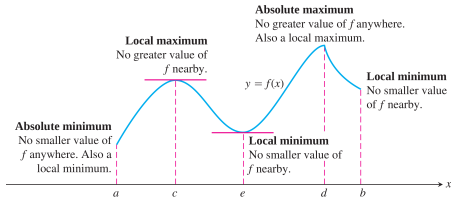


# Week 06, Lecture 1 Maxima and Minima

**Dr Niall Madden**

University of Galway

Tuesday, 21 October, 2025



# Today, we'll max out on...

1 Info: Survey, Assignments, etc

2 Higher-order Derivatives

3 Maxima and minima

- Overview

- Critical points

4 The First Derivative Test

- Increasing/decreasing

- Derivatives

- Example

- The test

- Summary

5 Exercises

**See also:** Section 4.3 (Maxima and Minima) of **Calculus** by Strang & Herman: [https://math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

# Info: Survey, Assignments, etc

- ▶ The module survey for MA140 has started. Please take a few minutes to complete it. See [https://universityofgalway.instructure.com/courses/46734/discussion\\_topics/189325](https://universityofgalway.instructure.com/courses/46734/discussion_topics/189325)
  - ▶ It only takes a few minutes
  - ▶ We take in the input seriously, and will update you on the main findings and the actions we will take.
  - ▶ Try to mix positive comments with suggestions for improvements.
- ▶ **Assignment 3** I added an extra 23 hours (for reasons...). Now due tomorrow (21st) at 17:00
- ▶ **Assignment 4** is open and is due next Tuesday (28th) at 17:00.
- ▶ **Assignment 5** will be posted soon.
- ▶ Grades for the class test have been posted. There were some updates/corrections. Grades are now final (I hope!). Answers have been posted to <https://universityofgalway.instructure.com/courses/46734/files?preview=2948747>

# Higher-order Derivatives

At the end of the last class, we started learning about higher-order derivatives.

- ▶ if  $f(x)$  is a function, then  $f'(x)$  is a function whose value at  $x$  is the derivative of  $f$  at that point.
- ▶ So, since  $f'(x)$  is a function, and we can differentiate functions, we can differentiate  $f'$  itself.
- ▶ The **derivative of the derivative** of  $f$  is called that **second derivative** of  $f$ .
- ▶ It is denoted as  $f''(x)$  or  $\frac{d^2y}{dx^2}$  or  $f^{(2)}(x)$ .
- ▶ We can continue this process to get third derivatives, fourth derivatives, etc, etc. However, the most important are the 1st and 2nd:  $f'$  and  $f''$  provide valuable information about the function and its graph, particularly concerning local or global maxima, local/global minima and points of inflection.

# Higher-order Derivatives

## Example

Find the **second** derivative of the functions

(i)  $f_1(x) = 3x^2 + 2x + 1$

(iii)  $f_3(x) = \ln(x)$

(ii)  $f_2(x) = e^x$

(iv)  $f_4(x) = \sin(x)$

This section of MA140 is concerned with using techniques of differentiation to finding where a function is

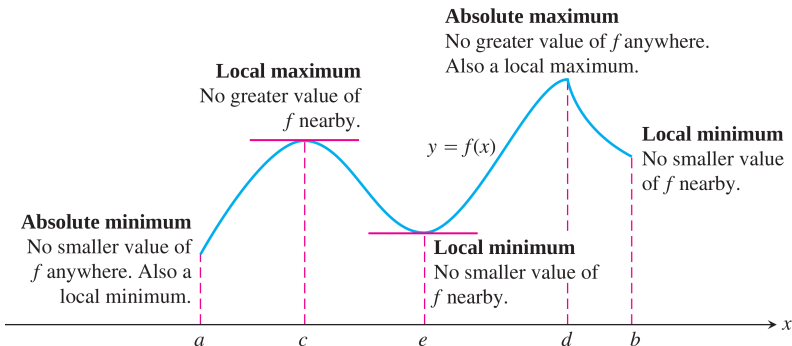
- ▶ Increasing
- ▶ Decreasing
- ▶ Has its maximum value
- ▶ Has its minimum value

Along the way we'll learn about **critical values** and the **first derivative test**.

### Mathematical English

- ▶ The plural of **maximum** is **maxima**;
- ▶ The plural of **minimum** is **minima**;
- ▶ An **extremum** a maximum or a minimum.
- ▶ The plural of **extremum** is **extrema**.

Given an interval  $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ , consider the function  $f : [a, b] \rightarrow \mathbb{R}$  whose graph is given below. It illustrates local and absolute (= "global") maxima and minima. Collectively, these are called **extrema**.



**Definition: critical points**

Let  $c$  in an point in the domain of a function  $f$ . We say that  $x = c$  is a **critical point** of  $f(x)$  if either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

**Important:** If  $f$  has an extremum at  $x = c$ , then  $c$  must be a **critical point** of  $f$  (This is called “Fermat’s Theorem”).

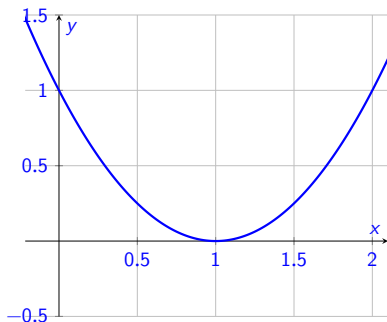
So, to find a maximum or minimum of  $f$ , it is enough to check at the critical points.

**Warning:** All extrema are at critical points, but not all critical points correspond to a extrema.



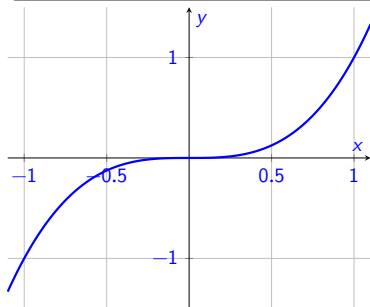
**Example**

$f(x) = x^2 - 2x + 1$  has one critical point. Find it. Does it correspond to an extremum?



**Example**

Find all critical points of  $f(x) = x^3$ . Do they correspond to extrema?



**Definition (Increasing/Decreasing)**

Let  $f$  be a function whose domain includes the interval  $[a, b]$ . Let  $x_1$  and  $x_2$  be any two points in  $[a, b]$  with  $x_1 < x_2$ .

- ▶ If  $f(x_1) < f(x_2)$ , then  $f$  is said to be *increasing* on  $[a, b]$ .
- ▶ If  $f(x_1) > f(x_2)$ , then  $f$  is said to be *decreasing* on  $[a, b]$ .

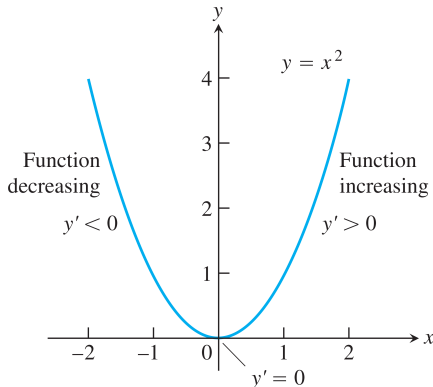
**Example**

The function  $f(x) = x^2$  is decreasing on  $(-\infty, 0]$ , and increasing on  $[0, \infty)$ .

**Theorem**

Suppose that  $f$  is differentiable on an interval  $[a, b]$ .

- ▶ If  $f'(x) > 0$  at each point  $x \in [a, b]$ , then  $f$  is increasing.
- ▶ If  $f'(x) < 0$  at each point  $x \in [a, b]$ , then  $f$  is decreasing.



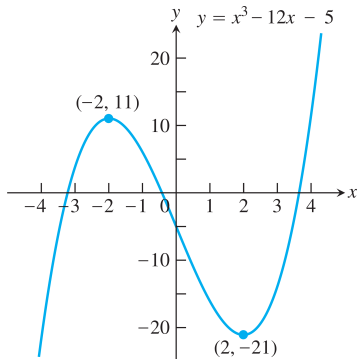
### Example

Find the critical points of  $f(x) = x^3 - 12x - 5$  and identify the intervals on which  $f$  is increasing and decreasing

**Idea:** find  $f'(x)$  and then solve for  $f'(x) = 0$ .

The critical points  $c = -2$  and  $c = 2$  of  $f(x) = x^3 - 12x - 5$  subdivide the domain of  $f$  into intervals  $(-\infty, -2)$ ,  $(-2, 2)$  and  $(2, \infty)$  on which  $f'$  is either positive or negative. We determine the sign of  $f'$  by evaluating  $f$  at a convenient point in each subinterval.

		-2		2	
$3(x + 2)$	-	•	+	•	+
$x - 2$	-	•	-	•	+
$f'(x)$	+	•	-	•	+



### Important:

- ▶ If  $f(x)$  has a local minimum of  $f(x)$  at  $x = c$ , then it switches from **decreasing** to **increasing**. That means,  $f'(x)$  changes sign at  $x = c$ . Therefore,  $f'(c) = 0$ .
- ▶ If  $f(x)$  has a local maximum at  $x = c$ , we have that  $f'(c) = 0$ .



**First Derivative Test for local maxima and minima**

Suppose that  $c$  is a critical point of a differentiable function  $f$ .

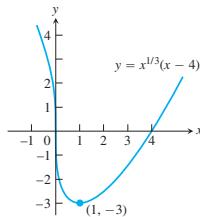
- ▶ If  $f'$  changes from negative to positive through  $c$ , then  $f$  has a local minimum at  $c$ .
- ▶ If  $f'$  changes from positive to negative through  $c$ , then  $f$  has a local maximum at  $c$ .
- ▶ If  $f'$  does not change sign through  $c$  (that is,  $f'$  is positive on both sides of  $c$  or negative on both sides), then  $f$  does not have a local maximum or minimum at  $c$ .

**Example**

Find the critical points of  $f(x) = x^{\frac{1}{3}}(x - 4)$ . Identify the local maxima and minima (if any).

First find  $f'(x)$ , and then where it is either zero or undefined:

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$4(x - 1)$	$-$	$-$	$+$
$3x^{2/3}$	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$+$



## Review

If a function  $g$  is differentiable on an interval  $[a, b]$ , then

- ▶  $g'(x) > 0$  for all  $x \in [a, b] \Leftrightarrow g$  increasing on  $[a, b]$ .
- ▶  $g'(x) < 0$  for all  $x \in [a, b] \Leftrightarrow g$  decreasing on  $[a, b]$ .

Similarly, if  $g'$  is also differentiable on  $[a, b]$ , then

- ▶  $(g')'(x) = g''(x) > 0$  for all  $x \in [a, b] \Leftrightarrow g'$  increasing on  $I$ .
- ▶  $(g')'(x) = g''(x) < 0$  for all  $x \in [a, b] \Leftrightarrow g'$  decreasing on  $I$ .

# Exercises

## Exercise 6.1.1

Let  $f(x) = x^2 e^x$ . Find  $f'(x)$ ,  $f''(x)$  and  $f'''(x)$ .

## Exercise 6.1.2 : 23/24 Exam, Q3(a)

Let  $f(x) = \ln(x^2 + 1)$ .

- (i) Find all critical point(s) of  $f$  and determine whether  $f$  has a local minimum, local maximum or neither.
- (ii) Determine the interval on which  $f$  is increasing.
- (iii) Determine the interval on which  $f$  is decreasing.
- (iv) Find all point(s) of inflection of  $f$ , justifying your answer.