3.0 Started & finished 3/mar/2023

MA378 Chapter 3: Numerical Integration

§3.3 Precision and Composition

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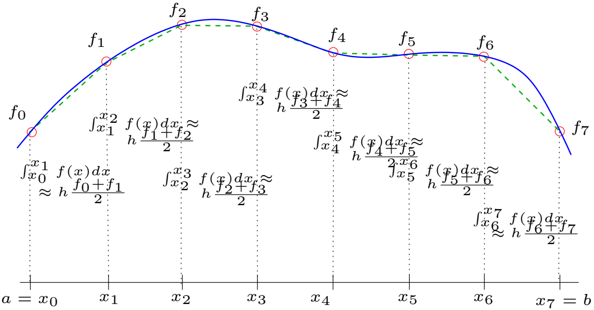


3.1 Simpson's, again

We know that Simpson's Rule applied to approximating $\int_0^1 x^3 dx$ yields exactly the right answer (i.e., the error is zero). In class, it was argued – very roughly – that Simpson's Rule is exact for **any** polynomial of degree 3 or less.

Since that was just scribbled on the board, here it is again.

1. We'll restrict our focus to the interval $[-1, 1]$. (See below for the general case). So $Q_2(f)$ approximates $\int_{-1}^{1} f(x)dx$.
2. We know that, if p_2 is any quadratic polynomial, then approximating $Q_2(p_2) = \int_{-1}^{1} p_2(x)dx$.
3. We want to show that, in fact, $Q_2(p_3) = \int_{-1}^{1} p_3(x)dx$ for any cubic polynomial, p_3.

3.1 Simpson's, again

4. First write p_3 as

$$p_3(x) = c_3x^3 + c_2x^2 + c_1x + c0 = c_3x^3 + p_2(x),$$

   for some quadratic $p_2(x)$.
5. $\int_{-1}^{1} p_3(x)dx = \int_{-1}^{1} c_3x^3 + p_2(x)dx =$ $c_3\int_{-1}^{1} x^3dx + \int_{-1}^{1} p_2(x)dx$. But since $\int_{-1}^{1} x^3dx = 0$, we have that $\int_{-1}^{1} p_3(x)dx = \int_{-1}^{1} p_2(x)dx$.
6. Similarly, $Q_2(p_3)dx = Q_2(c_3x^3) + Q_2(p_2) = c_3Q_2(x^3) + Q_2(p_2)$. But since $Q_2(x^3) = \frac{2}{6}\big((-1)^3 + 40^3 + 1^3\big) = 0$, we have that $\int_{-1}^{1} p_3(x)dx = \int_{-1}^{1} p_2(x)dx$.
7. So, we have that

$$\int_{-1}^{1} p_3(x)dx = \int_{-1}^{1} p_2(x)dx = Q_2(p_2) = Q_3(p_3).$$

We won't do though this section in class, but please read it carefully.

We now claim that Simpson's Rule is exact for *any* polynomial of degree 3 or less, and on *any* interval.

Denote by $Q_2(f)$ the approximation of $\int_a^b f(x)dx$ with Simpson's Rule:

$$Q_2(f) = \frac{b-a}{6}\Big(f(a) + 4f\big(\frac{a+b}{2}\big) + f(b)\Big).$$

Since the method can be derived by integrating the quadratic that interpolates $f(x)$ at the three points a, $(a+b)/2$, and b, it is clearly exact for all quadratics.

Let $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Then

$$\int_a^b f(x)dx = \int_a^b \left(c_0 + c_1 x + c_2 x^2\right)dx + c_3 \int_a^b x^3 dx = \\ \int_a^b \left(c_0 + c_1 x + c_2 x^2\right)dx + c_3 \frac{b^4 - a^4}{4}.$$

Also,

$$Q_2(f) = Q_2(c_0 + c_1 x + c_2 x^2) + Q_2(c_3 x^3) = \\ \int_a^b \left(c_0 + c_1 x + c_2 x^2\right)dx \\ + c_3\left(\frac{b-a}{6}\right)\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right). \quad (1)$$

With a bit of symbolic manipulation we get that

$$\frac{b^4 - a^4}{4} = (\frac{b-a}{6})\left(a^3 + 4(\frac{a+b}{2})^3 + b^3\right),$$

as required.

In Section 3.2, we discussed a nïeve attempt to derive an upper bound for the error in Simpson's rule leading to

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3.$$

This is not wrong – just not sharp. For example it does not give that Simpson's Rule is exact for all cubic. The sharp result is

Theorem 3.1

$$|E_2(x)| = \Big| \int_a^b f(x)dx - Q_2\big(f(x)\big)\Big| \leq \frac{(b-a)^5}{2880} M_4.$$

For the proof see the text book (Theorem 7.2 of Suli and Mayers). Instead of working through it in class we'll prove a more general version of a consequence it.

3.2 Precision

Definition 3.2 (Precision of a Quadrature Rule)

A quadrature rule has **precision** n if it is exact for all polynomials of degree n or less. That is, the rule $Q(f)$ has precision n if

$$Q(p_n) = \int_a^b p_n(x)dx \quad \text{for all } p_n \in \mathcal{P}_n.$$

Example 3.3

By construction, the $(n+1)$-point Newton-Cotes rule has precision n.

Important: to verify if a method has precision n (or not!) it is enough to check if it is exact for $f(x)=1$, $f(x)=x$, ..., $f(x)=x^n$ and any interval, eg $[-1,1]$ or $[0,1]$

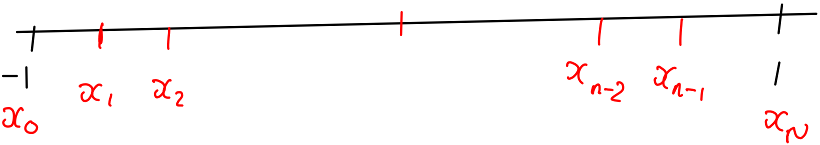
Theorem 3.4

If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k+1$ points, then $Q_{2k}(\cdot)$ has in fact precision $2k+1$.

Proof: Let p_{n+1} be a polynomial of degree $n+1$. We wish to show that $Q_n(p_{n+1}) = \int_a^b p_{n+1}(x)dx$. We can take $a=-1$, $b=1$ because a simple linear transformation can be used to map to an arbitrary interval.

Also, since the quadrature points are equally spaced on $[-1,1]$ we have that $x_i = -x_{n-i}$.

Furthermore (see exercises) the quadrature weights are symmetric: $q_i = q_{n-i}$.



$x_0 = -x_n$

$x_1 = -x_{n-1}$

$x_2 = -x_{n-2}$

Also $q_0 = q_N$

$q_1 = q_{N-1}$ . . .

Why?

3.3 Composite Rules

Suppose that we want to estimate $\int_a^b f(x)dx$ and the Trapezium rule is not sufficiently accurate. We could try Simpson's Rule, which should be better. Failing that, we could try a 4-point rule, based on integrating the p_4 interpolant, or a five-point rule, based on integrating p_5.

However, quite apart from the fact that it might be tedious to derive these rules, we know (Runge's example again!) that high-order polynomial interpolation can be very inaccurate.

3.3 Composite Rules

It is better to use a **Composite Rule**. This is analogous to the idea behind piecewise linear.

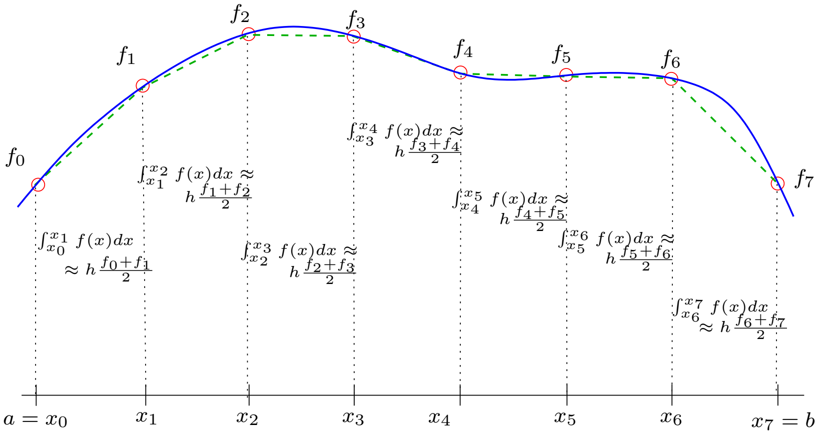
For the **Composite Trapezium Rule** we divide $[a, b]$ into N intervals of size $h = (b-a)/N$. Applying the Trapezium Rule on each interval $[x_{i-1}, x_i]$ we get

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx h\frac{f_{i-1} + f_i}{2}.$$

Summing for the n intervals we get

$$\int_a^b f(x)dx \approx \frac{b-a}{N}\Big(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-1} + \frac{f_N}{2}\Big). \qquad (2)$$

3.3 Composite Rules



3.4 Exercises

Exercise 3.1

Explain clearly, with an example, why in general it is not true that
$Q_n(f) \to \int_a^b f(x)dx$ as $n \to \infty$.

Exercise 3.2

(i) Deduce an error estimate for the Composite Trapezium Rule (2).

(ii) Taking $N = 10$, give an upper bound for the error in the Composite Trapezium Rule when approximating $\int_1^2 \ln(x)dx$.

(iii) What value of n would you have to take to ensure that the error was less that 10^{-5}?

Exercise 3.3

(i) Deduce the formula for the *composite Simpson's Rule*.

(ii) Derive an error estimate for the *composite Simpson's Rule*.

(iii) What value of N would you have to take to ensure that the error in the estimate of $\int_1^2 \ln(x)dx$ is less that 10^{-6}?

(iv) Denote the $(N+1)$-point Composite Simpson's Rule by $S_N(f) \approx \int_a^b f(x)dx$. Show that, for sufficiently smooth $f(x)$,

$$\lim_{n\to\infty} S_N(f) = \int_a^b f(x)dx.$$

Exercise 3.4 (Assignment)

Determine the precision of the following schemes for estimating $\int_0^1 f(x)dx$.

(i) $Q(f) = f(\frac{1}{2})$.

(ii) $Q(f) = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.

(iii) $Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{\mathbf{2}}{\mathbf{3}})$.

Exercise 3.5 (Assignment)

Consider the rule (which is not, strictly speaking, a Newton-Cotes rule):

$$R(f) = q_0 f(1/3) - f(\frac{1}{2}) + q_2 f(\frac{3}{4})$$

for approximating $\int_0^1 f(x)dx$.

1. Determine values of q_0 and q_2 that ensure this rule has precision 2.
2. What is the maximum precision of $R(\cdot)$ with the values of q_1 and q_2 that you have determined?