This is a sample paper for 2526-MA385. It is similar to the final Semester 1 exam paper in the following ways:

- It features 5 questions; all to be attempted.
- Questions 1 and 2 are based on material from Section 1 (may have some over-lapping content. E.g., Newton's Method, or FPI could feature on both).
- Questions 3, 4 and 5 are based on Sections 2, 3, and 4 respectively with minimal overlap (and only in so far as Sections 3 and 4 overlap a little)
- Questions feature a mixture of definitions, theory and calculations.
- The questions on the exam will, of course, be different. However, if you can attempt this paper unseen, you are well prepared.

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Q1. Suppose we wish to find $\tau \in [a, b]$ such that $f(\tau) = 0$ for some nonlinear function $f(x)$.

(a) State the **Secant Method** for this problem. Provide a justification for it.

(b) Suppose that $f(x) = 2x^2 - 5$. Show that $f(x) = 0$ has a solution in $[1, 2]$.

(c) Taking $x_0 = 1$ and $x_1 = 2$, carry out **three** iterations of the Secant Method to estimate the solution to $2x^3 - \mathbf{5} = 0$. Show your calculations to 4 decimal places.

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Q2. (a) What does it mean for a function, g, to be a contraction on an interval $[a, b]$?

(b) Suppose that we have a fixed point iteration (FPI) method $x_{k+1} = g(x_k)$, and that g is known to be a contraction, with a fixed point τ. Show that the sequence generated by the method, $\{x_0, x_1, x_2, \dots\}$ converges *at least linearly* to τ.

(c) Suppose that we want to solve $2x^2 - 5 = 0$ using FPI, in order to approximate $\tau = \sqrt{10}/2$. That is, we choose a function $g = g(x)$, and initial guess $x_0 \in [1, 2]$, and set $x_{k+1} = g(x_k)$ for $k = 0, 1, 2, \dots$. Consider the following functions:

$$g_1(x) = 2x^2 + x - 5, \qquad g_2(x) = x/2 + 5/(4x), \qquad g_3(x) = x^2/5 - 1/2.$$

For each of these, determine whether or not it is a suitable choice of g in the FPI.

(d) Show that Newton's method for solving $f(x) = 0$ can be considered as a FPI method. What FPI method does it yield when we use it to solve $f(x) = 2x^2 - 5 = 0$?

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Q3. Consider the general two-stage Runge-Kutta (RK2) method: $y_{i+1} = y_i + f\Phi(t_i, y_I; h)$, where

$$k_1 = f(t_i, y_i), \qquad\qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1)$$

and

$$\Phi(t_i, y_i; h) = ak_1 + bk_2.$$

For a specific method we can take

$$\alpha = 2, \text{ and } a = 1/4. \tag{1}$$

(a) What does it mean for a one-step method to be *consistent*? Determine the value of b_2 for the method to be consistent.

(b) Suppose y solves the initial value problem

$$y(1) = 1, \qquad y'(t) = 2t \quad \text{for} \quad t > 1.$$

Explain why the RK2 method should compute the exact solution. Use this fact to determine the value for α.

(c) Suppose that we attempt to solve

$$y'(t) = \lambda y(t) \qquad y(0) = 1,$$

with a RK2 method. Use the fact that the RK2 solution should agree with the Taylor series for $y(t_{i+1})$ about t_i, up to terms of order h^2, to find a value of β.

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Q4. (a) Let $L \in \mathbb{R}^{n\times n}$ be a non-singular lower triangular matrix, and $\boldsymbol{b} \in \mathbb{R}^n$ be such that $b_i = 0$ for $i = 1, \dots, k \leq n$. If $\boldsymbol{y}$ solves $L\boldsymbol{y} = \boldsymbol{b}$, show that $y_i = 0$ for $i = 1, \dots, k \leq n$.

Hence or otherwise, show that the inverse of a nonsingular lower triangular matrix is also lower triangular.

(b) Define the LU factorization of a matrix. What assumptions must be made on the matrix to ensure that such a factorization exists?

(c) Find the LU-factorisation of

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix}.$$

Use this factorization to solve $Ax = b$, where $b = (4, -4, 0, 8)^T$.

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Q5. (a) Recall the definition of the Euclidean norm on $\mathbb{R}^n$: $\|\boldsymbol{u}\|_2 = \sqrt{\boldsymbol{u}^T\boldsymbol{u}}$. Prove the Cauchy-Schwarz inequality:

$$|\sum_{i=1}^{n} u_i v_i| \leq \|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_2 \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n.$$

Hence show that $\|\cdot\|_2$ satisfies the triangle inequality.

(b) Let A be any matrix in $\mathbb{R}^{n\times n}$. What are the *singular values* of A? Show that they are real and non-negative.

Define the *subordinate matrix norm* on $\mathbb{R}^{n\times n}$ associated with $\|\cdot\|_2$ and show that $\|A\|_2$ is the largest singular value of A.

(c) State the Gerschgorin First Circle Theorem, and use it to find an upper bound on $\|A\|_2$ when

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix}.$$