MA378 Chapter 2: Splines

§2.1 Linear Interpolating Splines

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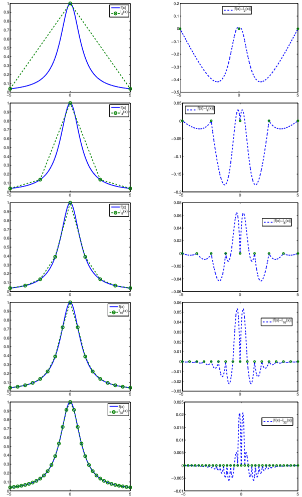
(W03.2)



1.0 A linear list of topics:

1.1 Introduction

In Section 1.5 (Convergence and Runge's Example), we learned that it is not always a good idea to interpolate functions by a high-order polynomial at equally spaced points. However, it is possible to obtain very good approximations using a very simple method. The trick is to use a **spline**: a ***piecewise polynomial interpolating function***.

We'll consider three important example of splines:

1. ***linear splines***
2. ***(natural) cubic splines***.
3. ***Hermite*** piecewise cubics.

For more details about splines, have a look at Chap. 11 of Süli and Mayers, and Lectures 10 and 11 Stewart's "Afternotes goes to Grad School".

1.1 Introduction

In this section, we always have N equally spaced intervals (and so $N+1$ equally spaced points). Let $h = (b-a)/N$, then

$$a = x_0, \quad b = x_N \quad \text{and} \quad \boxed{x_i = x_0 + ih} \text{ for } \; i = 0, 1, \ldots, N.$$

Often these are referred to as *knots points* (or simply as *knots*), and denote the set of knot points by $\underline{\omega^N} := \{x_i\}_{i=0}^N$.

$x_0 = a$

$x_1 = a + h$

$x_2 = x_1 + h = a + 2h$

$x_3 = a + 3h$

1.2 Linear Interpolating Splines

We first study the ***piecewise linear interpolant***, also called a ***linear spline***. We will see that they have important properties, including

(a) they are easy to construct and analyse;

(b) the bound on the error decreases as the number of interpolation points increases;

(c) the error we get using a linear spline is no more than twice the error using the best possible (piecewise linear) approximation;

(d) of all the interpolants to f at a given set of points, the linear spline is the one with the **smallest first derivative**.

Definition 1.1

Let f be a function that is continuous on $[a, b]$. The *linear spline interpolant* to f is the continuous function, l, such that

(i) $l(x_i) = f(x_i)$ for each $i = 0, 1, \ldots, N$,

(ii) l is a linear function l_i on each interval $[x_{i-1}, x_i]$. That is,

$$l(x) = \begin{cases} l_1(x) & x_0 \leq x \leq x_1 \\ l_2(x) & x_1 < x \leq x_2 \\ \ldots & \\ l_N(x) & x_{N-1} < x \leq x_N \end{cases}$$

where each $l_i \in \mathcal{P}_1$.

(i) l interpolates f at $x_0, x_1, \ldots x_n$

(ii) l is piecewise linear.

1.3 Construction on linear splines

It is easy to write down a formula for the l_i, based on Lagrange polynomials:

$h := x_i - x_{i-1}$

- Set $h = \underbrace{(b-a)/N}$.
- For each $i = 1, 2, \dots, N$, define

$$l_i(x) = f(x_{i-1})\frac{x_i - x}{h} + f(x_i)\frac{x - x_{i-1}}{h}, \quad x \in [x_{i-1}, x_i]. \quad (1)$$

Note that each $l_i(x)$ is in $\mathbb{P}_1$

- $l_i(x_{i-1}) = f(x_{i-1}) \underbrace{\frac{x_i - x_{i-1}}{x_i - x_{i-1}}}_{1} + f(x_i) \underbrace{\frac{x_{i-1} - x_{i-1}}{h}}_{0}$
  $= f(x_{i-1})$
- $l_i(x_i) = f(x_i)$ (Finished here W03.2)

Example 1.2

Write down the linear spline interpolant to $f(x) = e^x$ at the knot points $\{-1, 0, 1\}$.

$x_0 = -1 \qquad x_1 = 0 \qquad x_2 = 1 \qquad \Rightarrow h = 1$

$$l(x) = \begin{cases} l_1(x) & x_0 \leq x \leq x_1 \\ l_2(x) & x_1 < x \leq x_2 . \end{cases}$$

$$l_1(x) = f(x_0)\frac{x_1 - x}{h} + f(x_1)\frac{x - x_0}{h} = e^{-1}(-x) + (1)(x+1)$$

$$l_2(x) = f(x_1)\frac{x_2 - x}{h} + f(x_2)\frac{x - x_1}{h} = (1 - x) + e\,x .$$

1.4 Analysis

We know that if p_N is the polynomial of degree N that interpolates f at N equally spaced points, it does **not** follow that $p_N \to f$ as $N \to \infty$. But as we will see, the piecewise linear interpolant to f converges to f, albeit slowly.

This is verified in the following theorem, which is a direct consequence of Cauchy's theorem.

 $M_2 = \|f''\|_\infty$

Theorem 1.3

Suppose that f, f' and f'' are all continuous and defined on the interval $[a, b]$. Let l be the linear spline interpolant to f on the $N+1$ equally spaced points $a = x_0 < x_1 < \cdots < x_N = b$ with $h = x_i - x_{i-1} = (b-a)/N$. Then

$$\|f - l\|_\infty \le \frac{h^2}{8}\|f''\|_\infty, = \frac{N^{-2}}{(b-a)\,8} \|f''\|_0$$

(Here $\|g\|_\infty := \max_{a \le x \le b} |g(x)|$.)

Proof: We know from Cauchy's Theorem: If P_1 is the polynomial of degree $n=1$ interpolating f at x_0 & x_1 then

$$f(x) - P_1(x) = \frac{f''(c_1)}{2}(x - x_0)(x - x_1) \quad c_1 \in (x_0, x_1)$$

So $$|f(x)-l_1(x)| \leq \frac{1}{2} \max_{x_0 \leq x \leq x_1} |f''(x)| \, |(x-x_0)(x-x_1)|$$

$$\leq \frac{1}{2} \max_{a \leq x \leq b} |f''(x)| \, |\underbrace{(x-x_0)}_{\leq h/2} \underbrace{(x-x_1)}_{\leq h/2}|$$

$\Rightarrow$

$$|f(x)-l_1(x)| \leq \frac{h^2}{8} \|f''(x)\|_\infty$$

In fact this is true for any l_i, $i=1,\dots,N$. That is

$$|f(x)-l_i(x)| \leq \frac{h^2}{8} \|f''(x)\|_\infty$$

So

$$\|f-l\|_\infty \leq \frac{h^2}{8} \|f''(x)\|_\infty \quad \blacksquare$$

1.4 Analysis

It follows directly from this theorem that

$$\lim_{N\to\infty} \|f - l\|_\infty = 0.$$

This is because, as $N \to \infty$, we have $h \to 0$ since $h = \frac{b-a}{N}$

So $h^2 \underbrace{\left(\frac{\|f''\|_\infty}{8}\right)}_{\text{constant.}} \to 0$

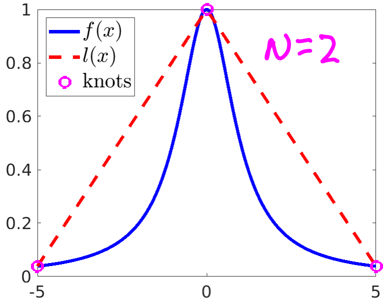
1.4 Analysis

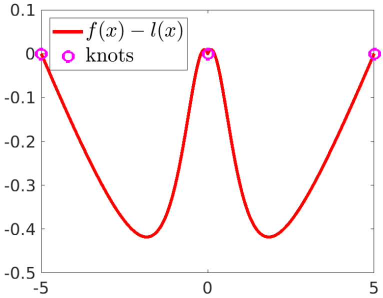
Example 1.4

The figure below shows linear spline interpolations of Runge's example:

$$f(x) = \frac{1}{1+x^2} \text{ on } [-5, 5].$$

These diagrams appear to support our assertion that the error tends to zero as $N \to \infty$.





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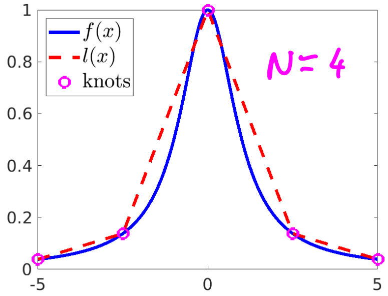

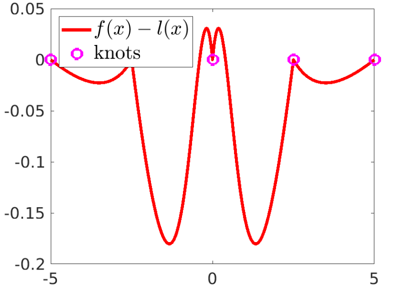

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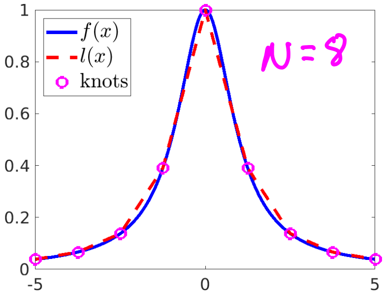

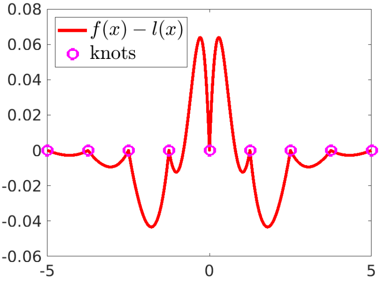

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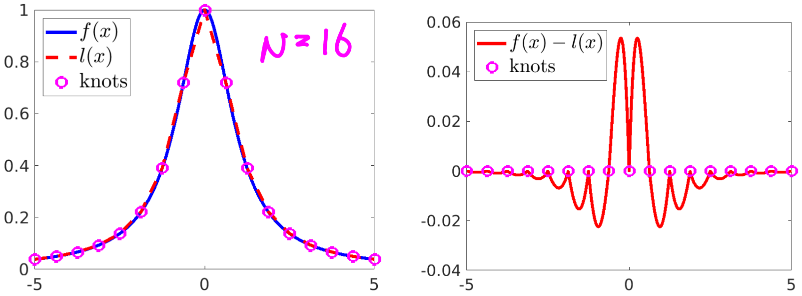

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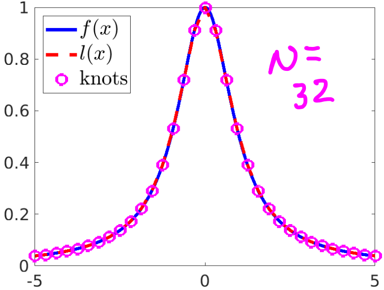
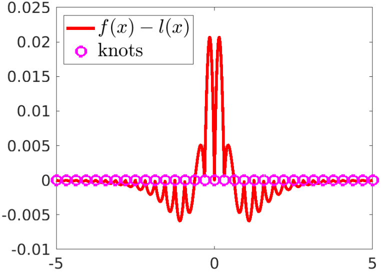




Example 1.5

Suppose we interpolate $f(x) = e^x$ with linear splines on $N+1$ equally spaced points between $x_0 = -1$ and $x_N = 1$. What value of N would we have to take to ensure that the maximum error is less than 10^{-2}?

Since $\|f - l\|_\infty \leq \frac{1}{8} h^2 \|f''\|_\infty$

we want to choose h so that

$$\frac{1}{8} h^2 \|f''\|_\infty \leq 10^{-2}.$$

$f(x) = e^x \Rightarrow f''(x) = e^x \Rightarrow$

$$\|f''\|_\infty := \max_{-1 \leq x \leq 1} |e^x| = e \approx 2.7183.$$

Example 1.5

Suppose we interpolate $f(x) = e^x$ with linear splines on $N+1$ equally spaced points between $x_0 = -1$ and $x_N = 1$. What value of N would we have to take to ensure that the maximum error is less than 10^{-2}?

So need h such that

$$\frac{1}{8} h^2 (2.7183) \leq 10^{-2}$$

This will give $h^2 \leq 0.02943$

Then, with $h = \frac{2}{N}$

we'll get $N \geq 11.655$.

ANS: take $N = 12$.

1.5 Best approximation

For the next part of the analysis it will help to think of piecewise linear interpolation as an *operator*. Then we can compare the linear spline to all the other piecewise linear approximations.

First, observe that one can define an infinite number of piecewise linear functions on a given set of $N + 1$ knot points, denoted ω^N. We'll call the set of these functions $\mathcal{L}$.

1.5 Best approximation

Definition 1.6

For a fixed set of knot points ω^N, let L be the operator that maps the continuous function f to its linear spline interpolant $l \in \mathcal{L}$.

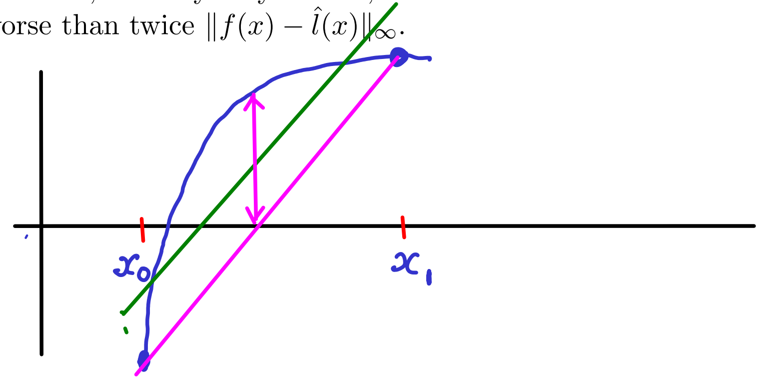
Now suppose that $g \in \mathcal{L}$. Then $\mathrm{L}(g) = g$. That is L is a *projection*: $\mathrm{L}(\mathrm{L}(f)) = \mathrm{L}(f)$.

1.5 Best approximation

It is not hard to see that one could find a different function $\hat{l} \in \mathcal{L}$ that is a better approximation of f in sense that

$$\max_{x_0 \leq x \leq x_n} |f(x) - \hat{l}(x)| < \max_{x_0 \leq x \leq x_n} |f(x) - l(x)|.$$

However, l is very easy to find, and the associated error is no worse than twice $\|f(x) - \hat{l}(x)\|_\infty$.



Theorem 1.7 (Stewart's "Afternotes goes to grad school", Lecture 10)

Let $l = \mathrm{L}(f)$. *For any* $\hat{l} \in \mathcal{L}$,

$$\|f - l\|_\infty \leq 2\|f - \hat{l}\|_\infty.$$

In the proof of this, we need several facts about L:

- L is a projection: $L(f) = L\big(L(f)\big)$.
- L is a linear: $L(f + g) = L(f) + L(g)$.
- L is $\|\cdot\|_\infty$-stable: $\|L(f)\|_\infty \leq \|f\|_\infty$.

Finished here Wednesday (W04.1)
before doing the proof.

Proof:

$$\|f - l\|_\infty = \|f - \hat{l} + \hat{l} - l\|_\infty$$

for any $\hat{l} \in \mathcal{L}$. By the triangle inequality

$$\|f - l\|_\infty \leq \|f - \hat{l}\|_\infty + \|\hat{l} - l\|_\infty$$

$$\Rightarrow \|f - l\|_\infty \leq \|f - \hat{l}\|_\infty + \|L(\hat{l}) - L(l)\|_\infty$$

because L is a projection

$$\leq \|f - \hat{l}\|_\infty + \|L(\hat{l} - f)\|_\infty \text{ (projection)}$$

$$\leq \|f - \hat{l}\|_\infty + \|\hat{l} - f\|_\infty \qquad (\|\cdot\|_\infty \text{ stability})$$

1.6 Minimum Energy

The final interesting property of l that we will study is called the *minimum energy property*.

Definition 1.8

Let u be a function that is continuous and defined on the interval $[a, b]$ except, maybe, at the (countable set) ω^N of knot points[a] Then the 2-norm of u is

$$\|u\|_{2,[a,b]} := \Big(\int_a^b u^2(x)dx\Big)^{1/2}.$$

Usually we just write this as $\|u\|_2$.

[a] More precisely, we should say "everywhere, except on a set of measure zero". However, since not everyone is familiar with the terminology, we'll skip the details.

1.6 Minimum Energy

Let H^1 be the set of all functions u that are continuous on $[a, b]$ and have $\|u'\|_2 < \infty$. Note that $l \in H^1$, even though we have not properly defined l' at the mesh points ω^N.

Theorem 1.9 (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l be the linear spline interpolant of f. Then

(continued)

$$\|l'\|_2 \leq \|w'\|_2.$$

But $\int_a^b (w-l)' l' \, dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (w-l)' l' \, dx.$

And, by Integration By Parts:

$$\int_{x_{i-1}}^{x_i} (w-l)' l' \, dx = l'(w-l) \Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (w-l) l'' \, dx$$

which is zero since $l''(x) \equiv 0$ and

$$l'(w-l)\Big|_{x_{i-1}}^{x_i} = l'(x_i)\big(f(x_i) - f(x_i)\big) - l'(x_{i-1})\big(f(x_{i-1})$$

1.6 Minimum Energy

Theorem 1.9 (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l be the linear spline interpolant of f. Then

$$\|l'\|_2 \le \|w'\|_2.$$

$$l'(w-l)\Big|_{x_{i-1}}^{x_i} = l'(x_i)\big(f(x_i) - f(x_i)\big)$$

$$- l'(x_{i-1})\big(f(x_{i-1}) - f(x_{i-1})\big)$$

$$= 0.$$

So now we have

$$\|w'\|_2^2 = \|(w-l)'\|_2^2 + \|l'\|_2^2$$

$$\ge \|l'\|_2^2$$

Theorem 1.9 (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l be the linear spline interpolant of f. Then

$$\|l'\|_2 \leq \|w'\|_2.$$

Proof: For any w that interpolates $f(x)$ at the ω^N points..

$$\|w'\|_2^2 = \int_a^b (w')^2\,dx = \int_a^b \big((w'-l') + l'\big)^2 dx$$

$$= \int_a^b (w'-l')^2\,dx + 2\int_a^b (w'-l')\,l'\,dx + \int_a^b (l')^2\,dx.$$

$$= \|w'-l\|_2^2 + \|l'\|_2^2 + 2\int_a^b (w'-l')\,l'\,dx.$$

1.7 Looking ahead

Piecewise linear interpolation is one of the most standard tools in computational science, engineering and statistics.

Its major drawback is that it can't represent the *curvature* of the function it is interpolating. In the next section we'll investigate how to do that using *cubic* splines.

Exercise 1.1

Page 28 of the Department of Education's old Mathematics Tables ("The *Log Tables*") reports that $\ln(1) = 0$, $\ln(1.5) = 0.4055$ and $\ln(2) = 0.6931$.

(i) Write down the linear spline l that interpolates $f(x) = \ln(x)$ at the points $x_0 = 1$, $x_1 = 1.5$ and $x_2 = 2$.

(ii) Use this to estimate $\ln(x)$ at $x = 1.2$. How does this compare to the value in the tables, which is 0.1823?

(iii) Give an estimate for the maximum error:

$$\max_{1 \leq x \leq 2} |f(x) - l(x)|.$$

(iv) What value of n would you choose to ensure that $|f(x) - l(x)| \leq 0.001$ for all $x \in [1, 2]$.

Exercise 1.2

One can also define the linear spline interpolant to a function is as a linear combination of a set of piecewise linear basis functions $\{\psi_i\}_{i=0}^N$:

$$\psi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

They are depicted in Figure 1.

(i) Write down a formula for the $\psi_i(x)$;

(ii) derive a formula for $l(x)$ in terms of the ψ_i.

This exercise is useful: we'll use these basis functions (called "hat" functions) in the final section of the course.