2323-MA378: Class Test in Week 7 (Friday, 24 Feb) ANS (with solutions)

The following fact (Cauchy's theorem) may be useful in answering some of these questions. *Let p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a,b]$ there is a $\tau \in (a,b)$ such that*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \tag{1}$$

where $\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ denotes the nodal polynomial. In addition, if S is the cubic spline interpolant the function f at N equally spaced points $\{a = x_0 < x_1 < \cdots < x_N = b\}$ with $x_i - x_{i-1} = (b-a)/N =: h$, then

$$\|f - S\|_\infty := \max_{a \le x \le b} |f(x) - S(x)| \le \frac{5h^4}{384} \max_{a \le x \le b} |f^{(4)}(x)|. \tag{2}$$

In all the questions below, the function f is $f(x) = (x^2 - 1)e^x$.

Q1. (40 marks)

(a) Write down the Lagrange form for the polynomial, $p_2(x)$, that interpolates f at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

Answer: [15 Marks] The Lagrange for an interpolant of degree 2 to f is

$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

For this problem

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{x(x-1)}{(-1)(-2)} = \frac{1}{2}x(x-1),$$
$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x+1)(x-1)}{(1)(-1)} = -(x+1)(x-1),$$
$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x+1)x}{(2)(1)} = \frac{1}{2}(x+1)x.$$

Using that $f(x_0) = f(-1) = 0$, $f(x_1) = f(0) = -1$ and $f(x_2) = f(1) = 0$, we get that the Lagrange form is

$$p_2(x) = (-1)(-(x+1)(x-1)) = x^2 - 1.$$

Notes: Since $L_0(x)$ and $L_2(x)$ are not needed you would get full marks even if you did not write them down. Similarly, simplifying $p_2(x) = x$ is not required.

(b) Evaluate $p_2(1/2)$. What is the exact value of $|f(1/2) - p_2(1/2)|$?
ANS [5 Marks] $|f(1/2) - p_2(1/2)| = |-3e^{1/2}/4 - (-3/4)| = |3/4(1 - e^{1/2})| \approx 0.48654$.

(c) What bound does (1) give for $|f(1/2) - p_2(1/2)|$?

Answer: [15 Marks] Equation (1) gives $f(1/2) - p_2(1/2) = \dfrac{f'''(\tau)}{3!}\pi_3(1/2)$, for some (unknown) $\tau \in (-1,1)$. So the bound on $|f(1/2) - p_2(1/2)|$ is

$$|f(1/2) - p_2(1/2)| \le \frac{\max_{-1 \le x \le 1} |f'''(x)|}{3!} |\pi_3(1/2)|.$$

Differentiating f, we get $f'''(x) = e^x(x^2 + 6x + 5)$. Since both e^x and $x^2 + 6x + 5$ are positive functions that are increasing on $[-1,1]$, we get

$$\max_{-1 \le x \le 1} |f'''(x)| = f'''(1) = 12e.$$

Also, $1/(3!) = 1/6$ and $\pi_3(x) = -3/8$. So

$$|f(1/2) - p_2(1/2)| \le \frac{12e}{6}\frac{3}{8} = 3e/4 \approx 2.3781.$$

(d) How do you account for the discrepency between the answers in Parts (b) and (c)?

Answer: [5 Marks] The solution in (b) is exact, but the one in (c) is an inexact upper bound. It is inexact because there is no way of knowning what value of τ to use in (1), so we take the worst possible case.

Q2. (40 marks)

(a) Give a formula for the piecewise linear interpolant, $l(x)$, that interpolates f, at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

Answer: [15 Marks]

$$l(x) = \begin{cases} f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} & x_0 \le x \le x_1 \\ f(x_1)\frac{x-x_2}{x_1-x_2} + f(x_2)\frac{x-x_1}{x_2-x_1} & x_1 < x \le x_2 \\ 0 & \text{otherwise} \end{cases}$$

Using that $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $f(x_0) = 0$, $f(x_1) = -1$ and $f(x_2) = 0$, this simplifies as

$$l(x) = \begin{cases} (-1)\frac{x-(-1)}{0-(-1)} & -1 \le x \le 0 \\ (-1)\frac{x-1}{0-1} & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} -x-1 & -1 \le x \le 0 \\ x-1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Evaluate $l(1/2)$. What is the exact value of $|f(x) - l(x)|$ for $x = 1/2$?

ANS [5 Marks] $|f(1/2) - l(1/2)| = |-(3/4)e^{1/2} - (-1/2)| \approx 0.73654$.

(c) Use (1) to give an upper bound for $|f(x) - l(x)|$ **at** $x = 1/2$.

Answer: [15 Marks] To use (1) we need to recognise that we are seeking the error in the p_1 where that is the linear interpolant to f at the points $x = 0$ and $x = 1$. That is, on $[0,1]$, $f(x) - l(1/2) = \dfrac{f''(\tau)}{2!}(x-0)(x-1)$, for some (unknown) $\tau \in (0,1)$. Thus the bound on $|f(1/2) - l(1/2)|$ is

$$|f(1/2) - l(1/2)| \le \frac{\max_{0\le x\le 1} |f''(x)|}{2!} |(1/2)(-1/2)|.$$

Differentiating f, we get $f''(x) = e^x(x^2 + 4x + 1)$. Since both e^x and $x^2 + 4x + 1$ are positive functions that are increasing on $[0,1]$, we get

$$\max_{0\le x\le 1} |f''(x)| = f''(1) = 6e.$$

So

$$|f(1/2) - l(1/2)| \le \frac{6e}{2}\frac{1}{4} = 3e/4 \approx 2.03871.$$

(d) How do you account for the discrepency between the answers in Parts (b) and (c)?

Answer: [5 Marks] The solution in (b) is exact, but the one in (c) is an inexact upper bound. It is inexact because there is no way of knowning what value of τ to use in (1), so we take the worst possible case.

Q3. (20 marks) Suppose that S is the cubic spline interpolant the function f at the $N+1$ equally spaced points $\{x_0 = -1 < x_1 < \cdots < x_N = 1\}$. What value of N should one take to ensure that $\|f - S\|_\infty$ is no more than 10^{-6}?

Answer: We'll use (2):

$$\|f - S\|_\infty := \max_{-1\le x\le 1} |f(x) - S(x)| \le \frac{5h^4}{384} \max_{-1\le x\le 1} |f^{(4)}(x)|.$$

We wish to choose h so that $\frac{5h^4}{384} M_4 \le 10^{-6}$, where $M_4 := \max_{-1\le x\le 1} |f^{(4)}(x)| \le 10^{-6}$. That is, we need h to satisfy $h^4 \le \frac{384}{5M_4} 10^{-6}$. To compute M_4, calculate the 4th derivative of f, finding that $f^{(4)}(x) = e^x(x^2 + 8x + 11)$. Since this is a positive, increasing function (because it is the product of positive, increasing functions), we get that $M_4 = f(1) = 20e$. So now we know that we need h to satisfy $h \le (\frac{384}{100e} 10^{-6})^{1/4} \approx (1.41266 \times 10^{-6})^{1/4} \approx 3.4475 \times 10^{-2}$. Then, since $h = (1 - (-1)/N$, we get than N must be at least 58.0124. However, since N is an integer, we should take $N \ge 59$.