Chapter 0 (Taylor's Theorem) and Chapter 1 (Solving nonlinear equations)

Please submit carefully written solutions to the following exercises: Exercises 0.2 1.8, 1.10, 1.11, and 1.15.

Exercise 0.1. Write down the formula for the Taylor Polynomial for

(i) $f(x) = 3x^2 + 3x - 12$

(ii) $f(x) = \sqrt{1+x}$ about the point $a = 0$,

(iii) $f(x) = \log(x)$ about the point $a = 1$.

Exercise 0.2 ($\star$ Homework problem)**.** Write out the Taylor polynomial at x, about $a = 0$, of degree 5 for $f(x) = \sin(x)$. How does its derivative compare to the corresponding Taylor polynomial for $f(x) = \cos(x)$?

The purpose of the next exercise is to demonstrate that, usually, the closer x is to a, the better the Taylor polynomial approximates that function's value.

Exercise 0.3. Write out the Taylor Polynomial about $a = 1$ of degree 4 and corresponding remainder for $f(x) = \ln(x)$. Give an upper bound for this remainder when $x = 2$, $x = 1.1$ and $x = 1.01$.

The purpose of the next exercise is to demonstrate that some functions do not have sensible Taylor polynomials.

Exercise 0.4. Write out the Taylor polynomial about $a = 0$, of degree 4, for $f(x) = e^{-1/x^2}$.
Hint: $\lim_{x\to 0} e^{-1/x^2}x^{-p} = 0$ for any positive, finite p.

Exercise 0.5. Prove the *Integral Mean Value Theorem*: there exists a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Exercise 1.1. Does Proposition 1.1.1 mean that, if there is a solution to $f(x) = 0$ in $[a, b]$ then $f(a)f(b) \leqslant 0$? That is, is $f(a)f(b) \leqslant 0$ a *necessary* condition for their being a solution to $f(x) = 0$? Give an example that supports your answer.

Exercise 1.2. Suppose we want to find $\tau \in [a, b]$ such that $f(\tau) = 0$ for some given f, a and b. Write down an estimate for the number of iterations K required by the bisection method to ensure that, for a given ε, we know $|x_k - \tau| \leqslant \varepsilon$ for all $k \geqslant K$. In particular, how does this estimate depend on f, a and b?

Exercise 1.3. How many (decimal) digits of accuracy are gained at each step of the bisection method? (If you prefer, how many steps are needed to gain a single (decimal) digit of accuracy?)

Exercise 1.4. Let $f(x) = e^x - 2x - 2$. Show that there is a solution to the problem: *find* $\tau \in [0, 2]$ *such that* $f(\tau) = 0$. Taking $x_0 = 0$ and $x_1 = 2$, use 6 steps of the bisection method to estimate τ. You may use a computer program to do this, but please note that in your solution.
Give an upper bound for the error $|\tau - x_6|$.

Exercise 1.5. We wish to estimate $\tau = \sqrt[3]{4}$ numerically by solving $f(x) = 0$ in $[a, b]$ for some suitably chosen f, a and b.

(i) Suggest suitable choices of f, a, and b for this problem.

(ii) Show that f has a zero in $[a, b]$.

(iii) Use 6 steps of the bisection method to estimate $\sqrt[3]{4}$. You may use a computer program to do this, but please note that in your solution.

(iv) Use Theorem 1.3 to give an upper bound for the error $|\tau - x_6|$.

Exercise 1.6. Suppose we define the Secant Method as follows.

> *Choose any two points* x_0 *and* x_1.
> *For* $k = 1, 2, \dots,$ *set* x_{k+1} *to be the point where the line through* $\big(x_{k-1}, f(x_{k-1})\big)$ *and* $\big(x_k, f(x_k)\big)$ *that intersects the* x*-axis.*

Show how to derive the formula for the secant method.

Exercise 1.7. (i) Is it possible to construct a problem for which the bisection method will work, but the secant method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which the secant method will work, but bisection will fail? If so, give an example.

Exercise 1.8 ($\star$ Homework problem)**.** Write down the equation of the line that is tangential to the function f at the point x_k. Give an expression for its zero. Hence show how to derive Newton's method.

Exercise 1.9. (i) Is it possible to construct a problem for which the bisection method will work, but Newton's method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which Newton's method will work, but bisection will fail? If so, give an example.

Exercise 1.10 ($\star$ Homework problem)**.** (i) Let q be your student ID number. Find k and m where $k-2$ is the remainder on dividing q by 4, and $m-2$ is the remainder on dividing q by 6.

(ii) Show how Newton's method can be applied to estimate the postive real number $\sqrt[k]{m}$. That is, state the nonlinear equation you would solve, and give the formula for Newton's method, simplified as much as possible.

(iii) Do three iterations by hand of Newton's Method for this problem.

Exercise 1.11 ($\star$ Homework problem)**.** Suppose we want apply to Newton's method to solving $f(x) = 0$ where f is such that $|f''(x)| \leqslant 10$ and $|f'(x)| \geqslant 2$ for all x. How close must x_0 be to τ for the method to converge?

Exercise 1.12. Here is (yet) another scheme called *Steffenson's Method*: Choose $x_0 \in [a, b]$ and set

$$x_{k+1} = x_k - \frac{\big(f(x_k)\big)^2}{f\big(x_k + f(x_k)\big) - f(x_k)} \quad \text{for } k = 0, 1, 2, \dots.$$

It is remarkable because its convergence is quadratic, like Newton's, but does not require derivatives of f.

Show how the method can be derived from Newton's Method, using the formal definition of the derivative.

Exercise 1.13. (This is Exercise 1.6 from Süli and Mayers) The proof of the convergence of Newton's method given in Theorem 1.10 uses that $f'(\tau) \neq 0$. Suppose that it is the case that $f'(\tau) = 0$.

(i) Starting from the Newton Error formula, show that

$$\tau - x_{k+1} = \frac{(\tau - x_k)}{2} \frac{f''(\eta_k)}{f''(\mu_k)},$$

for some μ_k between τ and x_k. (*Hint: try using the MVT*).

(ii) What does the above error formula tell us about the convergence of Newton's method in this case?

Exercise 1.14. Is it possible for g to be a contraction on $[a, b]$ but not have a fixed point in $[a, b]$? Give an example to support your answer.

Exercise 1.15 ($\star$ Homework problem)**.** Show that $g(x) = \ln(2x + 1)$ is a contraction on $[1, 2]$. Give an estimate for L. (Hint: Use the Mean Value Theorem).

Exercise 1.16. Suppose we wish to numerically estimate the famous *golden ratio*, $\tau = (1 + \sqrt{5})/2$, which is the positive solution to $x^2 - x - 1$. We could attempt to do this by applying fixed point iteration to the functions $g_1(x) = x^2 - 1$ or $g_2(x) = 1 + 1/x$ on the region $[3/2, 2]$.

(i) Show that g_1 is *not* a contraction on $[3/2, 2]$.

(ii) Show that g_2 *is* a contraction on $[3/2, 2]$, and give an upper bound for L.

Exercise 1.17. Consider the function $g(x) = x^2/4 + 5x/4 - 1/2$.

(i) It has two fixed points – what are they?

(ii) For each of these, find the largest region around them such that g is a contraction on that region.

Exercise 1.18. (i) Prove that if $g(\tau) = \tau$, and the fixed point method given by

$$x_{k+1} = g(x_k),$$

converges to the point τ (where $g(\tau) = \tau$), and

$$g'(\tau) = g''(\tau) = \cdots = g^{(p-1)}(\tau) = 0,$$

then it converges with order p. (Hint: you don't have to prove that the method converges; you can assume that. Also, use a Taylor Series).

(ii) We can think of Newton's Method for the problem $f(x) = 0$ as fixed point iteration with $g(x) = x - f(x)/f'(x)$. Use this, and Part (i), to show that, if Newton's method converges, it does so with order 2, providing that $f'(\tau) \neq 0$.