2425-MA140 Engineering Calculus

Week 03, Lecture 1
The Squeeze Theorem & one-sided limits

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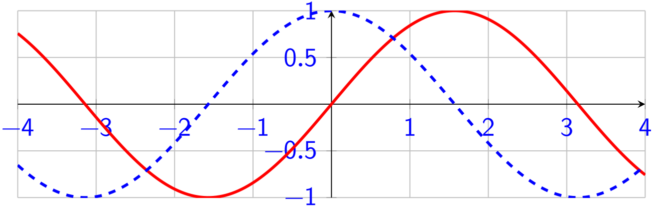

This version of the slides are by Niall Madden. Some are based on original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Section 7.8.1 (Limit of a function of a real number) in *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

However, even better is Section 2.2 (Limit of a Function) from **Calculus** by Gil Strang and Jed Herman, published by the non-profit OpenStax. See https://openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function

Reminder

- **Assignment 1** has a deadline of 5pm, Friday. You can access it on Canvas... 2425-MA140... Assignments.
- The **Tutorial Sheet** is available at https://universityofgalway.instructure.com/files/2040359/download?download_frd=1
- A new assignment will be posted later this week.

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For help with the assignment, attend a tutorial. The schedule is on the Canvas "Course Information" page: https://universityofgalway.instructure.com/courses/35693/pages/2425-ma140-information

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Support is also available at **SUMS**.

Recall... the Squeeze Theorem

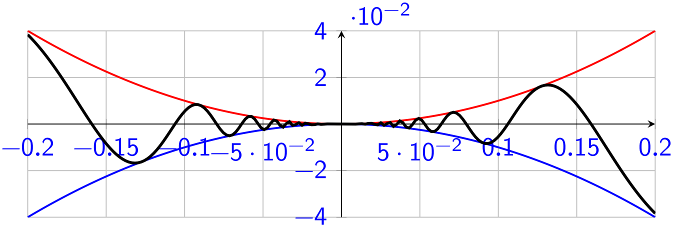
Last Thursday, we finished with...

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that for functions f, g and h in a given interval I:

$$g(x) \leqslant f(x) \leqslant h(x) \quad \underline{\text{and}} \quad \lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\boxed{\lim_{x\to c} f(x) = L.}$

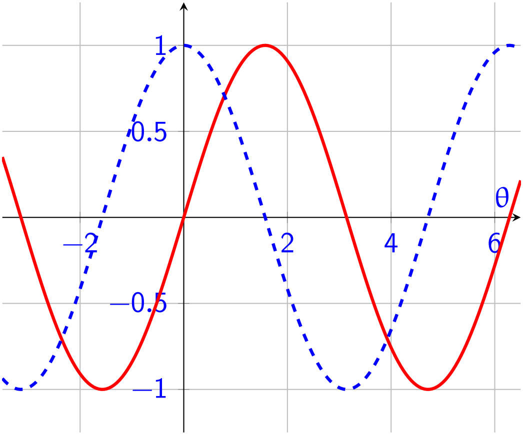


We'll use the Squeese Theorem to explain that $\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$

First, we few facts about trigonometric functions.

- **In this module, we only every use radians** (never degrees).
- Given the triangle drawn below, $\sin\theta = \frac{b}{h}$, $\cos\theta = \frac{a}{h}$, $\tan\theta = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$
- Area of a sector of a circle is $\frac{1}{2}r^2\theta$ where r is the radius of the circle, and θ is the angle subtended by the sector.

Here are plots of $\sin\theta$ (**red**) and $\cos\theta$ (**blue**).

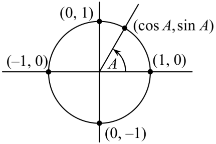


Various other facts are summarised in the State Examination Commission's Tables:

Triantánacht **Trigonometry**

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	0°	90°	180°	270°	30°	45°	60°	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	–1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	–1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. ≈ 57.296° 1° ≈ 0.01745 rad.

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Foirmlí uillinneacha comhshuite **Compound angle formulae**

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

Foirmlí uillinneacha dúbailte **Double angle formulae**

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\sin 2A = 2\sin A\cos A$$

$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)$$

$$\sin^2 A = \tfrac{1}{2}(1-\cos 2A)$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

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Iolraigh a thiontú ina suimeanna agus ina ndifríochtaí **Products to sums and differences**

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

Suimeanna agus difríochtaí a thiontú ina n-iolraigh **Sums and differences to products**

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

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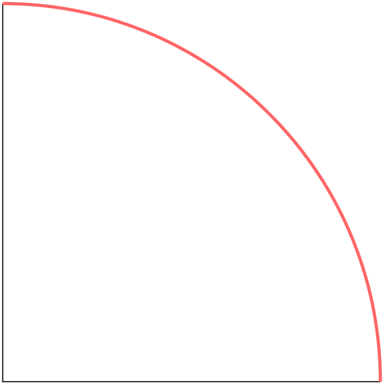
$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$$



Example

Evaluate $\lim\limits_{x\to 0} \dfrac{\tan 3x}{\sin 2x}$

Example

Evaluate $\displaystyle\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2}$

Infinite Limits

So far, we've had lots of examples that are a little like:

$$\lim_{x \to 1} \frac{x^3 - x^2 - x + 1}{(x-1)^2} = 2.$$

(Check that this is correct).

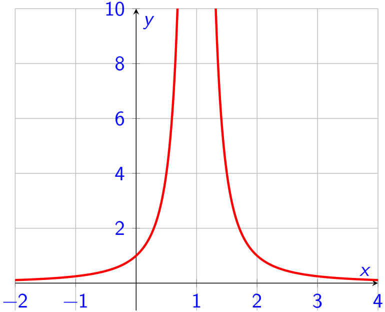
But what about

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = ???$$

Let's plot it and see:

Infinite Limits

$$\boxed{f(x) = \frac{1}{(x-1)^2}}$$



As x get closer and closer to 1, the value of $f(x)$ gets larger and larger. In fact, it becomes infinite.

For this we write $\boxed{\lim_{x\to 1} f(x) = \infty}$.

Digression: How fast can an object travel

- Q: Is there any limit to the speed at which an object can travel?
- A: Yes! (Assuming you believe Einstein)

Thanks to Einstein ($E = mc^2$), Lorenz and others, it is known that the mass of a moving charged particle behaves like

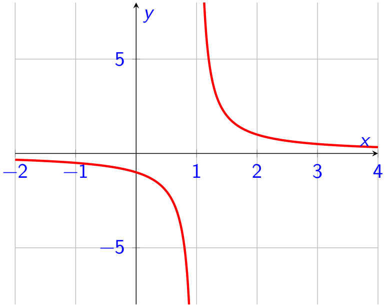
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is its mass at rest, c is the speed of light, and v is the particles current speed. What happens as $v \to c$?

One-sided Limits

Let's consider a motivating example, very similar to the one where we introduced ∞.

$$\boxed{f(x) = \frac{1}{x-1}}$$



As x get closer and closer to 1, then $f(x) \to -\infty$ or $f(x) \to \infty$, depending on whether x approaches 1 from the left or right.

To express this, we need the concept of a **one-sided limit**

$\lim_{x \to a^-} f(x)$ is: **limit of f as x approaches a from the left**

$\lim_{x \to a^+} f(x)$ is: **limit of f as x approaches a from the right**

In the previous example, with $f(x) = \dfrac{1}{x-1}$, we have

- $\displaystyle\lim_{x \to 1^-} f(x) = -\infty$
- $\displaystyle\lim_{x \to 1^+} f(x) = \infty$

In many important examples, we encounter functions that have different definitions in different regions. The most classic example is the **absolute value function**:

$$|x| = \begin{cases} -x & x < 0 \\ x & x > 0. \end{cases}$$

Care has to be taken when evaluating the limits of such functions....

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x \leqslant 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$$

Find $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$.

Empty and Full Circle Notation:

In the previous sketch, we use the convention that

- If the end point of a line segment is **not** included in its definition, it terminates with an **open circle**, $\circ$
- If the end point of a line segment **is** included in its definition, it terminates with an **closed circle** $\bullet$.

One-sided Limits

$\lim_{x\to a} f(x)$ **exists** if and only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

So if $\lim_{x\to a} f(x) = L$ exists, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

[but not necessarily $= f(a)$!]

Note: One-sided limits can be introduced formally by using the ϵ/δ approach.

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if $\lim_{x \to 2} f(x)$ exists.

Exercises

Exercise 3.1.1 (from 2023/24 Q1(b))

Evaluate

$$\lim_{\theta \to 0} \frac{2\sin(\theta)}{\theta + 3\tan(\theta)}$$