MA385: Tutorials 2+3 ANS with solutions to Q1, and Q2(a)–(c)

These exercises are for Tutorials 2 and 3 (Weeks 6 and 7). You do not have to submit solutions to these questions. However, you do have to submit solutions to related questions on Assignment 1

Q1. Suppose that we have a fixed point method $x_{k+1} = g(x_k)$ which we know to be converges to fixed point of g, denoted τ. Show that, if $g'(\tau) = g''(\tau) = 0$, then convergence of the method is at least Order **3**.

> **Answer:** From Definition 1.3.2 (Order of Convergence) from Section 1.3 (Secant Method), we know that method which generates the sequence $\{x_0, x_1, x_2, \dots\}$ converges with *at least order* q if we have error bounds $\varepsilon_k \leqslant |\tau - x_k|$ such that $\varepsilon_k \to 0$ as $k \to 0$, (i.e., the method converges) and $\frac{\varepsilon_{k+1}}{\varepsilon_k^q} \to \mu$ as $k \to \infty$. We are already told that the method converges, so we just need to find μ such that $\frac{\varepsilon_{k+1}}{\varepsilon_k^3} \to \mu$. Let's write out a Taylor series for $g(x)$ about τ:
>
> $$g(x) = g(\tau) + (x-\tau)g'(\tau) + \frac{1}{2}(x-\tau)^2 g''(\tau) + \frac{1}{3!}(x-\tau)^3 g'''(\eta)$$
>
> for some $\eta \in [x, \tau]$. Since $g'(\tau) = g''(\tau) = 0$, this simplifies:
>
> $$g(x) = g(\tau) + \frac{1}{3!}(x-\tau)^3 g'''(\eta).$$
>
> Take $x = x_k$, so this becomes $g(x_k) - g(\tau) = (x_k - \tau)^3 \frac{g'''(\eta_k)}{3!}$ for some $\eta_k \in [x_k, \tau]$. Since $x_{i+1} = g(x_k)$, and $g(\tau) = \tau$, we now have $|x_{i+1} - \tau| = (|x_k - \tau|)^3 \frac{g'''(\eta_k)}{3!}$. To finish, since we are told the method converges, we know that $x_k \to \tau$. So $\eta_k \to \tau$. Thus $g'''(\eta_k) \to g'''(\tau)$, which is a constant. We can take $\varepsilon_k = |\tau - x_k|$. Then we have
>
> $$\frac{\varepsilon_{k+1}}{\varepsilon_k^3} \to \mu \qquad \text{where } \mu = |g'''(\tau)|/6.$$

Q2. About 2,000 years ago, in Alexandria (Egypt), Hero proposed the following iterative method for estimating $\sqrt{n}$ for any $n > 0$:

$$x_{k+1} = \frac{x_k}{2} + \frac{n}{2x_k}. \tag{1}$$

(a) If this is a fixed point method, what is g?

> **Answer:** Here $g(x) = x/2 + n/(2x)$

(b) For the method to (provably) work we need to determine if there is a region around $\sqrt{n}$ for which it is a contraction. First show that $1 \leqslant g(x) \leqslant n$ for all $x \in [1, n]$. Then determine a region around $x = \sqrt{n}$ for which $|g'(x)| < 1$.

Answer: To see that $1 \leqslant g(x) \leqslant n$ for any $x \in [1, n]$, first check the end-points, and observe that $g(1) = (1 + n)/2 > 1$ since $n \geqslant 1$. Also $g(n) = (1 + n)/2 < n$.
Finally check for a local max/min in $[1, n]$. Since $|g'(x)| = (1-n/x^2)/2$ we have $g'(\sqrt{n}) = 0$. That is, g has an extreme point at $x = \sqrt{n}$. At that point (of course) $g(\sqrt{n}) = \sqrt{n}$, which is between 1 and n, we get what is required.

Next we determine a region around $x = \sqrt{n}$ for which $|g'(x)| < 1$ for all $x \in [1, n]$. Again we use that $g'(x) = (1 - n/x^2)/2$. This is a monotonically *increasing* function on $[1, n]$. To find the left-end point of the interval for which $|g'(x)| < 1$, we solve $g(x) = -1$ for x. That gives $x = \sqrt{n/3}$.
Next we observe that $g'(n) = (1 - n/n^2)/2 < 1/2$. So, certainly, $|g'(x)| < 1$ for any x in $[\sqrt{n/3}, n]$.
Consequently, it is a contraction on this region.

(c) Show that it is equivalent to Newton's Method, for a suitably defined function f, where $f(\sqrt{n}) = 0$.

(d) Show that it converges (at least) quadratically (i.e., with Order 2).

(e) Does it converge cubically (i.e., with Order 3)?

Q3. Edmund Halley is famous for analysing the orbit of the comet which is now named after him. Another of his discoveries is the following method for solving nonlinear equations:

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2\big(f'(x_k)\big)^2 - f(x_k)f''(x_k)}. \tag{2}$$

Write down the associated Fixed Point method for estimating $\sqrt{2}$. Show that this is the same as the method given by $g_3(x)$ in Lab 1.

(Extra: if you really want, you can show that $g_3'(\sqrt{2}) = g_3''(\sqrt{2}) = 0$, but it is a little tedious).