

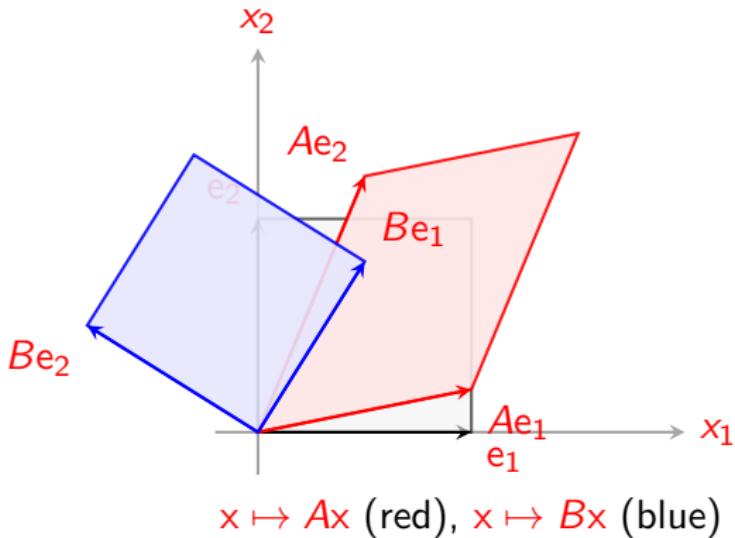
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MA385 Part 4: Linear Algebra 2

## 4.2: Matrix Norms

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# 1. Outline Section 4.2

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For more, see Section 2.7 of Suli and Mayers:

<https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG>

**Vector norms** are related to the magnitude of the entries of the vector.

Now we want to generalise to the concept of a **matrix norm**. In a sense, we can just consider the magnitude of the matrix's entries.

However, if we think of a matrix as a linear transformation, or simply as a function that maps (via matrix multiplication) from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , we should think about how much it changes a vector.

**Definition 4.2.1**

Given any (vector) norm  $\|\cdot\|$  on  $\mathbb{R}^n$ , there is a **subordinate matrix norm** on  $\mathbb{R}^{n \times n}$  defined by

$$\|A\| = \max_{v \in \mathbb{R}_*^n} \frac{\|Av\|}{\|v\|}, \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $\mathbb{R}_*^n = \mathbb{R}^n / \{0\}$ .

We define a matrix norm like this because we think of  $A$  as an *operator* on  $\mathbb{R}^n$ : if  $v \in \mathbb{R}^n$  then  $Av \in \mathbb{R}^n$ . So the norm of  $A$  gives us information on how much the matrix can change the size of a vector.

### 3. Computing Matrix Norms

It is not obvious from the above definition how to calculate the norm of a given matrix. We'll see that

- ▶ The  $\infty$ -norm of a matrix is also the largest absolute-value row sum.
- ▶ The 1-norm of a matrix is also the largest absolute-value column sum.
- ▶ The 2-norm of the matrix  $A$  is the square root of the largest eigenvalue of  $A^T A$ .

## 4. The max-norm on $\mathbb{R}^{n \times n}$

### Theorem 4.2.1

For any  $A \in \mathbb{R}^{n \times n}$  the subordinate matrix norm associated with  $\|\cdot\|_\infty$  on  $\mathbb{R}^n$  can be computed by

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|.$$

## 4. The max-norm on $\mathbb{R}^{n \times n}$

A similar result holds for the 1-norm, the proof of which is left as an exercise.

**Theorem 4.2.2**

For any  $A \in \mathbb{R}^{n \times n}$  the subordinate matrix norm associated with  $\|\cdot\|_\infty$  on  $\mathbb{R}^n$  can be computed by

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^n |a_{i,j}|. \quad (2)$$

Computing the 2-norm of a matrix is a little harder than computing the 1- or  $\infty$ -norms. However, later we'll need estimates not just for  $\|A\|$ , but also  $\|A^{-1}\|$ . And, unlike the 1- and  $\infty$ -norms, we can estimate  $\|A^{-1}\|_2$  without explicitly forming  $A^{-1}$ .

We begin by recalling some important facts about eigenvalues and eigenvectors.

### Definition 4.2.2

Let  $A \in \mathbb{R}^{n \times n}$ . We call  $\lambda \in \mathbb{C}$  an *eigenvalue* of  $A$  if there is a non-zero vector  $x \in \mathbb{C}^n$  such that

$$Ax = \lambda x.$$

We call any such  $x$  an *eigenvector of  $A$  associated with  $\lambda$* .

## 5. Computing $\|A\|_2$

Eigenvalues

- (i) If  $A$  is a real symmetric matrix (i.e.,  $A = A^T$ ), its eigenvalues and eigenvectors are all real-valued.
- (ii) If  $\lambda$  is an eigenvalue of  $A$ , then  $1/\lambda$  is an eigenvalue of  $A^{-1}$ .
- (iii) If  $x$  is an eigenvector associated with the eigenvalue  $\lambda$  then so too is  $\eta x$  for any non-zero scalar  $\eta$ .
- (iv) An eigenvector may be *normalised* as  $\|x\|_2^2 = x^T x = 1$ .

## 5. Computing $\|A\|_2$

Eigenvalues

- (v) There are  $n$  eigenvectors  $\lambda_1, \lambda_n, \dots, \lambda_n$  associated with the real symmetric matrix  $A$ . Let  $x^{(1)}, x^{(2)}, \dots, x^{(n)}$  be the associated normalised eigenvectors. The eigenvectors are linearly independent and so form a basis for  $\mathbb{R}^n$ . That is, any vector  $v \in \mathbb{R}^n$  can be written as a linear combination:

$$v = \sum_{i=1}^n \alpha_i x^{(i)}.$$

- (vi) Furthermore, these eigenvectors are *orthogonal* and *orthonormal*:

$$(x^{(i)})^T x^{(j)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Here is a useful consequence of (v) and (vi), which we will use repeatedly.

The *singular values* of a matrix  $A$  are the square roots of the eigenvalues of  $A^T A$ . They play a very important role in matrix analysis, applied linear algebra, and statistics (principal component analysis).

Our interest here is in their relationship to  $\|A\|_2$ .

But first we'll prove a theorem about certain matrices (so called, "normal matrices").

**Theorem 4.2.3**

For any matrix  $A \in \mathbb{R}^{n \times n}$ , the eigenvalues of  $A^T A$  are real and non-negative.

## 5. Computing $\|A\|_2$

Eigenvalues

Part of the above proof involved showing that, if  $(A^T A)x = \lambda x$ , then

$$\sqrt{\lambda} = \frac{\|Ax\|_2}{\|x\|_2}.$$

This at the very least tells us that

$$\|A\|_2 := \max_{x \in \mathbb{R}_*^n} \frac{\|Ax\|_2}{\|x\|_2} \geq \max_{i=1,\dots,n} \sqrt{\lambda_i}.$$

With a bit more work, we can show that if  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  are the eigenvalues of  $B = A^T A$ , then

$$\|A\|_2 = \sqrt{\lambda_n}.$$

**Theorem 4.2.4**

Let  $A \in \mathbb{R}^{n \times n}$ . Let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , be the eigenvalues of  $B = A^T A$ . Then

$$\|A\|_2 = \max_{i=1,\dots,n} \sqrt{\lambda_i} = \sqrt{\lambda_n},$$

Here is the main idea. For full details, see the text-book.

## 5. Computing $\|A\|_2$

Eigenvalues

## 6. Exercises

### Exercise 4.2.1

Show that, for any subordinate matrix norm on  $\mathbb{R}^{n \times n}$ , the norm of the identity matrix is 1.

### Exercise 4.2.2

Prove that

$$\|A\|_1 = \max_{j=1, \dots, n} \sum_{i=1}^n |a_{i,j}|.$$

*Hint:* Suppose that  $\sum_{i=1}^n |a_{ij}| \leq C$ , for  $j = 1, 2, \dots, n$ . Show that for any vector  $x \in \mathbb{R}^n$

$$\sum_{i=1}^n |(Ax)_i| \leq C\|x\|_1.$$

Now find a vector  $x$  such that  $\sum_{i=1}^n |(Ax)_i| = C\|x\|_1$ . Now deduce the result.