MA211

Lecture 17: Techniques of Integration

Wed 5^{th} Nov 2008

Today...

1. Integration
   - The Mathematical Tables
2. Method of Substitution
   - Examples
   - Definite Integrals
   - Some more exercises

Section 5.5 of Stewart has more examples.

But first...

Grouped Student Evaluation

of Teaching.

But first...

REMINDER

Homework exercises from Problem Set 3 are due on Friday.

Solutions must be carefully written and, if on more than one page, stapled together.

Integration

In this section of the course, we are trying to solve problems of the form

Given a function $f : \mathbb{R} \to \mathbb{R}$, find a function g such that $g'(x) = f(x)$. That is, find the antiderivative of f.

Usually we will write it as *find the integral of f*, i.e.,

$$\text{Evaluate} \quad \mathcal{I} = \int f(x)dx.$$

For many fundamental functions we can simply lookup their antiderivatives pages 41 and 42 of the Mathematical Tables.

DIFREÁIL (DIFFERENTIATION)

$f(x)$	$f'(x) \equiv \frac{d}{dx}[f(x)]$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$

SUIMEÁIL (INTEGRATION)

Glactar $a>0$ agus fágtar tairisigh na suimeála ar lár.

We take $a>0$ and omit constants of integration.

$f(x)$	$\int f(x)\,dx$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln \lvert \sec x \rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$
$\operatorname{cosec} x$	$\ln \lvert \tan \frac{x}{2} \rvert$
$\cot x$	$\ln \lvert \sin x \rvert$
e^x	e^x

$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$	e^{ax}	$\frac{1}{a}e^{ax}$
$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$	a^x	$\frac{a^x}{\ln a}$
$\operatorname{cosec}^{-1}\frac{x}{a}$	$-\frac{a}{x\sqrt{x^2-a^2}}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert$
$\cot^{-1}\frac{x}{a}$	$-\frac{a}{a^2+x^2}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\sinh x$	$\cosh x$		
$\cosh x$	$\sinh x$		
$\tanh x$	$\operatorname{sech}^2 x$	$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\coth x$	$-\operatorname{cosech}^2 x$		
$\operatorname{sech} x$	$-\operatorname{sech} x\tanh x$		
$\operatorname{cosech} x$	$-\operatorname{cosech} x\coth x$		
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$

$\coth^{-1} x$	$-\dfrac{1}{x^2-1}$
$\text{sech}^{-1} x$	$-\dfrac{1}{x\sqrt{1-x^2}}$
$\text{cosech}^{-1} x$	$-\dfrac{1}{x\sqrt{x^2+1}}$

Torthaí agus Líonta:
Products and Quotients:

$$y = uv\ ;\ \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = \frac{u}{v}\ ;\ \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\coth x$	$\ln \lvert \sinh x \rvert$
$\text{sech}\, x$	$\tan^{-1}(\sinh x)$
$\text{cosech}\, x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\cos^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sin 2x]$
$\sin^2 x$	$\frac{1}{2}[x - \frac{1}{2}\sin 2x]$
$\cosh^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sinh 2x]$
$\sinh^2 x$	$\frac{1}{2}[-x + \frac{1}{2}\sinh 2x]$
$\dfrac{1}{x\sqrt{a^2-x^2}}$	$-\dfrac{1}{a}\,\text{sech}^{-1}\dfrac{x}{a}$
$\dfrac{1}{x\sqrt{x^2+a^2}}$	$-\dfrac{1}{a}\,\text{cosech}^{-1}\dfrac{x}{a}$

Example

Evaluate the following integral: $\int \tan^2(x)dx$.

In most cases, we can't just look-up the answer in a table. We may have to simplify the express, e.g., using Partial Fractions, or (more often) using a **Substitution**.

Method of Substitution

The method of substitution comes from the *Chain Rule of differentiation* and is summarised as

Substitution

Let $u = g(x)$. The $du = g'(x)dx$. So

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C.$$

Example

Evaluate the indefinite integrals

$$(i)\quad \mathcal{I} = \int \sqrt{x+3}\,dx. \qquad (ii)\quad \mathcal{I} = \int \frac{1}{\sqrt{x+3}}\,dx.$$

Example

Evaluate the indefinite integral $\mathcal{I} = \int \frac{x}{x^2 + 1} dx.$

Example (Using Trigonometric Identities)

Evaluate the following integral: $\mathcal{I} = \int \sec^4(x)dx.$

Hint: $sec^2(x) = 1 + tan^2(x)$.

Example

Evaluate the indefinite integral $\mathcal{I} = \int \frac{\sin(3 \ln x)}{x} dx$.

Exercise (17.1)

Evaluate the following integrals:

(i) $\displaystyle\int \frac{1+x}{\sqrt{1+x}}\,dx.$

(ii) $\displaystyle\int e^{(2x-2)}\,dx.$

(iii) $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx.$

(iv) $\displaystyle\int e^{\sin(x)}\cos(x)\,dx$

Exercise (17.2)

Use a suitable substitution to show that

$$\int \frac{1}{\tan(x)}\,dx = \ln|\sin(x)|.$$

Hint:

If $g(a) = A$ and $f(b) = B$ then

$$\int_a^b f\left(g(x)\right)g'(x)dx = \int_A^B f(u)du.$$

Example

Evaluate

$$\int_0^8 \frac{\cos\left(\sqrt{x+1}\right)}{\sqrt{x+1}}dx$$

Exercise (17.3)

Evaluate the following integrals:

(i) $$\int_0^4 \frac{x^3}{\sqrt{x^2+1}}\,dx.$$

(ii) $$\int_1^{\sqrt{e}} \frac{\sin\left(\pi \ln(x)\right)}{x}\,dx.$$

(iii) $$\int_e^{e^2} \frac{1}{x\ln(x)}\,dx.$$

Exercise (17.4)

Evaluate the following integrals:

(i) $\displaystyle\int xe^{x^2}\,dx.$

(ii) $\displaystyle\int \frac{\cos(x)}{4+\sin^2(x)}\,dx.$

(iii) $\displaystyle\int e^{2x}\sin(e^{2x})\,dx$

(iv) $\displaystyle\int \frac{\ln(x)}{x}\,dx$

(v) $\displaystyle\int \frac{e^x+1}{e^x-1}\,dx.$