MA378: §1 Interpolation

§2 Lagrange Interpolation

Dr Niall Madden

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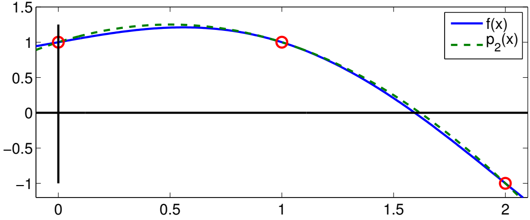
Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics.

2.1 Finding the polynomial

Example 2.1

Show that the polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ is $p_2 = -x^2 + x + 1$.



To verify, first note that degree of p_2 is 2.

$x_0 = 0 \quad f(x_0) = 1 - 0 + \sin(0) = 1 \quad p_2(0) = 0 + 0 + 1 = 1$ ✓

$x_1 = 1 \quad f(x_1) = 1 - 1 + \sin(\pi/2) = 1 \quad p_2(1) = -1 + 1 + 1 = 1$ ✓

$x_2 = 2 \quad f(x_2) = 1 - 2 + \sin(\pi) = -1 \quad p_2(-2) = -4 + 2 + 1 = -1$ ✓

How to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

As a first step, we'll prove the following:

Theorem 2.2

If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all x).

Proof: Say $p_n(x) = 0$ for $x = x_0$, $x = x_1$, .. $x = x_n$

So $p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) q(x)$.

Multiplying out we would get that

$p_n(x) = q(x) x^{n+1} + [\text{stuff}]$

Since $p_n \in P_n$, so $q(x) = 0$ ((coef of x^{n+1} is zero).

Theorem 2.3 (There is a unique solution to the PIP)

There is at most one polynomial of degree $\leq n$ that interpolates the $n+1$ points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) where x_0, x_1, ..., x_n are distinct.

Suppose both $p(x)$ and $q(x)$ solve the same PIP. Let $r(x) = p(x) - q(x)$. Since both $p, q \in P_n$ so too $r \in P_n$

Also, since $p(x_i) = y_i$ and $q(x_i) = y_i$.
So $r(x_i) = y_i - y_i = 0$ for $i = 0, 1, \ldots, n$.
So r is a polynomial of degree $\leq n$ with $n+1$ zeros. By Thm 2.2, $r \equiv 0$.

2.2 The Vandermonde matrix method

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find p_2 such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1x + a_2x^2$, this just amounts the finding the values of the coefficients a_0, a_1, and a_2. One might be tempted to solve for them using the system of equations

$$\begin{aligned} a_0 + a_1x_0 + a_2x_0^2 &= y_0 \\ a_0 + a_1x_1 + a_2x_1^2 &= y_1 \\ a_0 + a_1x_2 + a_2x_2^2 &= y_2 \end{aligned}$$

This is known as the *Vandermonde System*.

2.2 The Vandermonde matrix method

Writing

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

in matrix-vector format we get

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \quad \text{or} \quad V\mathrm{a} = \mathrm{y}. \qquad (1)$$

But this may not be a good idea. (*Unfortunately, to see exactly why, you needed to have studied MA385. If you didn't, you can skip the next bit*).

2.2 The Vandermonde matrix method

In MA385 we learned about the relationship between the *condition number* of a matrix, V, and the relative error in the (numerical) solution to a matrix-vector equation with V as the coefficient matrix. The condition number is $\kappa(V) = \|V\|\|V^{-1}\|$, for some subordinate matrix norm $\|\cdot\|$.

Example 2.4 (Stewart's "Afternotes...", Lecture 19)

Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

2.3 Lagrange Interpolation

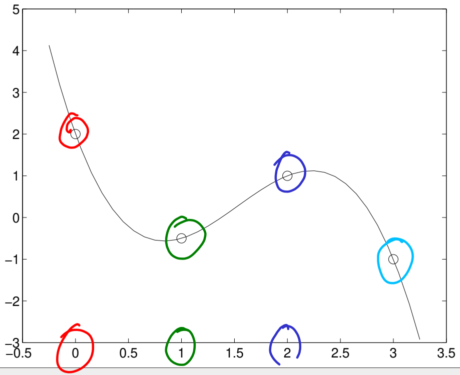
We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (Here "constructive" means that we'll prove it exists by actually computing it).

2.3 Lagrange Interpolation

Example

Consider the problem: *find* $p_3 \in \mathcal{P}_3$ *such that*

$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$

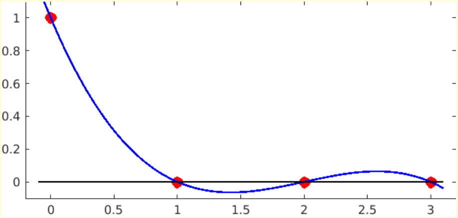


2.3 Lagrange Interpolation

Here is an easier problem to solve: Find $L_0 \in \mathcal{P}_3$ such that

$$L_0(0) = 1, \;\; L_0(1) = 0,$$

$$L_0(2) = 0, \;\; L_0(3) = 0.$$

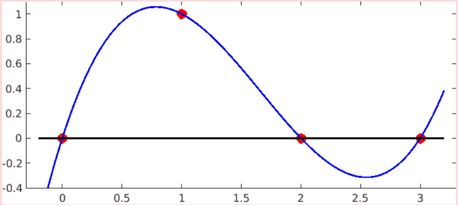


Because L_0 is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$.
Choosing C so that $L_0(0) = 1$, we get

$$L_0(x) = \frac{(x-1)(x-2)(x-3)}{(-1)\;(-2)\;(-3)}$$

Similarly, find $L_1 \in \mathcal{P}_3$ such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

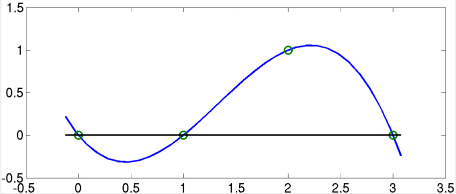


Then

$$L_1(x) = \frac{x(x-2)(x-3)}{(1)(-1)(-2)}$$

2.3 Lagrange Interpolation

In the same style, let $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



$$L_2(x) = \frac{x(x-1)(x-3)}{2(1)(-1)}$$

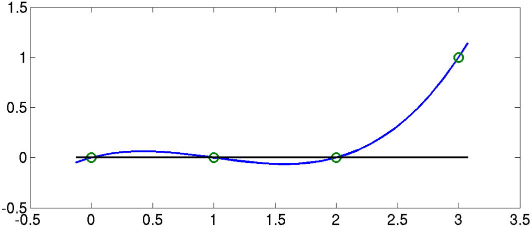
2.3 Lagrange Interpolation

Finally, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$$L_3(x) = \frac{(x-0)(x-1)(x-\cancel{3}\,2)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j\neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$



2.3 Lagrange Interpolation

Because each of L_0, L_1, L_2, and L_3 is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (\frac{1}{2})L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. Furthermore...

$$p_3(0) = 2L_0(0) - \frac{1}{2}L_1(0) + L_2(0) - L_3(0)$$

$$= 2(1) - 0 + 0 + 0$$

$$= 2 \checkmark$$

2.3 Lagrange Interpolation

$$
\begin{array}{rcllll}
p_3(0) & = & 2L_0(0) & -(1/2)L_1(0) & +(1)L_2(0) & +(-1)L_3(0) \\
& = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
& = & 2, & & & \\
p_3(1) & = & 2L_0(1) & -(1/2)L_1(1) & +(1)L_2(1) & +(-1)L_3(1) \\
& = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
& = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
& = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
& = & 1, & & & \\
p_3(3) & = & 2L_0(3) & -(1/2)L_1(3) & +(1)L_2(3) & +(-1)L_3(3) \\
& = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
& = & -1. & & &
\end{array}
$$

Thus p_3 solves the problem!

2.4 The Lagrange Form

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation.

Definition 2.5 (Lagrange Polynomials)

The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2b}$$

Definition 2.6

The **Lagrange form of the Interpolating Polynomial**

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3a}$$

or

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{3b}$$

Take care not to confuse

- the *Lagrange Polynomials*, which are the L_i with
- the *Lagrange Interpolating Polynomial*, which is the p_n defined in (3).

 "Existence theorem".

Theorem 2.7 (Lagrange)

There exists a solution to the Polynomial Interpolation Problem and it is given by

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

Proof. Each of the $L_i \in P_n$, so too is p_n.

Also $p_n(x_j) = \sum_{i=1}^{n} y_i L_i(x_j) = y_j$ so p_n solve the PIP.

since $L_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$.

Example 2.8 (Süli and Mayers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

$x_0 = -1 \qquad x_1 = 0 \qquad x_2 = 1$

$y_0 = e^{-1} \qquad y_1 = 1 \qquad y_2 = e$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{x(x-1)}{(-1)(-2)} = \frac{1}{2}x(x-1)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x+1)(x-1)}{(+1)(-1)} = -(x^2 - 1)$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x+1)(x)}{(2)(1)} = \frac{1}{2}x(x+1).$$

Example 2.8 (Süli and Mayers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

So

$$p_2(x) = \frac{e^{-1}}{2} x(x-1) + (1-x^2) + \frac{e}{2} x(x+1).$$

2.5 Example

The figure below shows the solution to Example 2.8 (top) and the difference between the function e^x and its interpolant (bottom). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.