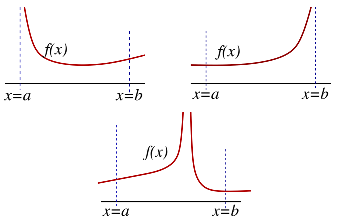
MA211

Lecture 19: Improper Integrals

Wed 12th Nov 2008

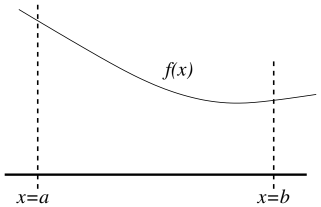


Proper Integrals

So far, the definite integrals we have considered:

$$\int_a^b f(x)dx,$$

have all been *Proper*: they are integrals of bounded functions on closed, finite intervals.

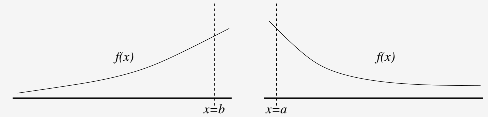


So we when we think of the integral as the area between the graph of the function and the x-axis, it is clear that that is well-defined.

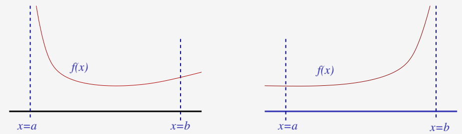
Improper Integrals

A definite integral $\int_a^b f(x)dx$ is *Improper* if:

Type I: if $a = -\infty$ or $b = \infty$



Type II: if $f(x)$ is unbounded (infinite) near a or b.



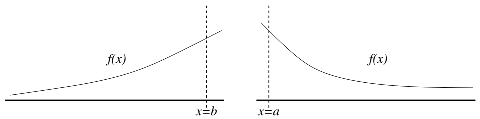
Improper Integrals

- *Some* improper integrals evaluate as a real, finite number. These are are said to **converge**, or to be *convergent* or **to exist**.
- Those that don't evaluate to a finite number are said to **diverge**, or to be *divergent* or **not to exist**.

Improper Integrals of Type I

Improper Integrals of Type I are of the form

$$\int_a^\infty f(x)\,dx \quad \text{or} \quad \int_{-\infty}^b f(x)\,dx.$$

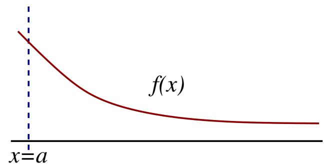


To evaluate these, note that $\int_a^\infty f(x)dx = \lim_{t=\infty} \int_a^t f(x)dx$. So:

- Evaluate $\mathcal{I}(t) = \int_a^t f(x)dx$;
- and then compute $\lim_{t=\infty} \mathcal{I}(t)$.

Improper Integrals of Type I $\int_a^\infty f(x)dx$

$$\int_a^\infty f(x)dx$$



1. Evaluate $\mathcal{I}(t) = \int_a^t f(x)dx$;
2. and then compute $\mathcal{I} = \lim_{t=\infty} \mathcal{I}(t)$.
3. If the limit exists, call it L and write $\int_a^\infty f(x)\,dx = L$. We say that $\int_a^\infty f(x)\,dx$ **converges to** L.
4. If no such limit exists, $\int_a^\infty f(x)\,dx$ is said to **diverge**.

Example

Evaluate $\mathcal{I} = \int_1^\infty \frac{1}{x^2}dx$

Example

Evaluate the improper integral $\mathcal{I} = \int_1^\infty \frac{dx}{x}$

Example

Evaluate $\mathcal{I} = \int_1^\infty \frac{1}{\sqrt{x}}dx$

$\int_1^\infty 1/x^p\,dx$ converges for $p > 1$, and diverges for $p \leq 1$.

Proof: If $p = 1$ then

$$\int_1^t x^{-p}dx = \int_1^t \frac{1}{x}dx = \ln(x)\Big|_1^t = \ln(t) - \ln(1) = \ln(t).$$

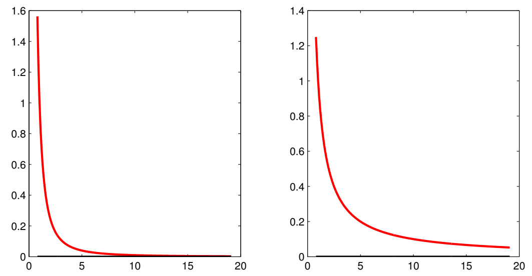
But $\lim_{t\to\infty} \ln(t)$ does not exists, so $\int_1^t \frac{1}{x}dx$ diverges.

If $p \neq 1$ then $\int_1^t x^{-p}dx = \frac{x^{1-p}}{1-p}\Big|_1^t = \frac{t^{1-p}-1}{1-p}$.

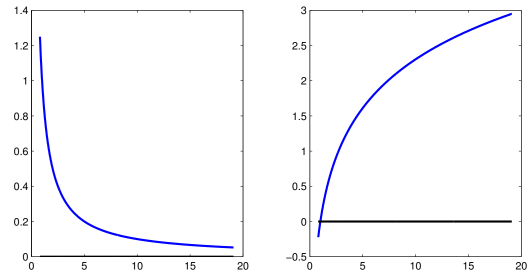
If $p < 1$ then $1 - p > 0$ so the limit $\lim_{t\to\infty} t^{1-p}$ does not exist, so the integral diverges in that case.

If however $p > 1$ then $1 - p < 0$ and $\lim_{t\to\infty} t^{1-p} = 0$, so the integral converges to $\frac{-1}{1-p}$.

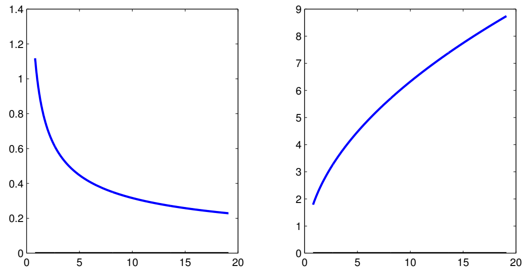
Example: $\int_a^\infty x^{-2}dx$



Example: $\int_a^\infty x^{-1}dx$



Example: $\int_a^\infty x^{-1/2} dx$

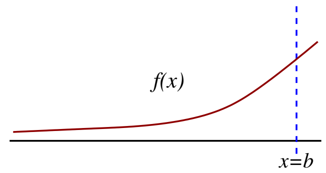


Example

Evaluate the integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}dx$

For problems of the form:

$$\int_{-\infty}^b f(x)dx$$



1. Evaluate $\mathcal{I}(t) = \displaystyle\int_t^b f(x)dx$;
2. and then compute $\mathcal{I} = \displaystyle\lim_{t=-\infty} \mathcal{I}(t)$.
3. If the limit exists, call it $\mathcal{I}$ and write $\int_{-\infty}^b f(x)\,dx = L$. We say that the integral **converges to** L.
4. If no such limit exists, it is said to **diverge**.

Example

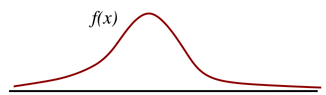
Evaluate $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2}$

Example

Show that $\displaystyle\int_{-\infty}^0 e^x dx$ converges, but that $\displaystyle\int_0^\infty e^x dx$ diverges.

We also have to deal with the case where *both* limits of integration are at infinity:

$$\int_{-\infty}^\infty f(x)\,dx$$



To do this we recall that

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx.$$

So $\displaystyle\int_{-\infty}^\infty f(x)dx$ converge if and only if *both* $\displaystyle\int_{-\infty}^0 f(x)\,dx$ and $\displaystyle\int_0^\infty f(x)\,dx$ converge.

Improper Integrals of Type I $\int_{-\infty}^{\infty} f(x)dx$

Example

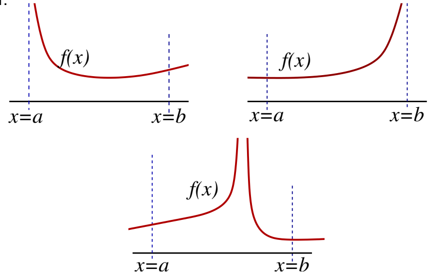
Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$

Improper Integrals: Type 2

Finally we consider integrals of the form

$$\int_a^b f(x)\,dx$$

where $f(x)$ may be unbounded at a or b, or at some point in between.



Improper Integrals: Type 2

$f(x)$ unbounded at $x = a$

When function $f(x)$ is defined for $a < x \leq b$ then evaluate $\mathcal{I}(t) = \displaystyle\int_t^b f(x)dx$ and then use that:

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx.$$

So:

1. Evaluate $\mathcal{I}(t) = \displaystyle\int_t^b f(x)dx$
2. Compute the limit $L = \displaystyle\lim_{t\to a^+} \mathcal{I}(t)$
3. If L is finite then $\displaystyle\int_a^b f(x)\,dx = L$, and we can say that $\int_a^b f(x)\,dx$ **converges to** L.
4. If L is *not* finite, then integral is said to diverge.

Improper Integrals: Type 2

Example

Does the integral $\displaystyle\int_0^1 \frac{dx}{x}$ converge?

Improper Integrals: Type 2

Example

Evaluate the improper integral $\displaystyle\int_0^1 \frac{dx}{x^2}$

Improper Integrals: Type 2

$\displaystyle\int_0^1 x^{-p}dx$ will *converge* when $p < 1$, and **diverge** for $p \geq 1$.

Proof: If $p = 1$ then

$\displaystyle\int_t^1 x^{-p}dx = \int_t^1 \frac{1}{x}dx = \ln(x)\Big|_t^1 = \ln(t) - \ln(1) = \ln(t).$

But $\displaystyle\lim_{t\to 0} \ln(t)$ does not exists, so $\displaystyle\int_0^1 \frac{1}{x}dx$ diverges.

If $p \neq 1$ then $\displaystyle\int_t^1 x^{-p}dx = \frac{1}{1-p}x^{1-p}\Big|_t^1 = \frac{1-t^{1-p}}{1-p}.$

If $p < 1$ then $1 - p > 0$ so the limit $\displaystyle\lim_{t\to 0} t^{1-p} = 0$. So the integral converges to $\dfrac{1}{1-p}$.

If however $p > 1$ then $1 - p < 0$ and $\displaystyle\lim_{t\to 0} t^{1-p}$ does not exist, so the integral **diverges**.

Improper Integrals: Type 2

If f is defined on $[a, b)$ and $\lim_{t \to b^-} \int_a^t f(x)\,dx$ exists, call the limit L and write

$$\int_a^b f(x)\,dx = L.$$

Again, $\int_a^b f(x)\,dx$ is said to **converge to** L. If no such limit exists, the integral is divergent.

Improper Integrals: Type 2

Example

Does the $\int_0^4 \frac{dx}{\sqrt{4-x}}$ converge or diverge?

Improper Integrals: Type 2

If a function f is defined on $[a, b]$ except at some point c in (a, b) at which f is *unbounded*, then use that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The integral converges if and only if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ **both** converge.

Example

Does the improper integral $\int_{-1}^1 \frac{dx}{x}$ converge or diverge?

The Comparison Test

Often, we just want to know if some integral converges or diverges – and not necessarily evaluate the integral.

In that case we can compare the integral with one that we know. This is helpful because:

Comparison Test

Suppose f and g are defined on $[a, \infty)$ and

$$0 \leq f(x) \leq g(x) \text{ for all } x \in [a, \infty).$$

$$\text{Then } \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx. \text{ Therefore}$$

1. If $\int_a^\infty g(x)\,dx$ **converges**, so does $\int_a^\infty f(x)\,dx$
2. if $\int_a^\infty f(x)\,dx$ *diverges*, so does $\int_a^\infty g(x)\,dx$

The Comparison Test

Comparison Test

Suppose f and g are defined on $[a, \infty)$ and

$$0 \leq f(x) \leq g(x) \text{ for all } x \in [a, \infty).$$

Then

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx.$$

Therefore

1. If $\int_a^\infty g(x)\,dx$ **converges**, so does $\int_a^\infty f(x)\,dx$
2. if $\int_a^\infty f(x)\,dx$ *diverges*, so does $\int_a^\infty g(x)\,dx$

There are corresponding results for the other types of improper integrals.

The Comparison Test

Example

Does the integral $\int_1^\infty \frac{dx}{x^2 + x^3}$ converge or diverge?

The Comparison Test

Example

$$\int_1^{\infty} \frac{dx}{x+x^2}$$

The Comparison Test

Example

Does the improper integral $\int_0^1 \frac{dx}{2x^2+3x^3}$ converge or diverge?

The Comparison Test

Example

Establish if $\int_0^1 \frac{dx}{2\sqrt{x}+x^2}$ is convergent or divergent.

The Comparison Test

Example

Test for convergence of the following integral:

$$\int_1^{\infty} \frac{\cos x \, dx}{1+x^2}$$