2425-MA140 Engineering Calculus

Week 08, Lecture 2 (L23)
Areas between curves

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The areas we'll cover today:

For more reading, see Sections **7.1** (Integration by Parts, briefly) and **6.1** (Areas Between Curves; mainly) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Recall: Definite integrals

Long, long ago (Tuesday of last week) we introduced the definite integral as follows:

Definition: definite integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral of** f from a to b is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{i=0}^{n-1} f(x_i),$$

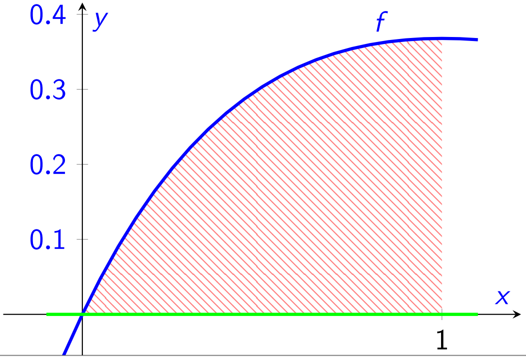
where $h = (b-a)/n$ and $x_i = a + ih$, provided the limit exists. Moreover, it is the area of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$ and $x = b$.

We'll now revisit this idea, and then extend it.

Integration by Parts for Definite Integrals

$$\int_a^b u dv = (uv)\Big|_a^b - \int_a^b v\, du$$

Example: First estimate $\int_0^1 xe^{-1}\, dx$ from the graph of xe^{-x}



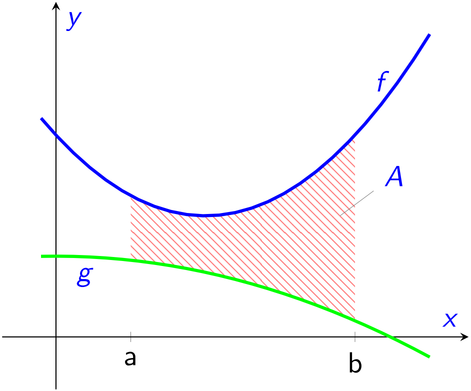
Definite Integrals with IbP

Now use *Integration By Parts* to actually evaluate $\int_0^1 xe^{-x}\,dx$.

Areas Between Curves

We know that $\int_a^b f(x)\,dx$ evaluates as the area of the region between $x = a$ and $x = b$, and between $y = f(x)$ and $y = 0$.

But what if we wanted to evaluate the area between two curves?



Area Between Curves

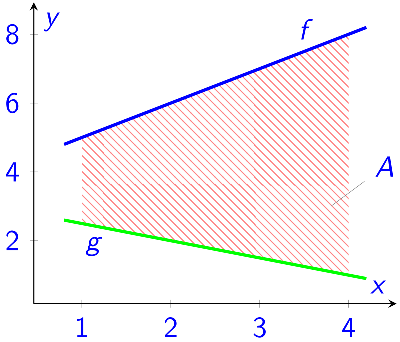
Let f and g be continuous functions with $f(x) \geqslant g(x)$ throughout the interval $[a, b]$. Then the area A of the region over $[a, b]$ and between the curves $y = f(x)$ and $y = g(x)$ is the integral of $f(x) - g(x)$ from $x = a$ to $x = b$; that is

$$A = \int_a^b (f(x) - g(x))\, dx \,.$$

Areas Between Curves

Example

Find the area of the region bounded above by the graph of $f(x) = x + 4$, and below by the graph of $g(x) = 3 - x/2$ over the interval $[1, 4]$

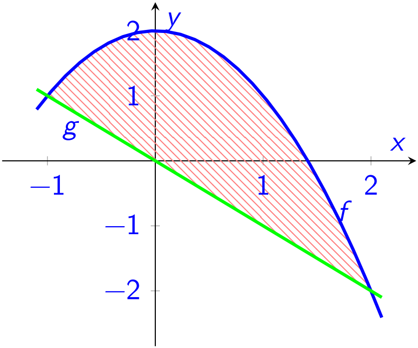


Areas Between Curves

Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect.

Example

Find the area of the region enclosed by the parabola $f(x) = 2 - x^2$ and the line $g(x) = -x$.



Areas Between Curves

Areas Between Curves

Example

Find the area enclosed between the two curves $f(x) = 6 - 2x^2$ and $g(x) = 4x$.

Compound Regions

In all previous examples, we assumed that $f(x) \geqslant g(x)$ for all $x \in [a, b]$. But what if f and g cross in the domain?

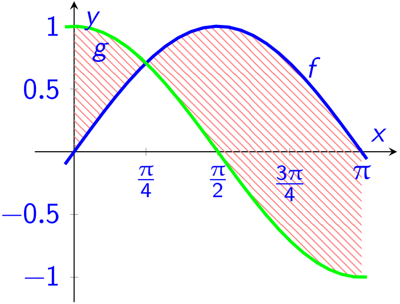
Areas between curves, without $f(x) \geqslant g(x)$

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ Then A, the area of the region between the graphs of $f(x)$ and $g(x)$, and between $x = a$ and $x = b$, is given by

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Example

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$, from $x = 0$ to $x = \pi$.



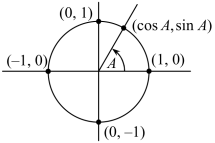
Compound Regions

It will help to consult p13 of the "log" tables.

Triantánacht | **Trigonometry**

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	$0°$	$90°$	$180°$	$270°$	$30°$	$45°$	$60°$	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. $\approx 57.296°$ $\qquad$ $1° \approx 0.01745$ rad.

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Compound Regions

Exercises

Exer 8.2.1 (From 2023/2024 exam)

Evaluate $\int_0^{\pi/2} x\cos(x)\,dx$.

Exer 8.2.2 (From 2019/2020 exam)

The functions $f(x) = 1/x$ and $g(x) = x^2$ intersect at $x = 1$. Calculate the area between their graphs on $[1, 2]$

Exer 8.2.3 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Exer 8.2.4 (From 23/24 exam)

Find the area bounded by the curves $f(x) = x^2 - 4x$ and $g(x) = 2x - 5$.