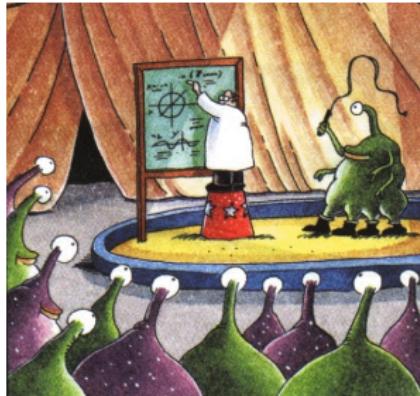


Week 08, Lecture 1 (REVISED) Techniques of Integration

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Assignments, etc

- ▶ **Problem Set 6** is open, and will be covered in tutorials this week. Deadline is 5pm next Monday (10 November).
- ▶ **Problem Set 7** opens after this week.
- ▶ The final weekly assignment, will open next week.
- ▶ Reminder: The second **class test** takes place November 18.

This morning, I think we will think about...

- 1 Substitution
 - Definite Integrals
- 2 Rational functions
 - Partial Fractions
- 3 Integration by Parts
 - When to use
 - The Rule
 - Easy example
 - Why it works
 - Examples
 - Choosing u and dv
 - Repeated application
- 4 Definite Integrals
- 5 Exercises

See also Section 5.5 (Substitution) of **Calculus** by Strang & Herman:
[math.libretexts.org/Bookshelves/Calculus/Calculus_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)) Maybe also
Section 7.1 (Integration by Parts).

Substitution

Suppose we want to evaluate an integral of the form

$$\int e^{x^3+x^2} (3x^2 + 2x) \, dx.$$

At first, this looks tricky: there is nothing like this in our table of integrals.

However, there is something a little unusual about it: it features both the function $x^3 + x^2$, and its derivative $3x^2 + 2x$.

It turns out that such problems are quite common (at least in textbooks and on exams!). Moreover, there is a handy technique called **substitution** for evaluating them. In this case:

Substitution

Method of substitution

When integrating an *integrand* of the form $\int f(g(x))g'(x)dx$, set $u = g(x)$, and then use that

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Equivalently: $\int f(u) \frac{du}{dx} dx = \int f(u) du$. (For a proof of why this works, see Section 5.5 of the textbook).

After this substitution, our task is reduced to evaluating $\int f(u) du$ which, we hope, is easier.

Substitution

Example

Evaluate the integral $\int 3x^2 \sin(x^3) dx$.

Notice that $3x^2$ is the derivative of x^3 .

Let's try integration by substitution with $u = x^3$.

If $u = x^3$, then $\frac{du}{dx} = 3x^2$, so

$$du = \frac{du}{dx} dx = 3x^2 dx.$$

Thus,

$$\begin{aligned}\int \sin(x^3)3x^2 dx &= \int \sin(u) du \\&= -\cos(u) + C \\&= -\cos(x^3) + C.\end{aligned}$$

Substitution

Example

Evaluate $\int 2x\sqrt{1+x^2} dx$.

Notice that $2x$ is the derivative of $1+x^2$.

Let's try integration by substitution with $u = 1+x^2$:

Substitution

Example

Evaluate $\int \cos(4x - 7) dx$.

Idea: think of this as $\frac{1}{4} \int \cos(4x - 7)4 dx$.

Substitution

Example

Show that $\int \sin^3(x) \cos(x) dx = \frac{1}{4} \sin^4(x) + C.$

Substitution can be used with **definite integrals**. However, this may require a change to the limits of integration.

Substitution with Definite Integrals

Let $u = g(x)$, with g' continuous on $[a, b]$, and f continuous over the range of $u = g(x)$. Then,

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u) du.$$

This allows us to apply the FTC2, without having to invert the substitution.

Example

$$\text{Evaluate } I = \int_0^1 x^2(1 + 2x^3)^2 dx$$

Example

$$\text{Evaluate } I = \int_{-1}^0 xe^{x^2} dx$$

Recall: Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Before trying to find an antiderivative of a rational function

$$f(x) = \frac{p(x)}{q(x)}$$
:

Step 1: If $\deg(p(x)) \geq \deg(q(x))$, divide $p(x)$ by $q(x)$.

Step 2: Check if integration by substitution might work.

Step 3: Factorise the denominator as far as possible.

Step 4: Write the rational function as sum of **partial fractions** to simplify.

Example

Evaluate the integral

$$\int \frac{x}{x^2 + 1} dx.$$

In this case we can use substitution.

Example

Evaluate the integral

$$\int \frac{3x + 4}{x^2 + 7x + 12} dx.$$

In this case, we must factorise the denominator, and express the integrand as **partial fractions**. Factorise:

$$x^2 + 7x + 12 = (x + 4)(x + 3).$$

Express as Partial Fractions:

$$\frac{3x + 4}{x^2 + 7x + 12} = \frac{A}{x + 4} + \frac{B}{x + 3}$$

With a little work, we can find that $A = 8$ and $B = -5$. Therefore,

$$\frac{3x + 4}{x^2 + 7x + 12} = \frac{8}{x + 4} - \frac{5}{x + 3}.$$

We can now express the integral as

$$\begin{aligned}\int \frac{3x+4}{x^2+7x+12} dx &= \int \left(\frac{8}{x+4} - \frac{5}{x+3} \right) dx \\ &= \underbrace{\int \frac{8}{x+4} dx}_{I_1} - \underbrace{\int \frac{5}{x+3} dx}_{I_2}.\end{aligned}$$

First, we evaluate I_1 .

$$I_1 = \int \frac{8}{x+4} dx = 8 \int \frac{1}{x+4} dx.$$

If we let $u = x + 4$, then $du = dx$ and, hence,

$$I_1 = 8 \int \frac{1}{x+4} dx = 8 \int \frac{1}{u} du = 8 \ln|u| + C_1 = 8 \ln|x+4| + C_1.$$

Similarly, we find that: $I_2 = \int \frac{5}{x+3} dx = 5 \ln|x+3| + C_2$.

To conclude:

$$\begin{aligned}\int \frac{3x+4}{x^2+7x+12} dx &= \int \left(\frac{8}{x+4} - \frac{5}{x+3} \right) dx \\&= \int \frac{8}{x+4} dx - \int \frac{5}{x+3} dx \\&= I_1 - I_2 \\&= (8 \ln|x+4| + C_1) - (5 \ln|x+3| + C_2) \\&= 8 \ln|x+4| - 5 \ln|x+3| + C.\end{aligned}$$

We use **substitution** when integrating the product of two functions, $f(x)g'(x)$, where $f(x) = f(g(x))$, which is a very specialised situation.

Integration by parts is also applied when we are integrating the **product** of a function and a derivative $f(x) = u(x)v'(x)$, but with no other constraint.

Integration by Parts (IBP) is, by some distance, the most important technique for integration, in both theory and practice.

Integration by Parts

If u and v are differentiable functions in the variable x , then

$$\int u(x) dv = u(x)v(x) - \int v(x) du.$$

It can also be written as

$$\int uv' dx = uv - \int vu' dx.$$

The link between the two formulations is given by

$$dv = v' dx \quad \text{and} \quad du = u' dx.$$

It is good to check any new rule/method for a simple example we already know the answer to.

Example

Evaluate $I = \int x^3 dx$

We know that $I = \frac{1}{4}x^4$. But we can also take $u(x) = x^2$ and $dv = xdx$:

The Product Rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

We can use this rule to develop another integration technique after a little rearrangement. From the above we have

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

and integrating both sides gives

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

Example

Evaluate $I = \int x \cos(x) dx$

We'll take $u = x$ and $dv = \cos(x)dx$.

One of the challenges of Integration by Parts is knowing how to choose u and dv . When integrating $\int x \cos(x) dx$ we choose $u = x$, because its derivative, $u' = 1$ is simpler.

Suppose we had made the bad choice of

$$u(x) = \cos(x), \quad dv = x dx,$$

then we'd get:

More generally, given choices for u and dv , we proceed as follows:

1. Some functions are easy to differentiate (and maybe not so easy to integrate), and so make a good choice for u .
Important examples include **logarithms** and **inverse trigonometric** functions.
2. Some functions (such as polynomials) have simple(r) derivatives, so are also a good choice for u .
3. Trigonometric and exponential functions don't simplify if differentiated, but can be integrated. So they can be a good choice for dv .

Example (of choosing u)

Evaluate $I = \int \frac{\ln(x)}{x^2} dx.$

Example

Evaluate $I = \int \ln(x) dx$.

Since $\int \ln(x) dx$ can be written as $\int \ln(x) \cdot 1 dx$, we use integration by parts, with $u = \ln(x)$ and $dv = dx$.

Sometimes, we have to apply Integration by Parts more than once.

Example

Evaluate $I = \int x^2 e^x dx$.

Example of repeated IbP

Evaluate $I = \int e^x \cos(x) dx.$

Definite Integrals

Integration by Parts for Definite Integrals

$$\int_a^b u dv = (uv) \Big|_a^b - \int_a^b v du$$

Example: Use Integration By Parts to evaluate $\int_0^1 xe^{-x} dx$.

Exercises

Exer 8.1.1

Evaluate the follow integrals

$$1. \int \sin(\ln x) \frac{1}{x} dx.$$

$$2. \int x^2(x^3 + 5)^9 dx$$

$$3. \int \frac{\sin(x)}{\cos^3(x)} dx$$

Exer 8.1.2

Evaluate $\int_0^1 x^2(x^3 + 5)^9 dx$

Exercises

Exer 8.1.3

Evaluate the follow integrals

$$1. \int xe^{2x} dx.$$

$$2. \int x^2 \cos(x) dx.$$

Exer 8.1.4

Evaluate $\int_1^e \ln(x^2) dx.$