* Wednesday and Friday of Week 9 (8 and 10, March)
* Wednesday of Week 10 (15 March)

MA378 Chapter 3: Numerical Integration

§3.5 Orthogonal Polynomials

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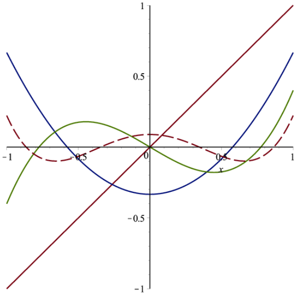


5.1 Orthogonal Polynomials

High order Newton-Cotes methods are of little use because of the problems associated with interpolation be high degree polynomials at equally spaced points. However, high-order Gaussian methods are very useful.

Driving such methods by undetermined coefficients is not practical, however. There is a simpler way, but some mathematical preliminaries are required, including the ideas of **vector spaces** and **inner products**.

5.2 Inner products

Definition 5.1 (Vector Space)

V is a *vector space* (a.k.a., a *linear space*) over a field F (e.g, the real or complex numbers) if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

(i) $\mathbf{u} + \mathbf{v} \in V$ (closed under addition)

(ii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)

(iii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)

(iv) V has a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(v) $-\mathbf{u} \in V$

(vi) $a\mathbf{u} \in V$

(vii) $a(b\mathbf{u}) = (ab)\mathbf{u}$

(viii) F contains 0 and 1 such that $1\mathbf{u} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$.

(ix) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Examples:

The vectors in $\mathbb{R}^2$ form a vector space.

Eg $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2 \Rightarrow \begin{bmatrix} a+c \\ b+d \end{bmatrix} \in \mathbb{R}^2$, etc.

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The set of polynomials of degree at most n forms a vector space

This is important because a vector space of dimension d has a basis consisting of d elements.

A basis for $\mathbb{P}^n$ is $\{1, x, x^2, \dots, x^n\}$.

Also: $\{L_0, L_1, L_2, \dots, L_n\}$ Lagrange Polys on $n+1$ points.

5.2 Inner products

Definition 5.2 (Inner Product)

Let V is a real vector space. An **Inner Product** (IP) is a real-valued function $(\cdot,\cdot)$ on $V \times V$ such that, for all $f, g, h \in V$,

(i) $(f+g,h) = (f,h) + (g,h)$,

(ii) $(\lambda f, g) = \lambda(f,g)$, for $\lambda \in \mathbb{R}$.

(iii) $(f,g) = (g,f)$,

(iv) $(f,f) \geq 0. \qquad (f,f) = 0 \Leftrightarrow f \equiv 0.$

5.2 Inner products

Example 5.3

Let $\mathbb{R}^n$ be our vector space, with $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$. Then the following is an inner product:

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i y_i,$$

 (iv) $(f,f) = \int_a^b (f(x))^2 dx \geq 0$

Example 5.4

The set of real-valued functions that are continuous and defined on the ionterval $[a, b]$, denoted $C[a, b]$, is a vector space. And

$$(f, g) := \int_a^b f(x)g(x)dx, \tag{1}$$

is an inner product.

Note: (i) $(f+g, h) = \int_a^b (f+g)\,h\,dx = \int_a^b fh + gh\,dx$

$= \int_a^b fh\,dx + \int_a^b gh\,dx = (f,h) + (g,h)$

(ii) For any $\lambda \in \mathbb{R}$ $(\lambda f, g) = \int_a^b \lambda f g\,dx = \lambda \int_a^b fg\,dx = \lambda (f,g)$

(iii) $(f,g) = \int_a^b fg\,dx = \int_a^b gf\,dx = (g,f)$

5.3 Sequence of Orthogonal Monic Polynomials

(See Lecture 23 of Stewart's "Afternotes" for more details).

Definition 5.5 (Monic Polynomial)

A polynomial is *monic* if the coefficient of its leading term is 1.

ie the highest degree term present.

Examples:

1, $1+x^2$, $3+2x^2+3x^3+x^5$ are all monic,

But x^{-1} is not (not a polynomial)

nor $1+2x^2+5x^5+x^3$

Definition 5.6

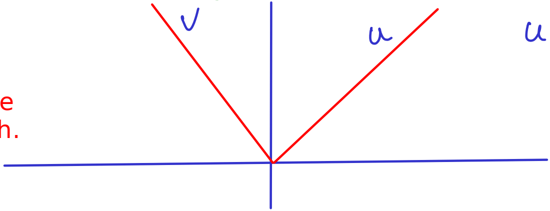
Two elements a, b, of a vector space are *orthogonal* with respect to a given inner product $(\cdot,\cdot)$ if $(a, b) = 0$.

Example:

In $\mathbb{R}^2$, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are orthogonal with respect to the IP

$$(u, v) = \sum_{i=1}^{n} u_i v_i .$$

So too are $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



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Example 5.7

Take the space of polynomials of degree 2 or less and the IP

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx.$$

Let $p(x) \equiv 1$, $q(x) \equiv x$, $r(x) \equiv x^2 - 1/3$, and $f(x) = 3x - 4$

We can check that $(r,p) = 0$, and $(r,q) = 0$. We can the verify that $(r,f) = 0$. **Details:**

$(r,p) = \int_{-1}^{1} x^2 - \frac{1}{3}\, dx = \left(\frac{1}{3}x^3 - \frac{1}{3}x\right)\Big|_{-1}^{1} = \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + \frac{1}{3} = 0.$

$(r,q) = \int_{-1}^{1} x^3 - \frac{1}{3}x\, dx = \left(\frac{1}{4}x^4 - \frac{2}{3}x^2\right)\Big|_{-1}^{1} = \frac{1}{4} - \frac{2}{3} - \frac{1}{4} + \frac{2}{3} = 0$

$(r,f) = (r, 3x - 4) = (r, 3x) - (r,4) = 3(r,q) - 4(r,p)$
$= 3(0) - 4(0) = 0$

5.3 Sequence of Orthogonal Monic Polynomials

As given above, a polynomial is **monic** if the coefficient of the leading term is 1:

$$p_n = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-1} + \cdots + c_1 x + c_0.$$

We'll now look at a sequence of such polynomials

$$\{\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2, \dots, \widetilde{p}_n, \dots\}$$

that have the property they are orthogonal to each other:

$$(\widetilde{p}_i, \widetilde{p}_j) := \int_a^b \widetilde{p}_i(x)\widetilde{p}_j(x)dx = 0 \quad \text{if } i \neq j.$$

We want to establish some important facts about monic polys:

(i)
- A set of monic polys of degrees $1, \dots, n$, forms a basis for $\mathcal{P}_n$.
- If the members of that set are orthogonal to each other, then they are orthogonal to *all* polynomials of lower degree.
- We can construct such as set.

Theorem 5.8

Let $\{\widetilde{p}_i\}_{i=0}^{n}$ be a sequence of polynomials where each p_i is monic an exactly of degree i. This sequence forms as basis for $\mathcal{P}_n$.

Proof: We want to show that, if p_n is any polynomial of degree n, then we can write p_n uniquely as a linear combination of $\{\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_n\}$. Proof is by induction.

First: $\hat{p}_0 = 1$ (since it is monic & of degree 0).

If $n=0$, then we can write any $p_0 \in \mathbb{P}_n$ in terms of 1.

Next assume this is true for $k=1, k=2, \ldots, k=n-1$.

For $n=k$, write p_k as

$$p_k = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0$$

5.3 Sequence of Orthogonal Monic Polynomials

Theorem 5.8 means that if q is a polynomial of degree n then it can be written uniquely as a linear combination of the $\widetilde{p}_i$:

$$q(x) = \sum_{i=0}^{n} a_i \widetilde{p}_i(x),$$

for some unique choice of the real coefficients a_i.

5.3 Sequence of Orthogonal Monic Polynomials

Definition 5.9

The sequence $\{\widetilde{p}_i\}_{i=0}^{n}$ is a sequence of *monic*, *orthogonal* polynomials if each $\widetilde{p}_i$ is monic and *exactly* of degree i and

$$(\widetilde{p}_i, \widetilde{p}_j) = 0 \quad \text{if } i \neq j.$$

Theorem 5.10

If $\widetilde{p}_j \in \{\tilde{p}_i\}_{i=0}^{\infty}$ then $\widetilde{p}_j$ is orthogonal to all polynomials of degree less than j.

Proof:

Let p_k be any polynomial of degree (at most) k. Since $\{\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_k\}$ is a basis for $\mathbb{P}_k$,

$$p_k(x) = \sum_{i=0}^{k} a_i \tilde{p}_i$$

Then, if $k < j$, $(p_k, \tilde{p}_j) = \sum_{i=0}^{k} a_i (\tilde{p}_i, \tilde{p}_j) = 0$

since $i < j$.

5.4 Constructing the Sequence

Theorem 5.11

The sequence $\{\tilde{p}_i\}_{i=0}^{\infty}$ *exists and can be constructed as follows: Let* α *and* β *be defined as*

$$\alpha_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_n)}{(\widetilde{p}_n, \widetilde{p}_n)}, \qquad \textit{and} \qquad \beta_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_{n-1})}{(\widetilde{p}_{n-1}, \widetilde{p}_{n-1})},$$

then the sequence is given by

$$\widetilde{p}_0(x) \equiv 1, \qquad \widetilde{p}_1(x) = x - \alpha_1$$

and

$$\widetilde{p}_{n+1}(x) = (x - \alpha_{n+1})\widetilde{p}_n(x) - \beta_{n+1}\widetilde{p}_{n-1}(x),$$

for $n \geq 1$.

The proof uses *Gram-Schmidt Orthogonalization*.

5.4 Constructing the Sequence

Proof : We want $(\tilde{p}_{n+1}, \hat{p}_n) = 0$

& $(\hat{p}_{n+1}, \hat{p}_{n-1}) = 0.$

The first of these gives

$$0 = \left((x - \alpha_{n+1})\hat{p}_n - \beta_{n+1}\hat{p}_{n-1}, \hat{p}_n \right)$$

$$= (x\hat{p}_n, \hat{p}_n) - \alpha_{n+1}(\hat{p}_n, \hat{p}_n) - \beta_{n+1}\underbrace{(\hat{p}_{n-1}, \hat{p}_n)}_{=0}$$

$$\Rightarrow \alpha_{n+1} = \frac{(x\hat{p}_n, \hat{p}_n)}{(\hat{p}_n, \hat{p}_n)}.$$

The second equation gives the formula for β_{n+1}.

5.4 Constructing the Sequence

Example 5.12

If we use the inner product $(f, g) := \int_{-1}^{1} f(x)g(x)$ then the first 3 polynomials in the sequence are:

$$\widetilde{p}_0 = 1, \quad \widetilde{p}_1 = x, \quad \text{and} \quad \widetilde{p}_2 = x^2 - 1/3.$$

These are p, q & r from Example 5.7.

Example 5.13

The zeros of $\widetilde{p}_2$ are ...

$-1/\sqrt{3}$, $\frac{1}{\sqrt{3}}$, which were x_0 & x_1 for the 2-point gaussian Rule G_2 on $[-1, 1]$.

5.5 Properties of the sequence

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One of the ways of constructing Gaussian Quadrature rule $G_n(\cdot)$ on $n+1$ is to take the quadrature points as the roots of $\widetilde{p}_{n+1}$. We know (from the fundamental theorem of algebra) a polynomial of degree $n+1$ has exactly $n+1$ roots in $\mathbb{C}$ up to multiplicity.

However, the polynomials $\widetilde{p}$ have the special properties, established in the following lemma. (*A slightly different proof of these facts is given in Thm 9.4 of Suli and Mayers.*).

(1) Roots are all real (not complex)

(2) No repeated roots (ie not like $x^2=0$)
because all quadrature points are distinct.

(3) For $\int_a^b f(x)\,dx$ need all points in $[a,b]$

$(u,v) = \int_a^b u(x)v(x)\,dx$

Theorem 5.14

Let $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\widetilde{p}_0, \widetilde{p}_1, \dots\}$ be the set of monic polynomials that are orthogonal with respect to the ([redacted]) inner product.

(i) *The zeros of each $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ are simple (not repeated).*

(ii) *All the zeros of $\widetilde{p}_i$ are real numbers in the interval $[a,b]$.*

Proof: (i) Suppose $\hat{p}_i$ has a repeated root at $x = q$. That is, we can write $\hat{p}_i(x)$ as

$$\hat{p}_i(x) = (x-q)^2 r(x)$$

where $r(x)$ has degree $i-2$. Since $\deg(r) < \deg(\hat{p}_i)$ we know that

$$0 = (\hat{p}_i, r) = \int_a^b \underbrace{(x-q)^2 r(x)}_{\tilde{p}_i(x)} r(x)\,dx = \int_a^b \left[(x-q)\,r(x)\right]^2 dx > 0$$

Consequently, we can say $\hat{p}_i$ has no repeated zero.

5.5 Properties of the sequence

See notes on board for Part (ii).

5.5 Properties of the sequence

5.6 Exercises

Exercise 5.1

$\mathcal{P}_n$, the space of polynomials of degree (at most) n forms a vector space. Is it true that the space of *monic* polynomials of degree n forms a vector space?

Exercise 5.2

(i) Using the Inner Product

$$(f, g) := \int_0^1 f(x)g(x)dx,$$

find $\widetilde{p}_0(x)$, $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$.

(ii) Find the zeros of $\widetilde{p}_2(x)$ and call them x_0 and x_1. Construct a quadrature rule for $\int_{-1}^{1} f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials.