

Solving Linear Systems

### §3.3 *LU*-factorisation

MA385/530 – Numerical Analysis 1

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*In these slides,*

- *LT* means “lower triangular”
- *UT* means “upper triangular”

The goal of this section is to demonstrate that the process of Gaussian Elimination applied to a matrix  $A$  is equivalent to factoring  $A$  as the product of a unit lower triangular and upper triangular matrix.

The Section 3.2 we saw that each elementary row operation in Gaussian Elimination involves replacing  $A$  with  $(I + \mu_{rs}E^{(rs)})A$ .

**Example:** For the  $3 \times 3$  case, this involved computing

$$(I + \mu_{32}E^{(32)})(I + \mu_{31}E^{(31)})(I + \mu_{21}E^{(21)})A.$$

In general we multiply  $A$  by a sequence of matrices

$$(I + \mu_{rs}E^{(rs)}),$$

all of which are unit lower triangular matrices.

When we are finished we have reduced  $A$  to an upper triangular matrix.

So we can write the whole process as

$$L_k L_{k-1} L_{k-2} \dots L_2 L_1 A = U, \tag{3}$$

where each of the  $L_i$  is a unit LT matrix.

But from Theorem 3.2.6, we know that the product of unit LT matrices is itself a unit LT matrix. So we can write the whole process described in (3) as

$$\tilde{L}A = U. \tag{4}$$

But Theorem 3.2.6 also tells us that the inverse of a unit LT matrix exists and is a unit LT matrix. So we can write (4) as

$$A = LU$$

where  $L$  is unit lower triangular and  $U$  is upper triangular.

This is called “**LU-factorisation**”.

**Definition 3.9**

The  **$LU$ -factorization** of the matrix is a unit lower triangular matrix  $L$  and an upper triangular matrix  $U$  such that  $LU = A$ .

**Example 3.10**

If  $A = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$  then:

**Example 3.11**

If  $A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & -4 \end{pmatrix}$  then:

We now want to work out formulae for  $L$  and  $U$  where

$$a_{i,j} = (LU)_{ij} = \sum_{k=1}^n l_{ik}u_{kj} \quad 1 \leq i, j \leq n.$$

Since  $L$  and  $U$  are triangular,

$$\text{If } i \leq j \quad \text{then} \quad a_{i,j} = \sum_{k=1}^i l_{ik}u_{kj} \quad (5a)$$

$$\text{If } j < i \quad \text{then} \quad a_{i,j} = \sum_{k=1}^j l_{ik}u_{kj} \quad (5b)$$

The first of these equations can be written as

$$a_{i,j} = \sum_{k=1}^{i-1} l_{ik} u_{kj} + l_{ii} u_{ij}.$$

But  $l_{ii} = 1$  so:

$$u_{i,j} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \quad \begin{cases} i = 1, \dots, j-1, \\ j = 2, \dots, n. \end{cases} \quad (6a)$$

And from the second:

$$l_{i,j} = \frac{1}{u_{jj}} \left( a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right) \quad \begin{cases} i = 2, \dots, n, \\ j = 1, \dots, i-1. \end{cases} \quad (6b)$$



**Example 3.12**

Find the  $LU$ -factorisation of

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -2 & -2 & 1 & 4 \\ -3 & -4 & -2 & 4 \\ -4 & -6 & -5 & 0 \end{pmatrix}$$

**Full details of Example 3.12:** First, using (6a) with  $i = 1$  we have

$$u_{1j} = a_{1j}:$$

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

Then (6b) with  $j = 1$  we have  $l_{i1} = a_{i1}/u_{11}$ :

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & l_{32} & 1 & 0 \\ 4 & l_{42} & l_{43} & 1 \end{pmatrix}.$$

Next (6a) with  $i = 2$  we have  $u_{2j} = a_{2j} - l_{21}u_{1j}$ :

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix},$$

then (6b) with  $j = 2$  we have  $l_{i2} = (a_{i2} - l_{i1}u_{12})/u_{22}$ :

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & l_{43} & 1 \end{pmatrix}$$

Etc....

Not every matrix has an  $LU$ -factorisation. So we need to characterise the matrices that do.

To prove the next theorem we need the Cauchy-Binet Formula:  
 $\det(AB) = \det(A) \det(B)$ .<sup>1</sup>

### Theorem 3.13

If  $n \geq 2$  and  $A \in \mathbb{R}^{n \times n}$  is such that every leading principal submatrix of  $A$  is nonsingular for  $1 \leq k < n$ , then  $A$  has an  $LU$ -factorisation.

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<sup>1</sup>Wikipedia disagrees with this attribution



**Exercise 3.9**

Many textbooks and computing systems compute the factorisation  $A = LDU$  where  $L$  and  $U$  are unit lower and *unit* upper triangular matrices respectively, and  $D$  is a diagonal matrix. Show such a factorisation exists, providing that if  $n \geq 2$  and  $A \in \mathbb{R}^{n \times n}$ , then every leading principal submatrix of  $A$  is nonsingular for  $1 \leq k < n$ .