Please submit carefully written solutions to the following exercises: Exercises 2.7 and 2.14 from this set, as well as 3.12 and 3.15 from Section 3.

Chapter 2: Initial Value Problems

Exercise 2.1. For the following functions show that they satisfy a Lipschitz condition on the corresponding domain, and give an upper-bound for L:

(i) $f(t, y) = 2yt^{-4}$ for $t \in [1, \infty)$,

(ii) $f(t, y) = 1 + t\sin(ty)$ for $0 \leqslant t \leqslant 2$.

Exercise 2.2. Many text books, instead of giving the version of the Lipschitz condition we use, give the following: *There is a finite, positive, real number* L *such that*

$$|\frac{\partial}{\partial y} f(t, y)| \leqslant L \qquad \text{for all } (t, y) \in D.$$

Is this statement *stronger than* (i.e., more restrictive then), *equivalent to* or *weaker than* (i.e., less restrictive than) the usual Lipschitz condition? Justify your answer.
Hint: the Wikipedia article on Lipschitz continuity is very informative.

Exercise 2.3. As a special case in which the error of Euler's method can be analysed directly, consider Euler's method applied to

$$y'(t) = y(t), \quad y(0) = 1.$$

The true solution is $y(t) = e^t$.

(i) Show that the solution to Euler's method can be written as

$$y_i = (1 + h)^{t_i/h}, \quad i \geqslant 0.$$

(ii) Show that

$$\lim_{h \to 0} (1 + h)^{1/h} = e.$$

This then shows that, if we denote by $y_n(T)$ the approximation for $y(T)$ obtained using Euler's method with n intervals between t_0 and T, then

$$\lim_{n \to \infty} y_n(T) = e^T.$$

Hint: Let $w = (1 + h)^{1/h}$, so that $\log w = (1/h)\log(1 + h)$. Now use l'Hospital's rule to find $\lim_{h \to \mathbf{0}} w$.

Exercise 2.4. An important step in the proof of Theorem 2.3.3, but which we didn't do in class, requires the observation that if $|\mathcal{E}_{i+1}| \leqslant |\mathcal{E}_i|(1 + hL) + h|T_i|$, then

$$|\mathcal{E}_i| \leqslant \frac{T}{L}\big[(1 + hL)^i - 1\big] \qquad i = 0, 1, \ldots, N.$$

Use induction to show that is indeed the case.

Exercise 2.5. Suppose we use Euler's method to find an approximation for $y(2)$, where y solves

$$y(1) = 1, \qquad y' = (t - 1)\sin(y).$$

(i) Give an upper bound for the global error taking $n = 4$ (i.e., $h = 1/4$).

(ii) What n should you take to ensure that the global error is no more that 10^{-3}?

Exercise 2.6. A popular RK2 method, called the *Improved Euler Method*, is obtained by choosing $\alpha = 1$.

(i) Use the Improved Euler Method to find an approximation for $y(4)$ when

$$y(0) = 1, \qquad y' = y/(1 + t^2),$$

taking $n = 2$. (If you wish, use MATLAB.)

(ii) Using a diagram similar to the one used to motivate the Modified Euler Method, justify the assertion that the Improved Euler Method is more accurate than the basic Euler Method.

(iii) Show that the method is consistent.

(iv) Write out what this method would be for the problem: $y'(t) = \lambda y$ for a constant λ. How does this relate to the Taylor series expansion for $y(t_{i+1})$ about the point t_i?

Exercise 2.7 ($\star$)**.** In his seminal paper of 1901, Carl Runge gave an example of what we now call a *Runge-Kutta 2 method*, where

$$\Phi(t_i, y_i; h) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\big).$$

(i) Show that it is consistent.

(ii) Show how this method fits into the general framework of RK2 methods. That is,

  (a) What are a, b, α, and β?

  (b) Do they satisfy the conditions

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b?$$

(iii) Use it to estimate the solution at the point $t = 2$ to $y(1) = 1$, $y' = 1 + t + y/t$ taking $n = 2$ time steps.

Exercise 2.8. We claim that, for RK4:

$$|\mathcal{E}_N| = |y(t_N) - y_N| \leqslant Kh^4.$$

for some constant K. How could you verify that the statement is true using the data of Table 2.3, at least for test problem in Example 2.4.2? Give an estimate for K.

Exercise 2.9. Recall the problem in Example 2.2.2: *Estimate* $y(2)$ *given that*

$$y(1) = 1, \qquad y' = f(t, y) := 1 + t + \frac{y}{t},$$

(i) Show that $f(t, y)$ satisfies a Lipschitz condition and give an upper bound for L.

(ii) Use Euler's method with $h = 1/4$ to estimate $y(2)$. Using the true solution, calculate the error.

(iii) Repeat this for the RK2 method of your choice (with $a \neq 0$) taking $h = 1/2$.

(iv) Use RK4 with $h = 1$ to estimate $y(2)$.

Exercise 2.10. Here is the tableau for a three stage Runge-Kutta method:

$$\begin{array}{c||ccc} 0 & & & \\ \alpha_2 & 1/2 & & \\ 1 & \beta_{31} & 2 & \\ \hline\hline & 1/6 & b_2 & 1/6 \end{array}$$

(i) Use that the method is consistent to determine b_2.

(ii) The method is exact when used to compute the solution to

$$y(0) = 0, \quad y'(t) = 2t, \ t > 0.$$

Use this to determine α_2.

(iii) The method should agree with an appropriate Taylor series for the solution to $y'(t) = \lambda y(t)$, up to terms that are $\mathcal{O}(h^3)$. Use this to determine β_{31}.

Exercise 2.11. Write down the Euler Method for the following 3rd-order IVP

$$y''' - y'' + 2y' + 2y = x^2 - 1,$$
$$y(0) = 1, y'(0) = 0, y''(0) = -1.$$

Exercise 2.12. Use a Taylor series to provide a derivation for the formula

$$\frac{\partial^2 u}{\partial x^2}(t_i, x_j) \approx \frac{1}{H^2}\big(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\big).$$

Exercise 2.13. Suppose that a 3-stage Runge-Kutta method tableaux has the following entries:

$$\alpha_2 = \frac{1}{3},\ \alpha_3 = \frac{1}{9},\ b_1 = 4,\ b_2 = \frac{15}{4},\ \beta_{32} = -\frac{2}{27}.$$

(i) Assuming that the method is *consistent*, determine the value of b_3.

(ii) Consider the initial value problem:

$$y(0) = 1,\ y'(t) = \lambda y(t).$$

Using that the solution is $y(t) = e^{\lambda t}$, write out a Taylor series for $y(t_{i+1})$ about $y(t_i)$ up to terms of order h^4 (use that $h = t_{i+1} - t_i$).

Using that your method should agree with the Taylor Series expansion up to terms of order h^3, determine β_{21} and β_{31}.

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Here are some entries for 3-stage Runge-Kutta method tableaux for Exercise 2.14.

Method 0: $\alpha_2 = 2/3$, $\alpha_3 = 0$, $b_1 = 1/12$, $b_2 = 3/4$, $\beta_{32} = 3/2$

Method 1: $\alpha_2 = 1/4$, $\alpha_3 = 1$, $b_1 = -1/6$, $b_2 = 8/9$, $\beta_{32} = 12/5$

Method 2: $\alpha_2 = 1/4$, $\alpha_3 = 1/2$, $b_1 = 2/3$, $b_2 = -4/3$, $\beta_{32} = 2/5$

Method 3: $\alpha_2 = 1/4$, $\alpha_3 = 1/3$, $b_1 = 3/2$, $b_2 = -8$, $\beta_{32} = 4/45$

Method 4: $\alpha_2 = 1$, $\alpha_3 = 1/4$, $b_1 = -1/6$, $b_2 = 5/18$, $\beta_{32} = 3/16$

Method 5: $\alpha_2 = 1$, $\alpha_3 = 1/5$, $b_1 = -1/3$, $b_2 = 7/24$, $\beta_{32} = 4/25$

Method 6: $\alpha_2 = 1$, $\alpha_3 = 1/6$, $b_1 = -1/2$, $b_2 = 3/10$, $\beta_{32} = 5/36$

Method 7: $\alpha_2 = 1/2$, $\alpha_3 = 1/7$, $b_1 = 7/6$, $b_2 = 22/15$, $\beta_{32} = -10/49$

Method 8: $\alpha_2 = 1/2$, $\alpha_3 = 1/8$, $b_1 = 4/3$, $b_2 = 13/9$, $\beta_{32} = -3/16$

Method 9: $\alpha_2 = 1/3$, $\alpha_3 = 1/9$, $b_1 = 4$, $b_2 = 15/4$, $\beta_{32} = -2/27$

Exercise 2.14 (Your own RK3 method $\star$)**.** Answer the following questions for Method K from the list above, where K is the last digit of your ID number. For example, if your ID number is 01234567, use Method 7.

(a) Assuming that the method is *consistent*, determine the value of b_3.

(b) Consider the initial value problem:

$$y(0) = 1,\ y'(t) = \lambda y(t).$$

Using that the solution is $y(t) = e^{\lambda t}$, write out a Taylor series for $y(t_{i+1})$ about $y(t_i)$ up to terms of order h^4 (use that $h = t_{i+1} - t_i$).

Using that your method should agree with the Taylor Series expansion up to terms of order h^3, determine β_{21} and β_{31}.

Exercise 2.15. (Attempt this exercises after completing Lab 3). Write a MATLAB program that implements your method from Exercise 2.14.

Use this program to check the order of convergence of the method. Have it compute the error for $n = 2$, $n = 4$, $\ldots$, $n = 1024$. Then produce a log-log plot of the errors as a function of n.