Week 07, Lecture 3
The Fundamental Theorem of Calculus

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Suimeálaithe
Tá tairisigh na suimeála fágtha ar lár.

Integrals
Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
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$\frac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
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Suimeáil na míreanna $\quad \int u\,dv = uv - \int v\,du \quad$ **Integration by parts**

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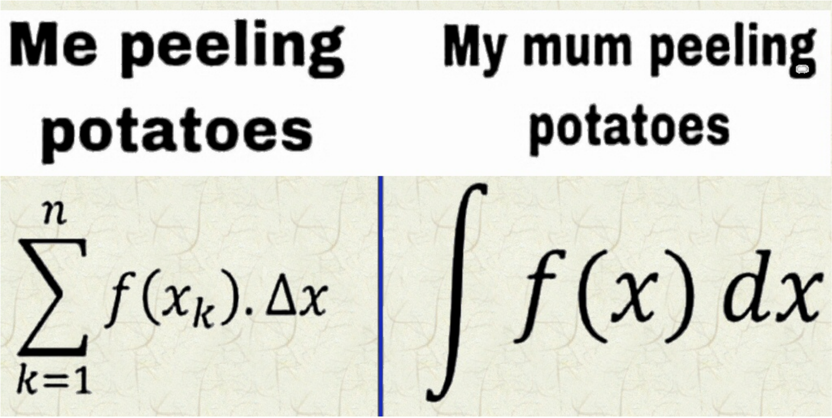
Today's joke (with thanks to Julie M).



The exciting topics that await us in today:

1. Recall from yesterday:
2. Fundamental Thm of Calculus: Part 1
3. FTC1+Chain Rule
4. Antiderivatives
   - Indefinite Integrals
   - Common functions
   - Properties
5. The Fundamental Thm of Calculus: Part 2
6. Exercises

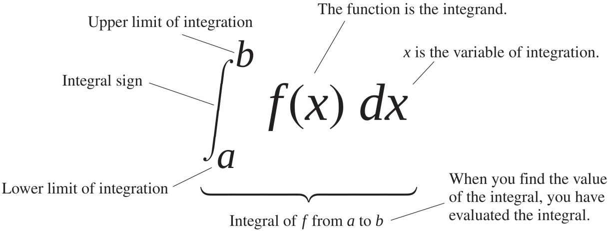
See also: Sections **4.10** (Antiderivatives) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Recall from yesterday:

Let $f(x)$ be function defined on an interval $[a, b]$. The **definite integral** of f from a to b is

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=0}^{n-1} hf(x_i),$$

where $h = (b-a)/n$ and $x_i = a + ih$. It is the **area** of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$, and $x = b$.



Recall from yesterday:

Given a function, f, we can define another, F as

$$F(x) = \int_a^x f(t)dt.$$

That is, the variable in F is the upper limit of integration on the right.

Recall from yesterday:

Example

Let $f(t) \equiv 1$, and $F(x) = \int_0^{x} f(t)dt$. Give a formula for $F(x)$, using the "area" meaning of the definite integral.

Fundamental Thm of Calculus: Part 1

Fundamental Theorem of Calculus: Part 1 (FTC1)

Let $f(x)$ be a continuous function on $[a, b]$. If as

$$F(x) = \int_a^x f(t)dt, \qquad \text{then} \qquad \frac{dF}{dx}(x) = f(x).$$

I.e., $F'(x) = f(x)$ for $x \in [a, b]$.

Roughly: $\boxed{f \text{ is the derivative its own integral}}$. You can find a proof in Section 5.3 of the textbook.

Fundamental Thm of Calculus: Part 1

Example

Let $g(x) = \int_1^x \frac{1}{t^3+1} dt$. Find $g'(x)$.

FTC1+Chain Rule

Sometimes the limit of integration is a more complicated function of x. In that case, we can apply the **Chain Rule**, along with the FTC1.

Example

Let $F(x) = \int_1^{\sqrt{x}} \sin(t)dt$. Find $F'(x)$.

Idea: Let $u(x) = \sqrt{x} = x^{1/2}$. So

- $F(u) = \int_1^{u} \sin(t)dt$, and
- $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$.

Then...

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = \sin(u(x))\left(\frac{1}{2\sqrt{x}}\right) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

Antiderivatives

Definition: Antiderivative

A function F is an **antiderivative** of f on $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$. Thus,

$$f \text{ is the derivative of } F \Leftrightarrow F \text{ is an antiderivative of } f.$$

Note: If F is an antiderivative of f, then the most general antiderivative of f is

$$F(x) + C$$

where C is an *arbitrary* constant, called a **constant of integration**.

- The word "arbitrary" here means that any choice is valid.
- The derivative of C is zero.

Antiderivatives

Examples:

- $F(x) = x + C$ is an antiderivative of $f(x) \equiv 1$.

- $F(x) = x^2 + C$ is an antiderivative of $f(x) =???$ ...

- $F(x) =???$ is an antiderivative of $f(x) = 3x^2$.

Antiderivatives

Examples

Find all antiderivatives of the following functions

(i) $f(x) = \dfrac{1}{x}$ for $x > 0$.

(ii) $f(x) = \sin(x)$

(iii) $f(x) = e^x$.

Definition: indefinite integral

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,\mathrm{d}x$$

is the general antiderivative of f. That is, if F is an antiderivative of f, then

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

Examples:

- $\int 2x\,dx = x^2 + C$
- $\int 3x^2\,dx = x^3 + C$

- $\int x \, dx = \frac{1}{2}x^2 + C$
- $\int x^2 \, dx = \frac{1}{3}x^3 + C$.

Spotting the pattern we can deduce...

Power Rule of Integration

If $n \neq -1$, then $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$

Note: For $n = -1$, we have

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C \, .$$

Here is a list of the antiderivatives of some common functions.

- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- $\int a^x\,dx = \frac{a^x}{\ln a} + C$
- $\int \sin(x)\,dx = -\cos(x) + C$
- $\int \cos(x)\,dx = \sin(x) + C$
- $\int \tan(x)\,dx = \ln|\sec(x)| + C$
- ...

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Properties of Integration

1. If k is a constant, then

$$\int kf(x)\,dx = k\int f(x)\,dx\,.$$

2. Integration is additive:

$$\int \left(f(x) \pm g(x)\right)\,dx = \int f(x)\,dx \pm \int g(x)\,dx\,.$$

Example

Evaluate the integral

$$\int 2x^2 + 9x^7 \, dx$$

Example

Evaluate the integral

$$\int \frac{4}{1+x^2}\,\mathrm{d}x.$$

The Fundamental Thm of Calculus: Part 2

Now that we know all about antiderivatves, we can see how the link to **definite integrals**

Theorem (The Fundamental Thm of Calculus, Part 2)

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Notation: We cal write $F(b) - F(a)$ as $F(x)\Big|_{x=a}^{x=b}$, or, more often, as $F(x)\Big|_a^b$.

The Fundamental Thm of Calculus: Part 2

Example: Show that $\displaystyle\int_{-1}^{1} (x^2 + 2)\,\mathrm{d}x = \frac{14}{3}$

The Fundamental Thm of Calculus: Part 2

Example: Show that $\int_{-1}^{1} (x^3 + x)\,\mathrm{d}x = 0$

Exercises

Exer 7.3.1

Let $F(x) = \int_x^{2x} t\,\mathrm{d}t$. Use the Fundamental Theorem of Calculus to evaluate $F'(x)$.

Hint: we can split this into two integrals:

$$F(x) = \int_x^{2x} t\,\mathrm{d}t = \int_x^0 t\,\mathrm{d}t + \int_0^{2x} t\,\mathrm{d}t = -\int_0^x t\mathrm{d}t + \int_0^{2x} t\mathrm{d}t.$$

Now apply the FTC to each term, including the Chain Rule for the second.

Exercises

Exer 7.3.2

Evaluate the following integrals.

1. $\int e^{2x} + \frac{1}{2x} \,\mathrm{d}x$
2. $\int \frac{3}{\sqrt{2 - x^2}} \,\mathrm{d}x$

Exer 7.3.3

Evaluate the definite integral $\int_1^e e^{2x} + \frac{1}{2x} \,\mathrm{d}x$

Exer 7.3.4

Find two values of q for which $\int_q^0 2x + x^2 \,\mathrm{d}x = 0$.