Week 08, Lecture 1 (L22)
Integration By Parts

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Assignments

- The grades for **Assignment 3** and **Assignment 3 (resit)** have been merged. Let me know if you think there is a problem...
- **Assignment 5** closed yesterday. The grades for Q2–Q8 have been posted. The full grades, including Q1, will be available next week.
- **Assignment 6** is open. Deadline is 5pm next Monday (4 November). There are 8 questions. You have 3 attempts for Questions 1–7. Q8 will be manually graded after the deadline.

Survey Feedback I

Many thanks to the 111 of you that completed the survey! Thanks for all the nice comments (very much appreciated). Here are a few things that came up, and what I'll do about it:

- "*Tutors could be more interactive*". Action: I've let the tutors know, and will meet them to discuss strategies for this.
- "*More interaction with class during lectures*". Action: I'll try... however, I've found that, since some of the class in a room elsewhere, trying to get more interaction can have a negative impact.
- "*More real world based examples*". Action: Fair point. I was saving much of this for the end of the semester, but will try to add more as I go along.
- Various issues with Venue B. Action: I'll feed this back to the College Office.

Survey Feedback II

- "*Extra optional problem sheets*". "*More questions to work on*", (etc). Action: This can lead to mix-ups with deadlines. But I'll increase the number of problems at the end of each set of slides.
- "*Post the annotated slides immediately after class*". Action: OK - will try harder!
- Various comments regarding references for Numbas over WeBWorK, and vice versa. Action: For boring reasons, isn't not feasible to bring back Numbas this semester, in a way that will ensure there are no problems with the grades, or at least can be resolved more easily.
- "*Post answers to the assignments*". Action: Good suggestion. Done! That is, the solutions to the Tutorial Sheets are now available. Go to `Modules` and then `Tutorial Sheets`.

Survey Feedback III

- "*Popcorn provided for students*". Action: Sorry - no food allowed!
- "*Past exam papers*". Action: Sorry. Due to COVID, the only recent ones that exist are from 2023/24 and 2019/2020. However, I'll set a "sample" paper at the end of the semester, and also provide solutions.
- "*Step-by-step solution for the questions at the end of the slides*". Action: Not sure I can promise this, but I have re-started providing answers (now up to Week 4).
- *Various requests to change tutorial times.* Action: Sorry: nothing I can do here (PS: have you seen your time-table???)
- "*Post the notes for the each day's lecture a day before the class*". Action: Sorry, sorry. Will try harder.

Survey Feedback IV

- *Comments about audio, and use of mics...* Action: will investigate...
- "*Record lectures*. Action: Sorry, that is against cyurrent university policy.
- "*Niall to teach us chemistry*". Action: That would not end well...

This part is about...

For more reading, see Sections **7.1** (Integration by Parts) and **6.1** (Areas Between Curves) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Integration by Parts

The Product Rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

We can use this rule to develop another integration technique after a little rearrangement. From the above we have

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

and integrating both sides gives

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

This method is called **Integration by Parts**: it is, by some distance, the most important technique for integration, in both theory and practice.

Integration by Parts

Integration by Parts

If u and v are differentiable functions in the variable x, then

$$\int uv' \, dx = uv - \int vu' \, dx \,.$$

Recall that we can write

$$du = u' \, dx \quad \text{and} \quad dv = v' \, dx \,.$$

Therefore, we can rewrite the formula for Integration by Parts as

$$\int u \, dv = uv - \int v \, du \,.$$

Example

Evaluate $\int x \cos(x)\, dx$

Lets take $u = x$ and $dv = \cos(x) dx$.

One of the challenges of Integration by Parts is knowing how to choose u and dv. When integrating $\boxed{\int x\cos(x)\,dx}$ we choose $u = x$, because its derivative, $u' = 1$ is simpler.

Suppose we had made the bad choice of

$$u(x) = \cos(x), \qquad dv = x\,dx,$$

then we'd get:

More generally, given choices for u and dv, we proceed as follows:

1. Some functions are easy to differentiate (and maybe not so easy to integrate), and so make a good choice for u. Important examples include **logarithms** and **inverse trig** functions.
2. Some functions (such as polynomials) have simple(r) derivatives, so are also a good choice for u.
3. Trig and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for dv.

Example (of choosing u

Evaluate $\int \frac{\ln(x)}{x^2}\, dx$.

Example

Evaluate $\int \ln(x)\, dx$.

Since $\int \ln(x) dx$ can be written as $\int \ln(x) \cdot 1 dx$, we use integration by parts, with $u = \ln(x)$ and $dv = dx$.

Sometimes, we have to apply Integration by Parts more than once.

Example

Evaluate $\int x^2 e^x \, dx$.

Example of repeated IbP

Evaluate $\int e^x \cos(x)\, dx$.

Definite Integrals

Integration by Parts for Definite Integrals

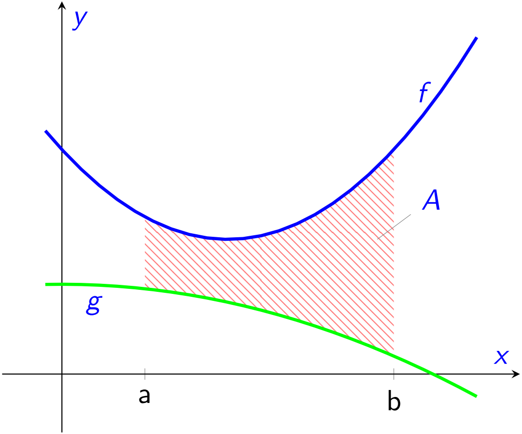
$$\int_a^b u\,dv = (uv)\Big|_a^b - \int_a^b v\,du$$

Example: Use Integration By Parts to evaluate $\int_0^1 xe^{-x}\,dx$.

Areas Between Curves

We know that $\int_a^b f(x)\,dx$ evaluates as the area of the region between $x = a$ and $x = b$, and between $y = f(x)$ and $y = 0$.

But what if we wanted to evaluate the area between two curves?



Areas Between Curves

Area Between Curves

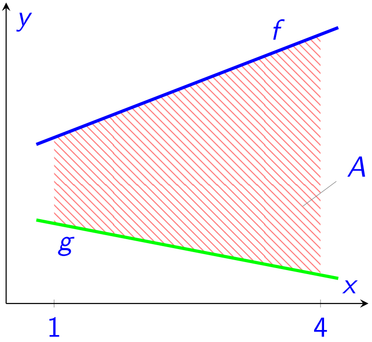
Let f and g be continuous functions with $f(x) \geqslant g(x)$ throughout the interval $[a, b]$. Then the area A of the region over $[a, b]$ and between the curves $y = f(x)$ and $y = g(x)$ is the integral of $f(x) - g(x)$ from $x = a$ to $x = b$; that is

$$A = \int_a^b (f(x) - g(x))\, dx\,.$$

Areas Between Curves

Example

Find the area of the region bounded above by the graph of $f(x) = x + 4$, and below by the graph of $g(x) = 3 - x/2$ over the interval $[1, 4]$

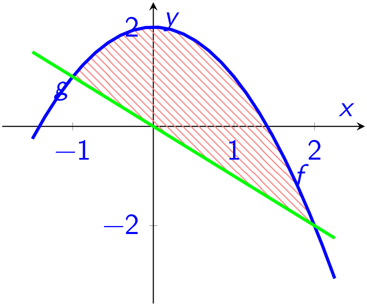


Areas Between Curves

Frequently, we need to work out the domain ourselves.

Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Areas Between Curves

Areas Between Curves

Example

Find the area enclosed between the two curves $f(x) = 6 - 2x^2$ and $g(x) = 4x$.

Exercises

Exer 8.1.1

Evaluate the follow integrals

1. $\int xe^{2x}\,dx$.
2. $\int x^3 e^{x^2}\,dx$. (Hint: take $u = x^2$, and then do substitution, like in Slide 12 from Week 7, Lecture 3).
3. $\int x^2 \cos(x)\,dx$.

Exer 8.1.2

Evaluate $\int_1^e \ln(x^2)\,dx$.