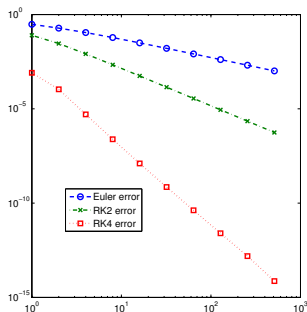


MA385 Part 2: Initial Value Problems

2.5: Runge-Kutta 4 (RK4)

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For more details, see Chapter 6 of [Süli and Mayers, *An Introduction to Numerical Analysis*](#). In particular, see Section 12.4 (Runge-Kutta methods)

1. A higher order method

It is possible to construct methods that have higher orders of accuracy than **RK2** methods. Of these, the most used are probably those that belong to the **Runge-Kutta 4 (RK4)** family, and have the property that

$$|y(t_n) - y_n| \leq Ch^4.$$

However, even writing down the general form of the RK4 method, and then deriving conditions on the parameters is rather complicated. Therefore, we'll focus on just one RK4 method, and use examples, rather than theory, to demonstrate that it is 4th-order.

1. A higher order method

“The RK4 Method”

$$k_1 = f(t_i, y_i),$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right),$$

$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right),$$

$$k_4 = f(t_i + h, y_i + hk_3),$$

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

1. A higher order method

The RK4 method can be interpreted as follows :

1. A higher order method

As the following example shows, RK4 can be much more accurate than the Euler or RK2 methods for small h (i.e., large n). For the RK4, doubling n reduces the error by a factor of 8 (compared with 2 and 4 for the Euler and RK2 methods, respectively).

1. A higher order method

Example 2.5.1

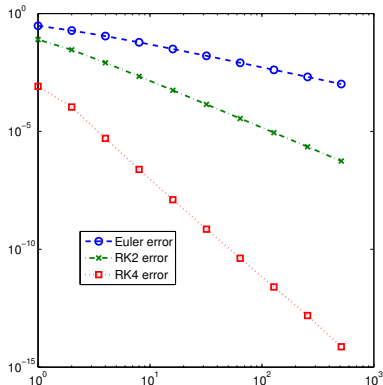
Compare Euler, Modified Euler, and RK4 for approximating $y(1)$ where: $y(0) = 1$, $y'(t) = y \log(1 + t^2)$.

Error: $ y(t_n) - y_n $			
n	Euler	Modified	RK4
1	3.02e-01	7.89e-02	8.14e-04
2	1.90e-01	2.90e-02	1.08e-04
4	1.11e-01	8.20e-03	5.07e-06
8	6.02e-02	2.16e-03	2.44e-07
16	3.14e-02	5.55e-04	1.27e-08
32	1.61e-02	1.40e-04	7.11e-10
64	8.13e-03	3.53e-05	4.18e-11
128	4.09e-03	8.84e-06	2.53e-12
256	2.05e-03	2.21e-06	1.54e-13
512	1.03e-03	5.54e-07	7.33e-15

1. A higher order method

Example 2.5.2

Compare Euler, Modified Euler, and RK4 for approximating $y(1)$ where: $y(0) = 1$, $y'(t) = y \log(1 + t^2)$.



2. Consistency and convergence of RK4

Although we won't do a detailed analysis of RK4, we can do a little. In particular, we would like to show it is

- (i) consistent,
- (ii) convergent and fourth-order, at least for some examples.

2. Consistency and convergence of RK4

Example 2.5.3

It is easy to see that RK4 is consistent:

2. Consistency and convergence of RK4

Example 2.5.4

In general, showing the rate of convergence is tricky. Instead, we'll demonstrate how the method relates to a Taylor Series expansion for the problem $y' = \lambda y$ where λ is a constant.

2. Consistency and convergence of RK4

3. The (Butcher) Tableau

Many (seemingly different) RK have been proposed and studied. A unified approach of representing them was developed by John Butcher: write an s -stage method as

$$\Phi(t_i, y_i; h) = \sum_{j=1}^s b_j k_j, \quad \text{where}$$

$$k_1 = f(t_i + \alpha_1 h, y_i),$$

$$k_2 = f(t_i + \alpha_2 h, y_i + \beta_{21} h k_1),$$

$$k_3 = f(t_i + \alpha_3 h, y_i + \beta_{31} h k_1 + \beta_{32} h k_2),$$

$$\vdots$$

$$k_s = f(t_i + \alpha_s h, y_i + \beta_{s1} h k_1 + \dots + \beta_{s,s-1} h k_{s-1}),$$

3. The (Butcher) Tableau

The most convenient way to represent the coefficients is in a tableau:

$$\begin{array}{c|ccc}
 \alpha_1 & & & \\
 \alpha_2 & \beta_{21} & & \\
 \alpha_3 & \beta_{31} & \beta_{32} & \\
 \vdots & & & \\
 \alpha_s & \beta_{s1} & \beta_{s2} & \cdots & \beta_{s,s-1} \\
 \hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
 \end{array}$$

The tableaux for basic Euler, Modified Euler, and RK4 are:

$$\begin{array}{c|c}
 0 & \\
 \hline
 & 1
 \end{array}
 \quad
 \begin{array}{c|c}
 0 & \\
 1/2 & 1/2 \\
 \hline
 & 0 \quad 1
 \end{array}
 \quad
 \begin{array}{c|ccc}
 0 & & & \\
 1/2 & 1/2 & & \\
 1/2 & 0 & 1/2 & \\
 1 & 0 & 0 & 1 \\
 \hline
 & 1/6 & 2/6 & 2/6 & 1/6
 \end{array}$$

4. Even higher-order methods?

A Runge Kutta method has s stages if it involves s evaluations of the function f . (That is, its formula features k_1, k_2, \dots, k_s).

We've seen a 1-stage method that is 1st-order.

We studied 2-stage methods that are 2nd-order.

In an exercise, you'll construct a 3-stage method that is 3rd order.

And, of course, we have just considered a four-stage method that is 4th-order.

4. Even higher-order methods?

It is tempting to think that we could get an Order s method by using s stages for any s .

However, it can be shown that, for example,

- ▶ for a method to have order $p = 5$, we need $s = 6$ stages (Kutta, 1901)
- ▶ for order $p = 6$, we need at least $s = 7$ stages (Butcher, 1964).
- ▶ for order $p = 7$, we need at $s = 9$ stages (Butcher, 1968).
- ▶ for order $p = 8$, we need at $s = 11$ stages (Curtis, 1970).
- ▶ for order $p = 10$, we need at $s = 17$ stages (Hairer, 1978).

The theory involved is both intricate and intriguing, and involves aspects of group theory, graph theory, and differential equations.

See [https://doi.org/10.1016/0168-9274\(95\)00108-5](https://doi.org/10.1016/0168-9274(95)00108-5)

5. Exercises

Exercise 2.5.1

We claim that, for RK4:

$$|\mathcal{E}_N| = |y(t_N) - y_N| \leq Kh^4.$$

for some constant K . How could you verify that the statement is true using the data of Table 2.3, at least for test problem in Example 2.4.2? Give an estimate for K .

5. Exercises

Exercise 2.5.2

Recall the problem in Example 2.2.2: Estimate $y(2)$ given that

$$y(1) = 1, \quad y' = f(t, y) := 1 + t + \frac{y}{t},$$

- (i) Show that $f(t, y)$ satisfies a Lipschitz condition and give an upper bound for L .
- (ii) Use Euler's method with $h = 1/4$ to estimate $y(2)$. Using the true solution, calculate the error.
- (iii) Repeat this for the RK2 method of your choice (with $a \neq 0$) taking $h = 1/2$.
- (iv) Use RK4 with $h = 1$ to estimate $y(2)$.

5. Exercises

Exercise 2.5.3

Here is the tableau for a three stage Runge-Kutta method:

0			
α_2	1/2		
1	β_{31}	2	
<hr/>			
	1/6	b_2	1/6

- (i) Use that the method is consistent to determine b_2 .
- (ii) The method is exact when used to compute the solution to

$$y(0) = 0, \quad y'(t) = 2t, \quad t > 0.$$

Use this to determine α_2 .

- (iii) The method should agree with an appropriate Taylor series for the solution to $y'(t) = \lambda y(t)$, up to terms that are $\mathcal{O}(h^3)$. Use this to determine β_{31} .