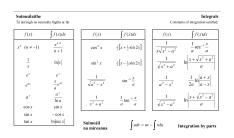
#### 2425-MA140 Engineering Calculus

# Week 07, Lecture 2 (L20) The Fundamental Theorem of Calculus

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# Tutorials, Assignments, etc

- ► I'm very sorry yesterday's 3pm tutorial in ENG-2034, for Teams 3 and 4, didn't take place. If possible, please attend a different tutorial, ideally on Friday.
- Assignment 3 (resit): Last reminder: send my your results summary if the result you got does not agree with what you think you scored.
- ➤ Assignment 5 is open. Deadline is 5pm next Monday (4 November). You have 3 attempts for each question. However, Q1 will be manually graded after the deadline.

# The exciting topics that await us in today:

See also: Sections 4.10 (Antiderivatives) and 5.3 (Fundamental Theorem of Calculus) of Calculus by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus\_(OpenStax)

# Recall from yesterday:

If f(x) is a function defined on an interval [a, b]

ightharpoonup The **definite integral** of f from a to b is

$$\int_a^b f(x)dx := \lim_{n \to \infty} \sum_{i=0}^{n-1} hf(x_i),$$

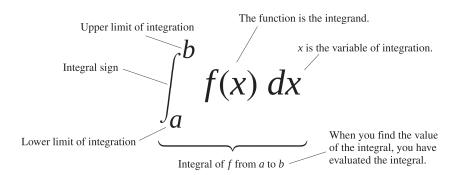
where h = (b - a)/n and  $x_i = a + ih$ .

- This is the area of the region in space bounded by y = 0, y = f(x), x = a, and x = b.
- ▶ Given a function, f, we can define another, F as

$$F(\mathbf{x}) = \int_{a}^{\mathbf{x}} f(t) dt.$$

That is, the variable in *F* is the upper limit of integration on the right. For an nice illustration, see <a href="https://www.geogebra.org/m/ugTmVRHj">https://www.geogebra.org/m/ugTmVRHj</a>

# Recall from yesterday:



#### Fundamental Thm of Calculus: Part 1

# Fundamental Theorem of Calculus: Part 1 (FTC1)

If f(x) is a continuous function on [a, b], and F(x) is defined as

$$F(x) = \int_{a}^{x} f(t)dt,$$

then F'(x) = f(x) for  $x \in [a, b]$ .

Roughly: the derivative of the integral of f is f. You can find a proof in Section 5.3 of the textbook.

# Fundamental Thm of Calculus: Part 1

# **Example**

Let 
$$g(x) = \int_1^x \frac{1}{t^3 + 1} dt$$
. Find  $g'(x)$ .

## FTC1+Chain Rule

Sometimes the limit of integration is a more complicated function of x. In that case, we can apply the **Chain Rule**, along with the FTC1.

# **Example**

Let 
$$F(x) = \int_{1}^{\sqrt{x}} \sin(t) dt$$
. Find  $F'(x)$ .

Idea: Let 
$$u(x) = \sqrt{x} = x^{1/2}$$
. So  $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ . Then...

$$F'(x) = \frac{dF}{du}\frac{du}{dx} = \sin(u(x))\left(\frac{1}{2\sqrt{x}}\right) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

### **Antiderivatives**

#### **Definition: Antiderivative**

A function F is an **antiderivative** of f on [a, b] if F'(x) = f(x) for all x in [a, b]. Thus,

f is the derivative of  $F \Leftrightarrow F$  is an antiderivative of f.

Note: If F is an antiderivative of f, then the most general antiderivative of f is

$$F(x) + c$$
,

where c is an arbitrary constant, called a **constant of integration**.

**Example:** For any x,  $F(x) = x^2 + c$  is an antiderivative of f(x) = 2x.

# **Antiderivatives**

**Example:** The *general* antiderivative of  $f(x) = 3x^2$  is  $F(x) = x^3 + c$ .

# **Antiderivatives**

# **Examples**

Find all antiderivatives of the following functions

(i) 
$$f(x) = \frac{1}{x}$$
 for  $x > 0$ .

(ii) 
$$f(x) = \sin(x)$$
  
(iii)  $f(x) = e^x$ .

(iii) 
$$f(x) = e^x$$
.

# **Definition:** indefinite integral

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x) \, \mathrm{d}x$$

is the general antiderivative of f. That is, if F is an antiderivative of f, then

$$\int f(x) \, \mathrm{d}x = F(x) + C.$$

#### **Examples:**

$$\int x^2 dx = \frac{1}{3}x^3 + C.$$

Spotting the pattern we can deduce...

# **Power Rule of Integration**

If  $n \neq -1$ , then

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C.$$

Note: For n = -1, we have

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C \, .$$

Here is a list of the antiderivatives of some common functions.

#### Suimeálaithe

Tá tairisigh na suimeála fágtha ar lár.

f(x)	$\int f(x)dx$
$x^n  (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln  x $
$e^x$	$e^x$
$e^{ax}$	$\frac{1}{a}e^{ax}$
a <sup>x</sup>	$\frac{a^x}{\ln a}$
cos x	$\sin x$
sin x	$-\cos x$
tan x	ln sec x

#### ealaithe

Integrals

Constants of integration omitted.

f(x)

f(x)	$\int f(x)dx$
$\cos^2 x$	$\frac{1}{2} \left[ x + \frac{1}{2} \sin 2x \right]$
$\sin^2 x$	$\frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right]$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

$$\int u dv = uv -$$

 $\ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right|$ 

 $\int f(x)dx$ 

Suimeáil  $\int u dv = uv - \int v du$ na míreanna

Integration by parts

# **Properties of Integration**

1. If k is a constant, then

$$\int kf(x) dx = k \int f(x) dx.$$

2. Integration is additive:

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

## **Example**

Evaluate the following integrals

1. 
$$\int 2x^2 + 9x^7 \, dx.$$
2. 
$$\int \frac{4}{1+x^2} \, dx.$$

$$2. \quad \frac{4}{1+x^2} \, \mathrm{d}x.$$

#### The Fundamental Thm of Calculus: Part 2

Now that we know all about antiderivatves, we can see how the link to **definite integrals** 

# Theorem (The Fundamental Thm of Calculus, Part 2)

If f(x) is continuous on [a, b], and F(x) is any antiderivative of f(x), then

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a).$$

**Notation:** We often write F(b) - F(a) as  $F(x)\Big|_{x=a}^{x=b}$ , or simply

$$F(x)\Big|_{a}^{b}$$

# The Fundamental Thm of Calculus: Part 2

**Example:** Show that 
$$\int_{-1}^{1} (x^2 + 2) dx = \frac{14}{3}$$

**Example:** Show that 
$$\int_{-1}^{1} (x^3 + x) dx = 0$$

### **Exercises**

#### Exer 7.2.1

Let  $F(x) = \int_{x}^{2x} t \, dt$ . Use the Fundamental Theorem of Calculus to evaluate F'(x).

Hint: we can split this into two integrals:

$$F(x) = \int_{x}^{2x} t \, dt = \int_{x}^{0} t \, dt + \int_{0}^{2x} t \, dt = -\int_{0}^{x} t \, dt + \int_{0}^{2x} t \, dt.$$

Now apply the FTC to each term, including the Chain Rule for the second.

#### Exer 7.2.2

Evaluate the following integrals.

$$1. \int e^{2x} + \frac{1}{2x} \, \mathrm{d}x$$

$$2. \int \frac{3}{\sqrt{2-x^2}} \, \mathrm{d}x$$

# **Exercises**

### Exer 7.2.3

Evaluate the definite integral  $\int_{1}^{e} e^{2x} + \frac{1}{2x} dx$ 

### Exer 7.2.4

Find two values of q for which  $\int_{a}^{0} 2x + x^{2} dx = 0$ .