2425-MA140 Engineering Calculus

Week 10, Lecture 3 (L30)
Moments and Centroids

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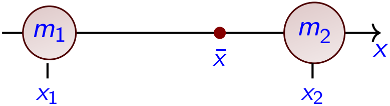


Assignments,etc

1. I'm (still) working on grading Q8 of **Assignment 6** results should be available by Friday.
2. Grades for **Assignment 7** will be posted by Monday (I hope!).
3. **Assignment 8** is open, and the tutorial sheet is available.

Today's lecture will be centred on...

For more, read Section **6.6** (Moments and Centres of Mass) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax).

Centre of Mass: over view

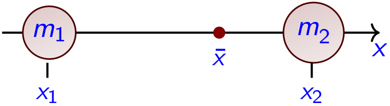
In this section, we what to study the **centre of mass** of an object, such as an irregularly shaped thin plate.

Intuitively, this is the point at which at which a plate could be perfectly balanced on the tip of a pin.

But first we study two one-dimensional, the first of which does not even require calculus. But all have the concept of "balance" at their heart.

Moments

Suppose we have a thin rod with negligible mass. We attach objects with mass m_1 and m_2, at the points x_1 and x_2. We want to find $\bar{x}$: the point at which the rod is balanced (e.g., if suspended from a string at that point).



Suppose $m_1 < m_2$. Then we know that x_1 should be further from $\bar{x}$ than x_2. More precisely, we need

$$m_1|x_1 - \bar{x}| = m_2|x_2 - \bar{x}|.$$

Moments

Starting from

$$m_1|x_1 - \bar{x}| = m_2|x_2 - \bar{x}|.$$

we can solve for $\bar{x}$:

Moments

- For this scenario, we have deduced that $\bar{x} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$.
- The quantity $m_1x_1 + m_2x_2$ is called the *(first) moment of the system (with respect to the origin)*.
- It can also be interpreted as: $\bar{x}(m_1 + m_2) = m_1x_1 + m_2x_2$.
  This means: "*if all the mass was concentrated at $x = \bar{x}$, the moment would not be changed*".
- If there are three masses, m_1, m_2 and m_3, at the points x_1, x_2 and x_3, the formula extends: $\bar{x} = \dfrac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$.
- And for n masses:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k}$$

Moments

Center of Mass of Objects on a Line

Let m_1, m_2, $\ldots$, m_n be point masses placed on a number line at points x_1, x_2, $\ldots$, x_n, respectively. The **total mass** of the system is $m = \sum_{k=1}^{n} m_k$.

Then the the **moment of the system**, with respect to the origin, is $M = \sum_{k=1}^{n} m_k x_k$.

And the **centre of mass** is $\bar{x} = \dfrac{M}{m}$.

Moments

Example:

Find the centre of mass of a system where a mass of $8kg$ is placed on the number line at $x = 1$, and a mass of $4kg$ is placed at $x = 4$,

Moments

Example:

We have a system where

- a mass of $8kg$ is placed on the number line at $x = 1$,
- a mass of $4kg$ is placed at $x = 4$,
- a mass of $6kg$ is placed at $x = x_3$.

If the centre of mass is at $x = 5$, find x_3.

A 1D rod with variable density

In our previous examples, we assumed the rod we were hanging masses from was itself mass-less. That was just to simplify calculations, as was the assumption that the masses were "point masses".

But suppose the rod does have mass, and it varies along the length. How do we find the centre of gravity?

First, it helps to understand that when we say that "the mass can vary", what we really mean is that the **density** (i.e., mass per unit length) can vary.

That is, there is a function $\rho(x)$, which is the density of the rod at x.

To get the total mass, we reason as follows.

- The mass of a "slice" from x_k to $x_k + \Delta x$ is $m_k = \rho(x_k)\Delta x$.
- Summing over all slices of such length we get the total mass is $m \approx \sum_{k=1}^{n} \rho(x_k)\Delta x$, where $\Delta x = (b-a)/n$, $x_0 = a$, and $x_k = x_0 + k\Delta x$.
- Doing our usual trick of letting $n \to \infty$, we get

$$\boxed{m = \int_a^b \rho(x)\,dx.}$$

Moments

For a discrete set of points (and masses), we know the moment is

$$M = \sum_{k=1}^{n} x_k m_k.$$

With our $\Delta x = (b-a)/n$ notation, this is

$$M \approx \sum_{k=1}^{n} x_k \rho(x_k) \Delta x.$$

Again, we let $n \to \infty$, and we get

$$\boxed{M = \int_a^b x\rho(x)\, dx.}$$

We can now conclude that the **centre of mass** of a rod on the x-axis with end-points at $x = a$ and $x = b$ (with $a < b$), and density $\rho(x)$ is

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx.}}$$

Example

Find the centre of mass of a rod with $a = 0$, $b = 1$, and $\rho(x) = x^2$.

Two dimensions

Suppose we are given a (positive) function $f(x)$, have a region in the plane bounded above by $y = f(x)$, below by $y = 0$, and left by $x = a$, and right by $y = b$. A thin plate defined by this region is sometimes called a **lamina**. Its area is $A = \int_a^b f(x)\,dx$.

We now want to consider how to find its **centre of mass ("centroid")**, which we denote $(\bar{x}, \bar{y})$.

Intuitively, (again): this is the point at which at which a cutout of the region could be **perfectly balanced** on the tip of a pin.

Again, the key idea we need is that of a **moment**. In a realistic setting, this is the **mass** of the lamina, times its distance from a reference point: usually $(0, 0)$.

In our setting, we'll just use the **area** of a region as a proxy for the mass. (This is physically reasonable if the limina has uniform density, and is very, very thin).

To start with, it is helpful to think of the moments (in x and y) of a thin rectangle:

Now let's get M_x, which is the moment about the x-axis, by summing the moments of all the rectangles, and taking the limit of the resulting Riemann sum:

$$M_x = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2}\big(f(x_i^\star)\big)^2 \Delta x = \int_a^b \frac{\big(f(x)\big)^2}{2}\, dx.$$

Similarly, we get M_y, which is the moment about the y-axis as

$$M_y = \lim_{n \to \infty} \sum_{k=1}^{n} x_i^\star f(x_i^\star) \Delta x = \int_a^b x f(x)\, dx.$$

If the centre of mass is the point $(\bar{x}, \bar{y})$, then we could think of the entire "area" as being centred there, but having the same moments.

That is

$$\bar{x}A = M_y, \quad \text{and } \bar{y}A = M_x.$$

giving...

Centroid of a planar region

If $f(x)$ is defined on $[a, b]$, then the **centroid** $(\bar{x}, \bar{y})$ of the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$ is given by

$$\bar{x} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx} \quad \text{and} \quad \bar{y} = \frac{\int_a^b [f(x)]^2\, dx}{2\int_a^b f(x)\, dx}$$

Example

Consider the plane region enclosed by the curve $y = \sqrt{x-2}$, the x-axis and the lines $x = 2$ and $x = 5$. Find

(1) the area of the region;

(2) the centroid of the region.

Exercises

Exer 10.3.1

Find the centre of mass, $\bar{x}$, of a system with thin rod of negligible mass, placed on the x-axis, with a mass of $m_1 = 1$ placed at $x_1 = -1$, and $m_2 = 3$ placed at $x_2 = 2$.

Exer 10.3.2

A system consists of a thin rod of negligible mass, placed on the x-axis, with a mass of $m_1 = 10$ placed at $x_1 = 0$, and $m_2 = 5$ placed at $x_2 = 2$, and a mass m_3 at $x_3 = 3$. If $\bar{x} = 1$, find m_3.

Exer 10.3.3

Find the centre of mass of a rod with density $\rho(x) = \sqrt{x}$, that is on the x-axis, with end points at $x = 0$ and $x = 1$.