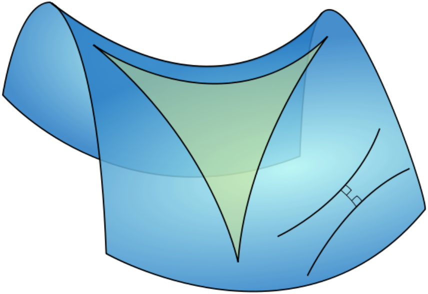
MA211

Lecture 8: Hyperbolic Functions

Wed, 01 October 2008



Reminder: Problem Set 1

Deadline for the homework exercises from Problem Set 1 is *11am, Monday, Oct 6th*.

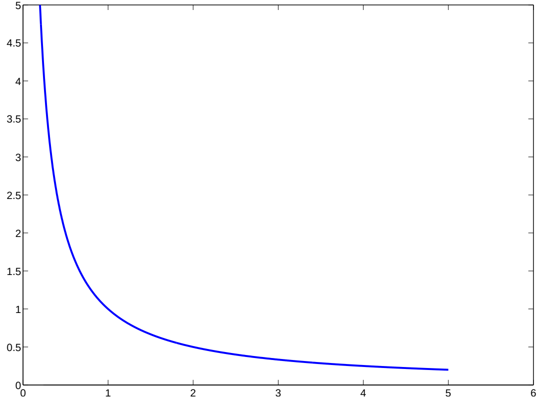
In today's class

1. Recall: The exponential function
   - Properties
2. Inverse Trigonometric functions
   - $\sin^{-1}(x)$
   - $\sin^{-1}, \cos^{-1}$ and $\tan^{-1}$
3. Euler Formula
4. The Hyperbolic Functions
   - Derivatives
   - Inverses

For more details, see **Sections 1.6, 3.5 and 3.11** of Stewart.

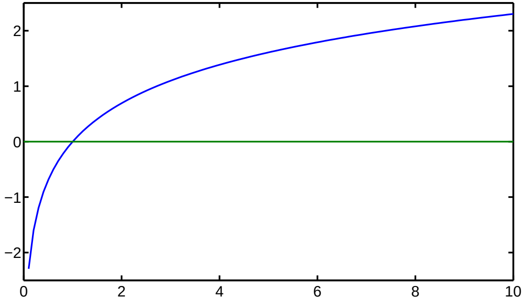
Last week... The Natural Logarithm

Last week we define the "*natural logarithm of x*", usually written $\ln(x)$, as follows: Let A be the area of the region from $t = 1$ to $t = x$ between the curve $1/t$ and the t-axis.



Then we define $\ln x$ as

$$\ln(x) = \begin{cases} A, & \text{for } x \geq 1 \\ -A, & \text{for } 0 < x < 1. \end{cases}$$



We then proved that, if $x > 0$ then

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Equivalently:

$$\int \frac{1}{x}dx = \ln(x) + C$$

Although it is not defined in the same way as other logarithmic functions, the Natural Log enjoys the same important properties:

(i) $\ln(xy) = \ln(x) + \ln(y)$

(ii) $\ln\left(\frac{1}{x}\right) = -\ln x$

(iii) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

(v) $\ln\left(x^y\right) = y\ln x$

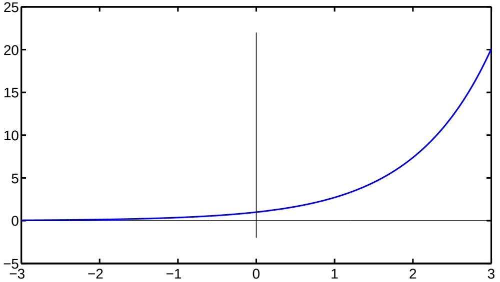
Recall: The exponential function

Next we defined the inverse of the **Exponential Function** as the inverse of the Natural Logarithmic Function

Definition (Exponential Function $\exp(x)$)

The function $\exp : (-\infty, \infty) \to (0, \infty)$ is the inverse of the natural log function $\ln : (0, \infty) \to (-\infty, \infty)$:

$$y = \ln(x) \iff x = \exp(y).$$



Recall: The exponential function

By definition:

$$\ln(\exp(x)) = x \qquad \text{for all } x \in \mathbb{R}$$

and

$$\exp(\ln(x)) = x \qquad \text{for all } x \in \mathbb{R}^+ = (0, \infty)$$

Recall: The exponential function

From the properties of $\ln(x)$, we can deduce that the exp function satisfies the usual properties on the exponential function $y = a^x$.

(i) $\exp(x+y) = \exp(x)\exp(y)$

(ii) $\exp(-x) = \dfrac{1}{\exp(x)}$

(ii) $\exp(x-y) = \dfrac{\exp(x)}{\exp(y)}$

(iv) $\exp(x)^y = \exp(xy)$

Perhaps the most important property:

$$\frac{d}{dx} e^x = e^x.$$

So the exponential function is its own derivative!

Because the derivative of e^x is e^x, we also get:

$$\int e^x dx = e^x + C$$

Example

Calculate the integral of $f(x) = Ae^{Bx}$, where A and B are constants.

Solution: $\displaystyle\int Ae^{Bx} dx = A\int e^{Bx} dx = \frac{A}{B} e^{Bx} + C.$

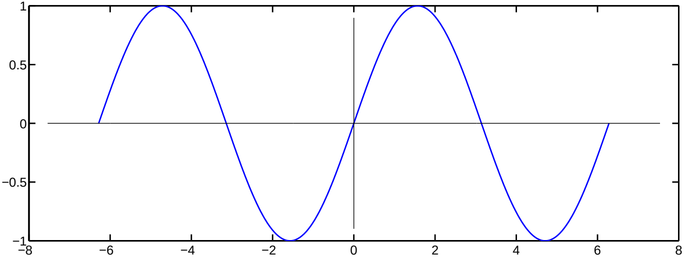
Example

Solve the Initial Value Differential Equation

$$f'(x) - f(x) = 0; f(0) = 2;$$

Solution:

Recall the function $\sin : (-\infty, \infty) \to [-1, 1]$:



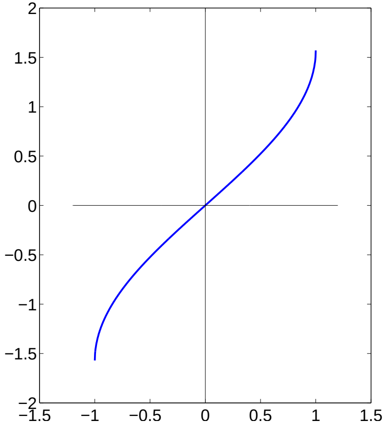
This function is *not* invertible, because it is not one-to-one.
However, if we restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then it is invertible.

Inverse sin function

The inverse of the sin function on $[-\pi/2, \pi/2]$ is denoted $\sin^{-1}(x)$ or $\arcsin(x)$

$$y = \sin(x) \iff x = \sin^{-1}(y).$$

The notation arcsin is still often used text books, but we'll use $\sin^{-1}$. Take care not to confuse this with $1/\sin(x)$.



Example

Simplify $\tan\left(\sin^{-1}(x)\right)$.

We can also define the inverse of the $\cos$ and $\tan$ functions.

Exercise (Q8.1)

(i) Show that $\cos(\tan^{-1}(x)) = \dfrac{1}{\sqrt{1+x^2}}$.

(ii) Simplify the expression $\sin(\tan^{-1}(x))$

(iii) Simplify the expression $\cos(2\tan^{-1}(x))$

The derivatives of the inverse trig functions are

$f(x)$	$\frac{d}{dx} f(x)$
$\sin^{-1}(x)$	
$\cos^{-1}(x)$	
$\tan^{-1}(x)$	

(See p41 in the Mathematical Tables).

However, we need to be able to work these out using the *Chain Rule*.

Example

Use the Chain Rule, and that $\cos^2(x) + \sin^2(x) = 1$ to find the derivative of $y = sin^{-1}(x)$

Exercise (Q8.2)

Show that

(i) $\dfrac{d}{dx}\Big(\sin^{-1}(x)\Big) = \dfrac{1}{\sqrt{1-x^2}}.$

(ii) $\dfrac{d}{dx}\Big(\cos^{-1}(\frac{x}{a})\Big) = \dfrac{-1}{\sqrt{a^2-x^2}}.$

(iii) $\dfrac{d}{dx}\Big(\tan^{-1}(\frac{x}{a})\Big) = \dfrac{1}{a^2+x^2}.$

Hint: Use that

- $\cos^2(x) + \sin^2(x) = 1,$
- $\sec(x) = 1/\cos(x),$
- $\sec^2(x) = 1 + \tan^2(x).$

Euler Formula

For complex numbers, it is possible to express e^x in terms of sin and cos:

$$e^{ix} = \cos(x) + i\sin(x), \quad \text{where } i = \sqrt{-1}.$$

This is known as *Euler's Formula*.

Euler Formula

Example

Use that

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

to find $\frac{d}{dx}\cos(x)$.

Solution:

Euler Formula

Exercise (Q8.3)

Use the Euler formula to show the following:

(i) $\sin(x) = \dfrac{-i}{2}\left(e^{ix} - e^{-ix}\right)$,

(ii) $\dfrac{d}{dx}\sin(x) = \cos(x)$

(iii) $\displaystyle\int \sin(x) = -\cos(x) + C$

(iv) $\sin^2(x) + \cos^2(x) = 1$

The Hyperbolic Functions

From Euler's formula, we can get the following definitions of sin and cos

$$\cos(x) = \frac{1}{2}\big(e^{ix} + e^{-ix}\big), \quad \text{and} \quad \sin(x) = \frac{-i}{2}\big(e^{ix} - e^{-ix}\big).$$

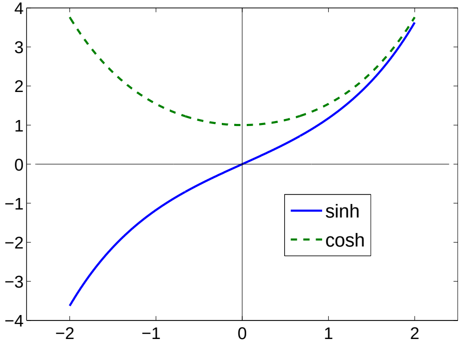
Based on these, can define their *Hyperbolic* analogs...

The Hyperbolic Functions

Definition (Hyperbolic Functions)

The **Hyperbolic cosine and sine functions** are defined as

$$\cosh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right), \quad \text{and} \quad \sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right),$$



Derivative of $\sinh(x)$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

Proof:

Exercise

Show that

$$\frac{d}{dx}(\cosh x) = \sinh x$$

Express $\sinh^{-1}(x)$ in terms of logarithms.

Answer:

Exercise

Show that

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$