Annotated slides from Thursday

2526-MA140: Week 02, Lecture 3 (L06)

Limits; The Squeeze Theorem

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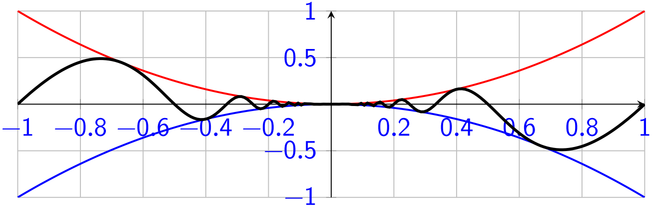


Outline

For more, see Chapter 2 (Limits) of Strang and Herman's Calculus, especially Section and 2.3 (Limit Laws).

Slides are on canvas, and at
niallmadden.ie/2526-MA140

Recall... Limits

Yesterday, we learned that

$$\lim_{x \to a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

$x \to a$ means "x goes (towards) a".

Properties of Limits

We finish with the following "Limit Laws": *Suppose that*

$$\lim_{x \to a} f_1(x) = L_1 \quad \text{and} \quad \lim_{x \to a} f_2(x) = L_2,$$

and $c \in \mathbb{R}$ is any constant. Then,

(1) $\lim_{x \to a} c = c, \; c \in \mathbb{R}$

(2) $\lim_{x \to a} x = a$

(3) $\lim_{x \to a} [c f_1(x)] = c L_1$

(4) $\lim_{x \to a} [f_1(x) + f_2(x)] = L_1 + L_2$ and $\lim_{x \to a} [f_1(x) - f_2(x)] = L_1 - L_2$

(5) $\lim_{x \to a} \big(f_1(x) f_2(x)\big) = L_1 L_2$

(6) $\lim_{x \to a} \big((f_1(x))^n\big) = (L_1)^n$

(7) $\lim_{x \to a} \left(\dfrac{f_1(x)}{f_2(x)}\right) = \dfrac{L_1}{L_2}$, providing $L_2 \neq 0$.

(8) $\lim_{x \to a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$

ie $\lim_{x \to a} \big(f_1(x) + f_2(x)\big) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x)$

Evaluating limits

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\boxed{\lim_{x\to a} x^n = a^n}$$

✓

Example

Evaluate the limit $\lim_{x\to 1}(x^3 + 4x^2 - 3)$

By Property (4):

$$\lim_{x\to 1}(x^3+4x^2-3) = \lim_{x\to 1} x^3 + \lim_{x\to 1} 4x^2 + \lim_{x\to 1}(-3)$$

(Rule 3) Rule 1

$$= \lim_{x\to 1} x^3 + 4\lim_{x\to 1} x^2 + (-3)$$

$$= (1)^3 + 4(1)^2 - 3 = 2.$$

Evaluating limits

Example

Evaluate $\lim_{x\to 1} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ using the Properties of Limits.

Property (7): $\lim_{x\to a} \dfrac{f_1(x)}{f_2(x)} = \dfrac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)}$

Providing $\lim_{x\to a} f_2(x) \neq 0$.

Here $f_2(x) = x^2+5 \Rightarrow \lim_{x\to 1} x^2+5 = 6 \neq 0$

So we can apply the Property:

$$\lim_{x\to 1} \frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to 1} x^4+x^2-1}{\lim_{x\to 1} x^2+5} = \frac{1}{6} \checkmark$$

Limits of rational functions

In many cases, evaluating limits is more complicated. In particular, we'll consider numerous examples where we want to evaluate $\lim_{x \to a} f(x)$ where a is not in the domain of f.

A typical example of this is when we evaluate a rational function:

$$\boxed{\lim_{x \to a} \frac{p(x)}{q(x)}}$$

where **both** $p(a) = 0$ and $q(a) = 0$.

Idea: Since we care about the value of p and q **near** $x = a$, but not actually at $x = a$, it is safe to factor out an $(x - a)$ term from both.

Limits of rational functions

Three examples

Evaluate the limits:

General form $\lim_{x \to a} \frac{p(x)}{q(x)}$

$$(a)\ \lim_{x\to 0} \frac{x}{x} \qquad (b)\ \lim_{x\to 0} \frac{x^2}{x} \qquad (c)\ \lim_{x\to 0} \frac{x}{x^2}$$

(a) as $x \to a$ both $p(x) \to 0$ & $q(x) \to 0$.

However, if $x \neq 0$ $\frac{x}{x} = 1$

So $\lim_{x\to 0} \frac{x}{x} = 1$ ✓

(b) as $x \to 0$ $\frac{x^2}{x} \to x$ $\Rightarrow \lim_{x\to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$.

(c) as $x \to 0$ $\frac{x}{x^2} \to \frac{1}{x}$. But $\lim_{x\to 0} \frac{1}{x}$ is <u>undefined</u>

Limits of rational functions

Example

Evaluate the limit
$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{p(x)}{q(x)}$$

where $p(x) = x^2+x-2$, $q(x) = x^2-x$.

Check: $p(1) = 1+1-2 = 0$ $q(1) = 1-1 = 0$.

So factorize (since both must have $x-1$ as a factor).

$p(x) = (x-1)(x+2)$ $q(x) = (x-1)x$.

So $x \neq 1$ $\frac{p(x)}{q(x)} = \frac{(x-1)(x+2)}{(x-1)x} = \frac{x+2}{x}$

So $\lim_{x\to 1} \frac{p(x)}{q(x)} = \frac{3}{1} = \boxed{3}$

Limits of rational functions

In that last example, we found that

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x}$$

But these are different functions:

They are different because they have different domains:

$x=1$ is not in the domain of $\dfrac{x^2+x-2}{x^2-x}$

but it is in the domain of $\dfrac{x+2}{x}$.

Limits of rational functions

Evaluate the limit

$$\lim_{x\to 2}\left(\frac{\frac{1}{2}-\frac{1}{x}}{x-2}\right) = \lim_{x\to 2} \frac{p(x)}{q(x)}$$

$p(x) = \frac{1}{2} - \frac{1}{x} = \frac{x-2}{2x}$ $\quad q(x) = x-2.$

Check: $p(2) = \frac{0}{4} = 0$ $\quad q(2) = 0.$

But, at $x \neq 2$, $\left(\frac{x-2}{2x}\right)\cdot\frac{1}{x-2} = \frac{1}{2x}.$

So $\lim_{x\to 2} \frac{\frac{1}{2}-\frac{1}{x}}{x-2} = \lim_{x\to 2} \frac{1}{2x} = \boxed{\frac{1}{4}}$

Completing the square

Very often, we'll evaluate limits of the form:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)}}$$

where f and g are not polynomials. Some of the same ideas still apply.

Example

Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2} = \lim_{x\to 0} \frac{p(x)}{q(x)}$$

$p(x) = \sqrt{1+x^2} - 1$

$q(x) = x^2$

$p(0) = \sqrt{1} - 1 = 0$

$q(0) = 0$

Completing the square

We have

$$\frac{\sqrt{1+x^2}-1}{x^2} = \frac{\left(\sqrt{1+x^2}-1\right)\left(\sqrt{1+x^2}+1\right)}{x^2\left(\sqrt{1+x^2}+1\right)}$$

$$= \frac{\left(\sqrt{1+x^2}\right)\left(\sqrt{1+x^2}\right) - \sqrt{1+x^2} + \sqrt{1+x^2} - 1}{x^2\left(\sqrt{1+x^2}-1\right)}$$

$$= \frac{(1+x^2)-1}{x^2\left(\sqrt{1+x^2}+1\right)} = \frac{x^2}{x^2\left(\sqrt{1+x^2}+1\right)} = \frac{1}{\sqrt{1+x^2}+1}$$

$$\lim_{x\to 0} \frac{1}{\sqrt{1+x^2}+1} = \boxed{\frac{1}{2}}$$

The Squeeze Theorem

There are various approaches to evaluating limits, including...

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that we have three functions f, g and h on some interval $[x_0, x_1]$, with

$$g(x) \leqslant f(x) \leqslant h(x),$$

" f is squeezed between g & h"

and

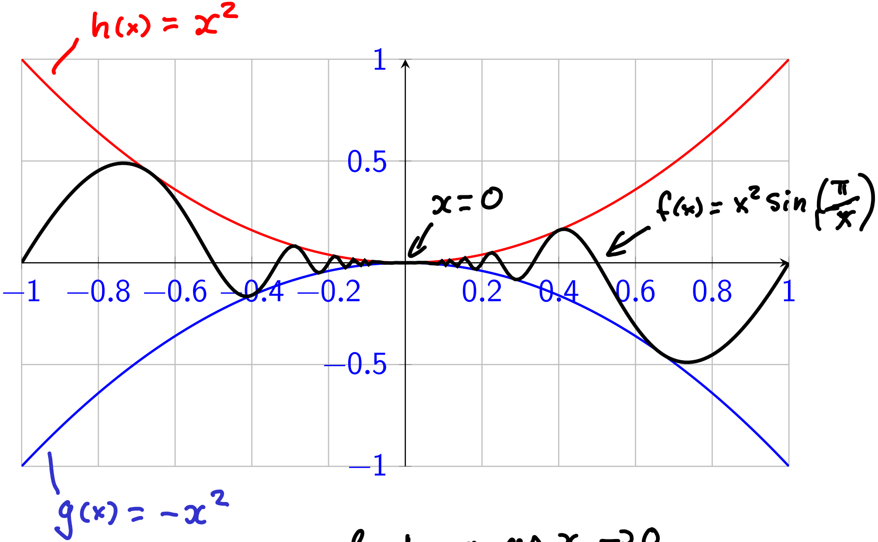
$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L,$$

h & g have some limit as x->a

for some $a \in [x_0, x_1]$. Then $\boxed{\lim_{x \to a} f(x) = L.}$

That is: if $\cancel{f}\,g(x)$ and $\cancel{g}\,h(x)$ have the same limit as $x \to a$, and $\cancel{h}\,f(x)$ is "squeezed" between them, then $\cancel{h}\,f(x)$ has the same limit as $x \to a$.

The Squeeze Theorem



Here $g \to 0$ as $x \to 0$ & $h \to 0$ as $x \to 0$

And $g(x) \leq f(x) \leq h(x)$.

Example

Suppose $f(x)$ is a function such that

$$1 - \frac{x^2}{4} \leqslant f(x) \leqslant 1 + \frac{x^2}{2}, \ \forall x \neq 0$$

Find $\lim_{x\to 0} f(x)$.

$g(x) = 1 - \frac{x^2}{4}$ $\quad\quad$ $\lim_{x \to 0} g(x) = 1 - \frac{0}{4} = 1.$

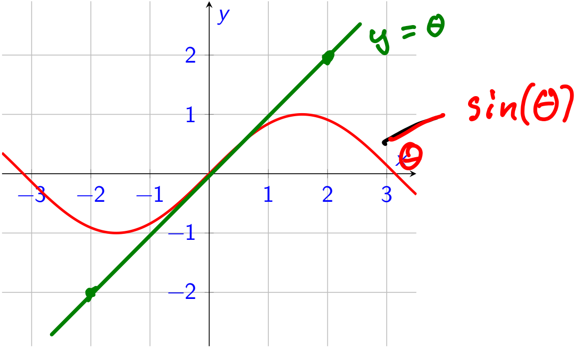
$h(x) = 1 + \frac{x^2}{2}$ $\quad\quad$ $\lim_{x \to 0} h(x) = 1$

So $\lim_{x \to 0} f(x) = 1$ as well.

Next week, we will use the Squeese Theorem to explain an important limit:

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

For now, let's just convince ourselves:



Finished here (!).

Exercises

Exercise 2.3.1

Evaluate the following limits

(a) $\displaystyle\lim_{x\to\frac{1}{2}} \frac{x-\frac{1}{2}}{x^2-\frac{1}{4}}$

(b) $\displaystyle\lim_{x\to-4} \frac{x^2+3x-4}{x^2+x-12}$

Exercise 2.3.2

(From 2023/2024 MA140 exam, Q1(a)) Evaluate the limit

$$\lim_{x\to 4} \frac{x-4}{(\sqrt{x}-2)(x+9)}$$

Exercises

Exercise 2.3.3

Suppose that $g(x) = 9x^2 - 3x + 1/4$, and $f(x)$ is such that $-g(x) \leqslant f(x) \leqslant g(x)$ for all x.

1. Can one use the Squeeze Theorem to determine $\lim_{x \to 1/3} f(x)$? If so, do so. If not, explain why.
2. Can one use the Squeeze Theorem to determine $\lim_{x \to 1/6} f(x)$? If so, do so. If not, explain why.

Exercises

Exercise 2.3.4 (from 2425-MA140 exam)

Let $f(x) = \dfrac{x^2 - 2x - 15}{3x^3 - 6x^2 - 45x}$. For each of the following, evaluate the limit, or determine that it does not exist.

(i) $\lim_{x \to -3} f(x)$

(ii) $\lim_{x \to 0} f(x)$

(iii) $\lim_{x \to 5} f(x)$