2425-MA140 Engineering Calculus

Week 03, Lecture 2
Vertical Asymptotes and Continuity

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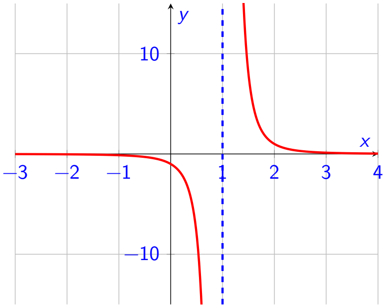
Wednesday, 1 October, 2025



This slides are by Niall Madden, with some content based on notes by Dr Kirsten Pfeiffer, and from Strang & Herman's "Calculus".

Outline

For more, see Chapter 2 (Limits) in **Calculus** by Strang & Herman. See openstax.org/books/calculus-volume-1/pages/2-introduction

In particular §2.2: One-sided limits (§2.2.4+§2.2.5) and vertical asymptotes (§2.2.7).

Reminders...

Reminders

- **Assignment 1** due 5pm, Monday 6 October. You may access it multiple times, by clicking on `Assignments` ... `Problem Set 1` ... and then, at the bottom of the page:
  `Load Problem Set 1 in a new window`
- The **Tutorial Sheet** is available at
  https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1
- Assignment 2 is also open; deadline is 5pm, 13 Oct.
- The first (of two) class tests will take place Tuesday, 14th October.
- If you wish to avail of Reasonable Accommodations for it tests, please complete this form:
  https://forms.office.com/e/HaAsrzaE3D by **10am Thursday 2nd Oct**.

Recall: One-sided Limits

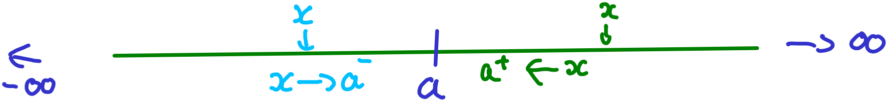
Yesterday we met the concept of **one-sided limits**:

$\lim_{x\to a^-} f(x)$ is: **limit of f as x approaches a from the left**

$\lim_{x\to a^+} f(x)$ is: **limit of f as x approaches a from the right**

These mean that

- if $\boxed{\lim_{x\to a^-} f(x) = L}$, then we can make $f(x)$ as close to L as we would like by taking $\boxed{x < a}$ as close to a as needed.
- If $\boxed{\lim \lim_{x\to a^+} f(x) = L}$, then we can make $f(x)$ as close to L as we would like by taking $\boxed{x > a}$ as close to a as needed.



Existence of a limit

$\lim_{x\to a} f(x)$ **exists** if and only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

So if $\lim_{x\to a} f(x) = L$ exists, we have

Right

Left $\lim_{x\to a^-} f(x) = \lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = L$

though it is not necessary that $f(a) = L$

That is, if we write $\lim_{x\to a}$ without any superscript on the a, we mean both left & Right.

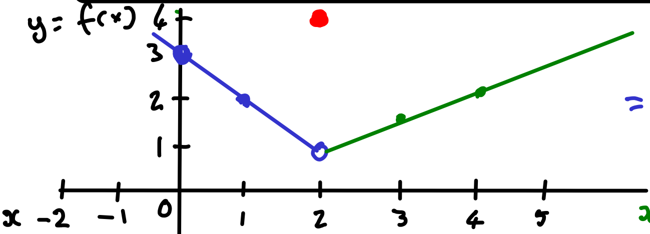
Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

$f(0) = 3$
$f(1) = 2$
$f(2) = 4$
$f(3) = \frac{3}{2}$
$f(4) = 2$

Determine if $\lim_{x \to 2} f(x)$ exists.



$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 1$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = 1$

So $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if $\lim_{x \to 2} f(x)$ exists.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 1$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = 1$$

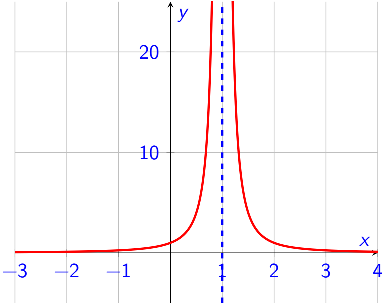
So $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$. Therefore $\lim_{x \to 2} f(x)$ exists.

Vertical Asymptotes

Let's revisit the following example from yesterday:

$$\boxed{f(x) = \frac{1}{(x-1)^2}}$$

$f(1) = \frac{1}{(1-1)^2} = \frac{1}{0}$
which is undefined



$\uparrow$
$x = 1$

Note that the points on the graph having x-coordinates very near to 1 are very close to the vertical line $x = 1$. That is, as x approaches 1, the points on the graph of $f(x)$ are closer to the line $x = 1$.

We call the line $x = 1$ a **vertical asymptote** of the graph.

Vertical Asymptotes

Definition: Vertical Asymptote

The vertical line $x = a$ is a **vertical asymptote** of $f(x)$ if any of $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, or $\lim_{x\to a} f(x)$ are ∞ or $-\infty$.

To find a vertical asymptote of a function $f(x) = \dfrac{p(x)}{q(x)}$, we find a value, a for which $p(a) \neq 0$ but $q(a) = 0$.

ie $f(a)$ evaluates as $\dfrac{p(a)}{0}$ which is undefined if $p(a) \neq 0$.

If, however $p(a) = q(a) = 0$ further investigations are needed.

Vertical Asymptotes

Example

Find any vertical asymptotes of

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

Here $p(x) = x^2 - x - 6$ and $q(x) = x + 1$

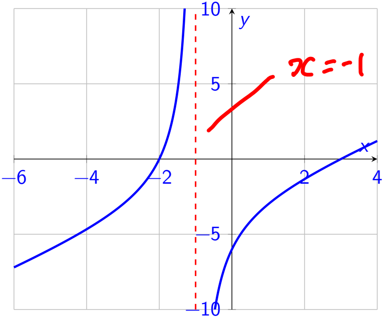
So $q(x) = 0$ if $x = -1$.

Since $p(-1) = (-1)^2 - (-1) - 6 = -4 \neq 0$

So there is a vertical asymptote at $x = -1$.

Vertical Asymptotes

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$



Vertical Asymptotes

Example

Find all vertical asymptotes of the graph of

$$g(x) = -\frac{8}{x^2-4}. = \frac{p(x)}{q(x)}$$

$p(x) = -8 \neq 0$ for any x.

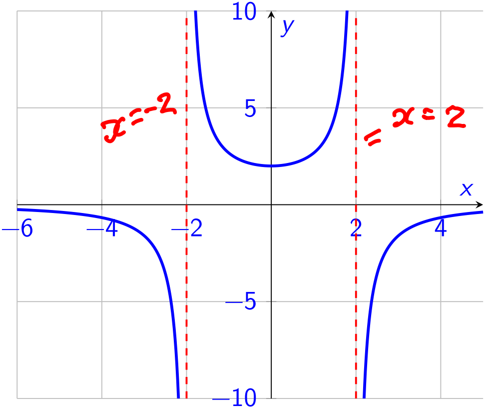
$q(x) = x^2 - 4. = (x-2)(x+2)$

so $q(2)=0$ & $q(-2)=0$. Therefore there are vertical asymptotes at

$x=2$ & $x=-2$.

Vertical Asymptotes

$$\boxed{f(x) = -\frac{8}{x^2 - 4}}$$



There is a related concept of a **horizontal asymptote**, but we'll save that for later, when we cover "limits at infinity".

Continuity

Many functions have the property that you can trace their graphs with pen and paper, without lifting the pen from the page. Such functions are called **continuous**.

Some other functions have points were you have to lift the pen occasionally. We say they have a **discontinutity** at such points.

Intuitively, a function is continuous at a particular point if there is no **break** (or "**jump**") in its graph at that point.

More formally, we define continuity in terms of **limits**.

Continuity

Definition

A function f is **continuous at** $x = a$ if

1. $f(a)$ is defined, i.e., a is in the domain of f,
2. $\lim_{x \to a} f(x)$ exists. → so left & right limits are equal.
3. $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$ we say it is **discontinuous** at $x = a$.

If f is continuous **at every point** in its domain, we say f **is continuous.**

"defined" meas "finite" (not ∞ or $-\infty$)

Many functions are continuous, e.g. all polynomial functions, most trigonometric functions (not tan), $|x|$, and so on.

Continuity

Example 1

Determine if $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$.

1. Is $f(2)$ defined?

$$f(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$$ which is not defined.

So 2 is not in the range of f.

So f is not continuous at $x = 2$.

Example 2

Determine if $f(x) = \begin{cases} 1 - x & x \leqslant 0 \\ 2 + x & x > 0 \end{cases}$ is continuous at $x = 0$.

1. $f(0) = 1 - 0 = 1$. so f is defined at $x = 0$.

2. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} 1 - x = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 + x = 2$.

Since $1 \neq 2$ the limit does not exist.
So f is <u>not</u> continuous at $x = 0$.

Continuity

Example 3

Determine if $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is continuous at $x = 0$.

1. $f(0) = 1$ so $f(0)$ is defined

2. for $x \neq 0$ $f(x) = \frac{\sin(x)}{x}$.

We know, from yesterday,

$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$. So $\lim_{x \to 0} f(x)$ Exists

3. Finally $\lim_{x \to 0} f(x) = f(0)$. So f is continuous

Continuity

Example

Consider the function

$$f(x) = \begin{cases} x+1, & x < 2 \\ b\,x^2, & x \geqslant 2 \end{cases}$$

For what value of b is f continuous at $x = 2$?

1. Note $f(2) = b x^2$ which is defined for any b.

2. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x+1) = 3.$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} b x^2 = 4b.$

So need $4b = 3 \Rightarrow \boxed{b = 3/4}$

Finished here 01/10/25

Example

For what values of x is $f(x) = \dfrac{2x+1}{2x-2}$ continuous?

Note $q(x) = 0$ at $x = 1$

But $q(x) \neq$ at any other x.

So $f(x)$ is continuous for all x except $x = 1$.

ie $x \in (-\infty, 1) \cup (1, \infty)$

$= \mathbb{R} / \{1\}$

Types of discontinuity

We have encountered three types of discontinuity.

- **Removable discontinuity**: $\lim_{x\to a} f(x)$ exists but
$$\lim_{x\to a} f(x) \neq f(a)$$

- **Jump discontinuity**: $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+ f(x)}$ both exist (and are finite), but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

- **Infinite discontinuity**: At least one of the one-sided limits does not exist.

Types of discontinuity

Example

Each of the following functions has a discontinuity at $x = 2$. Classify it.

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$
2. $g(x) = \dfrac{x^2}{x - 2}$
3. $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$
4. $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

Exercises

Exercise 3.2.1

Find all the vertical asymptotes of $f(x) = \dfrac{x+2}{x^2+2x-8}$.

Exercises 3.2.2 (Based on Q1(a), 23/24)

Let $g(x) = \begin{cases} 3 & x \leqslant 0 \\ 2x+1 & 0 < x < 1 \\ x^2 & x \geqslant 1. \end{cases}$

(i) Sketch the graph of $g(x)$ on the interval $[-3, 4]$, making use of the empty and full circle notation.

(ii) Compute $\lim_{x \to 1^-} g(x)$ and $\lim_{x \to 1^+} g(x)$. Is g continuous at $x = 1$. If not, classify the type of discontinuity.

Exercises

Exercise 3.2.3

For what values of b and c is $f(x) = \begin{cases} x^2+1 & x \leqslant -1 \\ x+b & -1 < x < 1 \\ cx^2 & x \geqslant 1. \end{cases}$

continuous at $x = -1$ and $x = 1$?