# MA385 Part 1: Solving nonlinear equations

# 1.4: Taylor's Theorem

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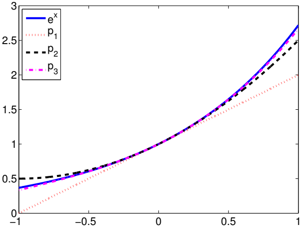


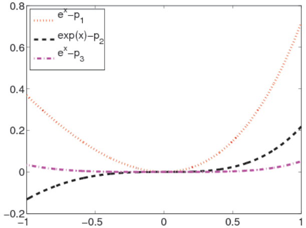

# 0. Outline



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Some notation for today:

$f^{(k)}(x)$ is short-hand for $\dfrac{d^k f}{dx^k}(x)$.

So $f(x) = f^{(0)}(x)$, $\quad f'(x) = f^{(1)}(x)$, $\quad f''(x) = f^{(2)}(x)$, etc.

# 1. Taylor's Theorem

**Taylor's Theorem** is perhaps the most important mathematical tool in Numerical Analysis. Providing we can evaluate the derivatives of a given function at some point, it gives us a way of approximating the function by a polynomial.

Working with polynomials, particularly ones of degree 3 or less, is much easier than working with arbitrary functions. For example, polynomials are easy to differentiate and integrate. Most importantly for the next section of this course, their zeros are easy to find.

*Brook Taylor, 1685 – 1731, England. He (re)discovered this polynomial approximation in 1712, though its full importance was not realised for another 50 years.*

In Section 1.3 we had the **Mean value theorem**: *If $f$ is function that is continuous and differentiable on the interval $[a, x]$, then there is a point $c \in [a, x]$ such that*

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

As noted, the MVT tells us that we can approximate the value of a function by a near-by value, with accuracy that depends on $f'$:

$$f(x) = f(a) + f'(c)(x - a).$$

What if we want a better approximation? We could replace our function, $f(x)$, with a quadratic polynomial. For example, let

$$p_2(x) = b_0 + b_1(x-a) + b_2(x-a)^2,$$

and solve for the coefficients $b_0$, $b_1$ and $b_2$ so that

$$f(x) = p_2(a), \qquad f'(x) = p_2'(a), \qquad f''(x) = p_2''(a).$$

Next, if we try to construct an approximating cubic of the form

$$\begin{aligned} p_3(x) &= b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3, \\ &= \sum_{k=0}^{3} b_k(x-a)^k, \end{aligned}$$

with the property that

$$\begin{aligned} p_3(a) &= f(a), \qquad & p_3'(a) &= f'(a), \\ p_3''(a) &= f''(a), \qquad & p_3'''(a) &= f'''(a). \end{aligned} \tag{1}$$

Again we find that

$$b_k = \frac{f^{(k)}(a)}{k!} \quad \text{for} \quad k = 0, 1, 2, 3.$$

**Definition 1 (Taylor Polynomial)**

The *Taylor Polynomial* of degree $k$ (also called the *Truncated Taylor Series*) that approximates the function $f$ about the point $x = a$ is

$$p_k(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^k}{k!}f^{(k)}(a).$$

**Example 2**

Write down the Taylor polynomial of degree $k$ that approximates $f(x) = e^x$ about the point $x = 0$.

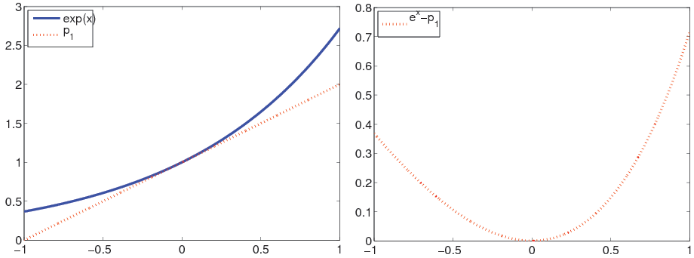


Figure 1: Taylor polys for $f(x) = e^x$ about $x = 0$ (left), and errors (right)

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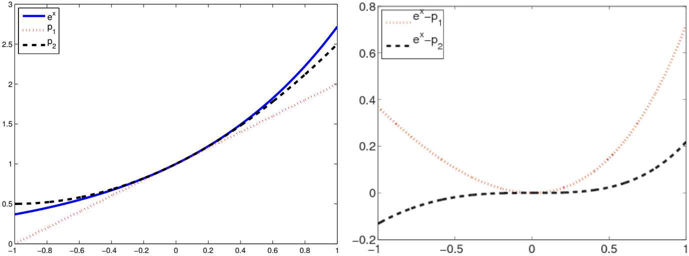


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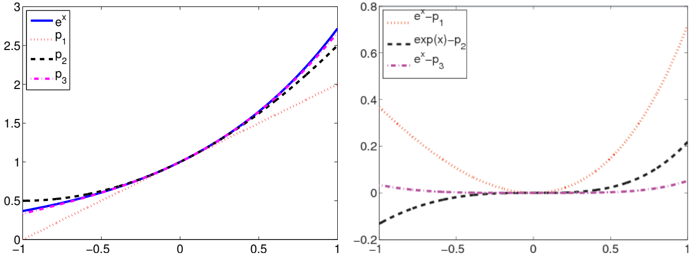


Figure 1: Taylor polys for $f(x) = e^x$ about $x = 0$ (left), and errors (right)

# 2. The Remainder

We now want to examine the *accuracy* of the Taylor polynomial as an approximation. In particular, we would like to find a formula for the *remainder* or *error*:

$$R_k(x) := f(x) - p_k(x).$$

With a little bit of effort one can prove that:

$$R_k(x) := \frac{(x-a)^{k+1}}{(k+1)!} f^{(k+1)}(\sigma), \text{ for some } \sigma \in [x, a].$$

*We won't prove this in class. But for the sake of completeness, I'll add it as an appendix to these notes when I get a chance!*

## Example 3

With $f(x) = e^x$ and $a = 0$, we get that

$$R_k(x) = \frac{x^{k+1}}{(k+1)!}e^{\sigma}, \text{ some } \sigma \in [0, x].$$

## Example 4

How many terms are required in the Taylor Polynomial for $e^x$ about $x = 0$ to ensure that the error at $x = 1$ is

- no more than $10^{-1}$?
- no more than $10^{-2}$?
- no more than $10^{-6}$?
- no more than $10^{-10}$?

# 3. An application of Taylor's Theorem

Taylor's theorem is important to MA385 because...

- We'll use it to derive, and analyse, Newton's method, which is the most important method for solving nonlinear equations (at least for problems where we can evaluate derivatives).
- It is the basis for the methods in Section 2 for solving initial value differential equations.

# 4. Exercises

**Exercise 1.4.1.** Write down the formula for the Taylor Polynomial for

(i) $f(x) = 3x^2 + 3x - 12$

(ii) $f(x) = \sqrt{1 + x}$ about the point $a = 0$,

(iii) $f(x) = \log(x)$ about the point $a = 1$.

**Exercise 1.4.2.** Write out the Taylor polynomial at $x$, about $a = 0$, of degree 7 for $f(x) = \sin(x)$. How does its derivative compare to the corresponding Taylor polynomial for $f(x) = \cos(x)$?

The purpose of the next exercise is to demonstrate that, usually, the closer $x$ is to $a$, the better the Taylor polynomial approximates that function's value.

**Exercise 1.4.3.** Write out the Taylor Polynomial about $a = 1$ of degree 4 and corresponding remainder for $f(x) = \ln(x)$. Give an upper bound for this remainder when $x = 2$, $x = 1.1$ and $x = 1.01$.

*The purpose of the next exercise is to demonstrate that some functions do not have sensible Taylor polynomials.*