2425-MA140 Engineering Calculus

Week 07, Lecture 1 (L19)
Introduction to Integration

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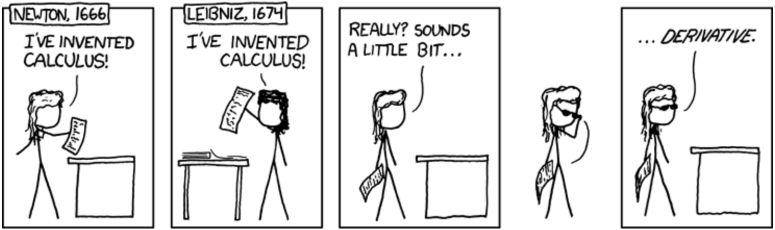

https://imgs.xkcd.com/comics/newton_and_leibniz.png

Assignments, etc

- **Assignment 3 (resit)**: If your grade does not correspond to what you expected, send Niall your **results summary**.
- The "optional" **Assignment 4** is due at 5pm today (Tuesday 29th October).
- **Assignment 5** is open. Deadline is 5pm next Monday (4 November). You have 3 attempts for each question. However, Q1 will be manually graded after the deadline.

Today, it is time we learned about...

1 Introduction

2 Sums

3 Approximating area
- The Riemann sum
- Area under the curve

4 The Definite Integral
- Properties
- Average value

5 Towards anti-derivatives

6 Exercises

See also: Sections **5.1** (Approximating Areas), **5.2** (The Definite Integral) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Introduction

Over the next few weeks, we'll learn all about *integration*.

It is an ancient topic: much, much older than the study of differentiation.

At its heart, integration is the study of **averages**, **areas** and **volumes**. Versions of when we'll learn were know the Greeks over 2000 years ago, and in China at least 1,700 years ago. The formulae we'll learn for integrating polynomials were known to scholars is the Middle East and Africa over 1,000 years ago.

The link between integration and differentiation was developed in the 17th century (Newton and Leibniz, again).

We'll start by studying estimation of areas, and then, ove rthe next few classes, move on the many methods for computing integrals.

Sums

We'll first learn about **integration** as a means for computing the area of regions is space ("area under the curve"). One approach to estimating areas is the divide the region into much smaller ones, whose areas we can compute, and then add those up. So we need notation for writing down long sums. For that we'll use the "**Sigma**" notation.

For example,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

can be written as $\displaystyle\sum_{i=1}^{10} i$.

Sums

More generally,

Sigma ($\sum$) notation

Given a list of numbers, a_1, a_2, ..., a_n,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

Examples:

Sums

Although we don't need to derive or prove the following facts, you should check they are correct for some small values of n

$$\sum_{i=1}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n.$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

(These results are not so significant, but we'll have reason to recall them tomorrow...).

Important: in the following slides, you should read to the "annotated" version because important details will be added in class.

Approximating area

Suppose we had a positive function $f(x)$, and wanted to estimate the area bounded by

- top: $y = f(x)$
- bottom: $y = 0$
- left: $x = a$, and
- right: $x = b$.

Then $A \approx (b - a)f(a)$ would be a crude approximation.

Approximating area

In the previous example, we approximated the area by the area of a single rectangle. We might be able to get a better estimate by using two smaller rectangles: $A \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f\left(\frac{a+b}{2}\right)$

Approximating area

We could do even better with, for example, 4 rectangles. However, the formulae would get more complicated, unless we introduce some notation:

- Let $h = \frac{b-a}{4}$. (Note: later, we'll call this quantity δx).
- Define four points: $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, and $x_3 = a + 3h$.
- Then $A \approx hf(x_0) + hf(x_1) + hf(x_2) + hf(x_3) = \sum_{i=0}^{3} hf(x_i)$

Approximating area

Now we can see that we can approximate the area using n rectangles, for any natural number n that you like:

- Let $h = \frac{b-a}{n}$.
- Define the points: $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $\ldots$ $x_i = a + ih$, $\ldots$
- Then $A \approx hf(x_0) + hf(x_1) + \cdots + hf(x_{n-1}) = \sum_{i=0}^{n-1} hf(x_i)$

The thing we've just been studying is called the **Riemann sum** (after Bernhard Riemann: look him up).

Riemann sum (kinda)

Let $f(x)$ be defined on the interval $[a,b]$. Let n be a positive integer, and set $h=(b-a)/n$. A **Riemann sum** for $f(x)$ is

$$h\sum_{i=0}^{n-1} f(x_i).$$

(The actual definition is a bit more general: see the text book for details).

What is important is: the larger n is, the better the approximation we get. And, as $n \to \textit{infinity}$ our estimate tends to the true value of the area!

Area under the curve

Let $f(x)$ be a non-negative function be defined on the interval $[a, b]$. Let n be a positive integer, and set $h = (b-a)/n$, Let $x_i = a + ih$. Then the area between $y = 0$ and $y = f(x)$, from $x = a$ to $a = b$ is

$$A = \lim_{n \to \infty} h \sum_{i=0}^{n-1} f(x_i)$$

Equivalently: $A = \lim_{h \to 0} h \sum_{i=0}^{n-1} f(x_i)$

The Definite Integral

It transpires that the concept we have just studied has many more applications that the previous discussion might suggest. These include

- Computing the distance travelled by an object knowing only its speed.
- The volume of a wine barrel (which involves the origin story of one of my areas of research: **quadrature**).
- Finding the centre of mass/gravity of an object.
- The chances of rolling a natural 20.
- Etc, etc.

The applications are so many and varied, that we eventually have to get go of the "area under the curve" idea. But we can keep the notation, to get the idea of a **definite integral**.

The Definite Integral

Definition: definite integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{i=0}^{n-1} f(x_i),$$

where $h = (b-a)/n$ and $x_i = a + ih$, provided the limit exists.

Notes:

- The $\int$ symbol is a stylised "S", meaning "sum"
- When reading "$\int_a^b f(x)dx$" out loud, we do so as "the integral of f of x, from a to b, with respect to x".
- We could use this definition to compute some definite integrals, but we won't!

1. $\displaystyle\int_a^a f(x)dx = 0.$
2. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\displaystyle\int_a^b f(x) + g(x)dx = \int_b^a f(x)dx + \int_a^b g(x)dx$
4. For a constant, c we have $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
5. For any points a, b and c,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Suppose we had a function $f(x)$ on $[a, b]$, and we wanted to find a simpler, constant function $g(x) = c$ so that area under $y = f(x)$ and $y = c$ are the same. Then...

Average value of a function

Let $f(x)$ be a continuous function on $[a, b]$. Then its **average value** is

$$f_{\mathrm{ave}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example

Find the average value of the function $f(x) = 3x$ on the interval $[0, 2]$

Towards anti-derivatives

For the rest of this week, we'll focus on being able to compute definite integrals of various functions. To do that, we'll need to see the link between integration and differentiation, given by the **fundamental theorem of calculus**.

First, we notice that, given a function, f, we can define another, F, as

$$F(x) = \int_a^x f(t)dt.$$

That is, the variable in F is the upper limit of integration on the right.

Towards anti-derivatives

Example: Let $F(x) = \int_0^x 1 dt$. Then ...

Exercises

Exer 7.1.1

Evaluate the following, by calculating the areas under the curve.

1. $\displaystyle\int_{-2}^{3} 3dx$
2. $\displaystyle\int_{0}^{2} (3-x)dx$
3. $\displaystyle\int_{-3}^{3} (3-|x|)dx$