MA378: §1 Interpolation

§1.1 Introduction to Interpolation

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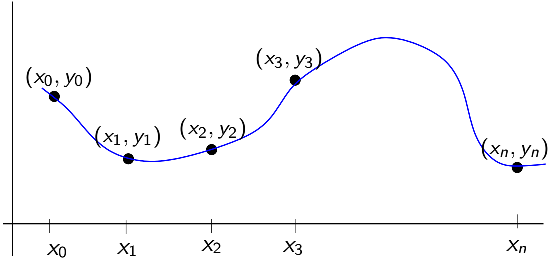
January 2024



1.1 Introduction

Suppose that we have a two sets of $n+1$ real numbers:

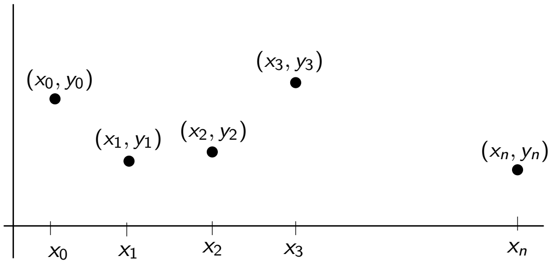
- $\{x_i\}_{i=0}^n$, which are are *strictly* increasing, meaning that $x_0 < x_1 < x_2 < \cdots < x_n$
- and $\{y_i\}_{i=0}^n$.

Interpolation problems are of the form: *Find a function, p, that is continuous and defined on* $[x_0, x_n]$, *such that*

$$p(x_k) = y_k, \quad \text{for} \quad k = 0, 1, \ldots, n.$$

We say: "*p interpolates* the points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n)".

1.1 Introduction



1.2 Why do this?

Why would one like to do this? There are several possibilities, including

- The points belong to an underlying, but unknown function, f. We wish to establish likely values of f at points other than x_0, x_1, ..., x_n. The values of f may have been obtained from physical experiments, or numerical procedures (e.g., Newton's method for initial value problems). Or it may be that some values of the function are easily available. For example $2! = 2$, and $3! = 6$, but what about $2\frac{1}{2}!$ or $\pi!$?
- We may know the function, but prefer to work with an interpolant to it. For example, in order to estimate derivatives or integrals of it.

Mathematics, from number theory to information theory, and nearly every aspect of numerical analysis, features many interpolation problem.

Elsewhere, the methods are used in fields ranging from aircraft design to computer animation.

The main reference for this section is Chapter 6 of Suli and Mayers, See also Lectures 18–20 of Stewart's *Afternotes on Numerical Analysis*.

Definition 1.1

$\mathcal{P}_n$ is the set of polynomials of degree at most n and real-valued coefficients, i.e., $p_n \in \mathcal{P}_n$ if

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_i \in \mathbb{R}$.

Examples:

1.3 Polynomial Interpolation

Exercise 1.3.1

Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

It is particularly important to note that if p_n and q_n both belong to $\mathcal{P}_n$, then so too does their sum.

The Polynomial Interpolation Problem comes in two forms.

The Polynomial Interpolation Problem I (PIP1)

Given is set of points $x_0 < x_1 < \cdots < x_n$, and a set of real numbers $y_0, y_1, \ldots, y_n$, find $p_n \in \mathcal{P}_n$ such that

$$p_n(x_k) = y_k, \quad \text{for } k = 0, 1, \ldots, n.$$

The Polynomial Interpolation Problem II (PIP2)

Given is set of points $x_0 < x_1 < \cdots < x_n$, and a function $f : [x_0, x_n] \to \mathbb{R}$, find $p_n \in \mathcal{P}_n$ such that

$$p_n(x_k) = f(x_k), \quad \text{for } k = 0, 1, \dots, n.$$

Clearly PIP2 is just PIP1 with $y_k = f(x_k)$.

The questions that we must ask (and answer) are

(i) Is there a solution to the polynomial interpolation problem.

(ii) Is it unique?

(iii) How do we find it?

(iv) How accurate is it? If f is the underlying function (i.e., $f(x_k) = y_k$), can we find an upper bound for

$$\max_{x_0 \leq x \leq x_n} \{|f(x) - p_n(x)|\}?$$

1.4 Exercises

Exercise 1.4.1

Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

Exercise 1.4.2

Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

1.4 Exercises

Exercise 1.4.3

(a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

§1 Interpolation

§1.2 Finding the interpolant

Dr Niall Madden, January 2024

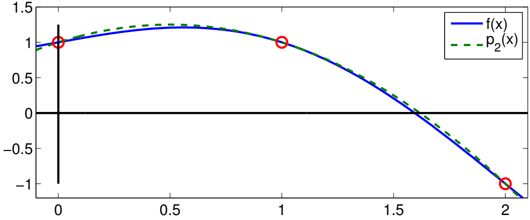
Source: http://jeff560.tripod.com/stamps.html

Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics, including *Calculus of Variations*.

2.5 Finding the polynomial

Example 2.2

Show that $p_2 = -x^2 + x + 1$ is a polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.



It is not hard to convince ourselves that $-x^2 + x + 1.$ is the solution to the above PIP. But how to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

As a first step, we'll prove the following:

Lemma 2.3

If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all x).

Theorem 2.4 (There is a unique solution to the PIP)

There is at most one polynomial of degree $\leq n$ *that interpolates the* $n+1$ *points* (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) *where* x_0, x_1, ..., x_n *are distinct.*

2.6 The Vandermonde matrix method

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find p_2 such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1 x + a_2 x^2$, this just amounts the finding the values of the coefficients a_0, a_1, and a_2. One might be tempted to solve for them using the system of equations

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

This is known as the *Vandermonde System*.

2.6 The Vandermonde matrix method

Writing

$$
\begin{aligned}
a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\
a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 &= y_2
\end{aligned}
$$

in matrix-vector format we get

$$
\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \quad \text{or} \quad V\mathrm{a} = \mathrm{y}. \tag{1}
$$

But this may not be a good idea. At the vest least, we'd have to solve a linear system of equations. Furthermore, the system is very *ill-conditioned*.

2.6 The Vandermonde matrix method

[*Slides 19 and* **??** *depend on material from MA385, so we'll skip them in class: please read in your own time.*]

In MA385 we learned about the relationship between the *condition number* of a matrix, V, and the relative error in the (numerical) solution to a matrix-vector equation with V as the coefficient matrix. The condition number is $\kappa(V) = \|V\|\|V^{-1}\|$, for some subordinate matrix norm $\|\cdot\|$.

2.6 The Vandermonde matrix method

Example 2.5 (Stewart's "Afternotes...", Lecture 19)

Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

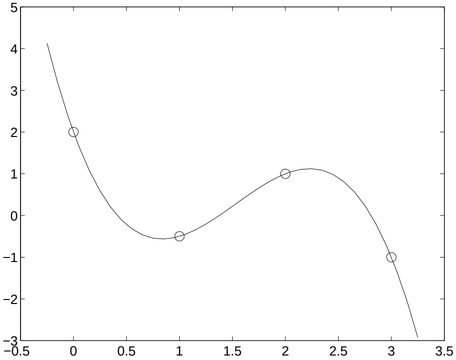
2.7 Lagrange Interpolation

We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (Here "constructive" means that we'll prove it exists by actually computing it).

2.7 Lagrange Interpolation

Example 2.6

Consider the problem: *find $p_3 \in \mathcal{P}_3$ such that*
$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$

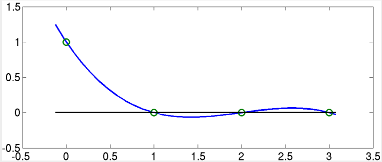


2.7 Lagrange Interpolation

Here is an easier problem to solve.

Find $L_0 \in \mathcal{P}_3$ such that

$$L_0(0) = 1, \quad L_0(1) = 0, \quad L_0(2) = 0, \quad L_0(3) = 0.$$



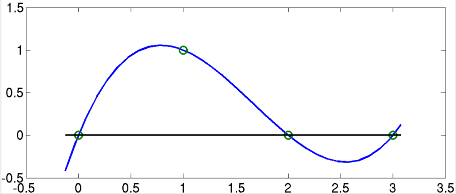
Because L_0 is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$. Choosing C so that $L_0(0) = 1$, we get

$L_0(x) =$

2.7 Lagrange Interpolation

Similarly, let $L_1 \in \mathcal{P}_3$ be the cubic polynomial such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

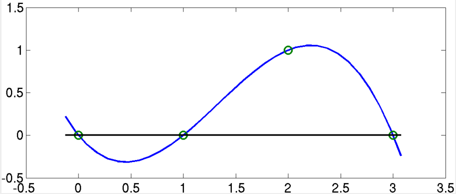


Then

$$L_1(x) =$$

2.7 Lagrange Interpolation

In the same style, let $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



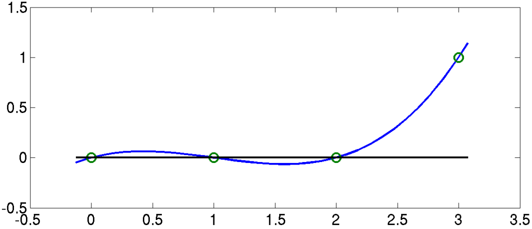
$$L_2(x) =$$

Finally, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j\neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$



2.7 Lagrange Interpolation

Because each of L_0, L_1, L_2, and L_3 is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (1/2)L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. Furthermore

$$\begin{array}{rcccccc}
p_3(0) & = & 2L_0(0) & -(1/2)L_1(0) & +(1)L_2(0) & +(-1)L_3(0) \\
 & = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
 & = & 2, & & & \\
p_3(1) & = & 2L_0(1) & -(1/2)L_1(1) & +(1)L_2(1) & +(-1)L_3(1) \\
 & = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
 & = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
 & = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
 & = & 1, & & & \\
p_3(3) & = & 2L_0(3) & -(1/2)L_1(3) & +(1)L_2(3) & +(-1)L_3(3) \\
 & = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
 & = & -1. & & &
\end{array}$$

Thus p_3 solves the problem!

2.8 The Lagrange Form

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation. We'll now look how to solve the general problem.

Definition 2.7 (Lagrange Polynomials)

The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2b}$$

2.8 The Lagrange Form

Definition 2.8

The **Lagrange form of the Interpolating Polynomial**

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3a}$$

or

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{3b}$$

Take care not to confuse the Lagrange Polynomials, which are the L_i with the Lagrange Interpolating Polynomial, which is the p_n defined in (3).

2.8 The Lagrange Form

Theorem 2.9 (Lagrange)

There exists a solution to the Polynomial Interpolation Problem and it is given by

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

2.9 Example

Example 2.10 (Süli and Mayers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

2.9 Example

The figure below shows the solution to Example 2.10 (top) and the difference between the function e^x and its interpolant (bottom). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.