Section 1: SPD Matrices (again). This lecture will be mainly about M-matrices, and so form a link between SPD matrices from Lecture 17, and the Perron–Frobenius theorem of Lecture 19. First, though, we'll finish the section on SPD matrices.

Recall that a real square matrix, A, is symmetric positive definite (SPD) if, for all non-zero vectors, x, we have that $x^T Ax > 0$.

We now finished establishing some of the following properties of SPD matrices.

(a) *Let A and B both be real $n \times n$ matrices. If B^{-1} exists, then A is SPD $\iff$ $B^T AB$ is SPD.* (See Lecture 17).

(b) *If A is SPD, then any principle submatrix of A is SPD.* (See Lecture 17).

(c) *A is SPD if and only if $A = A^T$ and all the eigenvalues of A are positive.* In Lecture 17 we proved that all the eigenvalues of an SPD matrix are positive. For the converse, note that we are trying to show that, if a symmetric matrix has only positive eigenvalues, then it is SPD. The proof uses a fact established in Lecture 6, and Exercise Sheet 1. Any symmetric matrix is unitarily diagonalisable. That is, there is a unitary matrix V whose columns are eigenvectors associated with eigevalues of A, such that $A = VDV^T$, where D is the diagonal matrix of eigenvalues of A. Then, for any vector x, we have that $x^T Ax = x^T (V^T DV)x = (Vx)^T D(Vx)$. Since V is unitary (and so has full rank), there is some y such that $y = Vx$. So $x^T Ax = y^T Dy = sum_{i=0}^{n} \lambda_i y^T y > 0$.

(d) *If A is SPD, then $a_{ii} > 0$ for all i, and $\max_{ij} |a_{ij}| = \max_i a_{ii}$.*

(e) *A is SPD of the determinant of every leading principal submatrix is positive.*

(f) *A is SPD $\iff$ there exists a unique lower triangular matrix L with positive diagonal entries such that $A = LL^T$.* This is called the Cholesky factorisation of A.

In this class we'll prove Part (d). For Part (e) it is easy to prove that the determinant of any principal submatrix of A is positive (the determinant of a matrix is the product of its eigenvalues). The converse is a little trickier, and usually involves an *interlacing* theorem.

Part (f) is very important in other contexts, but we probably won't get to it.

Definition 1 (Positive semi-definite)**.** A matrix, $A \in M_{n\times n}(\mathbb{R})$, is *symmetric positive semi-definite* if and only if $A = A^T$ (i.e., it is symmetric) and $\vec{x}^T A\vec{x} \geqslant 0$ for all vectors $\vec{x} \neq 0$.

We encounter semi-definite matrices in, for example, areas of graph theory.

Section 2: M-matrices There is remarkable number of different but equivalent definitions of an M-matrix. To try to break it down, we'll first define a Z-matrix, then a non-negative matrix, then an M-matrix.

Definition 2 (Z-matrix)**.** A square matrix A is a Z-matrix if all its non-diagonal entries are non-positive. That is,

$$a_{i,j} \leqslant 0 \qquad \text{for any } i \neq j.$$

Definition 3 (Non-negative matrix)**.** A square matrix A is non-negative all its are non-negative. That is,

$$a_{i,j} \geqslant 0 \qquad \text{for all } i, j.$$

Notation: we write this as $A \geqslant 0$.

Definition 4 (Diagonally dominant)**.** A matrix A is *diagonally dominant* if

$$|a_{ii}| \geqslant \sum_{j\neq i} |a_{ij}| \qquad \text{for all } i, j.$$

We'll postpone a formal definition of M-matrices briefly, in favour of an example. We shall learn that if A is a Z-matrix with positive diagonal entries, and that is diagonally dominant, then $A^{-1} \geqslant 0$.

Example 5. The following matrix is an M-Matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \text{since} \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

It is also SPD; its eigenvalues are $1, 2 \pm \sqrt{2}$.

Definition 6 (Spectral radius)**.** Denote the eigenvalues of the square matrix A as $\lambda_1, \lambda_2, \dots \lambda_n$. The *spectral* radius of A is

$$\rho(A) := \max_i |\lambda_i|.$$

Next week you'll learn about the Perron Frobenius theorem: if $A \geqslant 0$, then $\rho(A)$ is an eigenvalue of A, and its uniquely maximal one.

Our final fact that we need about the spectral radius of a matrix is that, of $\rho(T) < 1$, then $\|T^k\| \to 0$ as $k \to \infty$. (This is true for any norm). We say such a T is *convergent*. You can observe that $(I - T)^{-1}$ exists, and is given by $I + T + T^2 + T^3 + \dots$.

Finally, we can define an M-matrix.

Definition 7 (M-matrix)**.** A square matrix A is an M-matrix if it is a non-singular Z-matrix that can be written as

$$A = sI - B,$$

where $B \geqslant 0$ and $s > \rho(B)$.

Theorem 8. If A is an M-matrix, then $A^{-1} \geqslant 0$.

To show this, write A as $A = sI - B$. Let $T = B/s$, and note that $\rho(T) < 1$. Then

$$A^{-1} = \frac{1}{s}(I - T)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty} T^k.$$

Since $T \geqslant 0$, it must be that each $T^k \geqslant 0$, and so $A^{-1} \geqslant 0$.