Chapter 2 (Initial Value Problems) and Chapter 3 (Numerical Linear Algebra)

Outline solutions to homework assignment: Exercises 2.7, 2.14, 3.12 and 3.15.

Exercise 2.7 ($\star$). In his seminal paper of 1901, Carl Runge gave an example of what we now call a *Runge-Kutta 2 method*, where

$$\Phi(t_i, y_i; h) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\big).$$

(i) Show that it is consistent.

(ii) Show how this method fits into the general framework of RK2 methods. That is,

(a) What are a, b, α, and β?

(b) Do they satisfy the conditions

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b?$$

(iii) Use it to estimate the solution at the point $t = 2$ to $y(1) = 1$, $y' = 1 + t + y/t$ taking $n = 2$ time steps.

SOLUTION:

(i) The method is consistent if $\Phi(t_i, y_i; 0) = f(t_i, y_i)$ (that is, if we formally set $h = 0$ in the method, we recover the right-hand side of the differential equation) . In this case,

$$\Phi(t_i, y_i; 0) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\big(t_i + 0\frac{2}{3}, y_i + 0\frac{2}{3}f(t_i, y_i)\big) = \frac{1}{4}f(t_i, y_i) + \frac{3}{4}f\big(t_i, y_i\big) = f(t_i, y_i),$$

as required.

(ii) All RK-2 methods fall into the following framework:

$$k_1 = f(t_i, y_i), \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1), \qquad \Phi(t_i, y_i; h) = ak_1 + bk_2.$$

So we can see that $a = 1/4$ and $b = 3/4$. Furthermore, we see that $k_2 = f\big(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hk_1\big)$. That is, $\alpha = \beta = 2/3$. So, to answer the question,

(a) $a = 1/4$ and $b = 3/4$, $\alpha = \beta = 2/3$.

(b) Yes, since $a + b = (1 + 3)/4 = 1$ and $\alpha = \beta$ and $1/(2\alpha) = 3/4 = b$ as required.

(iii) For this problem, $f(x, t) = 1 + t + y/t$, $t_0 = 1$ and $y_0 = 1$. Proceed as follows.

Step 1: To solve for $y_1 \approx y(3/2)$, we compute

$$k_1 = f(t_0, y_0) = f(1, 1) = 3, \qquad \text{and} \qquad k_2 = f(1 + \frac{1}{3}, 1 + \frac{1}{3}f(1, 1)) = f(\frac{4}{3}, 2) = \frac{23}{6}.$$

Then

$$y_1 = y_0 + h(ak_1 + bk_2) = 1 + \frac{1}{2}(\frac{1}{4} \cdot 3 + \frac{3}{4} \cdot \frac{23}{6}) = \frac{45}{16}$$

Step 2: To Solve for $y_2 \approx y(2)$, we compute

$$k_1 = f(\frac{3}{2}, \frac{45}{16}) = \frac{35}{8}, \qquad \text{and} \qquad k_2 = f(\frac{3}{2} + \frac{1}{3}, \frac{45}{16} + \frac{1}{3} \cdot \frac{35}{8}) = f(\frac{11}{6}, \frac{205}{48}) = \frac{1363}{264}.$$

Then

$$y_2 = \frac{45}{16} + \frac{1}{2}(\frac{1}{4} \cdot \frac{35}{8} + \frac{3}{4} \cdot \frac{1363}{264}) = \frac{233}{44} \approx 5.2955.$$

Here are some entries for 3-stage Runge-Kutta method tableaux for Exercise 2.14.

Method 0: $\alpha_2 = 2/3,\ \alpha_3 = 0,\ b_1 = 1/12,\ b_2 = 3/4,\ \beta_{32} = 3/2$

Method 1: $\alpha_2 = 1/4,\ \alpha_3 = 1,\ b_1 = -1/6,\ b_2 = 8/9,\ \beta_{32} = 12/5$

Method 2: $\alpha_2 = 1/4,\ \alpha_3 = 1/2,\ b_1 = 2/3,\ b_2 = -4/3,\ \beta_{32} = 2/5$

Method 3: $\alpha_2 = 1/4,\ \alpha_3 = 1/3,\ b_1 = 3/2,\ b_2 = -8,\ \beta_{32} = 4/45$

Method 4: $\alpha_2 = 1,\ \alpha_3 = 1/4,\ b_1 = -1/6,\ b_2 = 5/18,\ \beta_{32} = 3/16$

Method 5: $\alpha_2 = 1,\ \alpha_3 = 1/5,\ b_1 = -1/3,\ b_2 = 7/24,\ \beta_{32} = 4/25$

Method 6: $\alpha_2 = 1,\ \alpha_3 = 1/6,\ b_1 = -1/2,\ b_2 = 3/10,\ \beta_{32} = 5/36$

Method 7: $\alpha_2 = 1/2,\ \alpha_3 = 1/7,\ b_1 = 7/6,\ b_2 = 22/15,\ \beta_{32} = -10/49$

Method 8: $\alpha_2 = 1/2,\ \alpha_3 = 1/8,\ b_1 = 4/3,\ b_2 = 13/9,\ \beta_{32} = -3/16$

Method 9: $\alpha_2 = 1/3,\ \alpha_3 = 1/9,\ b_1 = 4,\ b_2 = 15/4,\ \beta_{32} = -2/27$

Exercise 2.14 (Your own RK3 method $\star$)**.** *Answer the following questions for Method* K *from the list above, where* K *is the last digit of your ID number. For example, if your ID number is 01234567, use Method 7.*

(a) *Assuming that the method is* consistent, *determine the value of* b_3.

(b) *Consider the initial value problem:*

$$y(0) = 1,\ y'(t) = \lambda y(t).$$

Using that the solution is $y(t) = e^{\lambda t}$, *write out a Taylor series for* $y(t_{i+1})$ *about* $y(t_i)$ *up to terms of order* h^4 *(use that* $h = t_{i+1} - t_i$*).*

Using that your method should agree with the Taylor Series expansion up to terms of order h^3, *determine* β_{21} *and* β_{31}.

SOLUTION:

(a) A method is consistent if

$$\Phi(t_i, y_i; 0) = f(t_i, y_i),$$

For all these methods, RK3 if $h = 0$, then $k_1 = k_2 = k_3$. Since

$$\Phi\,(t_i, y_i; h) = b_1 k_1 + b_2 k_2 + b_3 k_3,$$

it follows that the method is consistent if $b_1 + b_2 + b_3 = 1$. You can use this to find your b_3

(b) The Taylor series expansion up to order h^4 is

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2!}y''(t_i) + \frac{(t_{i+1} - t_i)^3}{3!}y'''(t_i) + \frac{(t_{i+1} - t_i)^4}{4!}y''''(t_i).$$

Letting y_i approximate $y(t_i)$, and using that $y^{(k)}(t) = \lambda^t y(t)$, and $t_{i+1} - t_i = h$, we get

$$y(t_{i+1}) = y(t_i)\left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6}\right) + \mathcal{O}(h^4).$$

Next, we write out the Runge-Kutta 3 method:

$$\begin{aligned}
k_1 &= f(t_i + \alpha_1 h, y_i) = \lambda y_i,\\
k_2 &= f(t_i + \alpha_2 h, y_i + \beta_{21} h k_1) = \lambda(y_i + \beta_{21} h\lambda y_i),\\
k_3 &= f(t_i + \alpha_3 h, y_i + \beta_{31} h k_1 + \beta_{32} h k_2) = \lambda\left((y_i + \beta_{31} h\lambda y_i) + \beta_{32} h\lambda(y_i + \beta_{21} h\lambda y_i)\right).
\end{aligned}$$

The method is

$$y_{i+1} = y_i + h\,(b_1 k_1 + b_2 k_2 + b_3 k_3)$$

So

$$y_{i+1} = y_i + h\Big(b_1\lambda y_i + b_2\lambda(y_i + \beta_{21}h\lambda y_i) + b_3\lambda\big((y_i + \beta_{31}h\lambda y_i) + \beta_{32}h\lambda(y_i + \beta_{21}h\lambda y_i)\big)\Big)$$
$$= y_i\big(1 + h\lambda\,(b_1 + b_2 + b_3) + h^2\lambda^2\,(b_2\beta_{21} + b_3(\beta_{31} + \beta_{32})) + h^3\lambda^3\,(b_3\beta_{32}\beta_{21})\,\big)$$

For this to agree with the Taylor series expansion, we would need

(i) $(b_1 + b_2 + b_3) = 1$, which we have already.

(ii) $b_2\beta_{21} + b_3(\beta_{31} + \beta_{32}) = \frac{1}{2}$, and

(iii) $b_3\beta_{32}\beta_{21} = \frac{1}{6}$.

Since we already have b_3 and β_{32} we can use (iii) to calculate β_{21}, and then substitute this into (ii) to find β_{31}.

Table of coefficients

Method	b_3	β_{21}	β_{31}
0	1/6	2/3	-3/2
1	5/18	1/4	-7/5
2	5/3	1/4	1/10
3	15/2	1/4	11/45
4	8/9	1	1/16
5	25/24	1	1/25
6	6/5	1	1/36
7	-49/30	1/2	17/49
8	-16/9	1/2	5/16
9	-27/4	1/3	5/27

....

Exercise 3.12 ($\star$). *Show that, for any vector $x \in \mathbb{R}^n$, $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_2^2 \leqslant \|\mathbf{x}\|_1\|\mathbf{x}\|_\infty$. For each of these inequalities, give an example for which the equality holds. Deduce that $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2 \leqslant \|\mathbf{x}\|_1$.*

Solution: First recall the definitions of these norms:

$$\|\mathbf{x}\|_\infty = \max_{i=1,\dots,n} |x_i|, \quad \|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|, \quad \text{and} \quad \|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

To show that $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2$, let $\mathbf{x} \in \mathbb{R}^n$ be the vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, where x_j is such that $|x_j| \geqslant |x_i|$ for all i. Then,

$$\|\mathbf{x}\|_\infty^2 = x_j^2 \leqslant x_j^2 + (x_1^2 + x_2^2 + \cdots + \cdots + x_{j-1}^2 + \cdots + x_{j+1}^2 + \cdots + x_n^2) = \|\mathbf{x}\|_2,$$

where here we have used that each $x_i^2 \geqslant 0$.

Next, to show that $\|\mathbf{x}\|_2^2 \leqslant \|\mathbf{x}\|_1\|\mathbf{x}\|_\infty$ we will use that

$$\|\mathbf{x}\|_2^2 \;=\; x_1x_1 \,+\, x_2x_2 \,+\, \cdots \,+\, x_jx_j \,+\, \cdots \,+\, x_nx_n \;\leqslant\; (|x_1| + |x_2| + \cdots + |x_j| + \cdots + |x_n|)|x_j| \;=\; \|\mathbf{x}\|_1\|\mathbf{x}\|_\infty$$

where again we use that $\|\mathbf{x}\|_\infty = |x_j|$.

There are examples where equality holds. E.g., if $\mathbf{x} = (1, 0, \dots, 0)^T$, then $\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_2 = \|\mathbf{x}\|_1 = 1$. So, for this x, $\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_2^2 = \|\mathbf{x}\|_1\|\mathbf{x}\|_\infty$.

To finish, we need to show that $\|\mathbf{x}\|_2 \leqslant \|\mathbf{x}\|_1$. Clearly this is true for $\|x\| = 0$. Otherwise, Combine $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_2^2 \leqslant \|\mathbf{x}\|_1\|\mathbf{x}\|_\infty$, to get that

$$\|\mathbf{x}\|_2^2 \leqslant \|\mathbf{x}\|_1\|\mathbf{x}\|_2,$$

and so $\|\mathbf{x}\|_2 \leqslant \|\mathbf{x}\|_1$.

Exercise 3.15 ($\star$). *Prove that* $\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^n |a_{i,j}|$. *Hint: Suppose that* $\sum_{i=1}^n |a_{ij}| \leqslant C$, *for* $j = 1, 2, \dots n$, *and show that for* any *vector* $x \in \mathbb{R}^n$ $\sum_{i=1}^n |(Ax)_i| \leqslant C\|x\|_1$. *Now find a vector* x *such that* $\sum_{i=1}^n |(Ax)_i| = C\|x\|_1$. *Now deduce the result.*

SOLUTION: For a given $A \in \mathbb{R}^{n\times n}$ let

$$C_j = \sum_{i=1}^n |a_{i,j}|, \quad \text{and let } C = \max_{j=1,\dots,n} C_j.$$

That is, C is the sum of the largest column. Let $\mathbf{x}$ be the vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$. Then

$$\|A\mathbf{x}\|_1 = \sum_{i=1}^n |(Ax)_i| = \sum_{i=1}^n \left| \sum_{j=1}^n a_{i,j} x_j \right| \leqslant \sum_{i=1}^n \sum_{j=1}^n |a_{i,j}||x_j| = \sum_{j=1}^n |x_j| \sum_{i=1}^n |a_{i,j}| \leqslant \sum_{j=1}^n |x_j| C = C\|\mathbf{x}\|_1.$$

That is, for any x whatsoever, $\|Ax\|_1/\|x\|_1 \leqslant C$. It follows, since x is arbitrary that

$$\|A\|_1 := \max_{x\in\mathbb{R}/\{\mathbf{0}\}} \frac{\|Ax\|_1}{\|x\|_1} \leqslant C.$$

To get equality, we need to choose a x for which $\|Ax\|_1/\|x\|_1 = C$. Let q be such that column q of A has the largest sum. That is, $C_q = C$. Now let $\mathbf{x} = e^{(q)}$ (i.e., the vector whose only non-zero entry is $x_q = 1$; equivalently, it is column q of the identity matrix). For this vector $\|\mathbf{x}\|_1 = 1$. Moreover, $Ax = (a_{1,q}, a_{2,q}, \dots, a_{n,q})^T$; that is, it is the vector which is column q of A. So, $\|Ax\|_1 = C$. This gives, for this x,

$$\frac{\|Ax\|_1}{\|x\|_1} = \frac{C\|\mathbf{x}\|_1}{\|\mathbf{x}\|_1} = C = \max_{j=1,\dots,n} \sum_{i=1}^n |a_{i,j}|.$$