MA385: Assignment ANS with outline solutions (Due **5pm, Tuesday 28 October**)

Q1. Suppose that we have a fixed point iteration (FPI) method $x_{k+1} = g(x_k)$ which we know to be converges to fixed point of g, denoted τ. Show that the method converges with at least order p if

$$g'(\tau) = g''(\tau) = \dots g^{(p-1)} = 0$$

Answer: A method converges with at least order p if there is a constant $\mu \geqslant 0$ such that $\lim_{k\to\infty} \frac{|\tau - x_{k+1}|}{|\tau - x_k|^p} = \mu$. So, we need to find μ such that $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^p} \to \mu$. Let's write out a Taylor series for $g(x)$ about τ:

$$g(x) = g(\tau) + (x-\tau)g'(\tau) + \frac{1}{2}(x-\tau)^2 g''(\tau) + \dots + \frac{(x-\tau)^{p-1}}{(p-1)!} g^{(p-1)}(\tau) + \frac{(x-\tau)^p}{p!} g^{(p)}(\eta)$$

for some $\eta \in [x, \tau]$. Since $g'(\tau) = 0$, $g''(\tau) = 0$, ..., $g^{(p-1)} = 0$, this simplifies to

$$g(x) = g(\tau) + \frac{1}{p!}(x-\tau)^p g^{(p)}(\eta).$$

Now take $x = x_k$, so this becomes

$$\underbrace{g(x_k)}_{x_{k+1}} - \underbrace{g(\tau)}_{\tau} = \frac{(x_k - \tau)^p}{p!} g^{(p)}(\eta).$$

for some $\eta \in [x_k, \tau]$. Since $x_{i+1} = g(x_k)$, and $g(\tau) = \tau$, we now have

$$|x_{k+1} - \tau| = \frac{|x_k - \tau|^p}{p!} |g^{(p)}(\eta)|.$$

Rearranging:

$$\frac{|x_{k+1} - \tau|}{|x_k - \tau|^p} = \frac{|g^{(p)}(\eta)|}{p!}.$$

To finish, since we know the method converges, $x_k \to \tau$. So, since $\eta \in [x_k, \tau]$, it must be that $\eta \to \tau$, as $k \to \infty$. Thus $g^{(p)}(\eta_k) \to g^{(p)}(\tau)$, which is a constant.

$$\frac{|\tau - x_{k+1}|}{|\tau - x_k|^p} \to \mu \qquad \text{where } \mu = \frac{|g^{(p)}(\tau)|}{p!}.$$

Q2. Suppose we want to estimate $\sqrt{3}$, via FPI using a method of the form

$$x_{k+1} = a x_k + b/x_k, \qquad \text{for } k = 0, 1, 2, \dots. \tag{1}$$

Show that one needs to choose $b = 3 - 3a$.

Answer: What we need to do is show that $g(\sqrt{3}) = \sqrt{3}$ where $g(x) = ax - b/x$. In this case $g(x) = ax + (3-3a)/x$. Therefore,

$$g(\sqrt{3}) = a\sqrt{3} + \frac{3-3a}{\sqrt{3}} = a\sqrt{3} + \frac{3}{\sqrt{3}} - \frac{3a}{\sqrt{3}} = a\sqrt{3} + \sqrt{3} - \sqrt{3}a = \sqrt{3},$$

as required.

Q3. Take $a = 3/4$ in Equation (1).

(a) Show that the resulting g is a contraction on $[1, 3]$.

Answer: First we need that $1 \leqslant g(x) \leqslant 3$ for all $x \in [1, 3]$. Note that $g(1) = 3/2 > 1$ and $g(3) = 5/2 < 3$. Also, since $g'(x) = 3(1 - 1/x^2)/4 > 0$, this is a monotonically increasing function. So $1 \leqslant g(x) \leqslant 3$ for all $x \in [1, 3]$. Next, note that $1 - 1/x^2 \in (0, 1)$ so $|g'(x)| < 1$. Therefore it is a contraction.

(b) Take $x_0 = 1.5$, and compute the corresponding values of x_1, x_2 and x_3. Compute the errors $\mathcal{E}_k = |\tau - x_k|$ for $k = 0, 1, 2, 3$.

Answer:

$$x_1 = 1.625, \quad x_2 = 1.68029, \quad x_3 = 1.70657.$$
$$\mathcal{E}_0 = 0.23205, \quad \mathcal{E}_1 = 0.107051, \quad \mathcal{E}_2 = 0.051762, \quad \mathcal{E}_3 = 0.025483.$$

(c) Show that $\mathcal{E}_{k+1}/\mathcal{E}_k$ is roughly constant. What can we infer from that?

> **Answer:**
> $$\mathcal{E}_1/\mathcal{E}_0 = 0.4613249, \qquad \mathcal{E}_2/\mathcal{E}_1 = 0.4835306, \qquad \mathcal{E}_3/\mathcal{E}_2 = 0.4922986.$$
> This implies that there is a constant $\mu \in (0,1)$ such that $\mathcal{E}_{k+1}/\mathcal{E}_k \to \mu$. That is: it appears the methods converges almost linearly (i.e., order 1).

Q4. Determine the values of a and b in Eq (1) that would correspond to Newton's method applied to solving $x^2 - 3 = 0$. Show that this method converges with at least order 2.

> **Answer:** Taking $f(x) = x^2 - 3$, and $g(x) = x - f(x)/f'(x)$, we a bit of rearrangement we get $g(x) = x/2 + 3/(2x)$. That is $a = 1/2$ (and so $b = 3/2$).
> There are a few approaches to showing this converges with at least order 2. (Also, I should have made sense that you it should you could assume that it did converge). I'd accept if you checked that the conditions required by Theorem 1.5.2 were satisfied. I'd also accept a numerical demonstration. But what was really intended was for you to use the result in Q1 with $p = 2$. That is, show that $g'(\tau) = 0$. Here $g'(x) = (1 - 3/x^2)/2$. So $g'(\sqrt{3}) = (1 - 3/3)/2 = 0/2 = 0$.

Q5. Is it possible to determine values for a and b in Eq (1) for which the corresponding method converges with at least order 3 (using the result in Question 1)? Explain your answer.

> **Answer:** We'd need to be able to find values of a and b such that $g'(\tau) = g''(\tau) = 0$. However, $g''(x) = 2b/x^3$. There is no choice of b, other than $b = 0$ for which $g''(\tau) = 0$. And for $b = 0$, we don't have $g'(\tau) = 0$. So it is not possible.

Q6. When preparing this assignment, I asked a generative AI model to propose some fixed point methods for estimating $\sqrt{3}$. Its suggestions included taking

$$x_{k+1} = \frac{3}{x_k + 1} \qquad \text{and} \qquad x_{k+1} = x_k + \frac{3 - x_k^2}{2\sqrt{3}}.$$

Both of these are bad. Explain in one line (each) why this is so.

> **Answer:** For the first method, $g(\sqrt{3}) \neq g(\sqrt{3})$.
> For the second: to use this we already need to know $\sqrt{3}$. (Aside, if $x_k \approx \sqrt{3}$, we could use that instead – then the method becomes Newton's method).

Q7. In his seminal paper of 1901, Carl Runge gave the following example of what we now call a *Runge-Kutta 2 (RK2) method*:

$$\Phi(t_i, y_i; h) = \frac{1}{4} f(t_i, y_i) + \frac{3}{4} f\big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\big).$$

(i) Show that it is consistent.

> **Answer:** We want to show that $\Phi(i_i, y_i; 0) = f(t_i, y_i)$. Setting $h = 0$, we get
> $$\Phi(t_i, y_i; 0) = \frac{1}{4} f(t_i, y_i) + \frac{3}{4} f\big(t_i, y_i)\big) = f(t_i, y_i), + \frac{3}{4} f\big(t_i, y_i)\big),$$
> as required.

(ii) Show how this method fits into the general framework of RK2 methods. That is, what are a, b, α, and β? Do they satisfy the conditions

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b?$$

> **Answer:** The generic form is $k_1 = f(t_i, y_i)$, and $k_2 = f(t_i + \alpha h, y_i + \beta h k_1)$, and
> $$\Phi(t_i, y_i; h) = a k_1 + b k_2.$$
> So we can read off that $a = 1/4$, $b = 3/4$, $\alpha = 2/3$ and $\beta = 2/3$. Clearly $a + b = 1$, $\alpha = \beta$ and $1/(2b) = 1/(3/2) = 2/3$ we required.