

MA385 Part 4: Linear Algebra 2

## 4.3: Condition Numbers

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# 1. Outline Section 4.3

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For more, see Section 2.7 of Suli and Mayer:

<https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG>

## 2. Motivation

Numerical solutions to some linear systems are adversely affected by round-off errors.

This phenomenon is due to the *matrices* in the linear systems. Those matrices for which the issue is particularly prevalent are referred to as being **ill-conditioned**.

For any matrix, we can assign a numerical *score* that gives an indication of whether it is ill-conditioned. That score, called the **condition number**, is defined in terms of matrix norms, and is the subject of these section.

### 3. Consistency of matrix norms

Suppose we have a vector norm,  $\|\cdot\|$  and associated subordinate matrix norm. It is not hard to see that

$$\|A\mathbf{u}\| \leq \|A\|\|\mathbf{u}\| \quad \text{for any } \mathbf{u} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}.$$

Here is why:

### 3. Consistency of matrix norms

There is an analogous statement for the product of two matrices:

#### Definition 4.3.1 (Consistent matrix norm)

A matrix norm  $\|\cdot\|$  is **consistent** (or “*sub-multiplicative*”) if

$$\|AB\| \leq \|A\|\|B\|, \quad \text{for all } A, B \in \mathbb{R}^{n \times n}.$$

#### Theorem 4.3.1

Any subordinate matrix norm is consistent.

The proof is left to an exercise. That exercises also demonstrates that there are matrix norms which are *not* consistent.

## 4. Computer representation of numbers

**[Please read this slide in your own time!]**

Modern computers don't store numbers in decimal (base 10), but in binary (base 2) “floating point numbers” of the form :

$$x = \pm a \times 2^{b-M}.$$

Most use *double precision*, where 8 bytes (64 bits or *binary digits*) are used to store

- ▶ the sign (1 bit),
- ▶  $a$ , called the “significand” or “mantissa” (52 bits)
- ▶ and the exponent,  $b - 1023$  (11 bits)

Note that  $a$  has roughly 16 decimal digits.

(Some older computer systems sometimes use *single precision* where  $a$  has 23 bits — giving 8 decimal digits — and  $b$  has 7; so too do many new GPU-based systems).

## 4. Computer representation of numbers

When we try to store a real number  $x$  on a computer, we actually store the nearest floating-point number. That is, we end up storing  $x + \delta x$ , where  $\delta x$  is the “round-off” error.

But the quantity we are mainly interested in is the **relative error**:  $|\delta x|/|x|$ .

Since this is not a course on computer architecture, we'll simplify a little and just take it that single and double precision systems lead to a relative error of  $10^{-8}$  and  $10^{-16}$  respectively.

## 5. Condition Numbers

(See p68–70 of Süli and Mayers for a full development of the concept of a condition number).

Suppose we use, say,  $LU$ -factorization and back-substitution on a computer to solve

$$A\mathbf{x} = \mathbf{b}.$$

Because of the “round-off error” we actually solve

$$A(\mathbf{x} + \delta\mathbf{x}) = (\mathbf{b} + \delta\mathbf{b}).$$

Our problem now is, for a given  $A$ , if we know the (relative) error in  $\mathbf{b}$ , can we find an upper-bound on the relative error in  $\mathbf{x}$ ?

## 5. Condition Numbers

### Definition 4.3.2

The *condition number* of a matrix, with respect to a particular matrix norm  $\|\cdot\|_*$  is

$$\kappa_*(A) = \|A\|_* \|A^{-1}\|_*.$$

If  $\kappa_*(A) \gg 1$  then we say  $A$  is *ill-conditioned*.

**Example:** Find the condition number  $\kappa_\infty$  of

$$A = \begin{pmatrix} 10 & 12 \\ 0.08 & 0.1 \end{pmatrix}.$$

## 5. Condition Numbers

### Theorem 4.3.2

Suppose that  $A \in \mathbb{R}^{n \times n}$  is nonsingular and that  $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$  are non-zero vectors. If  $A\mathbf{x} = \mathbf{b}$  and  $A(\mathbf{x} + \delta\mathbf{x}) = (\mathbf{b} + \delta\mathbf{b})$  then

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A) \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}.$$

## 6. Calculating $\kappa_\infty$ and $\kappa_1$

### Example 4.3.1

Suppose we are using a computer to solve  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{pmatrix} 10 & 12 \\ 0.08 & 0.1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

But, due to round-off error, right-hand side has a relative error (in the  $\infty$ -norm) of  $10^{-6}$ . Give a bound for the relative error in  $\mathbf{x}$  in the  $\infty$ -norm.

## 6. Calculating $\kappa_\infty$ and $\kappa_1$

For every matrix norm we get a different condition number.

### Example 4.3.2

Let  $A$  be the  $n \times n$  matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

What are  $\kappa_1(A)$ , and  $\kappa_\infty(A)$ ?

First we compute  $\|A\|_1$  and  $\|A\|_\infty$ .

## 6. Calculating $\kappa_\infty$ and $\kappa_1$

For this very special example, it is easy to write down the inverse of  $A$ :

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

## 7. Estimating $\kappa_2$

To compute  $\kappa_1(A)$  and  $\kappa_\infty(A)$ , we need to know  $A^{-1}$ , which is usually not practical. However, for  $\kappa_2$ , we are able to *estimate* the condition number of  $A$  without knowing  $A^{-1}$ .

Recall that  $\|A\|_2 = \sqrt{\lambda_n}$  where  $\lambda_n$  is the largest eigenvalue of  $B = A^T A$ .

We can also show that  $\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_1}}$  where  $\lambda_1$  is the smallest eigenvalue of  $B$

- ▶  $A^T A$  and  $AA^T$  have the same eigenvalues (they are “*similar*”):
- ▶ For any non-singular matrix  $X$ , we have that  $(X^T)^{-1} = (X^{-1})^T$ .
- ▶  $(A^T A)^{-1} = (AA^T)^{-1}$

## 7. Estimating $\kappa_2$

So

$$\kappa_2(A) = \left( \lambda_n / \lambda_1 \right)^{1/2}.$$

.....  
Motivated by this, we'll finish MA385, by studying an easy way of estimating the eigenvalues of a matrix.

## 8. Exercises

### Exercise 4.3.1

- (i) Prove that, if  $\|\cdot\|$  is a subordinate matrix norm, then it is *consistent*, i.e., for any pair of  $n \times n$  matrices,  $A$  and  $B$ , we have  $\|AB\| \leq \|A\|\|B\|$ .
- (ii) One might think it intuitive to define the “max” norm of a matrix as follows:

$$\|A\|_{\infty} = \max_{i,j} |a_{ij}|.$$

Show that this is indeed a norm on  $\mathbb{R}^{n \times n}$ . Show that, however, it is not consistent.

## 8. Exercises

### Exercise 4.3.2

Let  $A$  be the matrix

$$A = \begin{pmatrix} 0.1 & 0 & 0 \\ 10 & 0.1 & 10 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Compute  $\kappa_{\infty}(A)$ . Suppose we wish to solve the system of equations  $A\mathbf{x} = \mathbf{b}$  on *single precision* computer system (i.e., the relative error in any stored number is approximately  $10^{-8}$ ). Give an upper bound on the relative error in the computed solution  $\mathbf{x}$ .