

CS4423: Networks

Week 6, Part 1: Centrality Measures

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These notes were revised after the lecture on 20 Feb, to correct for a mathematical error in Slide 13; see Slide 14.

These slides include material by Angela Carnevale.

Outline

Today's notes are split between these slides, and a Jupyter Notebook.

1 Centrality Measures

2 Degree Centrality

- Normalized

3 Eigenvector Centrality

- Eigenvalues and Eigenvectors

4 Centrality

5 Perron-Frobenius Theory

- Irreducible Matrix
- Non-negative matrix

6 The Theorem

- Erratum

Slides are at:

<https://www.niallmadden.ie/2425-CS4423>



Centrality Measures

What is it that makes a node in a network important?

Key nodes in networks can be identified through **centrality measures**: a way of assigning “scores” to nodes that represents their “importance”. However, what it means to be central depends on the context.

Examples

- ▶ In a friendship network, who is most popular?
- ▶ In a epidemiology network, who is most likely to get infected?
- ▶ In a banking, which institution poses the greatest danger to the system should it fail?

Accordingly, in the context of network analysis, a variety of different centrality measures have been developed.

Centrality Measures

Measures of centrality include:

- ▶ **Degree Centrality** which is just the degree of the node. It can be important in e.g., transport networks.
- ▶ **Eigenvector Centrality**, defined in terms of properties of the network's **adjacency matrix**.
- ▶ **Closeness Centrality**, defined in terms of a nodes **distance** to other nodes on the network.
- ▶ **Betweenness Centrality**, defined in terms of **shortest paths**.

Degree Centrality

Definition (Degree Centrality)

In a (simple) graph $G = (X, E)$, with $X = \{0, \dots, n-1\}$ and adjacency matrix $A = (a_{ij})$, the degree centrality c_i^D of node $i \in X$ is defined as

$$c_i^D = k_i = \sum_j a_{ij},$$

where k_i is the degree of node i .

Example:

In some cases, this measure can be misleading, since it depends—among other things—on the order of the graph. A better measure is then the following.

Normalized Degree Centrality

The **normalized degree centrality** C_i^D of node $i \in X$ is defined as

$$C_i^D = \frac{k_i}{n-1} = \frac{c_i^D}{n-1} \left(= \frac{\text{degree centrality of node } i}{\text{number of potential neighbors of } i} \right).$$

Note: in a directed graph one distinguishes between the in-degree and the out-degree of a node and defines the in-degree centrality and the out-degree centrality accordingly.

We now recall from important facts from **Linear Algebra**.

Eigenvalues and Eigenvectors

Let A be a square $n \times n$ matrix. An n -dimensional vector, \mathbf{v} , is called an **eigenvector** of A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar (number), λ , which is called an **eigenvalue** of A .

Example:

- ▶ When is a real-valued matrix, one usually finds that λ and \mathbf{v} are *complex valued*. However, if A is symmetric then they are *real valued*.
- ▶ A may have up to n eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_n$.
- ▶ The **spectral radius** of A is $\rho(A) := \max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\}$
- ▶ If \mathbf{v} is an eigenvector associated with the eigenvalue λ , so too is any non-zero multiple of \mathbf{v}

Centrality

The basic idea of eigenvector centrality is that **a node's ranking in a network should relate to the rankings of the nodes it is connected to.**

More specifically, up to some scalar λ , the centrality c_i^E of node i should be equal to the sum of the centralities c_j^E of its neighbouring nodes j .

In terms of the adjacency matrix $A = (a_{ij})$, this relationship is expressed as

$$\lambda c_i^E = \sum_j a_{ij} c_j^E,$$

which in turn, in matrix language is

$$\lambda c^E = A c^E,$$

for the vector $c^E = (c_i^E)$, which then is an eigenvector of A .

So c^E is an eigenvector of A ! (But which one???)

How to find c^E and/or λ ?

If the network is small, one could use the usual method (although it is almost never a good idea)

1. Find the *characteristic polynomial* $p_A(x)$ of A , as *determinant* of the matrix $xI - A$, where I is the $n \times n$ identity matrix);
2. Find the *roots* λ of $p_A(x)$ (i.e. scalars λ such that $p_A(\lambda) = 0$);
3. Find a *nontrivial solution* of the linear system $(\lambda I - A)c = 0$ (where 0 stands for an all-0 column vector, and $c = (c_1, \dots, c_n)$ is a column of *unknowns*).

For large networks, there are much better algorithms, such as the **Power Method** that we'll study later (in the Week 6 – Part 3 Jupyter Notebook).

Presently, we'll learn that the adjacency matrix always has one eigenvalue which is greater than all the others.

First, some definitions:

Irreducible Matrix

A matrix A is called **reducible** if, for some simultaneous permutation of its rows and columns, it has the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

A is **irreducible** if it is not reducible.

Important: The adjacency matrix of a simple graph G is irreducible if and only if G is connected.

Non-negative matrix

A matrix $A = (a_{ij})$ is **non-negative** if

$$a_{ij} \geq 0 \quad \text{for all } i, j.$$

For simplicity, we usually write $A \geq 0$.

Important: Adjacency matrices are examples non-negative matrices.

There are similar concepts of, say, positive matrices (nothing to do with positive definite!!), negative matrices.

Of particular importance are **positive vectors**: $v = (v_i)$ is positive if $v_i > 0$ for all i . We write $v > 0$.

The Theorem

Theorem (Perron-Frobenius Theorem 1907/1912)

*Suppose that A is a square, nonnegative, **irreducible** matrix. Then:*

- ▶ *A has a real eigenvalue $\lambda = \rho(A)$ and $\lambda \geq |\lambda'|$ for any other eigenvalue λ' of A . λ is called the **Perron root** of A*
- ▶ *λ is a simple root of the characteristic polynomial of A (so has just one corresponding eigenvector)*
- ▶ *There is an eigenvector, \mathbf{v} associated with λ , such that $\mathbf{v} > 0$.*

For us this means:

- (i) The adjacency matrix of a connected graph has an eigenvalue that positive; **no other eigenvalue is greater in magnitude.**
(See next page)
- (ii) It has an eigenvector, \mathbf{v} that is positive.
- (iii) \mathbf{v}_i is the Eigenvector Centrality node i .

CORRECTION

The version of Slide 13 shown in class had two errors:

- (a) In the statement of the theorem, it claimed that the Perron Root, λ , satisfied $|\lambda| > |\lambda'|$ where λ' is any other eigenvalue of A . It should have read $|\lambda| \geq |\lambda'|$.
- (b) This error was repeated in the first statement after the theorem. That has been corrected (see text in red)

Interestingly, the statement that $|\lambda| > |\lambda'|$ would be true if either of the following is true:

1. The matrix A is positive (which is never the case for the adjacency matrix of a simple graph: no loops means that the diagonal entry is always zero).
2. If there is some number k such that $A^k > 0$. That can happen. Will discuss more in class next week.