Annotated slides from Friday

MA313 : Linear Algebra I

Week 3: Spanning set; the Null and Column Spaces

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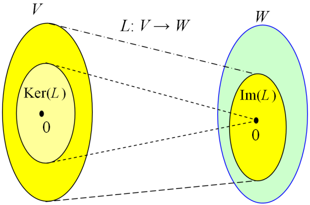

https://commons.wikimedia.org/wiki/File:KerIm_2015Joz_L2.png.

These slides are adapted (slightly) from ones by Tobias Rossmann.

Example

Show that $H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

Eg, if $a = 0 = b$, $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

If $a = 3, b = 1$
$\Rightarrow v = \begin{bmatrix} 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}$

If $a = 2.5$, $b = \pi = 3.14159..$

$v = \begin{bmatrix} 2.5 - 3\pi \\ \pi - 2.5 \\ \pi \\ 2.5 \end{bmatrix}$.

Example

Show that $H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

To answer this, there are two ways

(1) Show (a) $\vec{0} \in H$, (b) If $u, v \in H$, $u+v \in H$

(c) $u \in H$ & $c \in \mathbb{R} \Rightarrow cu \in H$.

(2) Find vectors that span H.

This is what we will do.

Example

Show that $H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

Answer:

We can write $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ b \\ 0 \\ b \end{bmatrix}$

$= a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$ Since $a, b \in \mathbb{R}$, so

$H = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

So H is a subspace of $\mathbb{R}^4$.

Example (From 2018/2019 exam paper)

Find vectors $u, v, w \in V$ with $V = \operatorname{span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$.

write $\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \\ -b \end{bmatrix} + \begin{bmatrix} -c \\ 0 \\ c \\ 0 \end{bmatrix}$

$= a\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$

$= a\,u + b\,v + c\,w$

So u, v, w are as shown.

Example: Care is required!

Is $\quad H = \left\{ \begin{bmatrix} 3s \\ 2+5s \end{bmatrix} : s \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$.

No! For example $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin H$. To see this

suppose $\begin{bmatrix} 3s \\ 2+5s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $3s = 0 \Rightarrow s = 0$.

But $2 + 5s = 0 \Rightarrow s = -2/5 \neq 0$.

s cannot have 2 values so $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin H$.

We now know that the span of any subset of vectors in a vectors space is itself a subspace (and, so, is a vector space). But...

Question

Is every subspace the span of some (collection of) vectors?

Answer: yes, if "finite".

We now know that the span of any subset of vectors in a vectors space is itself a subspace (and, so, is a vector space). But...

Question

Is every subspace the span of some (collection of) vectors?

We'll answer that question over the next week or so.

MA313
Week 3: Spanning set; the Null and Column Spaces

Start of ...

PART 3: Null spaces

Part 3: Null spaces

The big idea...

There are **two main ways of building f subspaces**:

- Spans of vectors ("bottom up").
- **Kernels** and **null spaces** of **linear transformations** ("top down").

The null space generalise sets of solutions to homogeneous systems of linear equations, which we'll look at now.

Eg

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned} \iff \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Homogeneous

(Zero Right hand side).

Part 3: Null spaces

Definition (NULL SPACE)

Let A be an $m \times n$ matrix. The **null space** of A is

$$\operatorname{Nul} A = \{x \in \mathbb{R}^n : Ax = 0\}.$$

Earlier, we did an example that showed that when we multiply a matrix by a vector, we are making a linear combination of the columns of A.

That is, for a matrix $A = [a_1 \cdots a_n]$ with columns $a_1, \ldots, a_n \in \mathbb{R}^m$ and a vector $x \in \mathbb{R}^n$, we have

$$Ax = x_1 a_1 + \cdots + x_n a_n.$$

Part 3: Null spaces

Nul A or Nul(A).

Example

Let

$$A = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 2 & -4 \end{bmatrix}, \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$x \in \mathrm{Nul}\, A \qquad \text{but} \qquad y \notin \mathrm{Nul}\, A.$$

Ans

$$\begin{bmatrix} 4 & -2 & -1 \\ 1 & 2 & -4 \end{bmatrix} \overset{x}{\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}} = \begin{bmatrix} 8-6-2 \\ 2+6-8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{So } x \in \mathrm{Nul}\, A$$

$$\begin{bmatrix} 4 & -2 & -1 \\ 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-2-1 \\ 1+2-4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad \text{So } y \notin \mathrm{Nul}\, A$$

Theorem

Let A be an $m \times n$ matrix.
Then $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

This follows from familiar properties of matrix multiplication.

1. $A\mathbf{0} = \mathbf{0}$
2. $A(x + y) = Ax + Ay$ and
3. $A(cx) = c(Ax)$

Suppose $x, y \in \operatorname{Nul} A$. So $Ax = 0$ $Ay = 0$.

Then $A(x+y) = Ax + Ay = 0 + 0 = 0$.

So $x + y \in \operatorname{Nul} A$.

In some cases, we want to compute vectors in $\mathrm{Nul}\,A$. However,

- Given a matrix A, it is very easy to test if a given vector x belongs to $\mathrm{Nul}\,A$.
- But how can we find non-zero vectors in $\mathrm{Nul}\,A$ or prove that none exist? (In the text-book, this is called "Finding an explicit description of $\mathrm{Nul}\,A$").

This should not be too surprising. We are, essentially, solving $Ax = \mathbf{0}$. And it is easier to check if a vector is a solution to a system of equations, then to find that solution.

But, also, some linear systems are much easier to solve than others. [See next examples]

Example (Some "easy" cases)

Find a vector, other than the zero vector, in the null space of each of the following, or show it does not exist.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ax = 0 \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives $x_1 = x_2 = x_3 = 0$, and no others.