2425-CS4423: Sample Exam Paper ANS with solutions

For more information on this paper, and how it relates to the actual example, refer to discussion in Class in Week 12.

Q1. (a) Give an example (e.g., by sketching) of a simple connected graph of order 6, and size 7.

Answer: Any graph with 6 nodes and 7 edges will do.

Is there any simple graph of order 6 and size 16? Explain your answer.

Answer: No. A simple graph on 6 nodes can have at most $\binom{6}{2} = 15$ edges.

Explain why there is no simple connected graph of order 6 and size 4.

Answer: For a graph with 6 nodes to be connected, it needs at least 5 edges.

(b) Consider the graph, G_1, shown in Figure 1. Write down the adjacency matrix, A_1, for G_1.

Answer:

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(c) Explain why G_1 is *not* bipartite. ANS It has 3-cycles: $1-3-4$ and $2-3-5$
Give an example of a subgraph of G_1 which is of order 7 and size 8 which *is* bipartite.

Answer: Subgraph obtained by deleting, e.g., edges $1-5$ and $2-4$; or edges $1-3$ and $2-3$, etc. But note that it is not enough just to delete any pair of edges. E.g., removing $2-3$ and $3-4$ still leaves a 3-cycle.

Give an example of a subgraph of G_1 which is of order 7 and is a tree.

Answer: Any options, e.g., graph obtained by deleting $1-5$, $2-3$, $3-4$, and $4-6$.

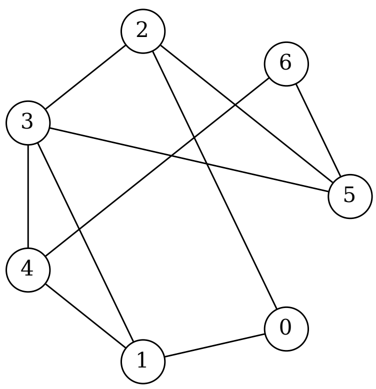
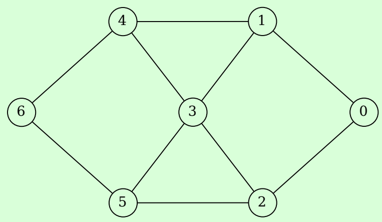


Figure 1: Graph G_1 from Question 1

Q2. (a) Sketch the graph with adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Answer: Actually, this is just the adjacency matrix of G_1 from Figure 1. But here is a nicer plot of it:



(b) Let A be the adjacency matrix of a graph G. Explain how one can compute the size of G as a function of the entries of A. ANS Sum the entries and divide by 2.

(c) Let A be the adjacency matrix of a graph G. Explain how one can compute the degree of the nodes of G as a function of the entries of A^2.

Answer: $(A^2)_{ii}$ is the degree of Node i.

(d) Let A be the adjacency matrix of a graph G. Explain how one can compute the number of triangles in G as a function of A^3.

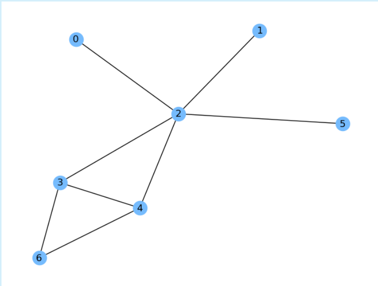
Answer: $\mathrm{trace}(A^3)/6$; see also Task 3 of Assignment 2 Part 2.

Q3. Consider the graph, G_3, generated by the following `networkx` instruction:

```
G3 = nx.Graph([[0,2], [1,2], [2,3], [2,4], [3,4], [2,5], [3,4], [3,6], [4,6]])
```

(a) Sketch G_3.

Answer:



(b) Calculate the *normalised degree centrality* of all nodes in G_3.

Answer: 0: 1/6; 1: 1/6; 2:5/6; 3:1/2, 4:1/2; 5:1/6; 6:1/3

(c) Determine both the radius and diameter of G_3.

Answer: For this, and the next question, it might help to note that the distance matrix is:

$$D = \begin{pmatrix} 0 & 2 & 1 & 2 & 2 & 2 & 3 \\ 2 & 0 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 & 0 & 3 \\ 3 & 3 & 2 & 1 & 1 & 3 & 0 \end{pmatrix}$$

The we see the radius is 2, and diameter is 3.

(d) Compute the *closeness centrality* of all nodes in G_3. ANS In order: 1/2., 1/2., 6/7, 2/9, 2/, 1/2, 6/13

(e) If one was to add another edge to G_3. Would that necessarily change both the degree centrality and closeness centrality of some of the nodes in G_3? If so, would they increase or decrease. Explain your answer.

Answer: They would change, assuming the new graph was still simple. Adding an edge will increase the degree centrality of 2 nodes. It mush also reduce the CC of two nodes

. . . .

Q4. (a) Describe Breadth First Search as an algorithm for computing distances between nodes in a (simple) graph. What is its input, what is its output, and what sequence of steps is taken to produce the output from the input?

> **Answer:** See lecture notes from, e.g., Week 5, Part 2.

(b) Consider the graph, G_4 shown in Figure 2. Show how to apply the Breadth First Search algorithm, starting at Node a, to determine, for every node, its *predecessors* on the shortest path between it and Node f. Use this information to list all shortest paths from Node a to Node f.

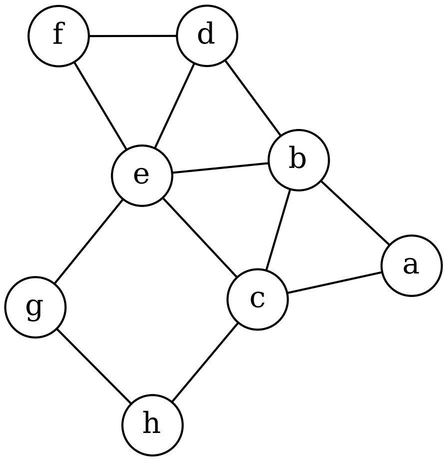
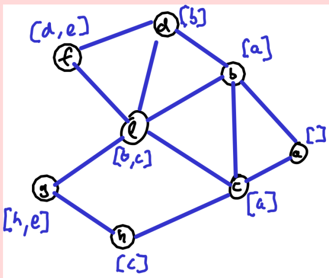


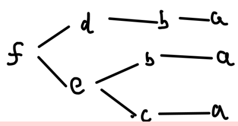
Figure 2: Graph G_4 from Question 4

Answer:



Node	Pred.	$d(a, x_i)$
a	∅	0
b	a	1
c	a	1
d	b	2
e	b, c	2
f	d, e	3
g	e, h	3
h	c	2

We'll list the paths from a to f by working back from f to a, in a tree-like way:



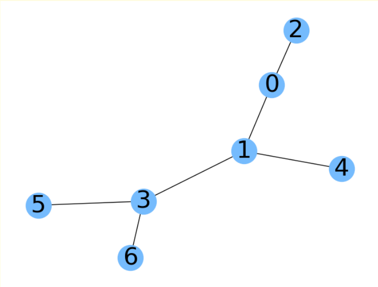
So there are 3 shortest paths:
a-b-d-f
a-b-e-f
a-c-e-f

Q5. (a) Let G_5 be the tree on the nodes $\{0,1,2,3,4,5,6\}$ that has as its *Laplacian matrix*

$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Sketch G_5.

Answer:



(b) How does one construct the Prüfer code for a tree? ANS See notes from Week 5

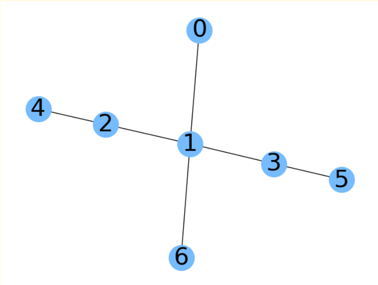
Compute the Prüfer code for G_5 from Part (a). ANS [0, 1, 1, 3, 3]

(c) How does a tree's Prüfer code relate to its degree sequence?

Answer: The degree of node i is 1 plus the number of times i occurs in the code

Construct the degree sequence for the tree on the nodes $\{0,1,2,3,4,5,6\}$ with Prüfer code $(1,2,1,3,1)$. Then construct and sketch the tree itself.

Answer: Degree sequence: $[1,4,2,2,1,1,1]$



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Q6. (a) Define the two Erdös-Rényi models, $G_{ER}(n, m)$ and $G_{ER}(n, p)$ of random graphs. ANS See lecture notes

(b) [5 Marks] In each model, what is the probability that a randomly chosen graph G has exactly m edges? Justify your answer.

> **Answer:** $G_{ER}(n, m)$ has exactly m edges with probability 1, by construction. $G_{ER}(n, p)$ has exactly m edges with probability $\binom{N}{m}p^m(1-p)^{N-m}$.

(c) A graph on 120 nodes is constructed by rolling a (fair) 6-sided die once for each possible edge: the edge is added only if the number shown is 3 or 6. What is the probability that a node chosen at random has degree 50? (You do not need to compute a numerical value. It is enough to give an explicit formula in terms of the given data).

> **Answer:** The probability that a specific node has degree k is
>
> $$\binom{n-1}{k}p^k(1-p)^{n-1-k}.$$
>
> In this case $p = 1/3$, $n = 120$ and $k = 50$. So the answer is $\binom{119}{50}(1/3)^{50}(2/3)^{69}$. You don't have to compute the value of that, but if you did you would find its about 0.01056.

Q7. (a) What is the *node clustering coefficient* of a node x in a graph G? What is the graph clustering coefficient C of G?

> **Answer:** See notes, especially Sections 4.2 and 4.3 of `https://www.niallmadden.ie/2425-CS4423/W10/CS4423-W10-Part-2.html`

(b) Determine the graph clustering coefficient C of a random graph in the $G_{ER}(n,p)$ model.

> **Answer:** For a node, i, with degree k, the social graph of i (i.e., the subgraph induced by the neighbours of i) has order k. Of the potential $\binom{k}{2}$ edges, one expects $p\binom{k}{2}$ to be present. Consequently, the clustering coefficient of i is
>
> $$c_i = \frac{p\binom{k}{2}}{\binom{k}{2}} = p.$$
>
> Then $C = \frac{1}{n}\sum_{i=1}^{n} c_i = \frac{1}{n}(np) = p.$.

How does C behave in the limit $n \to \infty$, when the average node degree is kept constant? What practical consequence does this observation have?

> **Answer:** $G_{ER}(n,p)$ has average degree k if $p = k/n$. So, if $k = np$ is some fixed constant, then, as $n \to \infty$ we must have $p \to 0$. Consequently, we get $C \to 0$. In practical terms, this means that $G_{ER}(n,p)$ will have a negligible number of triangles. Therefore, $G_{ER}(n,p)$ graphs to not exhibit the high clustering/transitivity observed in real small-world networks.

(c) Describe the *Watts-Strogatz small-world model* (WS model). What properties does a random graph sampled from the WS model have, that one wouldn't find in a random graph sampled from the $G_{ER}(n,p)$ model, or in an (n,d)-circle graph?

> **Answer:** See `https://www.niallmadden.ie/2425-CS4423/W11/CS4423-Week11-Part-1.html`