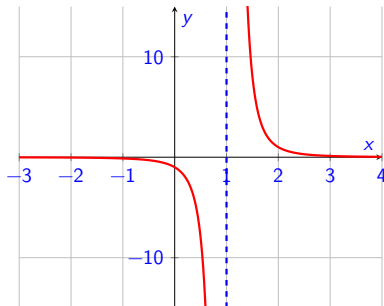


## Week 03, Lecture 2 Vertical Asymptotes and Continuity

Dr Niall Madden

University of Galway

Wednesday, 1 October, 2025



*This slides are by Niall Madden, with some content based on notes by Dr Kirsten Pfeiffer, and from Strang & Herman's "Calculus".*

# Outline

- 1 Reminders...
- 2 Recall: One-sided Limits
  - Notation
  - Existence of a limit
- 3 Vertical Asympotes
  - Horizontal Asymptotes
- 4 Continuity
- 5 Types of discontinuity
- 6 Exercises

For more, see Chapter 2 (Limits) in **Calculus** by Strang & Herman. See [openstax.org/books/calculus-volume-1/pages/2-introduction](https://openstax.org/books/calculus-volume-1/pages/2-introduction)

In particular §2.2: One-sided limits (§2.2.4+§2.2.5) and vertical asymptotes (§2.2.7).

### Reminders

- ▶ **Assignment 1** due 5pm, Monday 6 October. You may access it multiple times, by clicking on **Assignments ... Problem Set 1 ...** and then, at the bottom of the page:

Load Problem Set 1 in a new window

- ▶ The **Tutorial Sheet** is available at <https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1>
- ▶ Assignment 2 is also open; deadline is 5pm, 13 Oct.
- ▶ The first (of two) class tests will take place Tuesday, 14th October.
- ▶ If you wish to avail of Reasonable Accommodations for it tests, please complete this form: <https://forms.office.com/e/HaAsrzaE3D> by **10am Thursday 2nd Oct.**

Yesterday we met the concept of **one-sided limits**:

$\lim_{x \rightarrow a^-} f(x)$  is: **limit of  $f$  as  $x$  approaches  $a$  from the left**

$\lim_{x \rightarrow a^+} f(x)$  is: **limit of  $f$  as  $x$  approaches  $a$  from the right**

These mean that

- ▶ if  $\lim_{x \rightarrow a^-} f(x) = L$ , then we can make  $f(x)$  as close to  $L$  as we would like by taking  $x < a$  as close to  $a$  as needed.
- ▶ If  $\lim_{x \rightarrow a^+} f(x) = L$ , then we can make  $f(x)$  as close to  $L$  as we would like by taking  $x > a$  as close to  $a$  as needed.

**Existence of a limit**

$\lim_{x \rightarrow a} f(x)$  **exists** if and only if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

So if  $\lim_{x \rightarrow a} f(x) = L$  exists, we have

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

**though it is not necessary that**  $f(a) = L$

**Example**

Sketch the function

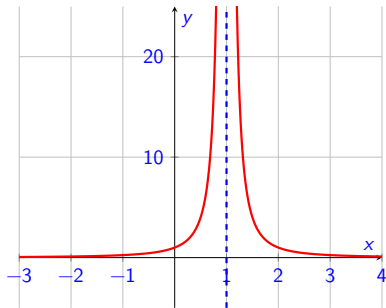
$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if  $\lim_{x \rightarrow 2} f(x)$  exists.

# Vertical Asymptotes

Let's revisit the following example from yesterday:

$$f(x) = \frac{1}{(x-1)^2}$$



Note that the points on the graph having  $x$ -coordinates very near to 1 are very close to the vertical line  $x = 1$ . That is, as  $x$  approaches 1, the points on the graph of  $f(x)$  are closer to the line  $x = 1$ .

We call the line  $x = 1$  a **vertical asymptote** of the graph.

# Vertical Asymptotes

## Definition: Vertical Asymptote

The vertical line  $x = a$  is a **vertical asymptote** of  $f(x)$  if any of  $\lim_{x \rightarrow a^-} f(x)$ ,  $\lim_{x \rightarrow a^+} f(x)$ , or  $\lim_{x \rightarrow a} f(x)$  are  $\infty$  or  $-\infty$ .

To find a vertical asymptote of a function  $f(x) = \frac{p(x)}{q(x)}$ , we find a value,  $a$  for which  $p(a) \neq 0$  but  $q(a) = 0$ .



# Vertical Asymptotes

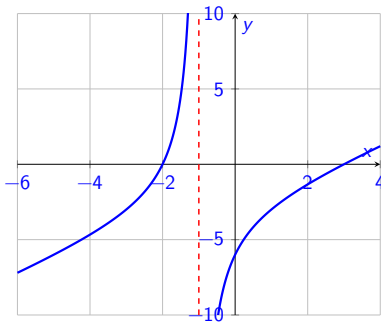
## Example

Find any vertical asymptotes of

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

# Vertical Asymptotes

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$



# Vertical Asymptotes

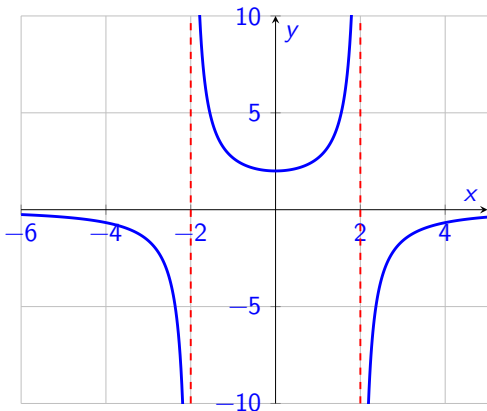
## Example

Find all vertical asymptotes of the graph of

$$g(x) = -\frac{8}{x^2 - 4}.$$

# Vertical Asymptotes

$$f(x) = -\frac{8}{x^2 - 4}$$



There is a related concept of a **horizontal asymptote**, but we'll save that for later, when we cover “limits at infinity”.

# Continuity

Many functions have the property that you can trace their graphs with pen and paper, without lifting the pen from the page. Such functions are called **continuous**.

Some other functions have points where you have to lift the pen occasionally. We say they have a **discontinuity** at such points.

Intuitively, a function is continuous at a particular point if there is no **break** (or “**jump**”) in its graph at that point.

More formally, we define continuity in terms of **limits**

# Continuity

## Definition

A function  $f$  is **continuous at**  $x = a$  if

1.  $f(a)$  is defined, i.e.,  $a$  is in the domain of  $f$ ,
2.  $\lim_{x \rightarrow a} f(x)$  exists.
3.  $\lim_{x \rightarrow a} f(x) = f(a)$ .

If  $f(x)$  is not continuous at  $x = a$  we say it is **discontinuous** at  $x = a$ .

If  $f$  is continuous **at every point** in its domain, we say  $f$  **is continuous**.

Many functions are continuous, e.g. all polynomial functions, **most** trigonometric functions (not **tan**),  $|x|$ , and so on.

## Example 1

Determine if  $f(x) = \frac{x^2 - 4}{x - 2}$  is continuous at  $x = 2$ .



## Example 2

Determine if  $f(x) = \begin{cases} 1 - x & x \leq 0 \\ 2 + x & x > 0 \end{cases}$  is continuous at  $x = 0$ .

## Example 3

Determine if  $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$  is continuous at  $x = 0$ .

## Example

Consider the function

$$f(x) = \begin{cases} x + 1, & x < 2 \\ bx^2, & x \geq 2 \end{cases}$$

For what value of  $b$  is  $f$  continuous at  $x = 2$ ?

# Continuity

## Example

For what values of  $x$  is  $f(x) = \frac{2x + 1}{2x - 2}$  continuous?

# Types of discontinuity

We have encountered three types of discontinuity.

- ▶ **Removable discontinuity:**  $\lim_{x \rightarrow a} f(x)$  exists but

$$\lim_{x \rightarrow a} f(x) \neq f(a)$$

- ▶ **Jump discontinuity:**  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist (and are finite), but  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$
- ▶ **Infinite discontinuity:** At least one of the one-sided limits does not exist.

# Types of discontinuity

## Example

Each of the following functions has a discontinuity at  $x = 2$ .  
Classify it.

1.  $f(x) = \frac{x^2 - 4}{x - 2}$

2.  $g(x) = \frac{x^2}{x - 2}$

3.  $h(x) = \begin{cases} x/2 & x < 2 \\ -2 & x = 2 \\ x^2 - 3 & x > 2. \end{cases}$

4.  $h(x) = \begin{cases} x/2 & x < 2 \\ x^2 - 2 & x > 2. \end{cases}$

# Exercises

## Exercise 3.2.1

Find all the vertical asymptotes of  $f(x) = \frac{x+2}{x^2+2x-8}$ .

## Exercises 3.2.2 (Based on Q1(a), 23/24)

$$\text{Let } g(x) = \begin{cases} 3 & x \leq 0 \\ 2x+1 & 0 < x < 1 \\ x^2 & x \geq 1. \end{cases}$$

- (i) Sketch the graph of  $g(x)$  on the interval  $[-3, 4]$ , making use of the empty and full circle notation.
- (ii) Compute  $\lim_{x \rightarrow 1^-} g(x)$  and  $\lim_{x \rightarrow 1^+} g(x)$ . Is  $g$  continuous at  $x = 1$ . If not, classify the type of discontinuity.

## Exercises

### Exercise 3.2.3

For what values of  $b$  and  $c$  is  $f(x) = \begin{cases} x^2 + 1 & x \leq -1 \\ x + b & -1 < x < 1 \\ cx^2 & x \geq 1. \end{cases}$   
continuous at  $x = -1$  and  $x = 1$ ?