

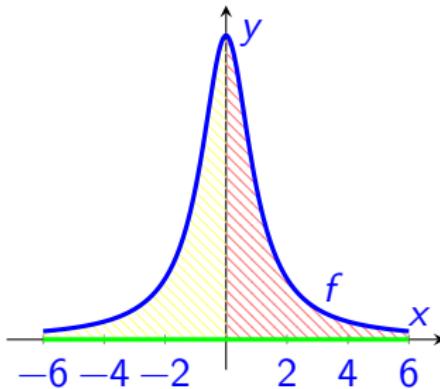
## Week 08, Lecture 3

# Areas between Curves, and Improper Integrals

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# Between now at 10.50...

- 1 Recall: Areas Between Curves
  - Finding  $a$  and  $b$
- 2 Compound Regions
- 3 Improper Integrals
  - Motivation: Areas (again)
- 4 Definitions
  - Example (convergent)
  - Example (divergent)
  - Convergent or Divergent?
  - Last example
- Exercises

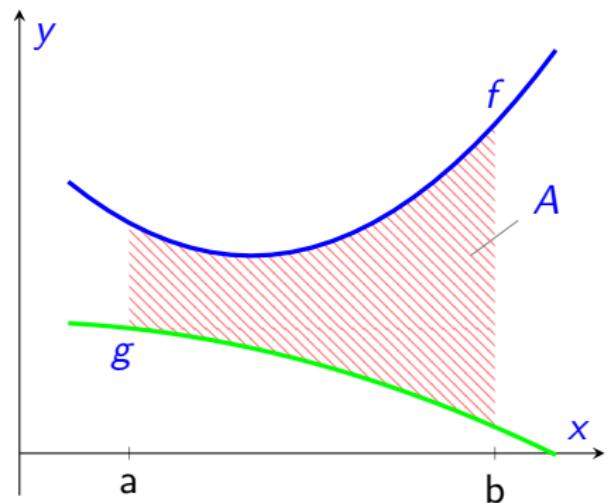
For more: Section 6.1 (Areas between Curves) and Section 7.7 (Improper Integrals) in the textbook:

[math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

## Recall: Areas Between Curves

If  $f(x) \geq g(x)$  for  $x \in [a, b]$ ,  
the area of the region  
between  $x = a$ , and  $x = b$ ,  
and between  $y = g(x)$  and  
 $y = f(x)$  is

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$
$$= \int_a^b (f(x) - g(x))dx.$$



Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect. That is, we have to find  $a$  and  $b$ .

### Example (from Q5(a) of 2024/2025 Exam paper)

Compute the region bounded by the curves  $f(x) = 3x + 4$  and the  $g(x) = 2x^2 + 2x + 1$ .

First we need to find the points where  $f(x)$  and  $g(x)$  intersect.  
That is, we solve  $f(x) = g(x)$ :

$$\begin{aligned}(3x + 4) - (2x^2 + 2x + 1) &= 0 \\ \Rightarrow -2x^2 + x + 3 &= 0 \\ \Rightarrow -2(x + 1)(x - 3/2) &= 0 \quad (1)\end{aligned}$$

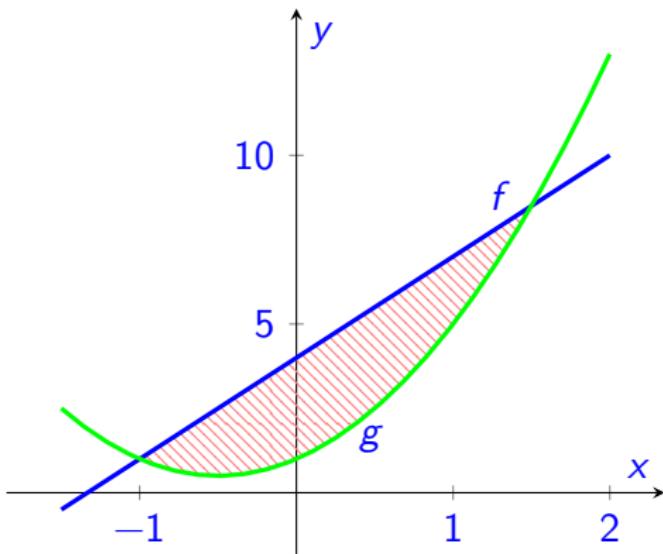
So they intersect at  $x = -1$  and  $x = 3/2$ .  
(Continued)

So the area is given by

$$\begin{aligned} & \int_{-1}^{3/2} f(x) - g(x) dx \\ &= \int_{-1}^{3/2} -2x^2 + x + 3 dx \\ &= \left( -\frac{2}{3}x^3 + \frac{1}{2}x^2 + 3x \right) \Big|_{-1}^{3/2} \\ &= \left( -\frac{2}{3}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right) \right) - \left( -\frac{2}{3}(-1) + \frac{1}{2}(1) + 3(-1) \right) \\ &= 125/24. \end{aligned}$$

# Recall: Areas Between Curves

## Finding $a$ and $b$



# Compound Regions

In the previous examples, we had  $f(x) \geq g(x)$  for all  $x \in [a, b]$ .  
But what if  $f$  and  $g$  cross in the domain?

## Areas between curves, without $f(x) \geq g(x)$

Let  $f(x)$  and  $g(x)$  be continuous functions over an interval  $[a, b]$ .  
Then  $A$ , the area of the region between the graphs of  $f(x)$  and  $g(x)$ , and between  $x = a$  and  $x = b$ , is given by

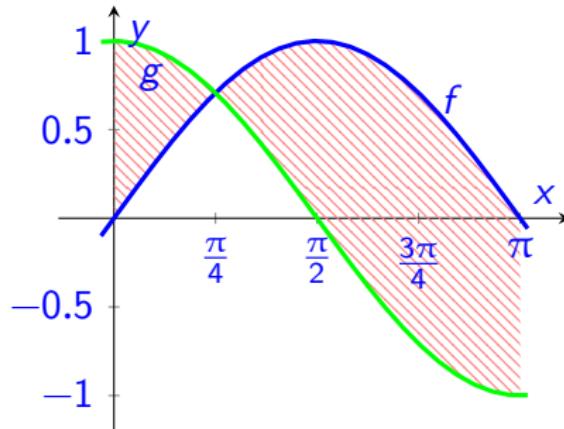
$$A = \int_a^b |f(x) - g(x)| dx.$$

In practice this involves finding the point  $c$  where the functions cross...

# Compound Regions

## Example [See Eg 6.1.3 in textbook]

Find the area between  $f(x) = \sin(x)$  and  $g(x) = \cos(x)$ , from  $x = 0$  to  $x = \pi$ .

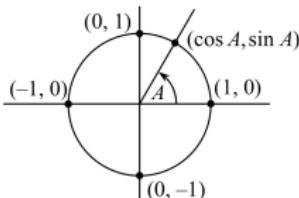


# Compound Regions

It will help to consult p13 of the “log” tables.

## Triantánacht

$$\begin{aligned}\tan A &= \frac{\sin A}{\cos A} & \cot A &= \frac{\cos A}{\sin A} \\ \sec A &= \frac{1}{\cos A} & \operatorname{cosec} A &= \frac{1}{\sin A}\end{aligned}$$



Nóta: Binn tan  $A$  agus sec  $A$  gan sainiú nuair  $\cos A = 0$ .

Binn cot  $A$  agus cosec  $A$  gan sainiú nuair  $\sin A = 0$ .

## Trigonometry

$$\begin{aligned}\cos^2 A + \sin^2 A &= 1 \\ \sec^2 A &= 1 + \tan^2 A \\ \cos(-A) &= \cos A \\ \sin(-A) &= -\sin A \\ \tan(-A) &= -\tan A\end{aligned}$$

Note: tan  $A$  and sec  $A$  are not defined when  $\cos A = 0$ .

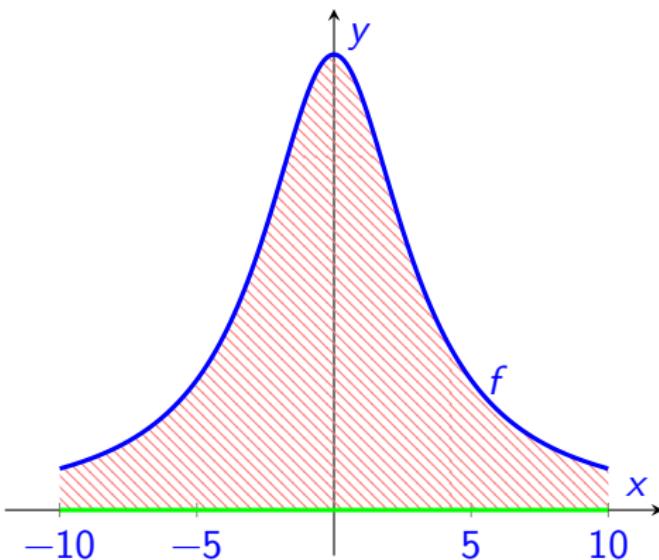
cot  $A$  and cosec  $A$  are not defined when  $\sin A = 0$ .

$A$ (céimeanna)	$0^\circ$	$90^\circ$	$180^\circ$	$270^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$A$ (degrees)
$A$ (raidiain)	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$A$ (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad.  $\approx 57.296^\circ$

$1^\circ \approx 0.01745$  rad.

Earlier we looked at how  $\int_a^b f(x) dx$  evaluates as the area of the region between  $y = f(x)$  and  $y = 0$ , and between  $x = a$  and  $x = b$ . But suppose we want to evaluate the area of the region between  $y = f(x)$  and  $y = 0$ , and between (say)  $x = -\infty$  and  $x = \infty$ ?



So far we have dealt with the definite integral  $\int_a^b f(x) dx$  for a continuous function  $f$  on a finite interval  $[a, b]$ , i.e. where  $a$  and  $b$  are both real numbers.

But sometimes the region in which we are interested is over an **infinite** interval, i.e. an interval of the form  $[a, \infty)$ ,  $(-\infty, b]$  or  $(-\infty, \infty)$ .

Let's consider how we might try to define an **improper integral** such as

$$\int_a^\infty f(x) dx .$$

## Definition (Improper Integral)

Let  $f$  be a continuous function on  $[a, \infty)$ . The **improper integral of  $f$  over  $[a, \infty)$**  is defined by

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx,$$

**provided that the limit exists.** If so, we say that the integral **convergent**. Otherwise, we say it is **divergent**.

Similarly, if  $g(x)$  is continuous  $(-\infty, b]$ , the improper integral  $\int_{-\infty}^b g(x) dx$  is **convergent** and given by

$$\int_{-\infty}^b g(x) dx = \lim_{t \rightarrow -\infty} \int_t^b g(x) dx$$

if that the limit exists; otherwise it is **divergent**.

Furthermore:

If  $f$  is a continuous function on  $\mathbb{R} = (-\infty, \infty)$  and the improper integrals

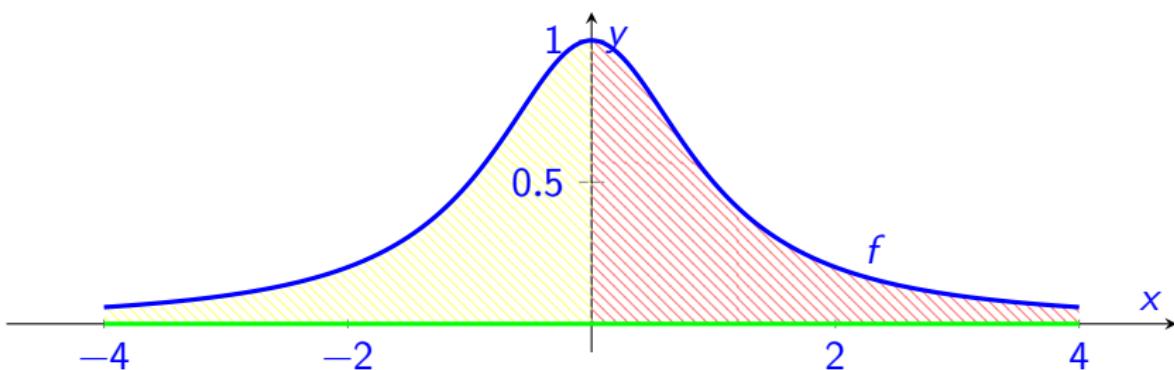
$$\int_{-\infty}^0 f(x) dx \quad \text{and} \quad \int_0^\infty f(x) dx$$

are both convergent, then the improper integral

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^\infty f(x) dx.$$

is also **convergent**. If not, we say it is **divergent**.

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx.$$



**Example**

Evaluate  $\int_1^\infty \frac{1}{x^2} dx$ .

Idea: Use the definition:

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx,$$

That is: set  $g(t) = \int_1^t f(x) dx$  and then evaluate  $\lim_{t \rightarrow \infty} g(t)$

Many improper integrals are divergent. Examples:

If  $f(x)$  is a positive function, for  $\int_a^\infty f(x) dx$  to exist, at the very least we need  $f(x)$  to be a decreasing function. But often that alone is not enough!

- ▶ We know that  $\int_1^\infty x^{-2} dx$  is convergent.
- ▶ From that we can deduce that  $\int_1^\infty x^{-n} dx$  is convergent for any  $n \geq 2$ . (Why?)
- ▶ And we know  $\int_1^\infty x^0 dx$  is divergent.
- ▶ But what about  $\int_1^\infty x^{-1} dx$ ?

**Example**

Determine whether the improper integral  $\int_1^\infty \frac{1}{x} dx$  is convergent or divergent.

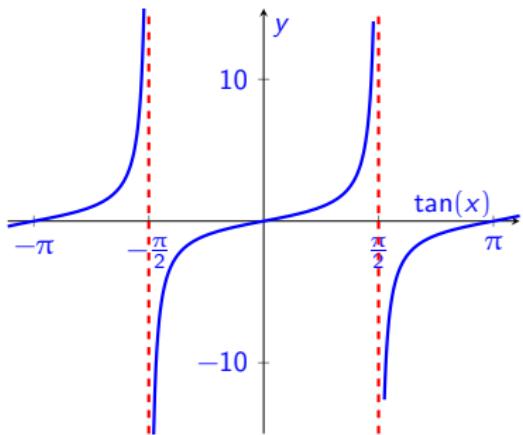
For  $t \geq 1$ , we have  $\int_1^t \frac{1}{x} dx = \ln(x) \Big|_1^t = \ln(t)$ .

Since  $\lim_{t \rightarrow \infty} \ln(t)$  does not exist, it follows that  $\int_1^\infty \frac{1}{x} dx$  is divergent.

In our next, and final example, we'll try to integrate

$f(x) = \frac{1}{1+x^2}$ . To follow the solution, you might find it useful to revise the fundamentals of **inverse trigonometric functions**. You can find that in Section 1.4 of the textbook:

[math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))



In the figure opposite, we see the graph of  $\tan(x)$ . Notice that it has vertical asymptotes at  $x = -\pi/2$  and  $x = \pi/2$ .

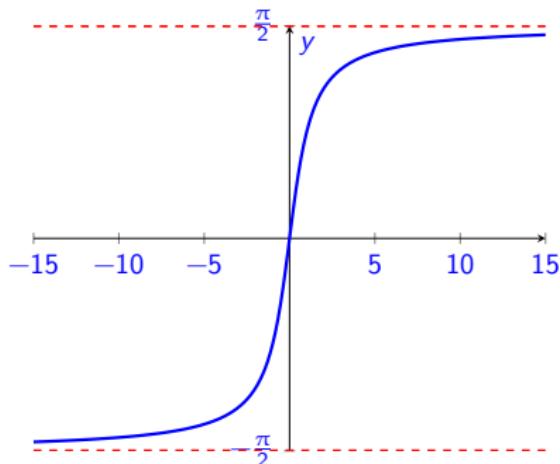
On the right is a plot of the **inverse of the  $\tan(x)$**  function, which is often written as either  $\tan^{-1}(x)$  or  $\arctan(x)$ . Notice that it has **horizontal** asymptotes at  $y = -\pi/2$  and  $y = \pi/2$ .

This means that

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2},$$

and

$$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$



**Example**

Evaluate the improper integral  $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$

## Exercises

### Exer 8.3.1 (From 2019/2020 exam)

The functions  $f(x) = 1/x$  and  $g(x) = x^2$  intersect at  $x = 1$ . Calculate the area between their graphs on  $[1, 2]$

### Exer 8.3.2 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves  $f(x) = \sqrt{x}$  and  $g(x) = x^2$ .

### Exer 8.3.3 (From 23/24 exam)

Find the area bounded by the curves  $f(x) = x^2 - 4x$  and  $g(x) = 2x - 5$ .

## Exercises

### Exer 8.3.4 (From 23/24 exam)

Evaluate  $\int_0^\infty \frac{x}{1+x^4} dx$  (*Hint: try substitution with  $u = x^2$* ).