MA378: Numerical Analysis II

§0.1 Introduction to MA378

Dr Niall Madden

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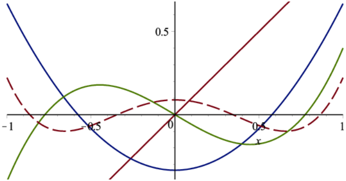

Outline

MA378 is a one-semester upper-level module on numerical analysis.

It complements MA385 (Numerical Analysis 1) and CS319 (Scientific Computing), but is independent of both those. In particular, you don't have to have taken MA385 to take this module.

The module covers several major topics in Numerical Analysis:

- Interpolation and approximation of functions;
- Numerical integration;
- Numerical solution of boundary value problems.

I'll explain more about these presently.

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There are about 45 students enrolled in MA378, from various programmes:

- 3rd and 4th Maths and Education
- 3rd and 4th Mathematical Science
- 3rd Science (various pathways)
- 4th Science (Applied Maths; any others?)
- MSc Mathematics
- ...

So we have people with diverse backgrounds in pure mathematics, applied mathematics, computer science, humanities...

Lectures: Wednesdays at 1pm in AC204.
Fridays at 10am in AC213.

Labs: There will be 3 MATLAB labs during the semester; we'll discuss the schedule presently;

Tutorials: There will be 7 problem-solving sessions during the semester; again, schedule TBA.

Assessment

- Three MATLAB labs, collectively worth 10%.
- Two written assignments: 10% each.
- An in-class test: 10%.
- A 2-hour exam at the end of the semester: 60%.

The labs provide an opportunity for you to implement the algorithms we study, as well as their extensions and limitations.

The written assignments promote in-depth engagement with specific topics, while the class test encourages one to take a broad view of the module.

Assessment Schedule

The schedule will be determined once labs are arranged. Rough plan:

- Week 4: Lab 1
- Week 5: Assignment 1
- Week 7: Class test
- Week 8: Lab 2.
- Week 10: Assignment 2 (Note: no Friday lecture in Week 10).
- Week 11: Lab 3

Main textbook

Primary text is: **Süli and Mayers, An Introduction to Numerical Analysis**. Available from the library at 519.4 MAY and online through the library portal: https://search.library.nuigalway.ie/permalink/f/1pmb91f/353GAL_ALMA_DS2182021360003626

We'll mainly use

- Chapter 6: Polynomial interpolation
- Chapter 11: Piecewise polynomial interpolation
- Chapter 7: Numerical Integration I
- Chapter 10: Numerical Integration II
- Chapter 13: Boundary value problems

Recommended

- G.W. Stewart, *Afternotes on Numerical Analysis*: Lectures 18–20, 21–23.
- G.W. Stewart, *Afternotes goes to Graduate School*: Lectures 10–11.
- Tobin Driscoll, *Learning MATLAB*.
  You'll find links to these on the library website.

The on-line content for the course will be hosted at NUIGalway.BlackBoard.com.

(I can mirror notes at a github repository, if anyone likes).

On blackboard you'll find:

- Announcements (1 per week)
- Information (where, when, what)
- These slides, posted in advance.
- Lab sheets.
- Problem sets.

If you are registered for MA378, you should be automatically enrolled onto the blackboard site.

Learning materials

The lecture slides contain most of the course material. They are arranged by topic, and will be posted in batches covered two or so weeks of material.

The slides contain most of the main ideas, statements of theorems, results and exercises. They don't contain proofs of theorems, examples, solutions to exercises, etc.

Please let me know of the typos and mistakes that you spot.

Each section of the notes has a set of exercises. *The homework assignments, class test, and final exam will be primarily based on these exercises*.

I'll annotate some slides during class, and post these later.

Lecture and Lab times

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					✗
11 – 12					
12 – 1					
1 – 2			✗		
2 – 3					
3 – 4					
4 – 5					
5 – 6					

Numerical analysis is the

- design
- mathematical **analysis**
- and computer implementation

of numerical algorithms that yield *exact* or *approximate* solutions to mathematical problems.

The specific problems we will study are

1. Interpolation I: **Polynomial interpolation**.
2. Interpolation II: **Piecewise polynomial “splines”**.
3. Numerical Integration I: **Newton-Cotes Quadrature**.
4. Numerical Integration II: **Gaussian Quadrature**.
5. Numerical solution of Boundary Value Problems by the Finite Element Method.

Although these might seem like diverse topics, we’ll see that there is a common thread running through them.

What is NA2 really about?

The big idea is...

Suppose we have a problem to solve, for which we know there is a solution, but that the solution is very hard to find. Or maybe impossible to find. We replace the problem with one that is easier, but has a similar solution, and solve that instead.

While there are many variations, there is a single core idea we will return to again and again.

For this to make sense, we have to...

- be able to come up with the "easier" problem;
- be able to solve that problem;
- quantify how "similar" this solution is to the one we originally tried to solve.

Designing a numerical method means coming up with an approach algorithm for solving a given problem. This is the concept of coming up with the "easier" problem mentioned earlier. It is probably the most creative part of Numerical Analysis.

For example:

Problem

Suppose there is a mathematical function, f of one variable, x. For "reasons", we can only check its value at certain points, e.g., $x = 0$, $x = 1$ and $x = 2$.
Problem: How can we estimate f at, say, $x = 1.5$?

Possible solution: find a quadratic polynomial that agrees with f at the known points, and the evaluate that at the desired points.

What is NA2 really about?

Now that we have designed solution technique, we have to use it.

In the previous example, that would involve devising a set of steps for computing the quadratic polynomial.

This is the "**implementation**" stage of Numerical Analysis, and results in an **algorithm**.

When this algorithm is implemented by hand, it can easily become very boring. Through the use of computers, it take on an important, creative role in the process.

Finally we have the "**analysis**" part: this is the most interesting and mathematically challenging aspect: *can we say how close our approximate solution is to true solution?*

That we can answer this question in a precise manner is a bit surprising. For how can I give an accurate estimate for how close my approximation is to the true solution, when I don't know what that true solution is?

. . . .

In MA378, we will cover each of "Design", "Implementation" and "Analysis". However, "analysis" will take most of our time. We will cover theory in some depth, and will prove numerous theorems.

What is NA2 really about?

1. We'll first study **polynomial interpolation**: the problem of computing a polynomial with known values at certain points.

2. We'll learn that polynomial interpolation is fraught with challenges for high degree. To get around this, we'll study **piecewise polynomial interpolation**: use low-order polynomials and glue them together.

3. **Quadrature**. Given any function $f = f(x)$, that is continuous on $a \leq x \leq b$, the integral $\int_a^b f(x)dx$ is known to exist. But finding the anti-derivative of f may be hard/impossible. "Quadrature" is the process of approximating such definite integral.

4. **Finite elements:** Suppose we know a function $u = u(x)$, solve

$$-u''(x) + b(x)u(x) = f(x) \qquad \text{on } a \leq x \leq b.$$

How can we approximate u?

Mathematical Preliminaries

The background for MA378 can be found in first and second year modules on analysis and algebra. The final section will be easier if you know a little about boundary value differential equations, but it is not essential.

Mathematical Preliminaries

If its been a while since you studied calculus or algebra, you will find it very helpful to revise the following:

- Polynomials (over real numbers).
- the Intermediate Value Theorem;
- **Rolle's Theorem** and the The Mean Value Theorem;
- Taylor's Theorem,
- and the triangle inequality: $|a+b| \leq |a| + |b|$.

You'll find them in any good text-book, e.g., Appendix 1 of Süli and Mayers.

The following ideas from linear algebra will also be very useful:

- Linear independence;
- Inner products;
- Finitely generated vector spaces. In particular, any sequence of n linearly independent vectors forms a basis for a vector space of degree n.