



MA385 Part 2: Initial Value Problems

## 2.2: Euler's Method

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For more details, see Chapter 6 of [Süli and Mayers, \*An Introduction to Numerical Analysis\*](#).

# 1. The goal

Our goal is to generate numerical solutions to initial value differential equations. The solutions to such problems are functions (usually, of one variable that we'll denote  $t$ ). Our approximation will give estimates of the values of this function at certain points. We'll denote the points we at which we are seeking approximations as

$$t_0 < t_1 < \cdots < t_n.$$

The methods we'll use are all **one-step** methods, and the first example we'll consider is **Euler's Method**.

Although it is not too important, we'll make the assumption that the points are equally spaced. So

$$t_{i+1} - t_i = \frac{t_n - t_0}{n} = h.$$

## 2. Euler's Method: motivation

The simplest method is **Euler's Method**. We motivate it as follows.

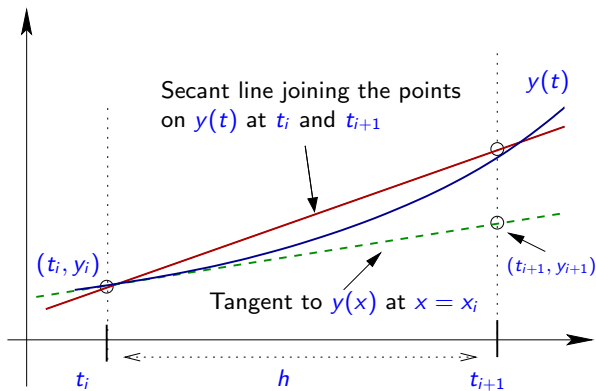
### Motivation

Suppose we know  $y(t_i)$ , and want to compute  $y(t_{i+1})$ . From the differential equation we can calculate the slope of the tangent to  $y$  at  $t_i$ . If this approximates the slope of the line joining  $(t_i, y(t_i))$  and  $(t_{i+1}, y(t_{i+1}))$ , then

$$y'(t_i) = f(t_i, y(t_i)) \approx \frac{y_{i+1} - y_i}{t_{i+1} - t_i}.$$

## 2. Euler's Method: motivation

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### 3. Euler's Method: formula

#### Euler's Method

Choose equally spaced points  $t_0, t_1, \dots, t_n$  so that

$$t_i - t_{i-1} = h = (t_n - t_0)/n \quad \text{for } i = 0, \dots, n-1.$$

We call  $h$  the “time step”. Let  $y_i$  denote the approximation for  $y(t)$  at  $t = t_i$ . Set

$$y_{i+1} = y_i + hf(t_i, y_i), \quad i = 0, 1, \dots, n-1. \quad (1)$$

## Example 2.2.1

Taking  $h = 1$ , estimate  $y(4)$  where

$$y'(t) = y/(1 + t^2), \quad y(0) = 1.$$

Choosing  $h = 1$  we get

▶  $i = 0$ :  $t_0 = 0$ ,  $y_0 = 1$ .

▶  $i = 1$ :  $t_1 = t_0 + h = 1$ .

$$y_1 = y_0 + hf(t_0, y_0) = 1 + \frac{1}{1+0^2} = 2.$$

▶  $i = 2$ :  $t_2 = t_0 + 2h = 2$ .

$$y_2 = y_1 + hf(t_1, y_1) = 2 + 1 \frac{2}{1+1^2} = 3.$$

▶  $i = 3$ :  $t_3 = t_0 + 3h = 3$ .

$$y_3 = y_2 + hf(t_2, y_2) = 3 + 1 \frac{3}{1+2^2} = 3.6$$

▶  $i = 4$ :  $t_n = t_4 = t_0 + 4h = 4$ .

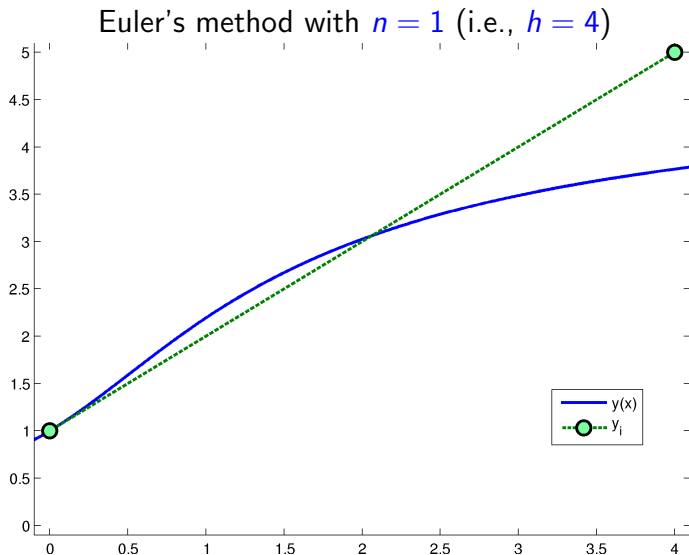
$$y_n = y_4 = y_3 + hf(t_3, y_3) = 3.6 + \frac{3.6}{1+3^2} = \mathbf{3.96}$$



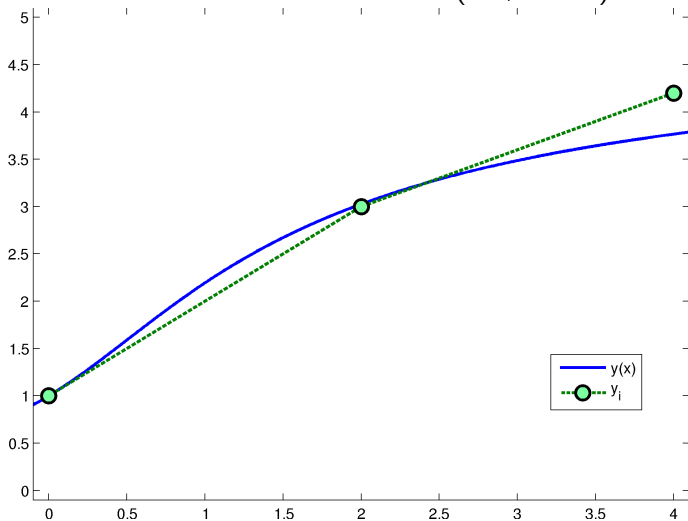


If we had chosen  $h = 4$  we would have only required one step:  
 $y_n = y_0 + 4f(t_0, y_0) = 5$ . However, this would not be very accurate.

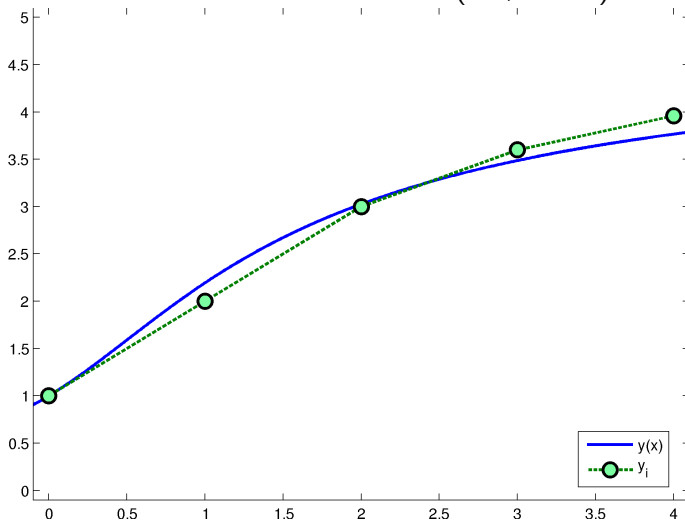
With a little work one can show that the solution to this problem is  $y(t) = e^{\tan^{-1}(t)}$  and so  $y(4) = 3.7652$ . Hence the computed solution with  $h = 1$  is much more accurate than the computed solution when  $h = 4$ . This is also demonstrated in next figure below, and in the follow table, where we see that the error seems to be proportional to  $h$ .



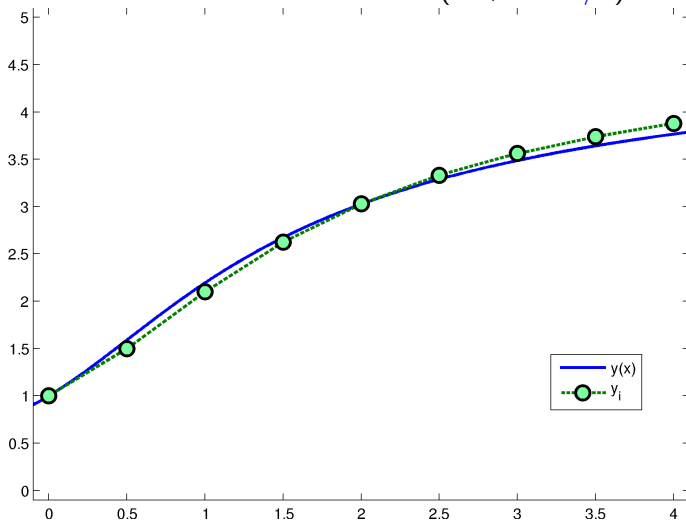
Euler's method with  $n = 2$  (i.e.,  $h = 2$ )

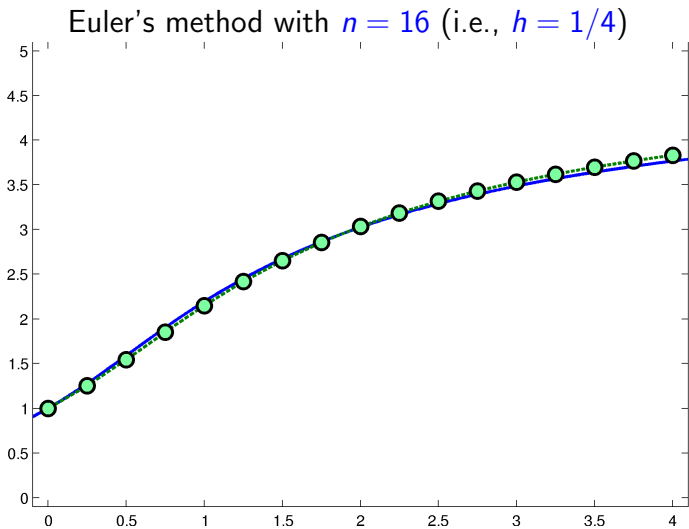


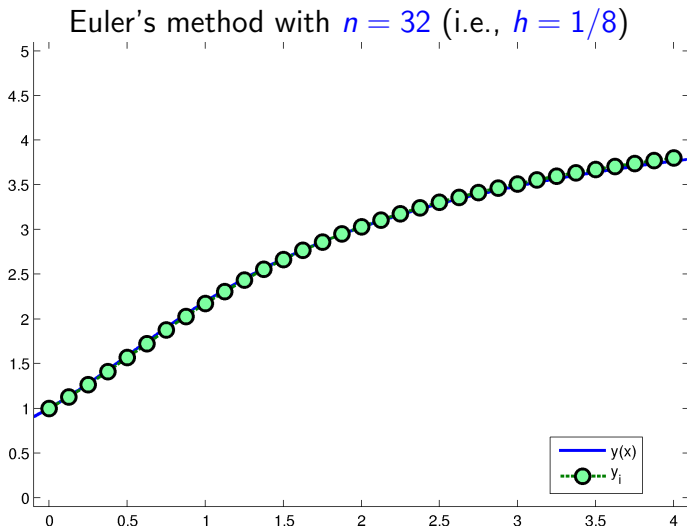
Euler's method with  $n = 4$  (i.e.,  $h = 1$ )



Euler's method with  $n = 8$  (i.e.,  $h = 1/2$ )









$n$	$h$	$y_n$	$ y(t_n) - y_n $
1	4	5.0	1.235
2	2	4.2	0.435
4	1	3.960	0.195
8	1/2	3.881	0.115
16	1/4	3.831	0.065
32	1/8	3.800	0.035

Table 1: Error in Euler's method for Example 8

## 4. Exercises

### Exercise 2.2.1

As a special case in which the error of Euler's method can be analysed directly, consider Euler's method applied to

$$y'(t) = y(t), \quad y(0) = 1.$$

The true solution is  $y(t) = e^t$ .

(i) Show that the solution to Euler's method can be written as

$$y_i = (1 + h)^{t_i/h}, \quad i \geq 0.$$

(ii) Show that

$$\lim_{h \rightarrow 0} (1 + h)^{1/h} = e.$$

This then shows that, if we denote by  $y_n(T)$  the approximation for  $y(T)$  obtained using Euler's method with  $n$  intervals between  $t_0$  and  $T$ , then

$$\lim_{n \rightarrow \infty} y_n(T) = e^T.$$

*Hint:* Let  $w = (1 + h)^{1/h}$ , so that  $\log w = (1/h) \log(1 + h)$ . Now use l'Hospital's rule to find  $\lim_{h \rightarrow 0} w$ .