2526-MA140 Engineering Calculus

Week 03, Lecture 3
Continuity Types; The Intermediate Value Theorem

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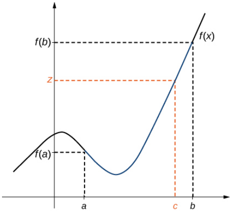
Thursday, 2 October, 2025



These slides are by Niall Madden. Some content is based on notes by Dr Kirsten Pfeiffer. And some more, such as the figure opposite, taken from Strang & Herman's "Calculus". However, all the typos are Niall's.

Today, in MA140...

For more, see Chapter 2 (Limits) in **Calculus** by Strang & Herman. See openstax.org/books/calculus-volume-1/pages/2-introduction. Section 2.4 (Continuity) relates to today's material.

Remembering the reminders...

Reminders

- **Assignment 1** due 5pm, Monday 6 October. You may access it multiple times, by clicking on `Assignments` ... `Problem Set 1` ... and then, at the bottom of the page: `Load Problem Set 1 in a new window`
- The **Tutorial Sheet** is available at https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1
- Assignment 2 is also open; deadline is 5pm, 13 Oct.
- The first (of two) class tests will take place Tuesday, 14th October.
- If you wish to avail of Reasonable Accommodations for it tests, please complete this form: https://forms.office.com/e/HaAsrzaE3D by **10am Thursday 2nd Oct**.

Types of discontinuity

We have encountered three types of discontinuity.

- **Removable discontinuity**: $\lim_{x\to a} f(x)$ exists but

$$\lim_{x\to a} f(x) \neq f(a)$$

So the left Right limits are equal.

- **Jump discontinuity**: $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+ f(x)}$ both exist (and are finite), but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

- **Infinite discontinuity**: At least one of the one-sided limits does not exist.

Types of discontinuity

Example

Each of the following functions has a discontinuity at $x = 2$. Classify it.

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$
2. $g(x) = \dfrac{x^2}{x - 2}$
3. $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$
4. $j(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

Types of discontinuity

1. $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2}$.

If $x \neq 2$ then $f(x) = x+2$.

Therefore $\lim\limits_{x \to 2^-} f(x) = 4$ $\lim\limits_{x \to 2^+} f(x) = 4$.

So the limit exists.

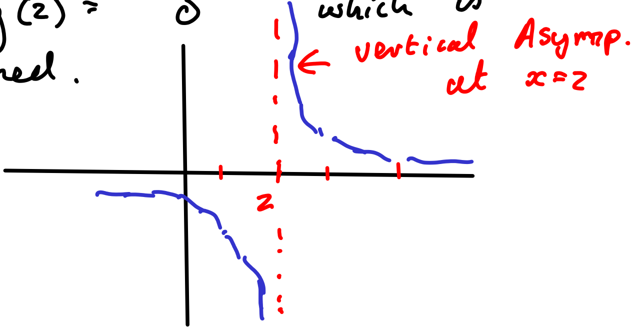
This is a <u>removable</u> discontinuity

Types of discontinuity

2. $g(x) = \dfrac{x^2}{x-2}$

This has an <u>infinite</u> discontinuity at $x = 2$

Since $g(2) = \dfrac{4}{0}$ which is not defined.



Types of discontinuity

3.
$$h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$$

Check: does the limit exist?

left: $\lim_{x \to 2^-} h(x) = \lim_{x \to 2^-} x/2 = 1$

Right: $\lim_{x \to 2^+} h(x) = \lim_{x \to 2^+} x^2 - 3 = 4 - 3 = 1.$

So limit does exist, but $\lim_{x \to 2} f(x) \neq f(2)$

So: we have a removable discontinuity!!

Types of discontinuity

4. $$j(x) = \begin{cases} x/2 & x < 2 \\ x^2-2 & x \geq 2 \end{cases}$$

So $$\lim_{x \to 2^-} j(x) = \lim_{x \to 2^-} \frac{x}{2} = 1$$

$$\lim_{x \to 2^+} j(x) = \lim_{x \to 2^+} x^2-2 = 2.$$

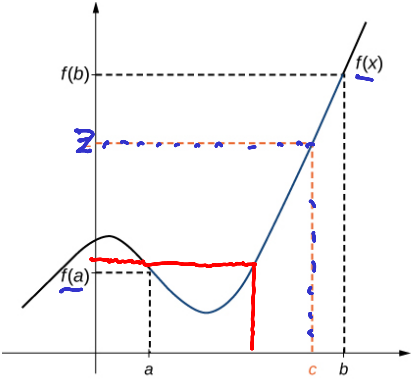
So the limit does not exist: jump discontinuity.

Intermediate Value Theorem

Continuous functions have numerous important properties, many of which we will study in MA140. The first of these is the **Intermediate Value Theorem**.

Intermediate Value Theorem (IVT)

Suppose that $f(x)$ is continuous on an interval $[a, b]$.
Let z be any real number between $f(a)$ and $f(b)$.
Then there exists a number $c \in [a, b]$ such that $f(c) = z$.



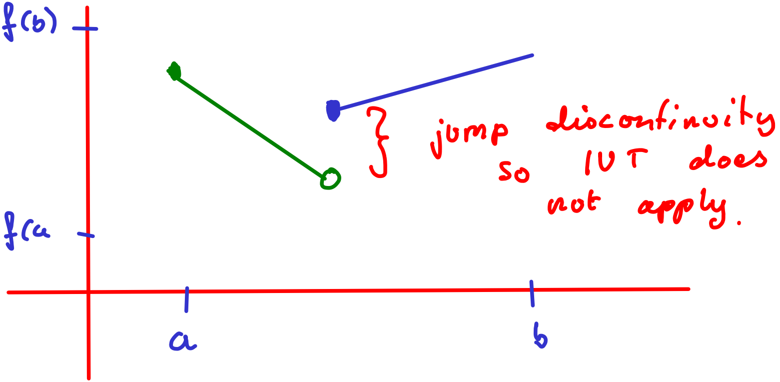
- If you travel by train from Galway to Athlone, then there must be a time when you are at Oranmore station, and a time when you are at Athenry, and at Woodlawn, etc.

- If your car/train/whatever accelerates from 0km/h to 100km/h, there was a time when it was travelling at 30 km/h.

- This morning my train ticket from Athenry to Galway cost €5.10. Suppose train fares increase next Thursday to €5.50. But there wasn't a day when they cost, say, €5.20, because the price had a jump discontinuity (so the IVT does not apply here).

Example

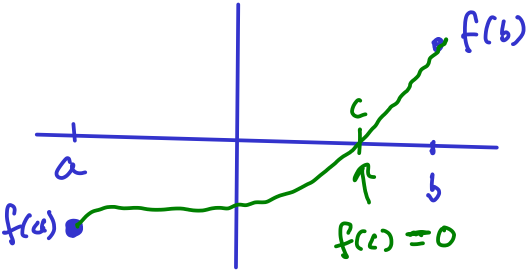
Sketch an example of a function for which the IVT does *not* hold.



One of the main applications of the IVT is in establishing if an equation as a solution:

Solutions to $f(x) = 0$

If $f(x)$ defined on $[a, b]$ is such that $f(a) < 0$ and $f(b) > 0$, then there must be a value $c \in [a, b]$ such that $f(x) = 0$.
More generally, if $f(a)f(b) \leqslant 0$, then $f(x)$ has at least one zero in $[a, b]$.



So, if f changes sign between a & b then there must be a solution to $f(x)=0$, ie $x=C$

Example

Show that $f(x) = x - \cos(x)$ has at least one zero.

Idea: $f(0) = 0 - 1 = -1$.

$f(2) = 2 - \cos(2) \geq 2 - 1 = 1$

(since $-1 \leq \cos(x) \leq 1$ for all x).

So f changes sign on $[0,2]$.

Therefore there is a solution to

$f(x) = 0$ in $[0,2]$.

Given a function $f(x)$,

- When we say c is a **zero** of a function, f, we mean that $f(c) = 0$.
- Many books and website also use the terminology "c is a **root** of f". This is particularly the case where $f(x)$ is a polynomial.
- If c is a zero of $f(x)$, then it is a solution to the equation $f(x) = 0$.

Example

How many solutions does $x^3 + 1 = 3x^2$ have?

Set $f(x) = x^3 - 3x^2 + 1$, and check for solutions to $f(x) = 0$

Then check some values

- $f(-1) = (-1) - 3 + 1 = -3 < 0$
- $f(0) = 0^3 - 3(0) + 1 = 1 > 0$
- $f(1) = 1 - 3 + 1 = -1 < 0$
- $f(2) = 8 - 12 + 1 = -3 < 0$
- $f(3) = 27 - 27 + 1 = 1 > 0$

$f(-1), f(0)$: $f(x) = 0$ has a solution in $[-1, 0]$

$f(0), f(1)$: Another solution in $[0, 1]$.

$f(2), f(3)$: And a 3^{rd} soln, between $x = 2$ & $x = 3$.

Example

Use the *Intermediate Value Theorem* to show that the equation

$$2x^3 + 3x^2 - 2x - 1 = 0$$

has three solutions in the range $-2 < x < 1$.

Let $f(x) = 2x^3 + 3x^2 - 2x - 1 = 0$.

(Check!)

$f(-2) = -1 < 0$

$f(-1) = 2 > 0$

$f(0) = -1 < 0$

$f(1) = 2 > 0$

By the IVT, there are solutions in $[-2, -1]$, $[-1, 0]$, $[0, 1]$.

Derivatives: the concept

The next section of MA140 is all about **derivatives** of function. The derivative of a function describes how quickly the function is changing.

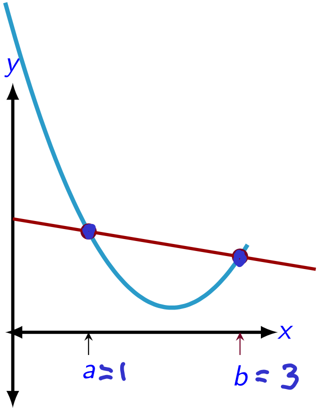
There are many, many applications: derivatives, and equations involving them are used everywhere: **speed/velocity** is the rate of change of displacement; **acceleration** is the rate of change of velocity.

We use derivatives to model how quickly a tumour is growing or shrinking, how pollutants are dispersed in a river, how pressure changes with depth, how inflation is changing in an economy. The list of applications is practically limitless.

Consider the graph opposite. It shows a function, f, and a secant line that intersects f at $a = 1$ and $b = a + 2$ (the actually values are not important).

If we wanted to summarised how f is changing between those two values, we could compute it as

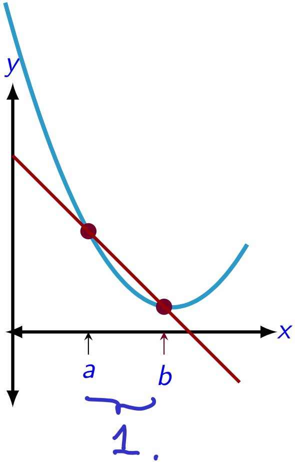
$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+2) - f(a)}{2}$$



Now we'll consider how f is changing over a shorter interval: from a to $b = a + 1$. Again, we sketch the secant line that intersects f at $x = a$ and $x = b$. The rate of change of f between these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+1) - f(a)}{1},$$

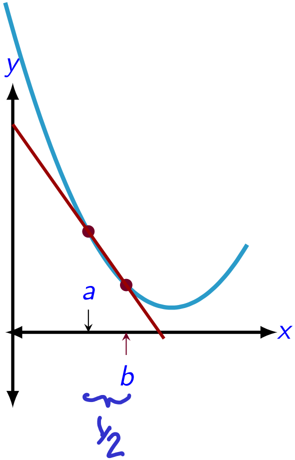
which, of course, is the slope of the secant line.



Next we shorter interval again: looking at how f changes from a to $b = a + \frac{1}{2}$, along with the secant line that intersects f at $x = a$ and $x = b$.

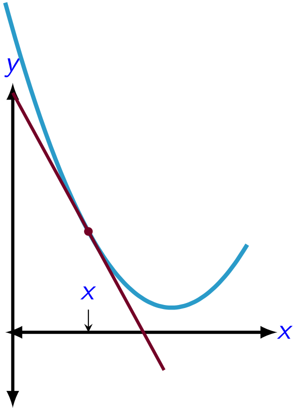
The rate of change of f between these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + \frac{1}{2}) - f(a)}{\frac{1}{2}}.$$



Finally, suppose we want to looking at the **instantaneous** rate of change of f at $x = a$. Hopefully, the preceding images have convinced you we could do this in two (equivalent) ways:

1. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
2. or as the slope of the tangent to f at $x = a$.



Finished here Thursday

Derivative at a point

The slope of the curve $y = f(x)$ at the point $P = (a, f(a))$ is given by the number (if it exists)

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, it is called **the derivative of f at $x = a$** and we denote it by $f'(a)$.

Definition: derivative at a point

Let $f(x)$ be a function that has $x = a$ in its domain. The **derivative** of the function $f(x)$ at a, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

Some terminology

$f'(a)$ exists then we say that function f is **differentiable at $x = a$**.

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the "limit" formula, we are doing "**differentiation from first principles"**.

Derivative at a point

Example

Use the limit definition of a derivative to compute the slope of the tangent to $f(x) = x^2$ at $x = 3$.

Exercises 3.3.1 (Based on Q1(a), 23/24)

Let $g(x) = \begin{cases} 3 & x \leqslant 0 \\ 2x+1 & 0 < x < 1 \\ x^2 & x \geqslant 1. \end{cases}$

(i) Sketch the graph of $g(x)$ on the interval $[-3, 4]$, making use of the empty and full circle notation.

(ii) Compute $\lim_{x \to 1^-} g(x)$ and $\lim_{x \to 1^+} g(x)$. Is g continuous at $x = 1$. If not, classify the type of discontinuity.

Exercise 3.3.2

For what values of b and c is $f(x) = \begin{cases} x^2+1 & x \leqslant -1 \\ x+b & -1 < x < 1 \\ cx^2 & x \geqslant 1. \end{cases}$ continuous at $x = -1$ and $x = 1$?

Exercises

Exercise 3.3.3 (23/24 exam)

Use the IVT to show that the equation $x^3 - 3x + 1 = 0$ has three solutions in the range $-2 < x < 2$.

Exercise 3.3.4

Use the (limit) definition of a derivative to differentiate the function $f(x) = x^2 + 2$.

Exercise 3.3.5

Use the (limit) definition of a derivative to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.