MA378 Chapter 1: Interpolation

§1.3 Interpolation Error Estimates

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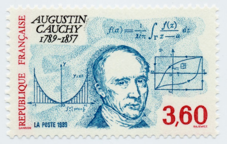


Source: http://jeff560.tripod.com/stamps.html

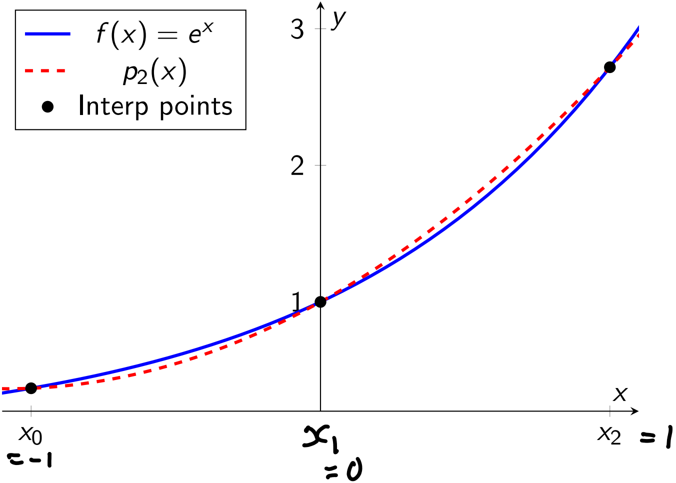
Augustin-Louis Cauchy (1789–1857), Paris, France. He was a pioneer of analysis, in particular in introducing rigour into calculus proofs. He founded the fields of complex analysis and the study of permutation groups.

3.0 Outline

Important: This section is based on Section 6.2 of the text-book (Suli and Mayers, Introduction to Numerical Analysis; or "[M+S]" for short). You can access the book from the Reading List on canvas. I have also posted Sections 6.1 and 6.2 to Canvas: https://universityofgalway.instructure.com/courses/46941/modules

3.1 Introduction

In our last example, we wrote down the polynomial of degree $n = 2$ interpolating $f(x) = e^x$ at $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.



3.1 Introduction

We now want to investigate how, in general, error in polynomial interpolation depends on

(i) the function (and its derivatives)

(ii) the number of points used (or, equivalently, degree of the polynomial used).

3.2 Rolle's Theorem

The main ingredient we need to the following theorem.

Theorem 3.1 (Rolle's Theorem)

Let g be a function that is continuous and differentiable on the interval $[a, b]$. If $g(a) = g(b)$, then there is at least one point c in (a, b) where $g'(c) = 0$

Our "proof" is by picture:[1]

$g'(c) = 0$

[1] One can easily deduce Rolle's Theorem from the Mean Value Theorem (MVT). But since the standard proof of the MVT uses Rolle's Theorem, that would be cheating.

3.3 Nodal Polynomial

The following is the most important theorem of NA2; it is used repeatedly through-out the semester. It's often called the *Polynomial Interpolation Error Theorem*, or *Cauchy's Theorem*.

First, we need to define an important polynomial.

Definition 3.2 (Nodal Polynomial)

The **Nodal Polynomial** π_{n+1} associated with the interpolation points that $a = x_0 < x_1 < \cdots < x_n = b$ is

$$\pi_{n+1}(x) := (x - x_0)(x - x_1)\dots(x - x_n) = \prod_{i=0}^{n}(x - x_i).$$

3.3 Nodal Polynomial

Example:

See board

Properties:

Eg $\pi_3(x) = (x-x_0)(x-x_1)(x-\cancel{x}_2)$

$$= x^3 + (?)x^2 + (?)x + ?$$

$$\pi_3'(x) = 3x^2 + 2(?)x + (?)$$

$$\pi_3''(x) = 6x + 2(?)$$

$$\pi_3'''(x) = 6 = 3!$$

And, in general, $\dfrac{d^{n+1}}{dx^{n+1}}\pi_{n+1}(x) = (n+1)!$

 Notation : $f^{(n+1)}(x) = \frac{d^{n+1}}{dx^{n+1}} f$

Theorem 3.3 (Cauchy, 1840)

Suppose that $n \geq 0$ and f is a real-valued function that is continuous and defined on $[a, b]$, such that the derivative of f of order $n+1$ exists and is continuous on $[a, b]$. Let p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a point $c \in (a, b)$ such that

$$\underbrace{f(x) - p_n(x)}_{\text{Error}} = \frac{f^{(n+1)}(c)}{(n+1)!}\pi_{n+1}(x). \tag{1}$$

Here is an outline of the proof; full details are in Theorem 6.2 of [S+M]. The crucial step is introducing an auxiliary function,

$$g(t) := f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\pi_{n+1}(t).$$

Proof: First note that if $x = x_i$ then $f(x_i) = p_n(x_i)$ (since p_n interpolates f at $x = x_i$). So $f(x_i) - p_n(x_i) = 0$

Also $\pi_{n+1}(x_i) = 0$. So the theorem is true when $x = x_i$.

Next, take x to be any point in $[a, b]$ but which is not an interpolation point. That is $x \neq x_i$. For such an x define

$$g(t) := f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t).$$

Proof:

Then
$$g(x_i) = f(x_i) - p_n(x_i) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(x_i)$$

which will be zero, since p_n interpolates f at x_i & $\pi_{n+1}(x_i) = 0$.

Also $g(x) = 0$ since
$$g(x) = f(x) - p_n(x) - \left(f(x) - p_n(x)\right)\frac{\pi_{n+1}(x)}{\pi_{n+1}(x)}$$
$$= f(x) - f(x) - p_n(x) + p_n(x) = 0$$

Proof:

So therefore $g(x)$ has $n+2$ zeros.

That is, it is zero at $x = x_i$,

and $x \neq x_i$.

Then we apply Rolle's Theorem:

$g(x)$ has $n+2$ zeros.

So $g'(x)$ has $n+1$ zeros

& $g'(x)$ has n zeros

And ... $g^{(n+1)}(x)$ has 1 zero.

3.4 Cauchy's Theorem

Proof: (Remaining details were done on the board. But, rougully, we now know that $g^{(n+1)}(c) = 0$ for some c.

So
$$0 = f^{(n+1)}(c) - p_n^{(n+1)}(c) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}^{(n+1)}(t).$$

However, $p_n^{(n+1)}(x) \equiv 0$ for all x

and $\pi_{n+1}^{(n+1)}(x) \equiv (n+1)!$ (Recall Slide 8)

So $$f^{(n+1)}(c) = \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n+1)!$$

Rearrange to finish the proof.

Example 3.4

In an earlier example, we wrote down the Lagrange form of the polynomial, p_2, that interpolates $f(x) = e^x$ at the points $\{-1, 0, 1\}$. Give a formula for $e^x - p_2(x)$.

Here $n=2$, $x_0=-1$, $x_1=0$, $x_2=1$.

Then
$$f(x) - p_2(x) = \frac{f'''(c)}{3!}(x+1)(x)(x-1)$$

Since $f(x)=e^x$, so $f'''(x) = e^x$.

So
$$f(x) - p_2(x) = \frac{e^c}{6} x(x^2-1)$$

Note: we don't know the value of c
(just that it exists...)

Usually (and as in the above example), we can't calculate $f(x) - p_n(x)$ exactly from Formula (1), because we have no way of finding c. However, we are typically not so interested in what the error is at some given point, but what is the maximum error over the whole interval $[x_0, x_n]$. That is given by:

Corollary 3.5

Define

$$M_{n+1} = \max_{x_0 \leq \sigma \leq x_n} |f^{(n+1)}(\sigma)|.$$

Then, for any x,

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|. \tag{2}$$

Example 3.6

Let p_1 be the polynomial of degree 1 that interpolates a function f at distinct points x_0 and x_1. Letting $h = x_1 - x_0$, show that

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

We know $|f(x) - p_1(x)| \leq \frac{M_2}{2} (x - x_0)(x - x_1)$

Also

$$\max_{x_0 \leq x \leq x_1} |(x - x_0)(x - x_1)| = |(x_m - x_0)(x_m - x_1)|$$

$$= \left| \left(\frac{h}{2}\right)\left(\frac{h}{2}\right) \right|$$

where $x_m = \dfrac{x_0 + x_1}{2}$

So $|f(x) - p_1(x)| \leq \dfrac{M_2}{2} \dfrac{h^2}{4}$

3.5 Exercises

Exercise 3.1

Read Section 6.2 of An Introduction to Numerical Analysis (Süli and Mayers). Pay particular attention to the proof of Thm 6.2 at https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=192.

Exercise 3.2

Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0 \leq x \leq x_2} |f(x) - p_2(x)| \leq \frac{1}{6}\frac{2}{3\sqrt{3}}h^3 M_3 = \frac{1}{9\sqrt{3}}h^3 M_3.$$

Hint: simplify the calculations by taking $t = x - x_1$, *writing* $(x - x_0)(x - x_1)(x - x_2)$ *in terms of* h *and* t.