MA385/MA530: Class Test
Thursday, 24 October 2019
Outline solutions

1. *Write out the Taylor Polynomial of degree 4, about $a = 0$, for $f(x) = 1/(1+x)$.*
   `Solution:` Then general form of the Taylor Polynomial of degree 4, about a, for a function f is

$$p_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4. \qquad (1)$$

In this case, $f(x) = (1+x)^{-1}$, so

$$f'(x) = -(1+x)^{-2}, \quad f''(x) = 2(1+x)^{-3}, \quad f^{(3)}(x) = -6(1+x)^{(-4)}, f^{(4)}(x) = 24(1+x)^{-5}.$$

Evaluating these derivatives at $x = 0$ gives

$$f(0) = 1, \quad f'(0) = -1, \quad f''(0) = 2, \quad f^{(3)}(0) = -6, \text{ and } f^{(4)}(0) = 24.$$

Taking these values, and $a = 0$ in (1), gives

$$p_4(x) = 1 - x + x^2 - x^3 + x^4.$$

. . . .

Give an expression for the remainder.
`Solution:` The formula for the remainder, associated with the Taylor Polynomial of degree 4, is

$$R_4 = \frac{(x-a)^5}{5!} f^{(5)}(\sigma) \quad \text{for some } \sigma \in (x, a).$$

Here $a = 0$, and $f^{(5)}(\sigma) = -120(1+\sigma)^{-6}$, so

$$R_4 = -\frac{x^5}{(1+\sigma)^6} \quad \text{for some } \sigma \in (x, a). \qquad (2)$$

. . . .

Give an upper bound for the remainder when $x = 1$, and when $x = 0.1$.
`Solution:` From (2) we get, for $x = 1$, that

$$|R_4| = \frac{x^5}{(1+\sigma)^6} \leq \max_{0 \leq x \leq 1} \frac{1}{(1+x)^6} \leq 1.$$

Similarly, when $x = 0.1$, (2) gives that

$$|R_4| = \frac{(0.1)^5}{(1+\sigma)^6} \leq 10^{-5} \max_{0 \leq x \leq 0.1} \frac{1}{(1+x)^6} \leq 10^{-5}.$$

2. *State* Newton's method *for solving the nonlinear equation* $f(x) = 0$.

`Answer:` *Newton's method* for solving the nonlinear equation $f(x) = 0$ is

Choose $x_0 \in [a, b]$. For $k = 0, 1, 2, \dots$ set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

[10 Marks]

Use a Taylor's series to show that if $f(\tau) = 0$*, and Newton's method generates a sequence of approximations* $\{x_0, x_1, \dots\}$*, then*

$$\tau - x_{k+1} = -\frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}, \qquad \text{for some } \eta_k \in [x_k, \tau]. \tag{3}$$

`Solution:` The relevant Truncated Taylor Series is

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{f''(\eta_k)}{2}(x - x_k)^2, \quad \text{for some } \eta \in [x, x_k].$$

Let $x = \tau$ (i.e., the solution to the nonlinear equation). Then

$$f(\tau) = f(x_k) + f'(x_k)(\tau - x_k) + \frac{f''(\eta_k)}{2}(\tau - x_k)^2, \quad \text{for some } \eta_k \in [x_k, \tau].$$

Using that $f(\tau) = 0$, and dividing by $f'(x_k)$ gives

$$0 = \frac{f(x_k)}{f'(x_k)} + (\tau - x_k) + \frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}, \quad \text{for some } \eta_k \in [x_k, \tau].$$

Now use the formula for Newton's method to get

$$0 = x_k - x_{k+1} + (\tau - x_k) + \frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}, \quad \text{for some } \eta_k \in [x_k, \tau].$$

Rearrange to get (3).

3. *Suppose we wish to find a solution to*

$$x + \frac{1}{x+1} = 2,$$

in the interval $[1,2]$.

(a) *Show that a solution to this problem exists.*

`Solution:` The equation $f(x) = 0$ must have (at least) one a solution in the interval $[a, b]$ if $f(a)f(b) \leq 0$. For this problem,

$$f(x) = x + \frac{1}{x+1} - 2, \tag{4}$$

and $[a, b] = [1, 2]$. Then

$$f(1) = 1 + \frac{1}{2} - 2 = -\frac{1}{2}, \quad \text{and } f(2) = 2 + \frac{1}{3} - 2 = \frac{1}{3}.$$

So $f(1)f(2) = -1/6 < 0$. So $f(x)$ has a root in $[1,2]$.

(b) *Say we take $x_0 = 1$. Use the Newton Error formula to give an upper bound for the error $|\tau - x_1|$.*

`Solution:` Suppose we take f as in (4). From (3), we know that

$$|\tau - x_1| =\leq \frac{1}{2}(\tau - x_0)^2 \frac{|f''(\eta_1)|}{|f'(x_0)|}, \qquad \text{for some } \eta_0 \in [x_0, \tau]. \tag{5}$$

We need to compute, or bound, the terms on the right-hand side on this equations. We do this as follows:

- We can't compute $|\tau - x_0|$ directly, since we don't know τ (or, don't need to know it). But we do know that $\tau \in [1, 2]$. Consequently, we can use the bound $|\tau - x_0| \leq 2 - 1 = 1$.
- We also can't compute $|f''(\eta_1)|$ exactly, since η_1 is some unknown value in the interval $[x_0, \tau]$. So We use that

$$|f''(\eta_1)| \leq \max_{x_0 \leq x \leq \tau} |f''(x)| \leq \max_{1 \leq x \leq 2} |f''(x)|.$$

  Then, since $f''(x) = 2/(x+1)^3$, which is a positive, decreasing function on $[1,2]$, we get that $|f''(\eta_1)| \leq |f''(1)| = 1/4$.
- Finally, since $f'(x) = 1 - 1/(x+1)^2$, and $x_0 = 1$, we get $|f'(x_0)| = 3/4$.

Putting this all together in (5), we get that $|\tau - x_1| \leq 1/6$.