2323-MA378: Class Test in Week 9 (Wed, 28 Feb)

Some useful formulae.

- Cauchy's theorem: If p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \tag{1}$$

where $\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ denotes the nodal polynomial.

- $\|g\|_\infty$ denotes $\max_{a \le x \le b} |g(x)|$.)
- If l be the linear spline interpolant to a function f on the equally spaced points $a = x_0 < x_1 \cdots < x_N = b$ with $h = x_i - x_{i-1} = (b-a)/N$, then

$$\|f - l\|_\infty \le \frac{h^2}{8}\|f''\|_\infty, \tag{2}$$

- If S is the Piecewise Cubic Hermite Interpolating Polynomial that interpolates the function f at the equally spaced points $\{a = x_0 < x_1 < \cdots < x_N = b\}$ with $x_i - x_{i-1} = (b-a)/N =: h$, then

$$\|f - S\|_\infty := \max_{a \le x \le b} |f(x) - S(x)| \le \frac{h^4}{384} \max_{a \le x \le b} |f^{(iv)}(x)|. \tag{3}$$

In all the questions below, the function f is

$$\boxed{f(x) = e^{x/2}}$$

Q1. (50 marks)

(a) Write down the Lagrange form for the polynomial, $p_2(x)$, that interpolates f at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(b) Evaluate $p_2(1/2)$.

(c) What bound does (1) give for $|f(1/2) - p_2(1/2)|$?

Q2. (30 marks)

(a) Give a formula for the piecewise linear interpolant, $l(x)$, that interpolates f, at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(b) Evaluate $l(1/2)$.

(c) Use (2) to give an upper bound for $\|f(x) - l(x)\|_\infty$.

(d) What value of N would you have to choose so that $\|f - l\|_\infty \le 10^{-6}$?

Q3. (18 marks) Suppose that S is the **PCHIP** interpolant to the function f at the $N+1$ equally spaced points $\{x_0 = -1 < x_1 < \cdots < x_N = 1\}$. What value of N should one take to ensure that $\|f - S\|_\infty$ is no more than 10^{-6}?

Q4. (2 marks) Could there ever be a situation where, if we use the same values of f and N in (2) and (3), the error bound for the linear spline interpolant could be *less* than that PCHIP interpolant? If so, suggest an example.