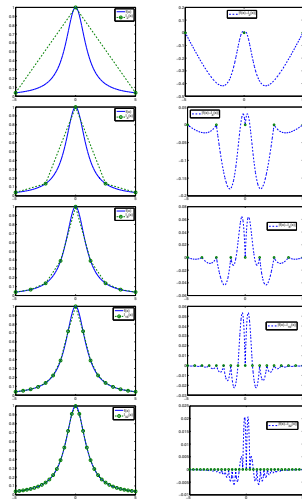


## MA378 Chapter 2: Splines

### §2.1 Linear Interpolating Splines

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## 1.1 Introduction

In Section 1.5 (Convergence and Runge's Example), we learned that it is not always a good idea to interpolate functions by a high-order polynomial at equally spaced points. However, it is possible to obtain very good approximations using a very simple method. The trick is to use a **spline**: a *piecewise polynomial interpolating function*.

We'll consider three important example of splines:

1. *linear splines*
2. *(natural) cubic splines*.
3. *Hermite* piecewise cubics.

For more details about splines, have a look at Chap. 11 of Süli and Mayers, and Lectures 10 and 11 Stewart's "Afternotes goes to Grad School".

## 1.1 Introduction

In this section, we always have  $N$  equally spaced intervals (and so  $N + 1$  equally spaced points). Let  $h = (b - a)/N$ , then

$$a = x_0, \quad b = x_N \quad \text{and} \quad x_i = x_0 + ih \quad \text{for } i = 0, 1, \dots, N.$$

Often these are referred to as *knots points* (or simply as *knots*), and denote the set of knot points by  $\omega^N := \{x_i\}_{i=0}^N$ .

## 1.2 Linear Interpolating Splines

We first study the *piecewise linear interpolant*, also called a *linear spline*. We will see that they have important properties, including

- (a) they are easy to construct and analyse;
- (b) the bound on the error decreases as the number of interpolation points increases;
- (c) the error we get using a linear spline is no more than twice the error using the best possible (piecewise linear) approximation;
- (d) of all the interpolants to  $f$  at a given set of points, the linear spline is the one with the **smallest first derivative**.

## 1.3 Construction on linear splines

### Definition 1.1

Let  $f$  be a function that is continuous on  $[a, b]$ . The *linear spline interpolant* to  $f$  is the continuous function,  $l$ , such that

- (i)  $l(x_i) = f(x_i)$  for each  $i = 0, 1, \dots, N$ ,
- (ii)  $l$  is a linear function  $l_i$  on each interval  $[x_{i-1}, x_i]$ . That is,

$$l(x) = \begin{cases} l_1(x) & x_0 \leq x \leq x_1 \\ l_2(x) & x_1 \leq x \leq x_2 \\ \dots & \\ l_N(x) & x_{N-1} \leq x \leq x_N \end{cases}$$

where each  $l_i \in \mathcal{P}_1$ .

## 1.3 Construction on linear splines

It is easy to write down a formula for the  $l_i$ , based on Lagrange polynomials:

- ▶ Set  $h = (b - a)/N$ .
- ▶ For each  $i = 1, 2, \dots, N$ , define

$$l_i(x) = f(x_{i-1}) \frac{x_i - x}{h} + f(x_i) \frac{x - x_{i-1}}{h}, \quad x \in [x_{i-1}, x_i]. \quad (1)$$

## 1.3 Construction on linear splines

### Example 1.2

Write down the linear spline interpolant to  $f(x) = e^x$  at the knot points  $\{-1, 0, 1\}$ .

## 1.4 Analysis

We know that if  $p_N$  is the polynomial of degree  $N$  that interpolates  $f$  at  $N$  equally spaced points, it does **not** follow that  $p_N \rightarrow f$  as  $N \rightarrow \infty$ . But as we will see, the piecewise linear interpolant to  $f$  converges to  $f$ , albeit slowly.

This is verified in the following theorem, which is a direct consequence of Cauchy's theorem.



## 1.4 Analysis

### Theorem 1.3

*Suppose that  $f$ ,  $f'$  and  $f''$  are all continuous and defined on the interval  $[a, b]$ . Let  $l$  be the linear spline interpolant to  $f$  on the  $N + 1$  equally spaced points  $a = x_0 < x_1 < \dots < x_N = b$  with  $h = x_i - x_{i-1} = (b - a)/N$ . Then*

$$\|f - l\|_{\infty} \leq \frac{h^2}{8} \|f''\|_{\infty},$$

(Here  $\|g\|_{\infty} := \max_{a \leq x \leq b} |g(x)|$ .)

# 1.4 Analysis

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## 1.4 Analysis

It follows directly from this theorem that

$$\lim_{N \rightarrow \infty} \|f - l\|_{\infty} = 0.$$

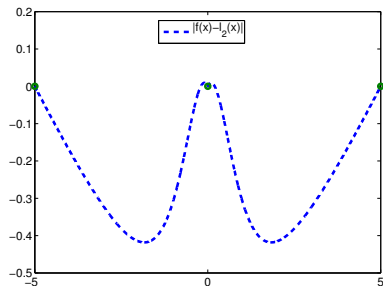
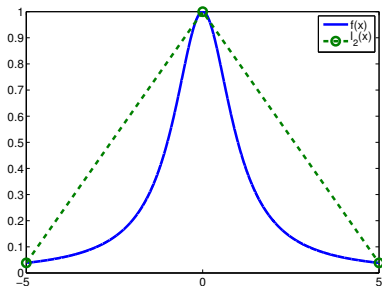
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### Example 1.4

The figure below shows linear spline interpolations of Runge's example:

$$f(x) = \frac{1}{1+x^2} \text{ on } [-5, 5].$$

These diagrams appear to support our assertion that the error tends to zero as  $N \rightarrow \infty$ .



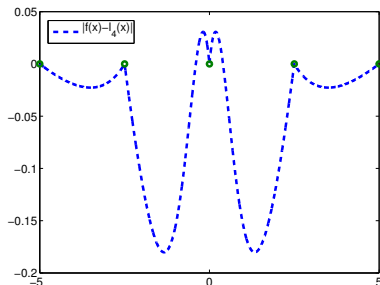
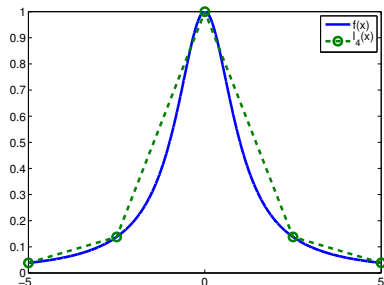
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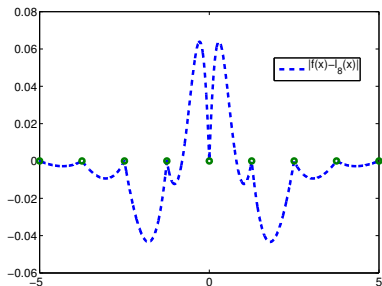
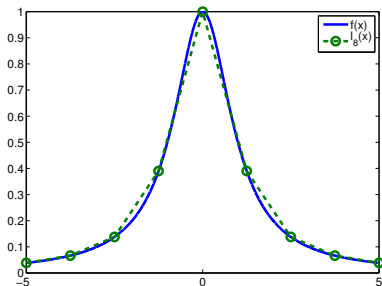
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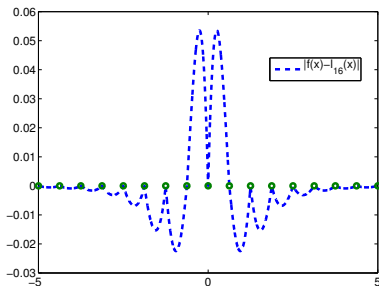
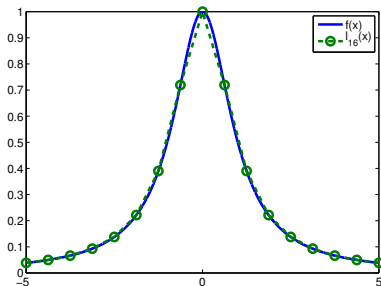
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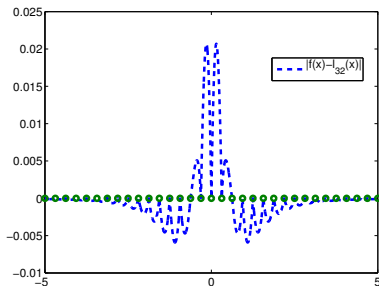
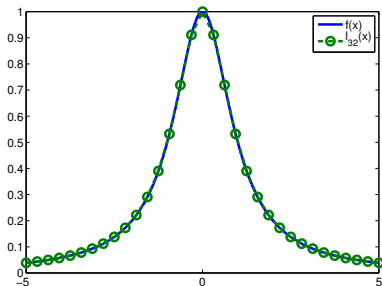
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## 1.4 Analysis

### Example 1.5

Suppose we interpolate  $f(x) = e^x$  with linear splines on  $N + 1$  equally spaced points between  $x_0 = -1$  and  $x_N = 1$ . What value of  $N$  would we have to take to ensure that the maximum error is less than  $10^{-2}$ ?

## 1.5 Best approximation

For the next part of the analysis it will help to think of piecewise linear interpolation as an *operator*. Then we can compare the linear spline to all the other piecewise linear approximations.

First, observe that one can define an infinite number of piecewise linear functions on a given set of  $N + 1$  knot points, denoted  $\omega^N$ . We'll call the set of these functions  $\mathcal{L}$ .

## 1.5 Best approximation

### Definition 1.6

For a fixed set of knot points  $\omega^N$ , let  $L$  be the operator that maps the continuous function  $f$  to its linear spline interpolant  $l \in \mathcal{L}$ .

Now suppose that  $g \in \mathcal{L}$ . Then  $L(g) = g$ . That is  $L$  is a *projection*:  $L(L(f)) = L(f)$ .

## 1.5 Best approximation

It is not hard to see that one could find a different function  $\hat{l} \in \mathcal{L}$  that is a better approximation of  $f$  in sense that

$$\max_{x_0 \leq x \leq x_n} |f(x) - \hat{l}(x)| < \max_{x_0 \leq x \leq x_n} |f(x) - l(x)|.$$

However,  $l$  is very easy to find, and the associated error is no worse than twice  $\|f(x) - \hat{l}(x)\|_\infty$ .

## 1.5 Best approximation

### Theorem 1.7 (Stewart's “Afternotes goes to grad school”, Lecture 10)

Let  $l = L(f)$ . For any  $\hat{l} \in \mathcal{L}$ ,

$$\|f - l\|_{\infty} \leq 2\|f - \hat{l}\|_{\infty}.$$

In the proof of this, we need several facts about  $L$ :

- ▶  $L$  is a projection:  $L(f) = L(L(f))$ .
- ▶  $L$  is a linear:  $L(f + g) = L(f) + L(g)$ .
- ▶  $L$  is a  $\|\cdot\|_{\infty}$ -stable:  $\|L(f)\|_{\infty} \leq \|f\|_{\infty}$ .

## 1.5 Best approximation

**Proof:**

## 1.6 Minimum Energy

The final interesting property of  $l$  that we will study is called the *minimum energy property*.

### Definition 1.8

Let  $u$  be a function that is continuous and defined on the interval  $[a, b]$  except, maybe, at the (countable set)  $\omega^N$  of knot points<sup>a</sup> Then the 2-norm of  $u$  is

$$\|u\|_{2,[a,b]} := \left( \int_a^b u^2(x) dx \right)^{1/2}.$$

Usually we just write this as  $\|u\|_2$ .

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<sup>a</sup>More precisely, we should say “everywhere, except on a set of measure zero”. However, since not everyone is familiar with the terminology, we’ll skip the details.

## 1.6 Minimum Energy

Let  $H^1$  be the set of all functions  $u$  that are continuous on  $[a, b]$  and have  $\|u'\|_2 < \infty$ . Note that  $l \in H^1$ , even though we have not properly defined  $l'$  at the mesh points  $\omega^N$ .



## 1.6 Minimum Energy

### Theorem 1.9 (Süli and Mayers, Thm. 11.2)

*Let  $w$  be any function in  $H^1$  that interpolates the function  $f$  at the points in  $\omega^N$ . Let  $l$  be the linear spline interpolant of  $f$ . Then*

$$\|l'\|_2 \leq \|w'\|_2.$$

## 1.7 Looking ahead

Piecewise linear interpolation is one of the most standard tools in computational science, engineering and statistics.

Its major drawback is that it can't represent the *curvature* of the function it is interpolating. In the next section we'll investigate how to do that using *cubic* splines.

## 1.8 Exercises

### Exercise 1.1

Page 28 of the Department of Education's old Mathematics Tables ("The *Log Tables*") reports that  $\ln(1) = 0$ ,  $\ln(1.5) = 0.4055$  and  $\ln(2) = 0.6931$ .

- (i) Write down the linear spline  $l$  that interpolates  $f(x) = \ln(x)$  at the points  $x_0 = 1$ ,  $x_1 = 1.5$  and  $x_2 = 2$ .
- (ii) Use this to estimate  $\ln(x)$  at  $x = 1.2$ . How does this compare to the value in the tables, which is 0.1823?
- (iii) Give an estimate for the maximum error:

$$\max_{1 \leq x \leq 2} |f(x) - l(x)|.$$

- (iv) What value of  $n$  would you choose to ensure that  $|f(x) - l(x)| \leq 0.001$  for all  $x \in [1, 2]$ .

## 1.8 Exercises

### Exercise 1.2

One can also define the linear spline interpolant to a function as a linear combination of a set of piecewise linear basis functions  $\{\psi_i\}_{i=0}^N$ :

$$\psi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

They are depicted in Figure 1.

- (i) Write down a formula for the  $\psi_i(x)$ ;
- (ii) derive a formula for  $l(x)$  in terms of the  $\psi_i$ .

This exercise is useful: we'll use these basis functions (called “hat” functions) in the final section of the course.

## 1.8 Exercises

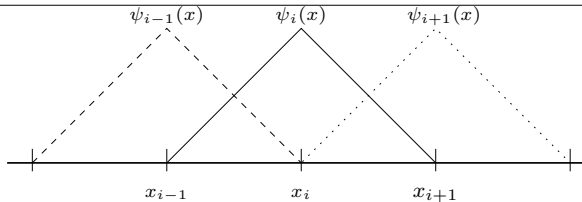


Figure: Some hat functions