2425-MA140 Engineering Calculus

Week 07, Lecture 3 (L21)
Techniques of Integration

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Thursday, 31 October, 2024

Assignments, etc

- **Assignment 5** is open. Deadline is 5pm next Monday (4 November). You have 3 attempts for each question. However, Q1 will be manually graded after the deadline.

This morning, I think we will think about...

1. Substitution
   - Definite Integrals
2. Rational functions
3. Exercises

See also Section **5.5** (Substitution) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Substitution

Suppose we want to evaluate an integral of the form

$$\int e^{x^3+x^2}(3x^2+2x)\,\mathrm{d}x.$$

At first, this looks tricky: there is nothing like this in our table of integrals.

However, there is something a little unusual about it: it features both the function x^3+x^2, and its derivative $3x^2+2x$.

It turns out that such problems are quite common (at least in textbooks and on exams!). Moreover, there is a handy technique called **substitution** for evaluating them. In this case:

Substitution

Method of substitution

If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))\, g'(x)\, \mathrm{d}x = \int f(u)\, \mathrm{d}u\,.$$

Equivalently: $\displaystyle\int f(u)\, \frac{du}{\mathrm{d}x} \mathrm{d}x = \int f(u)\, \mathrm{d}u$. (For a proof of why this works, see Section 5.5 of the textbook).

After this substitution, our task is reduced to evaluating $\displaystyle\int f(u)\, \mathrm{d}u$ which, we hope, is easier.

Substitution

Example

Evaluate the integral $\int 3x^2 \sin(x^3)\, dx$.

Notice that $3x^2$ is the derivative of x^3.

Let's try integration by substition with $u = x^3$.

If $u = x^3$, then $\frac{du}{dx} = 3x^2$, so

$$du = \frac{du}{dx}\, dx = 3x^2\, dx\,.$$

Thus,

$$\begin{aligned}\int 3x^2 \sin(x^3)\, dx &= \int \sin(u)\, du \\ &= -\cos(u) + C \\ &= -\cos(x^3) + C\,.\end{aligned}$$

Substitution

Example

Evaluate $\int 2x\sqrt{1+x^2}\,dx$.

Notice that $2x$ is the derivative of $1+x^2$.

Let's try integration by substition with $u = 1 + x^2$:

Substitution

Example

Evaluate $\int \cos(4x-7)\,dx$.

Idea: think of this as $\dfrac{1}{4}\int \cos(4x-7)4\,dx$.

Substitution

Example

Show that $\int \sin^3(x)\ \cos(x)\,\mathrm{d}x = \frac{1}{4}\sin^4(x) + C$.

Substitution can be used with **definite integrals**. However, this may requires a change to the limits of integration.

Substitution with Definite Integrals

Let $u = g(x)$, with g' continuous on $[a, b]$, and f continuous over the range of $u = g(x)$. Then Then,

$$\int_{x=a}^{x=b} f\big(g(x)\big)g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

This allows us to apply the FTC2, without having to invert the substitution.

Example

Evaluate $\int_0^1 x^2(1+2x^3)^2 dx$

Example

Evaluate $\int_{-1}^{0} xe^{x^2}\,dx$

Rational functions

Recall: Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Before trying to find an antiderivative of a rational function $f(x) = \dfrac{p(x)}{q(x)}$:

Step 1: If $\deg(p(x)) \geqslant \deg(q(x))$, divide $p(x)$ by $q(x)$.

Step 2: Check if integration by substitution might work.

Step 3: Factorise the denominator as far as possible.

Step 4: Write the rational function as sum of **partial fractions** to simplify.

Example

Evaluate the integral

$$\int \frac{x}{x^2+1}\,dx\,.$$

In this can we can use substitution.

Example

Evaluate the integral

$$\int \frac{3x+4}{x^2+7x+12}\,dx\,.$$

In this case, we must factorise the denominator, and express the integrand as partial fractions. Factorise:
$x^2+7x+12 = (x+4)(x+3)$.

Express as Partial Fractions:

$$\frac{3x+4}{x^2+7x+12} = \frac{A}{x+4} + \frac{B}{x+3}$$

With a little work, we can find that $A=8$ and $B=-5$. Therefore,

$$\frac{3x+4}{x^2+7x+12} = \frac{8}{x+4} - \frac{5}{x+3}\,.$$

We can now express the integral as

$$\int \frac{3x+4}{x^2+7x+12}\,dx = \int \left(\frac{8}{x+4} - \frac{5}{x+3}\right) dx$$
$$= \underbrace{\int \frac{8}{x+4}\,dx}_{I_1} - \underbrace{\int \frac{5}{x+3}\,dx}_{I_2}\,.$$

First, we evaluate I_1.

$$I_1 = \int \frac{8}{x+4}\,dx = 8\int \frac{1}{x+4}\,dx\,.$$

If we let $u = x+4$, then $du = dx$ and, hence,

$$I_1 = 8\int \frac{1}{x+4}\,dx = 8\int \frac{1}{u}\,du = 8\ln|u| + c_1 = 8\ln|x+4| + c_1\,.$$

Similarly, we find that: $I_2 = \displaystyle\int \frac{5}{x+3}\,dx = 5\ln|x+3| + c_2$.

Rational functions

To conclude:

$$
\begin{aligned}
\int \frac{3x+4}{x^2+7x+12}\,dx &= \int \left(\frac{8}{x+4} - \frac{5}{x+3}\right) dx \\
&= \int \frac{8}{x+4}\,dx - \int \frac{5}{x+3}\,dx \\
&= I_1 - I_2 \\
&= \left(8\ln|x+4| + c_1\right) - \left(5\ln|x+3| + c_2\right) \\
&= 8\ln|x+4| - 5\ln|x+3| + c\,.
\end{aligned}
$$

Exercises

Exer 7.3.1

Evaluate the follow integrals

1. $\displaystyle\int \sin(\ln x)\frac{1}{x}\,\mathrm{d}x.$
2. $\displaystyle\int x^2(x^3+5)^9\,\mathrm{d}x$
3. $\displaystyle\int \frac{\sin(x)}{\cos^3(x)}\,dx$

Exer 7.3.2

Evaluate $\displaystyle\int_0^1 x^2(x^3+5)^9\,\mathrm{d}x$