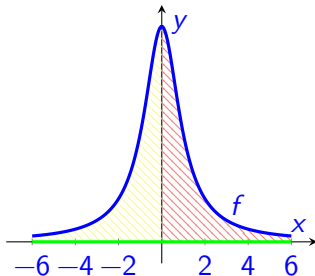


Week 08, Lecture 3
Areas between Curves, and Improper Integrals

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Between now at 10.50...

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| <ul style="list-style-type: none">1 Recall: Areas Between Curves<ul style="list-style-type: none">■ Finding a and b2 Compound Regions3 Improper Integrals<ul style="list-style-type: none">■ Motivation: Areas (again) | <ul style="list-style-type: none">■ Definitions■ Example (convergent)■ Example (divergent)■ Convergent or Divergent?■ Last example4 Exercises |
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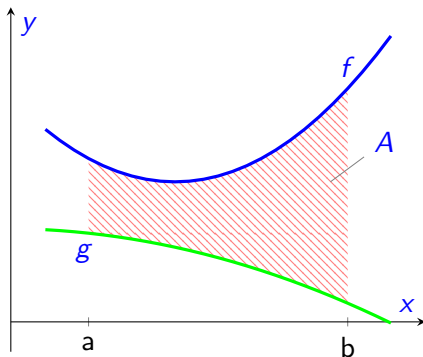
For more: Section 6.1 (Areas between Curves) and Section 7.7 (Improper Integrals) in the textbook:

[math.libretexts.org/Bookshelves/Calculus/Calculus_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

Recall: Areas Between Curves

If $f(x) \geq g(x)$ for $x \in [a, b]$, the area of the region between $x = a$, and $x = b$, and between $y = g(x)$ and $y = f(x)$ is

$$\begin{aligned} A &= \int_a^b f(x) dx - \int_a^b g(x) dx \\ &= \int_a^b (f(x) - g(x)) dx. \end{aligned}$$



Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect. That is, we have to find a and b .

Example (from Q5(a) of 2024/2025 Exam paper)

Compute the region bounded by the curves $f(x) = 3x + 4$ and the $g(x) = 2x^2 + 2x + 1$.

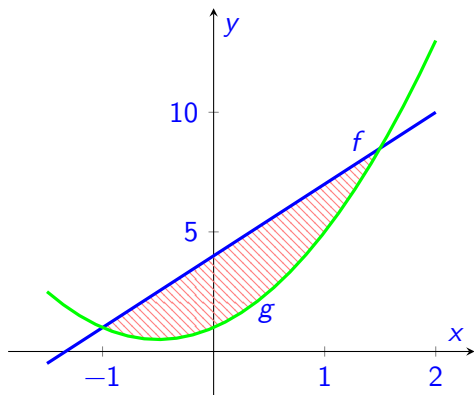
First we need to find the points where $f(x)$ and $g(x)$ intersect. That is, we solve $f(x) = g(x)$:

$$\begin{aligned}(3x + 4) - (2x^2 + 2x + 1) &= 0 \\ \implies -2x^2 + x + 3 &= 0 \\ \implies -2(x + 1)(x - 3/2) &= 0 \quad (1)\end{aligned}$$

So they intersect at $x = -1$ and $x = 3/2$.
(Continued)

So the area is given by

$$\begin{aligned} \int_{-1}^{3/2} f(x) - g(x) dx \\ &= \int_{-1}^{3/2} -2x^2 + x + 3 dx \\ &= \left(-\frac{2}{3}x^3 + \frac{1}{2}x^2 + 3x \right) \Big|_{-1}^{3/2} \\ &= \left(-\frac{2}{3}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right) \right) - \left(-\frac{2}{3}(-1) + \frac{1}{2}(1) + 3(-1) \right) \\ &= 125/24. \end{aligned}$$



Compound Regions

In the previous examples, we had $f(x) \geq g(x)$ for all $x \in [a, b]$.
But what if f and g cross in the domain?

Areas between curves, without $f(x) \geq g(x)$

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$.
Then A , the area of the region between the graphs of $f(x)$ and $g(x)$, and between $x = a$ and $x = b$, is given by

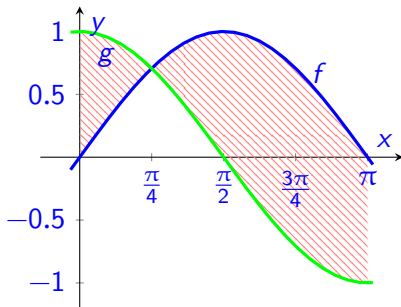
$$A = \int_a^b |f(x) - g(x)| dx.$$

In practice this involves finding the point c where the functions cross...

Compound Regions

Example [See Eg 6.1.3 in textbook]

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$, from $x = 0$ to $x = \pi$.



Compound Regions

It will help to consult p13 of the “log” tables.

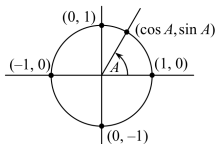
Triantánacht

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$

$$\sin(-A) = -\sin A$$

$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.

Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.

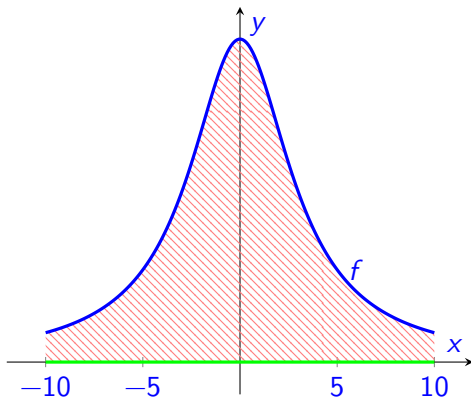
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	0°	90°	180°	270°	30°	45°	60°	A (degrees)
A (raidian)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

$$1 \text{ rad.} \approx 57.296^\circ$$

$$1^\circ \approx 0.01745 \text{ rad.}$$

Earlier we looked at how $\int_a^b f(x) dx$ evaluates as the area of the region between $y = f(x)$ and $y = 0$, and between $x = a$ and $x = b$. But suppose we want to evaluate the area of the region between $y = f(x)$ and $y = 0$, and between (say) $x = -\infty$ and $x = \infty$?



So far we have dealt with the definite integral $\int_a^b f(x) dx$ for a continuous function f on a finite interval $[a, b]$, i.e. where a and b are both real numbers.

But sometimes the region in which we are interested is over an **infinite** interval, i.e. an interval of the form $[a, \infty)$, $(-\infty, b]$ or $(-\infty, \infty)$.

Let's consider how we might try to define an **improper integral** such as

$$\int_a^{\infty} f(x) dx .$$

Definition (Improper Integral)

Let f be a continuous function on $[a, \infty)$. The **improper integral of f over $[a, \infty)$** is defined by

$$\int_a^{\infty} f(x) \, dx = \lim_{t \rightarrow \infty} \int_a^t f(x) \, dx,$$

provided that the limit exists. If so, we say that the integral **convergent**. Otherwise, we say it is **divergent**.

Similarly, if $g(x)$ is continuous $(-\infty, b]$, the improper integral $\int_{-\infty}^b g(x) \, dx$ is **convergent** and given by

$$\int_{-\infty}^b g(x) \, dx = \lim_{t \rightarrow -\infty} \int_t^b g(x) \, dx$$

if that the limit exists; otherwise it is **divergent**.

Furthermore:

If f is a continuous function on $\mathbb{R} = (-\infty, \infty)$ and the improper integrals

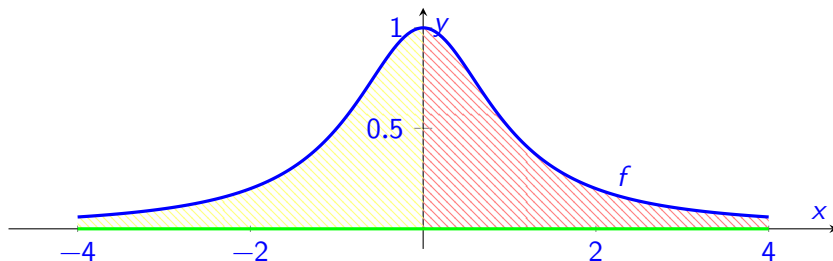
$$\int_{-\infty}^0 f(x) \, dx \quad \text{and} \quad \int_0^{\infty} f(x) \, dx$$

are both convergent, then the improper integral

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^0 f(x) \, dx + \int_0^{\infty} f(x) \, dx.$$

is also **convergent**. If not, we say it is **divergent**.

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^0 f(x) \, dx + \int_0^{\infty} f(x) \, dx.$$



Example

Evaluate $\int_1^{\infty} \frac{1}{x^2} dx$.

Idea: Use the definition:

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx,$$

That is: set $g(t) = \int_1^t f(x) dx$ and then evaluate $\lim_{t \rightarrow \infty} g(t)$

Many improper integrals are divergent. Examples:

If $f(x)$ is a positive function, for $\int_a^\infty f(x) dx$ to exist, at the very least we need $f(x)$ to be a decreasing function. But often that alone is not enough!

- ▶ We know that $\int_1^\infty x^{-2} dx$ is convergent.
- ▶ From that we can deduce that $\int_1^\infty x^{-n} dx$ is convergent for any $n \geq 2$. (Why?)
- ▶ And we know $\int_1^\infty x^0 dx$ is divergent.
- ▶ But what about $\int_1^\infty x^{-1} dx$?

Example

Determine whether the improper integral $\int_1^{\infty} \frac{1}{x} dx$ is convergent or divergent.

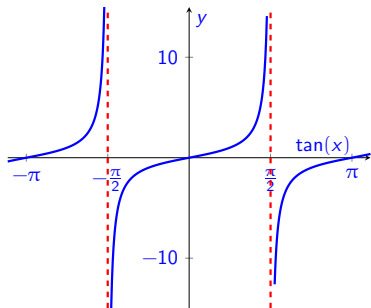
For $t \geq 1$, we have $\int_1^t \frac{1}{x} dx = \ln(x) \Big|_1^t = \ln(t)$.

Since $\lim_{t \rightarrow \infty} \ln(t)$ does not exist, it follows that $\int_1^{\infty} \frac{1}{x} dx$ is divergent.

In our next, and final example, we'll try to integrate

$f(x) = \frac{1}{1+x^2}$. To follow the solution, you might find it useful to revise the fundamentals of **inverse trigonometric functions**. You can find that in Section 1.4 of the textbook:

[math.libretexts.org/Bookshelves/Calculus/Calculus_\(OpenStax\)](http://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))



In the figure opposite, we see the graph of $\tan(x)$. Notice that it has vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$.

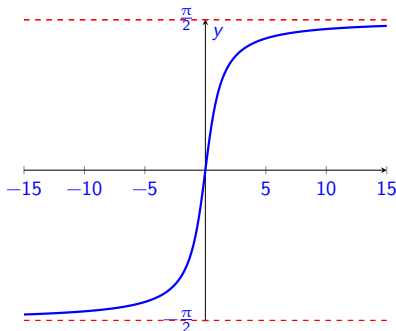
On the right is a plot of the **inverse of the $\tan(x)$** function, which is often written as either $\tan^{-1}(x)$ or $\arctan(x)$. Notice that it has **horizontal** asymptotes at $y = -\pi/2$ and $y = \pi/2$.

This means that

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2},$$

and

$$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$



Example

Evaluate the improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

Exercises

Exer 8.3.1 (From 2019/2020 exam)

The functions $f(x) = 1/x$ and $g(x) = x^2$ intersect at $x = 1$. Calculate the area between their graphs on $[1, 2]$

Exer 8.3.2 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Exer 8.3.3 (From 23/24 exam)

Find the area bounded by the curves $f(x) = x^2 - 4x$ and $g(x) = 2x - 5$.

Exercises

Exer 8.3.4 (From 23/24 exam)

Evaluate $\int_0^{\infty} \frac{x}{1+x^4} dx$ (Hint: try substitution with $u = x^2$).