MA378 Chapter 1: Polynomial Interpolation ANS Assignment Questions with solutions!

Exercises 1.4, 2.3, 2.5, 4.4. Also: Presentation [10 Marks]

Exer 1.4 ⋆ [30 Marks] For each of the following interpolation problems, determine (with explanation) if there is no solution, exactly one solution, or more than one solution. In all cases p_n denotes a polynomial of degree (at most) n. You are not required to determine p_n where it exists.

(a) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

> **Answer:** We want to interpolate the points $(-1, 0)$, $(0, -1)$ and $(1, 1)$ with a polynomial of degree 1. That is, we want to find a single straight line through these points. Since they are not co-linear, that is not possible: **no solution exists**.

(b) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = -2$.

> **Answer:** We want to interpolate $(-1, 0)$, $(0, -1)$ and $(1, -2)$ with a polynomial of degree 1. These *are* co-linear: $y_i = -1 - x_i$. So exactly one solution exists.

(c) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

> **Answer:** Here $n = 2$ and we have three distinct points: $(-1, 0)$, $(0, -1)$ and $(1, 1)$. So standard theory applies (e.g., Theorems 2.3 and 2.7: there exists exactly one solution.

(d) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

> **Answer:** Among the points we'd have to interpolate are $(x_0, y_0) = (-1, 0)$ and $(x_2, y_2) = (-1, 1)$. Since we can't have a polynomial with two different values at $x = 0$, there is **no** solution to this problem.

(e) Find $p_2(x)$ that interpolates (x_0, y_0) and (x_1, y_1) where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$.

> **Answer:** Since there are only two points, $(x_0, y_0) = (-1, 0)$ and $(x_1, y_1) = (1, -1)$, and $n = 2$, there is an infinite number of solutions. For example, let p_1 be the usual degree 1 interpolant of these points. Then any quadratic of the form $p_1(x) + C(x + 1)(x - 1)$, for any C also interpolates them.

Exer 2.3 ⋆ [15 Marks] Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \qquad \text{for all } x.$$

> **Answer:** One possible solution is: let
>
> $$q(x) = \sum_{i=0}^{n} L_i(x) - 1.$$
>
> Since each of the L_i is a polynomial of degree n, so too is q. Furthermore, for $j = 0, 1, \dots, n$,
>
> $$\begin{aligned} q(x_j) &= \sum_{i=0}^{n} L_i(x_j) - 1 \\ &= L_j(x_j) - 1 \quad (\text{since } L_i(x_j) = 0 \text{ if } i \neq j) \\ &= 1 - 1 \quad (\text{since } L_j(x_j) = 1) = 0. \end{aligned}$$
>
> That is q is a polynomial of degree n with $n + 1$ zeros. By Theorem 2.2, it follows that $q(x) \equiv 0$. Thus
>
> $$\sum_{i=0}^{n} L_i(x_j) - 1 = 0 \iff \sum_{i=0}^{n} L_i(x_j) = 1.$$
>
> Note: *If you didn't get full marks on this it is probably because you didn't appeal to the fact that the Lagrange interpolant to a function is unique, and consequently, every polynomial is its own interpolant.*

Exer 2.5 ⋆ [20 Marks] Show that all the following represent the polynomial $T_3(x) = 4x^3 - 3x$ (often called the "Chebyshev Polynomial of Degree 3"),

(a) Horner form: $H_3(x) := \big((4x + 0)x - 3\big)x + 0$.

> **Answer:** Multiply out the terms on the right of H_3 to get $H_3(x) = (4x)x^2 - 3x = 4x^3 - 3x$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3} \Bigg(\prod_{j=0, j\neq k}^{3} \frac{x - x_j}{x_k - x_j} \Bigg) (-1)^{k+1}$, where $x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$.

Answer: This is the Lagrange form of the polynomial of degree 3 that interpolates the four points $(-1,-1)$, $(-1/2,1)$, $(1/2,-1)$ and $(1,1)$. Check that $T_3(-1) = 4(-1) - 3(-1) = -1$; $T_3(-1/2) = 4(-1/8) - 3(-1/2) = -1/2 + 3/2 = 1$; $T_3(1/2) = 4(1/8) - 3(1/2) = 1/2 - 3/2 = -1$; and $T_4(1) = 4 - 3 = 1$. Since both these polynomials are of degree $n = 3$, and interpolate the same $n+1 = 4$ points, they are the same polynomials.
NOTE: *Most people got this right, but use a much more complicated approach involving writing down, explicitly, the formula above, and then simplifying.*

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and $T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

Answer: Note that $T_2 = 2xT_1 - T_0 = 2x^2 - 1$. Then $T_3 = 2xT_2 - T_1 = 2x(2x^2-1)^2 - x = 4x^3 - 2x - x = 4x^3 - 3x$.

Exer 4.4 ⋆ [25 MARKS] Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$.

Answer: $L_0(x) = 1 - x$ and $L_1(x) = x$. Using the formulae from Exer 4.3, we have that

$$H_0 = (L_0(x))^2(1 - 2L_0'(x))(x - x_0)$$
$$= (1-x)^2(1+2x) = 2x^3 - 3x^2 + 1.$$

$$H_1 = (L_1(x))^2(1 - 2L_1'(x))(x - x_1)$$
$$= x^2(1 - 2(x-1)) = -2x^3 + 3x^2.$$

$$K_0 = (L_0(x))^2(x - x_0) = (1-x)^2x = x^3 - 2x^2 + x.$$

$$K_1 = (L_1(x))^2(x - x_1) = x^2(x-1) = x^3 - x^2.$$

Also $f(0) = 0$, $f(1) = \sin(1/2) \approx 0.4794$, $f'(0) = 1/2$ and $f'(1) = \cos(1/2)/2 \approx 0.4388$. So Then

$$q_3 = (0.4794)H_1(x) + (1/2)K_0(x) + (0.4388)K_2(x).$$

If one wants to expand this, it can be written as

$$q_3(x) = -0.0201x^3 - 0.0005x^2 + x/2.$$

To give an upper bound for $|f(1/2) - q_3(1/2)|$... This is, perhaps, not a very sensible question. There are two valid approaches. First, one can calculate $f(1/2) = 0.2474039593$ and $q_3(x) = 0.2473638592$. Then we calculate $|f(1/2) - q_3(1/2)| \approx 4.01 \times 10^{-5}$.
The second is to use Thm 4.3, from which we can deduce that

$$|f(x) - q(x)| \leqslant \frac{f^{(iv)}(\tau)}{(4)!}[(x)(x-1)]^2,$$

where $\tau \in [0, 1]$. In this case $f^{(iv)}(x) = (1/16)\sin(x/2)$, so $|f^{(iv)}(\tau)| \leqslant |f^{(iv)}(1)| \leqslant 0.03$. Then

$$|f(\frac{1}{2}) - q(\frac{1}{2})| \leqslant \frac{0.03}{24}\left[(\frac{1}{2})(\frac{1}{2} - 1)\right]^2$$
$$= \frac{1}{12800} = 7.8125 \times 10^{-5}.$$

NOTE: *Most people did not get this fully correct. The main issue was finding the maximum of* $|f^{(iv)}(\tau)|$. *In particular,* $\max_{0 \leqslant x \leqslant 1} |f^{(iv)}(x)| = (1/16)\sin(1/2) = 0.029964 \approx 0.03$. *However, many instead wrote* $\max_{0,1} |f^{(iv)}(x)| \leqslant 1/16$. *While that is correct, since* $\sin(x/2) \leqslant 1$ *for any* x, *it is not sharp, and over estimates the error by a factor of 2.*