MA378: Assignment 2 (Version 2.0) ANS with solutions

Deadline: 5pm, Monday 20 March.

Your solutions must be clearly written, and neatly presented. You can submit an electronic copy, through blackboard, or a hard copy. If submitting a hard copy, please do so at the 10am lecture in the 10th. Also, make sure pages should be stapled together. Marks will be given for quality and clarity of exposition ([15 Marks]). Usual collaboration policy applies.

Chapter 2: Piecewise Polynomial Interpolation

Exer 3.2 [20 Marks] Let $f(x) = \ln(\mathbf{x^2}) - \mathbf{x^4}$. Let l and S be the piecewise linear and Hermite cubic spline interpolants (respectively) to f on $N+1$ equally spaced points $1 = x_0 < x_1 < \cdots < x_N = 2$. What value of N would you have to take to ensure that

(i) $\max_{1 \leqslant x \leqslant 2} |f(x) - l(x)| \leqslant 10^{-\mathbf{6}}$?

> **Answer:** From Thm 1.3 of Chapter 3, the error is bounded as
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> $$\|f - l\|_\infty \leqslant \frac{h^2}{8}\|f''\|_\infty.$$
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> Since $f''(x) = -2(6x^2 + x^{-2})$ is negative and decreasing for on $1 \leqslant x \leqslant 2$, $\|f - l\|_\infty = -f''(2) = 97/2 = 48.5$. So we need to choose h so that $(h^2)(48.5)/8 \leqslant 10^{-6}$. That gives $h \leqslant \sqrt{8 \times 10^{-6}/48.5} = 4.06 \times 10^{-4}$. Since $N = 1/h$, this gives $N \geqslant 2462.2$. As N must be an integer, we choose $N = 2463$.

(ii) $\max_{1 \leqslant x \leqslant 2} |f(x) - S(x)| \leqslant 10^{-\mathbf{6}}$?

> **Answer:** From Thm 3.2 of Chapter 3, the error is bounded as
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> $$\|f - S\|_\infty \leqslant \frac{h^4}{384}\|f^{(iv)}\|_\infty.$$
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> Since $f^{(iv)}(x) = -12(2 + x^{-4})$ is negative but increasing for on $1 \leqslant x \leqslant 2$, $\|f - l\|_\infty = -f''(1) = 36$. So we need to choose h so that $(h^4)(36)/384 \leqslant 10^{-6}$. That gives $h \leqslant \left(384 \times 10^{-6}/36\right)^{1/4} = 5.715 \times 10^{-2}$. Since $N = 1/h$, this gives $N \geqslant 17.498$. As N must be an integer, we choose $N = 18$.

Chapter 3: Numerical Integration

Exer 1.1 [10 Marks] (*For simplicity, you may assume that the quadrature rule is integrating f on the interval $[-1, 1]$.*) Let q_0, q_1, ..., q_n be the quadrature weights for the Newton-Cotes rule $Q_n(f)$. Show that $q_i = q_{n-i}$ for $i = 0, \ldots n$.

> **Answer:** There are a few possible ways of answering this one. Here is one. Recall that $q_i = \int_{-1}^{1} L_i(x)dx$, where L_i is the ith Lagrange polynomial associated with the points $-1 = x_0 < x_1 < \cdots < x_n = 1$. That is, $L_i(x)$ and $L_{n-i}(x)$ are the unique polynomials of degree n with the properties that
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> $$L_i(x_j) = \begin{cases} 1 & x_j = x_i \\ 0 & x_j \neq x_i, \end{cases} \qquad \text{and} \qquad L_{n-i}(x_j) = \begin{cases} 1 & x_j = x_{n-i} \\ 0 & x_j \neq x_{n-i}. \end{cases}$$
>
> Since the x_i are uniformly spaced on $[-1, 1]$ we can see that $x_i = -x_{n-i}$. Therefore,
> $L_{n-i}(-x_j) = \begin{cases} 1 & x_j = -x_{n-i} = x_i \\ 0 & x_j \neq -x_{n-i} = x_i. \end{cases}$ Thus $L_{n-i}(x) = L_i(-x)$. With the substitution $y = -x$, we can see that $q_{n-i} = \int_{-1}^{1} L_{n-i}(x)dx = \int_{-1}^{1} L_i(-x)dx = -\int_{1}^{-1} L_i(y)dy = \int_{-1}^{1} L_i(y)dy = q_i$ (note the change in the limits of integration). So $q_i = q_{n-i}$.

Exer 3.5 [15 Marks] Consider the rule (which is not, strictly speaking, a Newton-Cotes rule):

$$R(f) = q_0 f(\frac{1}{3}) - f(\frac{1}{2}) + q_2 f(\frac{3}{4})$$

for approximating $\int_0^1 f(x)dx$.

(a) Determine values of q_0 and q_2 that ensure this rule has precision 2.

> **Answer:** We need to find q_0 and q_2 so that $R(f) = \int_0^1 p_2(x)dx$ where p_2 is any polynomial of degree 2. Since that space of polynomials is spanned by the set $\{1, x, x^2\}$, we take q_0 and q_2 to satisfy the equations $q_0 - 1 + q_2 = 1$, $q_0/2 - 1/2 + q_2(3/4) = 1/2$, and $q_0/9 - 1/4 + q_2(9/16) = 1/3$. These equations are not linearly independent (since there are only two unknowns. Solving any pair of them should give $q_0 = 6/5$ and $q_2 = 4/5$. So $R(f) = \frac{6}{5}f(\frac{1}{3}) - f(\frac{1}{2}) + \frac{4}{5}f(\frac{3}{4})$.

(b) What is the maximum precision of $R(\cdot)$ with the values of q_1 and q_2 that you have determined?

> **Answer:** Could this method be exact for some higher degree polynomials? Checking with $f(x) = x^3$, we should find that $R(x^3) = 37/144 \neq \int_0^1 x^4 dx$. So the precision is at most 2.

Exer 3.4 [20 Marks] Determine the precision of the following schemes for estimating $\int_0^1 f(x)dx$.

> **Answer:** In the following solutions, $I(f) := \int_0^1 f(x)dx$. For each method the precision is n if $Q(x^k) = I(x^k)$ of $k = 0, \dots n$, but $Q(x^{n+1}) \neq I(x^{n+1})$.

(i) $Q(f) = f(\frac{1}{2})$.

> **Answer:** $Q(1) = 1 = I(1)$ and $Q(x) = 1/2 = I(x)$, but $Q(x^2)1/4 \neq I(x^2)$. So this method has precision 1. FYI, this is the so-called mid-point rule. It is the 1-point Gaussian Quadrature Rule.

(ii) $Q(f) = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.

> **Answer:** $Q(1) = 1 = I(1)$, $Q(x) = 1/2 = I(x)$, $Q(x^2)1/3 = I(x^2)$. But $Q(x^3) = 2/9 \neq I(x^3)$. So this method has precision 2.

(iii) $Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{3})$.

> **Answer:** $Q(x^k) = 1/(k+1) = I(x^k)$, for $k = 0, 1, 2, 3$. But $Q(x^4) = 41/216 \neq I(x^4)$. So $Q(\cdot)$ has precision 3.

Exer 4.3 [20 Marks] Derive a 3-point Gaussian Quadrature Rule to estimate $\int_{-1}^1 f(x)dx$. *Hint:* $x_1 = 0$.

> **Answer:** The method is $G_2(f) := w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$, and, since it has 6 degrees of freedom, it should be exact for each of the polynomials in $\{1, x, x^2, x^3, x^4, x^5\}$. These 6 polynomials will lead to 6 (nonlinear) equations. However, since we know that $x_1 = 0$, we need only 5. In any case the equations are
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> (i) $w_0 + w_1 + w_2 = 2$ (ii) $w_0x_0 + w_2x_2 = 0$ (iii) $w_0x_0^2 + w_2x_2^2 = 2/3$
> (iv) $w_0x_0^3 + w_2x_2^3 = 0$ (v) $w_0x_0^4 + w_2x_2^4 = 2/5$ (vi) $w_0x_0^5 + w_2x_2^5 = 0$
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> (ii) Gives that $w_0x_0 = -w_2x_2$. Substitute this into (iv) to get that $(-w_2x_2)x_0^2 - w_2x_2^3 = 0$. Since $x_2 \neq 0$, and $w_2 \neq 0$, we can deduce that $x_0^2 = x_2^2$. So $x_0 = -x_2$, because $x_0 < x_2$. Again using (ii) we get $w_0 = w_2$. Next use (iii) to see that $w_0x_0^2 = 1/3$, and (v) to give $w_0x_0^4 = 1/5$. Combining those leads to $x_0 2 = 3/5$. So now we have that $x_0 = -\sqrt{3/5}$ and $x_1 = \sqrt{3/5}$. Reusing $w_0x_0^2 = 1/3$, we have that $w_0 = 5/9 = w_2$. Finally, (i) gives $w_1 = 8/9$. That is, the method is
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> $$G_2(f) = \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}}).$$