2425-MA140 Engineering Calculus

Week 09, Lecture 1
Introduction to Volumes

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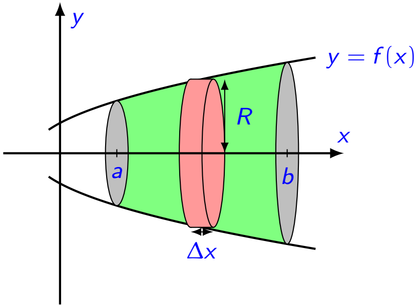
University of Galway

Tuesday, 11 November, 2025



Today's class revolves around:

For more: Section 6.2 (Determining Volumes by Slicing) in the textbook:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

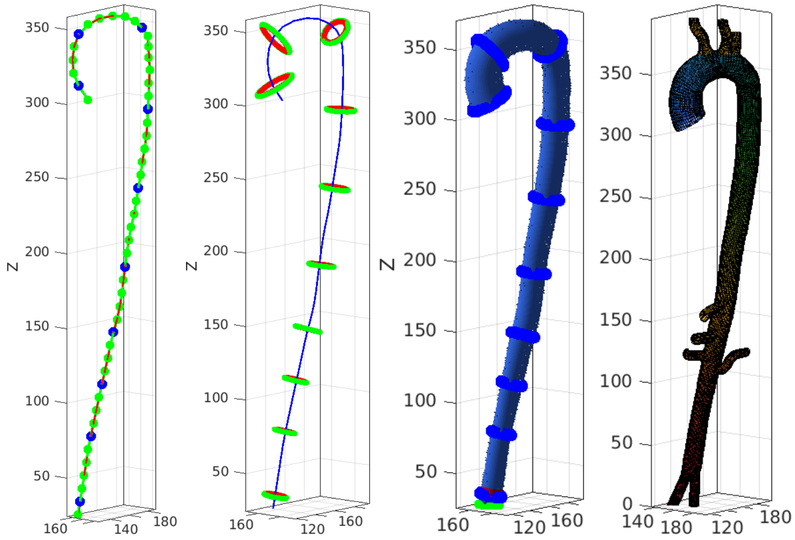
- **Problem Set 6** finished yesterday. Grades have been posted, as have solutions to the tutorial sheet.
- **Problem Set 7** is live, and will be worked on in tutorials this week. For more, see https://universityofgalway.instructure.com/courses/46734/assignments/132366
- **Problem Set 8** opens later this week.

- The second **class test** takes place Tuesday, 18 Nov at 10:00.
- **Topics:** anything from Weeks 4 (including the Chain Rule), 5, 6, 7 and 8. But not from material we cover this week.
- Test will be structured similar to the 1st Test:
  - 9 multiple choice question, each with a single correct answer.
  - 1 mark also given for participation (and entering your ID number correctly).
- Venues: **Teams 1–8** should go to ENG-G018.
  Teams 9–12 should go ENG-G017.
  Students with LENS/Accommodation arrangements will go to either MY231 or ENG-2052. See email with subject "*Venue for MA140 Class Test*" which will be sent on Monday.
- For more info on the rules, etc, check the info for the 1st Class Test: https://www.niallmadden.ie/2526-MA140/MA140-W05-1-ClassTest-Info.pdf

Some motivation

- The following images representing a human aorta. But the data are artificially generated (this is not from a real person).
- The images are generated by Kevin Moerman (biomechanical engineering)
- It is part of a project involving Dr Niamh Hynes (look her up!), and one of your tutors, Sean Tobin.
- The meaning of the images on the following slides, and significance to MA140, was discussed in class, but is not detailed in these notes.

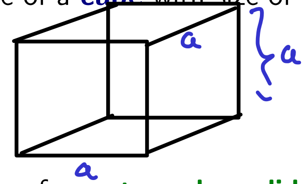
Some motivation



Computing Volumes

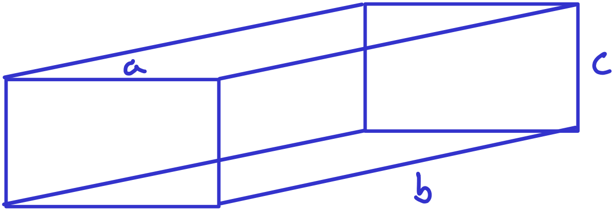
Last week, we used definite integrals to compute **areas**. Now we'll compute **volumes**. We already know how to compute the volumes of certain simple objects:

- Volume of a **cube** with size of length a:



Volume : $a \times a \times a = a^3$ ("units")

- Volume of a **rectangular solid**, length a, width b, and height c:



Volume : $(a)(b)(c)$

Computing Volumes

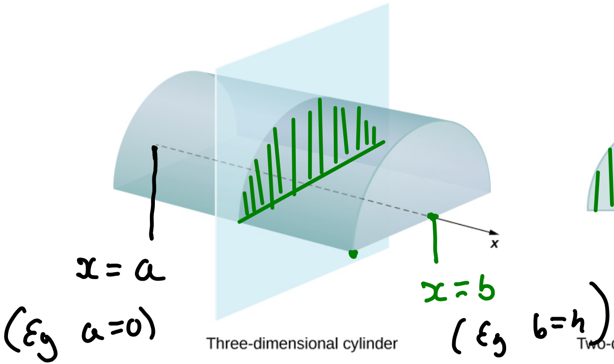
We also know formulae (e.g., from P10 of the Formulae and Tables booklet) for the volumes of a cylinder ($\pi r^2 h$), cone ($\pi r^2 h/3$), sphere ($\frac{4}{3}\pi r^3$), pyramid ($Ah/3$), etc.
We'll now see how these can be derived using integration.

$r =$ radius

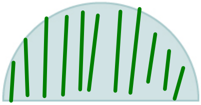
$h =$ height.

Usually, we think of a **cylinder** as something with a circular top and bottom, with the same radius. Furthermore, each cross section (parallel to top and bottom) is a circle of the same radius.

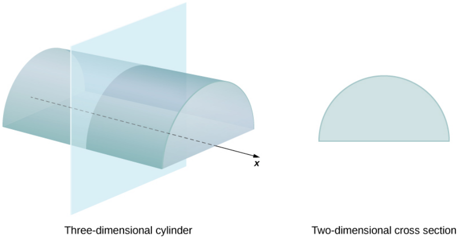
In mathematics, the term "cylinder" includes any object for which all cross-sections (in the same place) are the same.



Three-dimensional cylinder



Two-dimensional cross section



If the cross-sections all have area A, and the cylinder has length h, then the volume is $V = Ah$.

But we can go further, and study objects for cross-sections all have the same shape, but different areas (but we have a formula for the areas).

Volume of Cylinder

Suppose that we have an object for which every cross-section, perpendicular to the x-axis, through a given x has area $A(x)$.

Then the volume is $V = \int_a^b A(x)dx$. where x goes from a to b.

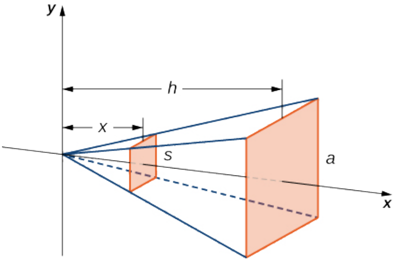
For our first example, we'll derive the formula for volume of a square-based pyramid.

Eg, if $A(x)$ is constant: $A(x) \equiv A$

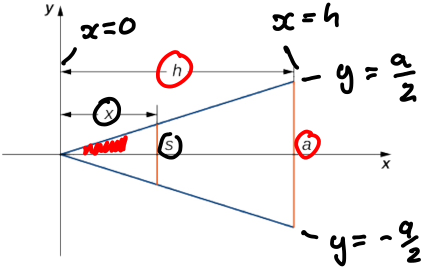
And if $a = 0$, $b = h$

$$V = \int_0^h A\,dx = Ax\Big|_0^h = Ah - A(0)$$
$$= Ah.$$

Consider a square-based pyramid, with height h, and base with sides of length a. We need to determine the length of the side of the cross-section which is a distance x from the vertex.



(a)



(b)

Reasoning from the side view in (b), we can see that

$$\frac{s}{x} = \frac{a}{h} \Rightarrow s = \frac{a}{h}x.$$

So the square cross-section at x has length $s = \frac{a}{h}x$.

So its area is $A(x) = s^2 = \frac{a^2}{h^2}x^2$.

Then the volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{a^2}{h^2}x^2\,dx$$

$$= \frac{a^2}{h^2}\int_0^h x^2\,dx = \frac{1}{3}\frac{a^2}{h^2}x^3\Big|_0^h$$

$$= \frac{1}{3}\frac{a^2}{h^2}\left(h^3 - 0\right) = \frac{1}{3}ha^2.$$

Slicing

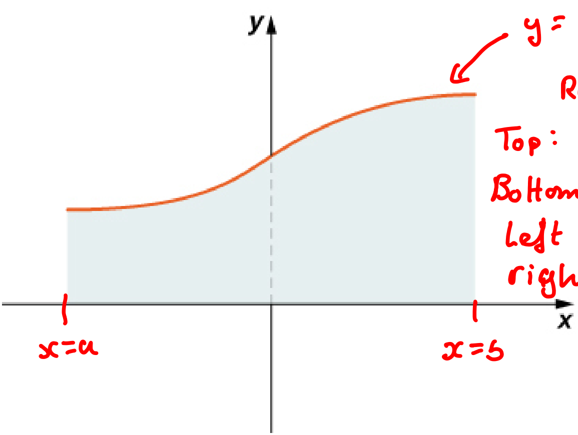
The method we just used is called **slicing**.

Next, we'll use it to calculate the volumes of **solids of revolution**.

Introducing "Solids of Revolution"

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**. Here is the idea...

1. Start with a region in the *xy*-plane.



Region bounded

Top: $y = f(x)$

Bottom: $y = 0$

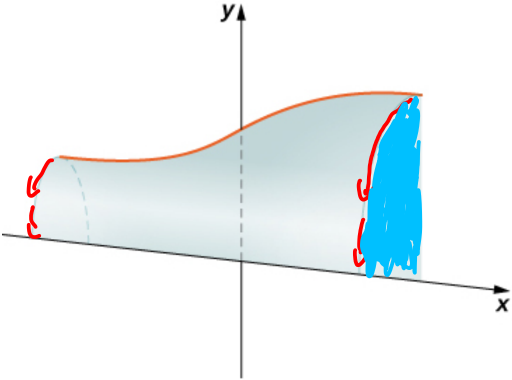
left: $x = a$

right: $x = b$

Introducing "Solids of Revolution"

2. Revolve the region abount the x-axis

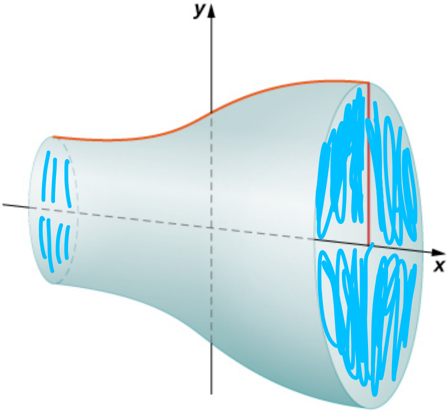
" Quarter rotation "



Introducing "Solids of Revolution"

3. Continue until you have produced a "solid of revolution"

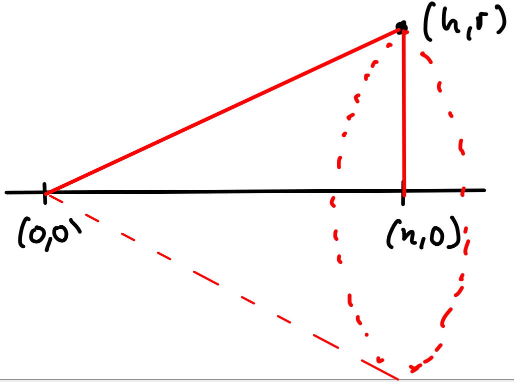
Complete revolution.



Introducing "Solids of Revolution"

Examples

1. What is the solid of revolution of a triangle with vertices $(0,0)$, $(h,0)$ and (h,r)?
2. ~~What is the solid of revolution of a semicircle with radius r?~~

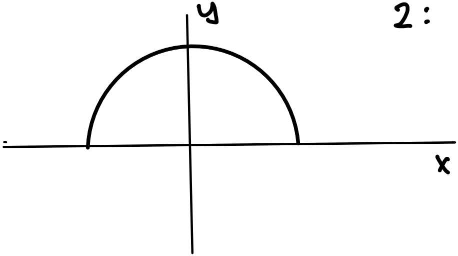


1. solid of revolution of this triangle is a cone

Introducing “Solids of Revolution”

Examples

1. What is the solid of revolution of a triangle with vertices $(0,0)$, $(h,0)$ and (h,r)?
2. What is the solid of revolution of a semicircle with radius r?

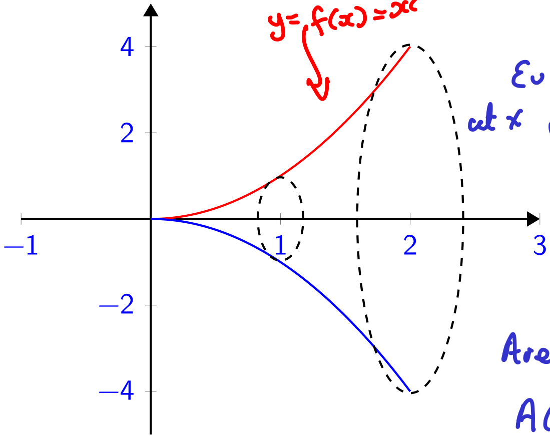


2: Ans Sphere.

Volumes of Solids of Revolution: slicing

Example

Find the volume of the solid of revolution that is bounded by the graphs of $f(x) = x^2$, $x = 0$ and $x = 2$



$y = 0$ = "x axis"

Every cross-section at x is a circle with radius $y = x^2$

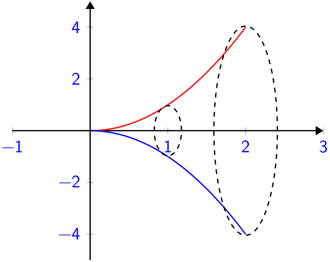
The area of a circle is πr^2 so

Area of cross-section

$A(x) = \pi x^4$

Volumes of Solids of Revolution: slicing

So, with $a = 0$, $b = 2$, and $f(x) = x^2$, the volume is...



$$V = \int_a^b A(x)\,dx$$

$$= \int_0^2 \pi x^4 \, dx$$

$$= \pi \frac{1}{5} x^5 \Big|_0^2$$

$$= \pi \frac{32}{5}$$

Solids of revolution: disk method

Since, for solids of revolution, each "slice" is actually a disk, it is often called the **disk method**. Furthermore, since, at a given x the disk has radius $f(x)$, and so area, $A(x) = \pi\big(f(x)\big)^2$, we can directly compute the volume

Solids of revolution: disk method

Let $f(x)$ be continuous and nonnegative. The volume of region formed by revolving the region between $f(x)$ and the x-axis, and between $x = a$ and $x = b$, about the x-axis is

$$V = \int_a^b \pi\big(f(x)\big)^2 dx.$$

Solids of revolution: disk method

Note: the following example is taken from the textbook, which has a nice animation of the process. Also try this link.

Example

Find the volume of the the solid of revolution generated by revolving the region between the graph of the function $f(x) = x^2 - 2x + 2$ and the x-axis over the interval $[-1, 3]$.

$$V = \pi \int_a^b (f(x))^2 \, dx = \pi \int_{-1}^{3} [x^2 - 2x + 2]^2 \, dx$$

$$= \pi \int_{-1}^{3} x^4 - 4x^3 + 8x^2 - 8x + 4 \, dx$$

$$= \pi \left[\frac{1}{5} x^5 - x^4 + \frac{8}{3} x^3 - 4x^2 + 4x \right] \Big|_{-1}^{3}$$

$$= \pi \left(\frac{78}{5} - \left(-\frac{178}{15} \right) \right) = \frac{412}{15} \pi.$$

Solids of revolution: disk method

Example

Use the disk method to verify that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Every cross-section is a circle

$x^2+y^2=r^2$.

Finish here.

So $f(x)=\sqrt{r^2-x^2}$

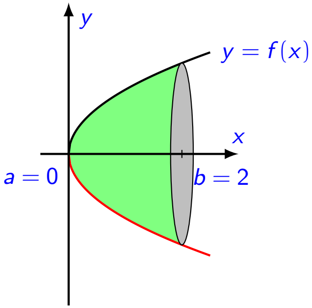
$$V = \pi\int_{-r}^{r} (f(x))^2\,dx = \pi\int_{-r}^{r} (r^2-x^2)\,dx$$

$$= \pi\left[r^2x - \tfrac{1}{3}x^3\right]_{-r}^{r} = \cdots = \tfrac{4}{3}\pi r^3.$$

Solids of revolution: disk method

Example:

Find the **volume** of the solid of revolution obtained by rotating $y = \frac{3}{\sqrt{2}}\sqrt{x}$, between $x = 0$ and $x = 2$, about the x-axis.



Solids of revolution: washer method

There are numerous other variations on this type of problem, such as

- Rotating the function about the y-axis; (easy: just give a function for x in terms of y).
- Rotating about a line that is not an axis (a little trickier: need to transform the problem).
- **rotating a region bounded by two functions**.

We'll look at the last of these, the method for which is sometimes called the "**washer method**".

However, it is not too hard: we apply the "disk" method to both functions, and then subtract.

Solids of revolution: washer method

Washer Method

Let $f(x)$ and $g(x)$ be continuous functions on $[a, b]$, with $f(x) \geqslant g(x) \geqslant 0$ for any $x \in [a, b]$. The volume of the solid obtained by rotating the region between $f(x)$ and $g(x)$, and $x = a$ and $x = b$, is

$$V = \pi \int_a^b \big(f(x)\big)^2 - \big(g(x)\big)^2 \, dx.$$

Solids of revolution: washer method

Example (from textbook: see Figure 6.2.12)

Consider the region in the plane bounded above by $y = \sqrt{x}$, below by $y = 1$, left by $x = 1$ and right by $x = 4$. If this region is rotated about the x-axis, show that the volume of the resulting solid of rotation is $\dfrac{9\pi}{2}$.

First we visualise: **the animation**

Exercises

Exer 9.1.1

Use the "slicing" method to derive the formula for the volume of a circular cone, of height h and base with radius r.

Exer 9.1.2

Use the "disk" method to derive the formula for the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$, $x = 1$ and $x = 2$.

Exer 9.1.3

Use the "washer" method to find the volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = x^2$ and $g(x) = x$, for $1 \leqslant x \leqslant 2$, about the x-axis.

Exercises

Exer 9.1.4

Find the volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2$ about the x-axis. (Hint: you need to find where the graphs of f and g intersect: these will be the points a and b).