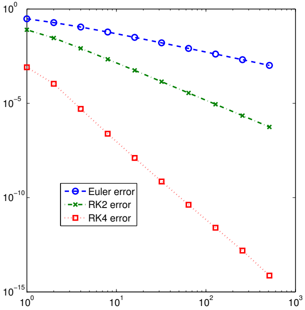
Initial Value Problems

§2.5 Runge-Kutta 4

MA385/530 – Numerical Analysis 1

October 2019

Annotated slides -- with annotations added long after the lecture.
So there may be some differences between these notes and those written in class.



It is possible to construct methods that have higher orders of accuracy than **RK2** methods. Of these, the most used are probably those that belong to the **Runge-Kutta 4 (RK4)** family, and have the property that

$$|y(t_n) - y_n| \leq Ch^4.$$

However, even writing down the general form of the RK4 method, and then deriving conditions on the parameters is rather complicated. Therefore, we'll focus on just one RK4 method, and use examples, rather than theory, to demonstrate that it is 4th-order.

"The RK4 Method"

$$
\begin{aligned}
k_1 &= f(t_i, y_i),\\
k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1),\\
k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2),\\
k_4 &= f(t_i + h, y_i + hk_3),\\
y_{i+1} &= y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}
$$

So, the method is $y_{i+1} + h\,\Phi(t_i, y_i; h)$

with $\Phi(t_i, y_i; h) = \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4$

The RK4 method can be interpreted as follows :

k_1 is the same as for Euler's method: it approximates the slope of the tangent to the solution curve at t_i.

Then k_2 uses this to approximate the slope at $t_i + h/3$.

k_3 refines this value, using k_2 instead of k_1.

k_4 uses k_3 to estimate the slope of the tangent at t_{i+1}. Then take a weighted average

As the following example shows, RK4 can be much more accurate than the Euler or RK2 methods for small h (i.e., large n). For the RK4, doubling n reduces the error by a factor of 8 (compared with 2 and 4 for the Euler and RK2 methods, respectively).

Example 2.12 (2.11 (again))

Compare Euler, Modified Euler, and RK4 for approximating $y(1)$ where: $y(0) = 1, \quad y'(t) = y \log(1 + t^2)$.

Notice that, for RK4 if we double n, the Error reduces by a factor of 8.

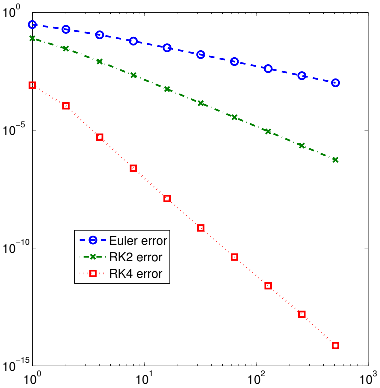
Error: $|y(t_n) - y_n|$

n	Euler	Modified	RK4
1	3.02e-01	7.89e-02	8.14e-04
2	1.90e-01	2.90e-02	$\boxed{1.08\text{e-}04}$
4	1.11e-01	8.20e-03	5.07e-06
8	6.02e-02	2.16e-03	2.44e-07
16	3.14e-02	5.55e-04	1.27e-08
32	1.61e-02	$\boxed{1.40\text{e-}04}$	7.11e-10
64	8.13e-03	3.53e-05	4.18e-11
128	4.09e-03	8.84e-06	2.53e-12
256	2.05e-03	2.21e-06	1.54e-13
512	1.03e-03	5.54e-07	7.33e-15

for examol, RK 4 with n=2 is more accurate than RK2 with n=32, or Euler with n=512.

Example 2.12 (2.11 (again))

Compare Euler, Modified Euler, and RK4 for approximating $y(1)$ *where:* $y(0) = 1, \quad y'(t) = y\log(1+t^2)$.



Although we won't do a detailed analysis of RK4, we can do a little. In particular, we would like to show it is

(i) consistent,

(ii) convergent and fourth-order, at least for some examples.

Example 2.13

It is easy to see that RK4 is consistent:

Show $\Phi(t_i, y_i, 0) = f(t_i, y_i)$.

If $h=0$, $k_1 = f(t_i, y_i)$ $k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) = f(t_i, y_i)$

& similarly $k_3 = f(t_i, y_i)$ & $k_4 = f(t_i, y_i)$

Then $\Phi(t_i, y_i; 0) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(6 f(t_i, y_i)) = f(t_i, y_i)$.

Example 2.14

In general, showing the rate of convergence is tricky. Instead, we'll demonstrate how the method relates to a Taylor Series expansion for the problem $y' = \lambda y$ where λ is a constant.

That is $f(t,y) = \lambda y$. This is easy to solve:

$$y(t) = e^{\lambda t} + \text{some constant}.$$

$$\left(\text{Then } y'(t) = \lambda e^{\lambda t} = \lambda y\right).$$

using $h = t_{i+1} - t_i$

To construct a Taylor's Series

$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t_i) + \frac{h^4}{24}y''''(t_i) + \frac{h^5}{120}y^{(5)}(\eta) \quad \text{some } \eta \in [t_i, t_{i+1}].$$

$$= y(t_i)\left[1 + h\lambda + \tfrac{1}{2}h^2\lambda^2 + \tfrac{1}{6}h^3\lambda^3 + \tfrac{1}{24}h^4\lambda^4\right] + \mathcal{O}(h^5).$$

"big Oh!"

Next write out the RK4 scheme for this problem:

$$\boxed{f(t,y) = \lambda y}$$

$$K_1 = f(t_i, y_i) = \lambda y_i$$

$$K_2 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{1}{2} h K_1) = \lambda \Big(y_i + \tfrac{1}{2} h \underbrace{\lambda y_i}_{K_1} \Big)$$

$$= y_i \left(\lambda + \tfrac{1}{2} h \lambda^2 \right).$$

$$K_3 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} K_2) = \lambda \Big(y_i + \tfrac{h}{2} \underbrace{y_i (\lambda + \tfrac{1}{2} h \lambda^2)}_{K} \Big)$$

$$= y_i \left(\lambda + \tfrac{1}{2} h \lambda^2 + \tfrac{1}{4} h^2 \lambda^3 \right).$$

$$K_4 = f(t_i + h, y_i + h K_3) = \lambda \Big(y_i + h \underbrace{y_i (\lambda + \tfrac{1}{2} h \lambda^2 + \tfrac{1}{4} h^2 \lambda^3)}_{K_3} \Big).$$

$$= y_i \left(\lambda + h \lambda^2 + \tfrac{1}{2} h^2 \lambda^3 + \tfrac{1}{4} h^3 \lambda^4 \right).$$

Many (seemingly different) RK have been proposed and studied. A unified approach of representing them was developed by John Butcher: write an s-stage method as

$$\Phi(t_i, y_i; h) = \sum_{j=1}^{s} b_j k_j, \quad \text{where}$$

$$\begin{aligned}
k_1 &= f(t_i + \alpha_1 h, y_i),\\
k_2 &= f(t_i + \alpha_2 h, y_i + \beta_{21} h k_1),\\
k_3 &= f(t_i + \alpha_3 h, y_i + \beta_{31} h k_1 + \beta_{32} h k_2),\\
&\vdots\\
k_s &= f(t_i + \alpha_s h, y_i + \beta_{s1} h k_1 + \dots \beta_{s,s-1} h k_{s-1}),
\end{aligned}$$

The most convenient way to represent the coefficients is in a tableau:

$$
\begin{array}{c||cccccc}
\alpha_1 & & & & & \\
\alpha_2 & \beta_{21} & & & & \\
\alpha_3 & \beta_{31} & \beta_{32} & & & \\
\vdots & & & & & \\
\alpha_s & \beta_{s1} & \beta_{s2} & \cdots & \beta_{s,s-1} & \\
\hline\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

The tableaux for basic Euler, Modified Euler, and RK4 are:

$$
\begin{array}{c||c}
0 & \\
\hline\hline
 & 1
\end{array}
\qquad
\begin{array}{c||cc}
0 & & \\
1/2 & 1/2 & \\
\hline\hline
 & 0 & 1
\end{array}
\qquad
\begin{array}{c||cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline\hline
 & 1/6 & 2/6 & 2/6 & 1/6
\end{array}
$$

A Runge Kutta method has s stages if it involves s evaluations of the function f. (That it, its formula features $k_1, k_2, \dots, k_s$).

We've seen a 1-stage method that is 1^{st}-order.

We studied 2-stage methods that are 2^{nd}-order.

In an exercise, you'll construct a 3-stage method that is 3rd order.

And, of course, we have just considered a four-stage method that is 4^{th}-order.

It is tempting to think that for any s we can get a method of order s using s stages. However, it can be shown that, for example, to get a 5^{th}-order method, you need at least 6 stages; for a 7^{th}-order method, you need at least 9 stages. The theory involved is both intricate and intriguing, and involves aspects of group theory, graph theory, and differential equations. Students in third year might consider this as a topic for their final year project.

Exercise 2.8

We claim that, for $RK4$:

$$|\mathcal{E}_N| = |y(t_N) - y_N| \le Kh^4.$$

for some constant K. How could you verify that the statement is true using the data of Table 2.3, at least for test problem in Example 2.4.2? Give an estimate for K.

Exercise 2.9

Recall the problem in Example 2.2.2: *Estimate $y(2)$ given that*

$$y(1) = 1, \qquad y' = f(t, y) := 1 + t + \frac{y}{t},$$

(i) Show that $f(t, y)$ satisfies a Lipschitz condition and give an upper bound for L.

(ii) Use Euler's method with $h = 1/4$ to estimate $y(2)$. Using the true solution, calculate the error.

(iii) Repeat this for the RK2 method of your choice (with $a \neq 0$) taking $h = 1/2$.

(iv) Use $RK4$ with $h = 1$ to estimate $y(2)$.

Exercise 2.10

Here is the tableau for a three stage Runge-Kutta method:

$$\begin{array}{c||ccc} 0 & & & \\ \alpha_2 & 1/2 & & \\ 1 & \beta_{31} & 2 & \\ \hline\hline & 1/6 & b_2 & 1/6 \end{array}$$

(i) Use that the method is consistent to determine b_2.

(ii) The method is exact when used to compute the solution to

$$y(0) = 0, \quad y'(t) = 2t, \; t > 0.$$

Use this to determine α_2.

(iii) The method should agree with an appropriate Taylor series for the solution to $y'(t) = \lambda y(t)$, up to terms that are $\mathcal{O}(h^3)$. Use this to determine β_{31}.