

Week 08, Lecture 2  
**Integration by Parts; Areas between Curves**

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## Assignments, etc

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- ▶ **Problem Set 6** is open, and will be covered in tutorials this well. Deadline is 5pm next Monday (10 November).
- ▶ **Problem Set 7** opens by tomorrow.
- ▶ The finally weekly assignment, will open next week.
- ▶ Reminder: The second **class test** takes place November 18.

# This part is about...

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| <ul style="list-style-type: none"><li><b>1</b> Integration by Parts<ul style="list-style-type: none"><li>■ Choosing <math>u</math> and <math>dv</math></li></ul></li><br/><li><b>2</b> Int by Parts: Repeated application<ul style="list-style-type: none"><li>■ Easy example</li></ul></li></ul> | <ul style="list-style-type: none"><li><b>3</b> Recall: Definite integrals</li><li><b>4</b> Definite Integrals with IbP</li><li><b>5</b> Areas Between Curves</li><li><b>6</b> Compound Regions</li><li><b>7</b> Exercises</li></ul> |
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See also Section **7.1** (Integration by Parts) and Section 6.1 (Areas between Curves) in the textbook:

[math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

# Integration by Parts

Yesterday, we learned about integration by parts:

## Integration by Parts

Let  $u$  and  $v$  be differentiable functions. Then

$$\int u \, dv = uv - \int v \, du.$$

One of the challenges of Integration by Parts is knowing how to choose  $u$  and  $dv$ .

In the last example from yesterday, when integrating

$\int x \cos(x) dx$  we choose  $u = x$ , because its derivative,  $u' = 1$  is simpler.

Suppose we had made the bad choice of

$$u(x) = \cos(x), \quad dv = x dx,$$

then we'd get:

To try to get good choices for  $u$  and  $dv$ , we proceed as follows:

1. Some functions are easier to differentiate than and so make a good choice for  $u$ . Important examples include **logarithms** and **inverse trigonometric** functions.
2. Some functions, such as polynomials, may be good choices for  $u$ , since  $u'(x)$  may be simpler than  $u(x)$ .
3. Trigonometric and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for  $dv$ .

**Example (of choosing  $u$ )**

Evaluate  $I = \int \frac{\ln(x)}{x^2} dx.$

**Example**

Evaluate  $I = \int \ln(x) dx$ .

Since  $\int \ln(x) dx$  can be written as  $\int (\ln(x))(1) dx$ , we use integration by parts, with  $u = \ln(x)$  and  $dv = dx$ .

## Int by Parts: Repeated application

Sometimes, we have to apply Integration by Parts more than once.

### Example

Evaluate  $I = \int x^2 e^x dx$ .

## Int by Parts: Repeated application

It is good to check any new rule/method for a simple example we already know the answer to. Now that we know about repeated application, we can do that:

### Example

We know that  $I = \int x^2 dx = (1/3)x^3$ . We can also use IbP.

Take  $u(x) = x$  and  $dv = xdx$ :



## Recall: Definite integrals

Last week we introduced the definite integral as follows:

### Definition: definite integral

If  $f(x)$  is a function defined on an interval  $[a, b]$ , the **definite integral of  $f$  from  $a$  to  $b$**  is given by

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h \sum_{i=0}^{n-1} f(x_i),$$

where  $h = (b - a)/n$  and  $x_i = a + ih$ , provided the limit exists. Moreover, it is the area of the region in space bounded by  $y = 0$ ,  $y = f(x)$ ,  $x = a$  and  $x = b$ .

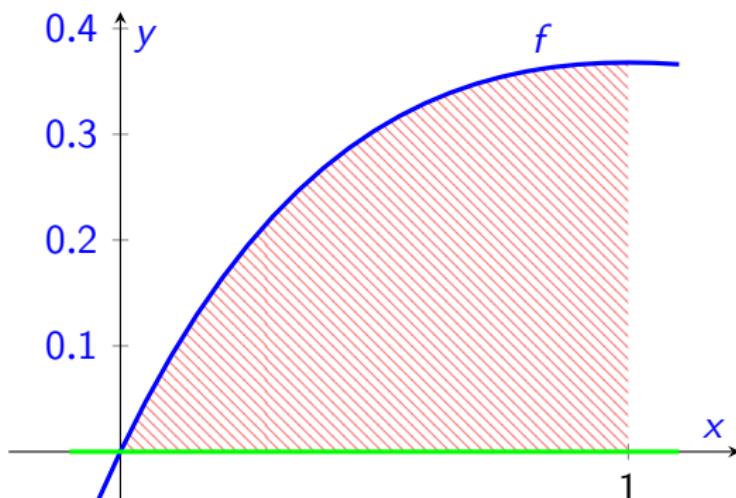
We'll now revisit this idea, and then extend it.

# Definite Integrals with IbP

## Integration by Parts for Definite Integrals

$$\int_a^b u dv = (uv) \Big|_a^b - \int_a^b v du$$

**Example:** First estimate  $\int_0^1 xe^{-x} dx$  from the graph of  $xe^{-x}$



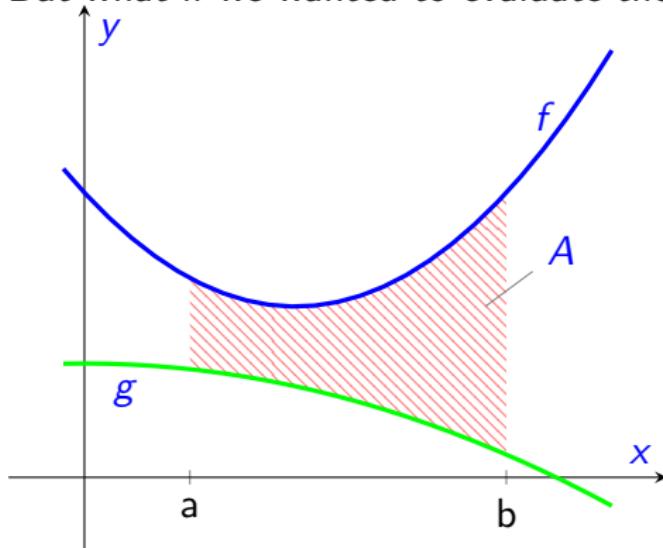
## Definite Integrals with IbP

Now use *Integration By Parts* to actually evaluate  $\int_0^1 xe^{-x} dx$ .

## Areas Between Curves

We know that  $\int_a^b f(x) dx$  evaluates as the area of the region between  $x = a$  and  $x = b$ , and between  $y = f(x)$  and  $y = 0$ .

But what if we wanted to evaluate the area between two curves?



## Area Between Curves

Let  $f$  and  $g$  be continuous functions with  $f(x) \geq g(x)$  throughout the interval  $[a, b]$ . Then the area  $A$  of the region that is

- ▶ bounded on the left by  $x = a$ , and on the right by  $x = b$ ,
- ▶ above by the curve  $y = f(x)$  and below by  $y = g(x)$

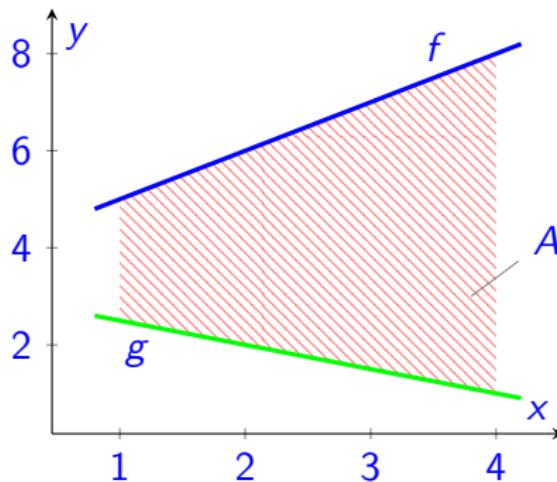
is given by

$$A = \int_a^b (f(x) - g(x)) dx.$$

# Areas Between Curves

## Example

Find the area of the region bounded above by the graph of  $f(x) = x + 4$ , and below by the graph of  $g(x) = 3 - x/2$  over the interval  $[1, 4]$



# Areas Between Curves

Frequently, we need to work out the domain ourselves, by finding where the graphs of the functions intersect. That is, we have to find  $a$  and  $b$ .

## Example (from Q5(a) of 2024/2025 Exam paper)

Compute the region bounded by the curves  $f(x) = 3x + 4$  and the  $g(x) = 2x^2 + 2x + 1$ .

First we need to find the points where  $f(x)$  and  $g(x)$  intersect.  
That is, we solve  $f(x) = g(x)$ :

$$(3x + 4) - (2x^2 + 2x + 1) = 0$$

$$\Rightarrow -2x^2 + x + 3 = 0$$

$$\Rightarrow -2(x + 1)(x - 3/2) = 0 \quad (1)$$

So they intersect at  $x = -1$  and  $x = 3/2$ .

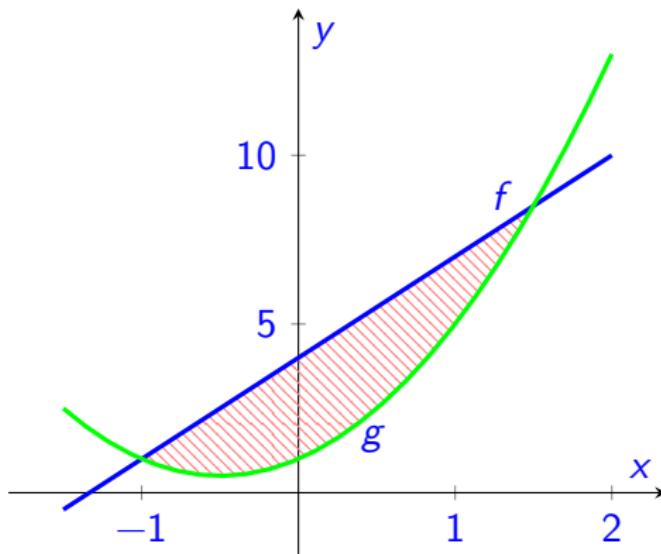
(Continued)

## Areas Between Curves

So the area is given by

$$\begin{aligned} & \int_{-1}^{3/2} f(x) - g(x) dx \\ &= \int_{-1}^{3/2} -2x^2 + x + 3 dx \\ &= \left( -\frac{2}{3}x^3 + \frac{1}{2}x^2 + 3x \right) \Big|_{-1}^{3/2} \\ &= \left( -\frac{2}{3}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right) \right) - \left( -\frac{2}{3}(-1) + \frac{1}{2}(1) + 3(-1) \right) \\ &= 125/24. \end{aligned}$$

## Areas Between Curves



## Compound Regions

In the previous examples, we had  $f(x) \geq g(x)$  for all  $x \in [a, b]$ .  
But what if  $f$  and  $g$  cross in the domain?

### Areas between curves, without $f(x) \geq g(x)$

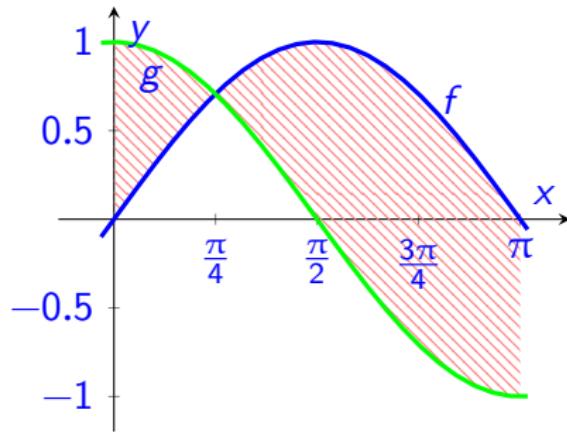
Let  $f(x)$  and  $g(x)$  be continuous functions over an interval  $[a, b]$ .  
Then  $A$ , the area of the region between the graphs of  $f(x)$  and  $g(x)$ , and between  $x = a$  and  $x = b$ , is given by

$$A = \int_a^b |f(x) - g(x)| dx.$$

# Compound Regions

## Example

Find the area between  $f(x) = \sin(x)$  and  $g(x) = \cos(x)$ , from  $x = 0$  to  $x = \pi$ .

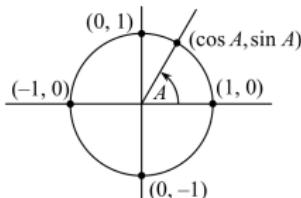


# Compound Regions

It will help to consult p13 of the “log” tables.

## Triantánacht

$$\begin{aligned}\tan A &= \frac{\sin A}{\cos A} & \cot A &= \frac{\cos A}{\sin A} \\ \sec A &= \frac{1}{\cos A} & \operatorname{cosec} A &= \frac{1}{\sin A}\end{aligned}$$



Nóta: Binn tan  $A$  agus sec  $A$  gan sainiú nuair  $\cos A = 0$ .

Binn cot  $A$  agus cosec  $A$  gan sainiú nuair  $\sin A = 0$ .

## Trigonometry

$$\begin{aligned}\cos^2 A + \sin^2 A &= 1 \\ \sec^2 A &= 1 + \tan^2 A \\ \cos(-A) &= \cos A \\ \sin(-A) &= -\sin A \\ \tan(-A) &= -\tan A\end{aligned}$$

Note: tan  $A$  and sec  $A$  are not defined when  $\cos A = 0$ .

cot  $A$  and cosec  $A$  are not defined when  $\sin A = 0$ .

$A$ (céimeanna)	$0^\circ$	$90^\circ$	$180^\circ$	$270^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$A$ (degrees)
$A$ (raidiain)	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$A$ (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad.  $\approx 57.296^\circ$

$1^\circ \approx 0.01745$  rad.

# Compound Regions

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## Exercises

### Exer 8.2.1 (From 2023/2024 exam)

Evaluate  $\int_0^{\pi/2} x \cos(x) dx$ .

### Exer 8.2.2 (From 2019/2020 exam)

The functions  $f(x) = 1/x$  and  $g(x) = x^2$  intersect at  $x = 1$ . Calculate the area between their graphs on  $[1, 2]$

### Exer 8.2.3 (From 2019/2020 exam)

Calculate the bounded area enclosed by the curves  $f(x) = \sqrt{x}$  and  $g(x) = x^2$ .

## Exercises

### Exer 8.2.4 (From 23/24 exam)

Find the area bounded by the curves  $f(x) = x^2 - 4x$  and  $g(x) = 2x - 5$ .