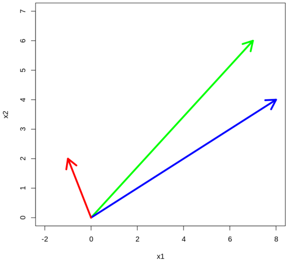
Week 10: Orthogonal Everything

Dr Niall Madden

8 and 11 November, 2022



R code

```
v <- c(7,6)
w <- c(8,4)
z <- c(-1,2)
plot(NULL, xlim=c(-2,8), ylim=c(0,7)),
   xlab="x1", ylab="x2")
arrows(0,0, v[1], v[2],lwd=4,col="green")
arrows(0,0, w[1], w[2],lwd=4,col="blue")
arrows(0,0, z[1], z[2],lwd=4,col="red")
```

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details,

- Section 6.1 (Inner Product, Length and Orthogonality) of the Lay et al text-book `https://nuigalway-primo.hosted.exlibrisgroup.com/permalink/f/1pmb9lf/353GAL_ALMA_DS5192067630003626`
- Chapters 6 and 9 of *Linear Algebra for Data Science* `https://shainarace.github.io/LinearAlgebra/norms.html` and `https://shainarace.github.io/LinearAlgebra/orthog.html`

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PART 1: Announcements and Preview of Week 10

Assignment 5

Assignment 5 opened on Thursday 10 Nov). Deadline is 5pm, Friday, 25th of November.

Communication Skills : Next steps...

- Instructions at https://www.niallmadden.ie/2223-MA313/22_23_Communication_Skills.pdf have been updated.
- Deadline is 5pm Friday, 18 November.
- Presentations will be during the week 21–25 November:
  - Monday at 12.00 in AC204 (i.e., MA335 class time)
  - Tuesday at 13.00 in Ac202 (i.e., MA313 class time)
  - Some other time ... (probably Thursday at 12).

The big ideas from this week will be **Orthogonality**.

- How to find the orthogonal projector of a vector onto a subspace
- What it means if the columns of a matrix are orthogonal to each other.

These are the essential ideas from recent lectures that you need for this week.

- A **BASIS** of a vector space V is the **smallest** sequence $(v_1, v_2, \ldots, v_r)$ of vectors which which spans V.
- The **DIMENSION** of V is the number of vectors in any basis for V.
- The vector space $\mathbb{R}^n$ has dimension n. Any sequence of n linearly independent vectors is a basis for $\mathbb{R}^n$.
- The **INNER PRODUCT** of vectors u and v in $\mathbb{R}^n$ is the real number given by

$$u \cdot v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n.$$

- $u \cdot v = u^\top v$.
- The **LENGTH** of a vector $v \in \mathbb{R}^n$ is $\|v\| := \sqrt{v \cdot v} = \sqrt{v_1^2 + \cdots + v_n^2}$.
- If $u, v \in \mathbb{R}^n$ are both be non-zero, then the **angle** $\measuredangle(u, v) \in [0, \pi]$ between u and v is defined by $\cos(\measuredangle(u, v)) = \dfrac{u \cdot v}{\|u\|\|v\|}$.
- We say $v \in \mathbb{R}^n$ is a **unit vector** if $\|v\| = 1$.
  The unit vector in the same direction as v is $v/\|v\|$.
- $u, v \in \mathbb{R}^n$ are **orthogonal** if $u \cdot v = 0$. We may write this as $u \perp v$.
- **Pythagorean Theorem:** If $u \perp v$, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

- If u and v are non-zero vectors in $\mathbb{R}^n$, then

$$w = u - \frac{u \cdot v}{v \cdot v} v$$

is orthogonal to v.

- **The Cauchy-Schwarz inequality:** $|u \cdot v| \leqslant \|u\|\|v\|$. And $|u \cdot v| = \|u\|\|v\|$, if and only if u and v are linearly dependent.

- The Cauchy-Schwarz inequality implies that, if $u \neq 0 \neq v$, then

$$-1 \leqslant \frac{u \cdot v}{\|u\|\|v\|} \leqslant 1.$$

Therefore, the definition of the angle between u and v via

$$\cos(\measuredangle(u, v)) = \frac{u \cdot v}{\|u\|\|v\|}$$

makes sense.

The Triangle inequality

$\|u + v\| \leqslant \|u\| + \|v\|$ for all $u, v \in \mathbb{R}^n$.

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PART 2: Orthogonal Projections

Part 2: Orthogonal Projections

Definition (ORTHOGONAL to a subspace)

Let W be a subspace of $\mathbb{R}^n$. We say that a vector $z \in \mathbb{R}^n$ is **orthogonal** to W if $z \perp w$ for all $w \in W$.

Example

$u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is orthogonal to the space $V = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Part 2: Orthogonal Projections

Orthogonal spaces

Two vector spaces, V and W are **orthogonal**, if, for every $v \in V$ is orthogonal to every $w \in W$. That is $v \cdot w = 0$.

Example

For any matrix A, its left null space is orthogonal to its column space.

Part 2: Orthogonal Projections

Definition (ORTHOGONAL COMPLEMENT)

The **orthogonal complement** of a vector space W, denoted $W^\perp$, is the set of vectors that are orthogonal to W. That is,

$$W^\perp = \{z \in \mathbb{R}^n : z \perp w \text{ for all } w \in W\}.$$

Part 2: Orthogonal Projections

Example

Let $W = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 2t \\ -t \end{bmatrix} : t \in \mathbb{R}\right\}$.

Give a basis for $W^\perp$.

Theorem: Unique representation/Orthogonal decomposition

Let W be a subspace of $\mathbb{R}^n$. Then:

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- If $W = \operatorname{span}\{w_1, \ldots, w_r\}$, then $W^\perp = \{z \in \mathbb{R}^n : z \perp w_1, \ldots, z \perp w_r\}$.
- Every vector $v \in \mathbb{R}^n$ has a **unique representation**

$$v = \hat{v} + z \qquad \text{for } \hat{v} \in W, \qquad \text{and } z \in W^\perp.$$

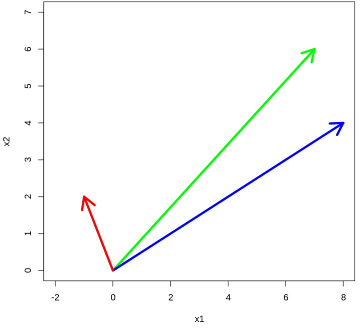
- The function $\operatorname{proj}_W \colon \mathbb{R}^n \to W, \quad v \mapsto \hat{v}$ is a linear transformation, called the **orthogonal projection** of $\mathbb{R}^n$ onto W.
- $W \cap W^\perp = \{0\}$.
- $\dim W^\perp = n - \dim W$.

Example

Let $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $W = \mathrm{span}\left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$. Then

The orthogonal projection of v onto W is $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$.

The component of v orthogonal to W is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



Proposition

Let $W = \mathrm{span}\{u\}$ be a subspace of $\mathbb{R}^n$, where $0 \neq u \in \mathbb{R}^n$. (That is, W is a line through the origin.)
Then the orthogonal projection $\hat{v} = \mathrm{proj}_W(v)$ of $v \in \mathbb{R}^n$ on W is

$$\hat{v} = \frac{v \cdot u}{u \cdot u}\, u.$$

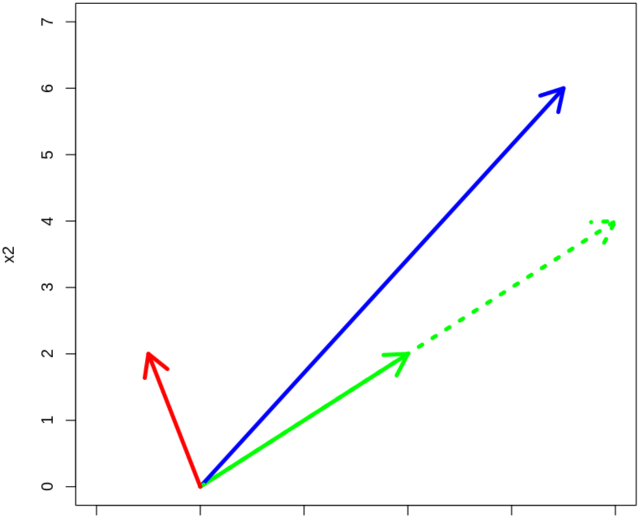
To see this, we have to show that

- $\hat{v} \in W$
- If $z := v - \hat{v}$, then $z \perp u$.

Example

Let $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$. Compute

- the orthogonal projection, $\hat{v}$, of v onto $\mathrm{span}\{u\}$;
- $z = v - \hat{v}$.
- Verify that $z \perp u$.


x2
0
1
2
3
4
5
6
7

In case you are interested, here is how to do this in R.

Solution in R

```
u <- c(4,2)
v <- c(7,6)
vhat = c((v %*% u)/(u %*% u))*u
z = v - vhat

plot(NULL, xlim=c(-2,8), ylim=c(0,7),
   xlab="x1", ylab="x2")
arrows(0,0, u[1], u[2],lwd=4,col="green")
arrows(0,0, v[1], v[2],lwd=4,col="blue")
arrows(0,0, vhat[1], vhat[2],lwd=4, lty=3, col="green")
arrows(0,0, z[1], z[2],lwd=4, col="red")
```

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PART 3: Orthogonal Bases

Part 3: Orthogonal Bases

In Part 2, we saw how to compute the projection of a vector orthogonal to a one-dimensional space.

Questions

Let W be an arbitrary subspace of $\mathbb{R}^n$, with $\dim W \geq 1$. How can we compute proj_W?
That is, given $v \in \mathbb{R}^n$, how can we find $\hat{v} = \mathrm{proj}_W(v)$?
Also, why bother?

Part 3: Orthogonal Bases

Definition (ORTHOGONAL BASIS)

- A sequence of vectors $u_1, \dots, u_p \in \mathbb{R}^n$ is **orthogonal** if $u_i \perp u_j$ for all $i \neq j$.
- An **orthogonal basis** of a subspace W of $\mathbb{R}^n$ is a basis of W which is orthogonal.

For $\mathbb{R}^n$, the standard basis is an example of an orthogonal basis. But there are others.

Proposition

If $u_1, \ldots, u_p$ is an orthogonal sequence of non-zero vectors, then these vectors are linearly independent.

Now we can generalise the idea on Part 2, Slide 17 which was for one-dimensional spaces.

Theorem

Let $(u_1, \ldots, u_p)$ be an orthogonal basis of a subspace W of $\mathbb{R}^n$. Then the orthogonal projection of $v \in \mathbb{R}^n$ onto W is given by

$$\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{v \cdot u_p}{u_p \cdot u_p} u_p.$$

Example

Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \mathrm{span}\,\{u_1, u_2\}$.

What is $\hat{v} = \mathrm{proj}_W(v)$?

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PART 4: Gram-Schmidt Process

Part 4: Gram-Schmidt Process

Question(s)

- Does every subspace of $\mathbb{R}^n$ have an orthogonal basis?
- If so, how do we construct it?

Part 4: Gram-Schmidt Process

Theorem: "Gram-Schmidt process"

Let $(v_1, \ldots, v_p)$ be a basis of a subspace W of $\mathbb{R}^n$.
Define vectors $u_1, \ldots, u_p$ via

- $u_1 := v_1$,
- $u_2 := v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$,
- $u_3 := v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$,
- $\vdots$
- $u_p := v_p - \frac{v_p \cdot u_1}{u_1 \cdot u_1} u_1 - \cdots - \frac{v_p \cdot u_{p-1}}{u_{p-1} \cdot u_{p-1}} u_{p-1}$.

Then $(u_1, \ldots, u_p)$ is an orthogonal basis of W.

Part 4: Gram-Schmidt Process

Example

Let $W = \mathrm{span}\{v_1, v_2\}$ for $v_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Construct an orthogonal basis of W.

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PART 5: Orthogonal Matrices

Definition: ORTHONORMAL

The vectors $u_1, \dots, u_p \in \mathbb{R}^n$ are **orthonormal** if they are orthogonal unit vectors. That is:

- $u_i \perp u_j$ for all $i \neq j$. Equivalently, $u_i \cdot u_j = 0$ for all $i \neq j$.
- $\|u_i\| = 1$ for all i.

Note: If $u_1, \dots, u_p$ are orthogonal and all non-zero, then $\frac{1}{\|u_1\|} u_1, \dots, \frac{1}{\|u_p\|} u_p$ are orthonormal.

Definition: ORTHONORMAL BASIS

An **orthonormal basis** of a subspace W of $\mathbb{R}^n$ is a basis of W that consists of orthonormal vectors.

Example: The standard basis of $\mathbb{R}^n$ is orthonormal.

Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent:

- The columns of A form an orthonormal basis of $\mathbb{R}^n$.
- $A^\top A = I_n = AA^\top$. (That is, A is invertible and $A^{-1} = A^\top$.)
- $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^n$.
- $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$

Definition: ORTHOGONAL MATRIX

An $n \times n$ matrix A is **orthogonal** if $A^\top A = I_n$ (in which case also $AA^\top = I_n$).

Example

- Reflections: e.g. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Rotations: $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ for $\vartheta \in \mathbb{R}$, e.g. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Exercises

Orthogonality

These exercises are taken from Section 6.2 and 6.1 of the textbook.

Orthogonal sets

1. *6.2.1–6.2.4* Determine which of the following sequences of vectors are orthogonal.

$$(a) \quad \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} \qquad (b) \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

$$(c) \quad \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \qquad (d) \quad \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$$

Exercises

Orthogonal projections

2. *6.2.11* Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line passing through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.
3. *6.2.12* Compute the orthogonal projection of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line passing through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.
4. *6.2.13* Let $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write y as the sum of two orthogonal vectors, one in span$\{u\}$ and one orthogonal to u.
5. *6.2.14* Let $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write y as the sum of two orthogonal vectors, one in span$\{u\}$ and one orthogonal to u.

Exercises

6. *6.3.2–6.3.6* In each of the following cases, verify that $\{u_1, u_2\}$ is an orthogonal set and find the orthogonal projection of y onto span$\{u_1, u_2\}$.

   6.1 $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

   6.2 $y = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

   6.3 $y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

   6.4 $y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Exercises

The Gram-Schmidt process

7. *6.4.9* Find an orthogonal basis for the column space of

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

8. *6.4.10* Find an orthogonal basis for the column space of

$$\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$