2526-MA140 Engineering Calculus

Week 10, Lecture 2
Cylindrical Shells, and Mean Values W10-2

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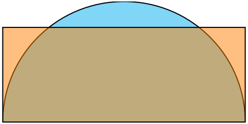


Assignments,etc

1. **Assignment 7** has finished and grades posted.
2. **Assignment 8** (last one!) is live and will be covered in tutorials this week. See https://universityofgalway.instructure.com/courses/46734/assignments/132796
3. Results for the class test will be available by next Tuesday.
4. There will be a change to the tutorial schedule next week...
5. Those tutorials will focus on revision and a sample paper, to be posted Monday.

Today, we mean to discuss

See also: Sections **6.3** (Volumes of Revolution - Cylindrical Shells) and the end of Section **5.2** (Average Value of a Function) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Assignment 8, Q2

In Q2 of the Tutorial Sheet for PS-8, we are asked:

Assignment 8, Q2

Find the volume of the solid formed by rotating the region enclosed by $x = 0$, $x = 1$, $y = 0$, and $y = 9 + x^6$, about the y-axis.

There are two ways (at least) of solving this:

- The “washer method” from W09-2;
- The Method of **Cylindirical Shells**, which we have yet to study.

We’ll now do each of these.

First we'll sketch the object in question: We should be able to convince ourselves that $V = V_1 - V_2$, where

- V_1 is the volume of $f(y) = 1$, from $y = 0$ to $y = 10$, rotated about the y-axis;
- V_2 is the volume of $f(y) = (y-9)^{1/6}$, from $y = 9$ to $y = 10$, rotated about the y-axis.

It should be easy to check that $V_1 = 10\pi$, and almost as easy to check that $V_2 = \frac{3}{4}\pi$. This gives that $V = \frac{37}{4}\pi$.

Cylindrical Shells

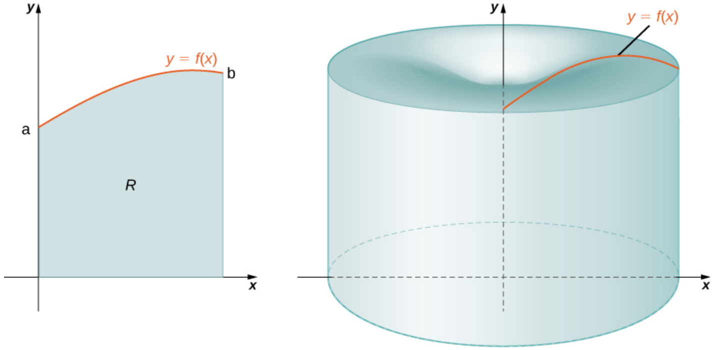
There is another approach, that is arguably easier.

So far, we've used the "disk method" for volumes of rotation. This came from the idea that every "slice" is a disk, whose area we can compute. Then integrating over the domain, we get the "sum" of all the disks: which is the volume.

An alternative approach, is to construct think of the solid as an infinite sum of cylinders...

Cylindrical Shells

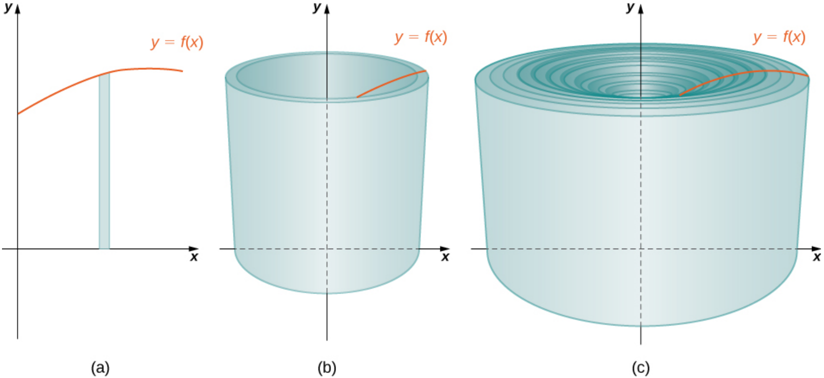
We state the problem as follows: *Find the volume of the solid obtained by rotating the region between $y = f(x)$ and $y = 0$, and $x = 0$ and $x = b$, around the y-axis*



(a) (b)

Cylindrical Shells

We can think of the region as made up of many small rectangles. When rotated about the y-axis, these become cylinders.



Cylindrical Shells

We can visualise this for a problem related to "Gabriel's Horn" from earlier, but over a finite region.

> *Let $f(x) = 1/x$. Construct a solid of revolution by rotating the region between $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$ about the y-axis.*

The visualisation from the textbook may be found at this link.

Cylindrical Shells

Very roughly, a cylinder with height $f(x)$, and thickness $\Delta x_i = x_i - x_{i-1}$ has volume

$$\begin{aligned} V_i = \pi(x_i^2 - x_{i-1}^2) f(x_i) &= \pi f(x_i)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i) \frac{x_i + x_{i-1}}{2} \Delta x_i \approx 2\pi f(x_i) x_i \Delta x_i \end{aligned}$$

Then

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i) x_i \Delta x_i = 2\pi \int_a^b x f(x)\, dx.$$

See text-book for more details!

Cylindrical Shells

The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Rotate about the y-axis, the region bounded above by $y = f(x)$, below by $y = 0$, on the left by $x = a$, and on the right by $x = b$.
Then the volume of the resulting solid of revolution is

$$V = 2\pi \int_a^b x f(x)\, dx.$$

Assignment 8, Q2

Use the Method of Cylindrical Shells to find the volume of the solid formed by rotating the region enclosed by $x = 0$, $x = 1$, $y = 0$, and $y = 9 + x^6$, about the y-axis.

Average values of functions

In many applications we wish to know the "average" (or **mean**) value of a continuously varying quantity, which is represented by a function.

We are already familiar with this concept when dealing with the mean of a set of n values: $\{x_1, x_2, \ldots, x_n\}$. There are two (equivalent) ways of thinking about this:

1. The mean of the set of values is
$$\bar{x} = \frac{x_1 + x_2 + \cdots x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k.$$
2. $\bar{x}$ is the mean of the set of values of $\{x_1, x_2, \ldots, x_n\}$.

That is, if we replaced each of the x_i with the constant value $\bar{x}$, the sum would not change.

We can extend both these ideas to defining the "average value of a function", on the interval $[a, b]$, getting the same result.

First, suppose we take n subintervals of $[a, b]$, and denote their end-points $\{x_0, x_1, \dots, x_n\}$. Note that $x_k = x_0 + k\Delta x$, where $\Delta x = (b - a)/n$.

Now take the average of the n sampled values:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$
$$= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(x_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\,\Delta x$$

If $n \to \infty$ (or $\Delta x \to 0$), we get the **average value of $f(x)$ on** $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The second version is more insightful, I think:

Average value of a function

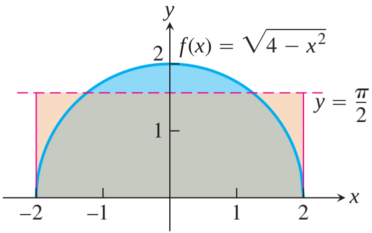
The constant $\bar{f}$ is the **average** value of $f(x)$ on $[a, b]$, if

$$\int_a^b \bar{f}\,dx = \int_a^b f(x)\,dx.$$

To see this is equivalent:

Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.



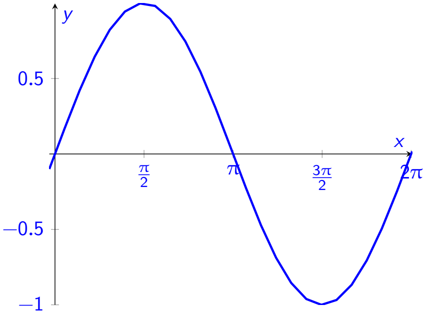
Example

Find the average value of the function $f(x) = x^2 - x - 2$ on $[-3, 3]$.

Average of $\sin(x)$

Find the average values of $f(x) = \sin(x)$ on

1. $[a, b] = [0, \pi]$
2. $[a, b] = [0, 2\pi]$



Root-Mean-Square Values

In some contexts, the **average value** of a function is a useful summary statistic. But it can be misleading too, as the last example showed.

Notable examples of this include

- The average value of an alternating current is zero;
- The average motion of a piston is zero.

Therefore (especially in power electronics) we need another measure to summarise a function

Root Mean Squared (RMS)

The **root mean square (RMS)** of a function $f(x)$ is

$$f_{\mathrm{RMS}} := \left(\frac{1}{b-a} \int_a^b [f(x)]^2 \, dx \right)^{1/2}$$

Root-Mean-Square Values

Example

An electric current $i(\theta)$ is given by $i(\theta) = I_{\text{peak}} \sin(\theta)$ where I_{peak} is a constant. Find the root mean square of $i(\theta)$ over the interval $[0, 2\pi]$.

(*Hint: use that* $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$).

Root-Mean-Square Values

Exercises

Exer 10.2.1 (from textbook)

Let $f(x) = 1/x$. Find the volume of the solid of revolution by rotating the region between $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$ about the y-axis.

Exer 10.2.2

Find the average value of $f(x) = \dfrac{1}{1 - 4x^2}$ for $0 \leqslant x \leqslant 1/4$.

Exer 10.2.3

Find $b > 0$ such that the average value of $f(x) = x^2 - 2x + 3/4$ on the interval $[0, b]$ is zero.
Compute the root mean squared of $f(x)$ on the same interval.