MA313 : Linear Algebra I

Review: Sample Exam Questions

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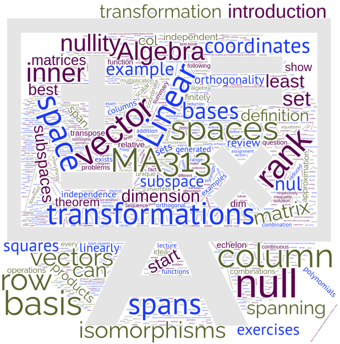

Q1. Q1(a) [5 Marks] Give an example of a 3-dimensional subspace of a 5-dimensional vector space.

Q1(b) [5 Marks] Find vectors $u, v, w \in V$ with $V = \mathrm{Span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} a + b - 2c \\ 3a \\ c - b \\ 3a - 12c \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$.

Q1(c) [10 Marks] Let

$$A = \begin{bmatrix} -3 & -9 & 1 & 2 \\ -4 & 12 & 1 & 0 \\ 2 & -6 & 1 & 2 \\ -1 & 3 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Determine, with justification, if $x \in \operatorname{Nul} A$, and if $x \in \operatorname{Col} A$.

Hint: $rref([A|b]) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 0 & 1/9 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 5/2 \end{bmatrix}$

Q1(d) [5 MARKS] Decide (with justification) whether

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 ; x^2 + y^2 \leq 0 \right\},$$

is a subspace of $\mathbb{R}^2$.

Q2. Q2(a) [5 Marks] Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} 2p - 2q \\ 2p + 3q \\ 2q \\ 5r \end{bmatrix} : p, q, r \in \mathbb{R} \right\}$$

of $\mathbb{R}^4$.

Q2(b) [10 marks] Show that $\mathcal{B} = \left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \right)$ is a basis of $\mathbb{R}^3$. Moreover, find the coordinate vector of $y = \begin{bmatrix} -10 \\ -5 \\ 4 \end{bmatrix}$ relative to $\mathcal{B}$.

$\left(\text{Check} = \text{solution is } (42, -22, -13)^T\right)$

Q2(c) [10 Marks] Find bases of the null space and the column space of

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 5 & 13 & 3 & 5 \\ 1 & -1 & -3 & 1 \end{bmatrix}.$$

Q3. Q3(a) [5 MARKS] ~~Explain the meaning of the following statement:~~
~~*Linear transformations can be regarded as generalisations of matrices.*~~

Q3(b) (i) What is the largest possible rank of an 20×10 matrix?

(ii) If the null space of a 12×4 matrix A is 2-dimensional, what is the dimension of its column space?

(iii) Give an example of a 4×3 matrix A with nullity $A = 2$. [5 marks]

Q3(c) [5 Marks] ~~Briefly indicate how vector spaces and linear transformations arise in signal processing.~~

Q3(d) [10 Marks] Recall that $\mathbb{P}_n$ denotes the vector space of polynomials $p(t)$ of degree at most n. Find the matrix of the linear transformation

$$T\colon \mathbb{P}_3 \to \mathbb{P}_3, \quad p(t) \mapsto 3p(t) - 2p'(t) + p''(t)$$

relative to the basis $(1, t, t^2, t^3)$ of $\mathbb{P}_3$.

Q4. Q4(a) [5 Marks] Give an example of a vector in $\mathbb{R}^3$, none of whose entries are zero and is length 3, or explain why this cannot be done.
Give an example of a vector in $\mathbb{R}^3$, none of whose entries are zero and is length 0, or explain why this cannot be done.

Q4(b) [4 Marks] Find the orthogonal projection of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ onto the line passing through $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and the origin in $\mathbb{R}^2$.

Q4(c) [5 Marks] Explain the meaning of the following statement:
Orthogonal matrices preserve angles.

Q4(d) [10 marks] Find a least-squares solution of the system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 50 \\ 0 \\ -25 \end{bmatrix}.$$

What is the length of the residual?