MA378 Chapter 4: Finite Element Methods

§4.2 The FEM

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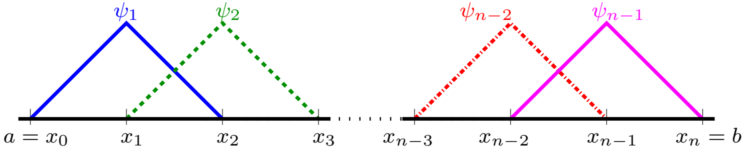


2.1 On Infinite- and Finite-Dimensional Spaces

A *space* is a collection of functions. The *dimension* of that space is smallest number of pieces of information that is required to uniquely describe any of its members.

Example 2.1

The set of points in $\mathbb{R}^2$. Each member can be written as (x_1, x_2) so we need two pieces of information ("coordinates") to describe the point. So $\mathbb{R}^2$ has dimension 2.

2.1 On Infinite- and Finite-Dimensional Spaces

Example 2.2

The space of quadratic polynomials (over the reals). Each member can be written as $a_0 + a_1 x + a_2 x^2$. So we need to know a_0, a_1 and a_2 for each member of the set; the space has dimension 3.

Another way of thinking about this is considering that these space have certain members that can be combined is various ways to give us every other member; they form a *basis* for the space.

Three bases for $\mathcal{P}^2$:

2.1 On Infinite- and Finite-Dimensional Spaces

Example 2.3

All the solutions to the differential equation $-u''(x) + u(x) = 0$ (note: no boundary conditions) can be written in the form $u(x) = C_0 e^{-x} + C_1 e^{x}$. So $\{e^{-x}, e^{x}\}$ is a basis for this space which is of dimension 2.

2.1 On Infinite- and Finite-Dimensional Spaces

All of the spaces mentioned have *finite-dimension*. But consider the space of *all continuous functions on* $[a, b]$. This is an *infinite dimensional space*: we would have to write down an infinite number of values to describe a member uniquely.

Example 2.4

The space of functions $C^2(a, b)$ is infinite-dimensional, as is $C_0^2(a, b)$. So too is $H_0^1(a, b)$.

2.1 On Infinite- and Finite-Dimensional Spaces

Example 2.5

The space of functions that are of the form $g(x) = \gamma(x-a)(x-b)$ for any $\gamma \in \mathbb{R}$ is *finite-dimensional* (because it has dimension 1). But every member of this space also belongs to the infinite dimensional space $H_0^1(a,b)$; it is a *finite-dimensional subspace of* $H_0^1(a,b)$.

2.2 The Galerkin Basis Functions

Example 2.6

To get another **very important** example of a finite dimensional subspace of $H_0^1(a,b)$, first fix a "*mesh*" on $[a,b]$. This is just a set of points $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. Then consider the space of all functions that are *piecewise linear* on this mesh and that vanish at $x = a$ and $x = b$.

This is a a finite-dimensional sub-space of $H_0^1(a,b)$. A reasonable basis for this space would be the hat functions $\{\psi_1, \psi_2, \dots, \psi_{n-1}\}$ given by

$$\psi_i(x) = \begin{cases} (x - x_{i-1})/h & x_{i-1} \leq x < x_i \\ (x_{i+1} - x)/h & x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

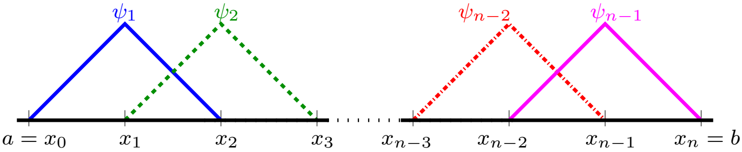
where $h = (b-a)/n$ is the distance between adjacent points.

2.2 The Galerkin Basis Functions

Then we can write any function u_h as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

This basis set, shown below, are often called *hat functions* or *Galerkin basis functions*. We met them before in Section 2.1 on piecewise linear interpolation.



2.3 The Discrete Variational Form

Recall that we are trying to solve the problem:

Find $u \in C^2(a,b)$ *such that*

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (a,b), \qquad u(a) = u(b) = 0.$$

We defined $\mathcal{A}(u,v) := (u',v') + (ru,v)$.

The variational form of the DE:

Find $u \in H_0^1(a,b)$ *such that*

$$\mathcal{A}(u,v) = (f,v) \quad \text{for all} \quad v \in H_0^1(a,b).$$

Imagine we try to solve this by taking a function from $H_0^1(a,b)$ and checking if this equation is true for all v in $H_0^1(a,b)$. This would take forever because there are an infinite number of candidates.

2.3 The Discrete Variational Form

So instead we restrict our attention to a **finite-dimensional** subspace S of $H_0^1(a, b)$. Now we can select a function u_h from S and "put it on trial" by testing it against (all) the functions v_h in S. This leads to the terminology of calling u_h a **trial** function and v_h a **test** function.

More exactly:

- We want to find the u_h
- It has $n - 1$ unknowns, so we need $n - 1$ equations
- Choose (carefully) $n - 1$ possible v_h to get these.

2.3 The Discrete Variational Form

This leads to...

Definition 2.7 (The Finite Element Method)

Let S be the finite dimensional subspace of $H_0^1(a,b)$ made up of the piecewise linear functions on a fixed mesh $a = x_0 < x_1 < \cdots < x_n = b$. Then the *Galerkin Finite Element method* is: *find $u_h \in S$ such that*

$$\mathcal{A}(u_h, v_h) = (f, v_h) \qquad \text{for all} \qquad v_h \in S. \tag{1}$$

We now want to look at how to turn Definition 2.7 into an algorithm (ideally, one that we can implement on a computer).

2.4 FE implementation

- Let S be the space of piecewise linear functions on the mesh $x_i = a + ih$, where $h = (b-a)/n$. As above, u_h is can be written as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

  So u_h has $n-1$ unknowns: $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$.
- To solve for these, we need $n-1$ equations. To get these, we just choose $n-1$ different (i.e., linearly independent) possible v_h, and substitute into (1).
- The most obvious, and (it turns out) sensible, choice for these $n-1$ equations are the $n-1$ hat functions ψ_1, $\psi_2, \dots, \psi_{n-1}$.
- This gives us $n-1$ equations to solve:

$$\mathcal{A}(u_h, \psi_i) = (f, \psi_i) \qquad \text{for } i = 1, \dots n-1. \tag{2}$$

2.4 FE implementation

It is now not too difficult to see that, if we write these equations as a matrix-vector equation, $Ax = F$, then

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j)$$

2.4 FE implementation

Some important observations:

- Any given "hat" function ψ_i is only non-zero on the region $[x_{i-1}, x_{i+1}]$.
- We have to compute

$$a_{ij} = \mathcal{A}(\psi_i, \psi_j) = \int_a^b \psi_i'(x)\psi_j'(x) + r(x)\psi_i(x)\psi_j(x)dx.$$

  But this will be non-zero only if there is overlap between $[x_{i-1}, x_{i+1}]$ and $[x_{j-1}, x_{j+1}]$.
- This gives that $\mathcal{A}(\psi_i, \psi_j) = 0$ if $|i - j| > 1$, and otherwise

$$\mathcal{A}(\psi_i, \psi_j) = \int_{x_{i-1}}^{x_{i+1}} \psi_i'(x)\psi_j'(x) + r(x)\psi_i(x)\psi_j(x)dx.$$

2.4 FE implementation

We call such a system **tridiagonal**. The left-hand side looks like this:

$$\begin{pmatrix}
\mathcal{A}(\psi_1,\psi_1) & \mathcal{A}(\psi_2,\psi_1) & 0 & 0 & \cdots & 0 \\
\mathcal{A}(\psi_1,\psi_2) & \mathcal{A}(\psi_2,\psi_2) & \mathcal{A}(\psi_3,\psi_2) & 0 & \cdots & 0 \\
0 & \mathcal{A}(\psi_3,\psi_2) & \mathcal{A}(\psi_3,\psi_3) & \mathcal{A}(\psi_3,\psi_4) & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \dots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \mathcal{A}(\psi_{n-1},\psi_{n-1}
\end{pmatrix}$$

There are two consequences to this:

(i) It takes less effort to set up the linear systems of equations than one might have thought.

(ii) It is relatively easy to solve.

We should also observe the the matrix, A, is **symmetric**.

In general a quadrature rule is used to compute in integrals

$$\int_{x_{i-1}}^{x_{i+1}} r(x)\psi_i(x)\psi_j(x).$$

The Gaussian Quadrature methods are the most popular for this.

2.5 An example

Use the FEM on the mesh $\{x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3\}$ *to find an approximate solution to*

$$-u'' + 3u = x \quad \text{on} \quad (0,3), \quad u(0) = u(3) = 0. \tag{3}$$

Solution: The FEM is: *Find* $u_h(x) \in S$ *such that*

$$\mathcal{A}(u_h, v_h) := (u_h', v_h') + 3(u_h, v_h) = (x, u_h)$$
$$\text{for all } v_h(x) \in S.$$

We have that $h = 1$ so let

$$\psi_1(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise,} \end{cases} \qquad \psi_2(x) = \begin{cases} x - 1 & 1 \le x < 2 \\ 3 - x & 2 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u_h(x) = \lambda_1 \psi_1(x) + \lambda_2 \psi_2(x).$$

2.5 An example

Our two equations are:

$$(\lambda_1\psi_1' + \lambda_2\psi_2', \psi_1') + 3(\lambda_1\psi_1 + \lambda_2\psi_2, \psi_1) = (x, \psi_1),$$

$$(\lambda_1\psi_1' + \lambda_2\psi_2', \psi_2') + 3(\lambda_1\psi_1 + \lambda_2\psi_2, \psi_2) = (x, \psi_2).$$

giving

$$\lambda_1\Big(\int_0^3 \psi_1'\psi_1' dx + 3\int_0^3 \psi_1\psi_1 dx\Big) + \\ \lambda_2\Big(\int_0^3 \psi_2'\psi_1' dx + 3\int_0^3 \psi_2\psi_1 dx\Big) = \int_0^3 x\psi_1 dx$$

$$\lambda_1\Big(\int_0^3 \psi_1'\psi_2' dx + 3\int_0^3 \psi_1\psi_2 dx\Big) + \\ \lambda_2\Big(\int_0^3 \psi_2'\psi_2' dx + 3\int_0^3 \psi_2\psi_2 dx\Big) = \int_0^3 x\psi_1 dx.$$

2.5 An example

We now need to evaluate these integrals. For example, from the 1st equation:

$$\int_0^3 \psi_1' \psi_1' dx = \int_0^1 (1)^2 dx + \int_1^2 (-1)^2 dx = 2,$$

$$\int_0^3 \psi_1 \psi_1 dx = \int_0^1 x^2 dx + \int_1^2 (2-x)^2 dx = 2/3,$$

so the coefficient of λ_1 is $2 + 3(2/3) = 4$. Also,

$$\int_0^3 \psi_2' \psi_1' dx = \int_1^2 \psi_2' \psi_1' dx = \int_1^2 (1)(-1) dx = -1,$$

$$\int_0^3 \psi_2 \psi_1 dx = \int_1^2 \psi_2 \psi_1 dx = \int_1^2 (x-1)(2-x) dx = 1/6.$$

So the coefficient of λ_2 is $-1 + 3(1/6) = -1/2$.

2.5 An example

For the right-hand side:

$$\int_0^3 x\psi_1 dx = \int_0^1 (x)(x)dx + \int_1^2 (x)(2-x)dx = 1/3 + 2/3 = 1.$$

Similarly, we can show that

$$\int_0^3 x\psi_2 dx = 2.$$

The final system is

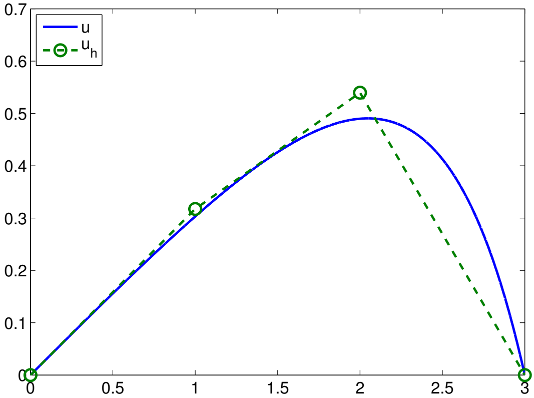
$$4\lambda_1 - \frac{1}{2}\lambda_2 = 1, \qquad -\frac{1}{2}\lambda_1 + 4\lambda_2 = 2.$$

Solving, we get $\lambda_1 = 20/63$ and $\lambda_2 = 34/63$.

2.5 An example

The FEM solution is

$$u_h(x) = \frac{20}{63}\psi_1(x) + \frac{34}{63}\psi_2(x).$$



2.6 A note on computation...

In more realistic settings, all these calculations are done by computer. The main steps are:

1. Choose a mesh. In higher dimensional spaces, we also need to choose a *triangulation*.
2. Choose a basis. The piecewise linear one we've studied is the most popular, but there are lots of alternatives.
3. **Assemble** the linear system
4. Solve the linear system.

There is lots of great software for doing some or all of these tasks, including FEniCS, Firedrake, deal.ii, MFEM, FreeFEM,... And you can write your own (I often do!). If inclined, please explore!

Our last task for MA378 is to convince ourselves that the method works...

2.7 Exercises

Exercise 2.1

Show that u_h solves (2) *if and only if* it solves (1).

Exercise 2.2

Consider the problem:

$$-u''(x) = 9x \qquad u(0) = 0, u(1) = 0.$$

Use the FEM to find an approximate solution on the mesh $\{0, 1/3, 2/3, 1\}$.
Also write down the true solution to this problem.

2.7 Exercises

Exercise 2.3

Suppose we want to use a finite element method to solve

$$-u''(x) + u(x) = 1 \text{ on } (0,1),$$

with $u(0) = u(1) = 0$, using the usual piecewise linear basis functions on the uniform mesh $\{x_0, x_1, \dots, x_n\}$. Let the resulting linear system is written as the matrix-vector equation $Au_h = F$.

(i) Show that the matrix A is symmetric (i.e. $a_{ij} = a_{ji}$).

(ii) Show that A is tridiagonal (i.e., if $|i - j| > 1$ then $a_{ij} = 0$).

(iii) Derive the formula for the entries of A in terms of $h = x_i - x_{i-1}$. That is, give an expression for $a_{i,i-1}$, $a_{i,i}$ and $a_{i,i+1}$.

Since this exercise is not covered in class, or in tutorials, solutions are given on the following slides.

2.7 Exercises

(i) Show A is symmetric

The entries in A are $a_{ij} = \mathcal{A}(\psi_j, \psi_i)$, where $\mathcal{A}$ is the bilinear form:

$$\mathcal{A}(u, v) = \int_0^1 u'(x)v'(x) + u(x)v(x)dx.$$

But $\mathcal{A}(v, u) = \int_0^1 v'(x)u'(x) + v(x)u(x)dx = \mathcal{A}(u, v)$. So $a_{ij} = \mathcal{A}(\psi_j, \psi_i) = \mathcal{A}(\psi_i, \psi_j) = a_{ji}$.

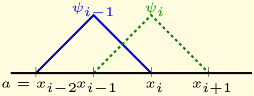
2.7 Exercises

(ii) Show that A is tridiagonal

The basis functions for the method, $\{\psi_1, \dots, \psi_{n-1}\}$, have the formulae

$$\psi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x_{i-1} \leq x < x_i \\ \frac{x_{i+1} - x}{h} & x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $h = x_i - x_{i-1}$. They are pictured below.



Note that $\psi_i(x)$ is only non-zero on $[x_{i-1}, x_{i+1}$. Therefore, if $i > j + 1$ or $j > i + 1$, then $\psi_i(x)\psi_j(x) = 0$ for all x. As mentioned above,

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j) = \int_0^1 \psi_i'(x)\psi_j'(x) + \psi_i(x)\psi)j(x)dx$$

So, if $|i - j| > 1$, then $a_{ij} = 0$.

2.7 Exercises

(iii) Derive formulae for the $a_{i,j}$ in terms of h.

First we'll compute $a_{i,i-1} = \mathcal{A}(\psi_i, \psi_{i-1})$.

$$\begin{aligned}\mathcal{A}(\psi_i, \psi_{i-1}) &= \int_0^1 \psi_i'(x)\psi_{i-1}'(x) + \psi_i(x)\psi_{i-1}(x)dx \\ &= \int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)dx + \int_{x_{i-1}}^{x_i} \psi_i(x)\psi_{i-1}(x)dx,\end{aligned}$$

because, as illustrated in Part (ii), the only interval where ψ_{i-1} and ψ_i are both non-zero is $[x_{i-1}, x_i]$. From (4) in Part (2),

$$\psi_i'(x) = \begin{cases} 1/h & x_{i-1} \leq x < x_i \\ -1/h & x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)dx = \int_{x_{i-1}}^{x_i} \frac{1}{h}\frac{-1}{h}dx = -\frac{x}{h^2}\Big|_{x_{i-1}}^{x_i} = -\frac{x_i - x_{i-1}}{h^2} = -\frac{1}{h}.$$

(iii) continued

For $\int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)dx$, we can simplify a little by setting $s = x - x_{i-1}$. Then $x_i - x = x_{i-1} + h - x = h - s$.

$$\int_{x_{i-1}}^{x_i} \psi_i(x)\psi_{i-1}(x)dx = \int_{x_{i-1}}^{x_i} \frac{x - x_i}{h}\frac{x_{i-1} - x}{h}dx = \int_0^h \frac{s}{h}\frac{h-s}{h}ds$$
$$= \frac{1}{h^2}\big(-\frac{1}{3}s^3 + \frac{1}{2}hs^2\big)\big|_0^h = \frac{h}{6}.$$

So $\mathbf{a_{i,i-1} = -1/h + h/6 = a_{i,i+1}}$.
Next we'll calculate $a_{i,i}$:

$$a_{ii} = \mathcal{A}(\psi_i, \psi_i) = \int_0^1 \psi_i'(x)\psi_i'(x) + \psi_i(x)\psi_i(x)dx = \int_{x_{i-1}}^{x_{i+1}} \big(\psi_i'(x)\big)^2 + \big(\psi_i(x)\big)^2 dx.$$

First,
$$\int_{x_{i-1}}^{x_{i+1}} \big(\psi_i'(x)\big)^2 dx = \int_{x_{i-1}}^{x_i} \big(\frac{1}{h}\big)^2 dx + \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)^2 dx = \frac{2}{h}.$$

For the $\int_{x_{i-1}}^{x_{i+1}} \big(\psi_i'(x)\big)^2 dx$ term, we'll again simplify by setting $s = x - x_{i-1}$.

2.7 Exercises

(iii) continued

This gives

$$\int_{x_{i-1}}^{x_{i+1}} (\psi_i(x))^2 dx = \int_0^h (\frac{s}{h})^2 ds + \int_h^{2h} (\frac{h-s}{h})^2 ds = \frac{s^3}{3h^2}|_0^h + \frac{-(h-s)^3}{3h^2}|_h^{2h} = \frac{h}{3} + \frac{h}{3}$$

To finish: $\mathbf{a_{i,i} = \int_{x_{i-1}}^{x_{i+1}} (\psi_i'(x))^2 dx + \int_{x_{i-1}}^{x_{i+1}} (\psi_i(x))^2 dx = \frac{2}{h} + \frac{4h}{6}}$.