Annotated slides

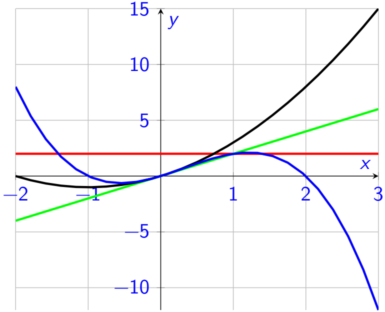
MA140: Engineering Calculus

Lecture 2: Functions

Dr Niall Madden

School of Mathematical and Statistical Sciences, University of Galway

18 September, 2024



This version of the slides are by Niall Madden, but are adapted from original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Chapter 2 of *Modern Engineering Mathematics*: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

MA313
Lecture 2: Functions

Start of ...

Section 1: Functions

The single most important concept in MA140 is that of a **function**.
For more, see Chapter 2 of "Modern Engineering Calculus (James).

We represent a function symbolically in two ways, either

"is mapped" $f : x \to y$ "f is a function mapping x to y"

or or "is sent to".

$y = f(x)$

Here x is in the set of X (or $x \in X$), and y is in the set of Y (or $y \in Y$).

$y = f(x)$ "y is f of x"

Later we'll also have $z = f(x, y)$.

Moreover $y = f(x)$ defines a point $(x, y) = (x, f(x))$

If f is a function from X to Y...

- The set X is called the **domain** of the function.
- The set Y is called the **codomain**.
- When we write $y = f(x)$, we say "x" is the **argument** of the function. ie $\underline{x}$ = input.
- When $y = f(x)$ for some $x \in X$, y is said to be the **image** of x under f. image = output.
- The set of all images $y = f(x), x \in X$, is called the **range** (or **image set**) of f. Range $\subseteq$ codomain.

f is a function from DOMAIN to the CODOMAIN

$\uparrow X$

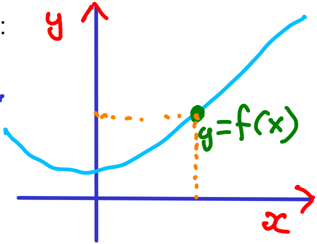
$\uparrow Y$

- While we could have functions between any pair of sets (e.g., a function from students in this class to their ID numbers), usually X and Y are *sets of numbers*.
- It is not necessary for all elements y of the codomain Y to be images under f.
- One element $y \in Y$ can serve as value $f(x)$ for several $x \in X$.

Uniqueness

A function can be represented in different ways:

1. **verbally** (by a description in *words*);
2. **numerically** (as a *table* of values);
3. **visually** (as a *graph*);
4. **algebraically** (by an explicit *formula*).



Often it is possible, and useful, to go from one way to another.

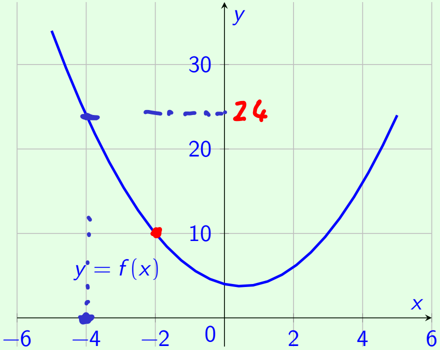
x	$f(x)$
-1	2
0	4
1	6
2	8
3	10.

eg $f(x) = 2x+4$

Graphical Representation

Graph $\to$ Table

A common way to *visualize* a function $f\colon X \to \mathbb{R}$ is its *graph* in the x, y-plane. In this example, $f(x) = x^2 - x + 4$.



$x = -4$

$f(-4) = (-4)^2 - (-4) + 4$
$= 16 + 4 + 4 = 24$

x	$f(x)$
-4	24
-2	10
0	4
2	6
4	16

Often, the domain of a function is not expilicitly stated.
In such a case the following **Domain Convention** applies.

The **domain** of a function f is the set of all numbers x for which $f(x)$ *makes sense* and gives a *real-number output*.

Example

1. Find the domain D of $f(x) = \dfrac{1}{x^2 - x}$.

Eg, if $x = 2$ $f(x) = \frac{1}{4-2} = \frac{1}{2}$

$x = 1$ $f(x) = \frac{1}{1-1} = \frac{1}{0}$ undefined.

$x = 0$ $f(x) = \frac{1}{0-0} = \frac{1}{0}$ undefined.

All other points are ok !!

Often, the domain of a function is not expilicitly stated.
In such a case the following **Domain Convention** applies.

The **domain** of a function f is the set of all numbers x for which $f(x)$ *makes sense* and gives a *real-number output*.

Example

1. Find the domain D of $f(x) = \dfrac{1}{x^2 - x}$.

Note.

- $\{a, b\}$ means the set with two points a and b.
- $[a, b]$ means all point x with $a \le x \le b$
- $(a, b]$ means $a < x \le b$
- (a, b) means $a < x < b$

$\cup$ = "union"

Often, the domain of a function is not expilicitly stated.
In such a case the following **Domain Convention** applies.

The **domain** of a function f is the set of all numbers x for which $f(x)$ *makes sense* and gives a *real-number output*.

Example

1. Find the domain D of $f(x) = \dfrac{1}{x^2 - x}$.

Answer: the Domain is
$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
union

$= \mathbb{R} / \{0, 1\}$

Example

Find the domain of the function $f(x) = \sqrt{x+2}$.

Since we want $f(x)$ to be real, we can't have $f(x) = \sqrt{\text{something negative}}$

So the domain is when $x+2 \geq 0$

That is $x \geq -2$

Equivalent Domain is $[-2, \infty)$

Example

Identify the domain, ~~codomain~~ and range of $f_1(x) = 3x^2 + 1$

Domain is $\mathbb{R} = (-\infty, \infty)$

Codomain is also $\mathbb{R}$.

Note that, for all x, $x^2 \geq 0$

So $3x^2 \geq 0$

And $3x^2 + 1 \geq 1$

So Range is $x \geq 1 \iff [1, \infty)$

Example

Identify the domain, codomain and range of $f_2(x) = \sqrt{(x+4)(3-x)}$

$= \mathbb{R}$.

Take a 60 second break

& think about this!

Ideas?? Need $(x+4)(3-x) \geq 0$.

So
- Both $x+4 \geq 0$, $3-x \geq 0$
- Both $x+4 \leq 0$ and $3-x \leq 0$.

Domain

Exer: finish.

Example

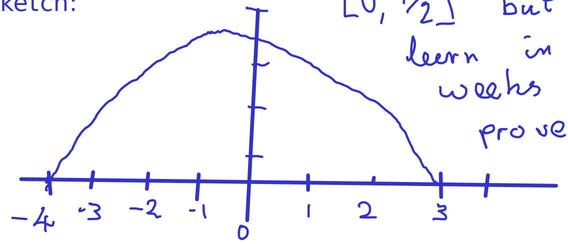
Identify the domain, ~~codomain~~ and range of $f_2(x) = \sqrt{(x+4)(3-x)}$

[These notes were added after class]

The domain of this function is [-4,3].
Outside of that range the function is not real valued because (x+4)(3-x) would be negative.

The range is $[0, 7/2]$ but we'll learn in a few weeks how to prove that.

Here is a sketch:



Example

Identify the domain, codomain and range of $f_3(x) = \dfrac{1}{x}$.

$= \mathbb{R}$

Domain: All values for which $\frac{1}{x}$ makes sense. So any x except $x = 0$.

ANS: $\mathbb{R}/\{0\}$ or $(-\infty, 0) \cup (0, +\infty)$.

Range: $\mathbb{R}/\{0\}$.

A Catalog of Functions

There are many *different types of functions* that can be used to *model relationships* between objects in the *real world*.

The most common types of functions (in MA140) are:

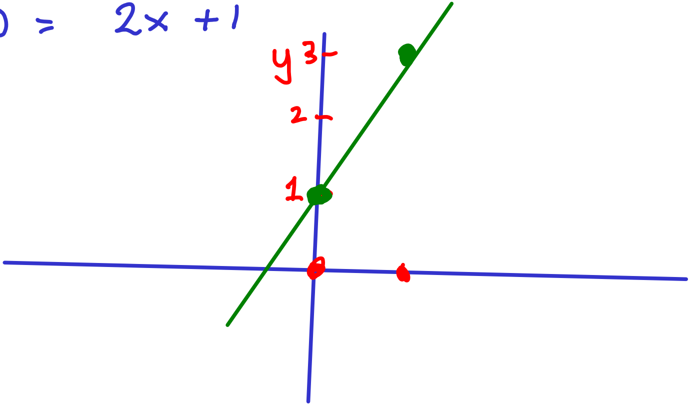
- *Linear* Functions, $3x-2$
- *Polynomial* Functions, $4x^4-5x^3+1$
- *Power* Functions, 5^x
- *Rational* Functions, $\frac{3x-2}{4x+1}$
- *Algebraic* Functions,
- *Trigonometric* Functions,
- *Exponential* Functions,
- *Logarithms*.

See tomorrow . . . .

Linear functions have formulae such as $f(x) = mx + c$, where m and c are some given numbers.

It is often represented graphically as a straight line of slope m through the point $(0, c)$.

$f(x) = 2x + 1$



 Finished

A **polynomial function** (or just **polynomial**) is a function of the form

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0, \quad x \in \mathbb{R},$$

where $a_0, a_1, \ldots, a_n$ are real numbers called the **coefficients** of the polynomial.
The number n is called the **degree** of the polynomial.

n is a finite integer.

Eg n=0 : Constant functions $f(x)=a_0$

n=1 : Linear functions

n=2 Quadratic eg x^2+2x-3.

Polynomials

Example: linear

$y = x$ is a **linear** polynomial with degree $n = 1$.

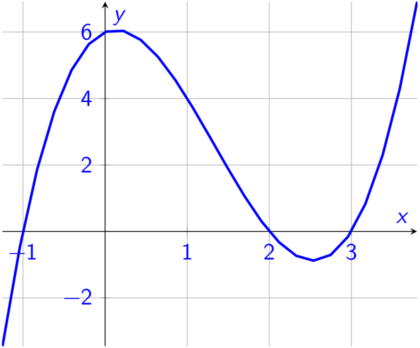
Polynomials

Example: quadratic

$y = x^2 - 4x + 3$ is a **quadratic** polynomial with degree $n = 2$.

Example

$y = x^3 - 4x^2 + x + 6$ is a **cubic** function with degree $n = 3$.



Fact: A polynomial function of grade n has **up to** $n-1$ bends,

Examples:

Break Time

During the break, think and talk about what you might do to sketch the graph of

$$y = -x^3 + x^2 + 2x$$

- To sketch the graph, first find the **intercepts**:
  - The **y-intercepts** can be found by letting $x = 0$.
  - The **x-intercepts** are called the **roots** (or **zeros**).
    To find the roots, set y equal to zero and solve for x.
- You don't have to use the same scale on the x- and on the y-axis.
- Do not use graph paper.

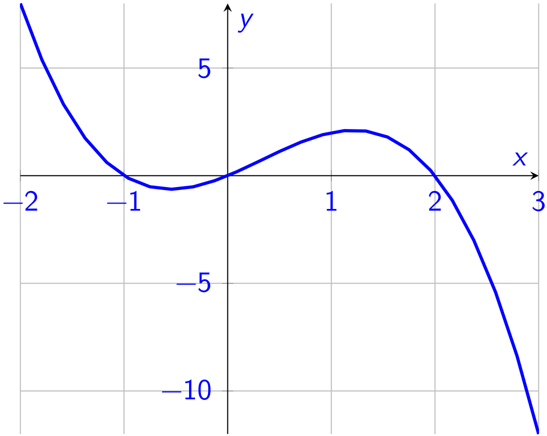
Example

Sketch the graph of

$$y = -x^3 + x^2 + 2x$$

How to sketch $y = -x^3 + x^2 + 2x$

Actual plot of $y = -x^3 + x^2 + 2x$



Exercises

Sketch the graphs of

(i) $y = 5x^2 - 7$

(ii) $y = x^2 - 4x + 3$

(iii) $y = x^3 - 6x^2 - 11x - 6$