a **subspace** of $V$.

## Null space

Given a $m \times n$ matrix, $A$, its **null space** is the set of all vectors for which $Ax = 0$. That is:

$$\operatorname{Nul} A = \{x \in \mathbb{R}^n : Ax = 0\}.$$

- For some matrices, the only vector in the null space is the zero vector.
- The null space of an $m \times n$ matrix is itself a vector space (and so a subspace of $\mathbb{R}^N$).

**MA313**
**Week 4: Spanning sets and column spaces**

***Start of ...***

# PART 2: Spanning Sets

# Part 2: Spanning Sets

## Definition (Spanning Set)

A **spanning set** of a vector space $V$ is a collection of vectors in $V$ whose span is all of $V$.

Equivalently, the set of vectors $\{v_1, \ldots, v_p\}$ in $V$ form a spanning set if and only if every vector in $V$ can be written as a linear combination of $v_1, \ldots, v_p$.

**Example (A spanning set for $\mathbb{R}^2$)**

The vectors

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

form a spanning set of $\mathbb{R}^2$.

$\mathbb{R}^2$ is the space of all vectors of the form $\begin{bmatrix}a\\b\end{bmatrix}$ where $a, b \in \mathbb{R}$.

These can be written as $a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

So they are a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

## Example (A spanning set for $\mathbb{R}^n$)

In the same way, for each $n \geqslant 1$, the vectors

$$\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \in \mathbb{R}^n$$

form a spanning set of $\mathbb{R}^n$.

Eg $n=3$ : set is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix},$

$v$

Recall that $\mathbb{P}_n$ is the vector space of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

of degree $n$ or less.

**Example**

$\mathbb{P}_n = \mathrm{span}\{1, t, \ldots, t^n\}$.

Recall: $M_{m\times n}$ is the vector space of all $m \times n$ matrices.

**Example**

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Any matrix in $M_{2\times 2}$ is of the form

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a, b, c, d \in \mathbb{R}$.

And

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So this is a spanning set for $M_{2\times 2}$.

**Important:** Spanning sets are (in general) not unique.

## Example (Another spanning set of $M_{2\times 2}$)

We also, have for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{u_3}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{u_4} \right\}.$$

We can write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$a\left(\tfrac{1}{2}u_1 + \tfrac{1}{2}u_4\right) + b u_2 + c u_3 + d\left(\tfrac{1}{2}u_1 - \tfrac{1}{2}u_4\right).$$

Note, eg, $\tfrac{1}{2}u_1 + \tfrac{1}{2}u_4 = \tfrac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$

**Important:** Spanning sets are (in general) not unique.

**Example (Another spanning set of $M_{2\times 2}$)**

We also, for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Note also

1. $M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

2. But $M_{2\times 2} \neq \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$

**Important:** Spanning sets are (in general) not unique.

**Example (Another spanning set of $M_{2\times 2}$)**

We also, for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Note also $V = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \neq M_{2\times 2}$,

since any matrix in $V$ is of the form $a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

**MA313**
**Week 4: Spanning sets and column spaces**

***Start of ...***

# PART 3: Column spaces

# Part 3: Column spaces

**Definition (COLUMN SPACE)**

Let $A = [a_1 \cdots a_n]$ be an $m \times n$ matrix, where $a_1, \ldots, a_n \in \mathbb{R}^m$. That is, $a_i$ is the $i$th column of $A$.

The **column space** of $A$ is

$$\mathrm{Col}\, A := \mathrm{span}\{a_1, \ldots, a_n\}.$$

Note that $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$.

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \end{bmatrix}$. then

$$\mathrm{Col}\,(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right).$$

Eg $\begin{bmatrix} 0 \\ -1 \end{bmatrix} \in \mathrm{col}\,(A)$ since $\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

**Example**

Let $I_n$ be the $n \times n$ identity matrix.
Then $\mathbb{R}^n = \mathrm{Col}\, I_n$.

Eg $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

And $\mathbb{R}^3 = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \mathrm{Col}\, I_3$

# Part 3: Column spaces

Here is another way of thinking about the column space: we have already seen that $Ax$ is a linear combination of the columns of $A$. So, ...

$$\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\}$$

and

$$\operatorname{Col} A = \{b \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \colon b = Ax\}.$$

$$I_3 x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Given a matrix $A$, we can construct two vector spaces:

### $\operatorname{Nul} A$

- Easy to test membership: does $x \in \mathbb{R}^n$ belong to $\operatorname{Nul} A$?
- Not as easy to produce a (finite) spanning set.

### $\operatorname{Col} A$

- Very easy to give a spanning set: it is how the space is defined!
- Not as easy to check to test membership.

**MA313**
**Week 4: Spanning sets and column spaces**

***Start of ...***

## **PART 4: Spanning sets of null spaces**

# Part 4: Spanning sets of $\operatorname{Nul} A$

## Question

Given an $m \times n$ matrix $A$, can we find a finite spanning set of $\operatorname{Nul} A$?

That is, can we find vectors $v_1, \ldots, v_p \in \mathbb{R}^n$ such that those vectors $x \in \mathbb{R}^n$ with $Ax = 0$ are precisely the linear combinations

$$c_1 v_1 + \cdots + c_p v_p,$$

where $c_1, \ldots, c_p \in \mathbb{R}$?

To see the answer, we'll recall that the $Ax = b$ is just another way of writing a linear system of equations.

"precisely" - includes all such vectors, but not anything else.

When we write

$$Ax = b$$

where $A$ is an $n \times n$ matrix, and $x, b \in \mathbb{R}^n$, we mean

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This is the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Remember that we used solve such systems using "row reduction" (a.k.a., Gaussian Elimination): we rearrange the equations to get

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ \hat{a}_{22}x_2 + \hat{a}_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \hat{a}_{33}x_3 + \cdots + \hat{a}_{3n}x_n &= b_2 \\ &\vdots \\ \hat{a}_{nn}x_n &= b_n \end{aligned}$$

This is done by so-called *elementary row operations*. And we do this because it is easy to solve this version.

## Elementary row operations

Performing an **elementary row operation** on a matrix means:

- Multiply some row by a non-zero scalar.
- Add a scalar multiple of some row to another row.
- Interchange (i.e., swap) two rows.

## Fact!

Let $A'$ be obtained from $A$ by performing an **elementary row operation**. The

$$\operatorname{Nul} A = \operatorname{Nul} A'.$$

## Definition (Row Echelon Form)

A matrix is in **row echelon form** if

- all non-zero rows are above all zero rows and
- the **leading entry** (or "*pivot*") in a row is in a column to the right of the leading entry in the row above it.
- All entries in a column below a leading entry are zero.

Eg
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

But not
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 7 & 0 & 8 \end{bmatrix}$$

Finished here Tuesday

### Definition (Reduced Row Echelon Form)

A matrix is in **reduced row echelon form** if it is in row echelon form, and also

- Each leading entry is one; ✓
- If a column contains a leading entry, all its other entries are zero.

**Theorem and Definition**

Using elementary row operations, *every* matrix $A$ can be row reduced to obtain a **unique** matrix $A'$ in reduced row echelon form. We call $A'$ **the reduced row echelon form** of $A$.

It turns out that we can read off a spanning set of $\mathrm{Nul}\, A$ from the reduced row echelon form of $A$.

## Example

Find a spanning set of $\operatorname{Nul} A$, where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

**MA313**
**Week 4: Spanning sets and column spaces**

***Start of ...***

# PART 5: Checking column space

# Part 5: Checking column space

**Question**

Given an $m \times n$ matrix $A$ and $b \in \mathbb{R}^m$, how can we decide if $b \in \operatorname{Col} A$?

Since $\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\}$, this problem is equivalent to deciding whether there exists a solution $x \in \mathbb{R}^n$ to the system of linear equations

$$Ax = b.$$

Again, **row reduction** (a.k.a. **Gaussian elimination**) can be used for this purpose.

# Part 5: Checking column space

**Example**

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$. Is $b \in \mathrm{Col}\, A$?

Details on board.... we see
x4=1/17
x3 is free, and then x2=-2 + 5x3 + 4/17.
Take x3=0, to get x2=-30/17.

Similarly, x1=5.

# Part 5: Checking column space

## Example (From 2018/19 exam paper)

Decide (with justification) if

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ belongs to the column space of } A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix}.$$

**Answer: No!**. Why? The RREF of

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ -1 & 3 & 5 & 4 & 2 \\ 2 & 1 & -3 & -1 & -1 \end{array}\right] \text{ is } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

So...

So now we know that, given an $m \times n$ matrix $A$, we can use **row reduction** to perform the following tasks:

- Construct a finite spanning set of $\mathrm{Nul}\, A$.
- Decide, for a given $b \in \mathbb{R}^m$, whether $b \in \mathrm{Col}\, A$.

But what has this to do with **vector spaces**?
Do these matrix computations (row reduction) and concepts (null spaces, column spaces) have analogues for general vector spaces?

Finished here Friday.