MA284 : Discrete Mathematics

Week 9: Convex Polyhedra

Dr Niall Madden

3rd and 5th November, 2021
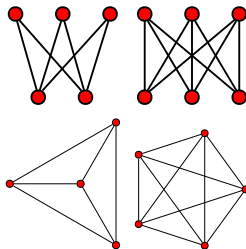
See also Section 4.3 of Levin's *Discrete Mathematics*.

This week:

- Tutorial: Tue at 12.00 - In person in Cairns Building CA117 (https://goo.gl/maps/qUAXAiMS3DdCFzhm6)
- Tutorial: Tue at 15.00 - Zoom: https://nuigalway-ie.zoom.us/j/ 95585742538?pwd=dEhmQWRnd01acUtzNHE2V054WUk4UT09
- Tutorial: Wednesday at 11.00 - In person in Martin Ryan Annex MRA201 (https://goo.gl/maps/knBXke5fw3VYQPvL9)
- **Lecture** Wednesday at 13:00 - Zoom: https://nuigalway-ie.zoom.us/j/ 92632490823?pwd=bGtjZTRyYTVacW1DOGVjc2s4dzJjZz09
- Tutorial: Wednesday at 14:00 - Zoom: https://nuigalway-ie.zoom.us/j/ 93105102422?pwd=blVtZ25WNk5aN0d5WXdLLzd3bU8vQT09
- Tutorial: Thursday at 15:00 - Zoom: https://nuigalway-ie.zoom.us/j/ 97389109686?pwd=NGdoeTQvOUV6eUJSZHV2Z0oxK2h2Zz09
- Tutorial: Thursday at 16.00 - In person AMB-G008 (https://goo.gl/maps/JvRxthuqfeQnBrwe7)
- **Lecture Friday @ 11:00** - **In person, O'Flaherty Lecture Theatre**. I may try to live-stream this...

- **Assignment 2** closed Tuesday (26 Oct). Your grades should be up on Blackboard.
- **Assignment 3** is now open. Deadline is *5pm, Tuesday 9 November*

. . . .

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Start of ...

PART 1: Non-Planar graphs

Recall: Planar graph

- If you can sketch a graph so that none of its edges cross, then it is a *PLANAR* graph.
- When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a *FACE*. The "exterior" of the graph is consider a face.
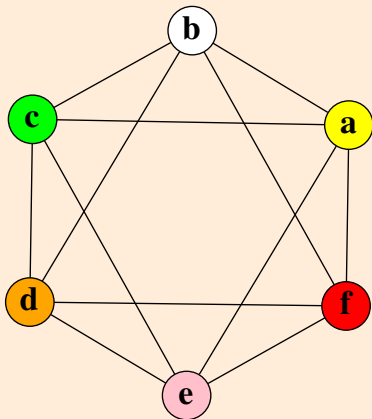
Euler's formula for planar graphs

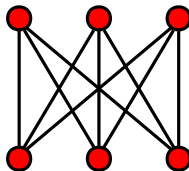For any (connected) planar graph with v vertices, e edges and f faces, we have
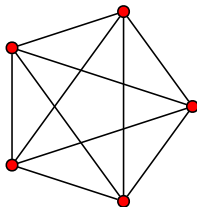
$$v - e + f = 2$$

Example

Give a planar representation of the following graph, and verify that Euler's Formula Holds.

Of course, most graphs do **not** have a planar representation. We have already met two that (we think) cannot be drawn so no edges cross: K_5 and K_{33}:
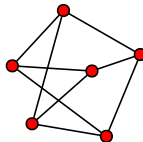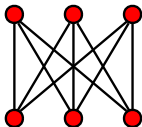

However, it takes a little work to *prove* that these are non-planar. While, through trial and error, we can convince ourselves these graphs are not planar, a proof is still required.

For this, we can use **Euler's formula for planar graphs** to *prove* they are not planar.
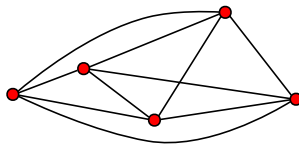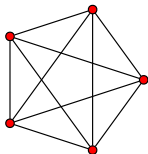
Theorem (Theorem 4.3.1 in textbook)

K_5 is not planar. (The proof is by *contradiction*).

Theorem ($K_{3,3}$ is not planar)

This is Theorem 4.2.2 in the text-book. Please read the proof there.

The proof for $K_{3,3}$ is somewhat similar to that for K_5, but also uses the fact that a bipartite graph has no 3-edge cycles.

This also means we have solved (negatively) the Utilities (Water-Power-Gas) problem from Week 7.

The understand the importance of K_5 and $K_{3,3}$, we first need the concept of *homeomorphic* graphs.

Recall that a graph G_1 is a *subgraph* of G if it can be obtained by deleting some vertices and/or edges of G.

A *SUBDIVISION* of an edge is obtained by "adding" a new vertex of degree 2 to the middle of the edge.

A *SUBDIVISION* of a graph is obtained by subdividing one or more of its edges.

Example:

Closely related: **SMOOTHING** of the pair of edges $\{a, b\}$ and $\{b, c\}$, where b is a vertex of degree 2, means to remove these two edges, and add $\{a, c\}$.

Example:

The graphs G_1 and G_2 are *HOMEOMORPHIC* if there is some subdivision of G_1 is isomorphic to some subdivision of G_2.

Examples:

There is a *celebrated* theorem due to Kazimierz Kuratowski. *The proof is beyond what we can cover in this module. But if you are interested in Mathematics, read up in it: it really is a fascinating result.*

Theorem (Kuratowski's theorem)

A graph is planar if and only if it does not contain a subgraph that is homeomorphic to K_5 or $K_{3,3}$.

What this *really* means is that *every* non-planar graph has some smoothing that contains a copy of K_5 or $K_{3,3}$ somewhere inside it.

Example

The Petersen graph is not planar https:
//upload.wikimedia.org/wikipedia/commons/0/0d/Kuratowski.gif
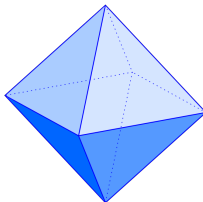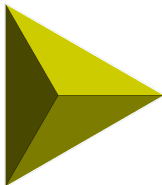
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END OF PART 1

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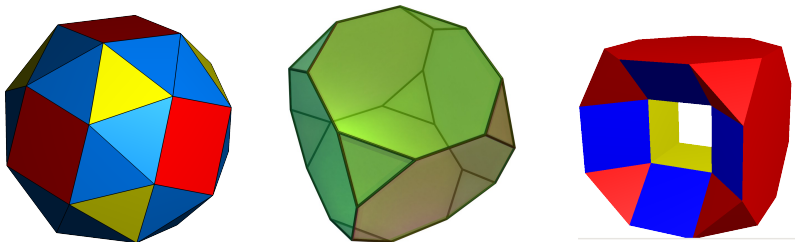## Week 9: Convex Polyhedra

Start of ...

PART 2: Polyhedra
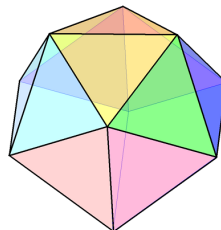
A *polyhedron* is a geometric solid made up of flat polygonal faces joined at edges and vertices.

A *convex polyhedron*, is one where any line segment connecting two points on the interior of the polyhedron must be entirely contained inside the polyhedron.
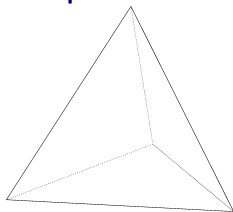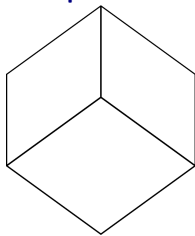
Examples:

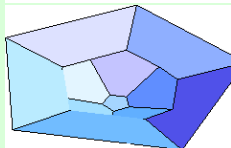Source: WikiMedia Uniform polyhedron-43-s012.png, Truncatedhexahedron.jpg and Excavated_truncated_cube.png

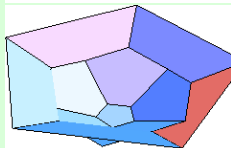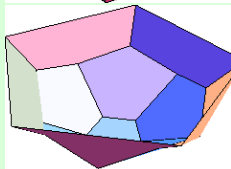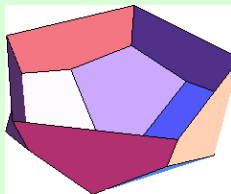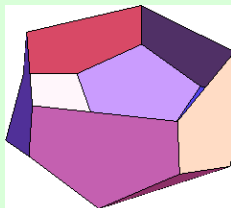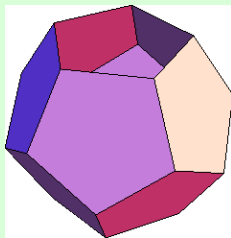A remarkable, and important fact, is that *every* convex polyhedron can be projected onto the plane without edges crossing.
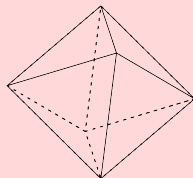
Example:

Example:

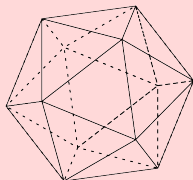## Example: the dodecahedron

Exercise

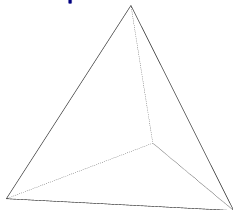Give a planar projection of each of the following polyhedra.

Now that we know every convex polyhedron can be represented as a planar graph, we can apply Euler's formula.
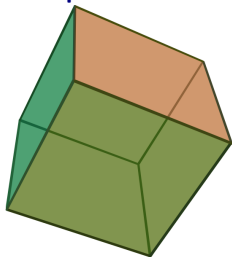
Euler's formula for polyhedra

If a convex polyhedron has v vertices, e edges and f faces, then
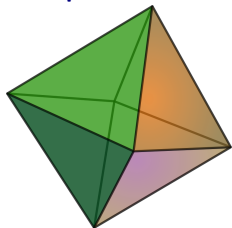
$$v - e + f = 2$$

Example: the tetrahedron.

Example: the cube

Example: the octahedron

We now have two very powerful tools for studying convex polyhedra:

- **Euler's formula:** If a convex polyhedron has v vertices, e edges and f faces, then $v - e + f = 2$
- (The Handshaking Lemma) **The sum of the vertex degrees is $2|E|$:** let $G = (V, E)$ be a graph, with vertices $V = v_1, v_2, \ldots, v_n$. Let $\deg(v_i)$ be the *"degree of v_i"*. Then

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|.$$

Example (See textbook, Section 4.2 (Polyhedra))

Show that there is no convex polyhedron with 11 vertices, all of degree 3?

See textbook, Example 4.2.3

Show that there is no convex polyhedron consisting of

- 3 triangles,
- 6 pentagons, and
- 5 heptagons (7-sided polygons).

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END OF PART 2
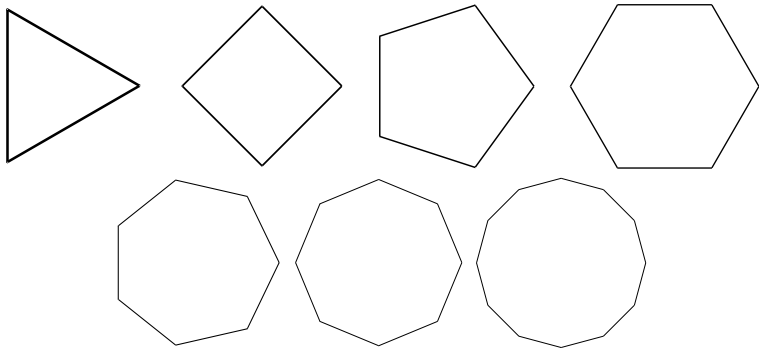
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PART 3: Platonic Solids

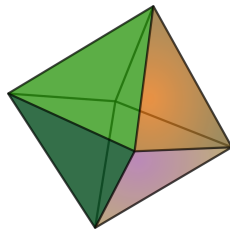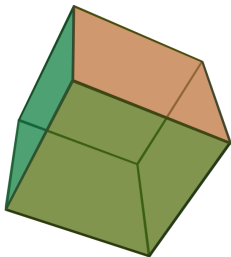*Regular polyhedra - they are surprisingly few of them!*

A **POLYGON** is a two-dimensional object. It is *regular* if all its sides are the same length:

A *polyhedron* with the following properties is called **REGULAR** if

- All its faces are identical regular polygons.
- All its vertices have the same degree.

The convex regular polyhedra are also called *Platonic Solids*. Examples:

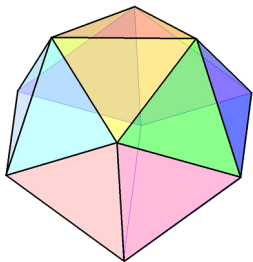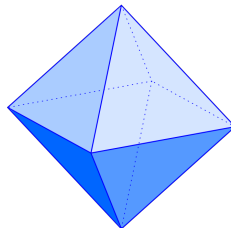**There are exactly 5 regular polyhedra**

This fact can be proven using Euler's formula.
For full details, see the proof in the text book.
Here is the basic idea: we will only look at the case of polyhedra with
triangular faces.

[All images here courtesy of Wikipedia]
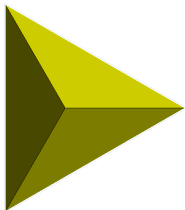
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END OF PART 3

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PART 4: Vertex Colouring

[From textbook, p184]. Here is a map of the (fictional) country "Euleria". Colour it so that adjacent regions are coloured differently. What is the fewest colours required?

There are maps that can be coloured with

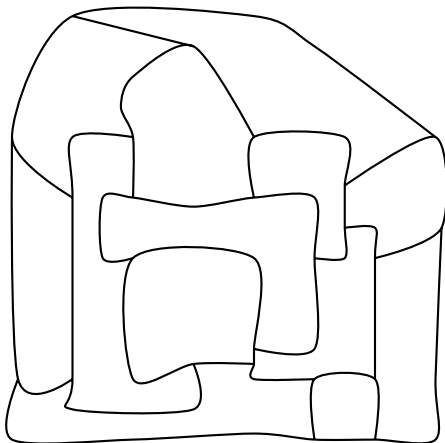- A single colour:

- Two colours (e.g., the island of Ireland):

- Three colours:

- **Four colours:**

It turns out that the is *no* map that needs more than 4 colours. This is the famous Four Colour Theorem, which was originally conjectured by the British/South African mathematician and botanist, Francis Guthrie who at the time was a student at University College London.

He told one of his mathematics lecturers, Augustus de Morgan, who, on **23 October, 1852**, wrote to friend William Rowan Hamilton, who was in Dublin:

From https://en.wikipedia.org/wiki/Four_color_theorem
de Morgan writes to Hamilton, 23 October, 1852..

From https://en.wikipedia.org/wiki/Four_color_theorem
de Morgan writes to Hamilton, 23 October, 1852..

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured-four colours may be wanted, but not more-the following is his case in which four are wanted.
Query: cannot a necessity for five or more be invented... What do you say? And has it, if true been noticed?

My pupil says he guessed it in colouring a map of England... The more I think of it, the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did...

De Morgan needn't have worried: a proof was not produced until **1976**. It is very complicated, and relies heavily on computer power.

To get a sense of *why* it might be true, try to draw a map that needs 5 colours.

. . . .
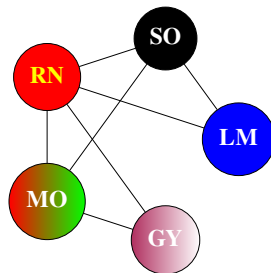
Our interest is not in trying to prove the Four Colour Theorem, but in how it is related to Graph Theory.

If we think of a map as a way of showing which regions share borders, then we can represent it as a *graph*, where

- A vertex in the graph corresponds to a region in the map;
- There is an edge between two vertices in the graph if the corresponds regions share a border.

Example:

Colouring regions of a map corresponds to colouring vertices of the graph.
Since neighbouring regions in the map must have different colours, so too
adjacent vertices in the graph must have different colours.

More precisely

Vertex Colouring: An assignment of colours to the vertices of a graph is called
a *VERTEX COLOURING*.

Proper Colouring: If the vertex colouring has the property that adjacent
vertices are coloured differently, then the colouring is called
PROPER.

Lots different proper colourings are possible. If the graph as v vertices, the
clearly at most v colours are needed. However, usually, we need far fewer.

Examples:
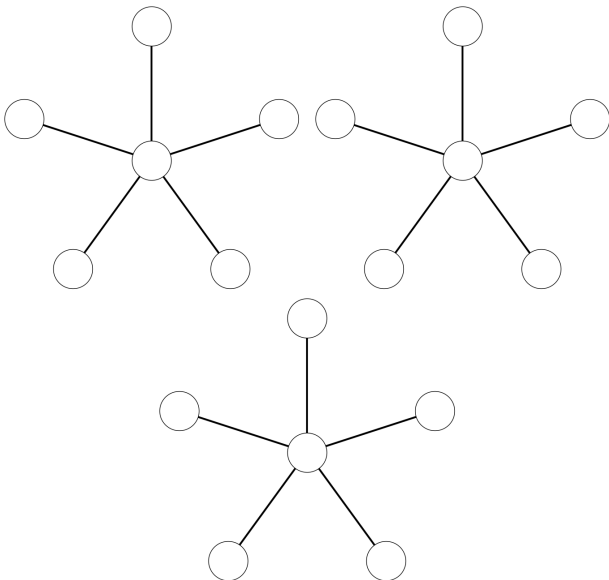
CHROMATIC NUMBER

The smallest number of colours needed to get a proper vertex colouring if a graph G is called the *CHROMATIC NUMBER* of the graph, written $\chi(G)$.

**Example:** Determine the Chromatic Number of the graphs C_2, C_3, C_4 and C_5.

Example: Determine the Chromatic Number of the K_n and $K_{p,q}$ for any n, p, q.

In general, calculating $\chi(G)$ is not easy. There are some ideas that can help. For example, it is clearly true that, if a graph has v vertices, then

$$1 \leq \chi(G) \leq v.$$

If the graph happens to be *complete*, then $\chi(G) = v$. If it is *not* complete the we can look at *cliques* in the graph.

> Clique: A *CLIQUE* is a subgraph of a graph all of whose vertices are connected to each other.
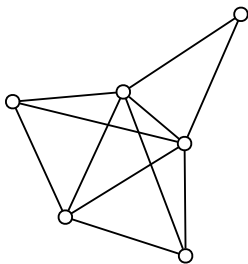
> The **CLIQUE NUMBER** of a graph, G, is the number of vertices in the largest clique in G.

From the last example, we can deduce that

> **LOWER BOUND**: The chromatic number of a graph is *at least* its clique number.

....

We can also get a useful upper bound. Let $\Delta(G)$ denote the largest degree of any vertex in the graph, G,

$$\textbf{UPPER BOUND: } \chi(G) \leq \Delta(G) + 1.$$

Why? This is called **Brooks' Theorem**, and is Thm 4.5.5 in the text-book:
http://discrete.openmathbooks.org/dmoi3/sec_coloring.html

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END OF PART 4

Most of these questions are taken from Levin's *Discrete Mathematics*.

Q1. Try to prove that $K_{3,3}$ is non-planar using *exactly* the same reasoning as that used to prove K_5 is non-planar. What does wrong? (The purpose of this exercise is to show that noting that $K_{3,3}$ has no 3-cycles is key. Also, we want to know that K_5 and $K_{3,3}$ are non-planar for different reasons).

Q2. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

Q3. The graph G has 6 vertices with degrees 2,2,3,4,4,5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

Q4. Euler's formula ($v - e + f = 2$) holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has k components?

Q5. Prove that any planar graph with v vertices and e edges satisfies $e \leq 3v - 6$.

Q6. Which of the graphs below are bipartite? Justify your answers.
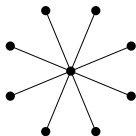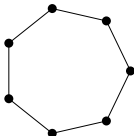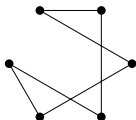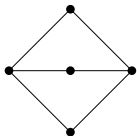
Q7. For which $n \geq 3$ is the graph C_n bipartite?

Q8. For each of the following, try to give two different unlabeled graphs with the given properties, or explain why doing so is impossible.

   (a) Two different trees with the same number of vertices and the same number of edges. (A tree is a connected graph with no cycles).

   (b) Two different graphs with 8 vertices all of degree 2.

   (c) Two different graphs with 5 vertices all of degree 4.

   (d) Two different graphs with 5 vertices all of degree 3.

Q9. Give a planar projection of each of the following polyhedra.
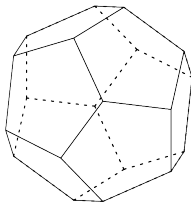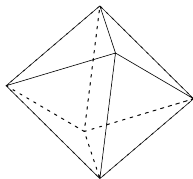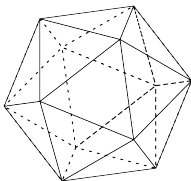
Q10. Show that there is only one regular convex polygon with square faces.

Q11. Show that there is only one regular convex polygon with pentagonal faces.

Q12. Could there be a regular polygon with faces that have more than 5 sides? Explain your answer.