

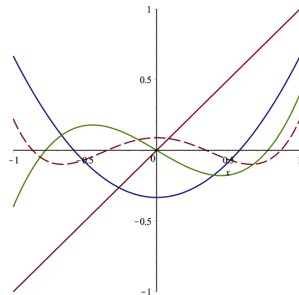
MA378 Chapter 3: Numerical Integration

§3.5 Orthogonal Polynomials

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5.1 Orthogonal Polynomials

High order Newton-Cotes methods are of little use because of the problems associated with interpolation by high degree polynomials at equally spaced points. However, high-order Gaussian methods are very useful.

Driving such methods by undetermined coefficients is not practical, however. There is a simpler way, but some mathematical preliminaries are required, including the ideas of **vector spaces** and **inner products**.

Definition 5.1 (Vector space: informal)

A **vector** space is a collection of objects, called **vectors**, where it makes sense to

- ▶ add two vectors to get another one;
- ▶ multiply a vector by a scalar to get another one.

5.2 Inner products

Definition 5.2 (Vector Space: formal)

V is a *vector space* (a.k.a., a *linear space*) over a field F (e.g, the real or complex numbers) if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

- (i) $\mathbf{u} + \mathbf{v} \in V$ (closed under addition)
- (ii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)
- (iii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)
- (iv) V has a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- (v) $-\mathbf{u} \in V$
- (vi) $a\mathbf{u} \in V$
- (vii) $a(b\mathbf{u}) = (ab)\mathbf{u}$
- (viii) F contains 0 and 1 such that $1\mathbf{u} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$.
- (ix) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

5.2 Inner products

Examples:

5.2 Inner products

Definition 5.3 (Inner Product)

Let V is a real vector space. An **Inner Product** (IP) is a real-valued function (\cdot, \cdot) on $V \times V$ such that, for all $f, g, h \in V$,

- (i) $(f + g, h) = (f, h) + (g, h)$,
- (ii) $(\lambda f, g) = \lambda(f, g)$, for $\lambda \in \mathbb{R}$.
- (iii) $(f, g) = (g, f)$,
- (iv) $(f, f) \geq 0$. $(f, f) = 0 \Leftrightarrow f \equiv 0$.

5.2 Inner products

Example 5.4

Let \mathbb{R}^n be our vector space, with $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$. Then the following is an inner product:

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n x_i y_i,$$

5.2 Inner products

Example 5.5

The set of real-valued functions that are continuous and defined on the interval $[a, b]$, denoted $C[a, b]$, is a vector space. And

$$(f, g) := \int_a^b f(x)g(x)dx, \quad (1)$$

is an inner product.

5.3 Sequence of monic polynomials

(See Lecture 23 of Stewart's "Afternotes" for more details).

Definition 5.6 (Monic Polynomial)

A polynomial is *monic* if the coefficient of its leading term is 1.

Examples:

5.3 Sequence of monic polynomials

Definition 5.7

Two elements a, b , of a vector space are *orthogonal* with respect to a given inner product (\cdot, \cdot) if $(a, b) = 0$.

Example:

5.3 Sequence of monic polynomials

Example 5.8

Take the space of polynomials of degree 2 or less and the IP

$$(f, g) = \int_{-1}^1 f(x)g(x)dx.$$

Let $p(x) \equiv 1$, $q(x) \equiv x$, $r(x) \equiv x^2 - 1/3$, and $f(x) = 3x - 4$

We can check that $(r, p) = 0$, and $(r, q) = 0$. We can the verify that $(r, f) = 0$. **Details:**

5.3 Sequence of monic polynomials

As given above, a polynomial is **monic** if the coefficient of the leading term is 1:

$$p_n = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1x + c_0.$$

We'll now look at a sequence of such polynomials

$$\{\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n, \dots\}$$

that have the property they are orthogonal to each other:

$$(\tilde{p}_i, \tilde{p}_j) := \int_a^b \tilde{p}_i(x)\tilde{p}_j(x)dx = 0 \quad \text{if } i \neq j.$$

We want to establish some important facts about monic polys:

- ▶ A set of monic polys of degrees $1, \dots, n$, forms a basis for \mathcal{P}_n .
- ▶ If the members of that set are orthogonal to each other, then they are orthogonal to *all* polynomials of lower degree.
- ▶ We can construct such as set.

5.3 Sequence of monic polynomials

Theorem 5.9

Let $\{\tilde{p}_i\}_{i=0}^n$ be a sequence of polynomials where each \tilde{p}_i is monic of exactly of degree i . This sequence forms as basis for \mathcal{P}_n .

Proof:

5.3 Sequence of monic polynomials

Theorem 5.9 means that if q is a polynomial of degree n then it can be written uniquely as a linear combination of the \tilde{p}_i :

$$q(x) = \sum_{i=0}^n a_i \tilde{p}_i(x),$$

for some unique choice of the real coefficients a_i .

5.3 Sequence of monic polynomials

Definition 5.10

The sequence $\{\tilde{p}_i\}_{i=0}^n$ is a sequence of *monic, orthogonal* polynomials if each \tilde{p}_i is monic and *exactly* of degree i and

$$(\tilde{p}_i, \tilde{p}_j) = 0 \quad \text{if } i \neq j.$$

5.3 Sequence of monic polynomials

Theorem 5.11

If $\tilde{p}_j \in \{\tilde{p}_i\}_{i=0}^{\infty}$ then \tilde{p}_j is orthogonal to all polynomials of degree less than j .

Proof:

5.4 Constructing the Sequence

Theorem 5.12

*The sequence $\{\tilde{p}_i\}_{i=0}^{\infty}$ exists and can be constructed as follows:
Let α and β be defined as*

$$\alpha_{n+1} = \frac{(x\tilde{p}_n, \tilde{p}_n)}{(\tilde{p}_n, \tilde{p}_n)}, \quad \text{and} \quad \beta_{n+1} = \frac{(x\tilde{p}_n, \tilde{p}_{n-1})}{(\tilde{p}_{n-1}, \tilde{p}_{n-1})},$$

then the sequence is given by

$$\tilde{p}_0(x) \equiv 1, \quad \tilde{p}_1(x) = x - \alpha_1$$

and

$$\tilde{p}_{n+1}(x) = (x - \alpha_{n+1})\tilde{p}_n(x) - \beta_{n+1}\tilde{p}_{n-1}(x),$$

for $n \geq 1$.

The proof uses *Gram-Schmidt Orthogonalization*.

5.4 Constructing the Sequence

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Example 5.13

If we use the inner product $(f, g) := \int_{-1}^1 f(x)g(x)$ then the first 3 polynomials in the sequence are:

$$\tilde{p}_0 = 1, \quad \tilde{p}_1 = x, \quad \text{and} \quad \tilde{p}_2 = x^2 - 1/3.$$

Example 5.14

The zeros of \tilde{p}_2 are ...

5.5 Properties of the sequence

One of the ways of constructing Gaussian Quadrature rule $G_n(\cdot)$ on $n + 1$ is to take the quadrature points as the roots of \tilde{p}_{n+1} . We know (from the fundamental theorem of algebra) a polynomial of degree $n + 1$ has exactly $n + 1$ roots in \mathbb{C} up to multiplicity.

However, the polynomials \tilde{p} have the special properties, established in the following lemma. (*A slightly different proof of these facts is given in Thm 9.4 of Suli and Mayers.*)

5.5 Properties of the sequence

Theorem 5.15

Let $\tilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\tilde{p}_0, \tilde{p}_1, \dots\}$ be the set of monic polynomials that are orthogonal with respect to the inner product

$(u, v) := \int_a^b u(x)v(x)dx$. Then:

- (i) The zeros of each $\tilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ are simple (not repeated).*
- (ii) All the zeros of \tilde{p}_i are real numbers in the interval $[a, b]$.*

5.5 Properties of the sequence

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5.6 Exercises

Exercise 5.1

\mathcal{P}_n , the space of polynomials of degree (at most) n forms a vector space. Is it true that the space of *monic* polynomials of degree n forms a vector space?

5.6 Exercises

Exercise 5.2 (★)

(i) Using the Inner Product

$$(f, g) := \int_0^1 f(x)g(x)dx,$$

find $\tilde{p}_0(x)$, $\tilde{p}_1(x)$, $\tilde{p}_2(x)$ and $\tilde{p}_3(x)$.

(ii) Find the zeros of $\tilde{p}_2(x)$ and call them x_0 and x_1 . Construct a quadrature rule for $\int_0^1 f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials.

Notes:

- ▶ An earlier version of this exercise had a typo in Part (ii), stating that the quadrature was on $[-1, 1]$.
- ▶ You can compute the weights using any method you like; although they are defined as the integrals of the relevant Lagrange polynomials, that is not the only way to compute them.