2526-MA140 Engineering Calculus

Week 11, Lecture 2
Numerical Integration

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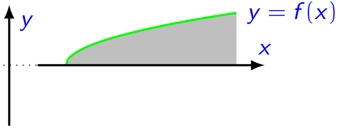
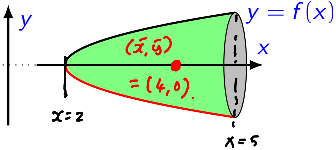
Wednesday, 26 November, 2025





Today's, we'll run the rule over...

1. Sorry!
2. Solids of Revolution
3. Numerical Integration
4. The Midpoint Rule
   - Python
   - MATLAB
   - Error Estimates
   - Other Methods

For more, read Section **7.6** (Numerical Integration) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax).

Sorry!

Tutorials

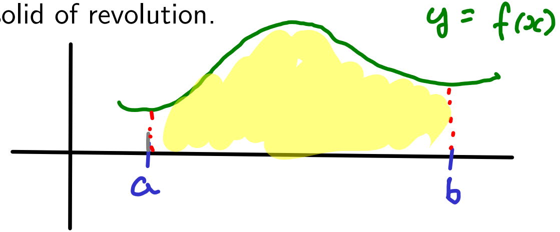
- Apologies to those who tried to attend the tutorial yesterday in MY243. It was booked for MA140, but another class who had been using it all semester thought they had it.
- If you've been inconvenienced by this, let me know and we'll try to find an alternative.
- Thursday's tutorials will go ahead in ENG-3035 and Aras Moyola MY129.

Solids of Revolution

We'll finish this section by considering how to find the centre of mass of a solid of revolution.

Suppose that $f(x) \geqslant 0$ on $[a, b]$ and consider the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$.

Recall that we can rotate this region about the x-axis to obtain a solid of revolution.



Intuitively, it is clear that the centroid of such a solid should lie on the x-axis because of symmetry, so $\bar{y} = 0$. So, we only need find $\bar{x}$.

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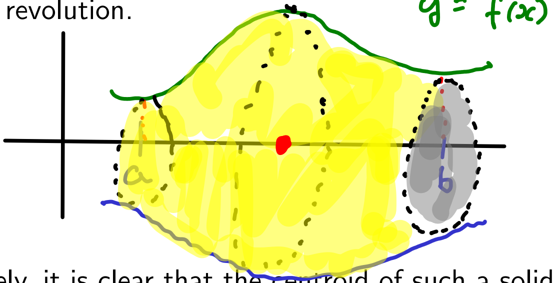

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Solids of Revolution

If the solid has uniform density, $\rho(x, y) \equiv 1$, then the total mass is the same as the volume.

We know already (Disk Method) that the volume of this region is

$$V = \pi \int_a^b f(x)^2 \, dx.$$

Solids of Revolution

To get the moment about the y-axis, we consider the moment of an individual disk of volume ΔV_r, at the point $x = x_r$, which is $x_r \Delta V_r$. If the solid is divided into N such rings:

$$M_y \approx \sum_{r=1}^{n} x_r \underline{\Delta V_r} = \sum_{r=1}^{n} x_r \left(\underbrace{\pi f(x_r)^2 \Delta x}_{} \right)$$

$f(x_r)$ is radius at $x = x_r$

Then, as we have seen repeatedly:

$$\boxed{M_y} = \lim_{n\to\infty} \sum_{r=1}^{n} x_r \left(\pi f(x_r)^2 \Delta x \right) = \boxed{\pi \int_a^b \underline{x} f(x)^2 \, dx}$$

Solids of Revolution

Putting all this together, and using that $M_y = V\bar{x}$, we get...

Centroid of a solid of revolution

If $f(x) \geqslant 0$ on $[a, b]$, then the **centroid**, $(\bar{x}, \bar{y})$ of the solid of revolution obtained by rotating the region enclosed by the curves $y = f(x)$, $y = 0$ and the lines $x = a$ and $x = b$ about the x-axis is

$$\bar{x} = \frac{M_y}{V} \quad \text{and} \quad \bar{y} = 0,$$

where

$$M_y = \pi \int_a^b x f(x)^2 \, dx \quad \text{and} \quad V = \pi \int_a^b f(x)^2 \, dx.$$

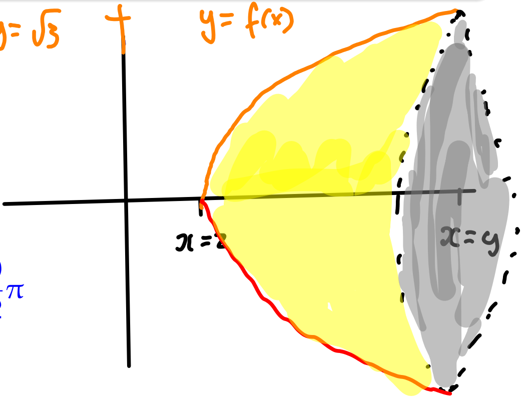
Solids of Revolution

Example

Consider the plane region enclosed by the curve $y = \sqrt{x-2}$, the x-axis and the lines $x = 2$ and $x = 5$. Find the centroid of the solid of revolution obtained by rotating this region about the x-axis.

Answer:

$$V = \pi \int_a^b (f(x))^{2} dx$$
$$= \pi \int_2^5 (x-2)\, dx$$
$$= \pi\left(\frac{1}{2}x^2 - 2x\right)\Big|_2^5$$
$$= \pi\left(\frac{25}{2} - \frac{20}{2} - 2 + 4\right) = \frac{9}{2}\pi$$



Solids of Revolution

$$M_y = \pi \int_a^b x\,[f(x)]^2\,dx = \pi \int_2^5 x\,(x-2)\,dx$$

$$= \pi \int_2^5 x^2 - 2x\,dx.$$

$$= \pi \left[\tfrac{1}{3}x^3 - x^2\right]\Big|_2^5$$

$$= \pi \left[\tfrac{125}{3} - 25 - \tfrac{8}{3} + 4\right]$$

So $M_y = 18\pi$.

So $\bar{x} = \dfrac{M_y}{V} = \dfrac{18\cancel{\pi}}{\tfrac{9}{2}\cancel{\pi}} = \; = \tfrac{2}{9} \cdot 18 = 4.$

Numerical Integration

Regrettably, I may have given you the impression that the best/only way to evaluate $I = \int_a^b f(x)\,dx$ is to find $F(x)$, the antiderivative of f, and then compute $I = F(b) - F(a)$.

That is rather unfortunate, the antiderivative of $f(x)$ maybe hard, or, indeed, **impossible**, to find!

This might be a little surprising, given all the examples we've studied. And also because, one can always find the derivative of $f(x)$, but matter how complicated $f(x)$ is. (Programming this is a major topic in computer science, and very important for modern AI).

Numerical Integration

However, there is good news!

- This is rather liberating, since it means we don't have to try (except for end-of-semester exams).
- There are **algorithms** which can compute $\int_a^b f(x)\,dx$ for you as accurately as you would like! And they are easy to code.
- The **motivation** for these algorithms is easy to understand: they try to estimate areas under curves.
- It is a fascinating area of mathematics.

The Midpoint Rule

Midpoint Rule

To estimate $\int_a^b f(x)\,dx$,

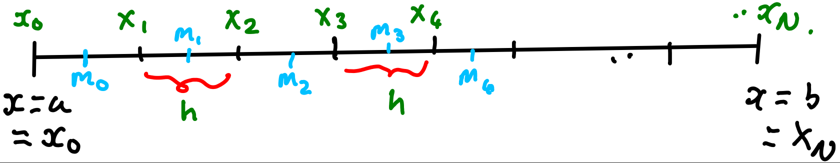
(i) Choose an integer ~~n~~ N, and set $h = (b-a)/N$.

(ii) Split $[a,b]$ in the intervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{N-1}, x_N]$, where $x_k = a + kh$.

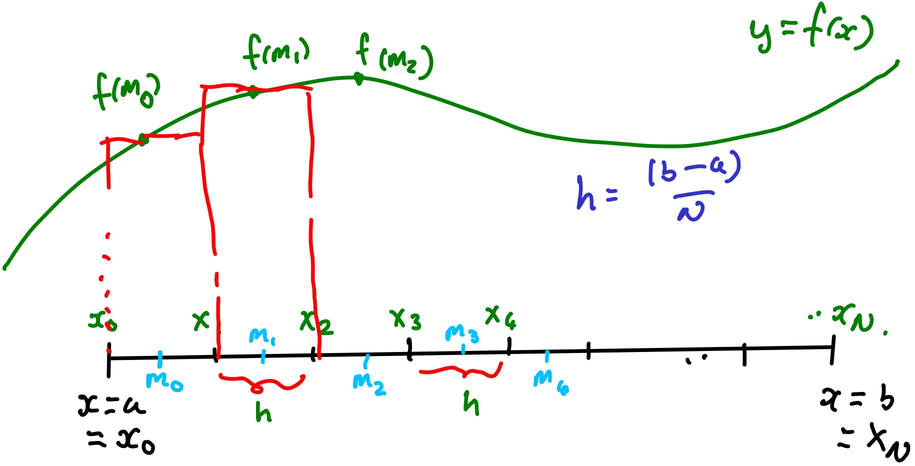
(iii) Let m_k be the midpoint of $[x_k, x_{k+1}]$; i.e., $m_k = x_k + h/2$.

(iv) Compute $M_N = h\big(f(m_0) + f(m_1) + \cdots + f(m_{N-1})\big)$.

This can be easily implemented in a few lines of code.



The Midpoint Rule



Each rectangle has area $f(m_k) \cdot h$.

Total: $\displaystyle\sum_{k=0}^{N-1} h\, f(m_k) = h \sum f(m_k) = \frac{b-a}{N} \sum f(m_k)$

The Midpoint Rule

MidPoint.py

```
import numpy as np
f = lambda x : np.sqrt(x) # function to be integrated
a,b=0,1     # limits of integration
N = 10; h=1/N;
x = np.linspace(0,1,N+1)
m = (x[0:-1] +x[1:])/2  # midpoints
M_N = h*np.sum(f(m))
print(f"N={N : 2d}, M_N={M_N : 8.4f}, Error={np.abs(
    M_N-2/3) : .3e}")
```

x[0]   x[1], ..   x[N]

```
def f(x):
    return np.sqrt(x)
```

MidPoint.m

```
f = @(x)sqrt(x); % function to be integrated
a=0; b=1;    % Limits of inetegration
N = 10; h=(b-a)/N;
x = linspace(0,1,N+1);
m = (x(1:end-1) + x(2:end))/2; % midpoints
M_N = h*sum(f(m));
fprintf("N=%2d, M_N=%8.4f, Error=%8.3e\n", N, M_N,
    abs(M_N-2/3));
```

We won't implement the method by hand, but from a mathematical perspective, it is important to note that it can be proved that, error, $\mathcal{E}_N$ is bounded like

$$\mathcal{E}_N := \left|\int_a^b f(x)\,dx - M_N\right| \leqslant h^2 \frac{b-a}{24} \max_{a\leqslant x\leqslant b} |f''(x)|$$

This has lots of consequences:

(i) If f is a constant or linear polynomial, then the error is zero.

(ii) For most reasonable f, it is clear that, as $N \to \infty$ (and so $h \to 0$) we get that $\mathcal{E}_N \to 0$.

(iii) More practically, it is possible to choose N so that $\mathcal{E}_N$ is as small as you would like.

(iv) The error is proportional to h^2 (or, equivalently, N^{-2}).

Example:

Suppose $f(x) = x^3$, and $a = 0$, $b = 1$.

(i) What is the maximum error expected with $N = 10$?

(ii) What N should you choose to ensure the error is no more than 10^{-4}

(i) $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x.$

So $\max_{0 \le x \le 1} |f''(x)| = 6.$

Also $h = \frac{b-a}{N} = \frac{1}{10} = 0.1$

$\mathcal{E}_{10} = h^2 \frac{|b-a|}{24} \max|f''(x)| = \frac{(0.1)^2}{24} \cdot 6$

$= 0.0025.$

Example:

Suppose $f(x) = x^3$, and $a = 0$, $b = 1$.

(i) What is the maximum error expected with $N = 10$?

(ii) What N should you choose to ensure the error is no more than 10^{-4}

(ii) We want $\mathcal{E}_N \leq 10^{-4}$.

So choose N (ie h) so that

$$h^2 \frac{|b-a|}{24} \max_{0 \leq x \leq 1} |f''(x)| \leq 10^{-4}$$

$$\Rightarrow (N^{-2}) \frac{6}{24} \leq 10^{-4} \Rightarrow N^2 \geq \frac{10^4}{4}$$

$$\Rightarrow N \geq 50.$$

With more time, we'd investigate methods that are more accurate, but require the same amount of effort, such as the Trapezoidal Rule, Simpson's Rule, Gaussian Quadrature, etc, etc. And their extension to higher dimensions.

If you are interested, read Section 7.6 of the textbook.