2425-MA140 Engineering Calculus

Week 07, Lecture 2 (L20)
The Fundamental Theorem of Calculus

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Suimeálaithe — Tá tairisigh na suimeála fágtha ar lár.

Integrals — Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ln a}$
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Suimeáil na míreanna $\quad \int u\,dv = uv - \int v\,du \quad$ **Integration by parts**

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Tutorials, Assignments, etc

- I'm **very sorry** yesterday's 3pm tutorial in ENG-2034, for Teams 3 and 4, didn't take place. If possible, please attend a different tutorial, ideally on Friday. .
- **Assignment 3 (resit)**: Last reminder: send my your **results summary** if the result you got does not agree with what you think you scored.
- **Assignment 5** is open. Deadline is 5pm next Monday (4 November). You have 3 attempts for each question. However, Q1 will be manually graded after the deadline.

The exciting topics that await us in today:

See also: Sections **4.10** (Antiderivatives) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Recall from yesterday:

If $f(x)$ is a function defined on an interval $[a, b]$

- The **definite integral** of f from a to b is

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=0}^{n-1} hf(x_i),$$

  where $h = (b-a)/n$ and $x_i = a + ih$.
- This is the area of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$, and $x = b$.
- Given a function, f, we can define another, F as

$$F(x) = \int_a^x f(t)dt.$$

  That is, the variable in F is the upper limit of integration on the right. For an nice illustration, see https://www.geogebra.org/m/ugTmVRHj

Recall from yesterday:

The function is the integrand.

Upper limit of integration

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

Fundamental Thm of Calculus: Part 1

Fundamental Theorem of Calculus: Part 1 (FTC1)

If $f(x)$ is a continuous function on $[a, b]$, and $F(x)$ is defined as

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$ for $x \in [a, b]$.

Roughly: $\boxed{\text{the derivative of the integral of } f \text{ is } f}$. You can find a proof in Section 5.3 of the textbook.

Fundamental Thm of Calculus: Part 1

Example

Let $g(x) = \int_1^x \frac{1}{t^3+1} dt$. Find $g'(x)$.

FTC1+Chain Rule

Sometimes the limit of integration is a more complicated function of x. In that case, we can apply the **Chain Rule**, along with the FTC1.

Example

Let $F(x) = \int_1^{\sqrt{x}} \sin(t)dt$. Find $F'(x)$.

Idea: Let $u(x) = \sqrt{x} = x^{1/2}$. So $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Then...

$$F'(x) = \frac{dF}{du}\frac{du}{dx} = \sin(u(x))\left(\frac{1}{2\sqrt{x}}\right) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

Antiderivatives

Definition: Antiderivative

A function F is an **antiderivative** of f on $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$. Thus,

$$f \text{ is the derivative of } F \Leftrightarrow F \text{ is an antiderivative of } f.$$

Note: If F is an antiderivative of f, then the most general antiderivative of f is

$$F(x) + c\,,$$

where c is an arbitrary constant, called a **constant of integration**.

Example: For any x, $F(x) = x^2 + c$ is an antiderivative of $f(x) = 2x$.

Antiderivatives

Example: The *general* antiderivative of $f(x) = 3x^2$ is $F(x) = x^3 + c$.

Antiderivatives

Examples

Find all antiderivatives of the following functions

(i) $f(x) = \dfrac{1}{x}$ for $x > 0$.

(ii) $f(x) = \sin(x)$

(iii) $f(x) = e^x$.

Definition: indefinite integral

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,\mathrm{d}x$$

is the general antiderivative of f. That is, if F is an antiderivative of f, then

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

Examples:

- $\int 2x\,dx = x^2 + C$
- $\int 3x^2\,dx = x^3 + C$

- $\int x\,dx = \frac{1}{2}x^2 + C$
- $\int x^2\,dx = \frac{1}{3}x^3 + C$.

Spotting the pattern we can deduce...

Power Rule of Integration

If $n \neq -1$, then

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C\,.$$

Note: For $n = -1$, we have

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C\,.$$

Here is a list of the antiderivatives of some common functions.

- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- $\int a^x\,dx = \frac{a^x}{\ln a} + C$
- $\int \sin(x)\,dx = -\cos(x) + C$
- $\int \cos(x)\,dx = \sin(x) + C$
- $\int \tan(x)\,dx = \ln|\sec(x)| + C$
- ...

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Suimeáil na míreanna

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Integration by parts

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Properties of Integration

1. If k is a constant, then

$$\int kf(x)\,dx = k\int f(x)\,dx\,.$$

2. Integration is additive:

$$\int (f(x) \pm g(x))\,\,dx = \int f(x)\,dx \pm \int g(x)\,dx\,.$$

Example

Evaluate the following integrals

1. $\displaystyle\int 2x^2 + 9x^7 \,\mathrm{d}x$.
2. $\displaystyle\int \frac{4}{1+x^2} \,\mathrm{d}x$.

The Fundamental Thm of Calculus: Part 2

Now that we know all about antiderivatves, we can see how the link to **definite integrals**

Theorem (The Fundamental Thm of Calculus, Part 2)

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Notation: We often write $F(b) - F(a)$ as $F(x)\Big|_{x=a}^{x=b}$, or simply $F(x)\Big|_a^b$.

The Fundamental Thm of Calculus: Part 2

Example: Show that $\int_{-1}^{1} (x^2 + 2)\,\mathrm{d}x = \frac{14}{3}$

Example: Show that $\int_{-1}^{1} (x^3 + x)\,\mathrm{d}x = 0$

Exercises

Exer 7.2.1

Let $F(x) = \int_x^{2x} t\,\mathrm{d}t$. Use the Fundamental Theorem of Calculus to evaluate $F'(x)$.

Hint: we can split this into two integrals:

$$F(x) = \int_x^{2x} t\,\mathrm{d}t = \int_x^0 t\,\mathrm{d}t + \int_0^{2x} t\,\mathrm{d}t = -\int_0^x t\mathrm{d}t + \int_0^{2x} t\mathrm{d}t.$$

Now apply the FTC to each term, including the Chain Rule for the second.

Exer 7.2.2

Evaluate the following integrals.

1. $\int e^{2x} + \frac{1}{2x}\,\mathrm{d}x$
2. $\int \frac{3}{\sqrt{2-x^2}}\,\mathrm{d}x$

Exercises

Exer 7.2.3

Evaluate the definite integral $\int_1^e e^{2x} + \frac{1}{2x}\,\mathrm{d}x$

Exer 7.2.4

Find two values of q for which $\int_q^0 2x + x^2\,\mathrm{d}x = 0$.