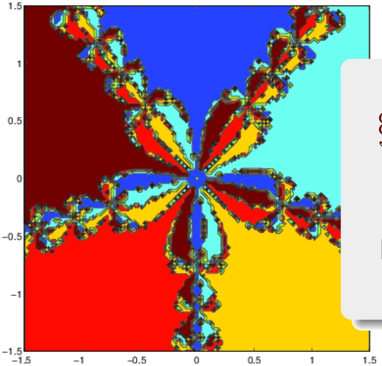

§1 Solving nonlinear equations

§1.5: Wrap up: what has Newton's method ever done for me?

MA385/530 – Numerical Analysis 1

October 2019

When studying a numerical method (or any piece of Mathematics) you should ask *why* you are doing this. For example, it might be

- because it will help you can understand other topics later;
- because it is interesting/beautiful in its own right; or
- (most commonly) because it is useful.

Here are some instances of each of these:

1. The analyses we have used in this section allowed us to consider some important ideas in a simple setting.

Examples include

- **Convergence**, including *rates of convergence*
- **Fixed-point theory**, and contractions. We'll be seeing analogous ideas in the next section (Lipschitz conditions).
- **The approximation of functions by polynomials** (Taylor's Theorem). This point will reoccur in the next section, and all through-out next semester.

2. Applications come from lots of areas of science and engineering. In financial mathematics, the **Black-Scholes equation** for pricing a put option can be written as

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0.$$

- $V(S,t)$ is the current value of the right (but not the obligation) to buy or sell ("put" or "call") an asset at a future time T;
- S is the current value of the underlying asset;
- r is the current interest rate (because the value of the option has to be compared with what we would have gained by investing the money we paid for it)
- σ is the volatility of the asset's price.

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0.$$

Often one knows S, T and r, but not σ. The method of *implied volatility* is when we take data from the market and then find the value of σ which, if used in the Black-Scholes equation, would match this data. This is a nonlinear problem and so Newton's method can be used. See Chapters 13 and 14 of Higham's "An Introduction to Financial Option Valuation" for more details.

(We will return to the Black-Scholes problem again at the end of the next section).

3. Some of these ideas are interesting *and* beautiful. The ***complex*** $n^{\rm th}$ ***roots of unity*** is the set of numbers $\{z_0, z_1, \dots, z_{n-1}\}$ who's $n^{\rm th}$ roots are 1. They can be expressed as

$$z_k = e^{i\theta} \quad \text{where } \theta = \frac{2k\pi}{n}$$

for $k \in \{0, 1, 2 \dots, n-1\}$ and $i = \sqrt{-1}$.

But suppose we wanted to estimate these numbers using Newton's method. We could try to solve $f(z) = 0$ with $f(z) = z^n - 1$. The iteration is:

$$z_{k+1} = z_k - \frac{(z_k)^n - 1}{n(z_k)^{n-1}}.$$

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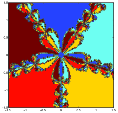
However, there are n possible solutions to

$$z^n - 1 = 0.$$

Given a particular starting point, which root with the method converge to? If we take a number of points in a region of space, iterate on each of them, and then colour the points to indicate the ones that converge to the same root, we get the famous Julia set, an example of a fractal.



A contour plot of a Julia set with $n = 5$, generated by the MATLAB script Julia.m, available from the MA385 website.

(Gaston Julia, French mathematician 1893–1978. The famous paper which introduced these ideas was published in 1918. Interest later waned until the 1970s when Mandelbrot's computer experiments reinvigorated