MA313 : Linear Algebra I

Week 5: Linear Independence and Bases

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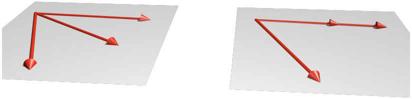
4th and 7th of October, 2022



"Linearly dependent vectors in $\mathbb{R}^3$ - 3D Visualisation"

https://commons.wikimedia.org/wiki/File:Vec-dep.png

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details, see

- Chapter 7 (Linear Independence) of Linear Algebra for Data Science: `https://shainarace.github.io/LinearAlgebra/linind.html`
- Section 4.3 of the Lay: `https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=5174425`

Assignment 3

- Opened on Monday (03 Oct, 2022).
- **Deadline:** 5pm, Monday 17 Oct 2022.
- It contributes 6% to the final grade for MA313.
- Tutorials continue Thursdays at 12 in IT206.

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Start of ...

PART 1: Recall...

1: Recall...

Linear combinations

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form $c_1 u_1 + \cdots + c_p u_p$ for scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

Span

The **span** of a set of vectors is the set of all possible liner combinations of them.
Given any set of a vectors in a vector space V, their span is a **subspace** of V.

1: Recall...

Null space

The **null space** of a $m \times n$ matrix, A, the set of all vectors in $\mathbb{R}^n$ for which $Ax = 0$.

Spanning Set

A **spanning set** of a vector space V is a collection of vectors in V whose span is all of V.

1: Recall...

Column space

There are three equivalent definitions of the **column space** of a $m \times n$ matrix, A.

- It is the set of all linear combinations of the vectors that make up the columns of A.
- It is the space spanned by the vectors that make up the columns of A.
- It is the set of all vectors in $\mathbb{R}^m$ that can be written as Ax for some vector $x \in \mathbb{R}^n$.

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Start of ...

PART 2: Linear Transformations

Definition (LINEAR TRANSFORMATIONS)

Let V and W be vector spaces. A **linear transformation** from V to W is a function $T\colon V \to W$ (i.e., a "rule" which assigns a unique $T(u) \in W$ to each $u \in V$) such that

- $T(u+v) = T(u) + T(v)$ for all $u, v \in V$ and
- $T(cu) = cT(u)$ for all $u \in V$ and $c \in \mathbb{R}$.

That is, a linear transformations is a function which "respects" (or "is compatible with") the vector space structures.

Example 1.

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Example 2.

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

Example 3.

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Example 4.

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + 2 \end{bmatrix}$$

Example

The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

defines a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

An important fact

Linear transformations preserve linear combinations: if $T\colon V \to W$ is a linear transformation, then

$$T(c_1 v_1 + \cdots + c_p v_p) = c_1 T(v_1) + \cdots + c_p T(v_p)$$

for all $v_1, \ldots, v_p \in V$ and $c_1, \ldots, c_p \in \mathbb{R}$.

Example (Matrices)

Let A be an $m \times n$ matrix. Define $T\colon \mathbb{R}^n \to \mathbb{R}^m$ via

$$T(x) = Ax \qquad (x \in \mathbb{R}^n).$$

Then T is a linear transformation.

Question

Are there any other linear transformations $\mathbb{R}^n \to \mathbb{R}^m$?

Answer: No! Linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ and $m \times n$ matrices are essentially the "same thing". What we mean is,

- Every $m \times n$ matrix defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can find a matrix that defines it.

The matrix of a linear transformation

Let e_i be the usual vector in $\mathbb{R}^n$ with 1 is row i, and zero everywhere else. Then the matrix for a given linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$A := [T(e_1) \cdots T(e_n)].$$

Example 1 from earlier.

Find the matrix of the *linear transformation*.

$$T_1\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_1-2x_2\\x_1+3x_2\end{bmatrix}$$

Example 3 from earlier

Find the matrix of the *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Since linear transformations are generalizations of matrices, we need the analogous idea of **null spaces** and **column spaces**.

Definition (KERNEL and RANGE of a linear transformation)

Let $T\colon V \to W$ be a linear transformation.

- The **kernel** of T is $\mathrm{Ker}\, T = \{u \in V : T(u) = 0\}$.
- The **range** (or *image*) of T is $\mathrm{Ran}\, T = \{\, T(u) : \; u \in V\}$.

Example

Let A be an $m \times n$ matrix. Let $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Ax$ be the associated linear transformation. Then:

- $\mathrm{Ker}\, T = \{x \in \mathbb{R}^n : T(x) = Ax = 0\} = \mathrm{Nul}\, A$.
- $\mathrm{Ran}\, T = \{T(x) = Ax : x \in \mathbb{R}^n\} = \mathrm{Col}\, A$.

Theorem

Let $T\colon V \to W$ be a linear transformation. Then:

- $\operatorname{Ker} T$ *is a subspace of* V.
- $\operatorname{Ran} T$ *is a subspace of* W.

Here is another result, though the importance might not be clear yet.

Theorem

Let V be a vector space and let $H \subseteq V$ be a subspace.
Then there are vector spaces U and W and linear transformations $S\colon U \to V$ and $T : V \to W$ such that

$$\operatorname{Ran} S = H = \operatorname{Ker} T.$$

We essentially get S for free...

But some new ideas would be required to produce T.

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Start of ...

PART 3: Linear independence of vectors

3: Linear Independence

Definition (LINEARLY INDEPENDENT)

Let V be a vector space and let $v_1, \ldots, v_p \in V$. We say that $v_1, \ldots, v_p$ are **linearly independent** if the equation

$$c_1 v_1 + \cdots + c_p v_p = 0$$

for $c_1, \ldots, c_p \in \mathbb{R}$ only has the "trivial solution"

$$c_1 = \ldots = c_p = 0.$$

In other words, the only linear combination of linear independent vectors that gives zero is the boring one.

Definition (LINEARLY DEPENDENT)

If a set of vectors is *not* linear independent, we say they are *linearly dependent*.

3: Linear Independence

Example (Example in $\mathbb{R}^3$: 1)

The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are linearly independent.

Example (Example in $\mathbb{R}^3$: 2)

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ are **not** linearly independent.

3: Linear Independence

Example (Slightly more complicated)

- A single vector $v \in V$ is linearly independent if and only if $v \neq 0$.
- No collection of vectors containing the zero vector is linearly independent.
- Let $u, v \in V$ with $u \neq 0 \neq v$. Then u, v are linearly independent if and only if $u \neq cv$ for any $c \in \mathbb{R}$.

Theorem

Let $v_1, \ldots, v_p \in V$. Then $v_1, \ldots, v_p$ are linearly dependent (= not linearly independent) ***if and only if*** *there exists an index j such that v_j is a linear combination of $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_p$.*

Example

Recall that $\mathbb{P}_n$ is the vector space of all polynomials $p(t)$ of degree at most n and that $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n$ is the vector space of all polynomials $p(t)$ (without any degree constraints).

Let $\quad p_1(t) = 1, \quad p_2(t) = 2t, \quad p_3(t) = 4 - 3t.$

Are $p_1(t), p_2(t), p_3(t)$ linearly independent in $\mathbb{P}$?

Example: Q2(a) 2021/22 exam paper

Determine if polynomials $p_1(t) = 1 - 2t$, $p_2(t) = 3 + 4t$, and $p_3(t) = 5$, are linearly independent in $\mathbb{P}_1$.

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PART 4: Bases

4: Bases

Definition (BASIS / BASES)

A sequence of vectors $(v_1, \ldots, v_p)$ in some vector space V is a **basis** for V if

- $v_1, \ldots, v_p$ are linearly independent.
- $V = \operatorname{span}\{v_1, \ldots, v_p\}$.

Note:

- Bases are not usually unique. That is, most vector spaces have many bases.
- Saying "A sequence of vectors $(v_1, \ldots, v_p)$ in some vector space V is a basis *for* V" is the same as saying it is a basis ***of*** V.
- We called $(v_1, \ldots, v_p)$ a **sequence** of vectors, which means we keep track of the order. Often, we'll set (say) $B = \{v_1, \ldots, v_p\}$, and call this set a basis.

Example (A basis for $\mathbb{R}^n$)

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right) \text{ is a basis of } \mathbb{R}^n.$$

It is called the **standard basis**.

Example

$\left(1, t, t^2, \ldots, t^n\right)$ is a basis of $\mathbb{P}_n$.

As mentioned before, a vector space can have many bases. (We say "the basis is not unique".)

Example

Show that $\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$ is a basis of $\mathbb{R}^2$.

The answer this, we have to show two things:

1. These two vectors are linearly independent;
2. They span $\mathbb{R}^2$.

Remark

We only considered *finite* spanning sets and bases of vector spaces and we only defined linear independence for finite collections of vectors.
All of these notions admit infinite generalisations. We will not pursue this (that is for a longer course).
Infinite bases are mathematically interesting, but they quickly lead to tricky foundational issues of **set theory**.

Questions

- Does every vector space have a basis?
- How can we find bases?
- What are bases good for?

Bases of null spaces

Let A be an $m \times n$ matrix.

Using row reduction, beginning with A, we obtain a (unique!) matrix A' in reduced row echelon form.

Recall:

- $\mathrm{Nul}\, A = \mathrm{Nul}\, A'$.
- We can read off a spanning set of $\mathrm{Nul} A$ from A'. (See Week 3, Part 5).

FACT

This method always produces a basis of $\mathrm{Nul}\, A$.

Example

Suppose that $A \rightsquigarrow A' = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ via row reduction.

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PART 5: Finitely generated vector spaces

5: Finitely generated vector spaces

Question: Does every vector space have a basis?

The way we defined them, bases are always **finite**.

It turns out that some vector spaces are so "large" that they don't admit (finite) bases.

Examples include:

- $\mathbb{P}$–the space of polynomials of arbitrary degree,
- $C(\mathbb{R})$–the space of continuous functions $\mathbb{R} \to \mathbb{R}$.

Rather than extend our concept of a basis to include such examples, we will now study those vector spaces that have (finite) bases in detail.

5: Finitely generated vector spaces

Definition (FINITELY GENERATED VECTOR SPACE)

A vector space V is **finitely generated** (or **finite-dimensional**) if

$$V = \operatorname{span}\{v_1, \ldots, v_p\}$$

for some $p \geqslant 0$ and some sequence $v_1, \ldots, v_p \in V$.
(Here, for $p = 0$, we write $\operatorname{span}\{\} := \{0\}$.)

Lemma (The "Casting out" lemma)

Suppose that $V = \operatorname{span}\{v_1, \ldots, v_p\}$ *and that some* v_k *is a linear combination of the other vectors*

$$v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p.$$

Then

$$V = \operatorname{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\}.$$

What this means is

1. The original set $\{v_1, \ldots, v_p\}$ was *not* linearly independent.
2. So we can write some v_k in terms of the other vectors.
3. Removing (casting out) v_k from the sequence, we still have a spanning set for V.

5: Finitely generated vector spaces

If a vector space has a basis, then it is spanned by that basis. So that means it is finitely generated. The converse is also true!

Theorem (A finitely generated vector space has a basis)

Let V be a finitely generated vector space with $V \neq \{0\}$.
Then V has a basis.

5: Finitely generated vector spaces

There is a method for constructing a basis of V:

- Write $V = \operatorname{span}\{v_1, \ldots, v_p\}$.
- If no v_k belongs to $\operatorname{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\}$, then $v_1, \ldots, v_p$ are linearly independent. In that case,

  $$(v_1, \ldots, v_p)$$

  is a basis of V and we stop.
- Otherwise, for some k, we have $v_k \in \operatorname{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\}$. We then discard v_k from our spanning set (this lowers p!) and repeat our procedure for the resulting smaller spanning set.
- After finitely many iterations, we will have found a basis of V.

(This is not especially practical. But it will do for now).

Example

Let $p_1(t) = 2t - t^2$, $p_2(t) = 2 + 2t$, and $p_3(t) = 6 + 16t - 5t^2$. Let $V = \mathrm{span}\{p_1(t), p_2(t), p_3(t)\}$, a subspace of $\mathbb{P}_2$.

Find a basis of V.

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Start of ...

PART 6: What's the point of all this?

And where's the data science you promised?

6: What's the point of all this?

Here is an informal summary of what we've learned:

- Suppose you have a data set whose elements can be thought of as vectors in a **vector space**.
- If you have a **spanning set** for the space, you can describe all your data in terms of that set.
- If you have a **basis** you have the "smallest" spanning set needed to describe the set.
- We'll investigate this more next week, taking an example from Chapter 8 of Linear Algebra for Data Science

Exercises

Q1. Let $T\colon V \to W$ be a linear transformation from a vector space V to a vector space W.

(a) Show that the kernel $\mathrm{Ker}\, T$ of T is a subspace of V.
(b) Show that the range $\mathrm{Ran}\, T$ of T is a subspace of W.

Q2. Recall that $M_{m\times n}$ denotes the vector space of $m \times n$ matrices with real entries. Further recall that $A^\top$ denotes the *transpose* of a matrix A. Define $T\colon M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^\top$.

(a) Show that T is a linear transformation.
(b) Show that the range of T consists precisely of those matrices $B \in M_{2\times 2}$ with $B = B^\top$. (Such matrices are called *symmetric*.)
(c) Describe the kernel of T.

Exercises

Q3. For each of the following collections of vectors, determine if it (i) is linearly independent, (ii) spans $\mathbb{R}^3$, and (iii) is a basis of $\mathbb{R}^3$.

$$(a)\ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad (b)\ \begin{bmatrix}-2\\3\\0\end{bmatrix}, \begin{bmatrix}6\\-1\\5\end{bmatrix}$$

$$(c)\ \begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

Q4. Find a basis for the null space of

$$\begin{bmatrix}1 & 0 & -2 & -2\\0 & 1 & 1 & 4\\3 & -1 & -7 & 3\end{bmatrix}.$$

Exercises

Q5. Find a basis for the null space of

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -8 & 0 & 16 \end{bmatrix}.$$

Q6. Find a basis for the subspace of $\mathbb{R}^3$ consisting of those vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with $x - 3y + 2z = 0$.

Q7. Find bases for $\mathrm{Nul}\, A$ and $\mathrm{Col}\, A$, where

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}.$$

Exercises

Q8. Find bases for $\mathrm{Nul}\, A$ and $\mathrm{Col}\, A$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}.$$

Q9. Find a basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$