2425-MA140 Engineering Calculus

Week 05, Lecture 1
Implicit Differentiation; Exponential Functions

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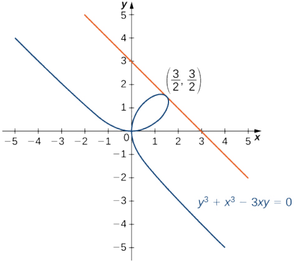
Tuesday, 15 October, 2024



Assignments

From now on, all assignments will have deadlines: **Monday at 17:00**

Assignment 3 is open. See https://universityofgalway.instructure.com/courses/35693/assignments/97067.

Deadline is 17:00, Monday 21 October. The associated **tutorial sheet** is at https://universityofgalway.instructure.com/courses/35693/files/2084087

Remaining Deadlines:

- Assignment 3: Monday 21 Oct (Week 6)
- Assignment 4: Monday 04 Nov (Week 8)
- Assignment 5: Monday 11 Nov (Week 9)
- Assignment 6: Monday 18 Nov (Week 10)
- Assignment 7: Monday 25 Nov (Week 11)

This lovely Tuesday morning, we'll discuss...

1. Implicit differentiation
2. Exponential functions
   - Properties
   - The number e
   - The derivative of e^x
3. Logarithms
   - Properties
   - The natural logarithm
   - Derivative of $\ln(x)$
   - Logarithmic differentiation
4. Exercises

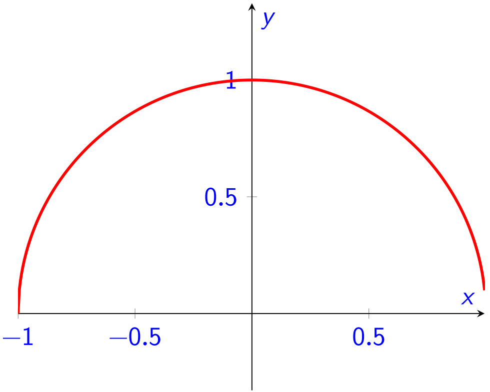
See also: Sections 3.8 (Implicit Differentiation) and 3.9 (Derivatives of Exponential and Logarithmic Functions) of **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

We'll also recap some basic information about exponential and logarithmic functions from Section 1.5 of that text.

Implicit differentiation

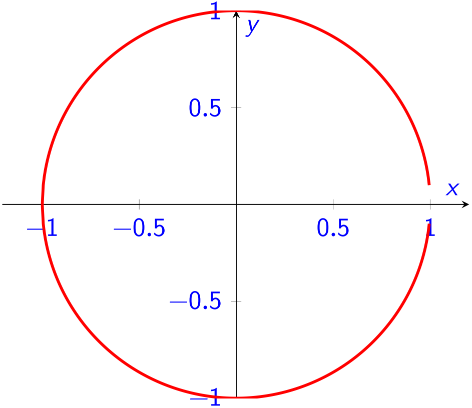
To date, most functions we have studied have been **explicitly** defined. Such functions and be written as $y = f(x)$: given a value of x we can substitute it into $f(x)$ to get the corresponding value of y

Example: $y = \sqrt{1 - x^2}$.



Implicit differentiation

However, sometimes we are given an equation involving x and y where these two terms are not "separated" entirely; e.g, $x^2 + y^2 = 1$. Here y is **implicitly** defined: for any pair (x, y) we can check if it is on the curve described by the equation.



Implicit differentiation

Since **implicit equations** define curves, we can use **implicit differentiation**, for example, finding tangents to these curves.

Method:

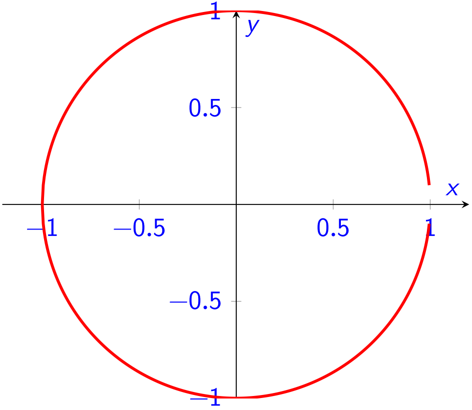
1. Differentiate both size of the equation, with respect to x, keeping in mind that y is a function of x, using the Chain Rule where needed.
2. Solve for dy/dx.

Implicit differentiation

If y is defined by $x^2 + y^2 = 1$, find $\dfrac{dy}{dx}$.

Implicit differentiation

Now we know that if $x^2 + y^2 = 1$, then $\dfrac{dy}{dx} = -\dfrac{x}{y}$. We can check that this relates to the slope of the tangents to this curve at various places:



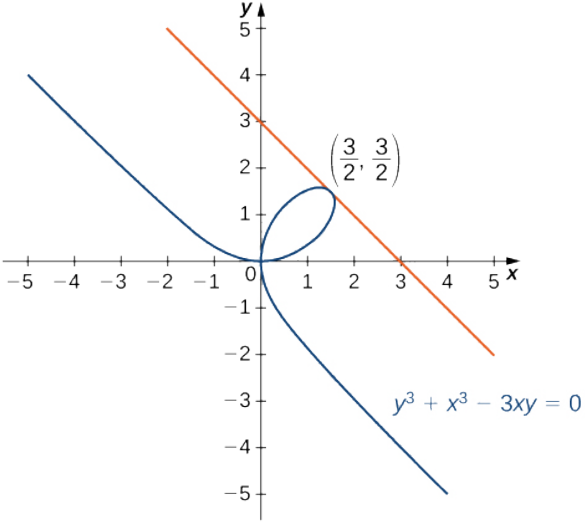
Implicit differentiation

Find the tangent to the curve $x^2 + y^2 = 25$, at the point $(3, -4)$.

Implicit differentiation

Find the tangent to the curve $y^3 + x^3 - 3xy = 0$, at the point $(3/2, 3/2)$.

Implicit differentiation



Exponential functions

Earlier in this course we met functions such as $y = x^2$; this is a **power** function.

Now we consider **exponential functions**, such as $\boxed{y = 2^x}$.

Such functions occur in many applications. For example: if I invest €100 with an annual interest rate of 20%, then after x years, I will have €$100 \times (1.2)^x$. **Why?**

Exponential functions

Exponential functions grow quite fast: if my investment is indeed worth $f(x) = 100 \times (1.2)^x$ euros after x years, then...

- After 1 year, I have €120
- After 10 years, I have €619.17
- After 20 years, I have €3,833.80
- After 25 years, I have €9,539.60
- After 50 years, and 190 days, I'll be a millionaire!

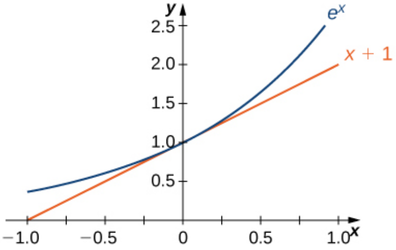
Here I remind you of some properties of exponents that you should already know: for any positive numbers a and b,

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x a^y$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{a^y}$

The number $e \approx 2.7182818284$. It is often called **Euler's Number** after Leonard Euler, who did not discover it: that was (probably) Jacob Bernoulli in 1683 while studying compound interest. Or maybe 100 years earlier by John Napier.

The Natural Exponential Function

The Natural Exponential Function is $f(x) = e^x$. It is special for many reasons, including the its tangent at $x = 0$ has slope 1.



Let's assume that e is the number for which, if $f(x) = e^x$, then $f'(0) = 1$. Using the limit definition of the derivative, this means

$$1 = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

From this can deduce that...

So now we know that

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

That is e^x is the function that is its own derivative!!!

Example

Compute the derivative of $f(x) = e^{\sin(x)}$

Logarithms

Suppose that $y = f(x)$ is an **exponential** function; that is: $y = b^x$ for some $b > 0$ (and excluding $x = 1$).

Its **inverse** is called a **logarithmic function**, denoted $\log_b$

$$\text{If } y = b^x \quad \text{then} \quad \log_b(y) = x.$$

Examples

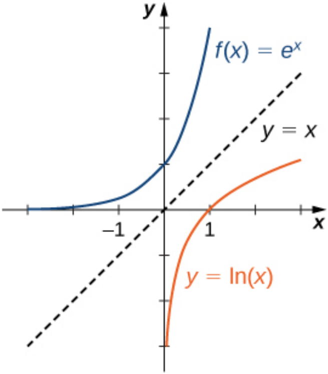
- $log_2(8) = 3$
- $\log_{10}(100) = 2$
- $\log_e(e^x) = x$

Properties of Logarithms

If $a, b, c > 0$ and $b \neq 1$m then

- $\log_b(ac) = \log_b(a) + \log_b(c)$
- $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$
- $\log_b(a^r) = r\log_b(a)$

We denote $\boxed{\log_e(x)}$ as $\boxed{\ln(x)}$



$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Why?

Example:

Find the derivative of $f(x) = \ln(x^2 + 2x + 3)$.

To finish we introduce the idea of **logarithmic differentiation**, which helps us differentiate functions with x, or a function of x in the exponent, such as $y = (2x)^{\sin(x)}$ or $y = x^x$.

Strategy:

- Take ln of both sides
- Simplify, using properties of logarithms.
- Differentiate.
- Solve for $\dfrac{dy}{dx}$

Example [2019 exam, Q2(b)(iii)]

Differentiate $f(x) = x^x$.

Exercises

Exercise 5.1.1

Find the equation of the tangent to the curve defined by $x^2 - y^2 = 16$ at the point $(5, 3)$.