MA385 Part 4: Linear Algebra 2

4.2: Matrix Norms

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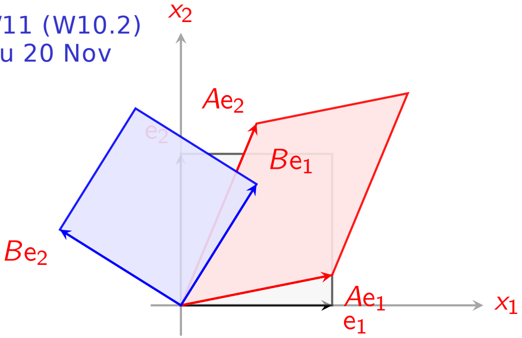
Started Thu 13/11 (W10.2) and finished Thu 20 Nov (W11.2)



$x \mapsto Ax$ (red), $x \mapsto Bx$ (blue)

1. Outline Section 4.2

For more, see Section 2.7 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG

Vector norms are related to the magnitude of the entries of the vector.

Now we want to generalise to the concept of a **matrix norm**. In a sense, we can just consider the magnitude of the matrix's entries.

However, if we think of a matrix as a linear transformation, or simply as a function that maps (via matrix multiplication) from $\mathbb{R}^n$ to $\mathbb{R}^n$, we should think about how much it changes a vector.

Definition 4.2.1

Given any (vector) norm $\|\cdot\|$ on $\mathbb{R}^n$, there is a **subordinate matrix norm** on $\mathbb{R}^{n\times n}$ defined by

$$\|A\| = \max_{v\in\mathbb{R}^n_\star} \frac{\|Av\|}{\|v\|}, = \max_{\substack{v\in\mathbb{R}^n \\ \|v\|=1}} \|Av\| \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$ and $\mathbb{R}^n_\star = \mathbb{R}^n/\{0\}$.

We define a matrix norm like this because we think of A as an *operator* on $\mathbb{R}^n$: if $v \in \mathbb{R}^n$ then $Av \in \mathbb{R}^n$. So the norm of A gives us information on how much the matrix can change the size of a vector.

"induced Matrix Norm" "Operator Norm".

3. Computing Matrix Norms

It is not obvious from the above definition how to calculate the norm of a given matrix. We'll see that

- The ∞-norm of a matrix is also the largest absolute-value row sum.
- The 1-norm of a matrix is also the largest absolute-value column sum.
- The 2-norm of the matrix A is the square root of the largest eigenvalue of $A^T A$.

4. The max-norm on $\mathbb{R}^{n\times n}$

Finished here W10.2. 13 Nov.

Theorem 4.2.1

For any $A \in \mathbb{R}^{n\times n}$ the subordinate matrix norm associated with $\|\cdot\|_\infty$ on $\mathbb{R}^n$ can be computed by

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|.$$

Eg $\quad A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -5 \\ 4 & -1 & -2 \end{pmatrix}$

Then $\displaystyle \max_{i=1\dots3} \sum_{j=1}^{3} |a_{ij}| = \max\{\underbrace{1+2+3}_{\substack{i=1\\6}},\ \underbrace{1+0+5}_{\substack{i=2\\6}},\ \underbrace{4+1+2}_{\substack{i=3\\7}}\}$

So $\|A\|_\infty = 7$

4. The max-norm on $\mathbb{R}^{n\times n}$

Proof: Let v be any vector in $\mathbb{R}^n/\{0\}$ and let $K = \|v\|_\infty = \max_i |v_i|$

The
$$(Av)_i = \sum_{j=1}^{n} a_{ij} v_j$$

So
$$|(Av)_i| = \left|\sum_{j=1}^{n} a_{ij} v_j\right| \le \sum_{j=1}^{n} |a_{ij}| \cdot \overbrace{|v_j|}^{\le K}$$

by the Triangle Inequality

$$|(Av)_i| \le K \sum_{j=1}^{n} |a_{ij}|$$

So
$$\max_i |(Av)_i| \le K \max_i \sum_{j=1}^{n} |a_{ij}|.$$

That is
$$\|Av\|_\infty \leq K \max_i \sum_j |a_{ij}|$$
So, since $K = \|v\|_\infty \neq 0$
$$\frac{\|Av\|_\infty}{\|v\|_\infty} \leq \max_i \sum_j |a_{ij}|.$$
for any vector v. Therefore
$$\|A\|_\infty \leq \max_i \sum_j |a_{ij}|$$
To finish we need a vector v such that
$\|Av\|_\infty = \max_i \sum_j |a_{ij}|$ & $\|v\|_\infty = 1$.

Say $\|A\|_\infty = \sum_j |a_{ij}|$ for a given i.

Let v be the vector with all entries $+1$ or -1, and

$$v_j = \text{sign}(a_{ij})$$

$(\text{so} \quad a_{ij} v_j \geq 0)$.

A similar result holds for the 1-norm, the proof of which is left as an exercise.

Theorem 4.2.2

For any $A \in \mathbb{R}^{n\times n}$ the subordinate matrix norm associated with $\|\cdot\|_\infty$ on $\mathbb{R}^n$ can be computed by

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{i,j}|. \qquad (2)$$

Computing the 2-norm of a matrix is a little harder that computing the 1- or ∞-norms. However, later we'll need estimates not just for $\|A\|$, but also $\|A^{-1}\|$. And, unlike the 1- and ∞-norms, we can estimate $\|A^{-1}\|_2$ without explicitly forming A^{-1}.

We begin by recalling some important facts about eigenvalues and eigenvectors.

Definition 4.2.2

Let $A \in \mathbb{R}^{n\times n}$. We call $\lambda \in \mathbb{C}$ an *eigenvalue* of A if there is a non-zero vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x.$$

We call any such x an *eigenvector of A associated with λ*.

(i) If A is a real symmetric matrix (i.e., $A = A^T$), its eigenvalues and eigenvectors are all real-valued.

(ii) If λ is an eigenvalue of A, the $1/\lambda$ is an eigenvalue of A^{-1}.

(iii) If x is an eigenvector associated with the eigenvalue λ then so too is ηx for any non-zero scalar η.

(iv) An eigenvector may be *normalised* as $\|x\|_2^2 = x^T x = 1$.

If $Ax = \lambda x$ and A^{-1} exists,

then $\underbrace{A^{-1}A}_{I} x = \lambda A^{-1} x$

$\Rightarrow x = \lambda A^{-1} x \Rightarrow A^{-1} x = \frac{1}{\lambda} x$

(v) There are n eigenvectors λ_1, λ_n, ..., λ_n associated with the real symmetric matrix A. Let $\mathsf{x}^{(1)}$, $\mathsf{x}^{(2)}$, ..., $\mathsf{x}^{(n)}$ be the associated normalised eigenvectors. The eigenvectors are linearly independent and so form a basis for $\mathbb{R}^n$. That is, any vector $v \in \mathbb{R}^n$ can be written as a linear combination:

$$\mathsf{v} = \sum_{i=1}^{n} \alpha_i \mathsf{x}^{(i)}.$$

(vi) Furthermore, these eigenvectors are *orthogonal* and *orthonormal*:

$$\left(x^{(i)}\right)^T x^{(j)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Here is a useful consequence of (v) and (vi), which we will use repeatedly.

(v) $$v = \sum_{i=1}^{n} \alpha_i x^{(i)}$$

$$Ax^{(i)} = \lambda_i x^{(i)}$$

(vi) $$\left(x^{(i)}\right)^T x^{(j)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then

$$v^T v = \left(\sum_{i=1}^{n} \alpha_i x^{(i)}\right)^T \sum_{j=1}^{n} \alpha_j x^{(j)}$$

$$= \sum_{i=1}^{n} \alpha_i^2$$

The *singular values* of a matrix A are the square roots of the eigenvalues of $A^T A$. They play a very important role in matrix analysis, applied linear algebra, and statistics (principal component analysis).

Our interest here is in their relationship to $\|A\|_2$.

But first we'll prove a theorem about certain matrices (so called, "normal matrices").

Theorem 4.2.3

For any matrix $A \in \mathbb{R}^{n \times n}$, the eigenvalues of $A^T A$ are real and non-negative.

Let $B = A^T A$. Then $B^T = (A^T A)^T = A^T (A^T)^T$

So $B^T = A^T A = B$ So B is symmetric

So any eigenvalue of B is real valued.

Let $Bx = \lambda x$.

So $(A^T A) x = \lambda x$

$\Rightarrow x^T (A^T A) x = \lambda x^T x \Rightarrow (x^T A^T)(Ax) = \lambda x^T x$.

$\Rightarrow (Ax)^T Ax = \lambda x^T x \Rightarrow \lambda = \dfrac{\|Ax\|_2^2}{\|x\|_2^2} \geq 0$

Part of the above proof involved showing that, if $(A^T A)\mathrm{x} = \lambda x$, then

$$\sqrt{\lambda} = \frac{\|A\mathrm{x}\|_2}{\|\mathrm{x}\|_2}.$$

This at the very least tells us that

$$\|A\|_2 := \max_{\mathrm{x}\in\mathbb{R}^n_\star} \frac{\|A\mathrm{x}\|_2}{\|\mathrm{x}\|_2} \geq \max_{i=1,\dots,n} \sqrt{\lambda_i}.$$

With a bit more work, we can show that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the the eigenvalues of $B = A^T A$, then

$$\|A\|_2 = \sqrt{\lambda_n}.$$

Theorem 4.2.4

Let $A \in \mathbb{R}^{n \times n}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, be the eigenvalues of $B = A^T A$. Then

$$\|A\|_2 = \max_{i=1,\dots,n} \sqrt{\lambda_i} = \sqrt{\lambda_n},$$

Here is the main idea. For full details, see the text-book.

Exercise 4.2.1

Show that, for *any* subordinate matrix norm on $\mathbb{R}^{n\times n}$, the norm of the identity matrix is 1.

Exercise 4.2.2

Prove that

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{i,j}|.$$

Hint: Suppose that $\displaystyle\sum_{i=1}^{n} |a_{ij}| \leq C$, for $j = 1, 2, \dots, n$. Show that for *any* vector $x \in \mathbb{R}^n$

$$\sum_{i=1}^{n} |(Ax)_i| \leq C\|x\|_1.$$

Now find a vector x such that $\sum_{i=1}^{n} |(Ax)_i| = C\|x\|_1$. Now deduce the result.