

# Week 07, Lecture 3

## The Fundamental Theorem of Calculus

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**Suimeáilte**

Tá turisigh na suimealaí fígha ar láir.

$f(x)$	$\int f(x)dx$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x $
$e^x$	$e^x$
$e^{ax}$	$\frac{1}{a}e^{ax}$
$a^x$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln \sec x $

**Integrals**

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$\cos^2 x$	$\frac{1}{2}[x + \frac{1}{2}\sin 2x]$
$\sin^2 x$	$\frac{1}{2}[x - \frac{1}{2}\sin 2x]$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \frac{x}{a}$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$

$f(x)$	$\int f(x)dx$
$\frac{1}{x\sqrt{x^2 - a^2}}$	$\frac{1}{a} \sec^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\ln \left  \frac{x + \sqrt{x^2 + a^2}}{a} \right $
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right $
$\frac{1}{\sqrt{x^2 - a^2}}$	$\ln \left  \frac{x + \sqrt{x^2 - a^2}}{a} \right $

**Suimeáil  
na mireanna**

$$\int u dv = uv - \int v du$$

**Integration by parts**

## Dad/Bad Joke of the Day

Today's joke (with thanks to Julie M).

**Me peeling  
potatoes**

**My mum peeling  
potatoes**

$$\sum_{k=1}^n f(x_k) \cdot \Delta x$$

$$\int f(x) dx$$

# The exciting topics that await us in today:

- 1 Recall from yesterday:
- 2 Fundamental Thm of Calculus: Part 1
- 3 FTC1+Chain Rule
- 4 Antiderivatives
  - Indefinite Integrals
  - Common functions
  - Properties
- 5 The Fundamental Thm of Calculus: Part 2
- 6 Exercises

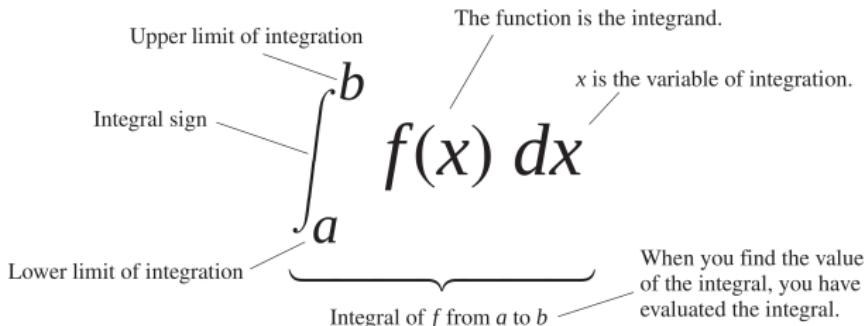
See also: Sections **4.10** (Antiderivatives) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman:  
[math.libretexts.org/Bookshelves/Calculus/Calculus\\_\(OpenStax\)](http://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

## Recall from yesterday:

Let  $f(x)$  be function defined on an interval  $[a, b]$ . The **definite integral** of  $f$  from  $a$  to  $b$  is

$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} h f(x_i),$$

where  $h = (b - a)/n$  and  $x_i = a + ih$ . It is the **area** of the region in space bounded by  $y = 0$ ,  $y = f(x)$ ,  $x = a$ , and  $x = b$ .



## Recall from yesterday:

Given a function,  $f$ , we can define another,  $F$  as

$$F(x) = \int_a^x f(t)dt.$$

That is, the variable in  $F$  is the upper limit of integration on the right.

Recall from yesterday:

## Example

Let  $f(t) \equiv 1$ , and  $F(x) = \int_0^x f(t)dt$ . Give a formula for  $F(x)$ , using the “area” meaning of the definite integral.

# Fundamental Thm of Calculus: Part 1

## Fundamental Theorem of Calculus: Part 1 (FTC1)

Let  $f(x)$  be a continuous function on  $[a, b]$ . If as

$$F(x) = \int_a^x f(t)dt, \quad \text{then} \quad \frac{dF}{dx}(x) = f(x).$$

I.e.,  $F'(x) = f(x)$  for  $x \in [a, b]$ .

Roughly:  **$f$  is the derivative its own integral**. You can find a proof in Section 5.3 of the textbook.

# Fundamental Thm of Calculus: Part 1

## Example

Let  $g(x) = \int_1^x \frac{1}{t^3 + 1} dt$ . Find  $g'(x)$ .

# FTC1+Chain Rule

Sometimes the limit of integration is a more complicated function of  $x$ . In that case, we can apply the **Chain Rule**, along with the FTC1.

## Example

Let  $F(x) = \int_1^{\sqrt{x}} \sin(t) dt$ . Find  $F'(x)$ .

Idea: Let  $u(x) = \sqrt{x} = x^{1/2}$ . So

- $F(u) = \int_1^u \sin(t) dt$ , and
- $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$ .

Then...

$$\frac{dF}{dx} = \frac{dF}{du} \frac{du}{dx} = \sin(u(x)) \left( \frac{1}{2\sqrt{x}} \right) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

## Definition: Antiderivative

A function  $F$  is an **antiderivative** of  $f$  on  $[a, b]$  if  $F'(x) = f(x)$  for all  $x$  in  $[a, b]$ . Thus,

$f$  is the derivative of  $F \Leftrightarrow F$  is an antiderivative of  $f$ .

**Note:** If  $F$  is an antiderivative of  $f$ , then the most general antiderivative of  $f$  is

$$F(x) + C$$

where  $C$  is an *arbitrary* constant, called a **constant of integration**.

- ▶ The word “arbitrary” here means that any choice is valid.
- ▶ The derivative of  $C$  is zero.

# Antiderivatives

## Examples:

- ▶  $F(x) = x + C$  is an antiderivative of  $f(x) \equiv 1$ .
- ▶  $F(x) = x^2 + C$  is an antiderivative of  $f(x) = ??? \dots$
- ▶  $F(x) = ???$  is an antiderivative of  $f(x) = 3x^2$ .

## Examples

Find all antiderivatives of the following functions

(i)  $f(x) = \frac{1}{x}$  for  $x > 0$ .

(ii)  $f(x) = \sin(x)$

(iii)  $f(x) = e^x$ .

**Definition: indefinite integral**

Given a function  $f$ , the **indefinite integral** of  $f$ , denoted

$$\int f(x) dx$$

is the general antiderivative of  $f$ . That is, if  $F$  is an antiderivative of  $f$ , then

$$\int f(x) dx = F(x) + C.$$

**Examples:**

- $\int 2x dx = x^2 + C$
- $\int 3x^2 dx = x^3 + C$

- $\int x \, dx = \frac{1}{2}x^2 + C$
- $\int x^2 \, dx = \frac{1}{3}x^3 + C.$

Spotting the pattern we can deduce...

### Power Rule of Integration

$$\text{If } n \neq -1, \quad \text{then} \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

**Note:** For  $n = -1$ , we have

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C.$$

Here is a list of the antiderivatives of some common functions.

- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \sin(x) dx = -\cos(x) + C$
- $\int \cos(x) dx = \sin(x) + C$
- $\int \tan(x) dx = \ln|\sec(x)| + C$
- ...

## Suimeálaithe

Tá tairisigh na suimeála fágtha ar lár.

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## Suimeáil na míreanna

$$\int u dv = uv - \int v du$$

## Integration by parts

## Properties of Integration

1. If  $k$  is a constant, then

$$\int kf(x) \, dx = k \int f(x) \, dx.$$

2. Integration is additive:

$$\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

**Example**

Evaluate the integral

$$\int 2x^2 + 9x^7 \, dx$$

**Example**

Evaluate the integral

$$\int \frac{4}{1+x^2} dx.$$

# The Fundamental Thm of Calculus: Part 2

Now that we know all about antiderivatives, we can see how the link to **definite integrals**

## Theorem (The Fundamental Thm of Calculus, Part 2)

If  $f(x)$  is continuous on  $[a, b]$ , and  $F(x)$  is any antiderivative of  $f(x)$ , then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

**Notation:** We can write  $F(b) - F(a)$  as  $F(x) \Big|_{x=a}^{x=b}$ , or, more often, as  $F(x) \Big|_a^b$ .

## The Fundamental Thm of Calculus: Part 2

**Example:** Show that  $\int_{-1}^1 (x^2 + 2) dx = \frac{14}{3}$

## The Fundamental Thm of Calculus: Part 2

**Example:** Show that  $\int_{-1}^1 (x^3 + x) dx = 0$

## Exercises

### Exer 7.3.1

Let  $F(x) = \int_x^{2x} t dt$ . Use the Fundamental Theorem of Calculus to evaluate  $F'(x)$ .

*Hint: we can split this into two integrals:*

$$F(x) = \int_x^{2x} t dt = \int_x^0 t dt + \int_0^{2x} t dt = -\int_0^x t dt + \int_0^{2x} t dt.$$

*Now apply the FTC to each term, including the Chain Rule for the second.*

## Exercises

### Exer 7.3.2

Evaluate the following integrals.

$$1. \int e^{2x} + \frac{1}{2x} dx$$

$$2. \int \frac{3}{\sqrt{2-x^2}} dx$$

### Exer 7.3.3

Evaluate the definite integral  $\int_1^e e^{2x} + \frac{1}{2x} dx$

### Exer 7.3.4

Find two values of  $q$  for which  $\int_q^0 2x + x^2 dx = 0$ .