Annotated slides

2425-MA140 Engineering Calculus

Week 04, Lecture 1
Introduction to Derivatives

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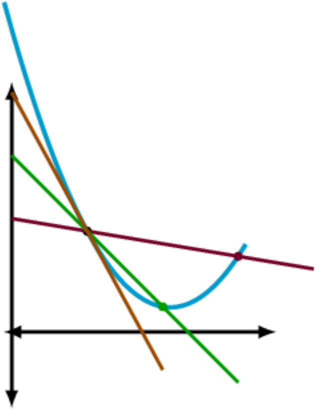


Assignments, etc

Assignment 2

- **Assignment 2** is open. See https://universityofgalway.instructure.com/courses/35693/assignments/96620.
  Deadline is 5pm, Friday, 11 October.
- The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2065926

What we'll study today

1. Derivative: the concept
   - Rate of change
2. Derivative at a point
   - The definition
   - Examples
3. Derivative as a function
4. Exercises

further reading:

- Section 8.1 of *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517
- Sections 3.1 and 3.2 of **Calculus** by Strang & Herman: https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative
- Nice animation: https://www.geogebra.org/m/MeMdCUEm

Derivative: the concept

The **derivative** of a function describes how quickly the function is changing.

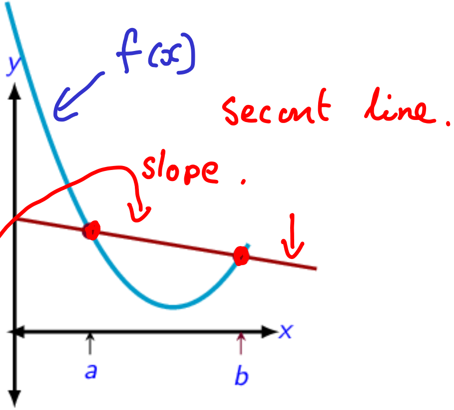
There are many, many applications: derivatives, and equations involving them are used everywhere: **speed/velocity** is the rate of change of displacement; **acceleration** is the rate of change of velocity.

We use derivatives to model how quickly a tumour is growing or shrinking, how pollutants are dispersed in a river, how pressure changes with depth, how inflation is changing in an economy. The list of applications is practically limitless.

Consider the graph opposite. It shows a function, f, and a secant line that intersects f at $a = 1$ and $b = a + 2$ (the actually values are not important).

If we wanted to summarised how f is changing between those two values, we could compute it as

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+2) - f(a)}{2}$$



Change in f is $f(x+2) - f(x)$.

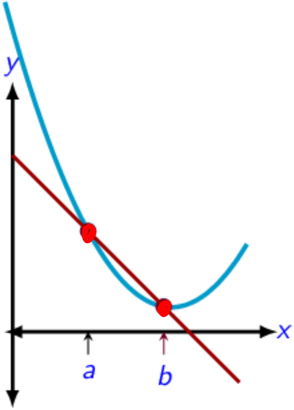
Rate of change is $\dfrac{\text{change in } f}{\text{change in } x}$.

Now we'll consider how f is changing over a shorter interval: from a to $b = a + 1$. Again, we sketch the secant line that intersects f at $x = a$ and $x = b$.

The rate of change of f between these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a+1) - f(a)}{1},$$

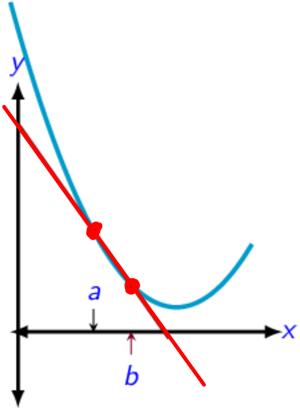
which, of course, is the slope of the secant line.



Next we shorter interval again: looking at how f changes from a to $b = a + \frac{1}{2}$, along with the secant line that intersects f at $x = a$ and $x = b$.

The rate of change of f between these two values is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + \frac{1}{2}) - f(a)}{\frac{1}{2}}.$$

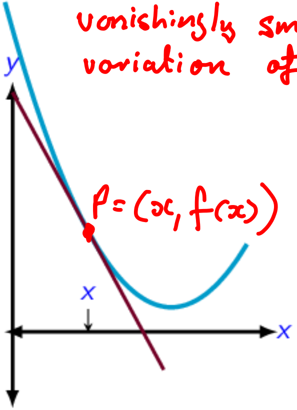


ie change over a vanishingly small variation of x.

Finally, suppose we want to looking at the **instantaneous** rate of change of f at $x = a$.

Hopefully, the preceding images have convinced you we could do this in two (equivalent) ways:

1. $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
2. or as the slope of the tangent to f at $x = a$.



The slope of the curve $y = f(x)$ at the point $P = (a, f(a))$ is given by the number (if it exists)

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Read "$f'(a)$" as "f prime of a", or just "derivative of f".

If this limit exists, it is called **the derivative of f at $x = a$** and we denote it by $f'(a)$.

Definition: derivative at a point

Let $f(x)$ be a function that has $x = a$ in its domain. The **derivative** of the function $f(x)$ at a, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

Some terminology

$f'(a)$ exists then we say that function f is **differentiable at $x = a$**.

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the "limit" formula, we are doing "**differentiation from first principles"**.

Example

Use the limit definition of a derivative to compute the slope of the tangent to $f(x) = x^2$ at $x = 3$.

We want to compute

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} = \lim_{h\to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h\to 0} (6+h) = 6.$$

$$\boxed{\text{Answer:} \quad f'(3) = 6.}$$

Example

Use the limit definition of a derivative to find the equation of the tangent to $f(x) = 1/x$ at $x = 2$.

To start we need to find the slope of the tangent to $f(x) = \frac{1}{x}$ at $x = 2$.

It is $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$.

$$= \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{1}{2+h} - \frac{1}{2}\right) = \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{2}{2(2+h)} - \frac{2+h}{2(2+h)}\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{\cancel{h}}\right)\left(\frac{-\cancel{h}}{2(2+h)}\right) = \lim_{h \to 0} \frac{-1}{4+2h} = -1/4$$

Example

Use the limit definition of a derivative to find the equation of the tangent to $f(x) = 1/x$ at $x = 2$.

So the slope is $m = 1/4$. Recall the equation for a line through (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$

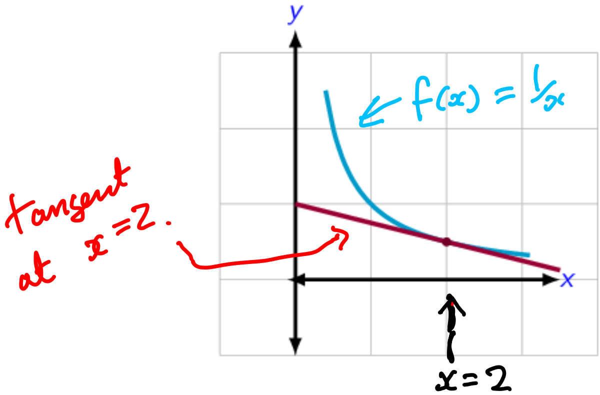
$\Rightarrow \quad y = y_1 + m(x - x_1)$

Since we want the tangent to f at $(2, f(2))$, take $x_1 = 2$ and $y_1 = 1/2$

So $y = 1/2 - (1/4)(x - 2)$.

Simplify to get . . . . .

$$\boxed{f(x) = \frac{1}{x}} \text{ and } \boxed{y = 1 - \frac{x}{4}}$$



Derivative as a function

We've seen how to compute $f'(a)$: the derivative of the function f at a given point, $x = a$.

But if $f'(a)$ has a value for all $x = a$ (in the domain of $f(x)$), we can think $f'(x)$ as a function itself!

Definition: derivative as a function

Let f be a function. The derivative function, denoted f'. is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative as a function

Terminology and notation

- We usually refer to f' simply as the derivative of $f(x)$.
- Where $y = f(x)$, we often we write f' as $\frac{dy}{dx}$, or y', or $\frac{d}{dx}(f)(x)$.

In Physics, if $f = f(t)$ where t is time, often the notation is $\dot{f}(t)$

 "From first Principles"

Example

Use the above definition to find the derivative of $f(x) = x^2$.

Solution

The derivative is defined as follows:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}.$$

Here $f(x+h) = (x+h)^2 = x^2 + h^2 + 2hx$, so we get:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(\cancel{x^2} + h^2 + 2hx) - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+2x)}{\cancel{h}} = \lim_{h\to 0} (h+2x) = 2x$$

Example

Use the "limit" definition to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Derivative as a function

Consider the absolute value function $f(x) = |x|$. What is its derivative at (i) $x = 2$, (ii) $x = -3$, or (iii) $x = 0$?

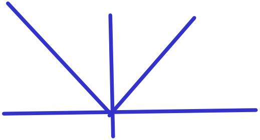
Recall $f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0. \end{cases}$

(i)

So $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{2+h-2}{h} = 1$

(ii)

$\lim_{h \to 0} \dfrac{f(-3+h) - f(-3)}{h} = \lim_{h \to 0} \dfrac{3-h-3}{h} = -1.$



(iii) Check that $\lim_{x \to 0^-} f(x) = -1$, but $\lim_{x \to 0^+} f(x) = 1$

So $f'(0)$ does not exist.

We didn't do this one in class, but the full details are here.

Show that $\dfrac{d}{dx}(\sin x) = \cos x$.

Solution: We need to evaluate

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h},$$

where $f(x) = \sin(x)$. From p5 of the "log" tables, we have that $\sin(A) - \sin(B) = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$.
Here $A = x+h$, and $B = x$, so
$\sin(x+h) - \sin(x) = 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$.
So now we evaluate

$$f'(x) = \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \lim_{h\to 0} \frac{2}{h}\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right).$$

Derivative as a function

We didn't do this one in class, but the full details are here.

But

$$\lim_{h\to 0} \frac{2}{h}\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right) = \left(\lim_{h\to 0} \frac{2}{h}\sin\left(\frac{h}{2}\right)\right)\left(\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right).$$

We learned last week that,

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Taking $\theta = h/2$, we get that

$$\lim_{h\to 0} \frac{2}{h}\sin\left(\frac{h}{2}\right) = 1.$$

And finally,

$$\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right) = \cos(x).$$

and we are done!

Exercises

Exercises 4.1.1 (Based on Q2(a), 2019/2020)

Use the (limit) definition of a derivative to differentiate the function $f(x) = x^2 + 2$.

Exercise 4.1.2

Use the (limit) definition of a derivative to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.