Annotated slides

2526-MA140 Engineering Calculus

Week 06, Lecture 1
Maxima and Minima

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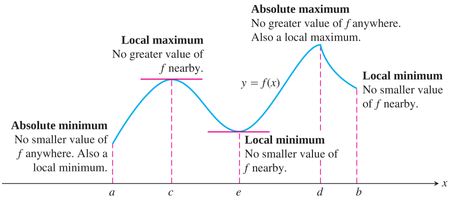
University of Galway

Tuesday, 21 October, 2025



Today, we'll max out on...

See also: Section 4.3 (Maxima and Minima) of **Calculus** by Strang & Herman: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Chap 4 : Applications of differentiation

Info: Survey, Assignments, etc

- The module survey for MA140 has started. Please take a few minutes to complete it. See https://universityofgalway.instructure.com/courses/46734/discussion_topics/189325
    - It only takes a few minutes
    - We take in the input seriously, and will update you on the main findings and the actions we will take.
    - Try to mix positive comments with suggestions for improvements.
- **Assignment 3** I added an extra 23 hours (for reasons...). Now due tomorrow (21st) at 17:00
- **Assignment 4** is open and is due next Tuesday (28th) at 17:00.
- **Assignment 5** will be posted soon.
- Grades for the class test have been posted. There were some updates/corrections. Grades are now final (I hope!). Answers have been posted to https://universityofgalway.instructure.com/courses/46734/files?preview=2948747

Higher-order Derivatives

At the end of the last class, we started learning about higher-order derivatives.

- if $f(x)$ is a function, then $f'(x)$ is a function whose value at x is the derivative of f at that point.
- So, since $f'(x)$ is a function, and we can differentiate functions, we can differentiate f' itself.
- The **derivative of the derivative** of f is called that **second derivative** of f.
- It is denoted as $f''(x)$ or $\frac{d^2y}{dx^2}$ or $f^{(2)}(x)$.
- We can continue this process to get third derivatives, fourth derivatives, etc, etc. However, the most important are the 1st and 2nd: f' and f'' provide valuable information about the function and its graph, particularly concerning local or global maxima, local/global minima and points of inflection.

$P(x) = x^n \quad P'(x) = n x^{n-1}$

Example

Find the **second** derivative of the functions

(i) $f_1(x) = 3x^2 + 2x + 1$

(ii) $f_2(x) = e^x$

(iii) $f_3(x) = \ln(x)$

(iv) $f_4(x) = \sin(x)$

(i) $f_1(x) = 3x^2 + 2x + 1$

$f_1'(x) = 3(2x) + 2$

$= 6x + 2$

$f_2''(x) = 6$

(ii) $f_2(x) = e^x$

$f_2'(x) = e^x \qquad f_2''(x) = e^x$

(iii) $f_3(x) = \ln(x)$ "Log"

$f_3'(x) = \frac{1}{x} = x^{-1}$

$f_3''(x) = (-1)x^{-2} = \frac{-1}{x^2}$

(iv) $f_4(x) = \sin(x)$

$f_4'(x) = \cos(x)$

$f_4''(x) = -\sin(x) = -f_4(x)$

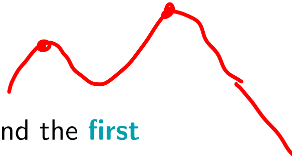
Maxima and minima

Overview

This section of MA140 is concerned with using techniques of differentiation to finding where a function is

- Increasing
- Decreasing
- Has its maximum value
- Has its minimum value

Section 4.3 of Text

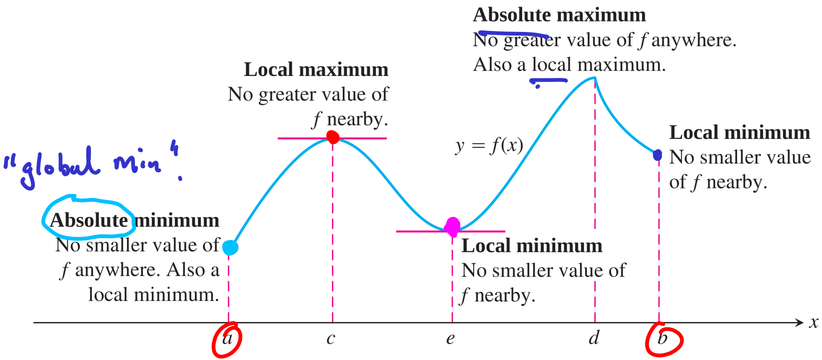


Along the way we'll learn about **critical values** and the **first derivative test**.

Mathematical English

- The plural of **maximum** is **maxima**;
- The plural of **minimum** is **minima**;
- An **extremum** a maximum or a minimum.
- The plural of **extremum** is **extrema**.

Given an interval $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$, consider the function $f : [a, b] \to \mathbb{R}$ whose graph is given below. It illustrates local and absolute (="global") maxima and minima. Collectively, these are called **extrema**.



Definition: critical points

Let c in an point in the domain of a function f. We say that $x = c$ is a **critical point** of $f(x)$ if either

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist.}$$

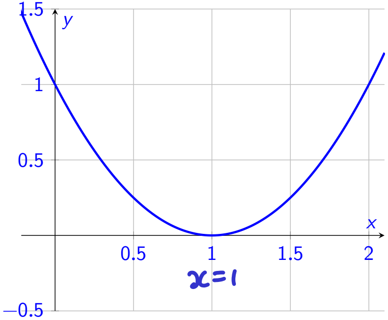
Important: If f has an extremum at $x = c$, then c must be a critical point of f (This is called "Fermat's Theorem").

So, to find a maximum or minimum of f, it is enough to check at the critical points.

Warning: All extrema are at critical points, but not all critical points correspond to a extrema.

Example

$f(x) = x^2 - 2x + 1$ has one critical point. Find it. Does it correspond to an extremum?



$f(x) = x^2 - 2x + 1$

$f'(x) = 2x - 2$

Solve for $f'(x) = 0$,

ie $2x - 2 = 0$

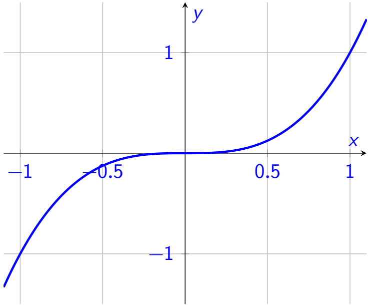
$\Rightarrow 2x = 2 \Rightarrow x = 1$

So there is a critical point at $x = 1$

We can observe this corresponds to a local (& global) minimum.

Example

Find all critical points of $f(x) = x^3$. Do they correspond to extrema?



$f(x) = x^3$

$f'(x) = 3x^2$

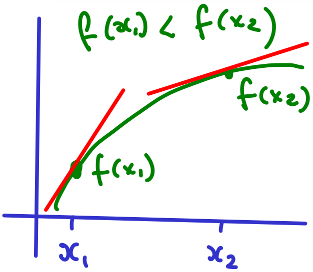
So $f'(x) = 0$ at $3x^2 = 0$ ie at $x = 0$. So there is a single critical point

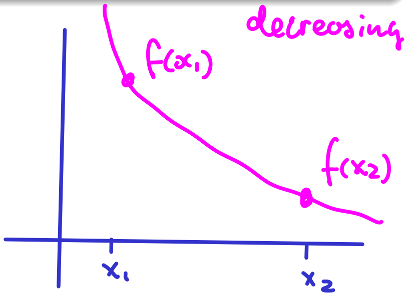
But we also notice: there is not an extremum at that point

Definition (Increasing/Decreasing)

Let f be a function whose domain includes the interval $[a, b]$. Let let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$.

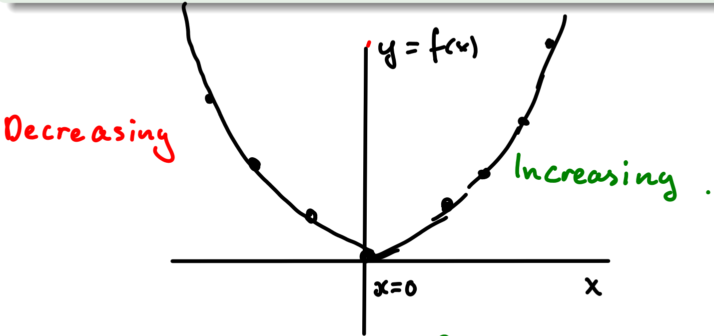
- If $\boxed{f(x_1) < f(x_2)}$, then f is said to be *increasing* on $[a, b]$.
- If $\boxed{f(x_1) > f(x_2)}$, then f is said to be *decreasing* on $[a, b]$.





Example

The function $f(x) = x^2$ is decreasing on $(-\infty, 0]$, and increasing on $[0, \infty)$.



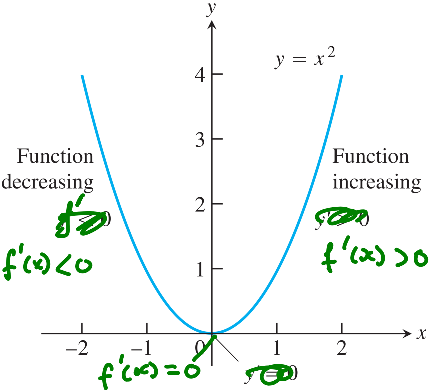
(Note also: $f'(x) = 2x \begin{cases} < 0 & x < 0 \\ > 0 & x > 0 \end{cases}$

The First Derivative Test

Theorem

Suppose that f is differentiable on an interval $[a, b]$.

- If $f'(x) > 0$ at each point $x \in [a, b]$, then f is increasing.
- If $f'(x) < 0$ at each point $x \in [a, b]$, then f is decreasing.



Example

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing

Idea: find $f'(x)$ and then solve for $f'(x) = 0$.

- find $f'(x)$:

$$f(x) = x^3 - 12x - 5$$
$$f'(x) = 3x^2 - 12.$$

- Find critical points : solve $f'(x) = 3x^2 - 12 = 0$

ie $3x^2 = 12 \Rightarrow x^2 = 4$ so $x = -2, 2$.

f has 2 critical points : at $x = -2$ and $x = 2$.

Example

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing

Idea: find $f'(x)$ and then solve for $f'(x) = 0$.

So we know $f'(x) = 3x^2 - 12$

We know $f'(-2) = 0$ $f'(2) = 0$

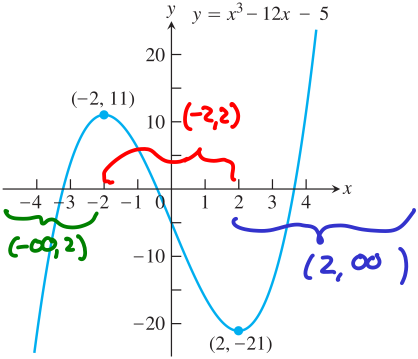
Check a point to the left of $x = -2$. Eg $x = -3$
$f'(-3) = 15 > 0$. So f is increasing to the left of $x = -2$

Similarly, check a point in $(-2, 2)$ eg $x = 0$
& a point to the right of $x = 2$, eg $x = 3$. .

The critical points $c = -2$ and $c = 2$ of $f(x) = x^3 - 12x - 5$ subdivide the domain of f into intervals $(-\infty, -2), (-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval.

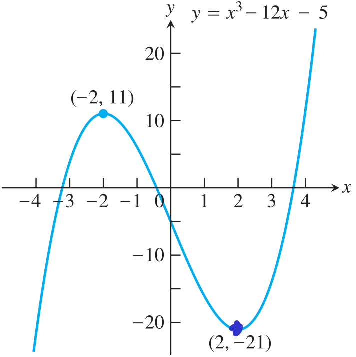
$$\begin{aligned} f'(x) &= 3x^2 - 12 \\ &= 3(x^2 - 4) \\ &= 3(x+2)(x-2) \end{aligned}$$

		-2	2
$3(x+2)$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$
	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$



The critical points $c = -2$ and $c = 2$ of $f(x) = x^3 - 12x - 5$ subdivide the domain of f into intervals $(-\infty, -2), (-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval.

		-2		2	
$3(x+2)$	$-$	•	$+$		$+$
$x-2$	$-$		$-$	•	$+$
$f'(x)$	$+$	•	$-$	•	$+$



The First Derivative Test

Important:

- If $f(x)$ has a local minimum of $f(x)$ at $x = c$, then it switches from **decreasing** to **increasing**. That means, $f'(x)$ changes sign at $x = 2$. Therefore, $f'(c) = 0$.
- If $f(x)$ has a local maximum at $x = c$, we have that $f'(c) = 0$.

The First Derivative Test

First Derivative Test for local maxima and minima

Suppose that c is a critical point of a differentiable function f.

- If f' changes from negative to positive through c, then f has a local minimum at c.
- If f' changes from positive to negative through c, then f has a local maximum at c.
- If f' does not change sign through c (that is, f' is positive on both sides of c or negative on both sides), then f does not have a local maximum or minimum at c.

ie an inflection point, as in $f(x) = x^3$

Example

Find the critical points of $f(x) = x^{\frac{1}{3}}(x-4)$. Identify the local maxima and minima (if any).

First find $f'(x)$, and then where it is either zero or undefined:

$f(x) = x^{1/3}(x-4). = u(x)\,v(x)$

Use the product rule

$u(x) = x^{1/3}$ $\qquad$ $v(x) = x - 4$

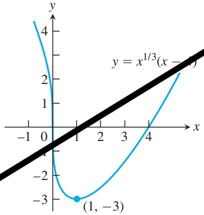
$\frac{du}{dx} = \frac{1}{3}x^{-2/3}$ $\qquad$ $\frac{dv}{dx} = 1.$

$f'(x) = \frac{df}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (x-4)\frac{1}{3x^{2/3}} + \frac{1}{x^3}$

So $f'(x) = \frac{[x-4]}{3x^{2/3}} - \frac{1}{x^3}.$

Bad example: ignore!

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$4(x-1)$	$-$	$-$	$+$
$3x^{2/3}$	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$+$



Bad example:
ignore!

Finished
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The First Derivative Test

Review

If a function g is differentiable on an interval $[a, b]$, then

- $g'(x) > 0$ for all $x \in [a, b] \Leftrightarrow g$ increasing on $[a, b]$.
- $g'(x) < 0$ for all $x \in [a, b] \Leftrightarrow g$ decreasing on $[a, b] I$.

Similarly, if g' is also differentiable on $[a, b]$, then

- $(g')'(x) = g''(x) > 0$ for all $x \in [a, b] \Leftrightarrow g'$ increasing on I.
- $(g')'(x) = g''(x) < 0$ for all $x \in [a, b] \Leftrightarrow g'$ decreasing on I.

Exercises

Exercise 6.1.1

Let $f(x) = x^2 e^x$. Find $f'(x)$, $f''(x)$ and $f'''(x)$.

Exercise 6.1.2 : 23/24 Exam, Q3(a)

Let $f(x) = \ln(x^2 + 1)$.

(i) Find all critical point(s) of f and determine whether f has a local minimum, local maximum or neither.

(ii) Determine the interval on which f is increasing.

(iii) Determine the interval on which f is decreasing.

(iv) Find all point(s) of inflection of f, justifying your answer.