

### MA378 Chapter 2: Splines

## §2.2 (Natural) Cubic Splines

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## 2.1 Introduction

The **Cubic Spline** is perhaps the most useful and popular interpolating function used in numerical analysis, automotive and aeronautical engineering, computer and film animation, digital photography, financial modelling, and as many other areas as there are human endeavours where continuous processes are modelled by discrete ones.

### History

It is generally accepted that (mathematical) splines were first described by Isaac Schoenberg (1903-1990) during WW2. However their physical realisation had been used in the ship-building and aircraft industries prior to that.

## 2.1 Introduction

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For more on cubic splines, see

1. Section 11.4 of “An Introduction to Numerical Analysis”.
2. Lecture 11 of “Afternotes Goes to Graduate School”.

## 2.1 Introduction

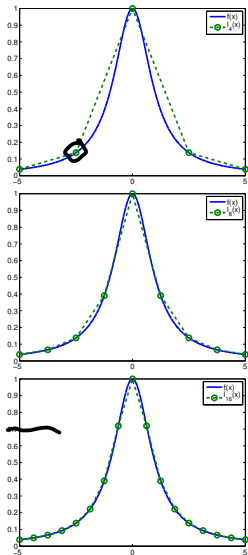
One could argue that the most fundamental short-coming of the linear spline interpolant to  $f$  is that they are not “*smooth*”: that is, although

$$l_i(x_i) = l_{i+1}(x_i),$$

it is generally the case that

$$l'_i(x_i) \neq l'_{i+1}(x_i).$$

Also, the interpolating function cannot capture the “*curvature*” of  $f$ . (If you think about this last statement, you’ll see that it can be expressed as  $l''(x) = 0$  for all  $x \in [a, b]$ .)



## 2.2 Definition

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Cubic splines try to balance the simplicity of the linear spline approach — by using a low-order polynomial on each interval — with the desire for curvature and more smoothness — by using cubics, and by forcing adjacent ones, and their first and second derivatives, to agree at knot points.

## 2.2 Definition

### Definition 2.1 (Cubic Spline)

Let  $f$  be a function that is continuous on  $[a, b]$ . The *cubic spline interpolant* to  $f$  is the continuous function  $S$  such that

- (i) for  $i = 1, \dots, N$ , on each interval  $[x_{i-1}, x_i]$  let  $S(x) = s_i(x)$ , where each of the  $s_i$  is a cubic polynomial.
- (ii)  $s_i(x_{i-1}) = f(x_{i-1})$  for  $i = 1, \dots, N$ , N equations.
- (iii)  $s_i(x_i) = f(x_i)$  for  $i = 1, \dots, N$ , N equations.
- (iv)  $\begin{cases} s'_i(x_i) = s'_{i+1}(x_i) \end{cases}$  for  $i = 1, \dots, N - 1$ , N-1 equations
- (v)  $\begin{cases} s''_i(x_i) = s''_{i+1}(x_i) \end{cases}$  for  $i = 1, \dots, N - 1$ . N-1 equations

Each  $s_i$  has 4 terms. So we've  
 $4N$  parameters to determine

And only  $4N - 2$  conditions to satisfy.

## 2.2 Definition

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So, we have defined the cubic spline  $S$  as a function that...

- (a) is a cubic polynomial on each of the  $n$  intervals  $[x_{i-1}, x_i]$ .  
That is, it is *piecewise cubic*.
- (b) interpolates  $f$  at  $N + 1$  points,
- (c) has a continuous first derivatives on  $[x_0, x_N]$
- (d) has a continuous second derivatives on  $[x_0, x_N]$

## 2.2 Definition

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We know that one can write a cubic as  $a_0 + a_1x + a_2x^2 + a_3x^3$ . So it takes 4 terms to uniquely define a single cubic. To define the spline we need  $4N$  terms. They can be found by solving  $4N$  (linearly independent) equations. But the definition only gives  $4N - 2$  equations.

So we are short two equations.



The “missing” equations can be chosen in a number of ways.

- (i) **By setting  $S''(x_0) = 0$  and  $S''(x_N) = 0$ . This is called a *natural spline*, and is the approach we'll take.**
- (ii) By setting  $S'(x_0) = 0$  and  $S'(x_N) = 0$ . This is called a *clamped spline*.
- (iii) By set  $S'(x_0) = S'(x_N)$  and  $S''(x_0) = S''(x_N)$ . This is the *periodic spline* and is used for interpolating, say, trigonometric functions.
- (iv) Only use  $N - 2$  components of the spline:  $s_2, \dots, s_{N-1}$ . But extend to two end ones so that  $s_2(x_0) = f(x_0)$  and  $s_{N-1} = f(x_N)$ . This is called the *not-a-knot* condition.

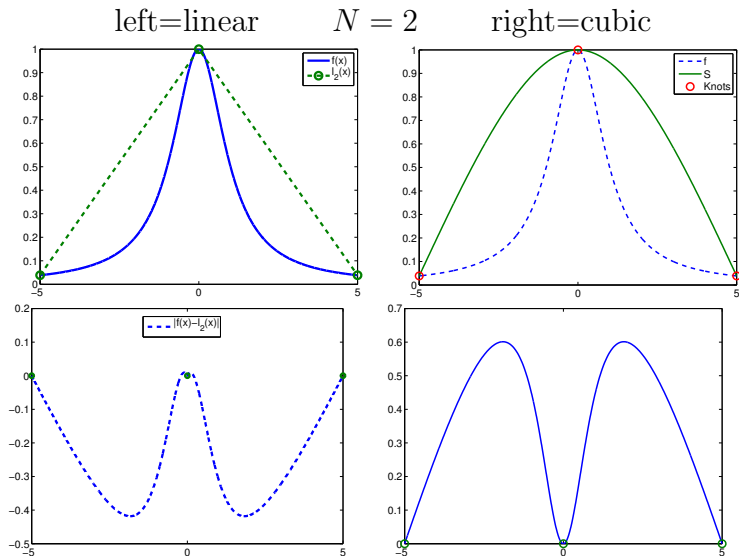
## 2.2 Definition

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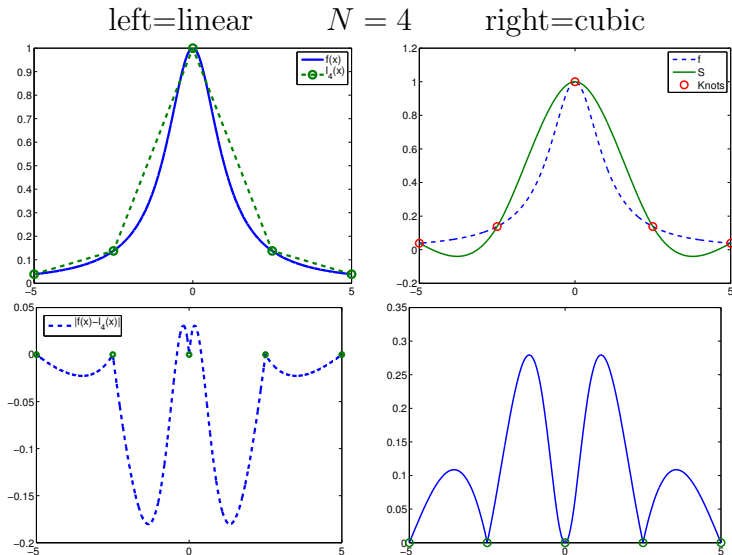
The following figures show the **linear spline** (left) and ***natural cubic spline*** (right), and errors, interpolants to  $f(x) = 1/(1 + x^2)$  (with  $a = -5$  and  $b = 5$  as usual) for various  $N$ .

Compared with linear splines, we seem to get significantly better approximation. Moreover, the rate at which the error decreases with respect to  $h$  is much faster.

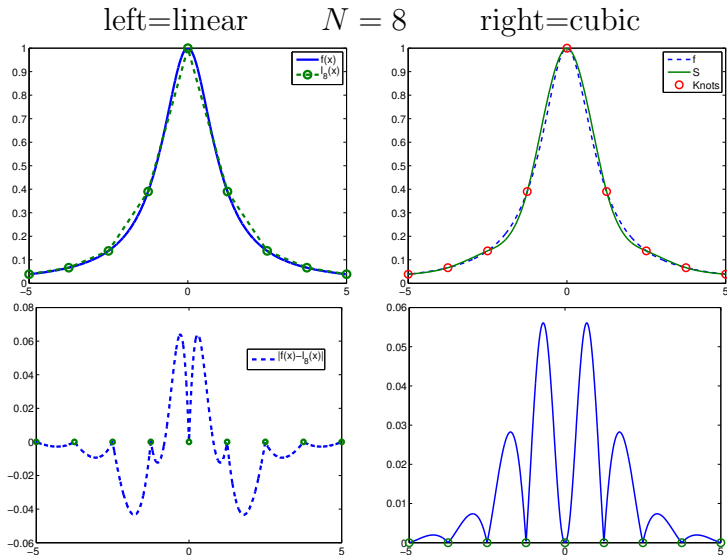
## 2.2 Definition



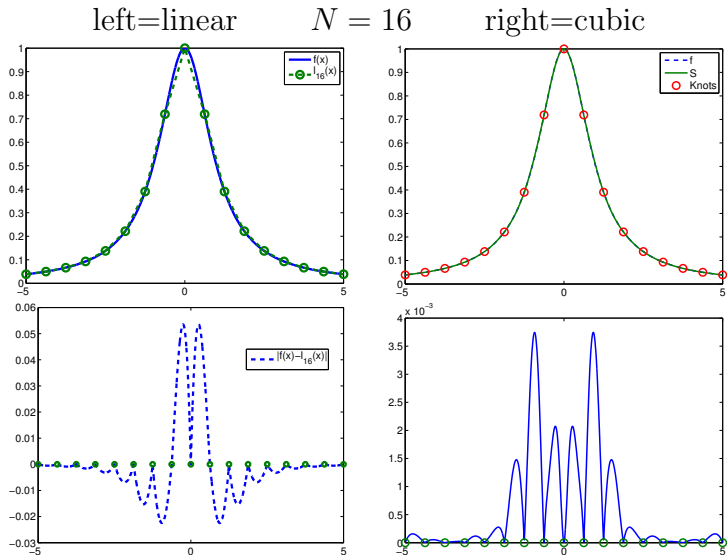
## 2.2 Definition



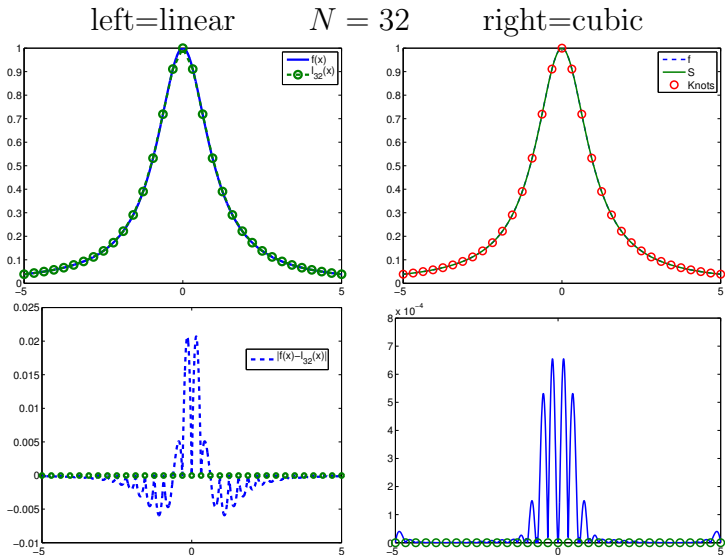
## 2.2 Definition



## 2.2 Definition



## 2.2 Definition



## 2.3 Constructing Cubic Splines

Some notation:

- ▶  $h = x_i - x_{i-1} = (x_N - x_0)/N$  for all  $i$ .
- ▶  $f_i := f(x_i)$ .

To construct the spline, first observe that if  $S$  is piecewise cubic, then  $S'$  is piecewise quadratic, and  $S''$  is piecewise linear.

- (i) Let  $\sigma_i = S''(x_i)$  for  $i = 0, \dots, N$ . pw  
//

Since,  $S(x) = S_i(x)$  on  $[x_{i-1}, x_i]$  is piecewise cubic, so  $S''(x) = S_i''(x)$  on  $[x_{i-1}, x_i]$  is pw linear. We know from Section 2.1 that

$$S_i''(x) = \sigma_{i-1} \frac{x_i - x}{h} + \sigma_i \frac{x - x_{i-1}}{h}$$

where the  $\sigma_i$  are, for now, unknown.



## 2.3 Constructing Cubic Splines

constant of  
integration

(ii) Integrate twice: (w.r.t.  $x$ )

$$S_i'(x) = -\frac{\sigma_{i-1}(x_i - x)^2}{2h} + \frac{\sigma_i(x - x_{i-1})^2}{2h} + A_i$$

$$S_i(x) = \sigma_{i-1} \frac{(x_i - x)^3}{6h} + \sigma_i \frac{(x - x_{i-1})^3}{6h} + A_i x + B_i$$

Since  $A_i$  and  $B_i$  are just some constants, we can write  $A_i x + B_i = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x)$

$$\text{So } \alpha_i - \beta_i = A_i, \quad -\alpha_i x_{i-1} + \beta_i x_i = B_i$$

We get

$$s_i(x) = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) + \frac{\sigma_{i-1}}{6h}(x_i - x)^3 + \frac{\sigma_i}{6h}(x - x_{i-1})^3, \quad (1)$$

for  $x \in [x_{i-1}, x_i]$ .

## 2.3 Constructing Cubic Splines

- (iii) The  $\alpha_i$  and  $\beta_i$  arose as the constants of integration. To find them:

Use that  $\underbrace{S_i(x_{i-1}) = f_{i-1}}_{\textcircled{1}} \text{ \& } \underbrace{S_i(x_i) = f_i}_{\textcircled{2}}$

$$\textcircled{1} \Rightarrow \beta_i (x_i - x_{i-1}) + \frac{\sigma_{i-1}}{6h} (x_i - x_{i-1})^3 = f_{i-1}$$

$$\Rightarrow \beta_i h + \frac{1}{6} h^2 \sigma_{i-1} = f_{i-1}$$

Similarly

$$\textcircled{2} \Rightarrow \alpha_i h + \frac{1}{6} h^2 \sigma_i = f_i$$

Note that  $f_{i-1}$  &  $f_i$  are given. So, if we know the  $\sigma_i, \sigma_{i-1}$  we can solve for  $\alpha_i, \beta_i$

## 2.3 Constructing Cubic Splines

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and so, for  $i = 1, 2, \dots, N$

$$\alpha_i = \frac{f_i}{h} - \frac{h}{6}\sigma_i, \quad \beta_i = \frac{f_{i-1}}{h} - \frac{h}{6}\sigma_{i-1}. \quad (2)$$

(Note that  $\beta_i = \alpha_{i-1}$ .)

## 2.3 Constructing Cubic Splines

- (iv) So, now we “just” need the equations for  $\sigma_i$ . Two of these come from the fact that this is a “natural” cubic spline, with  $S''(x_0) = 0$  and  $S''(x_N) = 0$ , therefore  $\sigma_0 = 0$  and  $\sigma_N = 0$ .

The remaining  $N - 1$  equations come from the fact that  $S'$  is continuous at  $x_1, x_2, \dots, x_{N-1}$ . So we get that

$$s'_i(x_i) = s'_{i+1}(x_i).$$

After some work (see exercises...), we can show this means that the system is:

$$\sigma_0 = 0, \quad \text{with } S''(x_0) = 0 \quad (3a)$$

$$\frac{1}{6}(\sigma_{i-1} + 4\sigma_i + \sigma_{i+1}) = \frac{1}{h^2}(f_{i-1} - 2f_i + f_{i+1}) \quad (3b)$$

for  $i = 1, \dots, N - 1$ ,

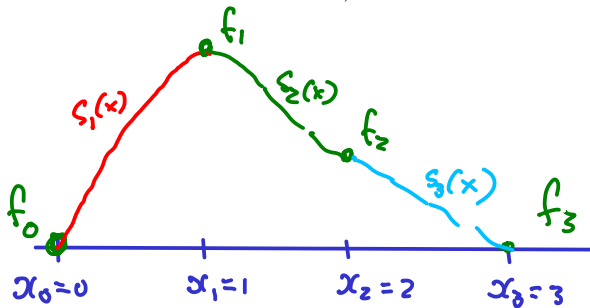
$$\sigma_N = 0, \quad \text{with } S''(x_N) = 0 \quad (3c)$$

## 2.4 A cubic spline example

### Example 2.2

Find the natural cubic spline interpolant to  $f$  at the points  $x_0 = 0$ ,  $x_1 = 1$ ,  $x_2 = 2$  and  $x_3 = 3$  where  $\underline{f_0 = 0}$ ,  $\underline{f_1 = 2}$ ,  $\underline{f_2 = 1}$  and  $\underline{f_3 = 0}$ .

[We won't do all the details in class, so there are included below]



## 2.4 A cubic spline example

### Solution:

From (1) we see we are looking for  $S(x)=$

$$S_1(x) = \alpha_1 x + \beta_1(1 - x) + \frac{\sigma_0}{6h}(1 - x)^3 + \frac{\sigma_1}{6h}x^3, \quad x \in [0, 1],$$

$$S_2(x) = \alpha_2(x - 1) + \beta_2(2 - x) + \frac{\sigma_1}{6h}(2 - x)^3 + \frac{\sigma_2}{6h}(x - 1)^3, \quad x \in [1, 2],$$

$$S_3(x) = \alpha_3(x - 2) + \beta_3(3 - x) + \frac{\sigma_2}{6h}(3 - x)^3 + \frac{\sigma_3}{6h}(x - 2)^3, \quad x \in [2, 3].$$

## 2.4 A cubic spline example

We first solve for the  $\sigma_i$  using (3):

$$\frac{1}{6} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\sigma_0 = 0, \quad \sigma_1 = -\frac{24}{5}, \quad \sigma_2 = \frac{6}{5}, \quad \sigma_3 = 0.$$

Now use (2) to get

$$\alpha_1 = \frac{14}{5}, \quad \alpha_2 = \frac{4}{5}, \quad \alpha_3 = 0.$$

and

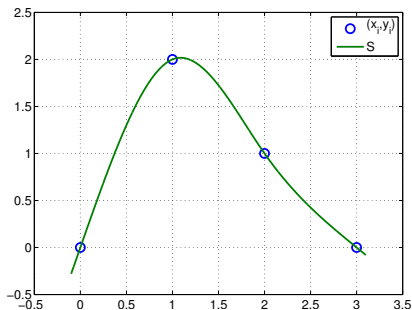
$$\beta_1 = 0, \quad \beta_2 = \frac{14}{5}, \quad \beta_3 = \frac{4}{5}.$$

## 2.4 A cubic spline example

The answer is

$$S(x) = \begin{cases} \frac{14}{5}x - \frac{4}{5}x^3, & x \in [0, 1], \\ \frac{4}{5}(x-1) + \frac{14}{5}(2-x) - \frac{4}{5}(2-x)^3 + \frac{1}{5}(x-1)^3, & x \in [1, 2], \\ \frac{4}{5}(3-x) + \frac{1}{5}(3-x)^3. & x \in [2, 3]. \end{cases}$$

This is shown below.





## 2.5 Error Estimates

We state the following error estimates without proof. You don't need to know these, or be able to prove them. But you should be able to use them if required.

### Theorem 2.3

If  $f \in C^4[a, b]$  and  $S$  is its cubic spline interpolant on  $N + 1$  equally spaced points, then

$$\begin{aligned}\|f - S\|_{\infty} &\leq \frac{5}{384} M_4 h^4, \\ \|f' - S'\|_{\infty} &\leq \frac{1}{24} M_4 h^3, \\ \|f'' - S''\|_{\infty} &\leq \frac{3}{8} M_4 h^2,\end{aligned}$$

where  $M_4 = \|f^{(iv)}\|_{\infty}$ .

## 2.5 Error Estimates

### Example 2.4

Give an upper bound on the error for the cubic spline interpolant to  $f = e^x$  on the interval  $[-1, 1]$  with  $N = 10$  mesh points.

$$f(x) = e^x, \quad \text{so} \quad f^{(iv)}(x) = e^x. \quad \text{So} \quad M_4 = e \approx 2.718.$$

$$h = \frac{x_N - x_0}{N} = \frac{1 - (-1)}{10} = 0.2$$

$$\begin{aligned} \text{So} \quad \|f - S\|_{\infty} &\leq \frac{5}{384} M_4 h^4 = \frac{5}{384} (2.718) (0.2)^4 \\ &= 5.6631 \times 10^{-5} \end{aligned}$$

Note, for linear splines we'd get

$$\|f - L\|_{\infty} \leq 1.4 \times 10^{-2}.$$

## 2.5 Error Estimates

on  $[-1, 1]$ .

### Example 2.5

What is the smallest value of  $N$  that you must take to ensure that, if interpolating  $f(x) = e^x$  on  $N$  equally sized intervals, the error is less than  $10^{-8}$ ? How does this compare with a linear spline interpolation (see corresponding example for linear splines).

We want

$$\frac{5}{384} (M_4) h^4 \leq 10^{-8}, \text{ where } M_4 = 2.718.$$

We should find we need  $h \leq 0.02306$ .

So  $N \geq 86.7$  i.e., take  $N = 87$ .

For linear splines, the answer is about 10,000.

## 2.5 Error Estimates

Recall the minimum energy property of linear splines. There is an analogous result for natural cubic splines.

### Theorem 2.6

*Let  $u$  be any function that interpolates  $f$  at  $\omega^N$ , and is such that  $u \in H^2(x_0, x_N)$ . Then*

$$\|S''\|_2 \leq \|u''\|_2.$$

The proof is not hard - it is analogous to the proof of Theorem 2.1.9.

## 2.6 Exercises

### Exercise 2.1

[Equation numbers given here refer to those for the slides for Section 2.2.] When deducing the system of equations for the natural cubic spline, we showed how to construct the formulation in (1). and the relationship between  $\sigma_i$ ,  $\alpha_i$  and  $\beta_i$  in (2). Now carefully show to deduce the system (3).

### Exercise 2.2

(For students who did MA385). Write the equations in (3) as a matrix-vector equation  $A\sigma = \mathbf{b}$ , where  $A$  is an  $n \times n$  matrix. Show that  $A$  is nonsingular, and hence that the system has a unique solution.

### Exercise 2.3

Find the natural cubic spline interpolant to  $f(x) = \sin(\pi x/2)$  at the nodes  $\{x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3\}$ . Calculate value of the interpolant at  $x = 2.5$ . What is the error at this point?

## 2.6 Exercises

### Exercise 2.4 (★)

Take  $f(x) = \ln(x)$ ,  $x_0 = 1$ ,  $x_N = 2$ . What value of  $N$  would you have to take to ensure that  $|\ln(x) - S(x)| \leq 10^{-4}$  for all  $x \in [1, 2]$ , where  $S$  is the natural cubic spline interpolant to  $f$ .

### Exercise 2.5

Suppose that  $S$  is a natural cubic spline on  $[0, 2]$  with

$$S(x) = \begin{cases} -3x + 2(1-x) + a(1-x)^3 + \frac{2}{3}x^3, & x \in [0, 1), \\ b(2-x) + c(2-x)^3 + d(x-1)^3, & x \in [1, 2]. \end{cases}$$

Find  $a$ ,  $b$ ,  $c$ , and  $d$ .

See also Exercise 2.6 on the "non-annotated" version of the slides.