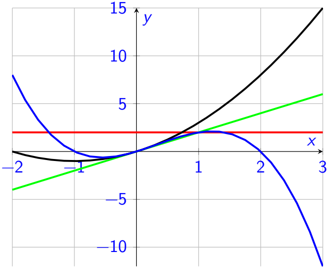
Functions

Dr Niall Madden

University of Galway

16 September, 2025



Outline

For more, see Sections 1.1 and 1.2 of `https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/01%3A_Functions_and_Graphs`

Functions: notation

Recall: This section is all about **functions**, which a "rule" for mapping inputs to outputs.

1. Writing $f : A \to B$ means the inputs come from the set A, and the outputs come from the set B. (A **set** is just a collection of things).
2. A is called the **domain**, and B is called the **co-domain**.
3. $y = f(x)$ means "x gets mapped to y according to the rule defined by f". We sometimes also say "y is the image of x".
4. The subset of B that contains all the images of the things in A is called the **range** of f.
5. When we write $x \in A$ we mean "x is an element of X, or "x belongs to A".

Often, the domain of a function is not expilicitly stated.
In such a case the following **Domain Convention** applies.

The **domain** of a function f is the set of all numbers x for which $f(x)$ *makes sense* and gives a *real-number output*.

Example

1. Find the subset of $\mathbb{R}$ that is the **domain** of $f_1(x) = \dfrac{1}{x^2 - x}$.

Example

Find the subset of $\mathbb{R}$ that is the **domain** of the function $f_2(x) = \sqrt{x+2}$.

Example

Given the function $f_3(x) = 3x^2 + 1$, find the largest subset of $\mathbb{R}$ that is the domain of f_3. What is the corresponding **range**?

Example

Identify the domain (in $\mathbb{R}$) and range of
$f_4(x) = \sqrt{(x+4)(3-x)}$

Example

Identify the domain and range of $f_5(x) = \dfrac{1}{x}$.

4 Ways to Represent a Function

A function can be represented in different ways:

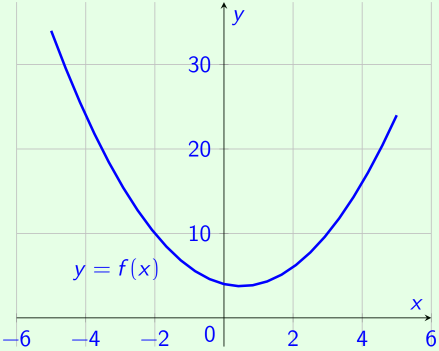
1. **verbally** (by a description in *words*);
2. **numerically** (as a *table* of values);
3. **visually** (as a *graph*);
4. **algebraically** (by an explicit *formula*).

Often it is possible, and useful, to go from one way to another.

Graphical Representation

Graph $\to$ Table

A common way to *visualize* a function $f\colon X \to \mathbb{R}$ is its *graph* in the x, y-plane. In this example, $f(x) = x^2 - x + 4$.



x	$f(x)$
-4	24
-2	10
0	4
2	6
4	16

A Catalog of Functions

There are many *different types of functions* that can be used to *model relationships* between objects in the *real world*.

The most common types of functions (in MA140) are:

- *Linear* Functions,
- *Polynomial* Functions,
- *Power* Functions,
- *Rational* Functions,
- *Algebraic* Functions,
- *Trigonometric* Functions,
- *Exponential* Functions,
- *Logarithms*.

Linear functions have formulae such as $f(x) = mx + c$, where m and c are some given numbers.

It is often represented graphically as a straight line of slope m through the point $(0, c)$.

Polynomials

Polynomials

A **polynomial function** (or just **polynomial**) is a function of the form

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0, \quad x \in \mathbb{R},$$

where $a_0, a_1, ..., a_n$ are real numbers called the **coefficients** of the polynomial.
The number n is called the **degree** of the polynomial.

There are special names for polynomials of low degree:

Polynomials

Example: Linear Polynomial

$y = 3x - 0.5$ is a **linear** polynomial: it has degree $n = 1$.

Polynomials

Example: quadratic

$x^2 - 2x - 3$ is a **quadratic** polynomial: it has degree $n = 2$.

There are many occasions when we want to **factorise** such quadratics, meaning we write them as the product of a pair of linear polynomials.

For example, we can **factorise** $x^2 - 2x - 3$ as

$$x^2 - 4x + 3 = (x - 3)(x + 1)$$

Polynomials

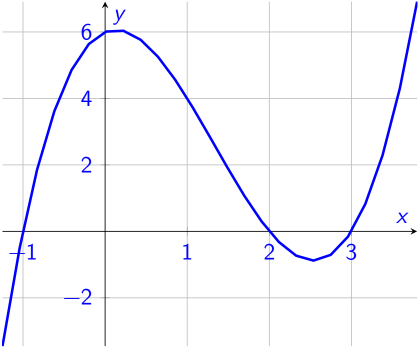
It is important to note that not all quadratic polynomials can be factorised as two linear polynomials. Such quadratics are called **irreducible**.

For example, $x^2 + 1$ is irreducible.

Polynomials

Example

$y = x^3 - 4x^2 + x + 6$ is a **cubic** function with degree $n = 3$.



Fact

A polynomial function of grade n has **up to** $n-1$ truning points ("bends").

Examples:

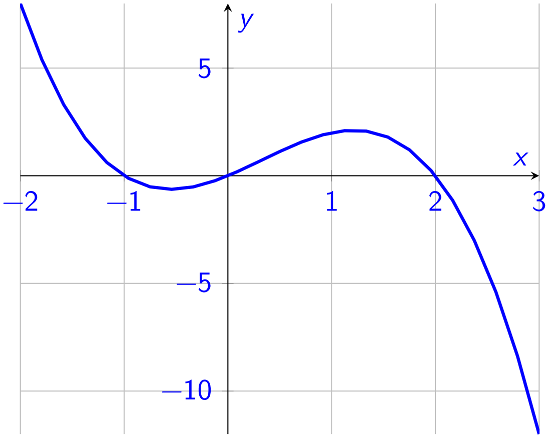
When sketching the graph of a function, we first find the **intercepts**:

- The y-**intercept** is where the graph of the function cuts the y-axis: found by letting $x = 0$.
- The x-**intercepts** are where the function's graph cuts the x-axis. These points are also called the **roots** (or **zeros**). To find them, set y equal to zero and solve for x.

Example

Sketch the graph of $y = -x^3 + x^2 + 2x$

Actual plot of $y = -x^3 + x^2 + 2x$



Rational Functions

Rational Functions have the general form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

- If degree of $p(x) <$ degree of $q(x)$,
  $f(x)$ is called a **strictly proper rational function**.
- If degree of $p(x) =$ degree of $q(x)$,
  $f(x)$ is called a **proper rational function**.
- If degree of $p(x) >$ degree of $q(x)$,
  $f(x)$ is called an **improper rational function**.

Rational Functions

An improper or proper rational function can always be expressed as a polynomial plus a strictly proper rational function, for example by algebraic division.

Example

$$\frac{4x^3 + 4x^2 + 4}{x^2 - 3} = 4x + 4 + \frac{12x + 16}{x^2 - 3}$$

For the previous example, we can work this out ourselves using **Long Division** to divide numerator by denominator:

Example 2.30 from text book

Use long division to show that

$$\frac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3} = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

Exercises

Exercise 1.2.1

Identify the largest possible subset of $\mathbb{R}$ that could be the domain and range of these functions:

1. $f(x) = (x-4)^2 + 5$
2. $f(x) = \sqrt{3x+2} - 1$
3. $f(x) = 3/(x-2)$.

(See Example 1.1.2 of the textbook).

Exercise 1.2.2

Sketch the graphs of

(i) $y = 5x^2 - 7$

(ii) $y = x^2 - 4x + 3$

(iii) $y = x^3 - 6x^2 - 11x - 6$