

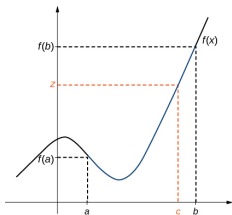
Week 03, Lecture 3

Continuity Types; The Intermediate Value Theorem

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These slides are by Niall Madden. Some content is based on notes by Dr Kirsten Pfeiffer. And some more, such as the figure opposite, taken from Strang & Herman's "Calculus". However, all the typos are Niall's.

Today, in MA140...

- 1 Remembering the reminders...
- 2 Types of discontinuity
- 3 Intermediate Value Theorem
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- Examples
- 4 Derivatives: the concept
 - Rate of change
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For more, see Chapter 2 (Limits) in **Calculus** by Strang & Herman. See openstax.org/books/calculus-volume-1/pages/2-introduction. Section 2.4 (Continuity) relates to today's material.

Remembering the reminders...

Reminders

- ▶ **Assignment 1** due 5pm, Monday 6 October. You may access it multiple times, by clicking on [Assignments ... Problem Set 1 ...](#) and then, at the bottom of the page:

[Load Problem Set 1 in a new window](#)

- ▶ The **Tutorial Sheet** is available at <https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1>
- ▶ Assignment 2 is also open; deadline is 5pm, 13 Oct.
- ▶ The first (of two) class tests will take place Tuesday, 14th October.
- ▶ If you wish to avail of Reasonable Accommodations for it tests, please complete this form: <https://forms.office.com/e/HaAsrzaE3D> by **10am Thursday 2nd Oct.**

Types of discontinuity

We have encountered three types of discontinuity.

- ▶ **Removable discontinuity:** $\lim_{x \rightarrow a} f(x)$ exists but

$$\lim_{x \rightarrow a} f(x) \neq f(a)$$

- ▶ **Jump discontinuity:** $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist (and are finite), but $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$
- ▶ **Infinite discontinuity:** At least one of the one-sided limits does not exist.

Types of discontinuity

Example

Each of the following functions has a discontinuity at $x = 2$.
Classify it.

1. $f(x) = \frac{x^2 - 4}{x - 2}$

2. $g(x) = \frac{x^2}{x - 2}$

3. $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$

4. $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

Types of discontinuity

Intermediate Value Theorem

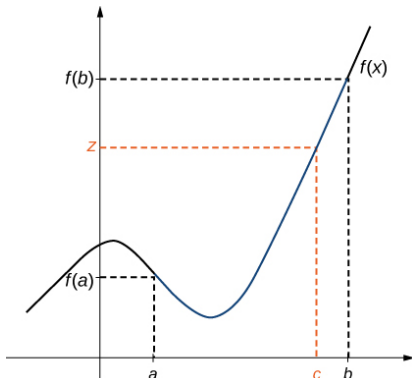
Continuous functions have numerous important properties, many of which we will study in MA140. The first of these is the **Intermediate Value Theorem**.

Intermediate Value Theorem (IVT)

Suppose that $f(x)$ is continuous on an interval $[a, b]$.

Let z be any real number between $f(a)$ and $f(b)$.

Then there exists a number $c \in [a, b]$ such that $f(c) = z$.



- ▶ If you travel by train from Galway to Athlone, then there must be a time when you are at Oranmore station, and a time when you are at Athenry, and at Woodlawn, etc.
- ▶ If your car/train/whatever accelerates from 0km/h to 100km/h , there was a time when it was travelling at 30 km/h .
- ▶ This morning my train ticket from Athenry to Galway cost €5.10. Suppose train fares increase next Thursday to €5.50. But there wasn't a day when they cost, say, €5.20, because the price had a jump discontinuity (so the IVT does not apply here).

Example

Sketch an example of a function for which the IVT does *not* hold.

One of the main applications of the IVT is in establishing if an equation has a solution:

Solutions to $f(x) = 0$

If $f(x)$ defined on $[a, b]$ is such that $f(a) < 0$ and $f(b) > 0$, then there must be a value $c \in [a, b]$ such that $f(c) = 0$.

More generally, if $f(a)f(b) \leq 0$, then $f(x)$ has at least one zero in $[a, b]$.

Example

Show that $f(x) = x - \cos(x)$ has at least one zero.

Given a function $f(x)$,

- ▶ When we say c is a **zero** of a function, f , we mean that $f(c) = 0$.
- ▶ Many books and website also use the terminology “ c is a **root** of f ”. This is particularly the case where $f(x)$ is a polynomial.
- ▶ If c is a zero of $f(x)$, then it is a solution to the equation $f(x) = 0$.

Example

How many solutions does $x^3 + 1 = 3x^2$ have?

Example

Use the *Intermediate Value Theorem* to show that the equation

$$2x^3 + 3x^2 - 2x - 1 = 0$$

has three solutions in the range $-2 < x < 1$.

Derivatives: the concept

The next section of MA140 is all about **derivatives** of function. The derivative of a function describes how quickly the function is changing.

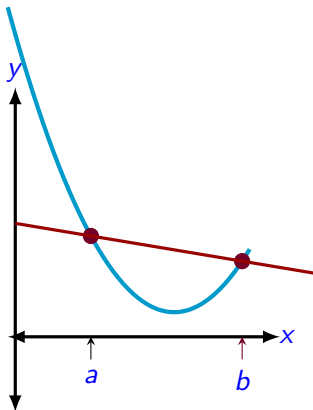
There are many, many applications: derivatives, and equations involving them are used everywhere: **speed/velocity** is the rate of change of displacement; **acceleration** is the rate of change of velocity.

We use derivatives to model how quickly a tumour is growing or shrinking, how pollutants are dispersed in a river, how pressure changes with depth, how inflation is changing in an economy. The list of applications is practically limitless.

Consider the graph opposite. It shows a function, f , and a secant line that intersects f at $a = 1$ and $b = a + 2$ (the actual values are not important).

If we wanted to summarised how f is changing between those two values, we could compute it as

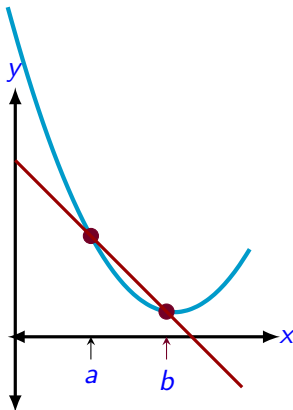
$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + 2) - f(a)}{2}$$



Now we'll consider how f is changing over a shorter interval: from a to $b = a + 1$. Again, we sketch the secant line that intersects f at $x = a$ and $x = b$. The rate of change of f between these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + 1) - f(a)}{1},$$

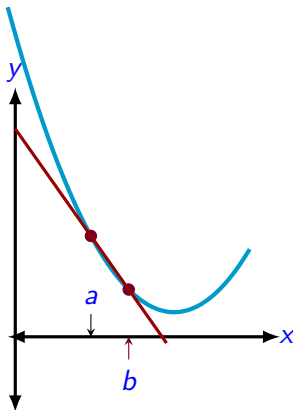
which, of course, is the slope of the secant line.



Next we shorten interval again:
looking at how f changes from
 a to $b = a + \frac{1}{2}$, along with the
secant line that intersects f at
 $x = a$ and $x = b$.

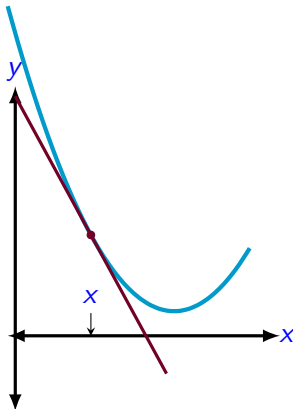
The rate of change of f between
these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + \frac{1}{2}) - f(a)}{\frac{1}{2}}.$$



Finally, suppose we want to looking at the **instantaneous** rate of change of f at $x = a$. Hopefully, the preceding images have convinced you we could do this in two (equivalent) ways:

1. $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$
2. or as the slope of the tangent to f at $x = a$.



The slope of the curve $y = f(x)$ at the point $P = (a, f(a))$ is given by the number (if it exists)

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, it is called **the derivative of f at $x = a$** and we denote it by $f'(a)$.

Definition: derivative at a point

Let $f(x)$ be a function that has $x = a$ in its domain. The **derivative** of the function $f(x)$ at a , denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

Some terminology

$f'(a)$ exists then we say that function f is **differentiable at $x = a$** .

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the “limit” formula, we are doing “**differentiation from first principles**”.

Example

Use the limit definition of a derivative to compute the slope of the tangent to $f(x) = x^2$ at $x = 3$.

Exercises

Exercises 3.3.1 (Based on Q1(a), 23/24)

$$\text{Let } g(x) = \begin{cases} 3 & x \leq 0 \\ 2x + 1 & 0 < x < 1 \\ x^2 & x \geq 1. \end{cases}$$

- (i) Sketch the graph of $g(x)$ on the interval $[-3, 4]$, making use of the empty and full circle notation.
- (ii) Compute $\lim_{x \rightarrow 1^-} g(x)$ and $\lim_{x \rightarrow 1^+} g(x)$. Is g continuous at $x = 1$. If not, classify the type of discontinuity.

Exercise 3.3.2

$$\text{For what values of } b \text{ and } c \text{ is } f(x) = \begin{cases} x^2 + 1 & x \leq -1 \\ x + b & -1 < x < 1 \\ cx^2 & x \geq 1. \end{cases}$$

continuous at $x = -1$ and $x = 1$?

Exercises

Exercise 3.3.3 (23/24 exam)

Use the IVT to show that the equation $x^3 - 3x + 1 = 0$ has three solutions in the range $-2 < x < 2$.

Exercise 3.3.4

Use the (limit) definition of a derivative to differentiate the function $f(x) = x^2 + 2$.

Exercise 3.3.5

Use the (limit) definition of a derivative to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.