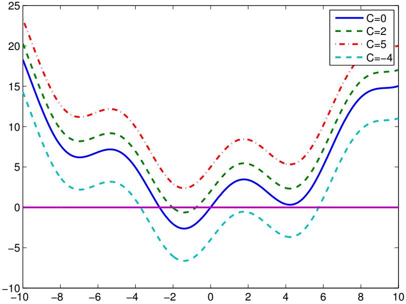
MA211

Lecture 6: Antiderivatives and Integrals

Wed 24 September 2008



Blackboard

From today (24/09/08) I won't be updating the pages at
`http://www.maths.nuigalway.ie/MA211/`

If you are registered for MA211, you should be able to access all course material through `http://blackboard.nuigalway.ie`

If for some reason you can't, then send me an email.

Problem Set 1

Problem Set 1 is available for down-load.
Write out, clearly and carefully, solutions to the selected exercises and submit them by 11am, Monday Oct 6th.

However, you should attempt **all** exercises. Some of them may feature on the final exam.

Tutorials take place

- Tuesday, 3pm, AC202
- Wednesday, 5pm, QA003 (Physiology lecture room)

In today's class...

Antiderivatives

See Stewart's *Calculus* 5.3

On Monday we considered problems of the form: *given a function f find it's derivative*. That is, find g such that $g(x) = \frac{d}{dx} f(x)$.

However, much of this course is related to the *inverse* of this problem: *given a function f find it's* **antiderivative**

Definition (Antiderivative)

Given a function f in an interval I, the function F is *an* **antiderivative** of f on I if

$$F'(t) = f(t) \quad \text{for all } x \in I.$$

Antiderivatives

Example

- $F(t) = t$ is an antiderivative of $f(x) = 1$.
- $F(t) = \frac{1}{2}t^2$ is an antiderivative of $f(t) = t$.
- $F(t) = -cos(t)$ is an antiderivative of $f(t) = \sin(t)$.

Note that $F(t) = -1/t$ is an antiderivative of $f(t) = 1/t^2$ (*on any interval that excludes* $t = 0$).

But so too is $F(t) = 5 - 1/t$ and $F(t) = -1/t - 3.1415$ and, indeed, any function of the form $F(t) = -1/t + C$ for some constant t.

When we write down the antiderivative of f and include the constant C we usually call it the **General Antiderivative** of f or, more commonly, the *The Indefinate Integral*.

Definition (Indefinite Integral)

The **Indefinite Integral** of $f(t)$ on the interval I is

$$\int f(t)dt = F(t) + C \quad \text{for } t \in I,$$

where $F'(t) = f(t)$ for all t in I.

We call C the *constant of integration*.

(*Next week we'll do* **definate integrals***, which have limits of integration:* $\int_a^b f(x)dx$).

1. $\displaystyle\int 1\ dx =$

2. $\displaystyle\int x\ dx =$

3. $\displaystyle\int x^2 dx =$

4. $\displaystyle\int x^n dx =$

5. $\int \sin(x)dx =$

6. $\int \cos(x)dx =$

It is neither important or necessary to memorise the antiderivatives of even reasonably common functions. However, you should be able to look them up on pages 41 and 42 of the Mathematical Tables.

Having p9 is also handy.

DIFREÁIL (DIFFERENTIATION)

$f(x)$	$f'(x) \equiv \frac{d}{dx}[f(x)]$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$
$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\operatorname{cosec}^{-1}\frac{x}{a}$	$-\frac{a}{x\sqrt{x^2-a^2}}$
$\cot^{-1}\frac{x}{a}$	$-\frac{a}{a^2+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

SUIMEÁIL (INTEGRATION)

Glactar $a>0$ agus fágtar tairisigh na suimeála ar lár.

We take $a>0$ and omit constants of integration.

$f(x)$	$\int f(x)\,dx$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert \sec x\rvert$
$\sec x$	$\ln\lvert \sec x + \tan x\rvert$
$\operatorname{cosec} x$	$\ln\left\lvert \tan\frac{x}{2}\right\rvert$
$\cot x$	$\ln\lvert \sin x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ln a}$
$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$

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$\coth^{-1} x$	$-\dfrac{1}{x^2-1}$
$\operatorname{sech}^{-1} x$	$-\dfrac{1}{x\sqrt{1-x^2}}$
$\operatorname{cosech}^{-1} x$	$-\dfrac{1}{x\sqrt{x^2+1}}$

$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\coth x$	$\ln \lvert \sinh x \rvert$
$\operatorname{sech} x$	$\tan^{-1}(\sinh x)$
$\operatorname{cosech} x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\cos^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sin 2x]$
$\sin^2 x$	$\frac{1}{2}[x - \frac{1}{2}\sin 2x]$
$\cosh^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sinh 2x]$
$\sinh^2 x$	$\frac{1}{2}[-x + \frac{1}{2}\sinh 2x]$
$\dfrac{1}{x\sqrt{a^2-x^2}}$	$-\dfrac{1}{a}\operatorname{sech}^{-1}\dfrac{x}{a}$
$\dfrac{1}{x\sqrt{x^2+a^2}}$	$-\dfrac{1}{a}\operatorname{cosech}^{-1}\dfrac{x}{a}$

Torthaí agus Líonta:
Products and Quotients:

$$y = uv\,;\quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = \frac{u}{v}\,;\quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Foirmlí áisiúla:
Useful formulae:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \quad (-\infty < x < \infty)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\frac{1+x}{1-x} \quad (-1 < x < 1)$$

Suimeáil trí mhíreanna:
Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

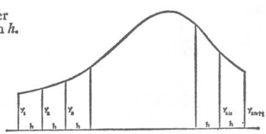
Teoragán Taylor (Taylor's Theorem):

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^r}{r!}f^r(x) + \dots$$

Riail Shimpson (Simpson's Rule):

Corr-uimhir ordanáidí iad $y_1, y_2, \dots y_{2n+1}$ fad h óna chéile.

$y_1, y_2, \dots y_{2n+1}$ is an odd number of ordinates at intervals of length h.



$$\text{Achar (Area)} \approx \tfrac{1}{3}h\{y_1 + y_{2n+1} + 2(y_3 + y_5 + \dots y_{2n-1}) + 4(y_2 + y_4 + \dots y_{2n})\}$$

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$\cos^2 A + \sin^2 A = 1$

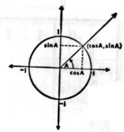
$\tan A = \dfrac{\sin A}{\cos A}$

$\sec^2 A = 1 + \tan^2 A = \dfrac{1}{\cos^2 A}$

$\cot A = \dfrac{1}{\tan A}$

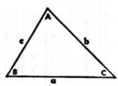
$\sec A = \dfrac{1}{\cos A}$

$\text{cosec } A = \dfrac{1}{\sin A}$



A	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
$\cos A$	1	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin A$	0	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan A$	0	0	gan sain-mhíniú not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

$\cos(-A) = \cos A$ $\qquad \sin(-A) = -\sin A$ $\qquad \tan(-A) = -\tan A$



Foirmle an tsín:
Sine formula: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Foirmle an chomhshínis:
Cosine formula: $a^2 = b^2 + c^2 - 2bc \cos A$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

$\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$e^{in\theta} = (\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin 2A = 2 \sin A \cos A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

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Example

1. $\displaystyle\int \left(x^3 - 5x^2 + 7\right) dx =$

2. $\displaystyle\int x^{-3} dx =$

Example

3. $\displaystyle\int x^{-n}dx =$

4. $\displaystyle\int x^{-1}dx =$

5. $\displaystyle\int \frac{\sin(t)}{\tan(t)}dt =$

Don't forget the constant of integration!

Exercise (Q6.1)

(i) $6t^2 - 1$,

(ii) $\dfrac{x+3}{x^{3/2}}$

(iii) $\int 6dx$

(iv) $\displaystyle\int x^{-2}dx$

(v) $\displaystyle\int \left(x^2 + \cos(x)\right)dx$

(vi) $\int \cos(t)\tan(t)dt$

(vii) $\displaystyle\int (A + Bx + Cx^2)dx$

(viii) $\displaystyle\int \cos(3x)dx$

Don't forget the constant of integration!

Differential equations

When we see a problem like:

$$\text{Evaluate} \int 3t^2 - 1 dt$$

we can think of it as

Find a function whose derivative (with respect to t) is $3t^2 - 1$.

Another equivalent way of asking the same question is:

Find a function f that solves the equation $f'(t) = 3t^2 - 1$.

This is an example of a simple *Differential Equation (DE)*, and we'll study much more of these as go through the course.

Differential equations

Our 1st Differential Equation is:

Example (1)

Find a function f that solves the equation $f'(t) = 3t^2 - 1$.

and its solution is of the form

$$f(t) = t^3 - t + C$$

for an *arbitrary* constant C.

Definition (General Solution)

The **general solution** of a differential equation is one that includes one or more arbitrary constants corresponding to constants of integration.

Differential equations

Example (2)

Find the general solution to the differential equation

$$f''(x) = x.$$

Differential equations

Example (3)

Show that the function

$$f(x) = C_1 x^3 + C_2/x$$

is a solution to the differential equation

$$x^2 f''(x) - x f'(x) - 3f(x) = 0.$$

Exercise (Q6.2)

(i) Show that, for any constants C_1 and C_2,

$$y(x) = C_1x^2 + C_2x^{-2}$$

is a solution to the differential equation

$$x^3y'''(x) + 6x^2y''(x) = 12y(x).$$

(ii) Write down a 2nd order differential equation that has $f(x) = x^2 - x$ as a solution.

Differential equations

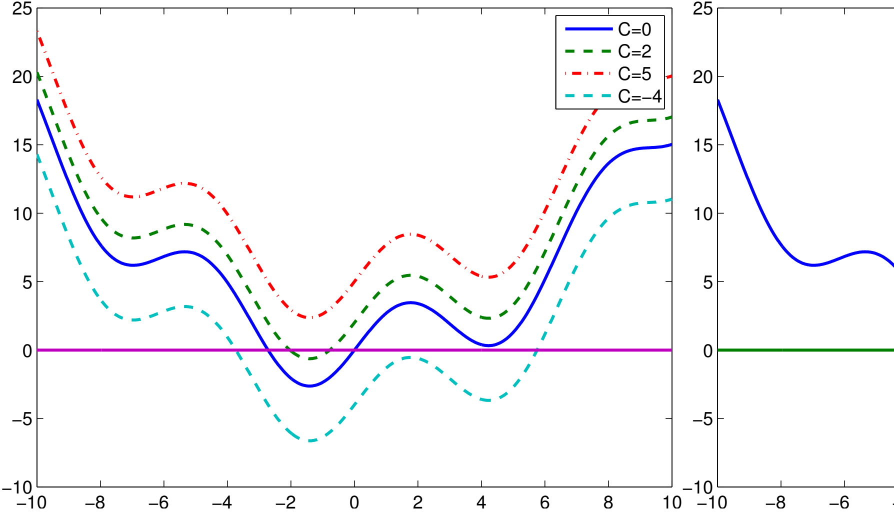
Example (4)

Write down the **general solution** to the following differential equation:

$$y'(x) = x/3 + 3\cos(x)$$

Differential equations

$$\frac{x^2}{6} + 3\sin(x) + C$$



General V Particular Solutions

The following is an example of a simple differential equation:

Q: If you travel east at a constant speed of 90km/hr for 1 hour, where are you?
A: 90km east of where we started!
This is the *general solution*.

An alternative problem is:

Q: If you travel east *from Galway* at a constant speed of 90km/hr for 1 hour, where are you?
A: Athlone.
This is a *particular solution*: the arbitrary constant is specified.

Particular Solutions

Example (5)

Find the solution to the differential equation

$$y'(x) = \frac{x}{3} + 3\cos(x),$$

given that $y(0) = 2$.

Particular Solutions

Exercise (Q6.3)

Find solutions to the following differential equations. If possible, gave a particular solution, otherwise, give the general solution.

(i) $y'(t) = x - 2$

(ii) $f'(x) = x^{-2} - x^{-3}$, subject to $f(-1) = 0$.

(iii) $y''(x) = x^3 - 1$, given that $y'(0) = 0, y(0) = 8$.