MA313 : Linear Algebra I

Week 3: The span of a set; the null space of a matrix

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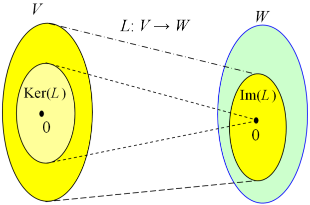
20 and 23 September, 2022



https://commons.wikimedia.org/wiki/File:KerIm_2015Joz_L2.png.

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details,

- LinAlg for Data Science: Chapter 7 for Linear Independence and Span
- Lay et al: Sections 4.1 and 4.2.

Assignment 1

Deadline is Tuesday, 20 Sept at 5pm.

Assignment 2

- Opened Monday, 19 Sep 2022.
- **Deadline:** 5pm, Friday 30 Sep 2022.
- It contributes 5% to the final grade for MA313.
- Topics: ...

Communication Skills

1. Topics and Info posted on Blackboard. Also at https://www.niallmadden.ie/teaching/2223-MA313/22_23_Communication_Skills.pdf
2. Select one that is not crossed out, or propose one of your own.
3. Confirm your topic by this Friday (23 September); do that by first emailing Niall with your choice and, if agreed, entering in on Blackboard.

Tutorials start this week.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12					
12 – 1				Tutorial IT206	Lecture
1 – 2		Lecture			
2 – 3					
3 – 4					
4 – 5					

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Start of ...

PART 1: Linear combinations

Part 1: Linear combinations

A question

Last week we learned how to check if a given space is indeed a subspace of some other vector space.

It is natural to wonder: *how can we make those subspaces in the first place?*

Equivalently: *How can we describe all subspaces of a given vector space?*

Part 1: Linear combinations

Example (Subspaces of $\mathbb{R}^2$)

There are precisely three *types* of subspaces of $\mathbb{R}^2$:

- $\{0\}$,
- $\mathbb{R}^2$,
- lines through the origin.

How we build subspaces?

There are two possible approaches.

- **Top down:** start with the full space, and look at all vectors that have “suitable properties”.
- **Bottom up:** start with some collection of vectors and consider the subspace that they “span”.

Definition (Linear combinations)

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form

$$c_1 u_1 + \cdots + c_p u_p$$

for scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

Example

In $\mathbb{R}^2$, $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example

Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is **not** linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -6 \end{bmatrix}$ in $\mathbb{R}^2$.

Example (Quadratic polynomials)

Which vectors in $\mathbb{P}_2$ (over t) are linear combinations of the vectors $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$?

Example (Polynomials again)

Which vectors in $\mathbb{P}_2$ (over t) are linear combinations of the vectors $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = 2t$?

Example

Define the 2×3 matrix

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

For any vector

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

the vector Ax is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

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Start of ...

PART 2: Spans

Part 2: Spans

Definition (SPAN)

Given vectors $u_1, \ldots, u_p$ in some vector space V, their **span** is

$$\operatorname{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

In other words, $\operatorname{span}\{u_1, \ldots, u_p\}$ is the set of all linear combinations of $u_1, \ldots, u_p$ within V.

Part 2: Spans

Theorem

$\mathrm{span}\{u_1, \ldots, u_p\}$ *is a subspace of* V*.*

In fact, more than this is true: one can show that $\mathrm{span}\{u_1, \ldots, u_p\}$ is the "smallest" subspace of V which contains each of $u_1, \ldots, u_p$.

Part 2: Spans

Immediate consequences

- Every choice of vectors $u_1, \ldots, u_p$ provides us with an example of a subspace of V.
  (However, ***different*** sequences of vectors may well span the ***same*** subspace!)
- If we can show a *subset* of V is the a **span of some set of vectors**, then we we have shown it is a subspace!

Example

Show that $$H = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$ is a subspace of $\mathbb{R}^4$.

Example (From 2018/2019 exam paper)

Find vectors $u, v, w \in V$ with $V = \mathrm{span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$.

Example: Care is required!

Is $\quad H = \left\{ \begin{bmatrix} 3s \\ 2+5s \end{bmatrix} : s \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$.

We now know that the span of any subset of vectors in a vectors space is itself a subspace (and, so, is a vector space). But...

Question

Is every subspace the span of some (collection of) vectors?

We'll answer that question over the next week or so.

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Start of ...

PART 3: Null spaces

Part 3: Null spaces

The big idea...

There are **two main ways of building f subspaces**:

- Spans of vectors (“bottom up”).
- **Kernels** and **null spaces** of **linear transformations** (“top down”).

The null space generalise sets of solutions to homogeneous systems of linear equations, which we’ll look at now.

Part 3: Null spaces

Definition (NULL SPACE)

Let A be an $m \times n$ matrix. The **null space** of A is

$$\operatorname{Nul} A = \{x \in \mathbb{R}^n : Ax = 0\}.$$

Earlier, we did an example that showed that when we multiply a matrix by a vector, we are making a linear combination of the columns of A.

That is, for a matrix $A = [a_1 \cdots a_n]$ with columns $a_1, \ldots, a_n \in \mathbb{R}^m$ and a vector $x \in \mathbb{R}^n$, we have

$$Ax = x_1 a_1 + \cdots + x_n a_n.$$

Part 3: Null spaces

Example

Let

$$A = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 2 & -4 \end{bmatrix}, \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$x \in \operatorname{Nul} A \qquad \text{but} \qquad y \notin \operatorname{Nul} A.$$

Theorem

Let A be an $m \times n$ matrix.
Then $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

This follows from familiar properties of matrix multiplication.

1. $A\mathbf{0} = \mathbf{0}$
2. $A(x + y) = Ax + Ay$ and
3. $A(cx) = c(Ax)$

In some cases, we want to compute vectors in $\operatorname{Nul} A$. However,

- Given a matrix A, it is very easy to test if a given vector x belongs to $\operatorname{Nul} A$.
- But how can we find non-zero vectors in $\operatorname{Nul} A$ or prove that none exist? (In the text-book, this is called "Finding an explicit description of $\operatorname{Nul} A$").

This should not be too surprising. We are, essentially, solving $Ax = \mathbf{0}$. And it is easier to check if a vector is a solution to a system of equations, then to find that solution.

But, also, some linear systems are much easier to solve than others. [See next examples]

Example (Some "easy" cases)

Find a vector, other than the zero vector, in the null space of each of the following, or show it does not exist.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercises

Q1. Let $u = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$.

(a) Is $w = \begin{bmatrix} 16 \\ -24 \end{bmatrix}$ a linear combination of u and v?

(b) Is $x = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ a linear combination of u and v?

Q2. (a) Determine if $\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \in \mathrm{Nul} \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$.

(b) Determine if $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \mathrm{Nul} \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}$.

Exercises

Q3. Construct a finite spanning set of each of the null space of each of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.