2425-MA140 Engineering Calculus

Week 09, Lecture 3 (L27)
Arc Lengths and Surface Areas

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Today, we'll discuss at length:

See also: Section **6.4** (Arc Length of a Curve and Surface Area) in **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Recall: Arc Length

Yesterday, we started on how to calculate the length of curve that is defined by a function, f. We finished by deriving that...

Arc length of a curve

If $f(x)$ is a differentiable function on the interval $[a, b]$, then the **arc length**, L, of the graph of $f(x)$, from $x = a$ to $x = b$, is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,.$$

(Note: *a few extra mathematical details have been added in Slides 18 and 19 from yesterday*).

Example

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}\,x^{\frac{3}{2}} - 1, \quad 0 \leqslant x \leqslant 1\,.$$

Recall: Arc Length

A example from Civil Engineering

Question! What do the following all have in common?

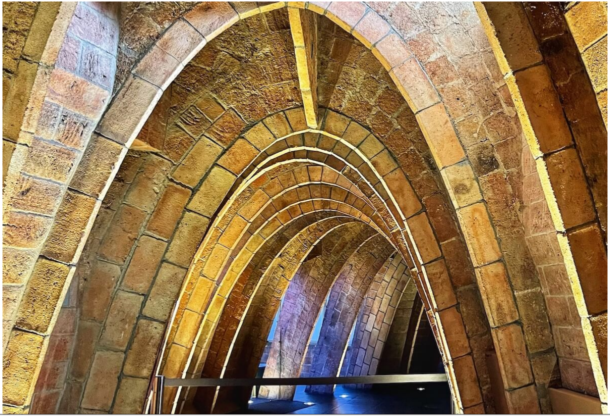







A example from Civil Engineering

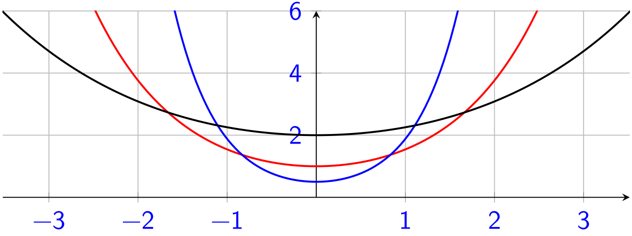
All the above are examples of **Catenary Arches**. This is the shape taken on by a free hanging chain. When used as an arch, it has an "optimal" shape in the sense that it can support its own weight. It has been known since ancient times. The example below is from Casa Milá in Barcelona.



A example from Civil Engineering

As a hanging chain, a catenary can be described by the function

$$f(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$



A example from Civil Engineering

Aside: the function

$$f(x) = \frac{e^{x/a} + e^{-x/a}}{2}$$

is also known as the **hyperbolic cosine**, or cosh function. That is

$$\cosh(x) = \frac{e^{x/a} + e^{-x/a}}{2}.$$

You can read about it in the textbook (Section 1.5). But we don't need that for the following example.

A example from Civil Engineering

Example (Example from Civil Engineering)

Metal posts have been installed $4m$ apart across a gorge. Find the length for rope bridge that follows the curve

$$f(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right).$$

A example from Civil Engineering

A note of caution

Computing the arc length of a function involves evaluating integrals where the integrand is of the form $\sqrt{1 + (g(x))^2}$. In some (rare) cases, this is easy. In others, it is possible to use a method called **trigonometric substitution**, which is not on our syllabus.

In the "real world" we actually use highly accurate numerical approximations. The details are beyond this course.
There is some interesting mathematics involved in determining how to take n large enough to ensure the error is small.

We'll mention this again briefly at the end of the semester.

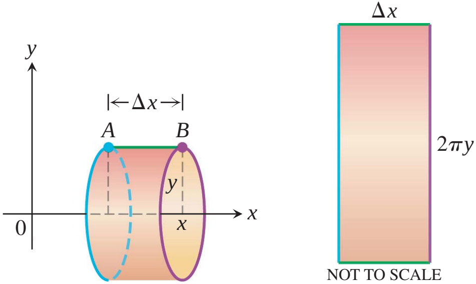
Surface area of a cylinder

Suppose we have the line $y = r$, for some $r > 0$, and two points $x = A$ and $x = B$. The length of the segment of the line between A and B is denoted $\Delta x = B - A$.

Now rotate the line segment about the $x-$axis, to make a cylinder with curved surface area (see p10 of the "Log Tables")

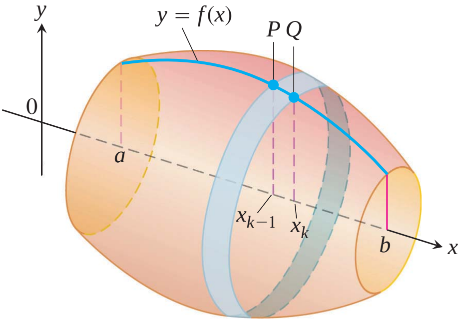
$$A = 2\pi r \Delta x.$$

This area is the same as that of a rectangle with side lengths Δx and $2\pi y$.



Areas of Rotation

We are interested in the **surface area of a solid generated by rotating a curve $y = f(x)$ about the x-axis**.



If the arc length from P to Q is L_k, then the surface area of the typical band is approximately

$$2\pi f(x_k)\, L_k\,.$$

Areas of Rotation

Summing over all the bands, we get that the surface area of the solid can be approximated as

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k)\, L_k\,.$$

We know that the arc length L_k can be computed as $L_k = \sqrt{1 + \left[\dfrac{\Delta y_k}{\Delta x_k}\right]^2} \cdot \Delta x_k\,.$ Thus, we can approximate the surface area of the solid as follows:

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k) \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x_k}\right]^2} \cdot \Delta x_k\,.$$

For $n \to \infty$, we get the Riemann sum

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi f(x_k) \sqrt{1 + \left[\frac{\Delta y_k}{\Delta x_k}\right]^2} \cdot \Delta x_k\,.$$

Areas of Rotation

We can now develop the following formula for the surface area of a solid obtained by rotating a curve.

Surface Area

If f is continuously differentiable on the closed interval $[a, b]$, then the surface area of the solid obtained by rotating the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

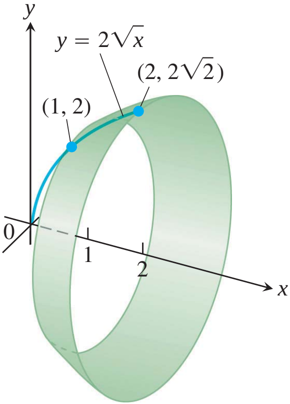
$$S = 2\pi \int_a^b y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx\,.$$

Areas of Rotation

Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leqslant x \leqslant 2$, about the x–axis



Areas of Rotation

Areas of Rotation

Example

Find the area of the surface generated by revolving the curve $y = \frac{x^3}{9}$ between $x = 0$ and $x = 2$ about the x-axis.

Areas of Rotation

Exercises

Exer 9.3.1

What is the arc length of the graphs of $f(x) = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 1$ to $x = 2$?

Exer 9.3.2 (A little tricky)

Find the length of the Catenary function $f(x) = e^{x/2} - e^{-x/2}$ from $x = -2$ to $x = 2$.

Exer 9.3.3

What is the area of the surface formed by rotating the curve of

$$f(x) = \frac{3\sqrt{x}}{\sqrt{2}}$$

between $x = 0$ and $x = 2$, about the x-axis?

Exercises

Exer 9.3.4

Use calculus to determine the area of the surface formed by rotating the curve of $f(x) = x$, between $x = 1$ and $x = 2$, about the x-axis.
Can you verify this using the formula for the surface area of a cone ($A = \pi r l$), where r is the radius of the base, and l is the length of the (sloping) side?