Annotated slides

2526-MA140 Engineering Calculus

Week 08, Lecture 2

Integration by Parts; Areas between Curves

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Assignments, etc

- **Problem Set 6** is open, and will be covered in tutorials this well. Deadline is 5pm next Monday (10 November).
- **Problem Set 7** ~~opens by tomorrow~~. is open.
- The finally weekly assignment, will open next week.
- Reminder: The second **class test** takes place November 18.

- PS-5 is closed.

Class Test in Week 10. Contact Niall if you require any accomodations (with LENS reports).

This part is about...

See also Section **7.1** (Integration by Parts) and Section 6.1 (Areas between Curves) in the textbook:

math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Integration by Parts

Yesterday, we learned about integration by parts:

Integration by Parts

Let u and v be differentiable functions. Then

$$\int u\,dv = uv - \int v\,du.$$

$f(x) = u(x)\,v'(x)$

Eg (yesterday) $f(x) = x\cos(x)$

we set $u(x) = x$ $\quad \frac{dv}{dx} = \cos(x)$

Etc.

One of the challenges of Integration by Parts is knowing how to choose u and dv.

In the last example from yesterday, when integrating $\boxed{\int x\cos(x)\,dx}$ we choose $u = x$, because its derivative, $u' = 1$ is simpler.

Suppose we had made the bad choice of

$$u(x) = \cos(x), \qquad dv = x\,dx,$$

then we'd get:

$u(x) = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$\Rightarrow du = -\sin(x)\,dx$

$\frac{dv}{dx} = x$

$v = \frac{1}{2}x^2$

$\int u\,dv = (uv) - \int v\,du = \frac{1}{2}x^2\cos(x) + \frac{1}{2}\int x^2\sin(x)\,dx$

To try to get good choices for u and dv, we proceed as follows:

1. Some functions are easier to differentiate than integrate and so make a good choice for u. Important examples include **logarithms** $\underbrace{}_{\ln(x)}$ and **inverse trigonometric** functions.

2. Some functions, such as polynomials, may be good choices for u, since $u'(x)$ may be simpler than $u(x)$.

3. Trigonometric and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for dv.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \qquad \int \ln(x)\,dx = x\ln(x) - x + C$$

Example (of choosing u

Evaluate $\mathrm{I} = \int \frac{\ln(x)}{x^2}\,dx$. $= \int \ln(x)\, x^{-2}\, dx$.

$u(x) = \ln(x)$ $\qquad$ $dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx$ $\qquad$ $v = -x^{-1}$

$$I = \int \underbrace{\ln(x)}_{u}\ \underbrace{x^{-2}\,dx}_{dv} = (uv) - \int v\,du$$

$$= -\frac{\ln(x)}{x} - \int (-x^{-1})\cdot(x^{-1})\,dx$$

$$= -\frac{\ln(x)}{x} + \int x^{-2}\,dx = -\frac{\ln(x)}{x} - \frac{1}{x} + C$$

Example

Evaluate $\mathrm{I} = \int \ln(x)\, dx$.

Since $\int \ln(x) dx$ can be written as $\int \Big(\ln(x)\Big)(1)\, dx$, we use integration by parts, with $u = \ln(x)$ and $dv = dx$.

$u = \ln(x)$ $\qquad dv = dx$

$du = x^{-1}\, dx$ $\qquad v = x.$

$$I = \boxed{\int u\, dv = (uv) - \int v\, du}$$

$$= x\ln(x) - \int x \cdot x^{-1}\, dx = x\ln(x) - \int 1\, dx$$

$$= x\ln(x) - x + C.$$

Int by Parts: Repeated application

Sometimes, we have to apply Integration by Parts more than once.

Example

Evaluate $\mathrm{I} = \int x^2 e^x \, dx$.

$u(x) = x^2$ $\qquad$ $dv = e^x \, dx$

$du = 2x \, dx$ $\qquad$ $v = e^x$

$$I = \int x^2 e^x \, dx = \int u \, dv = (uv) - \int v \, du$$

$$= x^2 e^x - \int e^x (2x) \, dx$$

So $\quad I = x^2 e^x - 2 \underbrace{\int e^x \, x \, dx}_{I_2}.$

Int by Parts: Repeated application

$I_2 = \int e^x \, x \, dx.$

Apply IbP again

$u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = e^x$

$I_2 = (xe^x) - \int e^x \, dx = xe^x - e^x + C.$

So finally we get

$I = x^2 e^x - 2(I_2) = x^2 e^x - 2xe^x + 2e^x + C$

It is good to check any new rule/method for a simple example we already know the answer to. Now that we know about repeated application, we can do that:

Example

We know that $\mathrm{I} = \int x^2 dx = \underline{(1/3)x^3}$. We can also use IbP.

Take $u(x) = x$ and $dv = xdx$:

$$I = \int \underbrace{x}_{u} \big(\underbrace{x\, dx}_{dv} \big)$$

$$u = x \qquad dv = x\,dx$$

$$du = dx \qquad v = \tfrac{1}{2} x^2$$

$$I = (uv) - \int v\,du = \tfrac{1}{2}x^3 - \int \left(\tfrac{1}{2} x^2 dx\right),$$

$$= \tfrac{1}{2}x^3 - \tfrac{1}{2}\int x^2 dx = \tfrac{1}{2}x^3 - \tfrac{1}{2} I$$

$$\Rightarrow I = \tfrac{1}{2}x^3 - \tfrac{1}{2} I \Rightarrow \tfrac{3}{2} I = \tfrac{1}{2}x^3 \Rightarrow I = \left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right)x^3$$

Use I by Parts Repeatedly to evaluate

$$\boxed{I = \int e^x \cos(x)\,dx\,.}$$

Let $u = e^x$ $\quad dv = \cos(x)\,dx$

$du = e^x\,dx$ $\quad v = \sin(x)$

$$I = (uv) - \int v\,du = e^x \sin(x) - \underbrace{\int \sin(x)\, e^x\,dx}_{I_2}$$

$$I_2 = \int \underbrace{e^x}_{u}\, \underbrace{\sin(x)\,dx}_{dv}$$

$u(x) = e^x$ $\quad dv = \sin(x)\,dx$

$du = e^x\,dx$ $\quad v = -\cos(x)$

$$I_2 = \int \underbrace{e^x}_{u} \underbrace{\sin(x)\,dx}_{dv}$$

$$u(x) = e^x \qquad dv = \sin(x)\,dx$$

$$du = e^x\,dx \qquad v = -\cos(x)$$

$$I_2 = (uv) - \int v\,du = -e^x\cos(x) + \underbrace{\boxed{\int \cos(x)e^x\,dx}}_{I}$$

So

$$I = e^x\sin(x) - I_2$$

$$= e^x\sin(x) + e^x\cos(x) - I$$

$$\Rightarrow 2I = e^x(\sin(x) + \cos(x))$$

$$\Rightarrow I = \frac{e^x}{2}(\sin(x) + \cos(x))$$

Recall: Definite integrals

Last week we introduced the definite integral as follows:

Definition: definite integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral of** f from a to b is given by

$$\int_a^b f(x)dx = \lim_{h \to 0} h \sum_{i=0}^{n-1} f(x_i),$$

where $h = (b-a)/n$ and $x_i = a + ih$, provided the limit exists. Moreover, it is the area of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$ and $x = b$.

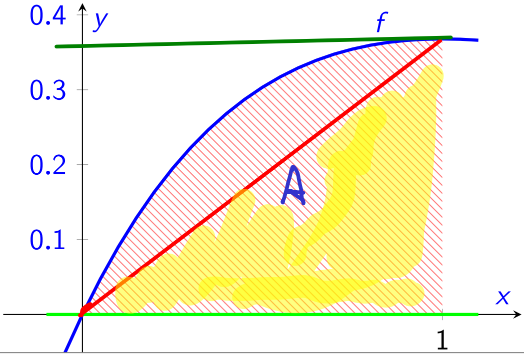
We'll now revisit this idea, and then extend it.

Definite Integrals with IbP

Integration by Parts for Definite Integrals

$$\int_a^b u dv = (uv)\Big|_a^b - \int_a^b v\, du$$

Example: First estimate $\int_0^1 xe^{-1}\, dx$ from the graph of xe^{-x}



$A \geq \frac{1}{2}(1)(0.3)$
≥ 0.15

$A \leq 0.4.$

So $0.15 \leq A \leq 0.4$

Definite Integrals with IbP

Now use *Integration By Parts* to actually evaluate $\int_0^1 xe^{-x}\,dx$.

Let $u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$I = \int_0^1 x\,e^{-x}\,dx = \int_0^1 u\,dv = (uv)\Big|_0^1 - \int_0^1 v\,du$$

$$= x(-e^{-x})\Big|_0^1 + \int e^{-x}\,dx$$

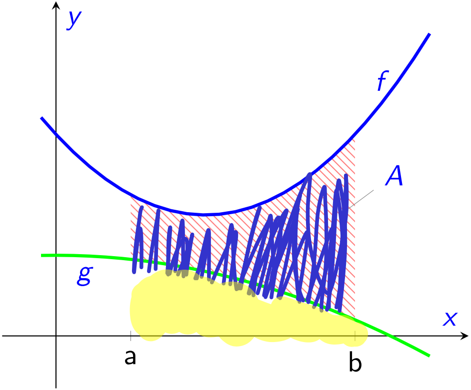
$$= \left(-x\,e^{-x}\right)\Big|_0^1 - e^{-x}\Big|_0^1$$

$$= \left[-x\,e^{-x} - e^{-x}\right]_0^1$$

$$= \left(-(1)e^{-1} - e^{-1}\right) - \left[(0)\,e^{0} - e^{0}\right] = \cdots = 0.2642$$

Areas Between Curves

We know that $\int_a^b f(x)\,dx$ evaluates as the area of the region between $x = a$ and $x = b$, and between $y = f(x)$ and $y = 0$.
But what if we wanted to evaluate the area between two curves?



Ans : Area

$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Areas Between Curves

Area Between Curves

Let f and g be continuous functions with $f(x) \geqslant g(x)$ throughout the interval $[a, b]$. Then the area A of the region that is

- bounded on the left by $x = a$, and on the right by $x = b$,
- above by the curve $y = f(x)$ and below by $y = g(x)$

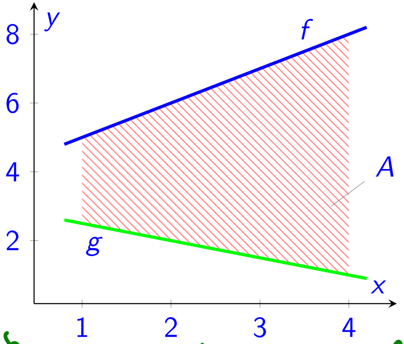
is given by

$$A = \int_a^b (f(x) - g(x))\, dx\,.$$

Areas Between Curves

Example

Find the area of the region bounded above by the graph of $f(x) = x + 4$, and below by the graph of $g(x) = 3 - x/2$ over the interval $[1, 4]$



$f(x) - g(x) =$

$x + 4 - (3 - x/2)$

$= 3x/2 + 1$

So Area $= \int_1^4 \frac{3}{2}x + 1 \, dx$

$= \left(\frac{3}{4}x^2 + x\right)\Big|_1^4$

$= \boxed{\frac{57}{4}}$

$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$

Areas Between Curves

Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect. That is, we have to find a and b.

Example (from Q5(a) of 2024/2025 Exam paper)

Compute the region bounded by the curves $f(x) = 3x + 4$ and the $g(x) = 2x^2 + 2x + 1$.

First we need to find the points were $f(x)$ and $g(x)$ intersect. That is, we solve $f(x) = g(x)$:

$$(3x + 4) - (2x^2 + 2x + 1) = 0$$
$$\implies -2x^2 + x + 3 = 0$$
$$\implies -2(x + 1)(x - 3/2) = 0 \quad (1)$$

So they intersect at $x = -1$ and $x = 3/2$.
(Continued)

Areas Between Curves

So the area is given by

$$\int_{-1}^{3/2} f(x) - g(x)dx$$

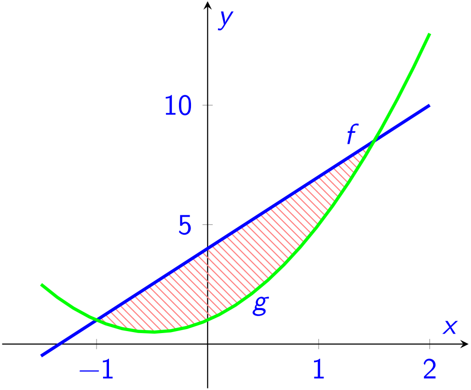
$$= \int_{-1}^{3/2} -2x^2 + x + 3dx$$

$$= \left(-\frac{2}{3}x^3 + \frac{1}{2}x^2 + 3x\right)\Big|_{-1}^{3/2}$$

$$= \left(-\frac{2}{3}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right)\right) - \left(-\frac{2}{3}(-1) + \frac{1}{2}(1) + 3(-1)\right)$$

$$= 125/24.$$

Areas Between Curves



Compound Regions

In the previous examples, we had $f(x) \geqslant g(x)$ for all $x \in [a, b]$. But what if f and g cross in the domain?

Areas between curves, without $f(x) \geqslant g(x)$

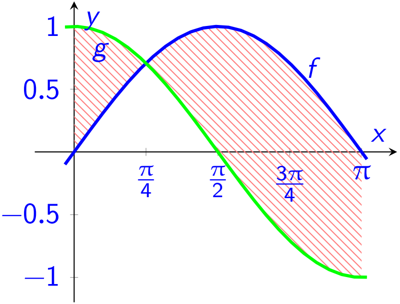
Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ Then A, the area of the region between the graphs of $f(x)$ and $g(x)$, and between $x = a$ and $x = b$, is given by

$$A = \int_a^b |f(x) - g(x)| \, dx.$$

Compound Regions

Example

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$, from $x = 0$ to $x = \pi$.



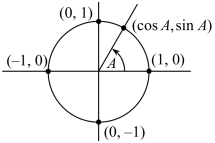
Compound Regions

It will help to consult p13 of the "log" tables.

Triantánacht		Trigonometry

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	$0°$	$90°$	$180°$	$270°$	$30°$	$45°$	$60°$	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. $\approx 57.296°$ $\qquad$ $1° \approx 0.01745$ rad.

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Compound Regions

Exercises

Exer 8.2.1 (From 2023/2024 exam)

Evaluate $\int_0^{\pi/2} x \cos(x)\, dx$.

Exer 8.2.2 (From 2019/2020 exam)

The functions $f(x) = 1/x$ and $g(x) = x^2$ intersect at $x = 1$. Calculate the area between their graphs on $[1, 2]$

Exer 8.2.3 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Exercises

Exer 8.2.4 (From 23/24 exam)

Find the area bounded by the curves $f(x) = x^2 - 4x$ and $g(x) = 2x - 5$.