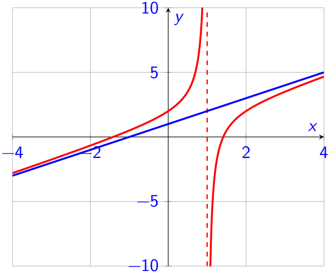
2526-MA140: Week 03, Lecture 2 (L05)

Inroduction to Limits

Dr Niall Madden

University of Galway

Wednesday, 24 September, 2025



Slides by Niall Madden, with some material adapted from textbooks, and original notes by Dr Kirsten Pfeiffer.

Annotated slides.
Note: first 20 minutes of the class were given by Dr Pfeiffer on SUMS, Diagnostic Test, and Digital Badge

Outline

1 Reminders

2 Towards Limits

3 Definition of a Limit

4 Properties of Limits
- Evaluating limits

5 Limits of rational functions

6 More limits

7 Exercises

For more, see Chapter 2 (Limits) of Strang and Herman's Calculus, especially Sections 2.2 (Limit of a Function) and 2.3 (Limit Laws).

Slides are on canvas, and at niallmadden.ie/2526-MA140

Reminders

- Tutorials started **this** week.
- Current assignment (for this week's tutorials) is `PS-0`. Just for practice. See https://universityofgalway.instructure.com/courses/46734/assignments/128373
- **Assignment 1** (`PS-1`) due 5pm, Monday 5 October. Will be covered in tutorials next week.
- Two class tests planned for this module, each worth 10% of the final grade.
  - Test 1: **Tuesday, 14 October** (Week 5)
  - Test 2: **Tuesday, 18 November** (Week 10)
  - Contact Niall if you have any concerns, or wish to avail of alternative arrangements, as provided by LENS reports.

Towards Limits

When we were considering the domain of a function, we looked at those x-values for which the function was not defined.

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1}$$

Neither f nor g are defined at $x = 1$. ("not def")

But what happens if x gets very closed to 1?

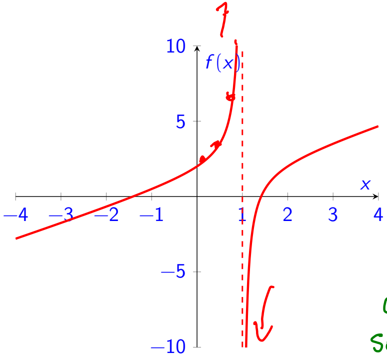
x	0.900	0.990	0.999	1	1.001	1.010	1.100
$f(x)$	11.9	101.99	1001.99	not def	-999.99	-99.9	-9.99
$g(x)$	1.9	1.99	1.999	not def	2.001	2.01	2.1

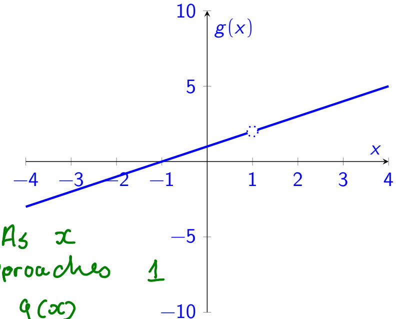
Let's look at the graphs of f and g.

Towards Limits

Example

$$f(x) = \frac{x^2 - 2}{x - 1} \qquad g(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1}$$





As x approaches 1 So $g(x)$ approaches 2.

Towards Limits

In the previous example, we saw that, although neither f nor g was defined at $x = 1$, they behaved very differently as x approaches 1.

To discuss this we'll need the concept of a **limit** which, roughly, relates to the value of function as it **approaches** a point (but not actually at that point).

$$\lim_{x \to a} f(x) = L$$

The concept of a limit is a prerequisite for a proper understanding of calculus, and numerical methods.

Definition of a Limit

Some conventions and terminology we'll use:

- x is a variable. (ie $x \in \mathbb{R}$)
- a is a fixed number. (ie some particular number in $\mathbb{R}$)
- ϵ is a **small** positive number (that we get to choose).
- δ is another **small** positive number (determined by ϵ).
- $|x - a| < \delta$ means that the distance between x and a is less than δ, i.e. very small.
- As x approaches a, so $f(x)$ approaches a number L.

When we write

$$\lim_{x \to a} f(x) = L,$$

" $x \to a$ " = " x goes to a " (but not $x = a$)

we read

"The limit of f, as x goes to a, is L".

Definition of a Limit

LIMIT: formal definition

$$\lim_{x \to a} f(x) = L,$$

means that, for every number $\epsilon > 0$, it is possible to find a number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

LIMIT: Informal explanation

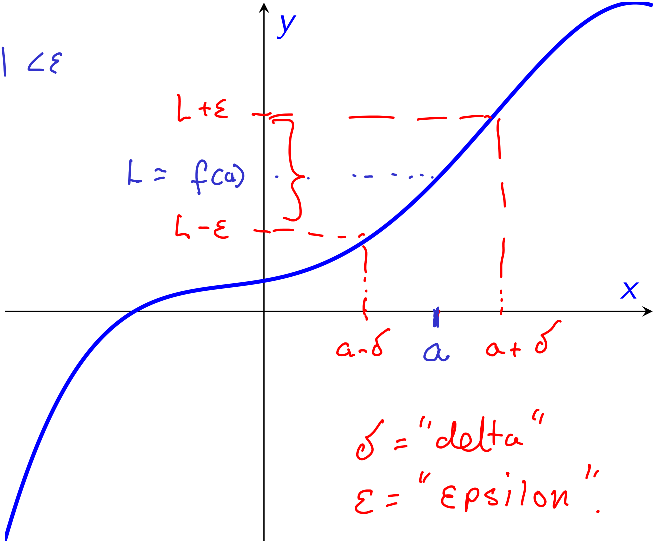
$$\lim_{x \to a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

Definition of a Limit

want

$|f(x) - L| < \varepsilon$



δ = "delta"

ε = "epsilon".

Definition of a Limit

Example

Prove formally that $\lim_{x \to 3} (4x - 5) = 7$.

That is, for arbitrary ϵ, find a δ such that

$$|(4x-5)-7| < \epsilon \quad \text{whenever} \quad |x-3| < \delta.$$

- We want

$$|(4x-5)-7| < \varepsilon$$

$$\Rightarrow \quad |4x-12| < \varepsilon \Rightarrow 4|x-3| < \varepsilon$$

$$\Rightarrow \quad |x-3| < \frac{\varepsilon}{4}$$

So take $\delta < \dfrac{\varepsilon}{4}$ ✓

Definition of a Limit

The approach we just used is technically correct, but not very practical in many cases.

Fortunately, there are other methods that can be used

- to show that a limit exists;
- find the limit of a function, $f(x)$ as $x \to a$.

Properties of Limits

See also...

... Section 2.3 of the textbook: "Limit Laws" (Link)

Suppose that $\lim_{x\to a} f_1(x) = L_1$, and $\lim_{x\to a} f_2(x) = L_2$ and $c \in \mathbb{R}$ is any constant. Then,

(1) $\lim_{x\to a} c = c$, $c \in \mathbb{R}$ That is, if the function is constant, $f(x) = c$ then the limit is alway c too.

(2) $\boxed{\lim_{x\to a} x = a}$ That is, if $f(x) = x$ then $\lim_{x\to a} f(x) = a$.

Properties of Limits

(3) $\lim\limits_{x\to a}[cf_1(x)] = cL_1$ We know $\lim\limits_{x\to a} f_1(x) = L$

Eg $\lim\limits_{x\to 3}(3x) = 3\lim\limits_{x\to 3}(x) = (3)(3) = 9.$

(4) $\lim\limits_{x\to a}[f_1(x) + f_2(x)] = L_1 + L_2$ and

$\lim\limits_{x\to a}[f_1(x) - f_2(x)] = L_1 - L_2$

$$\lim_{x\to a}\left(f_1(x) + f_2(x)\right) = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x)$$

$$= L_1 + L_2.$$

Properties of Limits

$\ast$ (5) $\lim_{x\to a}\left(f_1(x)f_2(x)\right) = L_1L_2$

So $\lim_{x\to a}\left(f_1(x)\,f_2(x)\right) = \lim_{x\to a} f_1(x)\cdot \lim_{x\to a} f_2(x)$

$= L_1\,L_2.$

(6) $\lim_{x\to a}\left((f_1(x))^n\right) = (L_1)^n$

$$\lim_{x\to a}\Big[\underbrace{\overset{L_1}{f_1(x)}\cdot \overset{L_1}{f_1(x)}\cdot \overset{L_1}{f_1(x)}\cdots \overset{L_1}{f_1(x)}}_{n \text{ times}}\Big] = L_1^{\,n}.$$

Properties of Limits

(7) $\lim_{x \to a} \left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$, providing $L_2 \neq 0$.

(8) $\lim_{x \to a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$

Finished here.

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\boxed{\lim_{x \to a} x^n = a^n}$$

Example

Evaluate the limit $\lim_{x \to 1} (x^3 + 4x^2 - 3)$

Example

Evaluate $\lim_{x\to 1} \dfrac{x^4+x^2-1}{x^2+5}$ using the Properties of Limits.

Limits of rational functions

In many cases it's more complicated. In particular, we'll consider numerous examples where we want to evaluate $\lim_{x \to a} f(x)$ where a is not in the domain of f.

A typical example of this is when we evaluate a rational function:

$$\boxed{\lim_{x \to a} \frac{p(x)}{q(x)}}$$

where **both** $p(a) = 0$ and $q(a) = 0$.

Idea: Since we care about the value of p and q **near** $x = a$, but not actually at $x = a$, it is safe to factor out an $(x - a)$ term from both.

Limits of rational functions

Three examples

Evaluate the limits:

$$(a)\ \lim_{x\to 0} \frac{x}{x} \qquad (b)\ \lim_{x\to 0} \frac{x^2}{x} \qquad (c)\ \lim_{x\to 0} \frac{x}{x^2}$$

Limits of rational functions

Example

Evaluate
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

Limits of rational functions

In that last example, we found that

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x}$$

But these are different functions:

Limits of rational functions

Evaluate the limit

$$\lim_{x \to 2} \left(\frac{\frac{1}{2} - \frac{1}{x}}{x - 2} \right)$$

More limits

Very often, we'll evaluate limits of the form:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)}}$$

where f and g are not polynomials. Some of the same ideas still apply.

Example

Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

Exercises

Exercise 2.2.1

Evaluate the following limits

(a) $\displaystyle\lim_{x\to\frac{1}{2}} \frac{x-\frac{1}{2}}{x^2-\frac{1}{4}}$

(b) $\displaystyle\lim_{x\to-4} \frac{x^2+3x-4}{x^2+x-12}$

Exercise 2.2.2 (from 2425-MA140 exam)

Let $f(x) = \dfrac{x^2-2x-15}{3x^3-6x^2-45x}$. For each of the following, evaluate the limit, or determine that it does not exist.

(i) $\displaystyle\lim_{x\to-3} f(x)$ (ii) $\displaystyle\lim_{x\to 0} f(x)$ (iii) $\displaystyle\lim_{x\to 5} f(x)$