MA385: Tutorial 1 ANS with outline solutions

These exercises are for Tutorial 1 (Week 4). You do not have to submit this work. However, you can expect similar questions on the final exam.

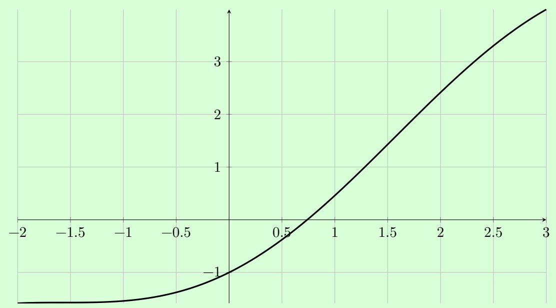
In the following questions, take

$$f(x) = x - \cos(x) \tag{1}$$

It may be useful to know that, if $f(\tau) = 0$ then $\tau \approx 0.7390851332$.

Q1. Show that there is a solution to $f(x) = 0$ in the interval $[0, 1]$.

Answer: $f(0) = -1$ and $f(2) \approx 2.41614$. Since f changes sign on $[0, 2]$, and is continuous on $[0, 2]$, by the IVT, there exists τ such that $f(\tau)| = 0$.
You don't have to plot the function, but if you did, it would look like this:



Q2. Suppose we want to implement the Bisection Method for this problem, with $x_0 = 0$ and $x_1 = 1$. If the sequence generated is denoted $\{x_0, x_1, x_2, \ldots\}$, what is the minimum number of iterations needed to ensure that $|x_k - \tau| \leqslant 10^{-3}$?

Answer: We know that $|\tau - x_k| \leqslant \left(\frac{1}{2}\right)^{k-1}|x_0 - x_1| = \left(\frac{1}{2}\right)^{k-1}$. So we need k large enough so that $(1/2)^{k-1} \leqslant 10^{-3}$. Rearranging, we should see that this means $k \geqslant \log_2(10^3) + 1 \approx 9.9658 + 1$. So we need to take $k = 11$ iterations.

Q3. Is it possible to use Theorem 1.5.2 to determine that Newton's method, applied to f in (1) will converge for any choice of $x_0 \in [0, 1]$? Can it be used to determine that Newton's method will converge for *any* $x_0 \in \mathbb{R}$?

Answer: Yes. We know from Q1 that $\tau \in [0, 1]$. So, although we don't (officially) know τ, we can take $\delta = \tau$, so that $I_\delta = [\tau - \delta, \tau + \delta] \subset [0, 1]$. Then we determine that

$$\frac{|f''(x)|}{|f'(y)|} = \frac{\cos(x)}{1 + \sin(y)} \leqslant \frac{\cos(0)}{1 + \sin(0)} = 1,$$

where we've taken $x = 0$ and $y = 0$ because they maximize and minimize $|f''(x)|$ and $|f'(y)|$ respectively. So we can take $A = 1$ in the theorem. Then the method should converge so long as $|\tau - x_0| \leqslant \min\{\delta, 1\}$, which is true for any $x_0 \in [0, 1]$.
No, we can't extend this to *any* $x_0 \in \mathbb{R}$. Suppose, for example, we tried to choose $x_0 = -\pi/2$. Then we'd have to allow for $f'(y) = 0$, meaning there is no finite A which bounds $|f''(x)|/|f'(y)|$.

Q4. Take $x_0 = 1$. Use Newton's method x_1, x_2, and x_3 as estimates for solutions to $x - \cos(x) = 0$. (If possible, evaluate the corresponding values of $f(x_k)$, and $|\tau - x_k|$, to convince yourself that the method is converging) .

Answer: You should get

(a) $x_1 = 0.75036, \quad f(x_1) = 1.8923 \times 10^{-2}, \quad |\tau - x_1| \approx 1.128 \times 10^{-2}.$

(b) $x_2 = 0.73912, \quad f(x_2) = 4.466 \times 10^{-5}, \quad |\tau - x_2| \approx 2.7758 \times 10^{-5}.$

(c) $x_3 = 0.739085, \quad f(x_3) = 2.847 \times 10^{-10}, \quad |\tau - x_3| \approx 1.7013 \times 10^{-10}.$

Q5. Let's suppose we don't know τ, only that it is located between $x = 0$ and $x = 1$. Taking $x_0 = 1$, give upper bounds for $|\tau - x_1|$, $|\tau - x_2|$, and $|\tau - x_3|$ using the *Newton Error Formula*. How does this compare with the corresponding bounds for the Bisection Method, and with the actual errors?

Answer: The NEF gives that

$$|\tau - x_{k+1}| = \frac{(\tau - x_k)^2}{2} \frac{|f''(\eta_k)|}{|f'(x_k)|}, \qquad \text{for some } \eta_k \in [x_k, \tau].$$

Since $f''(x) = \cos(x)$, and we know that $\tau \in [0, 1]$. So, even thought we don't know η_0, we know that $0 \leqslant \eta_0 \leqslant 1$. Therefore $|f''(\eta_k)| \leqslant 1$ for any k. Also, we've already computed x_0 and x_1, and we have $f'(x) = 1 + \sin(x)$. So we know that $f''(x_0) \approx 1.8415$ and $f''(x_1) \approx 1.6819$. Then the NEF gives:

$$|\tau - x_1| = \frac{(\tau - x_0)^2}{2} \frac{|f''(\eta_0)|}{|f'(x_0)|} \leqslant \frac{(1)^2}{2} \frac{1}{1.8415} \approx 0.42224.$$

Using it again we get

$$|\tau - x_2| = \frac{(\tau - x_1)^2}{2} \frac{|f''(\eta_1)|}{|f'(x_1)|} \leqslant \frac{0.4424^2}{2} \frac{1}{1.6819} \approx 0.0530.$$

And again:

$$|\tau - x_3| = \frac{(\tau - x_3)^2}{2} \frac{|f''(\eta_3)|}{|f'(x_2)|} \leqslant \frac{0.0530^2}{2} \frac{1}{1.6736} \approx 8.393 \times 10^{-4}.$$

We see that, in the worst case scenario, with three iterations Newton achieves an error that would take the Bisecton method 11 iterations. We also see that the bound given by the NEF can be very pessimistic. Essentially, that is due to our pessimistic estimate for $|\tau - x_0|$.