Annotated slides from 02+06 Oct

MA385 Part 2: Initial Value Problems

2.2: Euler's Method

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02 October 2025 (Week 4)

For more details, see Chapter 6 of Süli and Mayers, *An Introduction to Numerical Analysis*.

1. The goal

Our goal is to generate numerical solutions to initial value differential equations. The solutions to such problems are functions (usually, of one variable that we'll denote t). Our approximation will give estimates of the values of this function at certain points.

We'll denote the points we at which we are seeking approximations as

$$t_0 < t_1 < \cdots < t_n.$$

The methods we'll use are all **one-step** methods, and the first example we'll consider is **Euler's Method**.

Although it is not too important, we'll make the assumption that the points are equally spaced. So

$$t_{i+1} - t_i = \frac{t_n - t_0}{n} = h.$$

2. Euler's Method: motivation

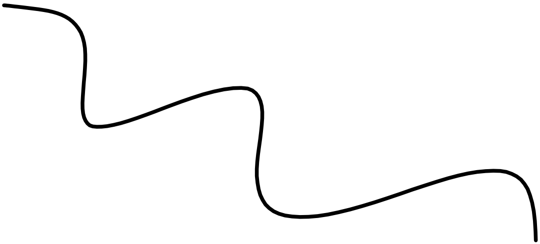
The simplest method is **Euler's Method**. We motivate it as follows.

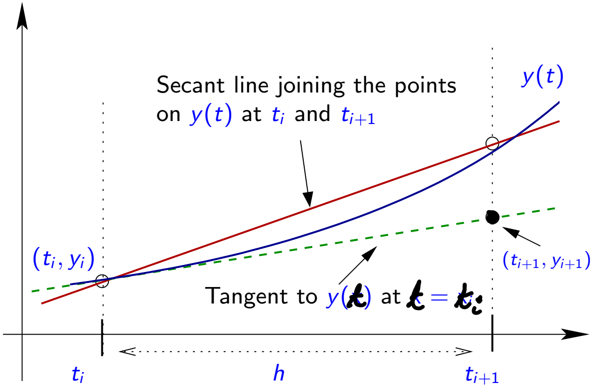
Motivation

Suppose we know $y(t_i)$, and want to compute $y(t_{i+1})$. From the differential equation we can calculate the slope of the tangent to y at t_i. If this approximates the slope of the line joining $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$, then

$$y'(t_i) = f(t_i, y(t_i)) \approx \frac{y_{i+1} - y_i}{t_{i+1} - t_i}.$$

2. Euler's Method: motivation





$$\frac{y_{i+1} - y_i}{t_{i+1} - t_i} \cong f(t, y_i) \Rightarrow y_{i+1} - y_i = h f(t_i, y_i)$$

$$\Rightarrow y_{i+1} \cong y_i + h f(t_i, y_i)$$

3. Euler's Method: formula

Euler's Method

Choose equally spaced points $t_0, t_1, \dots, t_n$ so that

$$t_i - t_{i-1} = h = (t_n - t_0)/n \quad \text{for } i = 0, \dots, n-1.$$

We call h the "time step". Let y_i denote the approximation for $y(t)$ at $t = t_i$. Set

$$y_{i+1} = y_i + hf(t_i, y_i), \quad i = 0, 1, \dots, n-1. \qquad (1)$$

Example 2.2.1

Taking $h = 1$, estimate $y(4)$ where

$t_0 = 0, \quad y_0 = 1$

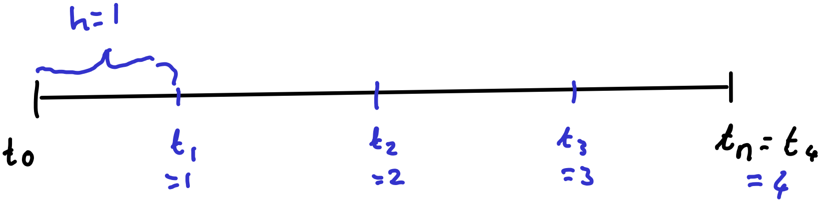
$$\underbrace{y'(t) = y/(1+t^2)}_{f(t,y) = \frac{y}{1+t^2}}, \qquad y(0) = 1.$$

Choosing $h = 1$ we get

Finished here Thursday

- $i = 0$: $t_0 = 0$, $y_0 = 1$.
- $i = 1$: $t_1 = t_0 + h = 1$.
  $y_1 = y_0 + hf(t_0, y_0) = 1 + \frac{1}{1+0^2} = 2.$
- $i = 2$: $t_2 = t_0 + 2h = 2$.
  $y_2 = y_1 + hf(t_1, y_1) = 2 + 1\frac{2}{1+1^2} = 3.$
- $i = 3$: $t_3 = t_0 + 3h = 3$.
  $y_3 = y_2 + hf(t_2, y_2) = 3 + 1\frac{3}{1+2^2} = 3.6$
- $i = 4$: $t_n = t_4 = t_0 + 4h = 4$.
  $y_n = y_4 = y_3 + hf(t_3, y_3) = 3.6 + \frac{3.6}{1+3^2} = \mathbf{3.96}$

Note :

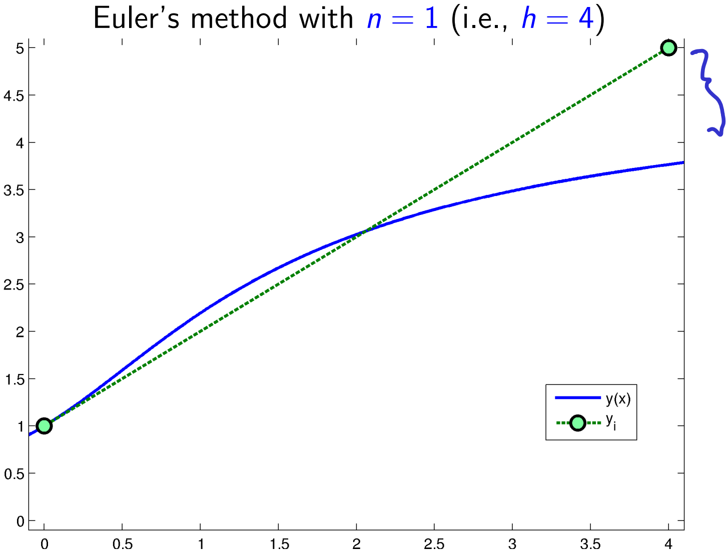


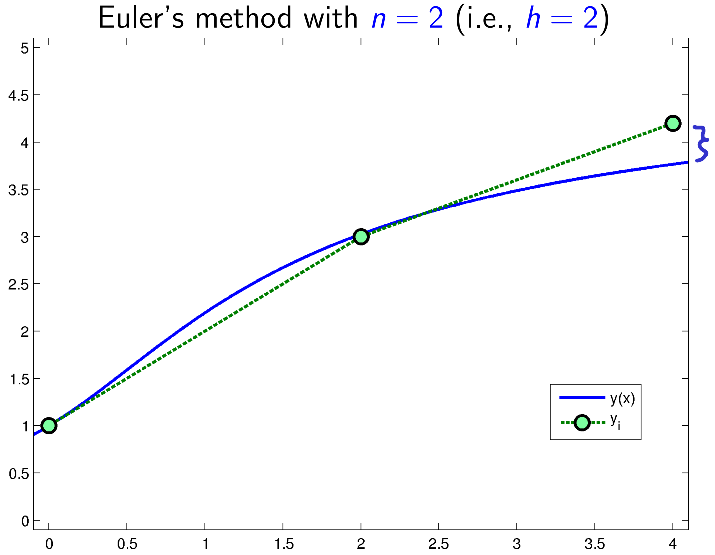
If we had chosen $h = 4$ we would have only required one step: $y_n = y_0 + 4f(t_0, y_0) = \mathbf{5}$. However, this would not be very accurate.

With a little work one can show that the solution to this problem is $y(t) = e^{\tan^{-1}(t)}$ and so $y(4) = 3.7652$. Hence the computed solution with $h = 1$ is much more accurate than the computed solution when $h = 4$. This is also demonstrated in next figure below, and in the follow table, where we see that the error seems to be proportional to h.

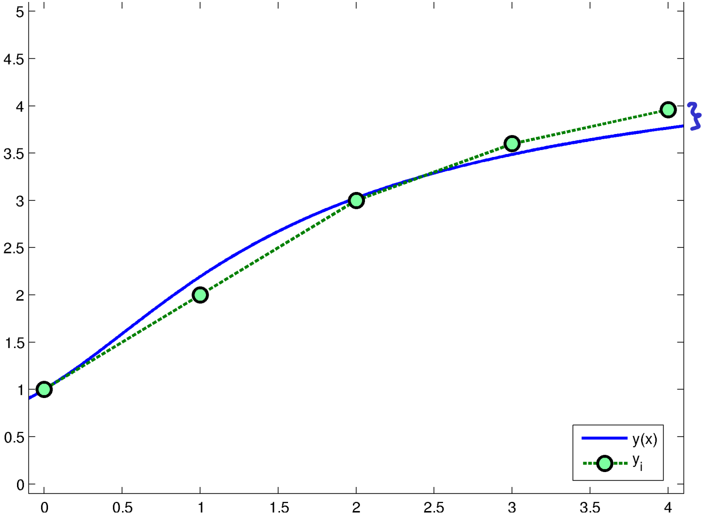
With $n=4$ (and $h=1$) Error ~ 0.2

With $n=1$ (and $h=4$) Error ~ 1.24.

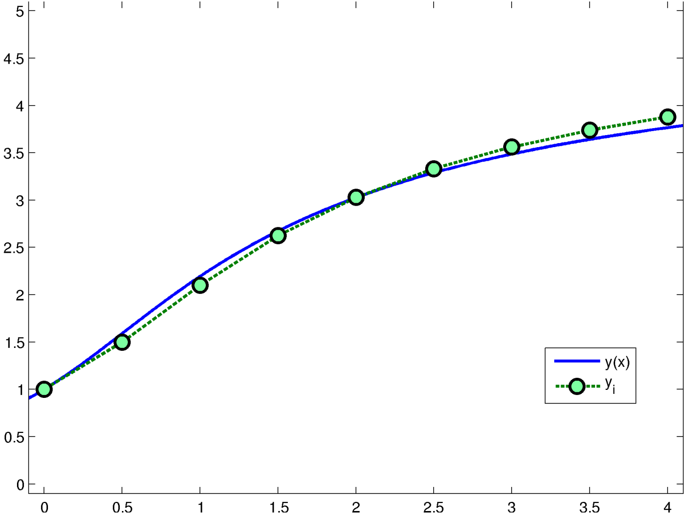

Euler's method with $n = 1$ (i.e., $h = 4$)
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
y(x)
y_i

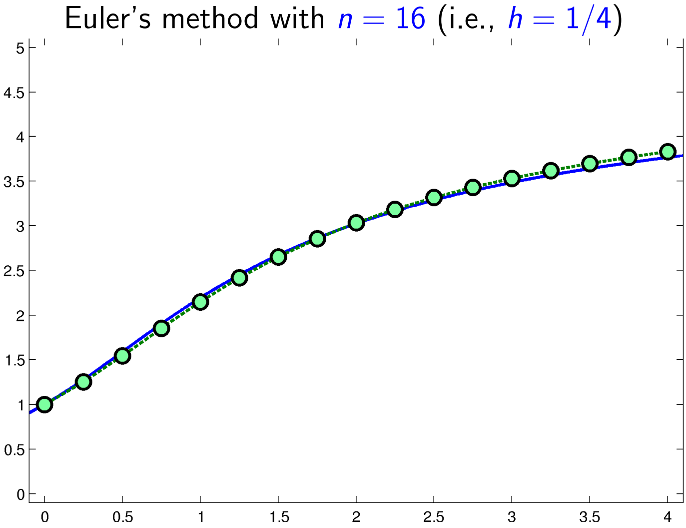

Euler's method with $n = 2$ (i.e., $h = 2$)
y(x)
y_i
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

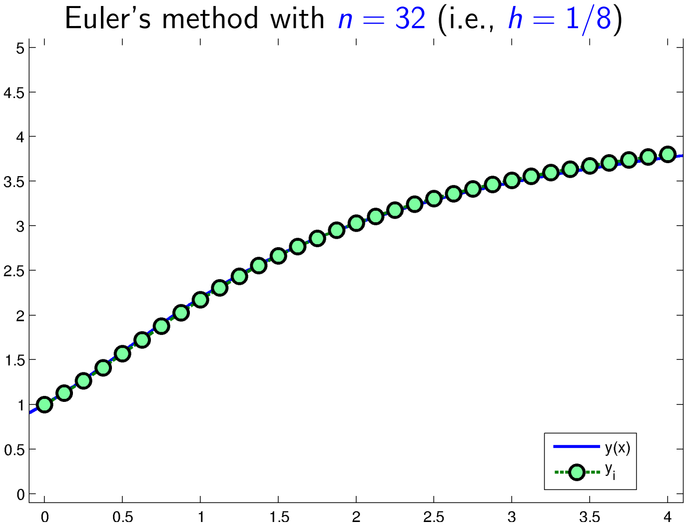
Euler's method with $n = 4$ (i.e., $h = 1$)



Euler's method with $n = 8$ (i.e., $h = 1/2$)




Euler's method with $n = 16$ (i.e., $h = 1/4$)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Euler's method with $n = 32$ (i.e., $h = 1/8$)
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
y(x)
y_i

n	h	y_n	$\lvert y(t_n) - y_n \rvert$
1	4	5.0	1.235
2	2	4.2	0.435
4	1	3.960	0.195
8	1/2	3.881	0.115
16	1/4	3.831	0.065
32	1/8	3.800	0.035

Table 1: Error in Euler's method for Example 8

- We see that, as h gets smaller, so too does the error.
- Looks like the error is (roughly) proportional to h.

Can we prove that??

Exercise 2.2.1

As a special case in which the error of Euler's method can be analysed directly, consider Euler's method applied to

$$y'(t) = y(t), \quad y(0) = 1.$$

The true solution is $y(t) = e^t$.

(i) Show that the solution to Euler's method can be written as

$$y_i = (1+h)^{t_i/h}, \quad i \geq 0.$$

(ii) Show that

$$\lim_{h \to 0} (1+h)^{1/h} = e.$$

This then shows that, if we denote by $y_n(T)$ the approximation for $y(T)$ obtained using Euler's method with n intervals between t_0 and T, then

$$\lim_{n \to \infty} y_n(T) = e^T.$$

Hint: Let $w = (1+h)^{1/h}$, so that $\log w = (1/h)\log(1+h)$. Now use l'Hospital's rule to find $\lim_{h \to 0} w$.