Annotated slides

2526-MA140: Week 03, Lecture 1 (L07)

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The Squeeze Theorem & one-sided limits

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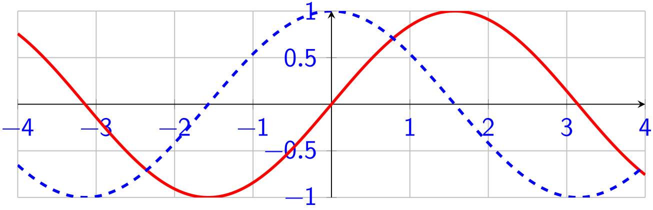
University of Galway

Tuesday, 30 September, 2025



Outline

1. News!
   - Assignments and Tutorials
   - Class test
2. Recall... the Squeeze Theorem
3. $\sin(\theta)/\theta$
   - Other examples
4. Infinite Limits
5. Digression: How fast can an object travel
6. One-sided Limits
   - Notation
   - Piecewise functions
   - Empty and full circle notation
   - Existence of a limit
7. Exercises

For more, see Section 2.2 (Limit of a Function) from **Calculus** by Gil Strang and Jed Herman, published by the non-profit OpenStax. See https://openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function

Reminder

- **Assignment 1** has a deadline of 5pm, Monday 6 October. You can access it on Canvas... 2526-MA140... Assignments. (Or directly, at this link).
- The **Tutorial Sheet** is available at https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1
- Assignment 2 is also open, with a deadline of 5pm, 13 Oct.

The first (of two) class tests will take place 2 weeks from now: Tuesday, 14th October.

- You will have 40 minutes to complete the test, which will be in the form of a Multiple Choice Test.
- Test will take place in one of ENG-G017 or ENG-G018.
- I need to gather information on Reasonable Accommodations for tests. If you want to avail of such, please complete this form: https://forms.office.com/e/HaAsrzaE3D by **10am Thursday 2nd Oct**.

Recall... the Squeeze Theorem

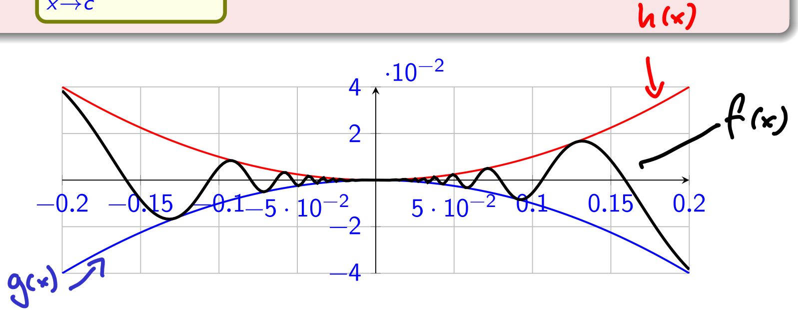
Last Thursday, we finished with...

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that for functions f, g and h in a given interval I:

$$g(x) \leqslant f(x) \leqslant h(x) \quad \textbf{and} \quad \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\boxed{\lim_{x \to c} f(x) = L.}$



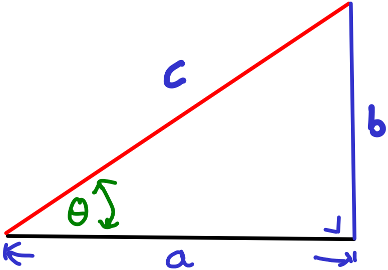
$\sin(\theta)/\theta$

Note $\sin(0) \approx 0$. So $\frac{\sin(\theta)}{\theta} \sim \frac{0}{0}$

We'll use the Squeese Theorem to explain that $\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$

First, we review some facts about trigonometric functions.

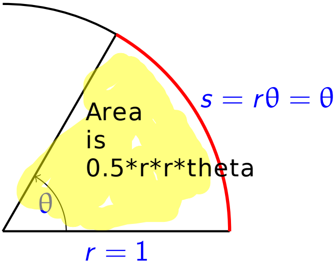
- **In this module, we only every use radians** (never degrees).
- Given the triangle drawn below, $\boxed{\sin \theta = \frac{b}{h}}$, $\cos \theta = \frac{a}{h}$, $\tan \theta = \frac{b}{a} = \frac{\sin \theta}{\cos \theta}$



$$\tan(\theta) = \frac{b}{a} = \frac{b}{h} \cdot \frac{h}{a}$$
$$= \sin(\theta) \cdot \frac{1}{\cos(\theta)}$$

$\sin(\theta)/\theta$

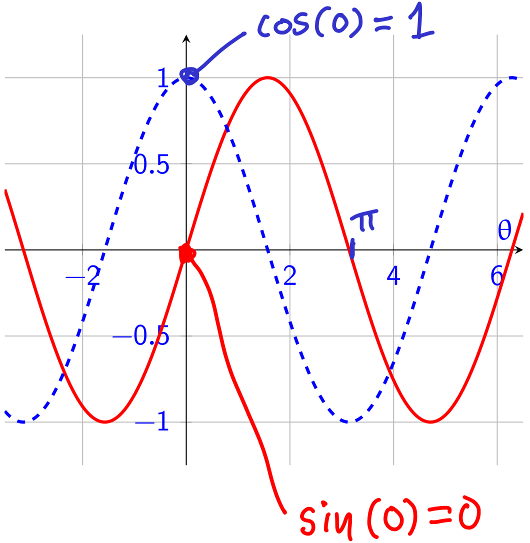
- The length of a sector, subtended by the angle θ, of a circle of radius r, is $s = r\theta$. In particular, for the unit circle **the angle (in radians) is the length of the arc.**.



- Area of a sector of a circle is $\frac{1}{2}r^2\theta$ where r is the radius of the circle, and θ is the angle subtended by the sector.

$\sin(\theta)/\theta$

- The $\sin$ (**red**) and $\cos$ (**blue**) functions look like this:



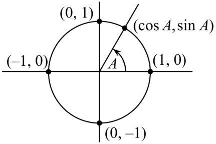
$\sin(\theta)/\theta$

Various other facts are summarised in the State Examination Commission's Tables:

Triantánacht **Trigonometry**

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$

$$\sin(-A) = -\sin A$$

$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	$0°$	$90°$	$180°$	$270°$	$30°$	$45°$	$60°$	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. $\approx 57.296°$ $\qquad$ $1° \approx 0.01745$ rad.

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$\sin(\theta)/\theta$

Foirmlí uillinneacha comhshuite — **Compound angle formulae**

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Foirmlí uillinneacha dúbailte — **Double angle formulae**

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

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$\sin(\theta)/\theta$

Iolraigh a thiontú ina suimeanna agus ina ndifríochtaí **Products to sums and differences**

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

Suimeanna agus difríochtaí a thiontú ina n-iolraigh **Sums and differences to products**

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

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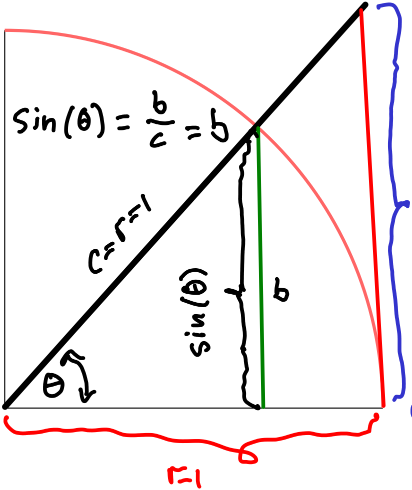
The radius of this circle is $r=1$

To show that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



- From this we should see that the length of the arc is θ, $\tan(\theta)$. and that

$$\sin(\theta) \le \theta \le \tan(\theta).$$

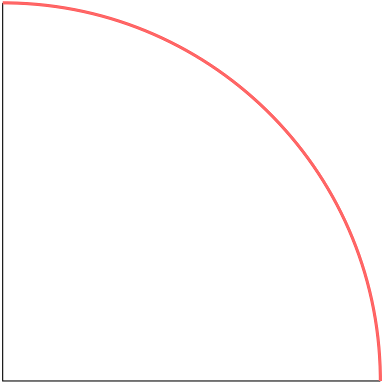
so $\sin(\theta) \le \theta \le \dfrac{\sin(\theta)}{\cos(\theta)}$

So, dividing by $\sin(\theta)$ we get

$$1 \le \frac{\theta}{\sin(\theta)} \le \frac{1}{\cos(\theta)}.$$

$\sin(\theta)/\theta$

To show that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$



Inverting the inequality we get

$$1 \geq \frac{\sin(\theta)}{\theta} \geq \cos(\theta).$$

So

$$\lim_{\theta\to 0} 1 = 1$$

$$\lim_{\theta\to 0} \cos(\theta) = \cos(0) = 1$$

So, by the Squeeze Thm, $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1.$

Example

Evaluate $\lim_{x\to 0} \dfrac{\tan(3x)}{\sin(2x)}$

First note $\tan(3x) = \dfrac{\sin(3x)}{\cos(3x)}$.

So $\dfrac{\tan(3x)}{\sin(2x)} = \dfrac{\sin(3x)}{1} \dfrac{1}{\cos(3x)} \cdot \dfrac{1}{\sin(2x)}$.

So $\lim_{x\to 0} \dfrac{\tan(3x)}{\sin(2x)} = \lim_{x\to 0} \left(\dfrac{\sin(3x)}{3x} \cdot \dfrac{1}{\cos(3x)} \cdot \dfrac{2x}{\sin(2x)} \cdot \dfrac{3}{2} \right)$

$= \lim_{x\to 0} \dfrac{\sin(3x)}{3x} \cdot \lim_{x\to 0} \dfrac{1}{\cos(3x)} \cdot \lim_{x\to 0} \dfrac{2x}{\sin(2x)} \cdot \dfrac{3}{2}$

$= (1) \times (1) \times (1) \times \left(\dfrac{3}{2}\right) = \dfrac{3}{2}$.

Example

Evaluate $\displaystyle\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}$

Try this yourself. .

Tip: $(1-\cos(\theta))(1+\cos(\theta))$

$= 1-\cos^2(\theta) = \sin^2(\theta)$

Notation $\cos^2(x)$ means $[\cos(x)]^2$

"cos squared x".

Infinite Limits

So far, we've had lots of examples that are a little like:

$$\lim_{x \to 1} \frac{x^3 - x^2 - x + 1}{(x-1)^2} = 2.$$

(Check that this is correct).

But what about

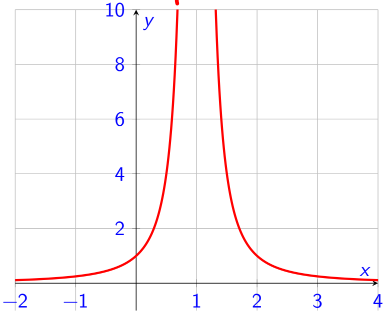
$$\lim_{x \to 1} \frac{1}{(x-1)^2} = ???$$

Let's plot it and see:

Infinite Limits

∞

$$f(x) = \frac{1}{(x-1)^2}$$



As x get closer and closer to 1, the value of $f(x)$ gets larger and larger. In fact, it becomes infinite.

For this we write

$$\boxed{\lim_{x \to 1} f(x) = \infty.}$$

Digression: How fast can an object travel

- Q: Is there any limit to the speed at which an object can travel?
- A: Yes! (Assuming you believe Einstein)

Thanks to Einstein ($E = mc^2$), Lorenz and others, it is known that the mass of a moving charged particle behaves like

$$m(v) = m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is its mass at rest, c is the speed of light, and v is the particles current speed. What happens as $v \to c$?

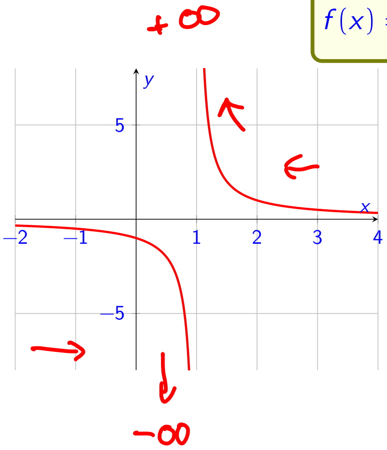
$$\lim_{v \to c} m(v) = \lim_{v \to c} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \to \frac{m_0}{\sqrt{1-1}} \Rightarrow \frac{m_0}{0}$$

$$\to \infty.$$

One-sided Limits

Let's consider a motivating example, very similar to the one where we introduced ∞.

$$\boxed{f(x) = \frac{1}{x-1}}$$



As x get closer and closer to 1, then $f(x) \to -\infty$ or $f(x) \to \infty$, depending on whether x approaches 1 from the left or right.

To express this, we need the concept of a **one-sided limit**

One-sided Limits

$\lim\limits_{x\to a^-} f(x)$ is: **limit of f as x approaches a from the left**

$\lim\limits_{x\to a^+} f(x)$ is: **limit of f as x approaches a from the right**

In the previous example, with $f(x) = \dfrac{1}{x-1}$, we have

- $\lim\limits_{x\to 1^-} f(x) = -\infty$
- $\lim\limits_{x\to 1^+} f(x) = \infty$

$\lim\limits_{x\to a^-} f(x)$

$\lim\limits_{x\to a^+} f(x)$

In many important examples, we encounter functions that have different definitions in different regions. The most classic example is the **absolute value function**:

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0. \end{cases}$$

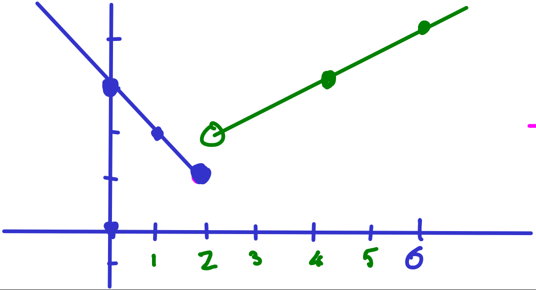
Care has to be taken when evaluating the limits of such functions....

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x \leqslant 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$$

Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.



$f(0) = 3 - 0 = 3$

$f(1) = 3 - 1 = 2$

$f(2) = 3 - 2 = 1$

$f(4) = 3$

$f(6) = 4$

Empty and Full Circle Notation:

In the previous sketch, we use the convention that

- If the end point of a line segment is **not** included in its definition, it terminates with an **open circle**, $\circ$
- If the end point of a line segment **is** included in its definition, it terminates with an **closed circle** $\bullet$.

Finished here Tuesday

Existence of a limit

$\lim_{x \to a} f(x)$ **exists** if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

So if $\lim_{x \to a} f(x) = L$ exists, we have

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = L$$

though it is not necessary that $f(a) = L$

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if $\lim_{x \to 2} f(x)$ exists.

Exercises

Exercise 3.1.1 (from 2023/24 Q1(b))

Evaluate

$$\lim_{\theta\to 0} \frac{2\sin(\theta)}{\theta + 3\tan(\theta)}$$