2323-MA378: Class Test in Week 7 (Friday, 24 Feb)

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The following fact (Cauchy's theorem) may be useful in answering some of these questions. *Let p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \tag{1}$$

where $\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ denotes the nodal polynomial.

In addition, if S is the cubic spline interpolant the function f at N equally spaced points $\{a = x_0 < x_1 < \cdots < x_N = b\}$ with $x_i - x_{i-1} = (b-a)/N =: h$, then

$$\|f - S\|_\infty := \max_{a \le x \le b} |f(x) - S(x)| \le \frac{5h^4}{384} \max_{a \le x \le b} |f^{(4)}(x)|. \tag{2}$$

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In all the questions below, the function f is

$$f(x) = (x^2 - 1)e^x. \tag{3}$$

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Q1. (`40 marks`)

(a) Write down the Lagrange form for the polynomial, $p_2(x)$, that interpolates f at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(b) Evaluate $p_2(1/2)$. What is the exact value of $|f(\mathbf{1/2}) - p_2(\mathbf{1/2})|$?

(c) What bound does (1) give for $|f(1/2) - p_2(\mathbf{1/2})|$?

(d) How do you account for the discrepency between the answers in Parts (b) and (c)?

Q2. (`40 marks`)

(a) Give a formula for the piecewise linear interpolant, $l(x)$, that interpolates f, at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(b) Evaluate $l(1/2)$. What is the exact value of $|f(x) - l(x)|$ **for** $\mathbf{x = 1/2}$?

(c) Use (1) to give an upper bound for $|f(x) - l(x)|$ **at** $\mathbf{x = 1/2}$.

(d) How do you account for the discrepency between the answers in Parts (b) and (c)?

Q3. (`20 marks`) Suppose that S is the cubic spline interpolant the function f at the $N+1$ equally spaced points $\{x_0 = -1 < x_1 < \cdots < x_N = 1\}$. What value of N should one take to ensure that $\|f - S\|$ is no more than 10^{-6}?