

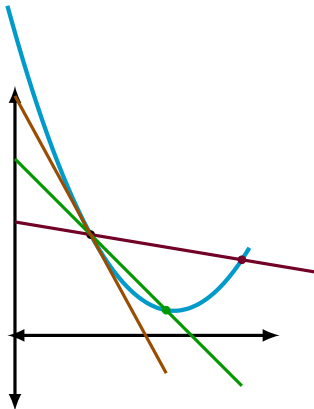
2526-MA140 Engineering Calculus

Week 04, Lecture 1
Differentiation Rules

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What we'll study today

- 1 Remember:
- 2 Derivative at a point
 - The concept
 - The definition
 - Example
- 3 Derivative as a function
- 4 Differentiation by rule
 - 1. The Constant Rule
 - 2. The Power Rule
 - 3. The constant multiple rule
 - 4. The Sum and Difference Rules
- 5 Tomorrow's Rules
- 6 Exercises

Further reading:

- ▶ Sections 3.1 and 3.2 of **Calculus** by Strang & Herman:
<https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative>
- ▶ Nice animation: <https://www.geogebra.org/m/MeMdCUEm>

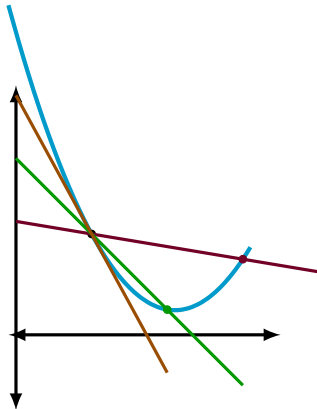
Remember:

Reminders

- ▶ **Assignment 1** finished yesterday at 5pm. Grades are available on Canvas
- ▶ Assignment 2 is open; deadline is 5pm, 13 Oct. You can access it at <https://universityofgalway.instructure.com/courses/46734/assignments/129715>. (Or: go to Canvas, click on Assignments ... Problem Set 2 ... the bottom of the page, click `Load Problem Set 2 in a new window`)
- ▶ This week's **Tutorial Sheet** is available at https://universityofgalway.instructure.com/courses/46734/files/2883465?module_item_id=943734
- ▶ The first (of two) class tests will take place next Tuesday, 14th October. I'll be in touch about accommodations for those who completed the request form.

At the end of the last class, we visualised a function, f , with a line intersecting it at the points $(a, f(a))$ and $(b, f(b))$, where $b = a + h$

We moved the point b closer and closer to a (by taking smaller and smaller h), until the intersecting line eventually became the tangent to f at $x = a$.



Conclude: the slope of the tangent to f at $x = a$ is the limit:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

The slope of the curve $y = f(x)$ at the point $P = (a, f(a))$ is given by the number (if it exists)

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, it is called the **derivative of f at $x = a$** and we denote it by $f'(a)$.

Definition: derivative at a point

Let $f(x)$ be a function that has $x = a$ in its domain. The **derivative** of the function $f(x)$ at a , denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

Some terminology

$f'(a)$ exists then we say that function f is **differentiable at $x = a$** .

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the “limit” formula, we are doing “**differentiation from first principles**”.

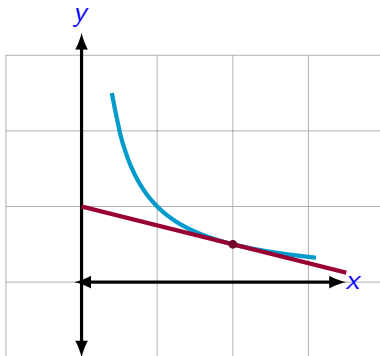
Example

Use the limit definition of a derivative to compute the slope of the tangent to $f(x) = x^2$ at $x = 3$.

Example

Use the limit definition of a derivative to find the equation of the tangent to $f(x) = 1/x$ at $x = 2$.

$$f(x) = \frac{1}{x} \quad \text{and} \quad y = 1 - \frac{x}{4}$$



Derivative as a function

We've seen how to compute $f'(a)$: the derivative of the function f at a given point, $x = a$.

But if $f'(a)$ has a value for all $x = a$ (in the domain of $f(x)$), we can think $f'(x)$ as a function itself!

Definition: derivative as a function

Let f be a function. The derivative function, denoted f' , is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative as a function

Terminology and notation

- ▶ We usually refer to f' simply as the derivative of $f(x)$.
- ▶ Where $y = f(x)$, we often we write f' as $\frac{dy}{dx}$, or y' , or $\frac{d}{dx}(f)(x)$.

Derivative as a function

Example

Use the above definition to find the derivative of $f(x) = x^2$.

Solution

The derivative is defined as follows:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Here $f(x+h) = (x+h)^2 = x^2 + h^2 + 2hx$, so we get:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x^2 + h^2 + 2hx) - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h + 2x)}{h} = \lim_{h \rightarrow 0} (h + 2x) = 2x \end{aligned}$$

Derivative as a function

Example

Use the “limit” definition to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

Derivative as a function

Consider the absolute value function $f(x) = |x|$. What is its derivative at (i) $x = 2$, (ii) $x = -3$, or (iii) $x = 0$?

Derivative as a function

Show that $\frac{d}{dx}(\sin x) = \cos x$.

Solution: We need to evaluate

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h},$$

where $f(x) = \sin(x)$. From p5 of the “log” tables, we have that

$$\sin(A) - \sin(B) = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right).$$

Here $A = x+h$, and $B = x$, so

$$\sin(x+h) - \sin(x) = 2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right).$$

So now we evaluate

$$f'(x) = \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{h} = \lim_{h \rightarrow 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right).$$

Derivative as a function

But

$$\lim_{h \rightarrow 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right) = \left(\lim_{h \rightarrow 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) \right) \left(\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \right).$$

We learned last week that,

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1.$$

Taking $\theta = h/2$, we get that

$$\lim_{h \rightarrow 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) = 1.$$

And finally,

$$\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) = \cos(x).$$

and we are done!

Differentiation by rule

We've seen we can compute derivatives of some functions using the “limit” definition (i.e., **differentiation from first principles**). However, that approach is tedious, and unnecessary in many case.

Instead we can use a set of “**rules**” which makes the process much more efficient. These rules are themselves derived from the “limit” definition – but we don't have to use that every time.

The Constant Rule

If f is a constant function, i.e. $f(x) = c$ for all x , then:

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Why:

We've already deduced that

- ▶ The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- ▶ The derivative of $f(x) = x^{1/2}$ is $f'(x) = \frac{1}{2}x^{-1/2}$

These are particular examples of the **Power Rules**

The Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Examples Calculate the derivatives of the following functions

1. $f(x) = x^6$

2. $f(x) = \sqrt[3]{x}$

The constant multiple rule

Let $f(x)$ be any differentiable function, and let k be constant, then

$$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x)).$$

Example: Find the derivative of $f(x) = 5x^4$.

The Sum and Difference Rules

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}(u(x) + v(x)) = \frac{d}{dx}(u(x)) + \frac{d}{dx}(v(x)).$$

Similarly,
$$\frac{d}{dx}(u(x) - v(x)) = \frac{d}{dx}(u(x)) - \frac{d}{dx}(v(x)).$$

Example: Find the derivative of $f(x) = 1 + x + x^2$.

Differentiation by rule 4. The Sum and Difference Rules

Actually, the “**Difference Rule**”, which states that

$$\frac{d}{dx}(u(x) - v(x)) = \frac{d}{dx}(u(x)) - \frac{d}{dx}(v(x)).$$

can be combined by combining the **Sum Rule** and the **Constant Multiple Rule**.

Differentiation by rule 4. The Sum and Difference Rules

Example

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

- (a) The derivative of $f(x)$;
- (b) The slope of the tangent line at $x = 2$;
- (c) The equation of the tangent at $x = 2$.

(a) $f'(x) = -15x^2 + 6x - 9$

(b) The slope of the tangent line at $x = 2$ is $f'(2)$:

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57.$$

Differentiation by rule 4. The Sum and Difference Rules

- (c) The equation of the line with slope M and passing through a point (x_1, y_1) is

$$y - y_1 = M(x - x_1)$$

The y coordinate at $x = 2$ is

$$\begin{aligned} f(2) &= -5(2)^3 + 3(2)^2 - 9(2) + 7 \\ &= -5(8) + 3(4) - 18 + 7 \\ &= -40 + 12 - 18 + 7 \\ &= -39. \end{aligned}$$

So the tangent line passes through the point $(2, -39)$ and the slope of the line is -57 .

Therefore, the equation of this line is $y + 39 = -57(x - 2)$

Ans: The equation of the tangent line is $x = 2$ is $y = 75 - 57x$.

Tomorrow's Rules

Tomorrow we'll focus on two more rules, with important applications:

- ▶ The **Product Rule** for computing the the derivative of the **product** of two functions.
- ▶ **The Quotient Rule** for differentiating the **ratio** of two functions.

Exercises

Exercises 4.1.1 (Based on Q2(a), 2019/2020)

Use the (limit) definition of a derivative to differentiate the function $f(x) = x^2 + 2$.

Exercise 4.1.2

Use the (limit) definition of a derivative to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.