OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Questions from the
MA378 (Numerical Analysis 2) Examination, 2022–2023

This version has some typos corrected.

Q1. [28 MARKS] *Give a proof of the statements in any two of subsections (a), (b), (c) or (d). All subsections carry equal marks. In all cases, where relevant, you may assume that* $f : [a, b] \to \mathbb{R}$ *and its derivatives (to what ever order required) are continuous.*

(a) Let p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \tag{1}$$

where $\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ denotes the nodal polynomial.

(b) Let $\mathcal{L}$ be the set of all functions that are piecewise linear on the intervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$. Let l be the function in $\mathcal{L}$ that interpolates f at the points $a = x_0 < x_1 < \cdots < x_n = b$. Show that,

$$\|f - l\|_{\infty,[a,b]} \leq 2\|f - \hat{l}\|_{\infty,[a,b]}, \quad \text{for any } \hat{l} \in \mathcal{L}.$$

(c) Let $\{\widetilde{p}_0, \widetilde{p}_1, \ldots, \widetilde{p}_n\}$ be a set of monic polynomials where each $\widetilde{p}_i$ has degree i, and that are orthogonal with respect to the inner product $(\cdot, \cdot)$ defined as

$$(u, v) := \int_a^b u(x)v(x)dx.$$

(i) Prove that the zeros of each of the $\widetilde{p}_i$ are simple (not repeated).

(ii) Prove that all the zeros of each $\widetilde{p}_i$ are real numbers in the interval $[a, b]$.

(d) Show that if u satisfies the differential inequality

$$\mathrm{L}u := -u''(x) + r(x)u(x) \geq 0 \quad \text{on } (0, 1), \qquad u(0) \geq 0, u(1) \geq 0,$$

where $r(x) > 0$ for all $x \in [0, 1]$, then $u \geq 0$. (That is, show that L satisfies a *maximum principle*.)

Use this to show that, for any function f in $(0, 1)$, there is at most one solution to the differential equation

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (0, 1), \qquad u(0) = u(1) = 0. \tag{2}$$

Q2. (a) [5 Marks] Let $f(x) = xe^{-x}$. Give the Lagrange form of p_2, the polynomial interpolant to f at $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(b) [15 Marks] Using (1), give an upper bound for

$$\max_{-1 \leq x \leq 1} |f(x) - p_2(x)|.$$

(c) [4 Marks] Let p_n be the polynomial of degree n that interpolates an arbitrary function f at the $n+1$ equidistant points

$$-1 = x_0 < x_1 < x_2 < \cdots < x_n = 1.$$

Can one expect that $\max_{-1 \leq x \leq 1} |f(x) - p_{\mathbf{n}}(x)| \to 0$ as $n \to \infty$? Explain your answer.

Q3. (a) [5 Marks] Let us denote the Newton-Cotes quadrature rule for approximating $\int_a^b f(x)dx$ as

$$Q_n(f) := \sum_{i=0}^{n} q_i f(x_i).$$

Show that $\sum_{i=0}^{n} q_i = b - a$.

(b) [3 Marks] What is meant by the *precision* of a quadrature rule?
Consider the rule

$$R(f) = q_0 f(0) + \frac{1}{3} f\left(\frac{1}{2}\right) + q_2 f\left(\frac{3}{4}\right)$$

for approximating $\int_0^1 f(x)dx$.

(i) [7 Marks] If this rule has precision 2, determine the values of q_0 and q_2.

(ii) [5 Marks] What is the maximum precision of $R(\cdot)$ with the values of q_1 and q_2 that you have determined?

(iii) [4 Marks] Why is $R(\cdot)$ not a Newton-Cotes rule, in the strictest sense?

Q4. Consider the following differential equation:

$$-u''(x) + u(x) = e^{-x} \quad \text{on} \quad (0,1) \quad \text{and} \quad u(0) = u(1) = 0. \tag{3}$$

(a) [10 Marks] State the variational formulation of the differential equation in (3).
Show that the solution to the variational problem is unique.

(b) Suppose one wanted to compute an approximation to (3) using the finite element method (FEM) on the uniform mesh $\{x_0, x_1, \ldots, x_{n-1}, x_n\}$. Further, denote the set of the usual piecewise linear Galerkin basis ("hat") functions as $\{\psi_1, \psi_2, \ldots, \psi_n\}$.

(i) [4 Marks] Sketch a typical basis function, ψ_i, and give a formula for it.

(ii) [4 Marks] The FEM applied to (3) leads to a linear system of equations, which we write as the matrix-vector equation $Ax = b$. Give an expression for the entries of A and b in terms of the ψ_i. (You do not have to derive an explicit formula for these entries).

(iii) [4 Marks] Explain why A is symmetric and tridiagonal.

(iv) [2 Marks] Give an example for an ODE for which the associated system matrix in the FEM is *not* symmetric, or explain why this is not possible.