MA140: Engineering Calculus

Week 1, Lecture 3: Polynomials and Partial Fractions

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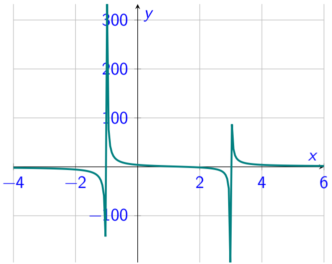


This version of the slides are by Niall Madden, but are adapted from original notes by Dr Kirsten Pfeiffer.

Outline

1. News!
   - Tutorials
   - Exercise sheet
2. Functions (again)
   - Recall...
3. Polynomials
   - Sketching polynomials
4. Rational Functions
   - Long division
5. Partial Fractions

For more, see Sections 2.4 (Polynomials) 2.5 (Rational Functions) of *Modern Engineering Mathematics*:
https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Tutorials start next week. Here is the schedule:

- Teams 1+2: Tuesday 15:00 ENG-**2003**
- Teams 3+4: Tuesday 15:00 ENG-**2034**
- Teams 9+10: Thursday 11:00 ENG-**2002**
- Teams 11+12: Thursday 11:00 ENG-**3035**
- Teams 5+6: Friday 13:00 Eng-**2002**
- Teams 7+8: Friday 13:00 Eng-**2035**

Note: I think the schedule is correct, but the venues are not confirmed... An announcement will be posted to Canvas on Monday confirming.

Would you be interested to taking a tutorial through Irish? If so, please complete this survey: `https://tinyurl.com/suirbhe1`

You don't have to complete a graded assignment next week. However, this is a "practice" one available. See https://universityofgalway.instructure.com/courses/35693/assignments/94873

During tutorials, the tutor will solve some similar questions. You can access the tutorial sheet at https://universityofgalway.instructure.com/courses/35693/files/2023552?module_item_id=650912

Yesterday, we learned that

- A **function** is a rule for mapping from elements of one set (the domain) to elements of another (the codomain).
- When we write $y = f(x)$, we say "x" is the **argument** of the function.
  $\approx$ input.
- When $y = f(x)$ for some $x \in X$, y is said to be the **image** of x under f.
  output
- The set of all images $y = f(x), x \in X$, is called the **range** of f.

Polynomials

A **polynomial function** (or just **polynomial**) is a function of the form

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0, \quad x \in \mathbb{R},$$

where $a_0, a_1, ..., a_n$ are real numbers called the **coefficients** of the polynomial.
The number n is called the **degree** of the polynomial.

Polynomials

Example: linear

$y = x$ is a **linear** polynomial with degree $n = 1$.

Eg $f(x) = 3x^3 + 2x^2 - 1$ is a polynomial of degree 3.

$f(x) = \pi x^2 + e x^4 + x$ is a poly of degree 4.

$f(x) = x^{1/2}$ and $f(x) = x^{-1} = \frac{1}{x}$ are not polys.

Polynomials

Example: quadratic

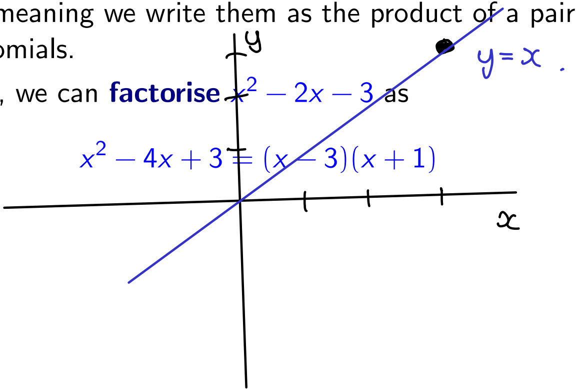
$x^2 - 2x - 3$. is a **quadratic** polynomial with degree $n = 2$.

ie $f(x) = x$ with $y = x$

There are many occasions when we want to **factorise** such quadratics, meaning we write them as the product of a pair of linear polynomials.

For example, we can **factorise** $x^2 - 2x - 3$ as

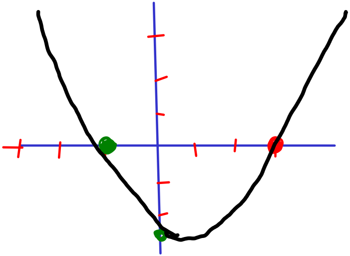
$$x^2 - 4x + 3 = (x - 3)(x + 1)$$



Polynomials

It is important to note that not all quadratic polynomials can be factorised as two linear polynomials. Such quadratics are called **irreducible**.

For example, $x^2 + 1$ is irreducible.



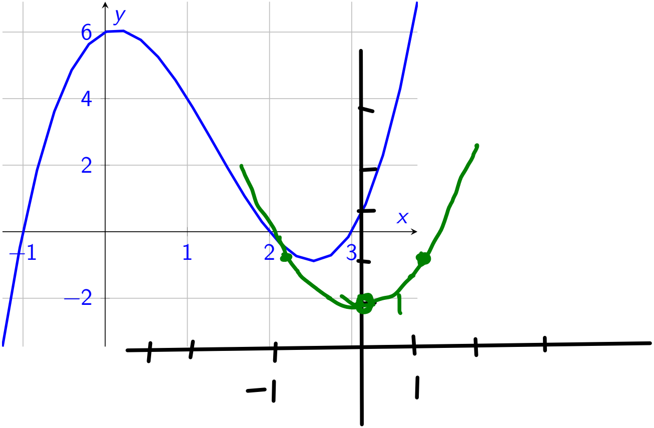
$= xx - 3x + x(1) + (-3)(1)$

$= x^2 - 2x - 3$ ✓

Polynomials

Example

$y = x^3 - 4x^2 + x + 6$ is a **cubic** function with degree $n = 3$.



Fact: A polynomial function of grade n has **up to** $n-1$ truning points ("bends").

Examples:

Note that a cubic always has at least one linear factor. This one has 3

So, since $f(-1) = f(2) = f(3) = 0$

$y = f(x) = (x+1)(x-2)(x-3)$,

Check!

" "

Break Time

During the break, think and talk about what you might do to sketch the graph of

Turning points

As we saw, $y = -x^3 + x^2 + 2x$

the cubic $x^3 - 4x^2 + x + 6$

has 2 turning points.

This can be useful when sketching.

- To sketch the graph, first find the **intercepts**:
  - The **y-intercepts** can be found by letting $x = 0$.
  - The **x-intercepts** are called the **roots** (or **zeros**). To find the roots, set y equal to zero and solve for x.
- You don't have to use the same scale on the x- and on the y-axis.
- Do not use graph paper.

Ideas:

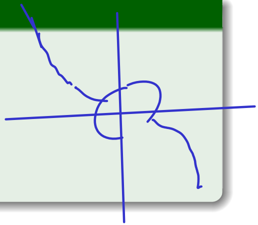
1 Check its value for several values of x.

Example

Sketch the graph of

$$y = -x^3 + x^2 + 2x$$

2 The shape will be like

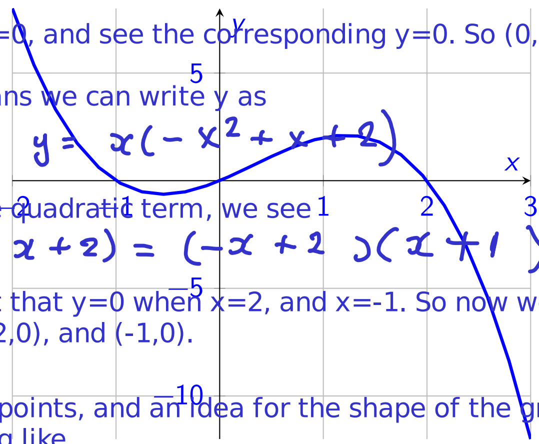


3 $y = x(\underbrace{-x^2 + x + 2}_{\text{factorize.}})$

C

How to sketch $y = -x^3 + x^2 + 2x$

Actual plot of $y = -x^3 + x^2 + 2x$



First we set x=0, and see the corresponding y=0. So (0,0) is a point on the graph.
That also means we can write y as

$$y = x(-x^2 + x + 2)$$

Factorising the quadratic term, we see

$$(-x^2 + x + 2) = (-x + 2)(x + 1)$$

we so also get that y=0 when x=2, and x=-1. So now we have two more points (2,0), and (-1,0).

With these 3 points, and an idea for the shape of the graph, we get something like ...

Exercises

Sketch the graphs of

(i) $y = 5x^2 - 7$

(ii) $y = x^2 - 4x + 3$

(iii) $y = x^3 - 6x^2 - 11x - 6$

Rational Functions

Rational Functions have the general form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

- If degree of $p(x) <$ degree of $q(x)$,
  $f(x)$ is called a **strictly proper rational function**.
- If degree of $p(x) =$ degree of $q(x)$,
  $f(x)$ is called a **proper rational function**.
- If degree of $p(x) >$ degree of $q(x)$,
  $f(x)$ is called an **improper rational function**.

Rational Functions

An improper or proper rational function can always be expressed as a polynomial plus a strictly proper rational function, for example by algebraic division.

Example

$$\frac{4x^3 + 4x^2 + 4}{x^2 - 3} = 4x + 4 + \frac{12x + 16}{x^2 - 3}$$

eg $\frac{x}{x^2 - 1}$

Eg $\frac{x^2}{x^2 - 1}$

$\frac{x^3}{x^2 - 1}$.

$\frac{10}{3} = 3 + \frac{1}{3}$

For the previous example, we can work this out ourselves using **Long Division** to divide numerator by denominator:

$$\frac{p(x)}{q(x)}$$

proper

$\deg(p) = 3 > 2 = \deg(q)$

Example 2.30 from text book

Use long division to show that

$$\frac{3x^4+2x^3-5x^2+6x-7}{x^2-2x+3} = 3x^2+8x+2-\frac{14x+13}{x^2-2x+3}$$

$$\begin{array}{r} 4x + 4 \\ x^2-3 \,\big)\, 4x^3 + 4x^2 + 4 \\ -\left(4x^3 - 12x\right) \\ \hline 4x^2 + 12x + 4 \\ -\left(4x^2 - 12\right) \\ \hline 12x + 16 \end{array}$$

$12x + 16 \leftarrow$ Remainder.

So $\quad 4x^3+4x^2+4 = (x^2-3)(4x+4) + 12x + 16$

$$\Rightarrow \quad \frac{4x^3+4x^2+4}{x^2-3} = 4x+4 + \frac{12x+16}{x^2-3}.$$

Partial Fractions

A (proper) rational function can often be written as a sum of simpler ones: **partial fractions.**

For example

$$\frac{8x-12}{x^2-2x-3}$$

can be written as

$$\frac{3}{x-3}+\frac{5}{x+1}$$

We verified this in class. Next week, we see how to compute partial fractions?

Finished here.

$=$

$$= \frac{3(x+1)}{(x-3)(x+1)} + \frac{5(x-3)}{(x+1)(x-3)}$$

$$= \frac{3x+3+5x-15}{x^2-2x-3}$$

$$= \frac{8x-12}{x^2-2x-3}.$$