Exercise 0.2 (* Homework Problem)

Write out the Taylor polynomial at x, about a = 0, of degree 5 for f(x) = sin(x).
How does its derivative compare to the corresponding Taylor polynomial for f(x) = cos(x)?

For any function, f, we can write its Taylor Polynomial of degree 5, about a, as

$$p_5(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(4)}(a) + \frac{(x-a)^5}{5!}f^{(5)}(a).$$

Here $a=0$, $f(a)=\sin(a)=0$, $f'(a)=\cos(0)=1$, $f''(a)=-\sin(0)=0$, $f'''(a)=-\cos(0)=-1$, $f^{(4)}(a)=\sin(0)=0$, $f^{(5)}(a)=\cos(0)=1$. So

$$\boxed{p_5(x) = x - x^3/6 + x^5/120.}$$

The Taylor Poly for $\cos(x)$ about $a=0$, of degree 5, is $1-\frac{x^2}{2}+\frac{x^4}{24}$. Since $p_5'(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$, in this case, the Taylor Poly of the derivative of $f(x)=\sin(x)$ is the same as the derivative of the Taylor Poly of $f(x)$.

Exercise 0.3

The Taylor Polynomial of degree 4 for $f(x) = \log(x)$, about $a=1$ is

$$p_4(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4.$$

The remainder is

$$R_4(x) = \frac{(x-1)^5}{5!} f^{(v)}(\eta) = \frac{(x-1)^5}{120}\cdot\frac{24}{\eta^5} = \frac{(x-1)^5}{5\cdot\eta^5}. \qquad \text{some } \eta \in (1,x)$$

Since the values of x of interest are greater than 1,

$$|R_4(x)| \le \frac{(x-1)^5}{5} \max_{1\le q\le x} \frac{1}{q^5} = \frac{(x-1)^5}{5}.$$

When $x=2$, $|R_4(2)| \le \frac{1}{5}$

When $x=1.1$, $|R_4(1.1)| \le \frac{(0.1)^5}{5} = 2\times 10^{-6}$

And when $x=0.01$, $|R_4(x)| \le \frac{0.01^5}{5} = 2\times 10^{-11}.$

Exer 1.1 No – it is not necessary that $f(a)f(b) \leq 0$ for there to be a solution to $f(x)=0$ in the interval $[a,b]$.

For example, suppose that $f(x) = x^2$, $a=-1$, and $b=1$.

Then there is a solution to $f(x)=0$, in $[-1,1]$, i.e., at $x=0$, even though $f(-1)f(1) = 1 \not\leq 0$.

Exer 1.2

We know that $|\tau - x_k| \leq \left(\frac{1}{2}\right)^{k-1} |b-a|$.

So, if $\left(\frac{1}{2}\right)^{k-1}|b-a| \leq \varepsilon$, then $|\tau - x_k| \leq \varepsilon$ too.

Therefore, we need k such that $\left(\frac{1}{2}\right)^{k-1} \leq \frac{\varepsilon}{|b-a|}$

If we express this as $2^{k-1} \geq |b-a|/\varepsilon$,

then we need $k-1 \geq \log_2\left(|b-a|/\varepsilon\right)$.

It follows that, if $K = \lceil \log_2\left(|b-a|/\varepsilon\right) \rceil + 1$

then $|\tau - x_k| \leq \varepsilon$ for all $k \geq K$.

This estimate is entirely independant of f (which was not the case with the Secant Method, or Newton's Method).

It does depend on a & b: if $|b-a|$ were doubled, one extra iteration would be required.

Exer 1.5

1. I'll choose $f(x) = x^3 - 4$, $a = 0$ and $b = 2$.
2. Since $f(0) = -4$, and $f(2) = 4$, and f is continuous, by the IVT, $\exists \tau \in [0,4]$ s.t. $f(\tau) = 0$
3. Using code from Lab 1

k	x(k)	|tau-x(k)|
1	0.000	1.587e+00
2	2.000	4.126e-01
3	1.000	5.874e-01
4	1.500	8.740e-02
5	1.750	1.626e-01
6	1.625	3.760e-02

4. $|\tau - x_6| \leq (\frac{1}{2})^5 (2-0) = 0.0625$

Exer 1.6

The equation is $y - f(x_k) = \left[\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right](x - x_k)$

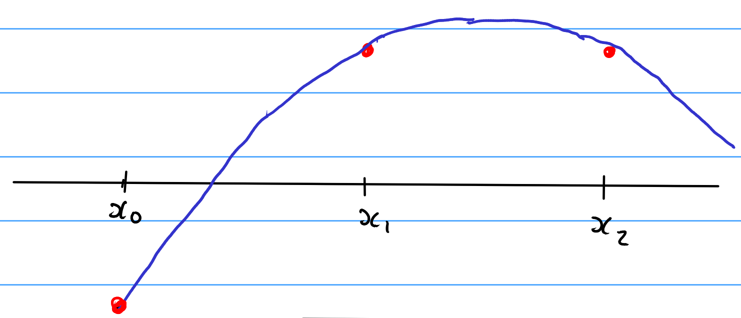
We take the zero of this line to be x_{k+1}. That is

$$0 - f(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} (x_{k+1} - x_k)$$

Rearrange to get $x_{k+1} = x_k - f(x_k) \cdot \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$.

Exer 1.7

(i) Yes. One way to do this is to choose a function f & interval $[a,b]$ such that f is continuous, $f(a)<0$, $f(b)>0$ but $f(b) = f(\frac{a+b}{2})$. Then bisection will work, but we'll find that $f(x_1) = f(x_2)$ so applying the Secant Method will lead to dividing by zero.



Eg try $f(x) = -x^2+3x-1$ with $a = x_0 = 0$, $b = x_1 = 1$.

Exer 1.7 (ii) This problem is really an exercise in articulating the assumptions that we are making regarding the function f. Further, there are two possible correct answers, providing you explain your reasoning correctly.

(a) If we assume that the functon f is continuous and defined on [a,b], and furthermore that f(a)f(b)≤0, then bisection must converge. So there is no such case where bisection will fail, but the secant succeed.

(b) Suppose that f(x)=0 has a solution in [a,b], but f(a)f(b)>0. In that case, bisection cannot even begin, but the secant method may work. For example, take

$$f(x) = x^2 - 2x + 1 \quad \text{on } [-1,2].$$

This has a (double) root at x=1, but f(-1)f(2)=(4)(1)>0. So we can't apply bisection. But one can verify that the Secant method will converge (though slowly).

Exer 1.8 The equation of the line through the point $(x_k, f(x_k))$, with slope $f'(x_k)$ is

$$y - f(x_k) = f'(x_k)(x - x_k).$$

Now take x_{k+1} to be the point where this line is zero. That is,

$$0 - f(x_k) = f'(x_k)(x_{k+1} - x_k).$$

Rearranging we get $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

Exercise 1.10 (⋆ Homework problem).

(i) Let q be your student ID number. Find k and m where k - 2 is the remainder on dividing q by 4, and m - 2 is the remainder on dividing q by 6.
(ii) Show how Newton's method can be applied to estimate the postive real number $\sqrt[k]{m}$
That is, state the nonlinear equation you would solve, and give the formula for Newton's method, simplified as much as possible.
(iii) Do three iterations by hand of Newton's Method for this problem, taking x0 = 1.

Solution. (i) Suppose my ID number is $q = 12345678$. The remainder on dividing q by 4 is 2, so $k=6$. The remaind on dividing q by 6 is 0. So $m=2$. So I want to estimate $2^{1/6}$.

(ii) I will solve $f(x) = x^6 - 2 = 0$. Newtons Method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - (x_k^6 - 2)/(6x_k^5) = \frac{5}{6}x_k + \frac{1}{3 \cdot x_k^5}$$

(iii) $x_1 = 1.16666$, $x_2 = 1.12644$, $x_3 = 1.122497$.

Exer 1.11

The Newton Error Formula is

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2}\frac{f''(\eta_k)}{f'(x_k)}.$$

In this case, $|f''(x)| \leq 10$ and $|f'(x)| \geq 2$, so $|\tau - x_{k+1}| \leq \frac{|\tau - x_k|^2}{2} \cdot \frac{10}{2}$.
In particular $|\tau - x_1| \leq \frac{5}{2}|\tau - x_0|^2$.
Since we need $|\tau - x_1| < |\tau - x_0|$, we will require that $|\tau - x_0| < \frac{2}{5}$ (notice the strict inequality). It will follow that $|\tau - x_{k+1}| < |\tau - x_k|$ for all k, and the method will converge.

Exer 1.12

(a) This method is sometimes called "the discrete Newton Method." First we write it as

$$x_{k+1} = x_k - f(x_k)\Big/\left[\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}\right]$$

Suppose it converges, then $\lim_{k\to\infty} x_k = \tau$ so $\lim_{k\to\infty} f(x_k) = 0$.
Recalling that

$$\lim_{\delta\to 0}\frac{f(x+\delta) - f(x)}{\delta} = f'(x),$$ we see that

$$\lim_{k\to\infty}\left[\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}\right] = f'(x_k).$$

So, as $k\to\infty$, the method converges to Newton's Method.

Exer 1.13 (i) f has a double root at τ.

(ii) We have that $\tau - x_{k+1} = -\frac{1}{2}(\tau - x_k)^2 \dfrac{f''(\eta_k)}{f'(x_k)}$ (Newton Error Formula)

Using the Mean Value Theorem, there is a point $\mu_k \in [x_k, \eta_k]$ such that

$$\frac{f'(\tau) - f'(x_k)}{\tau - x_k} = f''(\mu_k).$$

Since $f'(\tau) = 0$, we can substitute this into the Newton Error Formula to get

$$\tau - x_{k+1} = \tfrac{1}{2}(\tau - x_k)\cdot\frac{f''(\eta_k)}{f''(\mu_k)}$$

as required.

(iii) This formula tells us that Newton's method will converge more slowly in this case: the rate of convergence is (at least) linear, instead of quadratic.

Exercise 1.15 (⋆ Homework problem). Show that g(x) = ln(2x + 1) is a contraction on [1, 2]. Give an estimate for L. (Hint: Use the Mean Value Theorem).

Solution: The MVT states that, for any continuous, differentiable function, g, and points a and b, there is $c \in [a,b]$ such that

$$\frac{g(a) - g(b)}{a-b} = g'(c).$$

It follows that,

$$\frac{|g(a) - g(b)|}{|a-b|} = |g'(c)| \le \max_{a \le x \le b} |g'(x)|.$$

In this problem, $g(x) = \log(2x+1)$, so $g'(x) = \dfrac{2}{2x+1}$

Also, we have $a \in [1,2]$ and $b \in [1,2]$. So

$g'(x) \le \frac{2}{3}$ for all $x \in [1,2]$.

This gives $|g(a) - g(b)| \le L\,|a-b|$ where $L = 2/3$.

Exer 1.16 g is a contraction on $[a,b]$, if $|g(\alpha)-g(\beta)| < |\alpha-\beta|$ for all $\alpha,\beta \in [a,b]$

(i) $g_1 = x^2-1$ is not a contraction on $[3/2, 2]$ since, eg
$|g(2)-g(3/2)| = |4-1-9/4+1| = 5/4 > 1/2 = |2-3/2|$.

(ii) $g_2 = 1+1/x$ is a contraction. Since, for any α,β,
$$\frac{g(\alpha)-g(\beta)}{\alpha-\beta} = g'(x) \quad \text{some} \quad x \in [\alpha,\beta],$$
then $|g(\alpha)-g(\beta)| \le \max\limits_{\alpha\le x\le\beta} |g'(x)| \cdot |\alpha-\beta| \le \max\limits_{a\le x\le b} |g'(x)| \cdot |\alpha-\beta|$

Here $g'(x) = -x^{-2}$, so $\max\limits_{\frac{3}{2}\le x\le 2} |g'(x)| \le \frac{1}{(3/2)^2} = 4/9 < 1$, as required.

Exer 1.17

(i) Need to find points s.t.
$$\frac{x^2}{4} + \frac{5}{4}x - 1/2 = x \Rightarrow x^2 + x - 2 = 0$$
This has root $x=-2$ & $x=1$.

(ii) If $|g'(x)|<1$ on a region then, by the mean value thm, g is a contraction.
Here $g'(x) = \frac{x}{2} + 5/4$.
$$\frac{x}{2} + 5/4 > -1 \Rightarrow \frac{x}{2} > -9/4 \Rightarrow x > -9/2.$$
$$\frac{x}{2} + 5/4 < 1 \Rightarrow \frac{x}{2} < -1/4 \Rightarrow x < -1/2$$
So g is a contraction in the region $[-\frac{9}{2}, -\frac{1}{2}]$ about $x=-2$, and not at all about $x=1$.

1.18

(i) Write out the Taylor series for $g(x_{k+1})$ about $x=\tau$, truncating at the $\mathcal{O}(g^{(p)})$ term:

$$g(x_k) = g(\tau) + (x_k-\tau)g'(\tau) + \tfrac{1}{2}(x_k-\tau)^2 g''(\tau) + \dots$$
$$+ \frac{1}{(p-1)!}(x_k-\tau)^{p-1} g^{(p-1)}(\tau) + \frac{1}{p!}(x_k-\tau)^p g^{(p)}(\eta) \quad \text{some } \eta\in[x_k,\tau]$$

Using that $g(x_k) = x_{k+1}$, $g(\tau)=\tau$ and $g'(\tau)=g''(\tau)=\dots=g^{(p-1)}(\tau)=0$ we get

$$x_{k+1} = \tau + (x_k-\tau)^p \frac{g^{(p)}(\eta)}{p!}$$

So $$\frac{x_{k+1}-\tau}{(x_k-\tau)^p} = \frac{g^{(p)}(\eta)}{p!}$$

We are told that the method converges, so $\lim_{k\to\infty} x_k = \tau$.
Since $\eta\in[x_k,\tau]$, it follows that $\lim_{k\to\infty}\eta=\tau$.

Thus $\lim_{k\to\infty}\frac{x_k-\tau}{(x_k-\tau)^p} = \mu$, where μ is the constant $\frac{g^{(p)}(\tau)}{p!}$

(ii) If we write Newton's Method as $x_{k+1}=g(x_k)$ with $g(x) = x - f(x)/f'(x)$, to show that this is of order 2, we need (a) $g(\tau)=\tau$ (b) $g'(\tau)=0$.

For (a), use that $f(\tau)=0$, to get $g(\tau) = \tau - 0/f'(\tau) = \tau$ since $f'(\tau)\neq 0$.

For (b), differentiate g to get $$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$
$$= \frac{f(x)f''(x)}{(f'(x))^2}$$

Now use $f(\tau)=0$ to get $g'(\tau)=0$.

Therefore, we can apply Part (i) with $p=2$ to get the desired result.