MA385 Part 3: Linear Algebra 1

3.3 LU-factorisation

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In these slides,

- *LT means "lower triangular"*
- *UT means "upper triangular"*

1. Outline of Section 3.3

1. A formula for LU-factorisation
2. Existence of an LU-factorisation
3. Exercises

For more, see Section 2.3 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG

1. Outline of Section 3.3

The goal of this section is to demonstrate that the process of Gaussian Elimination applied to a matrix A is equivalent to factoring A as the product of a unit lower triangular and upper triangular matrix.

The Section 3.2 we saw that each elementary row operation in Gaussian Elimination involves replacing A with $(I + \mu_{rs}E^{(rs)})A$.

Example: For the 3×3 case, this involved computing

$$(I + \mu_{32}E^{(32)})(I + \mu_{31}E^{(31)})(I + \mu_{21}E^{(21)})A.$$

1. Outline of Section 3.3

In general we multiply A by a sequence of matrices

$$(I + \mu_{rs} E^{(rs)}),$$

all of which are unit lower triangular matrices.
When we are finished we have reduced A to an upper triangular matrix.
So we can write the whole process as

$$L_k L_{k-1} L_{k-2} \ldots L_2 L_1 A = U, \tag{1}$$

where each of the L_i is a unit LT matrix.

But from Theorem 3.2.6, we know that the product of unit LT matrices is itself a unit LT matrix. So we can write the whole process described in (1) as

$$\tilde{L}A = U. \tag{2}$$

But Theorem 3.2.6 also tells us that the inverse of a unit LT matrix exists and is a unit LT matrix. So we can write (2) as

$$A = LU$$

where L is unit lower triangular and U is upper triangular. This is called "LU-**factorisation**".

Definition 3.4.1

The LU-**factorization** of the matrix is a unit lower triangular matrix L and an upper triangular matrix U such that $LU = A$.

Example 3.4.1

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$ then:

Example 3.4.2

If $A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & -4 \end{pmatrix}$ then:

2. A formula for LU-factorisation

We now want to work out formulae for L and U where

$$a_{i,j} = (LU)_{ij} = \sum_{k=1}^{n} l_{ik} u_{kj} \qquad 1 \leq i,j \leq n.$$

Since L and U are triangular,

$$\text{If } i \leq j \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{i} l_{ik} u_{kj} \tag{3a}$$

$$\text{If } j < i \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{j} l_{ik} u_{kj} \tag{3b}$$

2. A formula for LU-factorisation

The first of these equations can be written as

$$a_{i,j} = \sum_{k=1}^{i-1} l_{ik} u_{kj} + l_{ii} u_{ij}.$$

But $l_{ii} = 1$ so:

$$u_{i,j} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \quad \begin{cases} i = 1, \dots, j-1, \\ j = 2, \dots, n. \end{cases} \tag{4a}$$

And from the second:

$$l_{i,j} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right) \quad \begin{cases} i = 2, \dots, n, \\ j = 1, \dots, i-1. \end{cases} \tag{4b}$$

Example 3.4.3

Find the LU-factorisation of

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -2 & -2 & 1 & 4 \\ -3 & -4 & -2 & 4 \\ -4 & -6 & -5 & 0 \end{pmatrix}$$

Full details of Example 10: First, using (4a) with $i = 1$ we have $u_{1j} = a_{1j}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

Then (4b) with $j = 1$ we have $l_{i1} = a_{i1}/u_{11}$:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & l_{32} & 1 & 0 \\ 4 & l_{42} & l_{43} & 1 \end{pmatrix}.$$

Next (4a) with $i = 2$ we have $u_{2j} = a_{2j} - l_{21}u_{2j}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix},$$

then (4b) with $j = 2$ we have $l_{i2} = (a_{i2} - l_{i1}u_{12})/u_{22}$:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & l_{43} & 1 \end{pmatrix}$$

Etc....

3. Existence of an LU-factorisation

Not every matrix has an LU-factorisation. So we need to characterise the matrices that do.

To prove the next theorem we need the Cauchy-Binet Formula: $\det(AB) = \det(A)\det(B)$.[1]

Theorem 3.4.1

If $n \geq 2$ and $A \in \mathbb{R}^{n\times n}$ is such that every leading principal submatrix of A is nonsingular for $1 \leq k < n$, then A has an LU-factorisation.

[1] Wikipedia disagrees with this attribution

3. Existence of an LU-factorisation

Exercise 3.4.1

Many textbooks and computing systems compute the factorisation $A = LDU$ where L and U are unit lower and *unit* upper triangular matrices respectively, and D is a diagonal matrix. Show such a factorisation exists, providing that if $n \geq 2$ and $A \in \mathbb{R}^{n \times n}$, then every leading principal submatrix of A is nonsingular for $1 \leq k < n$.