The Singular Value Decomposition. We now know that the SVD of an $m \times n$ matrix, A is the set of three matrices, U, Σ and V, such that

- U is an $m \times m$ unitary matrix;
- $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\min\{m,n\}})$ is a non-negative real $m \times n$ matrix, with $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{\min\{m,n\}}$;
- V is an $n \times n$ unitary matrix;

and, of course,

$$A = U\Sigma V^\star.$$

We call the σ_i the "*singular values*" of A. RQ presented a proof of the existence of and SVD, for *any* complex matrix, in Lecture 7. You can see a different proof in Theorem 4.1 of Trefethen and Bau (though it appeals to a "compactness argument").

Properties of the SVD. Now we want to study some key properties of the SVD. For more, see Lecture 5 of Trefethen and Bau.

Theorem 1. *The rank of A is r, the number of non-zero singular values of A.*

Theorem 2.

$$\|A\|_2 = \sigma,$$

and

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$$

Theorem 3. *A is the sum of the r rank-one matrices*

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^\star.$$

The next theorem is probably the most important: it tells us how best to approximate a matrix by one of lower rank.

Theorem 4. *Let A_ν be the rank-ν approximation to A*

$$A_\nu := \sum_{j=1}^{\nu} \sigma_j u_j v_j^\star.$$

Then

$$\|A - A_\nu\|_2 = \inf_{\operatorname{rank}(X) \leqslant \nu} \|A - X\|_2 = \sigma_{\nu+1},$$

where if $\nu = p = \min(m, n)$, we define $\sigma_{\nu+1} = 0$.

The analogous result holds for the $\|\cdot\|_F$ norm, though we won't prove it.

Theorem 5.

$$\|A - A_\nu\|_F = \inf_{\operatorname{rank}(X) \leqslant \nu} \|A - A_\nu\|_F.$$