2526-MA140 Engineering Calculus

Week 08, Lecture 1 (REVISED)
Techniques of Integration

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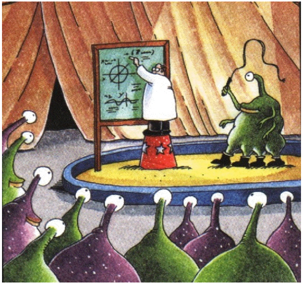
Tuesday, 04 November, 2025



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Assignments, etc

- **Problem Set 6** is open, and will be covered in tutorials this well. Deadline is 5pm next Monday (10 November).
- **Problem Set 7** opens this week.
- The finally weekly assignment, will open next week.
- Reminder: The second **class test** takes place November 18.

This morning, I think we will think about...

1. Substitution
   - Definite Integrals
2. Rational functions
   - Partial Fractions
3. Integration by Parts
   - When to use
   - The Rule
   - Why it works
   - Examples
   - Choosing u and dv
   - Repeated application
   - Easy example
4. Exercises

See also Section **5.5** (Substitution) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax) Maybe also Section **7.1** (Integration by Parts).

Substitution

Suppose we want to evaluate an integral of the form

$$\int e^{x^3+x^2}(3x^2+2x)\,\mathrm{d}x.$$

At first, this looks tricky: there is nothing like this in our table of integrals.

However, there is something a little unusual about it: it features both the function x^3+x^2, and its derivative $3x^2+2x$.

It turns out that such problems are quite common (at least in textbooks and on exams!). Moreover, there is a handy technique called **substitution** for evaluating them. In this case:

$u(x) = x^3 + x^2$

$u'(x) = 3x^2 + 2x$

$\frac{du}{dx} = 3x^2 + 2x$

$du = (3x^2+2x)\,dx$

$\int e^u (u')\,dx = \int e^u\,du$

Substitution

Method of substitution

When integrating an *integrand* of the form $\int f(g(x))g'(x)dx$, set $u = g(x)$, and then use that

$$\int f\big(g(x)\big)\, g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,.$$

Equivalently: $\boxed{\int f(u)\,\frac{du}{\mathrm{d}x}\mathrm{d}x} = \int f(u)\,\mathrm{d}u$. (For a proof of why this works, see Section 5.5 of the textbook).

After this substitution, our task is reduced to evaluating $\int f(u)\,\mathrm{d}u$ which, we hope, is easier.

Substitution

Example

Evaluate the integral $\int 3x^2 \sin(x^3)\, dx$. $= \int \sin(\underbrace{x^3}_{u}) \, (\underbrace{3x^2 dx}_{du})$

Notice that $3x^2$ is the derivative of x^3.

Let's try integration by substition with $u = x^3$.

If $u = x^3$, then $\frac{du}{dx} = 3x^2$, so

$$\boxed{du} = \frac{du}{dx}\, dx = \boxed{3x^2\, dx}.$$

Thus,

$$\begin{aligned}\int \sin(x^3) 3x^2\, dx &= \int \sin(u)\, du \\ &= -\cos(u) + C \\ &= -\cos(x^3) + C.\end{aligned}$$

Subs back in for x.

Substitution

Example

Evaluate $\int 2x\sqrt{1+x^2}\,dx.$ $= \int (1+x^2)^{1/2}\, 2x\, dx.$

Notice that $2x$ is the derivative of $1+x^2$.

Let's try integration by substition with $u = 1 + x^2$:

$$u = 1 + x^2 \qquad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2x\,dx.$$

So

$$\int (\underbrace{1+x^2}_{u})^{1/2}\,(\underbrace{2x\,dx}_{du}) = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$$

(by Power Rule)

$$= \frac{2}{3}(1+x^2)^{3/2} + C.$$

Example

Evaluate $\int \cos(4x - 7)\, dx$.

Idea: think of this as $\frac{1}{4}\int \cos(4x-7)4\, dx$.

$$\int \cos(4x-7)dx = \frac{1}{4}\int \cos(\underbrace{4x-7}_{u})\,(\underbrace{4dx}_{du})$$

$u = 4x-7 \qquad \frac{du}{dx} = 4 \quad \Rightarrow \quad du = 4\,dx.$

So

$$\int \cos(4x-7)dx = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u) + C$$

$$= \frac{1}{4}\sin(4x-7) + C$$

Example

Show that $\int \sin^3(x)\,\cos(x)\,\mathrm{d}x = \frac{1}{4}\sin^4(x) + C$.

$$I = \int [\underbrace{\sin(x)}_{u}]^3 \underbrace{\cos(x)\,dx}_{du}.$$

$u(x) = \sin(x) \qquad \frac{du}{dx} = \cos(x) \implies du = \cos(x)\,dx$

$$I = \int u^3\,du = \frac{1}{4}u^4 + C$$

$$= \frac{1}{4}(\sin(x))^4 + C.$$

Substitution can be used with **definite integrals**. However, this may requires a change to the limits of integration.

Substitution with Definite Integrals

Let $u = g(x)$, with g' continuous on $[a, b]$, and f continuous over the range of $u = g(x)$. Then,

$$\int_{x=a}^{x=b} f\big(g(x)\big)g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

This allows us to apply the FTC2, without having to invert the substitution.

Example

Evaluate $\mathrm{I} = \int_0^1 x^2(1+2x^3)^2 dx = \frac{1}{6}\int_0^1 (\underbrace{1+2x^3}_{u})^2 \underbrace{6(x^2 dx)}_{du}$

$u = (1+2x^3) \qquad du = 6x^2 dx$

At $x=0$, $u=1$
At $x=1$, $u=3$.

$$\mathrm{I} = \frac{1}{6}\int_{x=0}^{x=1} (\underline{1+2x^3})^2 (6x^2 dx)$$

$$= \frac{1}{6}\int_{u=1}^{u=3} u^2 \, du$$

$$= \frac{1}{6}\left(\frac{1}{3}u^3\right)\Big|_{u=1}^{u=3} = \frac{1}{18}(3^3 - 1^3) = \frac{1}{18}(26) = \frac{13}{9}$$

Example

Evaluate $\mathrm{I} = \int_{-1}^{0} x e^{x^2}\, dx \quad = \int_{-1}^{0} e^{(x^2)} \; (x\, dx)$

$u = x^2 \qquad du = 2\, \underline{x\, dx}$

$u(-1) = 1 \qquad u(0) = 0$

$\underline{I} = \frac{1}{2}\int_{-1}^{0} e^{(x^2)}\, x\, dx = \frac{1}{2}\int_{u=1}^{u=0} e^u\, du = \frac{1}{2} e^u \Big|_{u=1}^{u=0}.$

$\frac{1}{2} e^0 - \frac{1}{2} e^1 = \frac{1}{2}(1 - e).$

Rational functions

Recall: Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Before trying to find an antiderivative of a rational function $f(x) = \dfrac{p(x)}{q(x)}$:

Step 1: If $\deg(p(x)) \geqslant \deg(q(x))$, divide $p(x)$ by $q(x)$.
Step 2: Check if integration by substitution might work.
Step 3: Factorise the denominator as far as possible.
Step 4: Write the rational function as sum of **partial fractions** to simplify.

Rational functions

Example

Evaluate the integral

$$\int \frac{x}{x^2+1}\,dx. = \frac{1}{2}\int \underbrace{(x^2+1)}_{u}^{-1}\,\underbrace{(2x\,dx)}_{du}.$$

In this can we can use substitution.

$$u = x^2+1 \qquad du = 2x\,dx$$

$$\frac{1}{2}\int (x^2+1)^{-1}\,(2x\,dx) = \frac{1}{2}\int u^{-1}\,du$$

$$= \frac{1}{2}\ln(|u|) + C$$

$$= \frac{1}{2}\ln\left(|x^2+1|\right) + C.$$

Example

Evaluate the integral

$$\int \frac{3x+4}{x^2+7x+12}\,dx\,.$$

In this case, we must factorise the denominator, and express the integrand as **partial fractions**. Factorise:
$x^2+7x+12 = (x+4)(x+3)$.
Express as Partial Fractions:

$$\frac{3x+4}{x^2+7x+12} = \frac{A}{x+4} + \frac{B}{x+3}$$

With a little work, we can find that $A = 8$ and $B = -5$. Therefore,

Check!!

$$\frac{3x+4}{x^2+7x+12} = \frac{8}{x+4} - \frac{5}{x+3}\,.$$

We can now express the integral as

$$\int \frac{3x+4}{x^2+7x+12}\,dx = \int \left(\frac{8}{x+4} - \frac{5}{x+3}\right) dx$$
$$= \underbrace{\int \frac{8}{x+4}\,dx}_{\mathrm{I}_1} - \underbrace{\int \frac{5}{x+3}\,dx}_{\mathrm{I}_2}.$$

First, we evaluate I_1.

$$\mathrm{I}_1 = \int \frac{8}{x+4}\,dx = 8\int \frac{1}{x+4}\,dx.$$

If we let $\boxed{u = x+4}$, then $du = dx$ and, hence,

$$\mathrm{I}_1 = 8\int \frac{1}{x+4}\,dx = 8\int \frac{1}{u}\,du = 8\underline{\ln|u|} + C_1 = 8\ln|x+4| + C_1.$$

Similarly, we find that: $\mathrm{I}_2 = \displaystyle\int \frac{5}{x+3}\,dx = 5\ln|x+3| + C_2$.

To conclude:

$$\begin{aligned}
\int \frac{3x+4}{x^2+7x+12}\,dx &= \int \left(\frac{8}{x+4} - \frac{5}{x+3}\right) dx \\
&= \int \frac{8}{x+4}\,dx - \int \frac{5}{x+3}\,dx \\
&= \mathrm{I}_1 - \mathrm{I}_2 \\
&= (8\ln|x+4| + \underline{C_1}) - (5\ln|x+3| + \underline{C_2}) \\
&= 8\ln|x+4| - 5\ln|x+3| + C.
\end{aligned}$$

We use **substitution** when integrating the product of two functions, $f(x)g'(x)$, where $f(x) = f\big(g(x)\big)$, which is a very specialised situation.

Integration by parts is also applied when we are integrating the **product** of a function and a derivative $\underline{f(x)} = \underline{u(x)}\,\underline{v'(x)}$, but with no other constraint.

Integration by Parts (IBP) is, by some distance, the most important technique for integration, in both theory and practice.

Integration by Parts

If u and v are differentiable functions in the variable x, then

$$\int u(x)\, dv = u(x)v(x) - \int v(x)\, du\,.$$

It can also be written as $\displaystyle\int uv'\, dx = uv - \int vu'\, dx\,.$

The link between the two formulations is given by

$$dv = v'\, dx \quad \text{and} \quad du = u'\, dx\,.$$

$$v'(x) = \frac{dv}{dx}$$

$$\Rightarrow \quad v'(x)\, dx = dv$$

The Product Rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

We can use this rule to develop another integration technique after a little rearrangement. From the above we have

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v \qquad (*)$$

and integrating both sides gives

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

$$\int u\,dv = uv - \int v\,du$$

Example

Evaluate $\mathrm{I} = \int x \cos(x)\, dx$

Finished here
4 Nov

We'll take $u = x$ and $dv = \cos(x)dx$.

$u(x) = x \qquad dv = \cos(x)\,dx$

$\frac{du}{dx} = 1 \Rightarrow (du = dx) \qquad v = \int \cos(x)\,dx = \sin(x)$

$$I = \int x \cdot \cos(x)\,dx = \int u(x)\,dv$$

$$= (uv) - \int v\,du = x \cdot \sin(x) - \int \sin(x)\,dx$$

$$= x\sin(x) - (-\cos(x)) + C = x\sin(x) + \cos(x) + C.$$

One of the challenges of Integration by Parts is knowing how to choose u and dv. When integrating $\boxed{\int x\cos(x)\,dx}$ we choose $u = x$, because its derivative, $u' = 1$ is simpler.
Suppose we had made the bad choice of

$$u(x) = \cos(x), \qquad dv = x\,dx,$$

then we'd get:

More generally, given choices for u and dv, we proceed as follows:

1. Some functions are easy to differentiate (and maybe not so easy to integrate), and so make a good choice for u. Important examples include **logarithms** and **inverse trigonometric** functions.
2. Some functions (such as polynomials) have simple(r) derivatives, so are also a good choice for u.
3. Trigonometric and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for dv.

Example (of choosing u

Evaluate $\mathrm{I} = \int \frac{\ln(x)}{x^2}\, dx$.

Example

Evaluate $\mathrm{I} = \displaystyle\int \ln(x)\, dx$.

Since $\displaystyle\int \ln(x) dx$ can be written as $\displaystyle\int \Big(\ln(x) \Big) (1)\, dx$, we use integration by parts, with $u = \ln(x)$ and $dv = dx$.

Sometimes, we have to apply Integration by Parts more than once.

Example

Evaluate $\mathrm{I} = \int x^2 e^x \, dx$.

It is good to check any new rule/method for a simple example we already know the answer to. Now that we know about repeated application, we can do that:

Example

We know that $\mathrm{I} = \int x^2 dx = (1/3)x^3$. We can also use IbP.

Take $u(x) = x$ and $dv = xdx$:

Exercises

Exer 8.1.1

Evaluate the follow integrals

1. $\displaystyle\int \sin(\ln x)\frac{1}{x}\,\mathrm{d}x.$
2. $\displaystyle\int x^2(x^3+5)^9\,\mathrm{d}x$
3. $\displaystyle\int \frac{\sin(x)}{\cos^3(x)}\,dx$

Exer 8.1.2

Evaluate $\displaystyle\int_0^1 x^2(x^3+5)^9\,\mathrm{d}x$

Exercises

Exer 8.1.3

Evaluate the follow integrals

1. $\int xe^{2x}\,dx$.
2. $\int x^2\cos(x)\,dx$.

Exer 8.1.4

Evaluate $\int_1^e \ln(x^2)\,dx$.