2425-MA140 Engineering Calculus

Week 07, Lecture 1
Optimization and L'Hôpital's Rule

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Tuesday, 28 October, 2025

Today, MA140 is all about...

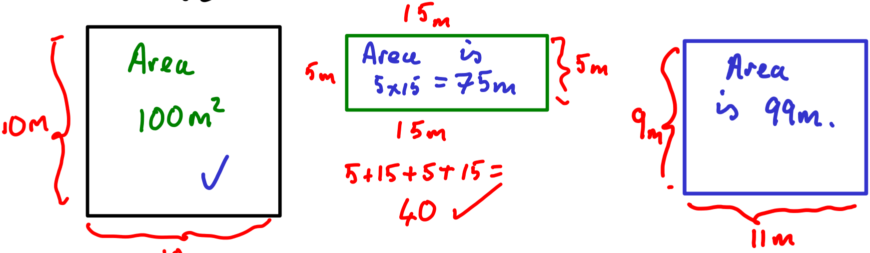
See also: Sections **4.7** (Applied Optimization Problems) and **4.8** (L'Hôpital's Rule) in **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Now that we know how to find maxima and minima of functions, we can solve **optimization** problems. Here is a classic example:

Example

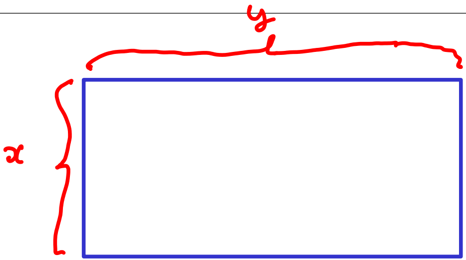
What is the largest area of a rectangular field we could enclose with 40m of fencing?

We can "solve" this problem by checking a few cases.



Optimal: 10x10 (square)

Now use calculus:



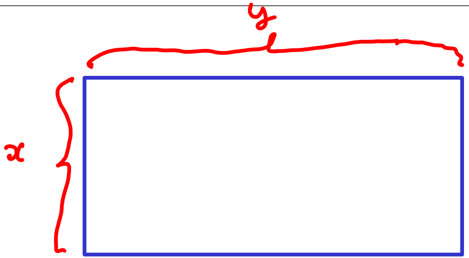
Let f be the area of a rectangle with sides of length x and y. So $f = xy$. We want to express f just in terms of x. We know that the perimtter is 40. So

$2x + 2y = 40 \Rightarrow 2y = 40 - 2x$

$\Rightarrow y = 20 - x$.

Then we write f as $f(x) = x(20 - x)$
$= 20x - x^2$.

Now use calculus:



So we have the Area is

$$f(x) = 20x - x^2.$$

To find a local max of f we solve $f'(x) = 0$.

$$\Rightarrow \quad 20 - 2x = 0 \quad \Rightarrow \quad 10 - x = 0 \quad \Rightarrow \quad x = 10.$$

So f has a critical point at $x = 10$. Also $f''(x) = -2 < 0$, so it is a local max.

$f(x) = 20x - x^2$ so $f(10) = 200 - 100 = 100$.

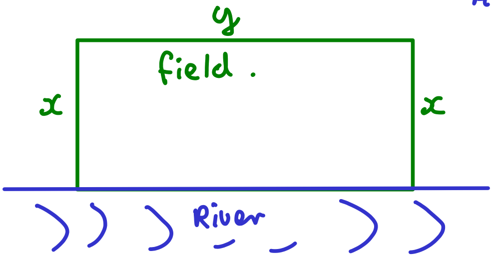
ANS: $100 m^2$

Here is a more general approach:

1. Write down a function, f, describing the quantity to be minimized/maximized. (ie the quantity to be optimized)
2. If f is in more than one variable, use other information, linking the variables, to reduce it to a function of one variable.
3. Differentiate f, and find its critical points. Determine which correspond to maxima and minima.
   (use 2^{nd} deriv test if needed).

Example:

A stretch of land is bordered by a (remarkably straight) river. What is the largest field we could enclose with 40m of fencing, if we don't have fence along by the river?



Again $f = xy$
The length of the fence is $2x + y = 40$
So $y = 40 - 2x$.
Then $f(x) = x(40 - 2x)$
$\Rightarrow f(x) = 40x - 2x^2$.

Then $f'(x) = 40 - 4x$. Solving $f'(x) = 0$
ie $40 - 4x = 0 \Rightarrow \boxed{x = 10}$ So $y = 20$
And the maximal f is $f(10) = 200$.

Sometimes, we are given the formula of the quantity to be optimised explicitly.

Example

Suppose that if a particular vehicle is been driven at a speed of x km/hr then its fuel usage, measured, in L/100km is given by

$$y = \frac{x^2}{1000} - \frac{1}{10}x + 10,$$

1. What speed should you drive at in order to minimise your fuel usage?
2. What is the fuel usage (in $L/100$km) at that speed?

Write the function to be optimized:

$$f(x) = \frac{x^2}{1000} - \frac{x}{10} + 10.$$

First, solve $f'(x)=0$ for x:

$f'(x) = \frac{x}{500} - \frac{1}{10}$. So solve $\frac{x}{500} - \frac{1}{10} = 0$

$\Rightarrow \frac{x}{500} = \frac{1}{10} \Rightarrow x = \frac{500}{10} = 50$

So, $f'(x)=0$ at $x=50$.

Also $f''(x) = \frac{1}{500} > 0$ so this is a local minimum

So our optimal speed is 50 km/hr.

2. The fule usage is $f(50) = 7.5$ L/100 km.

(Check!!)

L'Hôpital's Rule

Now that we've learned some differential calculus, we'll use a powerful tool for computing limits of the quotient of two functions: that is, something like

$$\lim_{x \to a} \frac{f_1(x)}{f_2(x)}.$$

We know (from Week 2, Lecture 2), that, if $\lim_{x \to a} f_1(x) = L_1$, and $\lim_{x \to a} f_2(x) = L_2 \neq 0$, then

$$\lim_{x \to a} \frac{f_1(x)}{f_2 x)} = \frac{L_1}{L_2}.$$

But what happens in both $L_1 = 0$ and $L_2 = 0$? This is called an **indeterminate form**, and some other methods are needed, e.g., if $f_1(x)$ and $f_2(x)$ are polynomials, we could factorise them.
But now we'll learn a powerful, more general approach...

L'Hôpital's Rule: the $\frac{0}{0}$ case

Suppose that f and g are both differentiable everywhere on an open interval containing a (except possibly at a). If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

providing the limit on the right exists, or is $\pm\infty$. This is true also for one-sided limits, or if $a = \pm\infty$.

Note about the spelling: L'Hôpital's Rule (1696) is named after Guillaume de l'Hospital, who spelled L'Hôpital as L'Hospital. But since then, French spelling has changed. Also, L'Hôpital's Rule was discovered (invented?) by Johann Bernoulli in 1694. Confused?

Example

Evaluate the limit

$$\lim_{x \to 1} \frac{\ln(x)}{x-1}$$

Check! $\frac{\ln(1)}{1-1} = \frac{0}{0}$ ✓ we can apply L'Hôpital's Rule.

$f(x) = \ln(x)$ $\qquad$ $g(x) = x - 1$

$f'(x) = \frac{1}{x}$ $\qquad$ $g'(x) = 1$

So $\lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1.$

One may apply L'Hôptial's Rule multiple times.

Example

Evaluate the limit $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

Let $f(x) = \sin(x) - x$ $\quad g(x) = x^3$

$f(0) = 0 - 0 = 0$ $\quad g(0) = 0^3 = 0$ ✓

$f'(x) = \cos(x) - 1$ $\quad g'(x) = 3x^2$

So $\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2}.$

But at $x = 0$, $\frac{\cos(0) - 1}{3(0)^2} = \frac{0}{0}$ (!)

One may apply L'Hôptial's Rule multiple times.

Example

Evaluate the limit $\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$

Apply the Rule Again (and again!)

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3} = \lim_{x \to 0} \frac{\cos(x) - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin(x)}{6x}$$

$$= \lim_{x \to 0} \frac{-\cos(x)}{6} = \boxed{\frac{-1}{6}}$$

L'Hôpital's Rule: the $\dfrac{\infty}{\infty}$ case

Suppose that f and g are both differentiable everywhere on an open interval containing a (except possibly at a). If $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

providing the limit on the right exists, or is $\pm\infty$. This is true also for one-sided limits, or if $a = \pm\infty$.

Example

Evaluate the limit $\lim_{x\to\infty} \frac{x^2}{e^x}$

$f(x) = x^2$ $g(x) = e^x$

$\lim_{x\to\infty} x^2 = \infty$ $\lim_{x\to\infty} e^x = \infty$

So, by the Rule (applied twice!)

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

In case you are wondering *why* L'Hôpital's Rule works. Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof: Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$
$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(x)} = \lim_{x\to a} \frac{f(x)-0}{g(x)-0} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

(Not on the exam!!)

Exercises

Exer 7.1.1 (taken from 2425-MA140 Exam Paper)

A car manufacturer estimates that the fuel efficiency of their car, when driven on a motorway at a given speed, can be determined by the formula

$$y = \frac{x^2}{1500} - \frac{1}{12}x + \frac{32}{5},$$

where x is the speed in km/h, and y is the fuel usage, measured in $L/100$km.

1. What speed should you drive at in order to minimise your fuel usage?
2. What is the fuel usage (in $L/100$km) at that speed?

Exercises

Exer 7.1.2

Use L'Hôpital's Rule to evaluate the following:

1. $\displaystyle\lim_{x\to 1} \frac{(x-1)^2}{\ln(x)}$.
2. $\displaystyle\lim_{x\to \infty} \frac{(x-1)^2}{\ln(x)}$.

Exer 7.1.3

Use L'Hôpital's Rule to evaluate $\displaystyle\lim_{x\to 0^+} x\ln(x)$. (*Hint:* write $\ln(x)$ as $\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ tend to either $-\infty$ or ∞ as $x \to 0$.)