MA211

Lecture 11: The case $D < 0$

Monday, 8^{th} October 2008

Class test on Wednesday

Reminder: There will be a 30 minute in-class test on Wednesday.

It will be worth approximately 5% for total for MA211.

Questions will be based on **Problem Set 2**.

In this class...

1. Recall...
   - $D > 0$
   - $D = 0$
2. $D < 0$
   - Simple Harmonic Motion
   - In general
3. Initial Value Problems
4. Boundary Value Problems

For more details, see **17.1** of Stewart.

Recall...

2nd Order, Constant Coefficient, Homogeneous DEs

Last week we started on solving problems of the form

$$ay''(x) + by'(x) + cy(x) = 0.$$

where a, b and c are constants (real numbers).

We introduced the *The Auxiliary Equation*:

$$aR^2 + bR + c = 0,$$

and the **Discriminant**, $D = b^2 - 4ac$.

We use a different approaches depending on if

(i) $D > 0$, (ii) $D = 0$ (iii) $D < 0$.

The easiest case is $D = b^2 - 4ac > 0$.

$D > 0$

If $D = b^2 - 4ac > 0$, then the auxiliary equation

$$ar^2 + br + c = 0$$

has two solutions:

$$R_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad R_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

and the general solution is

$$y(x) = Ae^{R_1 x} + Be^{R_2 x}.$$

The next easiest case is $D = b^2 - 4ac = 0$.

$D = 0$

If $D = b^2 - 4ac = 0$, then the auxiliary equation

$$ar^2 + br + c = 0$$

has just one solution:

$$R = \frac{-b}{2a},$$

and the general solution is

$$y(x) = Ae^{Rx} + Bxe^{Rx}.$$

$D < 0$

Finally, we consider the most complicated situation:

$D < 0$

$$D = b^2 - 4ac < 0,$$

so that the solutions to the auxiliary equation are *complex valued*.

But first we considered the simplest situation: when $a = 1$, $b = 0$ and $c = \omega^2 > 0$.

This describes *simple harmonic motion*

Example (Simple Harmonic Motion)

The general solution to the DE

$$y'' + \omega^2 y = 0.$$

is $y = \alpha e^{i\omega x} + \beta e^{i\omega x}$.
Using the Euler Forula, we can write this as

$$y = A\cos(\omega x) + B\sin(\omega x).$$

Then general case

If $D = b^2 - 4ac < 0$, then the auxiliary equation
$$ar^2 + br + c = 0$$
has two complex-valued solutions:
$$R_1 = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}, \qquad R_2 = \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}.$$

We write these as

$R_1 = k + i\omega, R_2 = k - i\omega$ where $k = \dfrac{-b}{2a}, \omega = \dfrac{\sqrt{4ac - b^2}}{2a}$.

Then one form of the general solution is:
$$y(x) = e^{kx}\left(\alpha e^{i\omega x} + \beta e^{-i\omega x}\right).$$

Now use the Euler Formula...

...

And by letting $A = \alpha + \beta$, $B = i(\alpha - \beta)$, we can rewrite this as

$$y(x) = e^{kx}\big(A\cos(\omega x) + B\sin(\omega x)\big).$$

(See also Exercise 11.2 from Problem Set 2.)

To summarise:

$D < 0$

If $D = b^2 - 4ac < 0$, then the auxiliary equation is
$$ar^2 + br + c = 0$$
It's solutions are
$$R_1 = k + i\omega, \qquad R_2 = k - i\omega$$
where
$$k = \frac{-b}{2a}, \qquad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$
Then the general solution can be expressed as
$$y(x) = e^{kx}\big(A\cos(\omega x) + B\sin(\omega x)\big).$$

Example

Find the general solution to the equation

$$y'' + y' + y = 0.$$

Solution:

Example

Solve the following differential equation:

$$y'' - 6y' + 13y = 0.$$

Solution:

Exercise (Q10.3)

Solve the following differential equations:

(i) $y'' = -2y$.

(ii) $y'' + 4y' + 13y = 0$.

(iii) $y'' + 2y' + 5y = 0$.

(iv) $8y'' + 12y' + 5y = 0$.

Exercise (Q11.2)

(*Here is an alternative way of dealing with the case $D < 0$ other than using Euler's formula.*)
Suppose we wish to find the general solution to the DE

$$ay'' + by' + cy = 0 \text{ with } b^2 < 4ac.$$

The roots of the auxiliary equation are $R = k \pm i\omega$ where $k = -b/(2a)$ and $\omega = \sqrt{4ac - b^2}/(2a)$.
Show that if $y(x) = e^{kx}u(x)$ then $u(x)$ satisfies

$$u''(x) + \omega^2 u(x) = 0.$$

Solve this equation to give an expression for $y(x)$.

Initial Value Problems

So far we have found *general solutions* to these equations: they involve arbitrary constants A and B.

If we are given more information, then we can solve for A and B.

Initial Values

If we are given the value of the solution and it's derivative at **the same point** these are called *Initial Values*.
We use the initial values to find *particular solutions*: that is, specific values of A and B for our problem.

Initial Value Problems

Example

Solve the DE: $$y'' + 2y' + 2y = 0,$$
with initial values: $$y(0) = 2, \quad y'(0) = -3.$$

Initial Value Problems

Exercise (11.3)

Solve the following initial value problems.

(i) $2y'' + 5y' - 3y = 0,\ y(0) = 1, y'(0) = 1.$

(ii) $y'' + 10y' + 25y = 0,\ y(1) = 0, y'(1) = 2.$

(iii) $y'' + 4y' + 5y = 0,\ y(0) = y'(0) = 2.$

Boundary Value Problems

Boundary Values

If we are given the value of the solution at two different points these are called *Boundary Values*.

Example

Find the particular solution to DE

$$y'' + y' - 2y = 0,$$

with boundary values

$$y(0) = 0, y(1) = 1.$$

Boundary Value Problems

Exercise (11.4)

Solve the following **boundary** value problems:

(i) $2y'' + 5y' - 3y = 0, \quad y(0) = 1, y(1) = e^{1/2}.$

(ii) $y'' + 10y' + 25y = 0, \quad y(0) = -1, y(1) = 0.$

(iii) $y'' + 9y = 0,\ y(0) = 2,\ y(\pi/2) = 3.$