CS4423: Networks

Lecture 8: Bipartite Networks: Colours and Computations

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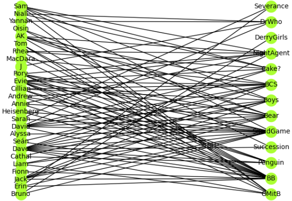


These slides include material by Angela Carnevale.

Outline

Today's notes are split between these slides, and a Jupyter Notebook.

1. The survey data...
   - A subgraph
   - Projections
2. Colouring
   - Bipartite graphs
3. Exercise(s)

Slides are at:

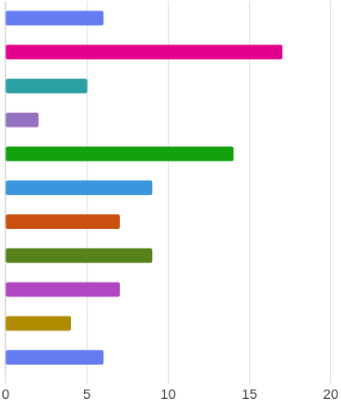
https://www.niallmadden.ie/2425-CS4423

The survey data...

This class is based around data we collected ina survey earlier this week. The final version is summarised as

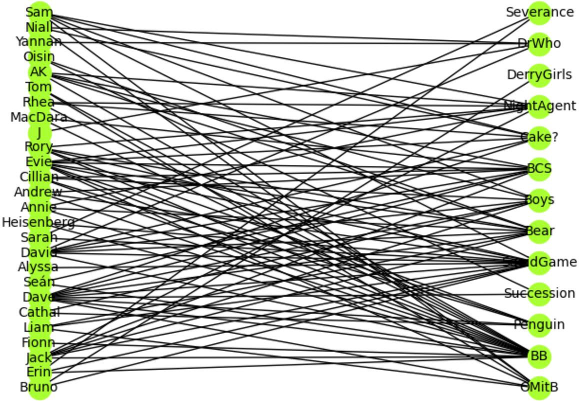
2. Which of the following do you watch?

Only Murders in the Building	6
Breaking Bad	17
The Penguin	5
Succession	2
Squid Game	14
The Bear	9
The Boys	7
Better Call Saul	9
Night Agent	7
Dr Who	4
Is it Cake?	6



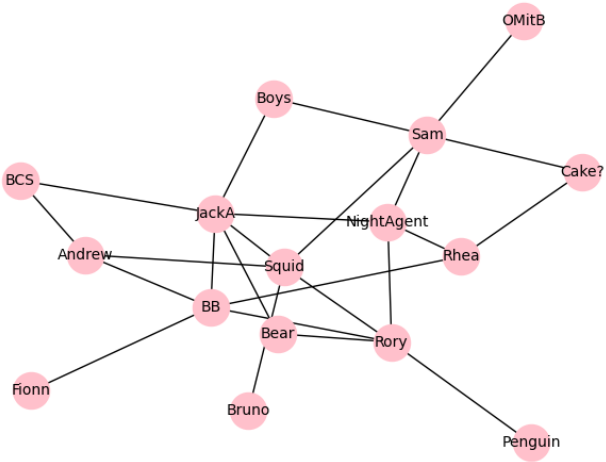
The survey data...

Here is what it looks like as a graph:



Its order is 39, and size is 87.

Here is **subgraph** of our survey network from yesterday. It is of order 16 and size 24, based on 7 randomly chosen people:



Yesterday, we also had this version of the adjacency matrix where the nodes for people are listed first:

$$A = \begin{pmatrix}
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0\\
1&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

And we had $B = A^2$:

$$B = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 5 & 1 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 4 & 1 & 6 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 1 & 3 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 3 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 3 & 1 & 1 & 3 & 2 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 2 & 1 & 3 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 1 & 1 & 5 & 2 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}$$

Since we know from Lecture 6 that $(A^k)_{ij}$ is the number of walks of length k between nodes i and j, we can see that, in this context:

- For the first **7** rows and columns, b_{ij} is the number programmes in common between person i and j. (This even works for $i = j$; but the number of programmes i has in common with them self is the number they watch!).
- For the last 9 rows and columns, b_{ij} is the number people who watch both programmes i and j.

It can be insightful to consider the submatrices of these blocks...

Given a bipartite graph, G, whose node set, V, has parts V_1 and V_2, and **projection** of G onto (for example) V_1, is the graph with

- node set V_1
- an edge between a pair of nodes in V_1 if they share a common neighbour in G

In the context of our example, a projection onto V_1 (people/actors) gives us the graph of people who share a common programme.

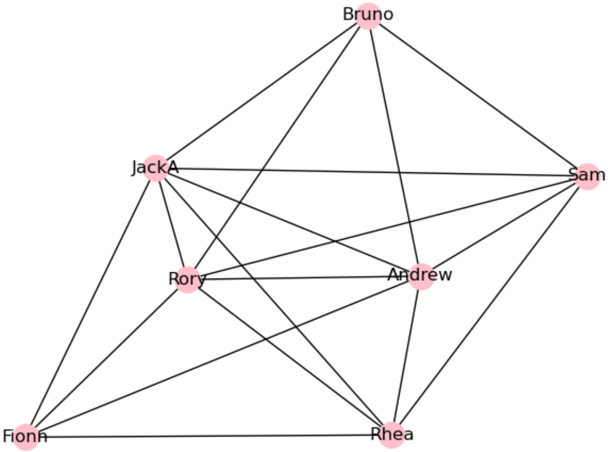
To make such a graph:

- Let A be the adjacency matrix of G.
- Let B be the submatrix of A^2 associated with the nodes in V_1.
- Let C be the (adjacency) matrix with the property

$$c_{ij} = \begin{cases} 1 & b_{ij} > 0 \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

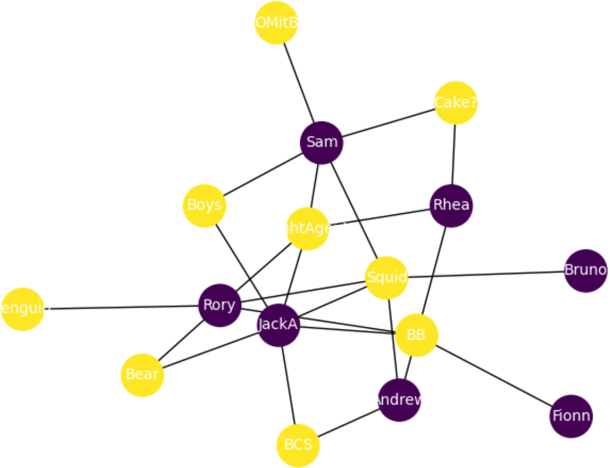
- Let G_{V_1} be the graph on V_1 with adjacency matrix C. Then G_{V_1} is the **projection of G onto V_1**.

Presently, we'll see how to compute G_{V_1} in `networkx`. But this is what it looks like:



Colouring

Our graph would look a bit better if we coloured the nodes, e.g.,



Colouring

For any bipartite graph, we can think of the nodes in the two sets as **coloured** with different colours. For instance, we can think of nodes in X_1 as white nodes and those in X_2 as black nodes.

Vertex colouring

- A **(vertex)-coloring** of a graph G is an assignment of (finitely many) colours to the nodes of G, so that any two nodes which are connected by an edge have **different** colours.
- A graph is called N-**colorable**, if it has a vertex coloring with (at most) N colors.
- The **chromatic number** of a graph G is *smallest* N for which a graph G is N-colourable.

FACT!

Let G be a graph. The following are equivalent:

- G is bipartite;
- G is 2-colorable;
- Each cycle in G has even length.

Later, we'll set how to get `networkx` to compute a colouring for us.

Now switch to the Jupyter notebook at ...

Exercise(s)

1. Let u be a vector with n entries. Let $D = \text{diag}(u)$. That is, $D = (d_{ij})$ is the diagonal matrix with entries

$$d_{ij} = \begin{cases} u_i & i = j \\ 0 & i \neq j. \end{cases}$$

   Verify that $PDP^T = \text{diag}(Pu)$.

2. In all the examples we looked at, we had a symmetric P. Is every permutation matrix symmetric? If so, explain why. If not, give an example.