MA378 Chapter 1: Interpolation

§1.2 Lagrange Interpolation

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Source: https://users.wfu.edu/kuz/Stamps/Lagrange/Lagrange.htm

Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics.

2.0 Contents

$\mathbb{P}_2$

Note

$f(0) = 1$

$f(1) = 1$

$f(2) = -1$

x_0 $\quad x_1$ $\quad x_2$

First, note that $p_2(x) \in \mathbb{P}_2$. Also

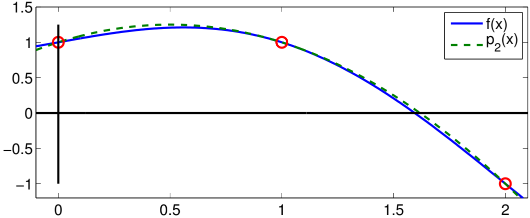
$p_2(0) = -0^2 + 0 + 1 = 1 = f(0)$ $\quad p_2(1) = -1+1+1 = 1 = f(1)$

$p_2(2) = -4 + 2 + 1 = -1 = f(2)$. ✓

2.1 Finding the polynomial

Example 2.1

Show that the polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ is $p_2 = -x^2 + x + 1$.



How to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

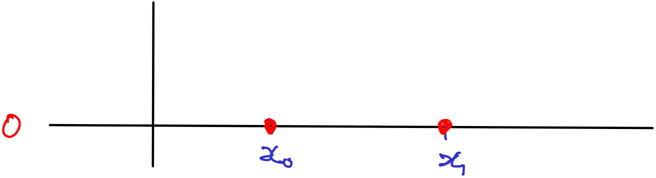
The answer is in the our first theorem!

Example

$n=1$. So $p_1(x) = a + bx$.

If p_1 has 2 zeros, then

$p(x_0) = 0$ $p(x_1) = 0$.



 $P(x) \equiv 0$ means

$P(x)=0$ for all x.

How to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

The answer is in the our first theorem!

Since p_n is a polynomial, and

$p_n(x_0)=0$, $p_n(x_1)=0$, ..., $p_n(x_n)=0$, we can write it as

$$p_n(x) = q(x)\,(x-x_0)(x-x_1)\cdots(x-x_n)$$

Howevere $(x-x_0)(x-x_1)\cdots(x-x_n)$ is a polynomial of degree $n+1$,

So the coeficent of x^{n+1} is $q(x)$

But p_n is of degree n. So $q(x) \equiv 0$.

Thus $p_n(x) \equiv 0$

Theorem 2.2

If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n = 0$ (i.e., $p_n(x) = 0$ for all x).

Suppose we have $n+1$ interpolation points $x_0, x_1, \ldots, x_n$, and a function f which is defined for all $x \in [x_0, x_n]$. Then there is at most one polynomial of degree n which interpolates f at those points.

Theorem 2.2

If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all x).

Proof: Suppose that $p(x) \in \mathbb{P}_n$, $q(x) \in \mathbb{P}_n$

And $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$ ...., $p(x_n) = f(x_n)$

$q(x_0) = f(x_0)$, $q(x_1) = f(x_1)$, ..., $q(x_n) = f(x_n)$.

Let $s(x) = p(x) - q(x)$

So $s(x_0) = 0$, $s(x_1) = 0$, ... $s(x_n) = 0$.

So $s(x) \in \mathbb{P}_n$ and has $n+1$ zeros,

$s(x) \equiv 0$. Thus $p(x) \equiv q(x)$

Theorem 2.3 (A solution to the PIP is unique)

There is at most one polynomial of degree $\leq n$ that interpolates the $n+1$ points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) where x_0, x_1, ..., x_n are distinct.

2.2 The Vandermonde matrix method

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find p_2 such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1x + a_2x^2$, this just amounts the finding the values of the coefficients a_0, a_1, and a_2. One might be tempted to solve for them using the system of equations

$$\begin{aligned} a_0 + a_1x_0 + a_2x_0^2 &= y_0 \\ a_0 + a_1x_1 + a_2x_1^2 &= y_1 \\ a_0 + a_1x_2 + a_2x_2^2 &= y_2 \end{aligned}$$

This is known as the *Vandermonde System*.

2.2 The Vandermonde matrix method

Writing

$$
\begin{aligned}
a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\
a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 &= y_2
\end{aligned}
$$

in matrix-vector format we get

$$
\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \quad \text{or} \quad V\mathrm{a} = \mathrm{y}. \tag{1}
$$

But this is usually not be a good idea. At the very least, we'd have to solve a linear system of equations. Furthermore, the system is very *ill-conditioned*.

2.2 The Vandermonde matrix method

[*Slides 9 and 10 depend on material from MA385, so we'll skip them in class: please read in your own time.*]

In MA385 you learned about the relationship between the *condition number* of a matrix, V, and the relative error in the (numerical) solution to a matrix-vector equation with V as the coefficient matrix. The condition number is $\kappa(V) = \|V\|\|V^{-1}\|$, for some subordinate matrix norm $\|\cdot\|$.

2.2 The Vandermonde matrix method

Example 2.4 (Stewart's "Afternotes...", Lecture 19)

Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

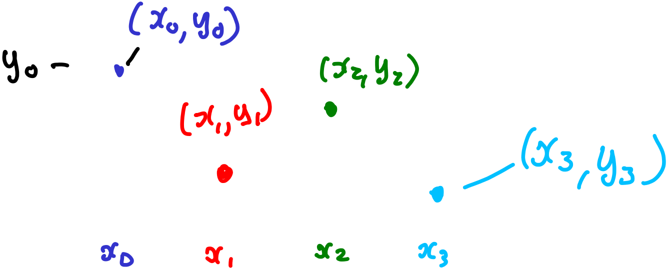
Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

2.3 Lagrange Polynomials

We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (A "constructive proof" is one that shows a thing exists by by actually computing it).



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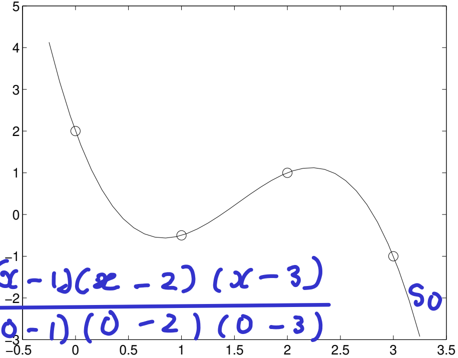
Example

Consider the problem: *find* $p_3 \in \mathcal{P}_3$ *such that*

$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$



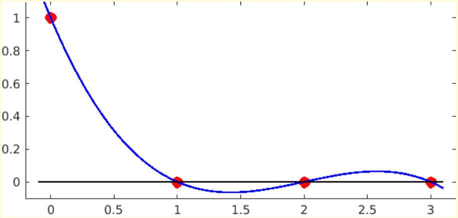
$$\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}$$

So $L_0(0) = 1$

Here is an easier problem to solve: Find $L_0 \in \mathcal{P}_3$ such that

$$L_0(0) = 1, \;\; L_0(1) = 0,$$

$$L_0(2) = 0, \;\; L_0(3) = 0.$$



Because L_0 is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$.
Choosing C so that $L_0(0) = 1$, we get
$L_0(x) =$

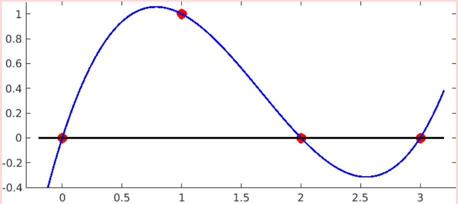
$$\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

$$= \frac{1}{(1)(-1)(-2)}(x-0)(x-2)(x-3) = \frac{1}{2}(\text{—})$$

2.3 Lagrange Polynomials

Similarly, find $L_1 \in \mathcal{P}_3$ such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

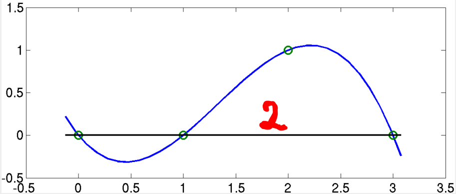


Then

$$L_1(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}$$

2.3 Lagrange Polynomials

In the same style, let $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



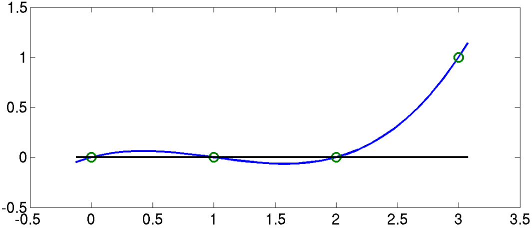
$$L_2(x) =$$

Finally, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j\neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$



Because each of L_0, L_1, L_2, and L_3 is a cubic polynomial, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (\frac{1}{2})L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is also a cubic polynomial. That is, $p_3 \in \mathcal{P}_3$.

Furthermore...

$$
\begin{array}{rclllll}
p_3(0) & = & 2L_0(0) & -(1/2)L_1(0) & +(1)L_2(0) & +(-1)L_3(0) \\
& = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
& = & 2, & & & \\
p_3(1) & = & 2L_0(1) & -(1/2)L_1(1) & +(1)L_2(1) & +(-1)L_3(1) \\
& = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
& = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
& = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
& = & 1, & & & \\
p_3(3) & = & 2L_0(3) & -(1/2)L_1(3) & +(1)L_2(3) & +(-1)L_3(3) \\
& = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
& = & -1. & & &
\end{array}
$$

Thus, as required,

$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$

So p_3 solves the problem!

2.4 Lagrange Interpolation

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation.

Definition 2.5 (Lagrange Polynomials)

The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$y_i = f(x_i)$

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2b}$$

Definition 2.6

The **Lagrange form of the Interpolating Polynomial**

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3a}$$

or

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{3b}$$

Proof: since each $L_i(x)$ is a polynomial of degree n, so too is $p_n(x)$

Take care not to confuse

Also for any point x_j

- the *Lagrange Polynomials*, which are the L_i with
- the *Lagrange Interpolating Polynomial*, which is the p_n defined in (3).

$$p_n(x_j) = y_0 \underbrace{L_0(x_j)}_{0} + y_1 \underbrace{L_1(x_j)}_{0} + \dots + y_j \underbrace{L_j(x_j)}_{1} + \dots + y_n \underbrace{L_n(x_j)}_{0}$$

$= y_j$. So p_n solves the PIP.

Theorem 2.7 (Lagrange's Interpolation Theorem)

There exists a solution to the Polynomial Interpolation Problem and it is given by

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

Here $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

we want $p_2(x)$ to interpolate $f(x) = e^x$ at x_0, x_1, x_2. Define the Lagrange Polys:

$$L_0(x) = \frac{(x)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}(x)(x-1).$$

Similarly

$$L_1(x) = 1 - x^2 \quad \& \quad L_2(x) = \frac{1}{2}(x+1)(x).$$

Then $p_2(x) = e^{-1}\frac{1}{2}x(x-1) + (1-x^2) + e\frac{1}{2}(x)(x+1)$

Example 2.8 (Süli and Mayers, E.g. 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

$f(x) - p_2(x)$

$f(x) = e^x$

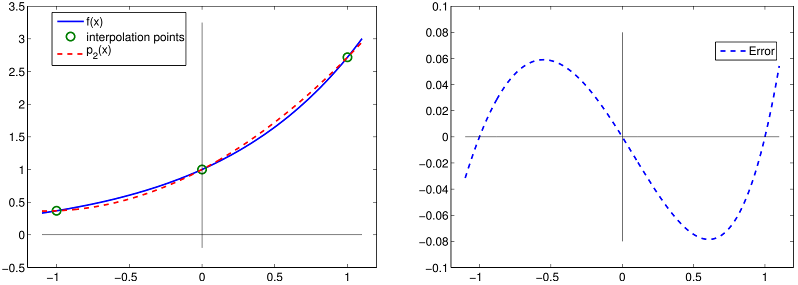
x_0 x_1 x_2

2.5 Example

The figure below shows the solution to Example 2.8 (top) and the difference between the function e^x and its interpolant (bottom). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.



Exercise 2.1

The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{0 \le i < j \le n} (x_j - x_i). \tag{4}$$

Verify (4) for the 2×2 and 3×3 cases.

(Note that from Formula (4) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \ldots, x_n$ are all distinct.)

Exercise 2.2

Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute

$$\max_{0 \leq x \leq a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayers.

2.6 Exercises

Exercise 2.3 ($\star$)

Suppose we have a set of distinct interpolation points $\{x_0, x_1, \dots, x_n\}$, and we define the associated Lagrange Polynomials $\{L_0, L_1, \dots, L_n\}$. For each of the following identities, either show that it is true for any set of interpolation points, or give an example where it is false.

1. $\displaystyle\sum_{i=0}^{n} L_i(x) = 1$ for all x.
2. $\displaystyle\sum_{k=0}^{n} x_k L_k(x) = x$ for all x.
3. $\displaystyle\sum_{k=0}^{n} \frac{1}{x_k} L_k(x) = \frac{1}{x}$ for all x.