MA385 Part 2: Initial Value Problems

2.3: Error Analysis of Euler's Method

Dr Niall Madden

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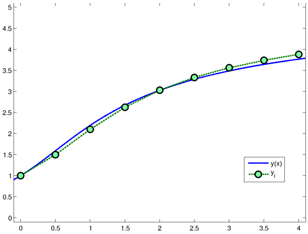

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For more details, see Chapter 6 of Süli and Mayers, *An Introduction to Numerical Analysis*.

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1. General one-step methods

Euler's method is an example of a **one-step method**, which have the *general* form:

$$y_{i+1} = y_i + h\Phi(t_i, y_i; h). \tag{1}$$

To get Euler's method, just take $\Phi(t_i, y_i; h) = f(t_i, y_i)$.

In the introduction, we motivated Euler's method with a geometrical argument. An alternative, more mathematical way of deriving Euler's Method is to use a *Truncated Taylor Series*:

- For Euler's Method,

$$\Phi(t_i, y_i; h) = f(t_i, y_i)$$

where the IVP is $y'(t) = f(t, y)$

- For Euler, $\Phi(t_i, y_i; h)$ is independent of h.
  But for all other method, Φ depends on h.

Lets derive Euler's Method again, this time from a Taylor series:

$$y(b) = y(a) + (b-a)\,y'(a) + \frac{1}{2}(b-a)^2 y''(\eta) \qquad \eta \in [a,b].$$

Take $a = t_i$, $b = t_{i+1}$. Note this gives $b - a = h$.

So
$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \frac{h^2}{2} y''(\eta_i) \qquad \eta_i \in [t_i, t_{i+1}]$$

Since $y'(t) = f(t,y)$ (from the DE)

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \underbrace{h^2\left[\tfrac{1}{2} y''(\eta_i)\right]}_{\text{Error}}.$$

Neglecting the $O(h^2)$ term:

$$y_{i+1} = y_i + h f(t_i, y_i)$$

This not only motivates Euler's formula, but also suggests that at each step the method introduces a (local) error of $h^2 y''(\eta)/2$. (More of this later).

where y_i denotes the approximation for $y(t_i)$.

2. Two types of error

Definition 2.1.1

Global Error: $\mathcal{E}_i = y(t_i) - y_i$.

Definition 2.1.2

Truncation Error:

$$T_i := \frac{y(t_{i+1}) - y(t_i)}{h} - \Phi(t_i, y(t_i); h). \tag{2}$$

It can be helpful to think of T_i as representing how much the difference equation differs from the differential equation. For Euler's method, it can be determined using a Taylor Series.

The relationship between the global error and truncation errors is explained in the following (important!) result (also, compare with Picard's Theorem).

Theorem 2.1.1 (Thm 12.1 in Süli & Mayers)

Let $\Phi()$ be *Lipschitz* with constant L. Then

$$|\mathcal{E}_n| \leq T\left(\frac{e^{L(t_n-t_0)}-1}{L}\right), \qquad (3)$$

where $T = \max_{i=0,1,\dots n} |T_i|$.

From the Definition of the truncation Error:

$$y(t_{i+1}) = y(t_i) + h\Phi(t_i; y(t_i); h) + hT_i$$

method $$y_{i+1} = y_i + h\Phi(t_i, y_i; h)$$

Subtracting: $$y(t_{i+1}) - y_{i+1} = y(t_i) - y_i + h\left[\Phi(t_i, y(t_i); h) - \Phi(t_i, y_i; h)\right] + hT_i$$

2. Two types of error

Using that $\mathcal{E}_i = y(t_i) - y_i$ we get

$$\mathcal{E}_{i+1} = \mathcal{E}_i + h\left[\Phi(t_i, y(t_i); h) - \Phi(t_i, y_i; h)\right] + hT_i$$

So

$$|\mathcal{E}_{i+1}| \leq |\mathcal{E}_i| + h\left|\Phi(t_i, y(t_i); h) - \Phi(t_i, y_i; h)\right| + hT$$

Using that $\Phi(t, y; h)$ is Lipschitz w.r.t. y:

$$|\Phi(t_i, y(t_i); h) - \Phi(t_i, y_i; h)| \leq L\,|y(t_i) - y_i|$$

So

$$|\mathcal{E}_{i+1}| \leq |\mathcal{E}_i| + hL\,|\mathcal{E}_i| + hT.$$

$$|\mathcal{E}_{i+1}| \leq |\mathcal{E}_i|\,(1 + hL) + hT.$$

Now see Exer 2.1.1 to finish.

For Euler's method, we get

$$T = \max_{0 \leq j \leq n} |T_j| \leq \frac{h}{2} \max_{t_0 \leq t \leq t_n} |y''(t)|.$$

Example 2.1.1

Given the problem:

$$y' = 1 + t + \frac{y}{t} \quad \textit{for } t > 1; \quad y(1) = 1,$$

find an approximation for $y(2)$.

(i) Give an upper bound for the global error taking $n = 4$ (i.e., $h = 1/4$)

(ii) What n should you take to ensure that the global error is no more that 0.1?

To answer these questions we need to use (3), which requires that we find L and an upper bound for T. In this instance, L is easy:

To find T we need an upper bound for $|y''(t)|$ on $[1, 2]$, even though we don't know $y(t)$...

With these values of L and T, using (3) we find $\mathcal{E}_n \leq 0.644$. In fact, the true answer is 0.43, so we see that (3) is somewhat pessimistic.

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To answer (ii): *What n should you take to ensure that the global error is no more that 0.1?* (We should get $n = 26$. This is not that sharp: $n = 19$ will do).

4. Convergence and Consistency

We are often interested in the *convergence* of a method. That is, is it true that

$$\lim_{h\to 0} y_n = y(t_n)?$$

Or equivalently that,

$$\lim_{h\to 0} \mathcal{E}_n = 0?$$

Given that the global error for Euler's method can be bounded:

$$|\mathcal{E}_n| \leq h\frac{\max|y''(t)|}{2L}\left(\mathrm{e}^{L(t_n-t_0)} - 1\right) = hK, \tag{4}$$

we can say it converges.

So now we know, for Euler's method, that $y_n \to y(t_n)$ as $n \to \infty$, but how quickly?

Definition 2.1.3

The **order of accuracy** of a numerical method is p if there is a constant K so that

$$|\mathcal{E}_n| \leq Kh^p.$$

So Euler's method is first-order.

The term **order of convergence** is often use instead of **order of accuracy**.

One of the requirements for convergence is *Consistency*:

Definition 2.1.4

A one-step method $y_{n+1} = y_n + h\Phi(t_n, y_n; h)$ is *consistent* with the differential equation $y'(t) = f\big(t, y(t)\big)$ if $f(t, y) \equiv \Phi(t, y; 0)$.

4. Convergence and Consistency

Next we'll try to develop methods that are of higher order than Euler's method; that is that we can show

$$|\mathcal{E}_n| \leq Kh^p \quad \text{for some} \quad p > 1.$$

Suppose we numerically solve some differential equation and estimate the error. If we think this error is too large we could redo the calculation with a smaller value of h. Or we could use a better method, for example **Runge-Kutta** methods. These are high-order methods that rely on evaluating $f(t, y)$ a number of times at each step in order to improve accuracy.

4. Convergence and Consistency

We'll first motivate one such method and then later look at the general framework.

The goal will be to develop some techniques to help us derive our own methods for accurately solving IVPs. Rather than using formal theory, we will reason based on carefully chosen examples.

Exercise 2.1.1

An important step in the proof of Theorem 2.3.3, but which we didn't do in class, requires the observation that if $|\mathcal{E}_{i+1}| \leq |\mathcal{E}_i|(1+hL)+h|T_i|$, then

$$|\mathcal{E}_i| \leq \frac{T}{L}\left[(1+hL)^i - 1\right] \qquad i = 0, 1, \ldots, N.$$

Use induction to show that is indeed the case.

Exercise 2.1.2

Suppose we use Euler's method to find an approximation for $y(2)$, where y solves

$$y(1) = 1, \qquad y' = (t-1)\sin(y).$$

(i) Give an upper bound for the global error taking $n = 4$ (i.e., $h = 1/4$).

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