MA385 Part 2: Initial Value Problems
2.1: **Introduction**
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Emile Picard: his fundamental work on differential equations was only one of his many contributions to mathematics

Olga Ladyzhenskaya: her extensive achievements include providing the first proof of the convergence of finite difference methods for the Navier-Stokes equations

0. Tutorials and Labs

1. Tutorials started next week... tutorial sheet is available at
   `https://www.niallmadden.ie/2526-MA385/`
   `MA385-Tutorial-1.pdf`.
2. Next week (Week 5) we'll have a lab.
   - That will be based on Python/Jupyter;
   - You will have to submit your work (worth 3.333%) by Monday 13 Oct.
   - Collaboration is encouraged.
   - Lab will take place
     - Mondays at 10 in AC-201
     - Thursday at 2 in ENG-3036.
     - Go to either/both/neither, as you prefer.

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Week 4 (29/10) : Tutorial 1 (Monday and Thursday)

Week 5 (06/10) : **Lab 1** (Monday and Thursday)

Week 6 (13/10) : Tutorial 2 (Monday and Thursday)

Week 7 (20/10) : Tutorial 3 (Monday and Thursday).
             Assignment due (OK?).

Week 8 (27/10) : No tutorials/labs. **Class test** Thursday at 3pm
             (OK?).

Week 9 (03/11) : **Lab 2** (Monday and Thursday)

Week 10 (10/11) : Tutorial 4 (Monday and Thursday).

Week 11 (17/11) : **Lab 3** (Monday and Thursday).

Week 12 (24/11) : Tutorial (Monday and Thursday).

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Discuss...

0. Outline of Section 1

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For more details, see Chapter 6 of Süli and Mayers, *An Introduction to Numerical Analysis*, and Chapter 12 of Epperson: https://search.library.nuigalway.ie/permalink/f/ 3b1kce/TN_cdi_askewsholts_vlebooks_9781118730966

1. Motivation

Motivation (See Chap 6 of Epperson)

The growth of some tumours can be modelled as

$$R'(t) = -\frac{1}{3}S_i R(t) + \frac{2\lambda\sigma}{\mu R + \sqrt{\mu^2 R^2 + 4\sigma}},$$

subject to the initial condition $R(t_0) = a$, where R is the radius of the tumour at time t.

Clearly, it would be useful to know the value of R as certain times in the future. Though it's essentially impossible to solve for R exactly, we can accurately estimate it. In this section, we'll study techniques for this.

2. IVPs

Initial Value Problems (IVPs)

Initial Value Problems (IVPs) are differential equations of the form: *Find $y(t)$ such that*

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = f(t, y) \ \text{ for } t > t_0, \qquad \text{and } y(t_0) = y_0. \qquad (1)$$

Here $y' = f(t, y)$ is the *differential equation* and $y(t_0) = y_0$ is the *initial value*.

Some IVPs are easy to solve. For example:

$$y' = t^2 \quad \text{with } y(1) = 1.$$

3. Lipschtiz

Most problems are much harder, and some don't have solutions at all. In many cases, it is possible to determine that a giving problem does indeed have a solution, even if we can't write it down. The idea is that the function f should be "Lipschitz", a notion closely related to that of a **contraction**.

3. Lipschtiz

Definition 2.1.1 (Lipschitz Condition)

A function f satisfies a **Lipschitz Condition** (with respect to its second argument) in the rectangular region D if there is a positive real number L such that

$$|f(t, u) - f(t, v)| \leq L|u - v| \tag{2}$$

for all $(t, u) \in D$ and $(t, v) \in D$.

3. Lipschtiz

Example 2.1.1

For each of the following functions f, show that is satisfies a
Lipschitz condition, and give an upper bound on the Lipschitz
constant L.

(i) $f(t, y) = y/(1 + t)^2$ for $0 \le t \le \infty$.

(ii) $f(t, y) = 4y - e^{-t}$ for all t.

(iii) $f(t, y) = -(1 + t^2)y + \sin(t)$ for $1 \le t \le 2$.

4. Existence

Theorem 2.1.1 (Picard's)

Suppose that the real-valued function $f(t, y)$ is continuous for $t \in [t_0, t_M]$ and $y \in [y_0 - C, y_0 + C]$; that $|f(t, y_0)| \leq K$ for $t_0 \leq t \leq t_M$; and that f satisfies the *Lipschitz condition* (2). If

$$C \geq \frac{K}{L}\left(e^{L(t_M - t_0)} - 1 \right),$$

then (1) has a unique solution on $[t_0, t_M]$. Furthermore

$$|y(t) - y(t_0)| \leq C \qquad t_0 \leq t \leq t_M.$$

You are not required to know this theorem for this course. However, it's important to be able to determine when a given f satisfies a Lipschitz condition.

5. Exercises

Exercise 2.1.1

For the following functions show that they satisfy a Lipschitz condition on the corresponding domain, and give an upper-bound for L:

(i) $f(t, y) = 2yt^{-4}$ for $t \in [1, \infty)$,

(ii) $f(t, y) = 1 + t\sin(ty)$ for $0 \leq t \leq 2$.

Exercise 2.1.2

Many text books, instead of giving the version of the Lipschitz condition we use, give the following: *There is a finite, positive, real number L such that*

$$|\frac{\partial}{\partial y} f(t, y)| \leq L \qquad \text{for all } (t, y) \in D. \qquad (3)$$

Is this statement *stronger than* (i.e., more restrictive then), *equivalent to* or *weaker than* (i.e., less restrictive than) the usual Lipschitz condition? Justify your answer.

5. Exercises

Answer: This statement in (3) is stronger (i.e., more restrictive), since it requires that f is differentiable everywhere. The usual definition does not require that (although it can be used to show that a derivative exists almost everywhere).