MA378 Chapter 3: Numerical Integration

§3.6 Gaussian Quadrature via Orthogonal Polynomials

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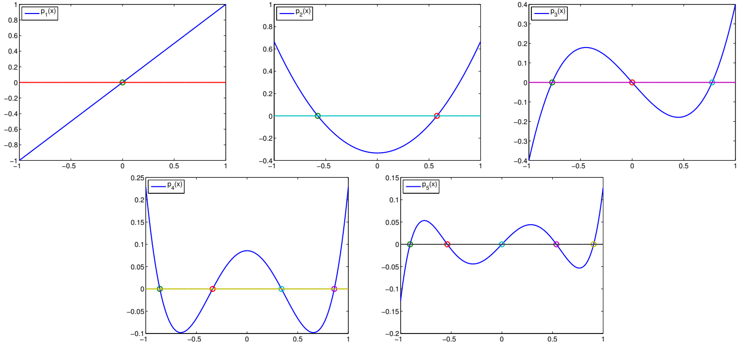


6.1 Gaussian Quadrature via Orthogonal Polys

At the start of this section, we introduced the Gaussian Quadrature technique for estimating integrals

$$\int_a^b f(x)dx \approx G_n(f) := \sum_{k=0}^{n} w_k f(x_k),$$

where the points x_k and weights w_k are chosen to maximise the precision of the method.

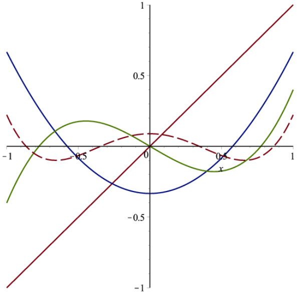
We now have an equivalent way of defining the method, $G_n(\cdot)$, and deriving the coefficients...

6.1 Gaussian Quadrature via Orthogonal Polys

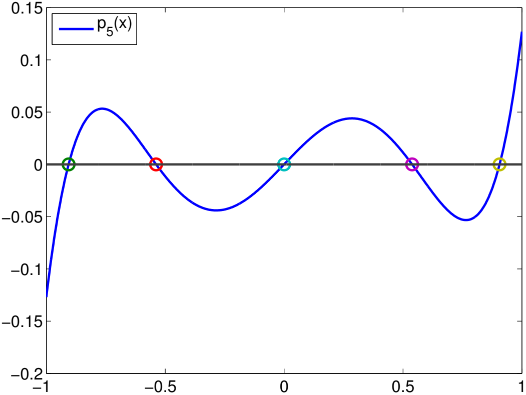
(a) Construct the set of monic polynomials $\{\widetilde{p}_0, \widetilde{p}_1, \ldots, \widetilde{p}_{n+1}\}$ that is orthogonal with respect to the inner product

$$(u, v) := \int_a^b u(x)v(x)dx.$$

Note that a and b, the limits of integration for the IP, are the same as for the integral we are trying to estimate.



(b) We know that $\widetilde{p}_{n+1}$ has $n+1$ zeros in the interval $[a, b]$. Let these be the quadrature points of the method.
For example, the polynomial $\widetilde{p}_5 = x^5 - (10/9)x^3 + (5/21)x$ is shown below, with its zeros highlighted.



$a=-1$ $b=1$

6.1 Gaussian Quadrature via Orthogonal Polys

(c) Take the quadrature weights to be $w_k = \int_a^b L_k(x)dx$, where the L_k are the usual Lagrange polynomials for this set of points.

So

$$G_n(f) = \sum_{k=0}^{n} \left(f(x_k) \int_a^b L_k(x)\,dx \right)$$

$$= \int_a^b \underbrace{\sum_{k=0}^{n} f(x_k) L_k(x)}\,dx.$$

$$= \int_a^b p_n(x)\,dx$$

where $p_n(x)$ is the polynomial interpolant of f at the x_i

So G_n has precision n.

6.1 Gaussian Quadrature via Orthogonal Polys

The key property of this method in stated in the following theorem.

Theorem 6.1

Let $x_0, \dots, x_n$ to be the zeros of $\widetilde{p}_{n+1}$, the $(n+1)$th polynomial in the sequence of orthogonal monic polynomials $\{\tilde{p}_i\}_{i=0}^{\infty}$. Set

$$G_n(f) = w_0 f(x_0) + \cdots + w_n f(x_n) \qquad \text{where } w_i = \int_a^b L_i(x)dx. \tag{1}$$

Then $G_n(f)$ has precision $2n+1$.

6.1 Gaussian Quadrature via Orthogonal Polys

Proof: Obviously $G_n(\cdot)$ has precision at least n. Suppose that f is a polynomial of degree at most $2n+1$, then we can write $f(x)$ as

$$f(x) = \widetilde{p}_{n+1}(x)q(x) + r(x) \qquad \deg(q), \deg(r) \leq n.$$

Note that $\widetilde{p}_{n+1}(x)$ is zero at $x = x_i$.

Hence

$$\begin{aligned} G_n(f) &= G_n\big(\widetilde{p}_{n+1}(x)q(x) + r(x)\big) \\ &= \sum_{i=0}^{n} w_i\big(\underbrace{p_{n+1}(x_i)}_{=0}q(x_i) + r(x_i)\big) \\ &= \sum_{i=0}^{n} w_i r(x_i) = G_n(r) \end{aligned}$$

Finishing the proof is an exercise...

6.2 Error estimates

Check!

Our final task associated with numerical integration is to prove that, as $n \to \infty$, so $G_n(f) \to \int_a^b f(x)dx$. We won't do this in full detail, but the key ideas will be presented.

We would like to prove the following error estimate, which is closely related to Theorem 1.5.2 (error in Hermitian interpolation):

Theorem 6.2

$$\int_a^b f(x)dx - G_n(f) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!}\int_a^b \pi_{n+1}(x)^2 dx.$$

Idea: show that Gaussian Quadrature is the same as integrating the Hermite interpolant to f, even though $f'(x_k)$ is not involved.

6.2 Error estimates

To do this we need to establish several facts. In each case we make use of Theorem 6.1, a consequence of which is that, if f is a polynomial of degree at most $2n-1$, then

$$\int_a^b f(x)dx = \sum_{k=0}^{n} w_k f(x_k).$$

Recall the basis functions that we use for **Hermite Interpolation**:

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Then $p_{2n+1}(x) = \sum_{i=0}^{n} H_i(x) f(x_i) + \sum_{i=0}^{n} K_i(x) f(x_i)$

is the Hermite interpolant of f.

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Similarly, can show

$$H_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

So $h_i(x_j) = H_i(x_j)$

6.2 Error estimates

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$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Also, note that K_i(x_j)=0 for all points x_j. And since the Gaussian Metho
is exact for this polynomial, that means its integral is zero.

6.2 Error estimates

Now we can deduce the error estimate: (Notes added here after class

$$G_n(f) = \sum_{j=0}^{n} w_j f(x_j)$$

$$= \sum_{j=0}^{n} \int_a^b H_j(x)\,dx\, f(x_j) + \sum_{j=0}^{n} \int_a^b K_j(x)\,dx\, f'(x_j)$$

$$= \int_a^b \sum_{j=0}^{n} H_j(x) f(x_j) + K_j(x) f'(x_j)\,dx$$

$$= \int_a^b p_{2n+1}(x)\,dx$$ where p_{2n+1} is the Hermite Interpolant of f.

The Error Estimate follows from Theorem 1.5.2.

6.2 Error estimates

Next we want to show that each of the w_k are positive. From (1) we have that the Gaussian Quadrature weights are $w_k = \int_a^b L_k(x)dx$ where the L_k are the usual Lagrange Polynomials:

$$L_k(x_j) = \begin{cases} 1 & k = j \\ 0 & k \neq j, \end{cases}$$

associated with the Gaussian interpolation points. Then in fact....

$L_k(x)$ has degree n, so

$[L_k(x)]^2$ has degree $2n$, so

$$0 < \int_a^b (L_k(x))^2 dx = G_n(L_k) = \sum_{j=0}^{n} w_j (L_k(x)_j)^2$$

$$= w_k \text{ , because } (L_k(x_j))^2 = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}$$

6.2 Error estimates

So, since

$$w_i = \int_a^b [L_i(x)]^2 dx,$$

we have that all the w_k are all positive. It follows directly that $0 < w_k < (b-a)$, for $k = 0, 1, 2, \ldots, n$:

Since $\sum_{k=0}^{n} w_k = b - a$ (See this by taking f(x)=1).

& $w_k > 0$ $\forall k$, Each $w_k < (b-a)$.

6.3 Convergence

Section 10.4 of Süli and Mayers also covers this, though from a different angle. One of the most interesting aspects of this theory is given in Theorem 10.2 of that book:

Theorem 6.3

$$\lim_{n\to\infty} G_n(f) = \int_a^b f(x)dx.$$

An outline of the proof is given below. Read it if you have time; we won't cover it in class and it is will not be on the MA378 exam.

6.3 Convergence

This section was not covered in class:
** read it in your own time*
** it won't be on the exam*

The **Weierstrass approximation theorem**, tells us that, for any $\epsilon > 0$, there exists a polynomial p such that $|f(x) - p(x)| \leq \epsilon$. Let n be the degree of this polynomial. Let $G_n(\cdot)$ be the $n+1$ point Gaussian Quadrature rule. Then

$$\int_a^b f(x)dx - G_n(f) = \int_a^b f(x) - p(x)dx + \int_a^b p(x)dx - G_n(p) + G_n(p) - G_n(f).$$

But because $G_n(\cdot)$ is exact for polynomials of degree n, $\int_a^b p(x)dx - G_n(p) = 0$. Using this, and the triangle inequality,

$$\left|\int_a^b f(x)dx - G_n(f)\right| \leq \left|\int_a^b f(x) - p(x)dx\right| + \left|G_n(p) - G_n(f)\right|.$$

But
$$\left|\int_a^b f(x) - p(x)dx\right| \leq \int_a^b \epsilon dx = \epsilon(b-a).$$

Also,

$$|G_n(f) - G_n(p)| = |G_n(p-f)| = \\ \left|\sum_{k=0}^{n} w_k\big(f(x_k) - p(x_k)\big)\right| = \\ \sum_{k=0}^{n} w_k\big|f(x_k) - p(x_k)\big|,$$

because all the w_i are positive. Then, using that $\sum_k w_k = b - a$ and $|f(x) - p(x)| \leq \epsilon$, we get

$$\big|G_n(f) - G_n(p)\big| \leq \epsilon(b-a).$$

6.3 Convergence

Combining these we have shown that for any ϵ, one can always find n such that

$$\left| \int_a^b f(x)dx - G_n(p) \right| \leq 2\epsilon(b-a).$$

. . . .

One of the key ingredients to this proof was that the w_i are all positive. That is not the case for Newton-Cotes methods—for large enough n their quadrature weights can be negative—so no similar result is possible.

6.4 Exercises

Exercise 6.1

Give a complete proof of Theorem 6.1.

Exercise 6.2

$\star$ Show that it is impossible to choose $n+1$ quadrature points and weights so that the $n+1$-point quadrature rule

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} w_k f(x_k)$$

has precision $2n+2$.

Hint: To show the method does not have precision $2n+2$, you just need to give a an example of a single polynomial p of degree exactly $2n+2$ for which $\int_a^b p(x)dx \neq \sum_{k=0}^{n} w_k f(x_k)$.