2425-MA140 Engineering Calculus

Week 2, Lectures 2 and 3
Limits

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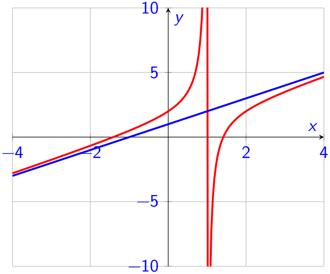

This version of the slides are by Niall Madden, but are adapted from original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Section 7.8.1 (Limit of a function of a real number) in *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Tutorials start **this** week. The schedule is:

- Teams 1+2: Tuesday 15:00 ENG-**2003**
- Teams 3+4: Tuesday 15:00 ENG-**2034**
- Teams 9+10: Thursday 11:00 ENG-**2002**
- Teams 11+12: Thursday 11:00 ENG-**3035**
- Teams 5+6: Friday 13:00 Eng-**2002**
- Teams 7+8: Friday 13:00 Eng-**2035**

If you are interested to taking a tutorial through Irish, please complete this survey: http://tinyurl.com/suirbhe1

- There is currently a "practice" assignment open. See https://universityofgalway.instructure.com/courses/35693/assignments/94873
- A new assignment will open...

During tutorials, the tutor will solve some similar questions. You can access the tutorial sheet at https://universityofgalway.instructure.com/courses/35693/files/2023552?module_item_id=650912

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In addition, each week I'll post a set of exercises related to the material covered. You don't have to submit your work for these, but you should try them: they are similar in style and standard to exam questions.

Limits

When we were considering the domain of a function, we looked at those x-values for which the function was not defined.

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad\qquad g(x) = \frac{x^2 - 1}{x - 1}$$

Neither f nor g are defined at $x = 1$.

But what happens if x gets very closed to 1?

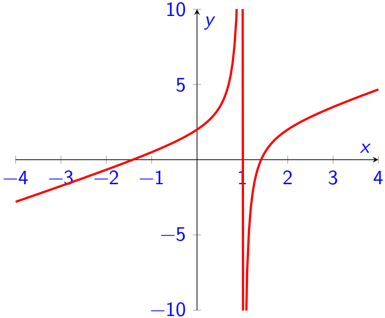
x	0.900	0.990	0.999	1	1.001	1.010	1.100
$f(x)$							
$g(x)$							

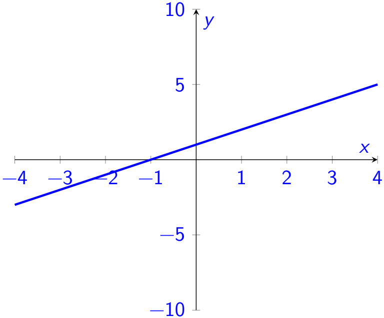
Let's look at the graphs of f and g.

Limits

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1}$$





Limits

In the previous example, we saw that, although neither f nor g was defined at $x = 1$, they behaved very differently as x approaches 1. To discuss this we need some terminology to help us articulate what it means to be really, really close to value, but not actually at x. We'll also need to be able to discuss what happens for very large or very small x-values.

To do that, we introduce the limit L of a function as x approaches some value $a \in \mathbb{R}$ and denote it by

$$\lim_{x \to a} f(x) = L$$

Note: The concept of a limit is a prerequisite for a proper understanding of calculus and numerical methods.

Definition of a Limit

Some conventions and terminology we'll use:

- x is a variable.
- a is a fixed number.
- ϵ is a small positive number (that we get to choose).
- δ is another small positive number (determined by ϵ).
- $|x - a| < \delta$ means that the distance between x and a is less than δ, i.e. very small.
- As x approaches a, so $f(x)$ approaches a number L.

When we write

$$\lim_{x \to a} f(x) = L,$$

we read

"The limit of f, as x goes to a, is L".

Definition of a Limit

LIMIT: formal definition

$$\lim_{x \to a} f(x) = L,$$

means that, for every number $\epsilon > 0$, it is possible to find a number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

LIMIT: Informal

$$\lim_{x \to a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

Example

Prove formally that $\lim_{x \to 3} (4x - 5) = 7.$

That is, for arbitrary ϵ, find a δ such that

$$|(4x - 5) - 7| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta.$$

Definition of a Limit

The approach we just used is technically correct, but not very practical in many cases.

Fortunately, there are other methods that can be used

- to show that a limit exists;
- find the limit of a function, $f(x)$ as $x \to a$.

Properties of Limits

Suppose that $\lim_{x \to a} f_1(x) = L_1$, and $\lim_{x \to a} f_2(x) = L_2$ and $c \in \mathbb{R}$ is any constant. Then,

(1) $\lim_{x \to a} c = c, \ c \in \mathbb{R}$

(2) $\lim_{x \to a} x = a$

(3) $\lim_{x \to a} [c f_1(x)] = cL_1$

Properties of Limits

(4) $\lim\limits_{x \to a} [f_1(x) + f_2(x)] = L_1 + L_2$ and
$\lim\limits_{x \to a} [f_1(x) - f_2(x)] = L_1 - L_2$

(5) $\lim\limits_{x \to a} \big(f_1(x) f_2(x)\big) = L_1 L_2$

(6) $\lim\limits_{x \to a} \big((f_1(x))^n\big) = (L_1)^n$

Properties of Limits

(7) $\lim_{x \to a} \left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$, providing $L_2 \neq 0$.

(8) $\lim_{x \to a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\boxed{\lim_{x \to a} x^n = a^n}$$

Example

Evaluate the limit $\lim_{x \to 1} (x^3 + 4x^2 - 3)$

Example

Evaluate $\lim_{x \to 1} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

Limits of rational functions

In many cases it's more complicated. In particular, we'll consider numerous examples where we want to evaluate $\lim_{x \to a} f(x)$ where a is not in the domain of f.

A typical example of this is when we evaluate a rational function:

$$\boxed{\lim_{x \to a} \frac{p(x)}{q(x)}}$$

Example

Evaluate Consider

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

Limits of rational functions

In that last example, we found that

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x}$$

But these are different functions:

Limits of rational functions

Evaluate the limit

$$\lim_{x\to 2}\left(\left(\frac{1}{2}-\frac{1}{x}\right)\left(\frac{1}{x-2}\right)\right)$$

Limits of rational functions

Example

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x^2}$$

There are various approaches to evaluating limits. One significant one is...

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that for functions f, g and h in a given interval I:

$$g(x) \leqslant f(x) \leqslant h(x) \quad \underline{\text{and}} \quad \lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\boxed{\lim_{x\to c} f(x) = L.}$

The Squeeze Theorem

Example

Suppose $f(x)$ is a function such that

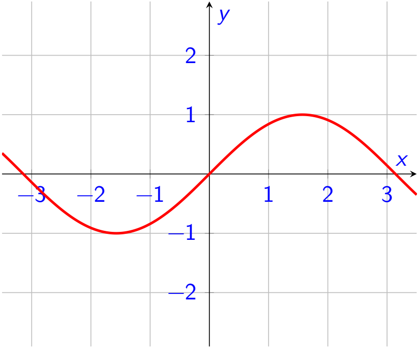
$$1 - \frac{x^2}{4} \leqslant f(x) \leqslant 1 + \frac{x^2}{2},\ \forall x \neq 0$$

Find $\lim_{x \to 0} f(x)$.

We use the Sandwich Theorem to explain an important limit:

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

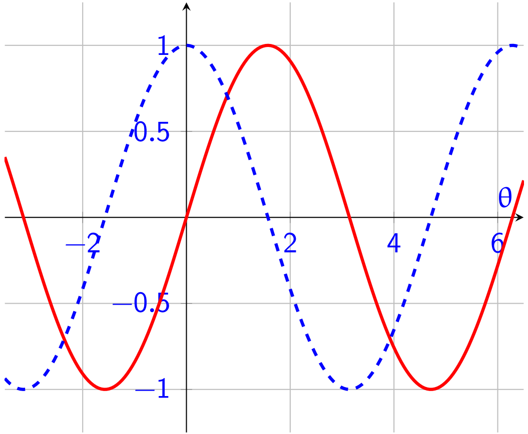
Before we show this is true, let's convince ourselves:



Before we ues the Squeeze Theorem, we need a few facts about trigonometric functions.

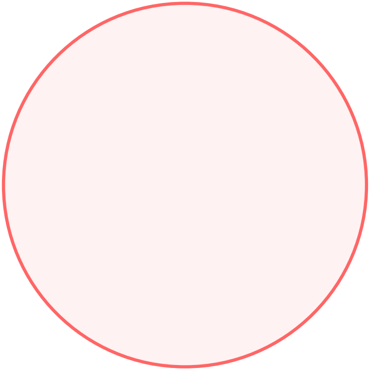
- **In this module, we only every use radians** (never, ever degrees).
- Given a triangle drawn below, $\sin\theta = \frac{b}{h}$, $\cos\theta = \frac{a}{h}$, $\tan\theta = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$
- Area of a sector of a circle is $\frac{1}{2}r^2\theta$ where r is the radius of the circle, and θ is the angle subtended by the sector.

Here are plots of $\sin\theta$ (**red**) and $\cos\theta$ (**blue**).



$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

Now let's reason more carefully:



The Squeeze Theorem
Sandwich Theorem

Exercises

Evaluate

(i)

$$\lim_{x \to 0} \frac{\tan 3x}{\sin 2x}$$

(ii)

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

Solution

Solution ctd.

1. When $x \longrightarrow 0$, then $\cos x \approx 1 - \frac{x^2}{2}$.
2. When $x \longrightarrow 0$, then $\sin x \approx x - \frac{x^3}{6}$.

Evaluate

$$\lim_{x \to 0} \frac{sinx - x}{x^3}$$

Solution

Often limit analysis end with

- $$\frac{\text{constant} \neq 0}{\text{infinity}} \quad \text{or} \quad \frac{0}{\text{constant} \neq 0} \quad \longrightarrow 0$$
- $$\frac{\text{infinity}}{\text{constant} \neq 0} \quad \text{or} \quad \frac{\text{constant} \neq 0}{\text{close to } 0} \quad \longrightarrow \pm\infty$$

Exercise

Evaluate

(i) $\lim_{x\to 0} \frac{1}{x}$

(ii) $\lim_{x\to 0} \frac{x+1}{x^3}$

Solution