

Week 4 of CS319: Scientific Computing (with C++)

## Lab 2: floats and numerical differentiation

**Goal:** To study the computer representation of numbers – particularly `floats` and `doubles`, and see the impact of their precision on numerical differentiation.

**Assignment:** Submit your work as a C++ program, and also a short report, as described on Slide 9. Upload it to Canvas... [2425-CS319](#) ... “Assignments... Lab 2”.

**Deadline:** 12pm (noon), Wednesday 12 Feb 2025.

It is **important** that your files include comments with `your name`, email address, and **ID number**.

The assignment will be graded taking into account if the program compiles, if achieves the task described in Slide 8, and if it includes the requested information.

**Collaboration policy.** Collaboration is encouraged. It is acceptable for two people to work together and submit exactly the same work. However.

- ▶ Both need to submit the code independently. (No submission, no score, no exceptions).
- ▶ Your submission must include comments with YOUR name and ID number, and also give the name of your collaborator.
- ▶ The use of generative AI is **strictly prohibited**. Any suspected cases will be subject to standard University procedures.

## Question 1

Compute the “machine epsilon” for `float`. That is: find the smallest `float`,  $x$ , such that we can distinguish between 1 and  $1 + x$ . Write a C++ program to do this. Some notes:

- ▶ The Wikipedia entry for machine epsilon is rather good.
- ▶ Compilers, and CPUs, often try to be clever, and may preform interim calculations at higher precision than asked. So:

```
if ( 1.0 + x/2.0 > 1.0 ) ...
```

can behave differently from

```
z=1.0+x/2.0;
```

```
if ( z > 1.0 ) ...
```

# Q1

- ▶ In C++, one can actually check the “epsilon” value of any datatype, by including the `limits` header, and outputting the value of `std::numeric_limits<T>::epsilon()` where `T` is replaced with the type of interest. Compare your value with the value from `limits`.
- ▶ When you are done, compare the results with those given in the solution:  
<https://www.niallmadden.ie/2425-CS319/lab2/Lab2-Q1.cpp>

## Q2

### Question 2

Write a C++ programme that computes the *machine epsilon* for the `double` data type. That is, repeat Q1 but using the `double` data type instead of `float`.

Do the results agree with the theory covered in Week 3 lectures?

## Question 3

**Numerical Differentiation** is the process of estimating the derivative of a function,  $f = f(x)$ , at some value of  $x$ . It is a hugely important topic, including in machine learning (for back-propagation steps in deep neural networks). Most modern approaches are based on **algorithmic** differentiation. However, we'll use a classic approach, called **finite differences**.

It is not hard to show (using truncated Taylor Series), that

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h) \quad (1)$$

This is called a “**two-point** method”.

It is implemented in the code at

[niallmadden.ie/2425-CS319/lab2/Lab2-Q3.cpp](http://niallmadden.ie/2425-CS319/lab2/Lab2-Q3.cpp)

That programme attempts to estimate  $f'(1.0)$  with  $f(x) = e^{-x}$  using the formula in (1) with  $h = 2^{-2} = 0.25$ . The correct answer, to 8 digits, is **-0.36787944**. The estimate it computes is **-0.32549858**, giving an error of **0.0424**.

### Q3

Suppose we decide we need a more accurate estimate. Then we have two choices:

- (a) Use a smaller value of  $h$ , since (1) suggests the error is proportional to  $h$ .
- (b) Use a better method such as the so-called *three-point* scheme:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2) \quad (2)$$

(Aside: *why might this be called a 3-point scheme, given that it involves only two points:  $x-h$  and  $x+h$ ? Thoughts?*)

Your task will be to try both these approaches, and establish what is the most accurate estimate that can be achieved, using both `floats` and `doubles`.

If there were no round-off errors, you could take  $h$  as small as needed to make the error as small as you'd like. However, that is not possible in practice.

More detail below...

## Assignment

**Task 1** Adapt the code in `Lab2-Q3.cpp` so that the formula in (1) is used to estimate  $f'(x_0)$  for smaller and smaller values of  $h$ . Initially, you should find that smaller  $h$  gives smaller error. However, eventually you should find round-off error starts to dominate.

**What is the smallest error you can achieve, and what is the corresponding value of  $h$ ?**

Tips:

- Try the algorithm for  $h = 1, h = 1/2, \dots, h = 2^{-n}, \dots$ . That is, at each iteration, reduce the size of  $h$  by a factor of 2.
- You can use a `for`-loop or a `while`-loop in your code, as you see fit. If using a `for`-loop, experiment to find the optimal number of iterations.

**Task 2** Repeat **Task 1**, but using the formula in (2). What is the smallest  $h$  you can use, and what is the corresponding error?

**Task 3** Repeat **Task 1**, but this time using a `double`. **Tip:** The numbers involved may be very small. Best display them with the `scientific` modifier to `cout`.



# What you upload

## What to upload

1. Upload your code that implements any **one** of **Tasks 1, 2 or 3**.  
Your code does not have to do all three, but should be easily tweaked to do any of them. (E.g., it might run for `float`, but using a search/replace function changed to `double`. Or, if you are feeling very determined, you could use `#define...` ).  
Make sure your code identifies you as the author, and names any collaborators.  
Your file must be in plain text, and should have a `.cpp` suffix. If you are not sure about this, talk to Niall.
2. Write a very short report, outlining your findings for each of the three cases. Maximum half a page. Again: include your name and ID number, and any collaborators.

## Further reading

The following is not part of the assignment.

- ▶ If one implements Algorithm (1) in a datatype that has precision  $\varepsilon$ , one can show the optimal  $h$ , and corresponding errors are, in theory:

$$h = 2\sqrt{\frac{\varepsilon}{M}}, \quad \text{Error} \sim 2\sqrt{\varepsilon M}.$$

- ▶ To read more about that, have a look at Lecture 24 of “Afternotes on Numerical Analysis”. You can find it online through the Library Catalogue: [https://search.library.nuigalway.ie/permalink/f/1pmb91f/353GAL\\_ALMA\\_DS5164291970003626](https://search.library.nuigalway.ie/permalink/f/1pmb91f/353GAL_ALMA_DS5164291970003626)
- ▶ Amazingly, there is a very similar scheme that does not suffer any of the same problems, called the “Complex Step Method”. You can read about it at <https://nhigham.com/2020/10/06/what-is-the-complex-step-approximation>