2425-MA140 Engineering Calculus

Week 08, Lecture 3
Areas between Curves, and Improper Integrals

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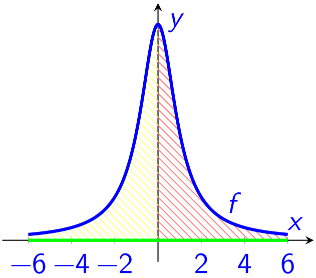


Between now at 10.50...

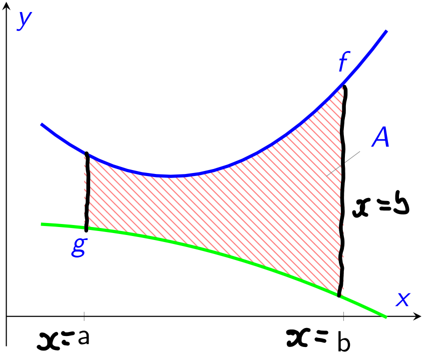
For more: Section 6.1 (Areas between Curves) and Section **7.7** (Improper Integrals) in the textbook:

math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Recall: Areas Between Curves

If $f(x) \geqslant g(x)$ for $x \in [a, b]$, the area of the region between $x = a$, and $x = b$, and between $y = g(x)$ and $y = f(x)$ is

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \boxed{\int_a^b \big(f(x) - g(x)\big)\,dx\,.}$$



Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect. That is, we have to find a and b.

Example (from Q5(a) of 2024/2025 Exam paper)

Compute the region bounded by the curves $f(x) = 3x + 4$ and the $g(x) = 2x^2 + 2x + 1$.

First we need to find the points were $f(x)$ and $g(x)$ intersect. That is, we solve $f(x) = g(x)$:, ie $f(x) - g(x) = 0$.

$$\underbrace{(3x+4)}_{f(x)} - (\underbrace{2x^2 + 2x + 1}_{g(x)}) = 0$$

$$\implies -2x^2 + x + 3 = 0 \quad \text{(factorize)}$$

$$\implies -2(x+1)(x-3/2) = 0 \quad (1)$$

So they intersect at $x = -1$ and $x = 3/2$. So $a = -1$, $b = 3/2$.

(Continued)

So the area is given by $\int_a^b f(x) - g(x)\,dx =$

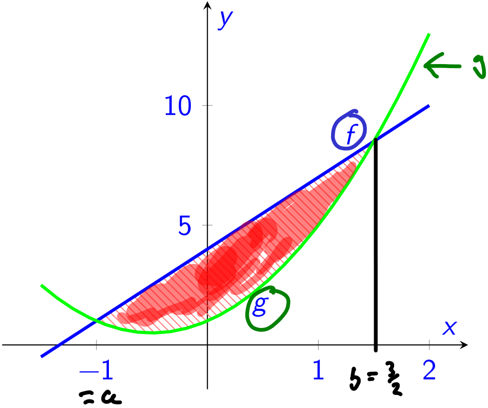
$$\int_{-1}^{3/2} f(x) - g(x)dx$$

$$= \int_{-1}^{3/2} -2x^2 + x + 3dx$$

$$= \left(-\frac{2}{3}x^3 + \frac{1}{2}x^2 + 3x\right)\Big|_{-1}^{3/2}$$

$$= \left(-\frac{2}{3}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right)\right) - \left(-\frac{2}{3}(-1) + \frac{1}{2}(1) + 3(-1)\right)$$

$$= 125/24.$$


y
10
5
x
−1
1
2
f
g
g
= a
b = 3/2

Compound Regions

In the previous examples, we had $f(x) \geqslant g(x)$ for all $x \in [a, b]$. But what if f and g cross in the domain?

Areas between curves, without $f(x) \geqslant g(x)$

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Then A, the area of the region between the graphs of $f(x)$ and $g(x)$, and between $x = a$ and $x = b$, is given by

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

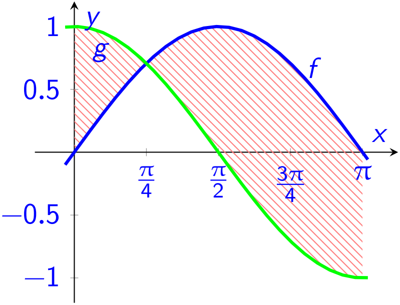
In practice this involves finding the point c where the functions cross... If $f(c) = g(c)$ & $f(x) \geq g(x)$ $x \leq c$
$f(x) \leq g(x)$ $x \geq c$

$$A = \int_a^c f(x) - g(x)\, dx + \int_c^b g(x) - f(x)\, dx$$

Compound Regions

Example [See Eg 6.1.3 in textbook]

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$, from $x = 0$ to $x = \pi$.



Note: at $x=0$
$\cos(0)=1$ $\sin(0)=0$
So $g(x) \geq f(x)$ on the left.

At $x=\pi$ $\cos(\pi)=-1$
$\sin(\pi)=0$
So $f(x) \geq g(x)$.

So where do they cross??

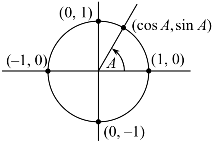
Compound Regions

It will help to consult p13 of the "log" tables.

Triantánacht **Trigonometry**

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \text{cosec}\, A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\text{cosec}\, A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\text{cosec}\, A$ are not defined when $\sin A = 0$.

A (céimeanna)	0°	90°	180°	270°	30°	45°	60°	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	−1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	−1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. ≈ 57.296° 1° ≈ 0.01745 rad.

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We see $\cos(\pi/4) = \sin(\pi/4)$. So $c = \pi/4$.

Compound Regions

Area is $\int_0^{\pi} |f(x)-g(x)|\,dx = \int_0^{\pi} |\sin(x) - \cos(x)|\,dx$

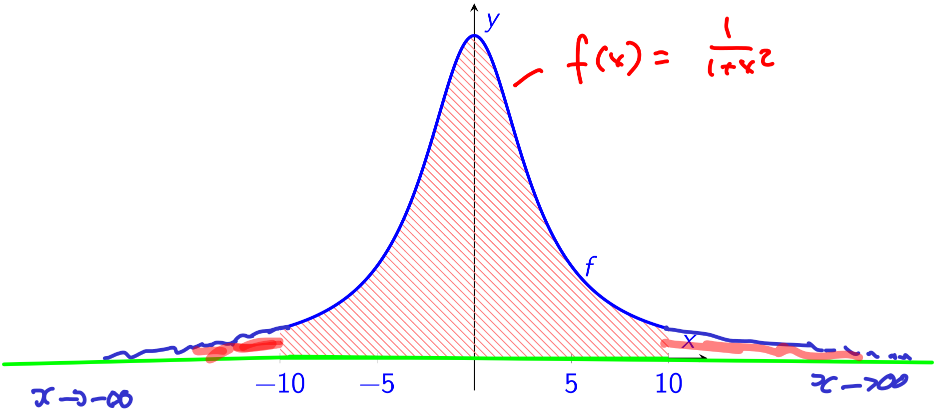
$$= \int_0^{\pi/4} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi} \sin(x) - \cos(x)\,dx.$$

$$= \Big[\sin(x) + \cos(x)\Big]\Big|_0^{\pi/4} + \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi}$$

$$= \Big[\sin(\pi/4) + \cos(\pi/4) - \sin(0) - \cos(0)\Big]$$
$$+ \Big[-\cos(\pi) - \sin(\pi) + \cos(\pi/4) + \sin(\pi/4)\Big]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + 1 - 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= 4/\sqrt{2} = \frac{2\cdot 2}{\sqrt{2}} = 2\,\frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}.$$

Earlier we looked at how $\int_a^b f(x)\,dx$ evaluates as the area of the region between $y = f(x)$ and $y = 0$, and between $x = a$ and $x = b$. But suppose we want to evaluate the area of the region between $y = f(x)$ and $y = 0$, and between (say) $x = -\infty$ and $x = \infty$?



Definition (Improper Integral)

Let f be a continuous function on $[a, \infty)$. The **improper integral of f over $[a, \infty)$** is defined by

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

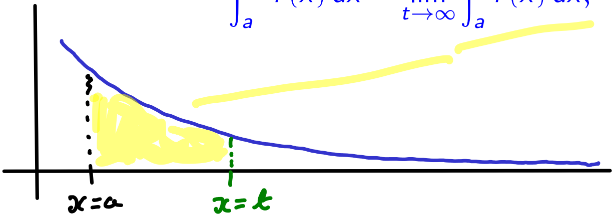
provided that the limit exists. If so, we say that the integral **convergent**. Otherwise, we say it is **divergent**.

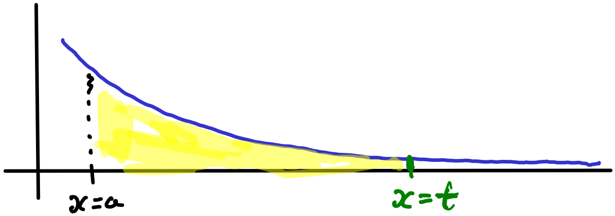
Similarly, if $g(x)$ is continuous $(-\infty, b]$, the improper integral $\int_{-\infty}^b g(x)\,dx$ is **convergent** and given by

$$\int_{-\infty}^b g(x)\,dx = \lim_{t\to-\infty} \int_t^b g(x)\,dx$$

if that the limit exists; otherwise it is **divergent**.

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$





Let $t \to \infty$

Furthermore:

If f is a continuous function on $\mathbb{R} = (-\infty, \infty)$ and the improper integrals

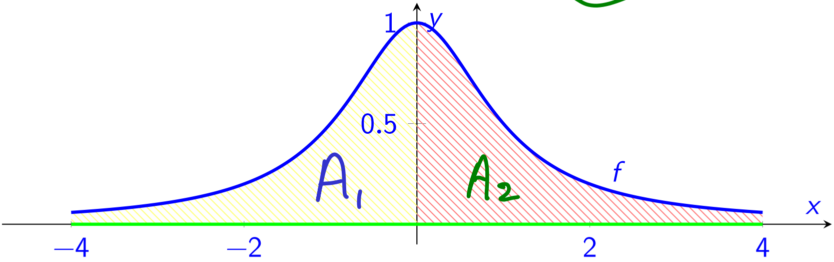
$$\int_{-\infty}^{0} f(x)\,dx \quad \text{and} \quad \int_{0}^{\infty} f(x)\,dx$$

are both convergent, then the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx\,.$$

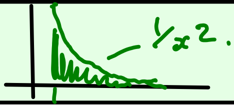
is also **convergent**. If not, we say it is **divergent**.

$$\int_{-\infty}^{\infty} f(x)\,dx = \overbrace{\int_{-\infty}^{0} f(x)\,dx}^{A_1} + \overbrace{\int_{0}^{\infty} f(x)\,dx}^{A_2}.$$



Example

Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.



Idea: Use the definition:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$

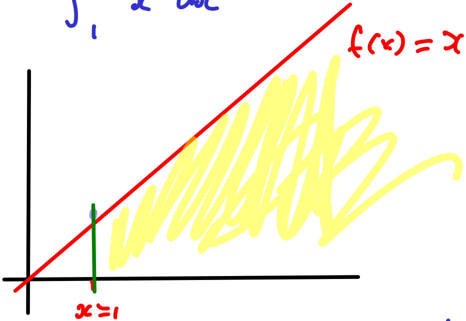
That is: set $g(t) = \displaystyle\int_1^t f(x)\,dx$ and then evaluate $\displaystyle\lim_{t\to\infty} g(t)$

$$g(t) = \int_1^t x^{-2}\,dx = -x^{-1}\Big|_1^t = -\left(\tfrac{1}{t}\right) - \left(-\tfrac{1}{1}\right) = 1 - 1/t.$$

Then $\displaystyle\lim_{t\to\infty} 1 - 1/t = 1$. So: Answer is 1.

Many improper integrals are divergent. Examples:

$$\int_1^\infty x \, dx$$



$$g(t) = \int_1^t x \, dx = \frac{1}{2}x^2 \Big|_1^t = \frac{1}{2}(t^2 - 1)$$

$$\text{So } \lim_{t \to \infty} g(x) = \lim_{t \to \infty} \frac{1}{2}(t^2 - 1) = \infty$$

Many improper integrals are divergent. Examples: $\int_1^\infty f(x)\,dx$

with $f(x) \equiv 1$.

So $g(t) = \int_1^t 1\,dx = [x]\Big|_1^t = t-1$

So $\lim_{t\to\infty} g(t) = \infty$.

So this integral is divergent too.

If $f(x)$ is a positive function, for $\int_a^\infty f(x)\,dx$ to exist, at the very least we need $f(x)$ to be a decreasing function. But often that alone is not enough!

- We know that $\int_1^\infty x^{-2}\,dx$ is convergent.
- From that we can deduce that $\int_1^\infty x^{-n}\,dx$ is convergent for any $n \geqslant 2$. (Why?)
- And we know $\boxed{\int_1^\infty x^0\,dx}$ is divergent. $x^0 \equiv 1$
- But what about $\int_1^\infty x^{-1}\,dx$?

Well, $x^n \geq x^2$ for any $n \geq 2$, $x \geq 1$

So $\dfrac{1}{x^n} \leq \dfrac{1}{x^2}$

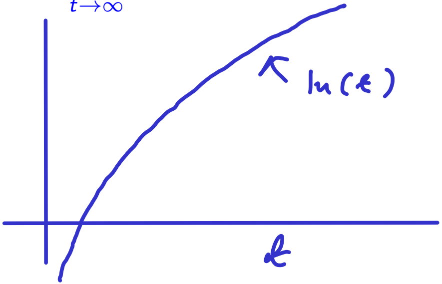
So $\displaystyle\int_1^\infty \frac{1}{x^n}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx.$ So convergent

Example

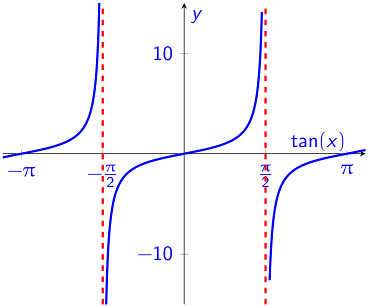
Determine whether the improper integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

For $t \geqslant 1$, we have $\displaystyle\int_1^t \frac{1}{x}\,dx = \ln(x)\Big|_1^t = \ln(t)$. since $\ln(1)=0$.

Since $\lim_{t\to\infty} \ln(t)$ does not exist, it follows that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.



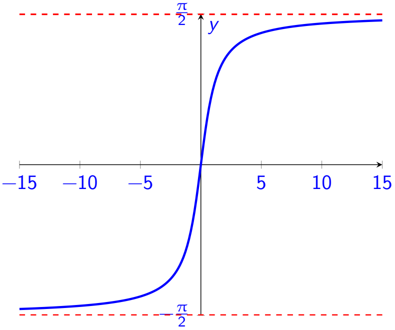
In our next, and final example, we'll try to integrate $f(x) = \dfrac{1}{1+x^2}$. To follow the solution, you might find it useful to revise the fundamentals of **inverse trigonometric functions**. You can find that in Section 1.4 of the textbook:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)



In the figure opposite, we see the graph of $\tan(x)$. Notice that it has vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

On the right is a plot of the **inverse of the** $\tan(x)$ function, which is often written as either $\tan^{-1}(x)$ or $\arctan(x)$. Notice that it has **horizontal** asymptotes at $y = -\pi/2$ and $y = \pi/2$.



This means that

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2},$$

and

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Example

Evaluate the improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Note ("log tables"): $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$.

So $\int_{t}^{0} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_{t}^{0} = \tan^{-1}(0) - \tan^{-1}(t)$

$= -\tan^{-1}(t)$

$\lim_{t\to-\infty} -\tan^{-1}(t) = -\left(-\pi/2\right) = \pi/2.$

Similarly $\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \pi/2.$

Example

Evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Therefore

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

$$= \frac{\pi}{2}. + \frac{\pi}{2}$$

$$= \pi.$$

Exercises

Exer 8.3.1 (From 2019/2020 exam)

The functions $f(x) = 1/x$ and $g(x) = x^2$ intersect at $x = 1$. Calculate the area between their graphs on $[1, 2]$

Exer 8.3.2 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Exer 8.3.3 (From 23/24 exam)

Find the area bounded by the curves $f(x) = x^2 - 4x$ and $g(x) = 2x - 5$.

Exercises

Exer 8.3.4 (From 23/24 exam)

Evaluate $\int_0^\infty \frac{x}{1+x^4}\,dx$ (*Hint: try substitution with* $u = x^2$).