CS4423: Problem Set 1 ANS with solutions

These exercises are to help you master material covered in classes. You don't have to submit your work. However, Assignment 1, which will be posted later in Week 4, will include questions based on some of these exercises.

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Q1. For what values of n is K_n bipartite? ANS $n = 1$ and $n = 2$ only

Q2. For what values of m and n is $K_{m,n}$ bipartite? ANS All, by definition!

Q3. For what values of n is P_n bipartite? ANS All n

Q4. For what values of n is C_n, the cycle graph on $n \geqslant 3$ nodes, bipartite? ANS Even n

Q5. Let G be the graph on the set of nodes $\{1, 2, 3, 4, 5, 6\}$ with edges $1-2$, $1-3$, 2–4, $3-4$, $3-6$, $4-5$, $4-6$. Draw the graph G. Is G bipartite? Justify your answer. (Note: writing "$a - b$ is an edge in G" is the same as saying (a, b) is an element of its edge set). ANS Will add solution later

Q6. At a party with $n = 5$ people, some people know each other already while others don't. Each of the 5 guests is asked how many friends they have at this party. Two report that they have one friend each. Two other guests have two friends each, and the fifth guest has three friends at the party. Understanding friendship as a symmetric relation, is this network possible? Why, or why not? (*Hint: recall that the sum of all node degrees is twice the number of edges in the graph*). ANS Not possible: degree sum must be even

Q7. We say two graphs are *equal* if they have the same node and edge sets. We say they are *isomorphic* if there is a relabling of their nodes that makes them equal. Verify that C_5 is isomorphic to its complement.

Q8. Convince yourself that C_n is always isomorphic to $L(C_n)$, the line graph of C_n.

> **Answer:** It is sufficient to check a few cases; formal proof not required.

Q9. Let G be any graph of order n. Let $\bar{G}$ be its compliment. Call their adjacency matrices A_G and $A_{\bar{G}}$, respectively. Let H be the graph with adjacency matrix $A_G + A_{\bar{G}}$. By what name is H more commonly known? ANS K_n

Q10. Let P_n be the path graph on $n \geqslant 2$ vertices. There is exactly one n for which P_n is isomorphic to its complement, $\bar{P_n}$. What value of n is that? Show that there are no other values of n for which P_n is isomorphic to $\bar{P_n}$

> **Answer:** P_4 is isomorphic to itself. One can check the result for $n = 2$ and $n = 3$. For any $n > 4$ you can verify that, for example, there are two nodes with degree at least 3.

Q11. Is the Petersen graph bipartite? Explain your answer. ANS No. Possible reasons include there being odd cycles

Q12. Write down the adjacency matrix, A of $K_{2,3}$. Compute A^2 and A^3. Use A^3 this to verify that $K_{2,3}$ has no triangles (3-cycles).

Answer: We should find that $A^3 = \begin{pmatrix} 0 & 0 & 6 & 6 & 6 \\ 0 & 0 & 6 & 6 & 6 \\ 6 & 6 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 \end{pmatrix}$. Since all diagonal entries are zero, there are no triangles.

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Theory for the Q13 and Q14 will be covered in lectures in Week 4.

Q13. Consider the graph, G, shown in Figure 1.

(a) Write down the node set, V, edge set E, and adjacency matrix A for this graph.

(b) Find a permutation matrix, P, such that PAP^T is structured like:

$$PAP^T = \begin{pmatrix} A_{11} & O_{12} \\ O_{12}^T & A_{22}. \end{pmatrix}$$

where O_{12} is a 5×3 matrix of zeros.

(c) Show that $(PAP^T)^k = \begin{pmatrix} A_{11}^k & O_{12} \\ O_{12}^T & A_{22}^k. \end{pmatrix}$ for all k, and conclude that G is not connected.

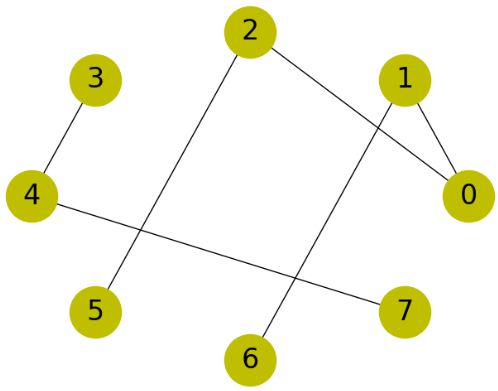

Figure 1: Graph for Q13

Q14. Consider the bipartite graph, shown in Figure 1. Construct the projection of it onto the sets $\{a, b, c, d, e\}$. Sketch the resulting graph.

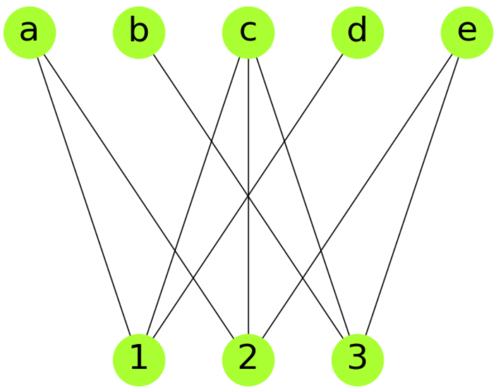

Figure 2: Graph for Q14