2.0 Started 10am, 12/1/24

MA378 Chapter 1: Interpolation

§1.2 Lagrange Interpolation

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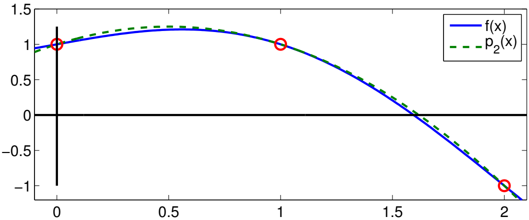
Source: http://jeff560.tripod.com/stamps.html

Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics.

2.1 Finding the polynomial

Example 2.1

Show that the polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ is $p_2 = -x^2 + x + 1$.



First note that $p_2(x) \in \mathbb{P}_2$

Next: and

$$\left.\begin{array}{l} f(0) = 1 - 0 - \sin\left(\frac{\pi 0}{2}\right) = 1 \\ p_2(0) = -0^2 + 0 + 1 = 1 \end{array}\right\} p_2(0) = f(0).$$

Similarly, (check!) $p_2(1) = f(1)$, $p_2(2) = f(2)$.

How to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

As a first step, we'll prove the following:

Theorem 2.2

If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all x).

We know that $p_n(x_0) = p_n(x_1) = \cdots = p_n(x_n) = 0$.
So we can write

$$p_n(x) = q(x)\left[(x-x_0)(x-x_1)\cdots(x-x_n)\right]$$

where q is some polynomial. Then q is the coeficient of x^{n+1}! Since $p_n \in P_n$, we have $q \equiv 0$. So $p_n(x) \equiv 0$.

Theorem 2.3 (There is a unique solution to the PIP)

There is at most one polynomial of degree $\leq n$ that interpolates the $n+1$ points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) where x_0, x_1, ..., x_n are distinct.

Proof. Suppose both $q(x)$ and $r(x)$ are polynomials of degree n that solve this interpolation problem. That is

$q(x_0) = y_0 \quad q(x_1) = y_1 \quad \dots \quad q(x_n) = y_n$

$r(x_0) = y_0 \quad r(x_2) = y_1 \quad \quad r(x_n) = y_n$

Let $s(x) = q(x) - r(x)$. So $s(x) \in P_n$

And $s(x_0) = s(x_1) = \dots = s(x_n) = 0$. So s has $n+1$ zero. Thus $s(x) \equiv 0$ and $q(x) = r(x)$.

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2.2 The Vandermonde matrix method

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find p_2 such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1 x + a_2 x^2$, this just amounts the finding the values of the coefficients a_0, a_1, and a_2. One might be tempted to solve for them using the system of equations

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

Solve for a_0, a_1, a_2

This is known as the *Vandermonde System*.

2.2 The Vandermonde matrix method

Writing

$$
\begin{aligned}
a_0 + a_1x_0 + a_2x_0^2 &= y_0\\
a_0 + a_1x_1 + a_2x_1^2 &= y_1\\
a_0 + a_1x_2 + a_2x_2^2 &= y_2
\end{aligned}
$$

in matrix-vector format we get

$$
\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}
=
\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}
\qquad \text{or} \qquad V\mathrm{a} = \mathrm{y}. \tag{1}
$$

But this is usually not be a good idea. At the very least, we'd have to solve a linear system of equations. Furthermore, the system is very *ill-conditioned*.

2.2 The Vandermonde matrix method

[*Slides 7 and 8 depend on material from MA385, so we'll skip them in class: please read in your own time.*]

In MA385 you learned about the relationship between the *condition number* of a matrix, V, and the relative error in the (numerical) solution to a matrix-vector equation with V as the coefficient matrix. The condition number is $\kappa(V) = \|V\| \|V^{-1}\|$, for some subordinate matrix norm $\| \cdot \|$.

Example 2.4 (Stewart's "Afternotes...", Lecture 19)

Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

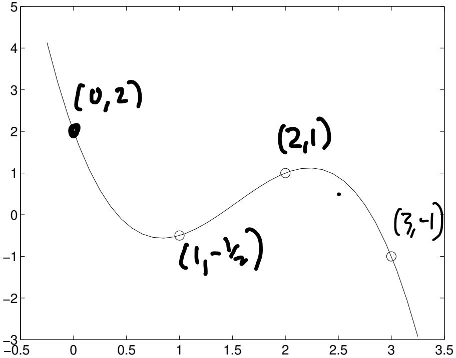
2.3 Lagrange Interpolation

We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (A "constructive proof" is one that shows a thing exists by by actually computing it).

Example

Consider the problem: *find* $p_3 \in \mathcal{P}_3$ *such that*

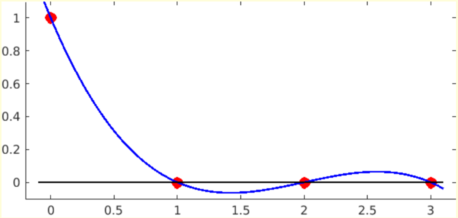
$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$



Here is an easier problem to solve: Find $L_0 \in \mathcal{P}_3$ such that

$$L_0(0) = 1, \;\; L_0(1) = 0,$$

$$L_0(2) = 0, \;\; L_0(3) = 0.$$



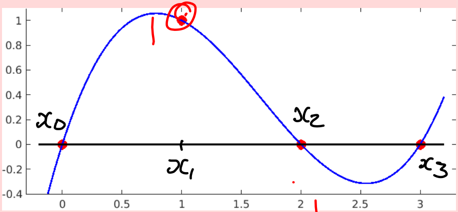
Because L_0 is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$.
Choosing C so that $L_0(0) = 1$, we get

$$L_0(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{-1}{6}(x-1)(x-2)(x-3)$$

2.3 Lagrange Interpolation

Similarly, find $L_1 \in \mathcal{P}_3$ such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

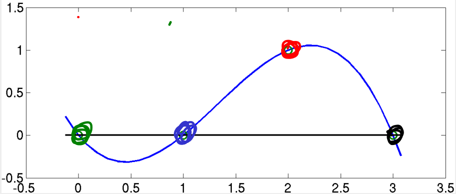


Then

$$L_1(x) = \frac{(x-0)(x-2)(x-3)}{\underset{(1)}{(1-0)}\ \underset{(-1)}{(1-2)}\ \underset{(-2)}{(1-3)}}$$

$$= \frac{1}{2}(x)(x-2)(x-3)$$

In the same style, let $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



$$L_2(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}$$

$$= -\frac{1}{2}(x)(x-1)(x-3)$$

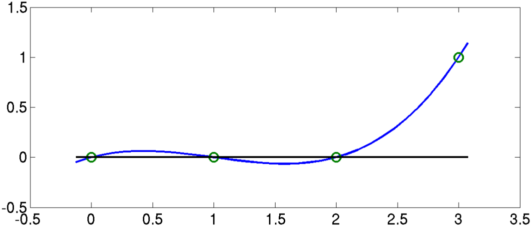
Finally, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

where $n = 3$

$$L_3(x) = \frac{(x-0)(x-1)(x-3)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j\neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$



Because each of L_0, L_1, L_2, and L_3 is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (\frac{1}{2})L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. ~~Furthermore...~~ Becase each of L_0, L_1, L_2 and L_3 are cubics.

$$\begin{array}{rcllll}
p_3(0) & = & 2L_0(0) & -(1/2)L_1(0) & +(1)L_2(0) & +(-1)L_3(0) \\
& = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
& = & 2, & & & \\
p_3(1) & = & 2L_0(1) & -(1/2)L_1(1) & +(1)L_2(1) & +(-1)L_3(1) \\
& = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
& = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
& = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
& = & 1, & & & \\
p_3(3) & = & 2L_0(3) & -(1/2)L_1(3) & +(1)L_2(3) & +(-1)L_3(3) \\
& = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
& = & -1. & & &
\end{array}$$

Thus p_3 solves the problem! That is p_3 is a cubic that interpolates $(0,2)$, $(1,-\frac{1}{2})$, $(2,1)$ and $(3,-1)$.

2.4 The Lagrange Form

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation.

Definition 2.5 (Lagrange Polynomials)

The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2b}$$

Definition 2.6

The **Lagrange form of the Interpolating Polynomial**

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3a}$$

or

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{3b}$$

Take care not to confuse

- the *Lagrange Polynomials*, which are the L_i with
- the *Lagrange Interpolating Polynomial*, which is the p_n defined in (3).

Theorem 2.7 (Lagrange)

There exists a solution to the Polynomial Interpolation Problem and it is given by

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

Proof: Since each L_i is a polynomial of degree n, so too is $p_n(x)$. Furthermore

$$p_n(x_j) = \sum_{i=0}^{n} y_i L_i(x_j) = y_j L_j(x_j)$$

since $L_i(x_j) = 0$ if $i \neq j$

$$= y_j$$

since $L_j(x_j) = 1$

So p_n solves the PIP. Consequently, it exists!

Example 2.8 (Süli and Mayers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

Here $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

We want $p_2(x_0) = e^{-1}$, $p_2(x_1) = e^0 = 1$, $p_2(x_2) = e$

Write down each of L_0, L_1, L_2

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x)(x-1)}{(-1)(-2)} = \frac{1}{2}(x)(x-1)$$

$L_1(x) = 1 - x^2$ $\qquad$ $L_2(x) = \frac{1}{2}(x)(x+1)$.

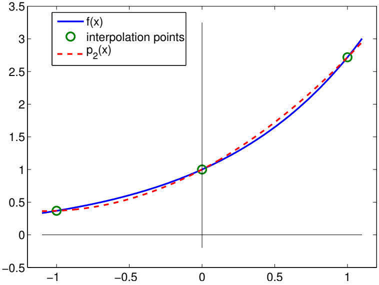
So $p_2(x) = e^{-1} L_0(x) + L_1(x) + e L_2(x)$ [Exer: write this out...]

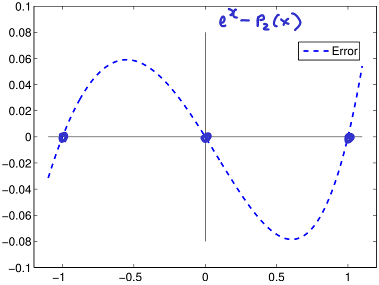
2.5 Example Finished here 17/02/2024

The figure below shows the solution to Example 2.8 (top) and the difference between the function e^x and its interpolant (bottom). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.





Exercise 2.1

The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{0 \le i < j \le n} (x_j - x_i). \tag{4}$$

Verify this for the 2×2 and 3×3 cases.

2.6 Exercises

(Note that from Formula (4) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \dots, x_n$ are all distinct.)

Exercise 2.2

Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute

$$\max_{0 \leq x \leq a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayers.

Exercise 2.3

Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \quad \text{for all } x.$$

Exercise 2.4

Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0, 3)$, $(1, 2)$ and $(2, 4)$.
Source: Chapter 2 of Stoer and Bulirsch.

Exercise 2.5

Show that all the following represent the same polynomial (usually called the "Chebyshev Polynomial of Degree 3"),
$T_3(x) = 4x^3 - 3x$.

(a) Horner form: $\big((4x+0)x-3\big)x+0$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3}\Bigg(\prod_{j=0,j\neq k}^{3}\frac{x-x_j}{x_k-x_j}\Bigg)(-1)^{k+1}$, where
$x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$.

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and
$T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

(d) Trigonometric form: $T_3(x) = \cos\big(3\cos^{-1}(x)\big)$.