MA385 Part 1: Solving nonlinear equations

1.3: The Secant Method

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Started Monday 15 Sep (W02.1)

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0. Outline

1 Motivation

2 Order of Convergence

3 Side quest: The Mean Value Theorem

4 Analysis of the Secant Method

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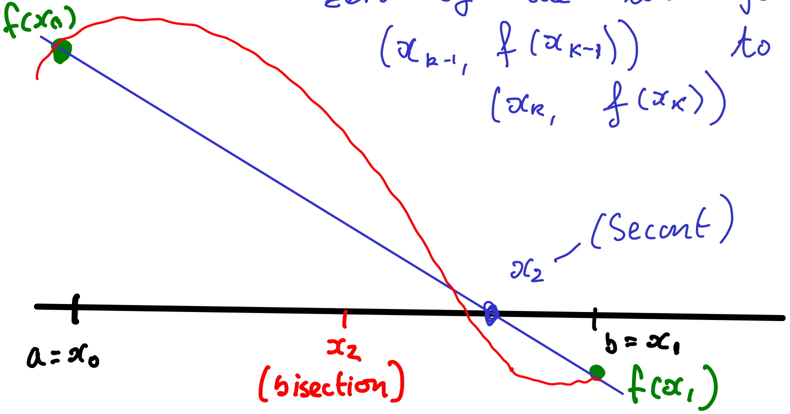
For more details, see Section 1.5 (The secant method) of Süli and Mayers, *An Introduction to Numerical Analysis*

1. Motivation

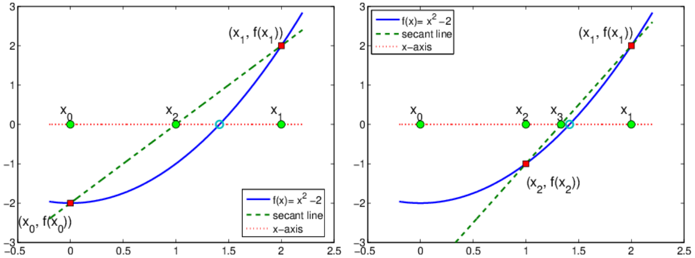
We'll start with considering, heuristically, how we could improve upon bisection:

Take x_{k+1} to be the zero of the line joining $(x_{k-1}, f(x_{k-1}))$ to $(x_k, f(x_k))$



Idea:

- Choose two points, x_0 and x_1.
- Take x_2 to be the zero of the line joining $(x_0, f(x_0))$ to $(x_1, f(x_1))$.
- Take x_3 to be the zero of the line joining $(x_1, f(x_1))$ to $(x_2, f(x_2))$.
- Etc.



The Secant Method

Choose x_0 and x_1 so that there is a solution in $[x_0, x_1]$. Then define

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}. \tag{1}$$

To derive this:

- Let L_k be the line joining $(x_k, f(x_k))$ & $(x_{k-1}, f(x_{k-1}))$:

$$y - f(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x - x_k)$$

$$\left(\text{"} y - y_1 = m(x - x_1) \text{"}\right)$$

- Let x_{k+1} be the value of x for which $y=0$. Solve.

Example 1.1

Use the Secant Method to solve $x^2 - 2 = 0$ in $[0, 2]$. Results are shown below. We see that, not only does the method appear to converge to the true solution, it seem to do so much more efficiently than Bisection. We'll return to why this is later.

	Secant		Bisection	
k	x_k	$\lvert x_k - \tau \rvert$	x_k	$\lvert x_k - \tau \rvert$
0	0.000000	1.41	0.000000	1.41
1	2.000000	5.86e-01	2.000000	5.86e-01
2	1.000000	4.14e-01	1.000000	4.14e-01
3	1.333333	8.09e-02	1.500000	8.58e-02
4	1.428571	1.44e-02	1.250000	1.64e-01
5	1.413793	4.20e-04	1.375000	3.92e-02
6	1.414211	2.12e-06	1.437500	2.33e-02
7	1.414214	3.16e-10	1.406250	7.96e-03
8	1.414214	4.44e-16	1.421875	7.66e-03

To compare different methods, we need the following concept.

Definition 2.1 (Linear Convergence)

Suppose that $\tau = \lim_{k\to\infty} x_k$. *We say that the sequence* $\{x_k\}_{k=0}^{\infty}$ *converges to* τ at least linearly *if there is a sequence of positive numbers* $\{\varepsilon_k\}_{k=0}^{\infty}$, *and* $\mu \in (0,1)$, *such that*

$$\lim_{k\to\infty} \varepsilon_k = 0, \tag{2a}$$

and

$$|\tau - x_k| \leq \varepsilon_k \quad \textit{for} \quad k = 0, 1, 2, \dots. \tag{2b}$$

and

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu. \tag{2c}$$

2. Motivation

For Example 1.1, the bisection method converges at least linearly. As we have seen, the Secant Method converge more quickly than bisection. Now we'll give a more precise description of what "*more quickly*" means.

Definition 2.2 (Order of Convergence)

Let $\tau = \lim_{k\to\infty} x_k$. *Suppose there exists* $\mu > 0$ *and a sequence of positive numbers* $\{\varepsilon_k\}_{k=0}^{\infty}$ *such that (2a) and and (2b) both hold. Then we say that the sequence* $\{x_k\}_{k=0}^{\infty}$ *converges with at least order* q *if*

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu.$$

→ For Bisection, $\mu = 1/2$

2. Motivation

$$\boxed{\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu.}$$

Two particular values of q are important to us:

(i) If $q = 1$, and we have that $0 < \mu < 1$, then the rate is **linear**.

(ii) If $q = 2$, the rate is **quadratic** for any $\mu > 0$.

Bisection

Eg, Newton's Method
→ Thursday.

The **Mean Value Theorem** (aka "MVT") is simple, and incredibly useful. The modern version is due to Cauchy (1823), but there is a version called Rolle's Theorem which applies to polynomials only from 1691, and one for the sin function to Parameshvara or Kerala (1380–1460).

Theorem 3.1 (Mean Value Theorem (MVT))

If $f : [a, b] \to R$ is continuous and differentiable, then there is a point $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

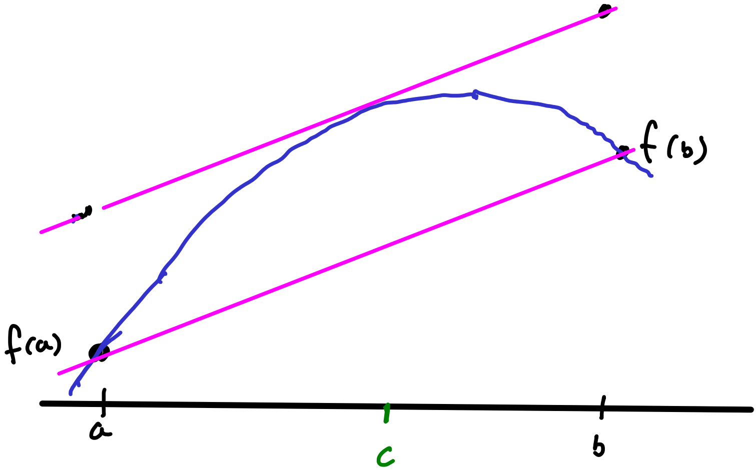
$(a, f(a))$.

Informally: *there is a point between a and b where the tangent to f is parallel to the line joining $(a, f(b))$ at $(b, f(b))$.*

3. Side quest: The Mean Value Theorem

A visual proof:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



3. Side quest: The Mean Value Theorem

From a numerical analysis point of view, the MVT tells us “how different $f(b)$ might be from $f(a)$”:

$$f(b) = f(a) + f'(c)(b - a)$$

gives that

$$|f(b) - f(a)| \leq \max_{a \leq x \leq b} |f'(x)|(b - a).$$

That is, in a computation, if we replace $f(a)$ with $f(b)$, the error we introduce depends on $b - a$ and also the magnitude of f'.

Theorem 4.1

Suppose that f and f' are real-valued functions, continuous and defined in an interval $I = [\tau - h, \tau + h]$ for some $h > 0$. If $f(\tau) = 0$ and $f'(\tau) \neq 0$, then the sequence (1) converges at least linearly to τ.

I want to get an estimate for $|\tau - x_{k+1}|$ in terms of $|\tau - x_k|$.

From the method:

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

So
$$\tau - x_{k+1} = \tau - x_k + f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

4. Analysis of the Secant Method

- We wish to show that $|\tau - x_{k+1}| < |\tau - x_k|$.
- From the MVT, there is a point $w_k \in [x_{k-1}, x_k]$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(w_k). \tag{3}$$

- Also by the MVT, there is a point $z_k \in [x_k, \tau]$ such that

$$\frac{f(x_k) - \overset{=0}{f(\tau)}}{x_k - \tau} = \frac{f(x_k)}{x_k - \tau} = f'(z_k). \tag{4}$$

Therefore $\quad f(x_k) = (x_k - \tau) f'(z_k).$

4. Analysis of the Secant Method

- Using (3) and (4), we can show that

$$\tau - x_{k+1} = (\tau - x_k)\Big(1 - f'(z_k)/f'(w_k)\Big).$$

Therefore

$$\frac{|\tau - x_{k+1}|}{|\tau - x_k|} = \Big|1 - \frac{f'(z_k)}{f'(w_k)}\Big|.$$

FInished here Monday, (W02.1)

As we see onthe next slide, we can have $\dfrac{f'(z_k)}{f'(w_k)} \leq \dfrac{5}{3}$

So then $|\tau - x_{k+1}| \leq \frac{2}{3}|\tau - x_k|$

- Suppose that $f'(\tau) > 0$. (If $f'(\tau) < 0$ just tweak the arguments accordingly). Saying that f' *is continuous in the region* $[\tau - h, \tau + h]$ means that, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f'(x) - f'(\tau)| < \varepsilon \text{ for any } x \in [\tau - \delta, \tau + \delta].$$

Take $\varepsilon = f'(\tau)/4$. Then $|f'(x) - f'(\tau)| < f'(\tau)/4$. Thus

$$\frac{3}{4} f'(\tau) \leq f'(x) \leq \frac{5}{4} f'(\tau) \quad \text{for any } x \in [\tau - \delta, \tau + \delta].$$

Then, so long as w_k and z_k are both in $[\tau - \delta, \tau + \delta]$

$$\left|\frac{f'(z_k)}{f'(w_k)}\right| \leq \frac{5}{3}.$$

4. Analysis of the Secant Method

Given enough time and effort we *could* show that the Secant Method converges faster that linearly. In particular, that the order of convergence is

$$q = (1 + \sqrt{5})/2 \approx 1.618.$$

$\Leftarrow q^2 = q + 1$

This number arises as the only positive root of $q^2 - q - 1$. It is called the **Golden Mean**, and arises in many areas of Mathematics, including finding an explicit expression for the Fibonacci Sequence:

$$f_0 = 1,$$
$$f_1 = 1,$$
$$f_{k+1} = f_k + f_{k-1} \text{ for } k = 2, 3, \ldots.$$

That gives, $f_0 = 1$, $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, $\ldots$.

The connection here is that it turns out that $\varepsilon_{k+1} \leq C\varepsilon_k\varepsilon_{k-1}$. Repeatedly using this we get:

- Let $r = |x_1 - x_0|$ so that $\varepsilon_0 \leq r$ and $\varepsilon_1 \leq r$,
- Then $\varepsilon_2 \leq C\varepsilon_1\varepsilon_0 \leq Cr^2$
- Then $\varepsilon_3 \leq C\varepsilon_2\varepsilon_1 \leq C(Cr^2)r = C^2r^3$.
- Then $\varepsilon_4 \leq C\varepsilon_3\varepsilon_2 \leq C(C^2r^3)(Cr^2) = C^4r^5$.
- Then $\varepsilon_5 \leq C\varepsilon_4\varepsilon_3 \leq C(C^4r^5)(C^2r^3) = C^7r^8$.
- And in general, $\varepsilon_k = C^{f_k-1}r^{f_k}$.

k^{th} Fibonacci Number.

Exercise 1.5.1. Suppose we define the Secant Method as follows.

Choose any two points x_0 and x_1.

For $k = 1, 2, \ldots$, set x_{k+1} to be the point where the line through $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ that intersects the x-axis.

Show how to derive the formula for the secant method.

Exercise 1.5.2. Another interpretation of the Secant Method, is that it computes x_{k+1} as a weighted average:

$$x_{k+1} = (1 - \sigma_k)x_k + \sigma_k x_{k-1},$$

where σ_k chosen to obtain fast convergence to the true solution. (This is something called a "relaxation method").

Derive a formula for σ_k in terms of $f(x_{k-1})$ and $f(x_k)$. Can you give a physical interpretation to σ_k and $1 - \sigma_k$ that might justify the claim that the method should be more efficient than bisection?