AARMS-CRM Workshop on NA of SPDEs, July 2016

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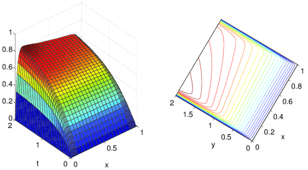
Short course on Numerical Analysis of Singularly Perturbed Differential Equations

Niall Madden, NUI Galway (Niall.Madden@NUIGalway.ie)

§5 Time-dependent problems

(Version 25.07.16)



Handout version.

Outline

	Monday, 25 July	Tuesday, 26 July
09:00	Welcome/Coffee	
09:15	1. Introduction to singularly perturbed problems	5. PDEs (i): time-dependent problems.
10:00	Break	
10:15	2. Numerical methods and uniform convergence; FDMs and their analysis.	6. PDEs (ii): elliptic problems 7. Finite Element Methods
12:00	*Lunch*	
14:00	3. Coupled systems	8. Convection-diffusion (Stynes)
15:00	Break	
15:15	3. Coupled systems (continued)	9. Nonlinear problems (Kopteva)
16:15	4. Lab 1	10. Lab 2 (PDEs)
17:30	*Finish*	

§5 PDEs Part 1: time-dependent problems

(45 minutes)

In this section, we study the robust numerical solution of problems of the form

$$u_t + \mathcal{L}u = f$$

where $\mathcal{L}$ is a singularly perturbed operator.

We'll use an implicit time-stepping scheme. So most of the work involves extending known results for steady-state problems.

However, we will consider some **graded meshes**: the analysis of these is more complicated than for piecewise uniform Shishkin meshes.

Primary references

This presentation is based around the analysis in [Linß and Madden, 2007] for the time-dependent reaction-diffusion problem:

$$\frac{\partial}{\partial t}u(x,t)\underbrace{-\varepsilon^2\frac{\partial^2}{\partial x^2}u(x,t)+b(x)u(x,t)}_{\mathcal{L}u(x,t)}=f(x,t)\text{ on }(0,1)\times(0,T].$$

The main goals are:

- to see how to extend the ideas for stationary problems to an unsteady setting;
- introduce the analysis of finite difference methods by using discrete Green's functions.

There are many other expositions available, e.g., [Miller et al., 1998] and [Clavero and Gracia, 2010].

For a more detailed exposition see [Linß and Stynes, 2009] and, especially, [Linß, 2010].

The approach here is largely based on an analysis of the corresponding steady-state [Linß, 2005], and the time-dependent convection-diffusion problem [Kopteva, 2001].

The problem...

$$\frac{\partial}{\partial t}u(x,t)\underbrace{-\varepsilon^2\frac{\partial^2}{\partial x^2}u(x,t)+b(x)u(x,t)}_{\mathcal{L}u(x,t)}=f(x,t) \text{ on } (0,1)\times(0,T]. \tag{1a}$$

subject to boundary conditions

$$u(0,t)=\gamma_0(t),\ u(1,t)=\gamma_1(t) \quad \text{in} \quad (0,T], \tag{1b}$$

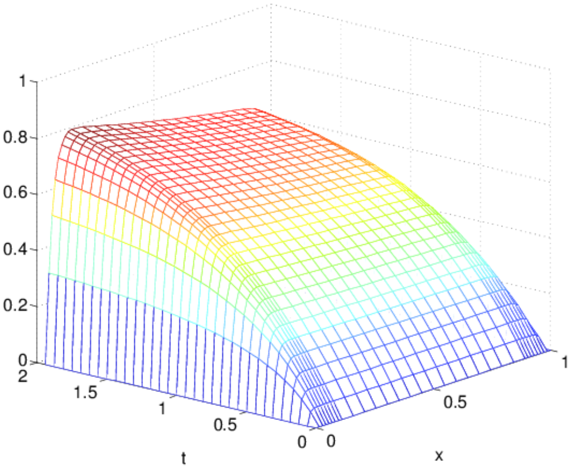
and initial condition

$$u(\cdot,0)=u^0 \qquad \text{in} \quad (0,1). \tag{1c}$$

As before, we assume $b(x)>\beta^2$ with some positive constant β.

The PDE

Solutions to this PDE typically exhibit layers along $x = 0$ and $x = 1$.

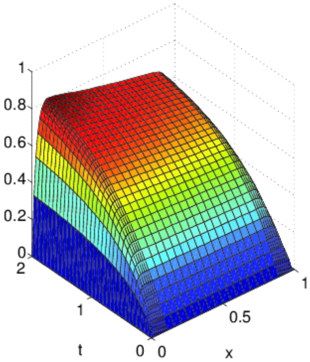


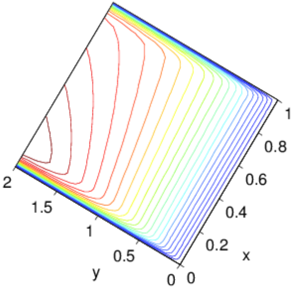
The PDE

Using techniques like those from Section 3, one can obtain the following bounds (subject to data being sufficiently smooth):

$$\left|\partial_x^\ell \partial_t^k u(x,t)\right| \le C\left\{\varepsilon^{\min\{0,2-\ell\}} + \varepsilon^{-\ell}\mathrm{e}^{-\beta x/\varepsilon} + \varepsilon^{-\ell}\mathrm{e}^{-\beta(1-x)/\varepsilon}\right\}$$

$$\text{for } \ell = 0,\dots,4,\ k = 0,1,2 \quad (2)$$





Some notation:

- We denote an arbitrary mesh in space as $\Omega_x^N := \{0 = x_0 < x_1 < \cdots < x_N = 1\}$.
  Let $h_i := x_i - x_{i-1}$ be the mesh diameter in the spatial dimension, and $\overline{h}_i := (h_i + h_{i+1})/2$.
- In time, set $\Omega_t^K := \{0 = t_0 < t_1 < \cdots < t_K = T\}$.
  Time step sizes are $\tau_j := t_j - t_{j-1}$, with $\tau := \max_{j=1,\ldots,K} \tau_j$.
- Let $\Omega^{N,K}$ be the tensor product of the one-dimensional meshes Ω_x^N and Ω_t^K.
- If g is a mesh function defined on $\Omega^{N,K}$, then $g_i^j = g(x_i, t_j)$.
- In some settings we consider a stationary problem on Ω_x^N, and let $g_i = g(x_i)$.
- Similarly, g^j denotes a one-dimensional mesh function om Ω_t^K at time-step j.

We use central differences in space and backward differences in time:

$$D_x^- v_i := \frac{v_i - v_{i-1}}{h_i}, \qquad D_x^+ v_i := \frac{v_{i+1} - v_i}{\bar{h}_i} \qquad \delta_x^2 v_i = D_x^+ D_x^- v_i,$$

and

$$D_t^- v^j = \frac{v^j - v^{j-1}}{\tau_j}.$$

The operator $\mathcal{L}$ is approximated by

$$\left[L^N v\right]_i := -\varepsilon^2 \delta_x^2 v_i + b_i v_i.$$

The numerical approximation U of (1) is the solution of the linear difference equation

$$\left[D_t^- + L^N\right] U_i^j = f_i^j \quad \text{for} \quad i = 1, \ldots, N-1, \quad j = 1, \ldots, K,$$

with the boundary and initial condition discretizations.

Consider the difference scheme

$$\left[L^N U\right]_i = f_i \quad \text{for} \quad i = 1, \ldots, N-1, \quad U_0 = \gamma_0,\ U_N = \gamma_1, \tag{3}$$

as a discretization of the stationary reaction-diffusion problem

$$\mathcal{L}u = -\varepsilon^2 u'' + bu = f \quad \text{in } (0,1), \quad u(0) = \gamma_0,\ u(1) = \gamma_1, \tag{4}$$

with $b > \beta^2$, $\beta > 0$.

We have already analysed the above method on a Shishkin mesh. Now we want to look at a more general approach – based on **discrete Green's functions** – that can yield results for the graded meshes we saw in §3.

First, let's recall from Section 2 that the solution u of (4), and its derivatives, can be bounded as

$$\left|u^{(l)}(x)\right| \leq C\left\{\varepsilon^{\min\{0,2-\ell\}} + \varepsilon^{-\ell}\mathrm{e}^{-\beta x/\varepsilon} + \varepsilon^{-\ell}\mathrm{e}^{-\beta(1-x)/\varepsilon}\right\} \quad \text{for } \ell = 0, \ldots, 4, \tag{5}$$

Let δ_j^i denote the usual Kronecker delta: it is the mesh function defined on ω_x^N such that for all $0 \le i, j \le N$,

$$\delta_j^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Discrete Green's function

The **discrete Green's function** G^i associated with L^N and the mesh node x_i is the solution to

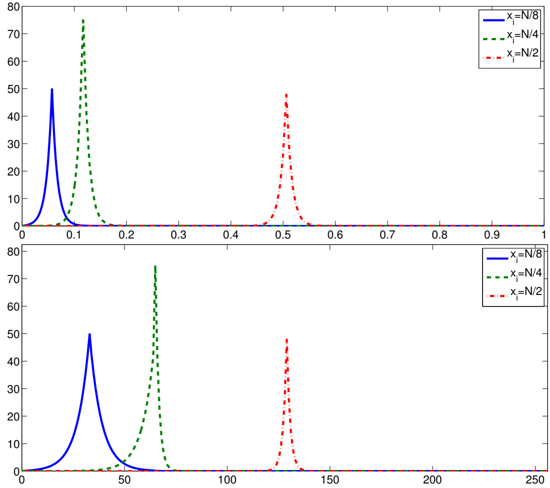
$$\left[L^N G^i\right]_j := \overline{h}_j^{-1} \delta_j^i \quad \text{for} \quad j = 1, \dots, N-1, \qquad G_0^i = G_N^i = 0.$$

Any mesh function $v = (v_0, \dots, v_N)$ with $v_0 = v_N = 0$ can be represented as

$$v_i = \sum_{k=1}^{N-1} \overline{h}_k G_k^i \left[L_\varepsilon v\right]_k \quad \text{for } i = 1, \dots, N-1.$$

Therefore the set $\{G^i\}$ forms a useful basis for expressing solutions to (3), and important properties of the discrete operator can be derived by studying the associated Green's functions.

Some discrete Green's functions, expressed in terms of x (top) and mesh index (bottom), for $\varepsilon = 10^{-2}$ and $N = 256$.



Each G^i is defined at the mesh nodes only, but can be interpreted as a piecewise linear function on the mesh Ω_x^N. Then we have the estimates [Linß, 2010]

$$\int_0^1 G^i(\xi)\, d\xi = \sum_{k=1}^{N-1} \overline{h}_k G_k^i \leq \frac{1}{\beta^2}. \tag{6}$$

An immediate consequence is the stability inequality

$$\|v\|_\infty := \max_{i=1,\ldots,N-1} |v_i| \leq \frac{1}{\beta^2} \|L_\varepsilon v\|_\infty , \tag{7}$$

which holds true for arbitrary mesh functions v with $v_0 = v_N = 0$.

By means of the Green's function, the error at the mesh node x_i can be written

$$
\begin{aligned}
(u-U)_i &= \sum_{k=1}^{N-1} \bar{h}_k G_k^i \left[L^N (u-U)\right]_k = \sum_{k=1}^{N-1} \bar{h}_k G_k^i \left[L^N u - \mathcal{L}u\right]_k \\
&= \varepsilon^2 \sum_{k=0}^{N-1} \frac{u_{k+1}-u_k}{h_{k+1}} \left(G_{k+1}^i - G_k^i\right) + \varepsilon^2 \sum_{k=1}^{N-1} \bar{h}_k G_k^i u_k'' \\
&= \varepsilon^2 \int_0^1 u'(x)(G^i)'(x)dx + \varepsilon^2 \sum_{k=1}^{N-1} \bar{h}_k G_k^i u_k'' \qquad ((G^i)' \text{ is pw constant}) \\
&= -\varepsilon^2 \int_0^1 u''(x)G^i(x)dx + \varepsilon^2 \sum_{k=1}^{N-1} \bar{h}_k G_k^i u_k''.
\end{aligned}
$$

Thus, with $\phi := -\varepsilon^2 u''$,

$$
(u-U)_i = \int_0^1 \left\{(G^i\phi)(x) - (G^i\phi)^I(x)\right\} dx. \tag{8}
$$

Some rather detailed calculations follow..., but eventually we arrive at

Given a mesh Ω^N, define the mesh characterisation function $\theta(\cdot)$

$$\theta\left(\Omega_x^N\right) := \max_{k=1,\ldots,N} \int_{x_{k-1}}^{x_k} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta t/2\varepsilon} + \varepsilon^{-1} \mathrm{e}^{-\beta(1-t)/2\varepsilon}\right) dt.$$

The right-hand side from (8) can now be bounded:

$$\left| \int_0^1 \left\{ (G^i \phi)(x) - (G^i \phi)^I(x) \right\} dx \right| \leq C\theta\left(\Omega_x^N\right)^2 \left(\varepsilon \int_0^1 D_x^- G^i(x)\, dx + \int_0^1 G^i(x)\, dx \right).$$

"Hence"

$$\|u - U\|_\infty \leq C\theta\left(\Omega_x^N\right)^2,$$

The significance of this result is that it does not depend on a particular mesh. It can be used to establish results for various meshes.

Stability of the discrete time-dependent operator is not hard to establish. Therefore, we need to determine the truncation error.

However, given the absence of significant ε-dependency in the time derivatives of u, one can establish the following result without recourse to particularly specialised techniques.

Theorem

$$\max_{j=0,\ldots,K} \left\| u^j - U^j \right\|_\infty \leq C\,T \left(\theta \left(\Omega_x^N \right)^2 + \tau \right).$$

So now we have the task of quantifying $\theta(\Omega_x^N)$.

It is not hard to check that:

- If Ω^N is a Shishkin mesh, with transition point $\tau = \min\left\{q, \frac{\sigma\varepsilon}{\beta} \ln N\right\}$, then

$$\theta\left(\omega_x^N\right) \le C\left\{N^{-\sigma/2} + N^{-1}\ln N\right\}.$$

- If Ω^N is a Bakhvalov mesh, generated by equidistributing the function

$$M_{Ba}(x) = \max\left\{1, \kappa\varepsilon^{-1}\mathrm{e}^{-\beta x/\varepsilon\sigma}, \kappa\varepsilon^{-1}\mathrm{e}^{-\beta(1-x)/\varepsilon\sigma}\right\},$$

  with positive constants κ and σ. We can conclude

$$\theta\left(\omega_x^N\right) \le C\left\{N^{-1}\right\}.$$

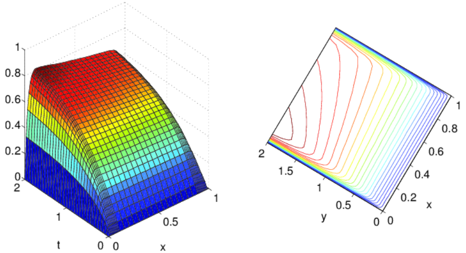
Example

$$u_t - \varepsilon^2 u_{xx}(x,t) + \sqrt{x+1}\,u(x,t) = 1 \quad (0,1)\times(0,1], \tag{9}$$

with

$$u(0,s) = u(1,s) = u(s,0) \quad \text{for } 0 \le s \le 1. \tag{10}$$

The exact solution is not available, so estimate the accuracy of numerical solutions by comparing them to solutions computed on a much finer mesh.



An example

The results below are for a Bakhvalov mesh, with $\kappa = 1$, $\sigma = 2.5$, $\beta = 0.99$ and $K = N$ uniform time steps.

ε^2	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	
1	4.61e-03	2.51e-03	1.31e-03	6.69e-04	3.38e-04	1.70e-04	$\eta_\varepsilon^{N,K}$
	0.88	0.94	0.97	0.98	0.99	—	$r_\varepsilon^{N,K}$
1e-02	4.67e-03	2.35e-03	1.18e-03	5.90e-04	2.95e-04	1.48e-04	$\eta_\varepsilon^{N,K}$
	0.99	1.00	1.00	1.00	1.00	—	$r_\varepsilon^{N,K}$
1e-04	4.71e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.47e-04	$\eta_\varepsilon^{N,K}$
	1.00	1.00	1.00	1.00	1.00	—	$r_\varepsilon^{N,K}$
1e-06	4.72e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04	$\eta_\varepsilon^{N,K}$
	1.00	1.00	1.00	1.00	1.00	—	$r_\varepsilon^{N,K}$
1e-08	4.72e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04	$\eta_\varepsilon^{N,K}$
	1.00	1.00	1.00	1.00	1.00	—	$r_\varepsilon^{N,K}$
1e-10	4.72e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04	$\eta_\varepsilon^{N,K}$
	1.00	1.00	1.00	1.00	1.00	—	$r_\varepsilon^{N,K}$
1e-12	4.72e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04	$\eta_\varepsilon^{N,K}$
	1.00	1.00	1.00	1.00	1.00	—	$r_\varepsilon^{N,K}$

Since $K = N$, and since the method is only first-order in time, the time error dominates.

An example

Next we set $K = 4N^2$, and observe second-order convergence.

ε^2	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	
1	7.67e-05	1.92e-05	4.80e-06	1.20e-06	3.00e-07	7.50e-08	$\eta_\varepsilon^{N,K}$
	2.00	2.00	2.00	2.00	2.00	—	$r_\varepsilon^{N,K}$
1e-02	3.27e-04	8.69e-05	2.23e-05	5.67e-06	1.43e-06	3.60e-07	$\eta_\varepsilon^{N,K}$
	1.91	1.96	1.98	1.99	1.99	—	$r_\varepsilon^{N,K}$
1e-04	5.47e-04	1.46e-04	3.79e-05	9.64e-06	2.44e-06	6.14e-07	$\eta_\varepsilon^{N,K}$
	1.91	1.94	1.97	1.98	1.99	—	$r_\varepsilon^{N,K}$
1e-06	5.90e-04	1.57e-04	4.10e-05	1.04e-05	2.64e-06	6.65e-07	$\eta_\varepsilon^{N,K}$
	1.91	1.94	1.97	1.98	1.99	—	$r_\varepsilon^{N,K}$
1e-08	5.97e-04	1.59e-04	4.14e-05	1.06e-05	2.67e-06	6.72e-07	$\eta_\varepsilon^{N,K}$
	1.91	1.94	1.97	1.98	1.99	—	$r_\varepsilon^{N,K}$
1e-10	5.97e-04	1.59e-04	4.15e-05	1.06e-05	2.68e-06	6.73e-07	$\eta_\varepsilon^{N,K}$
	1.91	1.94	1.97	1.98	1.99	—	$r_\varepsilon^{N,K}$
1e-12	5.98e-04	1.59e-04	4.15e-05	1.06e-05	2.68e-06	6.74e-07	$\eta_\varepsilon^{N,K}$
	1.91	1.94	1.97	1.98	1.99	—	$r_\varepsilon^{N,K}$

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