MA378: §1 Interpolation

§4 Hermite Interpolation

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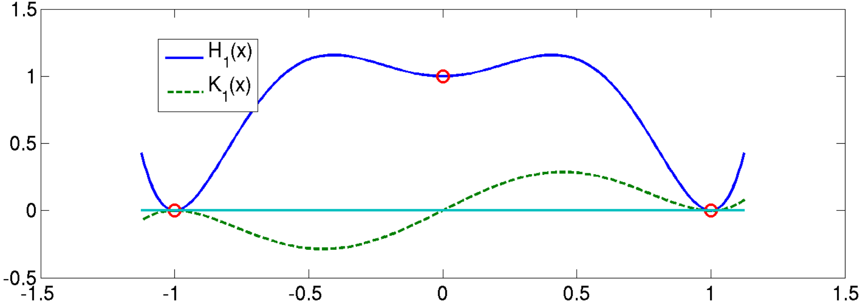


Charles Hermite

Charles Hermite, France, 1822–1901. Apart from this form of interpolation, his contributions to mathematics included the first proof that e is transcendental.

His methods were later used to show that π is transcendental.

Hermite interpolation is a variant on the standard Polynomial Interpolation Problem: we seek a polynomial that not only agrees with a given function f at the interpolation points, but its first derivative also matches f' at those points.

We are not that interested in this problem for its own sake, but the idea recurs again in the sections in piecewise polynomial interpolation and Gaussian quadrature.

Formally, the problem is

The Hermite Polynomial Interpolation Problem (HPIP) *Given a set of interpolation points* $x_0 < x_1 < \cdots < x_n$ *and a continuous, differentiable function* f*, find* $p_{2n+1} \in \mathcal{P}_{2n+1}$ *such that*

$$p_{2n+1}(x_i) = f(x_i) \quad \textit{and} \quad p'_{2n+1}(x_i) = f'(x_i).$$

One can prove that if there is a solution to this problem, then it is unique (see exercise).

Finished here Wed (25th Jan) at 1.50.

4.2 Constructing Hermite Interpolants

It is possible to solve this problem using an extension of the Lagrange Polynomial approach. Given the usual Lagrange Polynomials, $\{L_i\}$, for $i = 0, \dots, n$, let

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

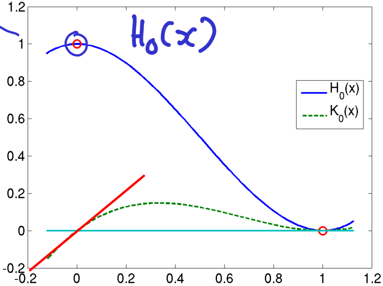
$$K_i(x) = [L_i(x)]^2(x - x_i).$$

$H_0(x_0) = 1$

$H_0'(x_0) = 0$

$K_0(x_0) = 0$

$K_0'(x_0) = 1$



$K_0(x_1) = 0.$

$K_0'(x_1) = 0$

$H_0(x_1) = 0$

$H'(x_1) = 0$

Hermite bases functions H_0 and K_0 for $n = 1$, $x_0 = 0$ and $x_1 = 1$

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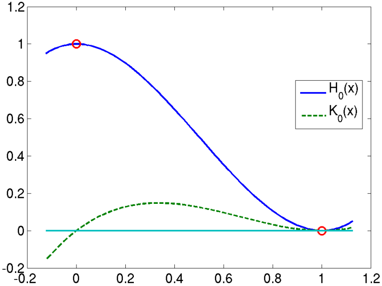
$$H_i(x) = [L_i(x)]^2 \big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2 (x - x_i).$$

Note that $L_i(x)$ is a poly of degree n. So $(L_i(x))^2$ has degree $2n$.

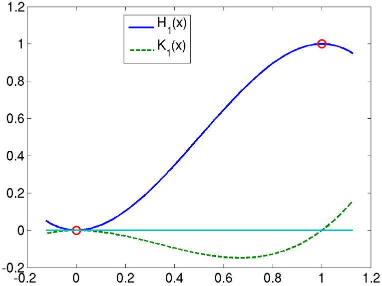


Hermite bases functions H_0 and K_0 for $n = 1$, $x_0 = 0$ and $x_1 = 1$

And $H_i(x)$ has degree $2n+1$.

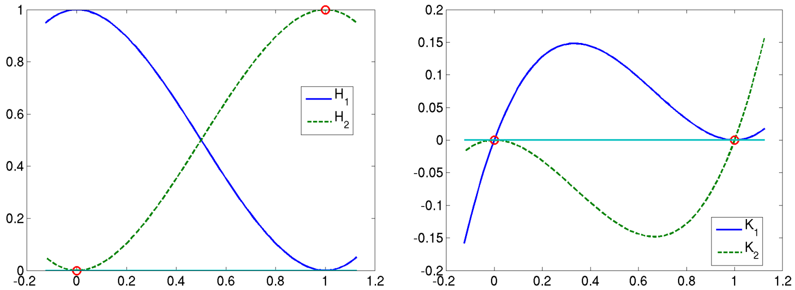
4.2 Constructing Hermite Interpolants

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$
$$K_i(x) = [L_i(x)]^2(x - x_i).$$



Hermite bases functions H_1 and K_1 (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

4.2 Constructing Hermite Interpolants



Hermite bases functions H_0, H_1 (left) and K_0, K_1 (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

The Hermite basis functions

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We can show that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad\qquad H_i'(x_k) = 0\ \forall k$$

Did this on the white board.

4.2 Constructing Hermite Interpolants

The Hermite basis functions

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Also, for $i, k = 0, 1, \dots n$,

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

This part is left as an exercise.

4.2 Constructing Hermite Interpolants

One can now show that the solution to the HPIP exists and is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i) H_i(x) + f'(x_i) K_i(x) \big).$$

Again, this part is left as an exercise.

Need to verify that

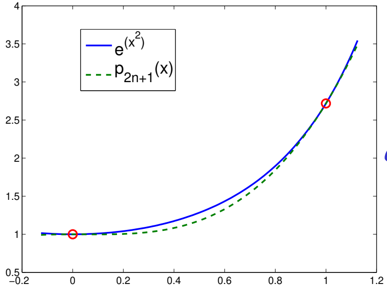
$$p_{2n+1}(x_i) = f(x_i)$$

and $$p'_{2n+1}(x_i) = f'(x_i)$$

4.2 Constructing Hermite Interpolants

Example 4.1

Find the polynomial of degree 3 that interpolates $\exp(x^2)$, and its first derivative, at $x_0 = 0$ and $x_1 = 1$. (See below).



Here

$$H_i(x) = (L_i(x))^2 (1 - 2L_i'(x_i)(x - x_i))$$

and $K_i(x) = (L_i(x))^2 (x - x_i)$

Here $L_0(x) = \dfrac{x-1}{(-1)} = 1 - x$.

$L_1(x) = x$.

Then $H_0(x) = (1-x)^2 (1 + 2(x))$

$H_1(x) = x^2 (1 - 2(x-1))$

PTO

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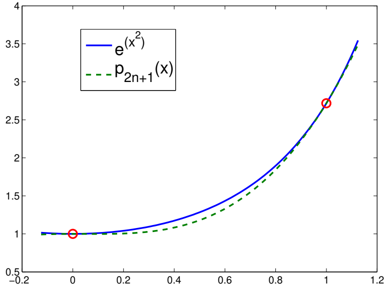

Similarly

$$K_0(x) = (x-1)^2 x$$
$$K_1(x) = (x-1)\, x^2$$

Then $p_3(x) = (x-1)^2(1+2x)\, f(0) + (x)^2(3-2x)\, f(1)$
$+ (x-1)^2 x\, f'(0) + (x-1)x^2 f(1)$

where $f(0)=1$, $f(1)=e$, $f'(0)=0$, $f'(1)=2e$

4.3 Error estimates

Theorem 4.2

Let f be a real-valued function that is continuous and defined on $[a, b]$, such that the derivatives of f of order $2n+2$ exist and are continuous on $[a, b]$. Let p_{2n+1} be the Hermite interpolant to f. Then, for any $x \in [a, b]$ there is an $\tau \in (a, b)$ such that

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!}[\pi_{n+1}(x)]^2.$$

We won't do a proof of this in class. However, later in this course we'll be interested in the particular example of finding p_3 the cubic Hermite Polynomial Interpolant to a function f at the points x_0 and x_1. Also, see exercises...

4.4 Exercises

Exercise 4.1

For *just* the case $n = 1$, state and prove an appropriate version of Theorem 4.2 (i.e., error in the Hermite interpolant). Use this to find a bound for $\|f - p_3\|_{[x_0,x_1]}$ in terms of f and $h = x_1 - x_0$. (Here $\|g\|_{[x_0,x_1]}$ is short-hand for $\max_{x_0 \leq x \leq x_1} |g(x)|$.)

Exercise 4.2

Let $n = 2$ and $x_0 = -1$, $x_0 = 1$ and $x_1 = 1$. Write out the formulae for H_i and K_i for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 4.3

Do Exercise 6.6 from from Süli and Mayers, *An Introduction to Numerical Analysis*.

4.4 Exercises

Exercise 4.4

Let L_0, L_1, ..., L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \dots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

and

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad\qquad H_i'(x_k) = 0.$$

Show that: $K_i(x_k) = 0$, for $k = 0, 1, \dots n$, and $K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \left(f(x_i)H_i(x) + f'(x_i)K_i(x) \right).$$

Exercise 4.5

Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$.

Exercise 4.6

(This exercise is based on Exer 6.5 from Süli and Mayers' *Introduction to Numerical Analysis*). Consider the following problem.

> *Take $n+1$ distinct interpolation points $x_0 < x_1 < \cdots < x_n$. Let p_{2n+1} be the polynomial of degree $2n+1$ with the property that*
>
> $$p_{2n+1}(x_i) = f(x_i),$$
>
> *and*
>
> $$p''_{2n+1}(x_i) = f''(x_i).$$

In general this problem does *not* have a unique problem.

(i) Explain briefly but carefully why the arguments, based on Rolle's Theorem, used to prove **uniqueness** of solutions to the HPIP, will not work here.

(ii) Show that there is no $p_5(x)$ that solves this problem when

- $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.
- $f(-1) = 1$, $f(0) = 0$, $f(1) = 1$.
- $f''(-1) = 0$, $f''(0) = 0$, $f''(1) = 0$.

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