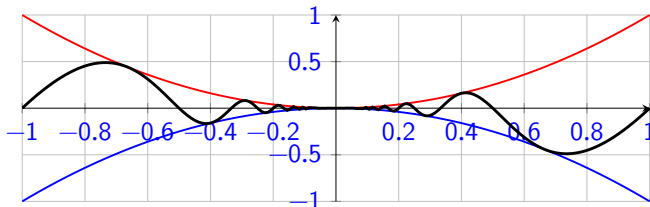


# Limits; The Squeeze Theorem

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# Outline

- 1 Recall... Limits
- 2 Properties of Limits
- 3 Evaluating limits
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- 6 The Squeeze Theorem
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For more, see Chapter 2 (Limits) of Strang and Herman's **Calculus**, especially Section and 2.3 (Limit Laws).

Slides are on canvas, and at [niallmadden.ie/2526-MA140](http://niallmadden.ie/2526-MA140)



## Recall... Limits

Yesterday, we learned that

$$\lim_{x \rightarrow a} f(x) = L,$$

means that we can make  $f(x)$  as close to  $L$  as we like, by taking  $x$  as close to  $a$  as needed.

# Properties of Limits

We finish with the following “Limit Laws”: *Suppose that*

$$\lim_{x \rightarrow a} f_1(x) = L_1 \quad \text{and} \quad \lim_{x \rightarrow a} f_2(x) = L_2,$$

*and*  $c \in \mathbb{R}$  *is any constant. Then,*

$$(1) \quad \lim_{x \rightarrow a} c = c, \quad c \in \mathbb{R}$$

$$(2) \quad \lim_{x \rightarrow a} x = a$$

$$(3) \quad \lim_{x \rightarrow a} [c f_1(x)] = c L_1$$

$$(4) \quad \lim_{x \rightarrow a} [f_1(x) + f_2(x)] = L_1 + L_2$$

and

$$\lim_{x \rightarrow a} [f_1(x) - f_2(x)] = L_1 - L_2$$

$$(5) \quad \lim_{x \rightarrow a} (f_1(x) f_2(x)) = L_1 L_2$$

$$(6) \quad \lim_{x \rightarrow a} ((f_1(x))^n) = (L_1)^n$$

$$(7) \quad \lim_{x \rightarrow a} \left( \frac{f_1(x)}{f_2(x)} \right) = \frac{L_1}{L_2},$$

providing  $L_2 \neq 0$ .

$$(8) \quad \lim_{x \rightarrow a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$$

# Evaluating limits

**Note:** we can combine these properties as needed. For example, (5) and (8) together give that

$$\lim_{x \rightarrow a} x^n = a^n$$

## Example

Evaluate the limit  $\lim_{x \rightarrow 1} (x^3 + 4x^2 - 3)$

# Evaluating limits

## Example

Evaluate  $\lim_{x \rightarrow 1} \frac{x^4 + x^2 - 1}{x^2 + 5}$  using the Properties of Limits.

# Limits of rational functions

**In many cases, evaluating limits is more complicated.** In particular, we'll consider numerous examples where we want to evaluate  $\lim_{x \rightarrow a} f(x)$  where  $a$  is not in the domain of  $f$ .

A typical example of this is when we evaluate a rational function:

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$$

where **both**  $p(a) = 0$  and  $q(a) = 0$ .

**Idea:** Since we care about the value of  $p$  and  $q$  **near**  $x = a$ , but not actually at  $x = a$ , it is safe to factor out an  $(x - a)$  term from both.

# Limits of rational functions

## Three examples

Evaluate the limits:

$$(a) \lim_{x \rightarrow 0} \frac{x}{x} \qquad (b) \lim_{x \rightarrow 0} \frac{x^2}{x} \qquad (c) \lim_{x \rightarrow 0} \frac{x}{x^2}$$



# Limits of rational functions

## Example

Evaluate the limit

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$$

# Limits of rational functions

In that last example, we found that

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x}$$

But these are different functions:

# Limits of rational functions

Evaluate the limit

$$\lim_{x \rightarrow 2} \left( \frac{\frac{1}{2} - \frac{1}{x}}{x - 2} \right)$$

# Completing the square

Very often, we'll evaluate limits of the form:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where  $f$  and  $g$  are not polynomials. Some of the same ideas still apply.

## Example

Evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2}$$

# Completing the square

# The Squeeze Theorem

There are various approaches to evaluating limits, including...

## The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that we have three functions  $f$ ,  $g$  and  $h$  on some interval  $[x_0, x_1]$ , with

$$g(x) \leq f(x) \leq h(x),$$

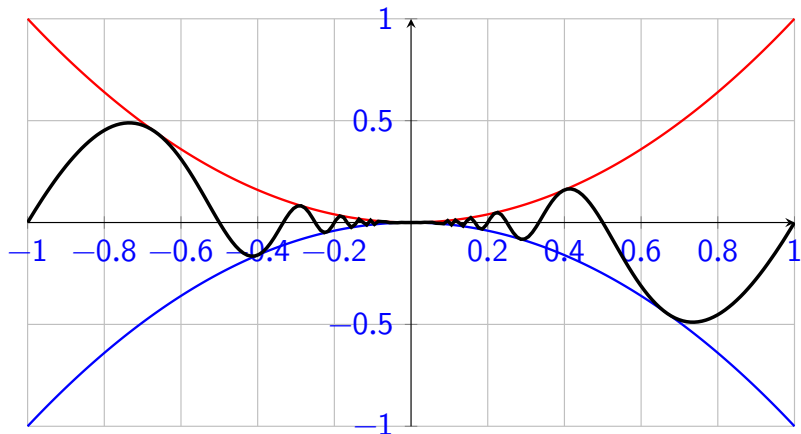
and

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L,$$

for some  $a \in [x_0, x_1]$ . Then  $\lim_{x \rightarrow a} f(x) = L$ .

That is: if  $g(x)$  and  $h(x)$  have the same limit as  $x \rightarrow a$ , and  $f(x)$  is “squeezed” between them, then  $f(x)$  has that same limit too as  $x \rightarrow a$ .

# The Squeeze Theorem



# The Squeeze Theorem

## Example

Suppose  $f(x)$  is a function such that

$$1 - \frac{x^2}{4} \leq f(x) \leq 1 + \frac{x^2}{2}, \quad \forall x \neq 0$$

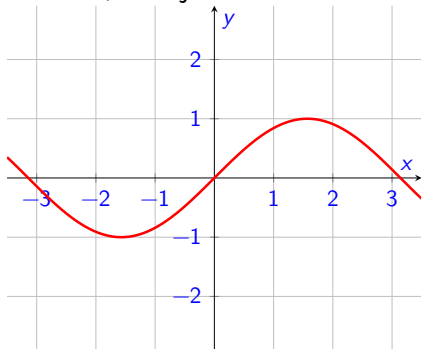
Find  $\lim_{x \rightarrow 0} f(x)$ .



Next week, we will use the Squeeze Theorem to explain **an important limit**:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

For now, let's just convince ourselves:



## Exercises

### Exercise 2.3.1

Evaluate the following limits

$$(a) \lim_{x \rightarrow \frac{1}{2}} \frac{x - \frac{1}{2}}{x^2 - \frac{1}{4}}$$

$$(b) \lim_{x \rightarrow -4} \frac{x^2 + 3x - 4}{x^2 + x - 12}$$

### Exercise 2.3.2

(From 2023/2024 MA140 exam, Q1(a)) Evaluate the limit

$$\lim_{x \rightarrow 4} \frac{x - 4}{(\sqrt{x} - 2)(x + 9)}$$

## Exercises

### Exercise 2.3.3

Suppose that  $g(x) = 9x^2 - 3x + 1/4$ , and  $f(x)$  is such that  $-g(x) \leq f(x) \leq g(x)$  for all  $x$ .

1. Can one use the Squeeze Theorem to determine  $\lim_{x \rightarrow 1/3} f(x)$ ? If so, do so. If not, explain why.
2. Can one use the Squeeze Theorem to determine  $\lim_{x \rightarrow 1/6} f(x)$ ? If so, do so. If not, explain why.

## Exercises

### Exercise 2.3.4 (from 2425-MA140 exam)

Let  $f(x) = \frac{x^2 - 2x - 15}{3x^3 - 6x^2 - 45x}$ . For each of the following, evaluate the limit, or determine that it does not exist.

(i)  $\lim_{x \rightarrow -3} f(x)$

(ii)  $\lim_{x \rightarrow 0} f(x)$

(iii)  $\lim_{x \rightarrow 5} f(x)$