Chapter 3: Numerical Linear Algebra

Exercise 3.1. Suppose you had a computer that computer perform 2 billion operations per second. Give a lower bound for the amount of time required to compute the determinant of a 10-by-10, 20-by-20, and 30-by-30, matrix using the method of minors.

Exercise 3.2. The Cauchy-Binet formula for determinants tells us that, if A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$. Using it, or otherwise, show that $\det(\sigma A) = \sigma^n \det(A)$ for any $A \in \mathbb{R}^{n\times n}$ and any scalar $\sigma \in \mathbb{R}$.

Note: this exercise gives us another reason to avoid trying to calculate the determinant of the coefficient matrix in order to find the inverse, and thus the solution to the problem. For example $A \in \mathbb{R}^{16\times 16}$ and the system is rescaled by $\sigma = 0.1$, then $\det(A)$ is rescaled by 10^{-16}. On standard computing platforms, like MATLAB, it would be indistinguishable from a singular matrix!

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Exercise 3.3. Every step of Gaussian Elimination can be thought of as a left multiplication by a unit lower triangular matrix. That is, we obtain an upper triangular matrix U by multiplying A by k unit lower triangular matrices: $L_k L_{k-1} L_{k-2} \dots L_2 L_1 A = U$, where each $L_i = I + \mu_{pq} E^{(pq)}$, and $E^{(pq)}$ is the matrix whose only non-zero entry is $e_{pq} = 1$. Give an expression for k in terms of n.

Exercise 3.4. Let L be a lower triangular $n \times n$ matrix. Show that $\det(L) = \prod_{j=1}^{n} l_{jj}$. Hence give a necessary and sufficient condition for L to be invertible. What does that tell us about *Unit* Lower Triangular Matrices?

Exercise 3.5. Let L be a lower triangular matrix. Show that each diagonal entry of L, l_{jj} is an eigenvalue of L.

Exercise 3.6. Prove Parts (i)–(iii) of Theorem 3.7.

Exercise 3.7. Prove Theorem 3.8.

Exercise 3.8. Construct an alternative proof of the first part of Theorem 3.7 (iv) as follows: Suppose that L is a non-singular lower triangular matrix. If $\mathbf{b} \in \mathbb{R}^n$ is such that $b_i = 0$ for $i = 1, \dots, k \leqslant n$, and $\mathbf{y}$ solves $L\mathbf{y} = \mathbf{b}$, then $y_i = 0$ for $i = 1, \dots, k \leqslant n$. (Hint: partition L by the first k rows and columns.)

Now use this to give a alternative proof of the fact that the inverse of a lower triangular matrix is itself lower triangular.

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Exercise 3.9. Many textbooks and computing systems compute the factorisation $A = LDU$ where L and U are unit lower and *unit* upper triangular matrices respectively, and D is a diagonal matrix. Show such a factorisation exists, providing that if $n \geqslant 2$ and $A \in \mathbb{R}^{n\times n}$, then every leading principal submatrix of A is nonsingular for $1 \leqslant k < n$.

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Exercise 3.10. Suppose that A has an LDU-factorisation (see Exercises 3.9). How could this factorization be used to solve $A\mathbf{x} = \mathbf{b}$?

Exercise 3.11. Prove that

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1), \quad \text{and}$$

$$1^2 + 2^2 + \cdots k^2 = \frac{1}{6}k(k+1)(2k+1).$$

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Exercise 3.12 ($\star$). Show that, for any vector $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_2^2 \leqslant \|\mathbf{x}\|_1 \|\mathbf{x}\|_\infty$. For each of these inequalities, give an example for which the equality holds. Deduce that $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2 \leqslant \|\mathbf{x}\|_1$.

Exercise 3.13. Show that if $x \in \mathbb{R}^n$, then $\|\mathbf{x}\|_1 \leqslant n\|\mathbf{x}\|_\infty$ and that $\|\mathbf{x}\|_2 \leqslant \sqrt{n}\|\mathbf{x}\|_\infty$.

Exercise 3.14. Show that, for *any* subordinate matrix norm on $\mathbb{R}^{n\times n}$, the norm of the identity matrix is 1.

Exercise 3.15 ($\star$)**.** Prove that

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{i,j}|.$$

Hint: Suppose that

$$\sum_{i=1}^{n} |a_{ij}| \leqslant C, \qquad j = 1, 2, \dots n,$$

show that for *any* vector $x \in \mathbb{R}^n$

$$\sum_{i=1}^{n} |(Ax)_i| \leqslant C\|x\|_1.$$

Now find a vector x such that $\sum_{i=1}^{n} |(Ax)_i| = C\|x\|_1$. Now deduce the result.

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Exercise 3.16. (i) Prove that, if $\|\cdot\|$ is a subordinate matrix norm, then it is *consistent*, i.e., for any pair of $n \times n$ matrices, A and B, we have $\|AB\| \leqslant \|A\|\|B\|$.

(ii) One might think it intuitive to define the "max" norm of a matrix as follows:

$$\|A\|_{\widetilde{\infty}} = \max_{i,j} |a_{ij}|.$$

Show that this is indeed a norm on $\mathbb{R}^{n\times n}$. Show that, however, it is not consistent.

Exercise 3.17. Let A be the matrix

$$A = \begin{pmatrix} 0.1 & 0 & 0 \\ 10 & 0.1 & 10 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Compute $\kappa_\infty(A)$. Suppose we wish to solve the system of equations $Ax = \mathbf{b}$ on *single precision* computer system (i.e., the relative error in any stored number is approximately 10^{-8}). Give an upper bound on the relative error in the computed solution x.

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Exercise 3.18. A real matrix $A = \{a_{i,j}\}$ is *Strictly Diagonally Dominant* if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{i,j}| \qquad \text{for } i = 1, \dots, n.$$

Show that all strictly diagonally dominant matrices are nonsingular.

Exercise 3.19. Let

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & -3 \end{pmatrix}$$

Use Gerschgorin's theorems to give an upper bound for $\kappa_2(A)$.