2526-MA140 Engineering Calculus

Week 09, Lecture 2
Volumes and Arc Lengths

Dr Niall Madden

University of Galway

Wednesday, 12 November, 2025

Clochán (Beehive Hut), Dingle Peninsula, Co Kerry.

Reminders

1. **Problem Set 7** is due by 5pm, Monday 17 Nov. You can access it here: https://universityofgalway.instructure.com/courses/46734/assignments/132366
   You'll also find the tutorial sheet at that link.
2. **Problem Set 8** is open. Find it on canvas: https://universityofgalway.instructure.com/courses/46734/assignments

Talking about a (solid of) revolution:

See also: Section **6.2** (Determining Volumes by Slicing) and Section **6.4** (Arc Length of a Curve and Surface Area) in **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

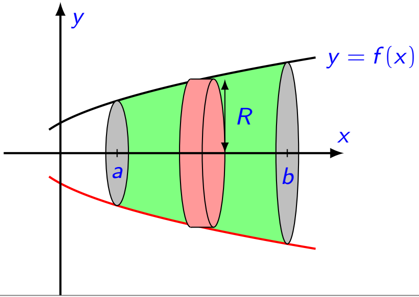
Recall "Solids of Revolution"

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**.

Often that region is bounded:

- above by $y = f(x)$, where y is some given nonnegative function;
- below by $y = 0$ (i.e., the x-axis)
- on the left by $x = a$, and on the right by $x = b$

The whole region is then **rotated about the x-axis**.



Recall "Solids of Revolution"

Since, every cross section of for a solids of revolution, is a disk with area $A(x) = \pi\big(f(x)\big)^2$, we can directly compute the volume, we get the following formula.

Solids of revolution: disk method

Let $f(x)$ be continuous and nonnegative. The volume of region formed by revolving the region between $f(x)$ and the x-axis, and between $x = a$ and $x = b$, about the x-axis is

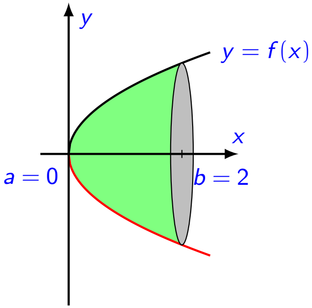
$$V = \pi \int_a^b \big(f(x)\big)^2 dx.$$

$$A(x) = \pi r^2 = \pi\,(f(x))^2$$

Recall "Solids of Revolution"

Example:

Find the **volume** of the solid of revolution obtained by rotating $y = \frac{3}{\sqrt{2}}\sqrt{x}$, between $x = 0$ and $x = 2$, about the x-axis.



$$f(x) = \frac{3}{\sqrt{2}} x^{1/2}$$

$$V = \pi \int_0^2 \left(f(x)\right)^2 dx$$

$$= \pi \int_0^2 \left[\frac{3}{\sqrt{2}} x^{1/2}\right]^2 dx$$

$$= \pi \int_0^2 \frac{9}{2} x \, dx$$

$$= \pi \left[\frac{9}{4} x^2\right]\Big|_0^2 = \pi\left(\frac{9}{4}(4) - 0\right) = 9\pi.$$

Solids of revolution: the "washer method"

There are numerous other variations on this type of problem, such as

- Rotating the function about the y-axis; (easy: just give a function for x in terms of y).
- Rotating about a line that is not an axis (a little trickier: need to transform the problem).
- **rotating a region bounded by two functions**.

We'll look at the last of these, the method for which is sometimes called the "**washer method**".

However, it is not too hard: we apply the "disk" method to both functions, and then subtract.

Washer Method

Let $f(x)$ and $g(x)$ be continuous functions on $[a, b]$, with $f(x) \geqslant g(x) \geqslant 0$ for any $x \in [a, b]$. The volume of the solid obtained by rotating the region between $f(x)$ and $g(x)$, and $x = a$ and $x = b$, is

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx.$$

Why? This is the difference of two volums

$$V_1 = \pi \int_a^b (f(x))^2 \, dx \quad [\text{Disk}]$$

$$V_2 = \pi \int_a^b (g(x))^2 \, dx \quad [\text{Hole}]$$

$$V = V_1 - V_2 = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx \neq \pi \int_a^b (f(x) - g(x))^2 \, dx$$

Solids of revolution: the "washer method"

Example (from textbook: see Figure 6.2.12)

Consider the region in the plane bounded above by $y = \sqrt{x}$, below by $y = 1$, left by $x = 1$ and right by $x = 4$. If this region is rotated about the x-axis, show that the volume of the resulting solid of rotation is $\dfrac{9\pi}{2}$.

First we visualise: **the animation**

$f(x) = x^{1/2} \qquad g(x) = 1$

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx = \pi \int_1^4 x - 1 \, dx$$

$$= \pi \left[\tfrac{1}{2} x^2 - x \right]_1^4 = \pi \left(\frac{16}{2} - 4 - \left(\tfrac{1}{2} - 1 \right) \right) = \frac{9}{2} \pi$$

Solids of revolution: the "washer method"

Example (finding a and b)

Find the volume of the solid obtained by rotating, around the x-axis, the region bounded by $f(x) = x$ and $g(x) = x^2$.

Note: for this example, we have to determine the values of a and b.

we have to find a and b : that is, where $f(x)$ & $g(x)$ intersect. So solve

$f(x) = g(x)$, ie $f(x) - g(x) = 0 \Rightarrow x - x^2 = 0$

$\Rightarrow x(1 - x) = 0$. This has two solutions: $x = 0$ and $x = 1$.

So take $a = 0$ and $b = 1$. <u>Note</u>: $a < b$.

Solids of revolution: the "washer method"

Now Evaluate

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx$$

$$= \pi \int_0^1 x^2 - x^4 \, dx$$

$$= \pi \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right] \Big|_0^1$$

$$= \pi \left[\tfrac{1}{3} - \tfrac{1}{5} - (0 - 0) \right] = \frac{2}{15} \pi.$$

Arc Length

Δ = "Delta" = (uppercase δ)

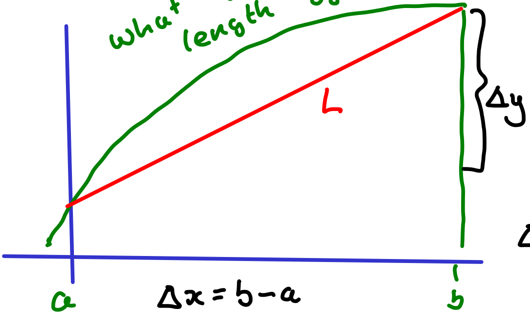
> "How long is a piece of string?"

In this section we will work out the **length of a curve**.

The method for doing this is a little surprising, since it involves both differentiation and integration.

However, it is easy, if the function is linear (so it's graph is a straight line):

what is the length of the curve?



The length of L is

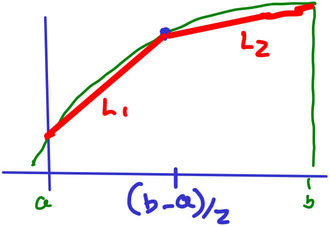
$$\sqrt{\Delta x^2 - \Delta y^2}$$

$\Delta x = b - a$

$\Delta y = f(b) - f(a)$

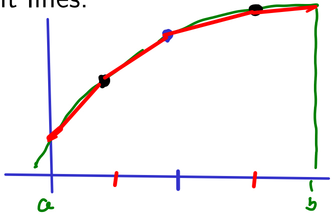
Arc Length

Given a curve, we can use this idea, to get an estimate for its length, by approximating it by a straight line:



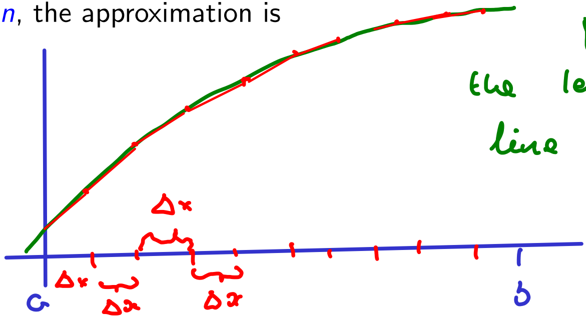
$L \approx L_1 + L_2.$

We can improve upon that by (say), approximating the curve by ~~two~~ 4 straight lines:



Arc Length

The more intervals we take, the better approximation we get. If we take n, the approximation is



L_k is the length of line k.

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

$$= \sum_{k=1}^{n} \sqrt{(\Delta x)^2 \left(1 + \left[\frac{\Delta y_k}{\Delta x}\right]^2\right)}.$$

Arc Length

Hence,

$$L \approx \sum_{k=1}^{n} \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x}\right]^2} \cdot \Delta x \,.$$

If we take an infinite number of intervals, then

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x}\right]^2} \cdot \Delta x \,.$$

Using the "limit" definition of the derivative, and the integral, we get that the **arc length**, L, of a curve of $y = f(x)$, from $x = a$ to $x = b$, is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} \, dx \,.$$

(If you need more convincing, read this later).

In the last step of the previous slide, there are several things going on.

- Since $\Delta x = \dfrac{b-a}{n}$, as $n \to \infty$, so $\Delta x \to 0$.
- Back in Week 4, Lecture 1, we defined

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Here, $\Delta x = h$, and $\Delta y = f(x_i + \Delta x) - f(x_i)$ where x_k is the start point of interval k. That is,

$$\lim_{n \to \infty} \frac{\Delta y_k}{\Delta x} = \lim_{h \to 0} \frac{f(x_k + h)}{h} = f'(x_k).$$

- In Week 7, Lecture 2, (Slide 19) we defined

$$\int_a^b g(x)\,dx = \lim_{n\to\infty} h \sum_{k=0}^{n-1} g(x_k).$$

Here, we let $g(x_k) = \sqrt{1+(f'(x_k))^2}$, our expression is

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (\Delta x) g(x_k) = \int_a^b g(x)\,dx$$

where again we are swapping the notation h and δx.

Arc length of a curve

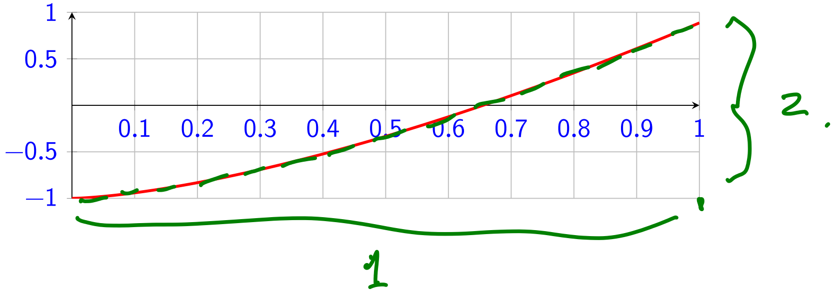
If $f(x)$ is a differentiable function on the interval $[a, b]$, then the **arc length**, L, of the graph of $f(x)$, from $x = a$ to $x = b$, is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,.$$

Example

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1, \quad 0 \leqslant x \leqslant 1\,.$$



$$f(x) = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$$f'(x) = \left(\frac{4\sqrt{2}}{3}\right)\frac{3}{2} x^{1/2} = 2\sqrt{2}\, x^{1/2}$$

$$[f'(x)]^2 = (2\sqrt{2}\, x^{1/2})^2 = 4 \cdot 2 \cdot x = 8x.$$

Then $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_0^1 \sqrt{1 + 8x}\,dx.$

Let $u = 1 + 8x$, $du = 8\,dx$. Also $u(0) = 1$ $u(1) = 9.$

So $L = \int_{u=1}^{u=9} u^{1/2} \left(\frac{1}{8}\right) du = \frac{1}{8}\frac{2}{3} u^{3/2}\Big|_1^9 = \frac{1}{2}\left(9^{3/2} - 1^{3/2}\right)$

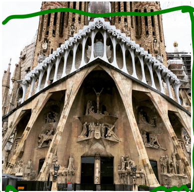
$= \cdots = 13/6.$

A example from Civil Engineering

Question! What do the following all have in common?



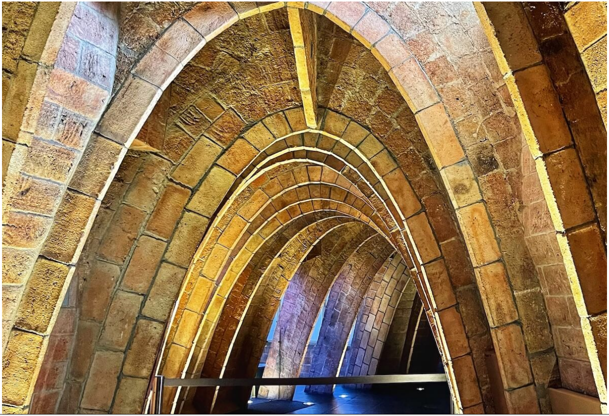
near Ventry





A example from Civil Engineering

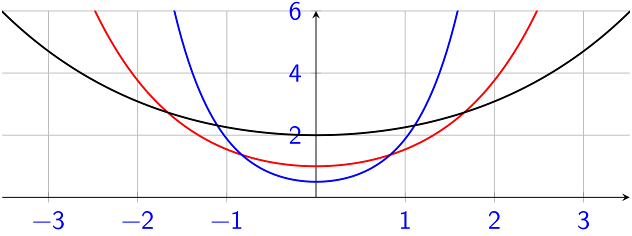
All the above are examples of **Catenary Arches**. This is the shape taken on by a free hanging chain. When used as an arch, it has an "optimal" shape in the sense that it can support its own weight. It has been known since ancient times. The example below is from Casa Milá in Barcelona.



A example from Civil Engineering

As a hanging chain, a catenary can be described by the function

$$f(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$



Finished Here,

A example from Civil Engineering

Aside: the function

$$f(x) = \frac{e^{x/a} + e^{-x/a}}{2}$$

is also known as the **hyperbolic cosine**, or cosh function. That is

$$\cosh(x) = \frac{e^{x/a} + e^{-x/a}}{2}.$$

You can read about it in the textbook (Section 1.5). But we don't need that for the following example.

A example from Civil Engineering

Example (Example from Civil Engineering)

Metal posts have been installed $4m$ apart across a gorge. Find the length for rope bridge that follows the curve

$$f(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right).$$

A example from Civil Engineering

A note of caution

Computing the arc length of a function involves evaluating integrals where the integrand is of the form $\sqrt{1+(g(x))^2}$. In some (rare) cases, this is easy. In others, it is possible to use a method called **trigonometric substitution**, which is not on our syllabus.

In the "real world" we actually use highly accurate numerical approximations. The details are beyond this course.
There is some interesting mathematics involved in determining how to take n large enough to ensure the error is small.

We'll mention this again briefly at the end of the semester.

Exercises

Exer 9.2.1

Use the “washer” method to find the volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = x^2$ and $g(x) = x$, for $1 \leqslant x \leqslant 2$, about the x-axis.

Exer 9.2.2

The volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = x^2$ and $g(x) = 1$, for $1 \leqslant x \leqslant b$, about the x-axis, is $4/3$. Find b. (Hint: b is an integer; this information will help you find b by inspection).

Exer 9.2.3

Find the volume of the solid of revolution formed by revolving the region between the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2$ about the x-axis. (Hint: you need to find where the graphs of f and g intersect: these will be the points a and b).

Exercises

Exer 9.2.4

What is the arc length of the graphs of $f(x) = \dfrac{1}{3}(x^2+2)^{3/2}$ from $x = 1$ to $x = 2$?