MA385 Part 4: Linear Algebra 2

4.1: Vector Norms

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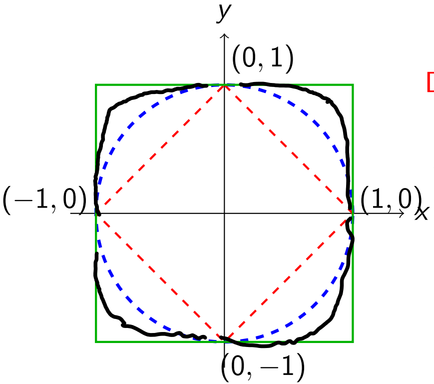


Diamond: $\|x\|_1 \leq 1$

Circle: $\|x\|_2 \leq 1$

Square: $\|x\|_\infty \leq 1$

1. Outline Section 4.1

1. Introduction
2. Three vector norms
3. $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$
4. $\|\cdot\|_2$ is a norm on $\mathbb{R}^n$
   - Cauchy-Schwarz
   - Applying C-S
5. Exercises

For more, see Section 2.7 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG

2. Introduction

This is the final section of MA385. It is kinda a direct continuation of Section 3 – and much of the material is from the same chapter as Section 3 in the text-book (though we'll also take some material from Chapter 5).

At its heart, is the task of bounding the eigenvalues and singular values of a matrix. Our motivation comes from doing an error analysis for LU-factorization. However, the applications are far more general than that.

2. Introduction

But for now, we'll just note that all computer implementations of algorithms that involve floating-point numbers (roughly, finite decimal approximations of real numbers) contain errors due to round-off error.

It transpires that computer implementations of LU-factorization, and related methods, lead to these round-off errors being greatly magnified: and we want to understand why.

2. Introduction

You might remember from earlier sections of the course that we had to assume functions where well-behaved in the sense that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L,$$

for some number L, so that our numerical schemes (e.g., fixed point iteration, Euler's method, etc) would work. If a function *doesn't* satisfy a condition like this, we say it is "ill-conditioned". One of the consequences is that a small error in the inputs gives a large error in the outputs.

We'd like to be able to express similar ideas about matrices: that $A(u - v) = Au - Av$ is not too "large" compared to $u - v$. To do this we used the notion of a "norm" to describing the relatives sizes of the vectors u and Au.

3. Three vector norms

When we want to consider the size of a real number, without regard to sign, we use the *absolute value*. Important properties of this function are:

1. $|x| \geq 0$ for all x.
2. $|x| = 0$ if and only if $x = 0$.
3. $|\lambda x| = |\lambda||x|$.
4. $|x + y| \leq |x| + |y|$ (triangle inequality).

This notion can be extended to vectors and matrices.

Definition 4.1.1

Let $\mathbb{R}^n$ be all the vectors of length n of real numbers. The function $\|\cdot\|$ is called a **norm** of $\mathbb{R}^n$ if, for all $u, v \in \mathbb{R}^n$

1. $\|v\| \geq 0$,
2. $\|v\| = 0$ if and only if $v = 0$.
3. $\|\lambda v\| = |\lambda|\|v\|$ for any $\lambda \in \mathbb{R}$,
4. $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).

Norms on vectors in $\mathbb{R}^n$ quantify the *size* of the vector. But there are different ways of doing this...

Eg $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\|x\| = 0$ then

$x = 0$ ie $x_1 = 0$, $x_2 = 0$

3. $\|-2x\| = 2\|x\|$

3. Three vector norms

Definition 4.1.2

Let $\mathrm{v} \in \mathbb{R}^n$: $\mathrm{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)^T$.

(i) The 1-*norm* (a.k.a. the *Taxi cab* norm) is

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|.$$

p-norm

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

(ii) The 2-*norm* (a.k.a. the *Euclidean norm*)

$$\|v\|_2 = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2}.$$

Note, if v is a vector in $\mathbb{R}^n$, then

$$\mathrm{v}^T\mathrm{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathrm{v}\|_2^2.$$

(iii) The ∞-*norm* (a.k.a. the max-*norm*) $\|v\|_\infty = \max_{i=1}^{n} |v_i|$.

Example: Compute the 1-, 2- and ∞-norm of $v = (-2, 4, -4)^T$

$$v = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$$

$$\|v\|_1 = |v_1| + |v_2| + |v_3| = |-2| + |4| + |-4|$$
$$= \underline{\underline{10}}.$$

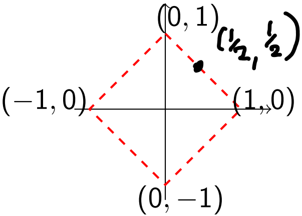
$$\|v\|_2 = \sqrt{(-2)^2 + 4^2 + (-4)^2} = \sqrt{4 + 16 + 16} = 6.$$

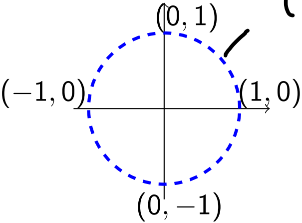
$$\|v\|_\infty = \max\{|-2|, |4|, |-4|\} = 4.$$

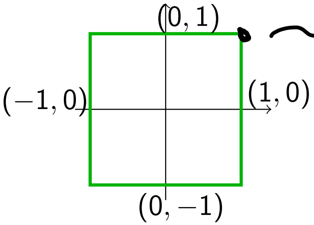
$\left\| \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\|_1 = 1.$

The unit balls in $\mathbb{R}^2$ given by $\|\cdot\|_1$ (top left), $\|x\|_2 = \sqrt{x_1^2 + x_2^2} = 1$ (top right), and $\|\cdot\|_\infty$.

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$







$\sim (1, 0)$

3. Three vector norms

It is easy to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms (see next slide).

And it is not hard to show that $\|\cdot\|_2$ satisfies conditions (1), (2) and (3) of Definition 4.1.1.
It takes a little bit of effort to show that $\|\cdot\|_2$ satisfies the triangle inequality; so we'll do that with care.

4. $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$

$$\|u\|_\infty := \max_{i=1,\dots,n} |u_i| \quad \text{where} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix}$$

Eg $n=3$

① since $\|u\|_\infty = \max\{|u_1|, |u_2|, |u_3|\}$

then $\|u\|_\infty \geqslant 0$ since each $|u_i| \geq 0$

② If $\|u\|_\infty = 0$ then each u_i is such that $|u_i| \leq 0$, ie $u_i = 0$

③ $\|\lambda u\|_\infty = \max\{|\lambda u_1| + |\lambda u_2| + |\lambda u_3|\}$

$= |\lambda| \max\{|u_1|, |u_2|, |u_3|\}$

4.

$$\|u+v\|_\infty =$$

$$\max\{|u_1+v_1|,\ |u_2+v_2|,\ |u_3+v_3|\}$$

$$\leq \max\{|u_1|+|v_1|,\ |u_2+v_2|,\ |u_3+v_3|\}$$

$$\leq \max\{|u_1|, |u_2|, |u_3|\} + \max\{|v_1|, |v_2|, |v_3|\}$$

$$= \|u\|_\infty + \|v\|_\infty.$$

As mentioned, it takes a little effort to show that $\|\cdot\|_2$ is indeed a norm on $\mathbb{R}^2$; in particular to show that it satisfies the triangle inequality, we need the Cauchy-Schwarz inequality.

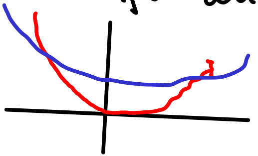
Lemma (Cauchy-Schwarz)

$$\left|\sum_{i=1}^{n} u_i v_i\right| \leq \|\mathrm{u}\|_2 \|\mathrm{v}\|_2, \qquad \forall \mathrm{u}, \mathrm{v} \in \mathbb{R}^n.$$

Idea: $0 \leq \|\lambda u + v\|_2^2$.

First: Suppose $ax^2 + bx + c \geq 0$ for all x.

Its roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$



So $b^2 - 4ac \leq 0$

So $b^2 \leq 4ac$.

For any $\lambda \in \mathbb{R}$, $u, v \in \mathbb{R}^n$ we have

$$0 \le \|\lambda u + v\|_2^2 \qquad (\text{since } \|x\|_2 \ge 0 \ \forall x)$$

$$\Rightarrow 0 \le \sum_{i=1}^{n} (\lambda u_i + v_i)^2 = \sum_{i=1}^{n} \left(\lambda^2 u_i^2 + 2\lambda u_i v_i + v_i^2\right).$$

$$\Rightarrow \lambda^2 \underbrace{\sum_{i=1}^{n} u_i^2}_{a} + \lambda \underbrace{2 \sum_{i=1}^{n} u_i v_i}_{b} + \underbrace{\sum_{i=1}^{n} v_i^2}_{c} \ge 0.$$

$(b^2 \le 4ac)$

$$\left(2 \sum u_i v_i\right)^2 \le 4 \|u\|_2^2 \|v\|_2^2$$

since $\left(\sum u_i u_i\right)^2 = \left(\left|\sum u_i v_i\right|\right)^2$ we get C.S.

Example: Pick two vectors in $\mathbb{R}^3$ and convince yourself they satisfy the Cauchy-Schwarz Inequality.

$$u = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix} \qquad v = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\left| \sum_{i=1}^{3} u_i v_i \right| = (-2)(-1) + (4)(1) + (-4)(-1)$$

$$= 2 + 4 + 4 = 10.$$

$$\|u\|_2 = 6 \qquad \|v\|_2 = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$= 1.7\ldots$$

So $\|u\|_2 \cdot \|v\|_2 = (6)(1.7\ldots) = 10.392\ldots$

So $\left| \sum u_i v_i \right| \leq \|u\|_2 \|v\|_2.$

Now can now apply Cauchy-Schwatz to show that

$$\|\mathrm{u}+\mathrm{v}\|_2 \leq \|\mathrm{u}\|_2 + \|\mathrm{v}\|_2.$$

This is because

$$\|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2$$
$$= (\|u\| + \|v\|)^2$$

$$\begin{aligned}
\|u+v\|_2^2 &= (u+v)^T(u+v) \\
&= u^T u + 2u^T v + v^T v \\
&\leq u^T u + 2|u^T v| + v^T v \quad \text{(by the triangle-inequality)} \\
&\leq u^T u + 2\|u\|\|v\| + v^T v \quad \text{(by Cauchy-Schwarz)} \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

It follows directly that

Corollary

$\|\cdot\|_2$ *is a norm.*

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