Chapter 2 (Initial Value Problems) and Chapter 3 (Numerical Linear Algebra)

Submit carefully written solutions to the following exercises: Exercises 2.7, 2.14, 3.12 and 3.15.

Attach the marking sheet to the front of your solutions using a stable. Don't use paper clips. No plastic covers/envelopes, please.

Exercise 2.7 ($\star$)**.** In his seminal paper of 1901, Carl Runge gave an example of what we now call a *Runge-Kutta 2 method*, where

$$\Phi(t_i, y_i; h) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\big).$$

(i) Show that it is consistent.

(ii) Show how this method fits into the general framework of RK2 methods. That is,

  (a) What are a, b, α, and β?

  (b) Do they satisfy the conditions

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b?$$

(iii) Use it to estimate the solution at the point $t = 2$ to $y(1) = 1$, $y' = 1 + t + y/t$ taking $n = 2$ time steps.

Here are some entries for 3-stage Runge-Kutta method tableaux for Exercise 2.14.

Method 0: $\alpha_2 = 2/3$, $\alpha_3 = 0$, $b_1 = 1/12$, $b_2 = 3/4$, $\beta_{32} = 3/2$

Method 1: $\alpha_2 = 1/4$, $\alpha_3 = 1$, $b_1 = -1/6$, $b_2 = 8/9$, $\beta_{32} = 12/5$

Method 2: $\alpha_2 = 1/4$, $\alpha_3 = 1/2$, $b_1 = 2/3$, $b_2 = -4/3$, $\beta_{32} = 2/5$

Method 3: $\alpha_2 = 1/4$, $\alpha_3 = 1/3$, $b_1 = 3/2$, $b_2 = -8$, $\beta_{32} = 4/45$

Method 4: $\alpha_2 = 1$, $\alpha_3 = 1/4$, $b_1 = -1/6$, $b_2 = 5/18$, $\beta_{32} = 3/16$

Method 5: $\alpha_2 = 1$, $\alpha_3 = 1/5$, $b_1 = -1/3$, $b_2 = 7/24$, $\beta_{32} = 4/25$

Method 6: $\alpha_2 = 1$, $\alpha_3 = 1/6$, $b_1 = -1/2$, $b_2 = 3/10$, $\beta_{32} = 5/36$

Method 7: $\alpha_2 = 1/2$, $\alpha_3 = 1/7$, $b_1 = 7/6$, $b_2 = 22/15$, $\beta_{32} = -10/49$

Method 8: $\alpha_2 = 1/2$, $\alpha_3 = 1/8$, $b_1 = 4/3$, $b_2 = 13/9$, $\beta_{32} = -3/16$

Method 9: $\alpha_2 = 1/3$, $\alpha_3 = 1/9$, $b_1 = 4$, $b_2 = 15/4$, $\beta_{32} = -2/27$

Exercise 2.14 (Your own RK3 method $\star$)**.** Answer the following questions for Method K from the list above, where K is the last digit of your ID number. For example, if your ID number is 01234567, use Method 7.

(a) Assuming that the method is *consistent*, determine the value of b_3.

(b) Consider the initial value problem:

$$y(0) = 1, \; y'(t) = \lambda y(t).$$

Using that the solution is $y(t) = e^{\lambda t}$, write out a Taylor series for $y(t_{i+1})$ about $y(t_i)$ up to terms of order h^4 (use that $h = t_{i+1} - t_i$).

Using that your method should agree with the Taylor Series expansion up to terms of order h^3, determine β_{21} and β_{31}.

Exercise 3.12 ($\star$)**.** Show that, for any vector $x \in \mathbb{R}^n$, $\|x\|_\infty \leqslant \|x\|_2$ and $\|x\|_2^2 \leqslant \|x\|_1 \|x\|_\infty$. For each of these inequalities, give an example for which the equality holds. Deduce that $\|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1$.

Exercise 3.15 ($\star$)**.** Prove that

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{i,j}|.$$

Hint: Suppose that

$$\sum_{i=1}^{n} |a_{ij}| \leqslant C, \qquad j = 1, 2, \dots n,$$

show that for *any* vector $x \in \mathbb{R}^n$

$$\sum_{i=1}^{n} |(Ax)_i| \leqslant C\|x\|_1.$$

Now find a vector x such that $\sum_{i=1}^{n} |(Ax)_i| = C\|x\|_1$. Now deduce the result.