Linear Algebra I - Assignment 4 ANS Solutions

Q1 [25 Marks] Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n, in the variable $t \in \mathbb{R}$. Which of the following are **subspaces** of $\mathbb{P}_3$? Explain your answers.

(a) $H_0 := \{\vec{0}\}$, where $\vec{0}$ is the zero vector in $\mathbb{P}_2$.

> **Answer:** Yes. $\mathbb{P}_2$ in a subspace of $\mathbb{P}_3$, the zero vector in $\mathbb{P}_2$ is also the zero vector for $\mathbb{P}_3$. Furthermore, the zero vector on its own constitutes a vector space. [4 Marks]

(b) $H_1 := \{\vec{0}, t, t^2, t^3\}$.

> **Answer:** No. This is a subset of $\mathbb{P}_3$, but not a subspace. For example, it does not include $t + t^2$. [3 Marks]

(c) $H_2 := \text{span}\{4t^2\}$.

> **Answer:** Yes, since the span of any set of vectors in a space is a subspace of it. [3 Marks]

(d) $H_3 := \text{span}\{t, t^3\}$

> **Answer:** Yes, since the span of any set of vectors in a space is a subspace of it. [3 Marks]

(e) $H_4 := \{p(t) \in \mathbb{P}_1\}$.

> **Answer:** Yes, since $\mathbb{P}_1$ is a subset of $\mathbb{P}_3$, and also closed under addition and scalar multiplication. However, I'd also accept "No", since the question is not correctly written: the set could be understood to mean a single vector from $\mathbb{P}_1$. [3 Marks]

(f) $H_5 := \{p(t) \in \mathbb{P}_2\}$.

> **Answer:** Same as (f). [3 Marks]

(g) $H_6 := \{p(t) \in \mathbb{P}_2 : p'(0) = 0\}$.

> **Answer:** Yes. Firstly, any polynomial in $\mathbb{P}_2$ also belongs to $\mathbb{P}_3$. Second, if $p'(0) = 0$, $q'(0) = 0$, and $r = p + q$, then $r'(0) = p'(0) + q'(0) = 0$. So it is closed under addition. Similarly it is closed under scalar multiplication. Finally, the zero vector, z has $z'(0) = 0$. [3 Marks]

(h) $H_7 := \{p(t) \in \mathbb{P}_2 : p(1) = 0\}$.

> **Answer:** Yes: same as (g). [3 Marks]

Tip: in Week 2 we saw that, in order to verify that H is a subspace of a real vector space V, we have to check:

- That every element of H is also an element of V;
- That the zero vector in V is also in H;
- If $u, v \in H$ then $u + v \in H$.
- If $u \in H$ then $cu \in H$ for any scalar $c \in \mathbb{R}$.

Q2 [15 Marks] (This is Q1(c) on the 2021/2022 exam paper). Let

$$A = \begin{bmatrix} -3 & 8 & 19 \\ 1 & -6 & -13 \\ 2 & -2 & -6 \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}.$$

Determine, with justification, if $x \in \text{Nul}\, A$, and if $x \in \text{Col}\, A$.

> **Answer:** Yes, $x \in \text{Nul}\, A$ since $Ax = 0$ [6 Marks]
>
> And, yes, $y \in \text{Col}\, A$, since $A \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = y$. This can be worked out by row-reduction, or just observation.[9 Marks]

Q3 [15 Marks] Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} p+q+r \\ p+q+r \\ p+2q-r \\ p+2q-r \end{bmatrix} : p, q, r \in \mathbb{R} \right\},$$

of $\mathbb{R}^4$ and give a basis for it.

Answer: And element of H can be written as $pu + qv + rw$, where $u = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $v = \begin{bmatrix}1\\1\\2\\2\end{bmatrix}$ and $w = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$. By observation (or row reduction) we can see that $w = 2u - 3v$. So w is linearly dependent on u and v. Also, u is not a scalar multiple of v, so u and v are linearly independent. We can concluded that (for example) $\{u, v\}$ is as basis for H, and H has dimension 2.

Q4 [20 Marks] (Based on Q3(b) on the 2021/2022 exam paper).

(a) What is the largest possible rank of an 10×5 matrix? ANS 5. [4 Marks]

(b) If the null space of a 10×8 matrix A is 1-dimensional, what are the dimensions of its column space, of its row space, and of its left null space?

Answer: For a $m \times n$ matrix, $\dim \text{Col}\, A + \dim \text{Nul}\, A = n$. So $\dim \text{Col}\, A = 8 - 1 = 7$. Also, $\dim \text{Col}\, A = \dim \text{Nul}\, A$, so $\dim \text{Nul}\, A = 7$. Finally, the dimension of the left null space, i.e., $\dim \text{Nul}\, A^T$ is $m - \dim \text{Col}\, A = 10 - 7 = 3$. [4 Marks]

(c) Give an example of a 4×3 matrix A with nullity $A = 2$.

Answer: Take any 4×3 matrix with column space of dimension 1. That is, each column should be a multiple of another. Even, $A = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ will do. [4 Marks]

(d) Suppose a $m \times n$ matrix has $x = \begin{bmatrix}-1\\1\end{bmatrix}$ in both its null and column space. What are m and n?

Answer: $\text{Nul}\, A \subset \mathbb{R}^n$, so $n = 2$. $\text{Col}\, A \subset \mathbb{R}^m$, so $m = 2$. [4 Marks]

(e) Give an example of a matrix that has $x = \begin{bmatrix}-1\\1\end{bmatrix}$ in its null space, and $y = \begin{bmatrix}1\\0\\1\end{bmatrix}$. in its column space.

Answer: $A = \begin{bmatrix}1&-1\\0&0\\1&1\end{bmatrix}$ There are other possibilities, but all are 3×2 matrices, have columns that are multiples of $[1, 0, 1]^T$, and sum to zero. [4 Marks]

Q5 [10 MARKS] (This is similar to Q2(a) on the 2021/2022 exam paper). Let $\mathbb{P}_n$ denote the vector space of polynomials of degree at most n. Determine if

$$p_1(t) = 1 - 2t, \quad p_2(t) = 3 + 4t, \quad \text{and} \quad p_3(t) = 5,$$

are linearly independent in $\mathbb{P}_1$. Give a basis for Span$\{p_1(t), p_2(t), p_3(t)\}$.

Answer: They are not linearly independent. Since $\mathbb{P}_1$ has dimension two, any linearly independent set can have at most 2 vectors. Alternative, observe that $p_3 = 2p_1 + p_2$. Or try to solve:

$$c_1(1 - 2t) + c_2(3 + 4t) + c_3(5) = 0.$$

That is actually two equations:

$$c_1 + 3c_2 + 5c_3 = 0, \text{ and } -2c_1 + 4c_2 = 0.$$

This has nontrivial solution $c_1 = 2, c_2 = 1, c_3 = -1$. [5 MARKS]
Since any pair of these polynomials are linearly independent, they will suffice as a basis. For example, take $\{p_1, p_2\}$. However, the usual basis $\{1, t\}$ is also OK (or, indeed, any pair of linearly independent polynomials.) [5 MARKS]

[15 MARKS] for clarity and correctness of exposition and presentation.

Answer: [15 MARKS] for well presented, clearly written and explained solutions.
[10 MARKS] if we can easily read and understand most of what is written, even if some details are not well explained.
[5 MARKS] for solutions not written with care, or are in parts unintelligible, or not easy to understand.
[0 MARKS] if we've no idea what you are on about.