2425-MA140 Engineering Calculus

Week 03, Lecture 2
Vertical Asymptotes and Continuity

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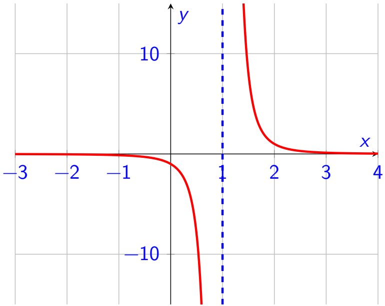

This slides are by Niall Madden, with some content based on notes by Dr Kirsten Pfeiffer, and from Strang & Herman's "Calculus".

Outline

For more, see Section 7.8.1 (Limit of a function of a real number) in *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

However, I *highly* recommend Chapter 2 (Limits) in **Calculus** by Strang & Herman. See openstax.org/books/calculus-volume-1/pages/2-introduction

Reminder

- **Assignment 1** has a deadline of 5pm, Friday. You can access it on Canvas... 2425-MA140... Assignments.
- The **Tutorial Sheet** is available at https://universityofgalway.instructure.com/files/2040359/download?download_frd=1
- A new assignment will be posted later this week.

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For help with the assignment, attend a tutorial. The schedule is on the Canvas "Course Information" page: https://universityofgalway.instructure.com/courses/35693/pages/2425-ma140-information. Note the change of venue for the Irish language tutorials (Tue at 1, AMB-G021).

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Support is also available at tutorials and **SUMS**.

Yesterday we met the concept of **one-sided limits**:

$\lim_{x\to a^-} f(x)$ is: **limit of f as x approaches a from the left**

$\lim_{x\to a^+} f(x)$ is: **limit of f as x approaches a from the right**

These mean that

- if $\boxed{\lim\limits_{x\to a^-} f(x) = L}$, then we can make $f(x)$ as close to L as we would like by taking x as close to a as needed, and that $\boxed{x < a}$.
- If $\boxed{\lim \lim\limits_{x\to a^+} f(x) = L}$, then we can make $f(x)$ as close to L as we would like by taking x as close to a as needed, with $\boxed{x > a}$.

Note: One-sided limits can be introduced formally by using the ϵ/δ approach, but we won't do that.

Existence of a limit

$\lim_{x \to a} f(x)$ **exists** if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

So if $\lim_{x \to a} f(x) = L$ exists, we have

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = L$$

though it is not necessary that $f(a) = L$

Example

Sketch the function

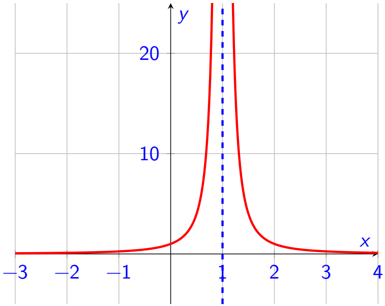
$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if $\lim_{x \to 2} f(x)$ exists.

Vertical Asymptotes

Let's revisit the following example from yesterday:

$$\boxed{f(x) = \frac{1}{(x-1)^2}}$$



Note that the points on the graph having x-coordinates very near to 1 are very close to the vertical line $x = 1$. That is, as x approaches 1, the points on the graph of $f(x)$ are closer to the line $x = 1$.

We call the line $x = 1$ a **vertical asymptote** of the graph.

Vertical Asymptotes

Definition: Vertical Asymptote

The vertical line $x = a$ is a **vertical asymptote** of $f(x)$ if any of $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, or $\lim_{x \to a} f(x)$ are ∞ or $-\infty$.

To find a vertical asymptote of a function $f(x) = \dfrac{p(x)}{q(x)}$, we find a value, a for which $p(a) \neq 0$ but $q(a) = 0$.

Vertical Asymptotes

Example

Find any vertical asymptotes of

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

Example

Find all vertical asymptotes of the graph of

$$g(x) = -\frac{8}{x^2 - 4}.$$

There is a related concept of a **horizontal asymptote**, but we'll save that for later, when we cover "limits at infinity".

Continuity

Many functions have the property that you can trace their graphs with pen and paper, without lifting the pen from the page. Such functions are called **continuous**.

Some other functions have points were you have to lift the pen occasionally. We say they have a **discontinutity** at such points.

Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

More formally, we define continuity in terms of **limits**

Continuity

Definition

A function f is **continuous at** $x = a$ if

1. $f(a)$ is defined, i.e., a is in the domain of f,
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$ we say it is **discontinuous** at $x = a$.

If f is continuous **at every point** in its domain, we say f **is continuous.**

Many functions are continuous, e.g. all polynomial functions, most trigonometric functions (not tan), $|x|$, and so on.

Continuity

Example 1

Determine if $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$.

Continuity

Example 2

Determine if $f(x) = \begin{cases} 1 - x & x \leqslant 0 \\ 2 + x & x > 0 \end{cases}$ is continuous at $x = 0$.

Continuity

Example 3

Determine if $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is continuous at $x = 0$.

Continuity

Example

Consider the function

$$f(x) = \begin{cases} x+1, & x < 2 \\ bx^2, & x \geqslant 2 \end{cases}$$

For what value of b is f continuous at $x = 2$?

Continuity

Example

For what values of x is $f(x) = \dfrac{2x+1}{2x-2}$ continuous?

Exercises

Exercise 3.2.1

Find all the vertical asymptotes of $f(x) = \dfrac{x+2}{x^2+2x-8}$.