MA385 Part 1: Solving nonlinear equations

1.6: Fixed Point Iteration

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Newton's method can be considered to be a special case of a very general approach called *Fixed Point Iteration* (**FPI**) or *Simple Iteration*.

The basic idea is:

> *If we want to solve* $f(x) = 0$ *in* $[a, b]$, *find a function* $g(x)$ *such that, if* τ *is such that* $f(\tau) = 0$, *then* $g(\tau) = \tau$. *Choose* x_0 *and set* $x_{k+1} = g(x_k)$ *for* $k = 0, 1, 2, \dots$.

0. News!

1. Week 4: Tutorials start next week (week beginning Monday, 29 Sep).
2. A tutorial sheet is available at https://www.niallmadden.ie/2526-MA385/MA385-Tutorial-1.pdf. The tutor will work with you on that. Questions will be similar in style to the final exam.
3. Tutorials are Mondays at 10 in AC-201 and Thursday at 2 in ENG-3036. Go to either. If available, please go to the Monday class (larger room).
4. Week 5: we'll have a lab, using Python/Jupyter.

0. Outline

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For more details, see Section 1.4 (Relaxation and Newton's method) of Süli and Mayers, *An Introduction to Numerical Analysis*

Also, Chapter 3 of Epperson: https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9781118730966

1. Introduction

Yet again, we want to solve

Given a function $f(x)$, find $\tau \in [a, b]$ such that

$$f(\tau) = 0.$$

Again, we'll try to find a sequence $\{x_0, x_1, \dots, x_k, \dots\}$, such that $x_k \to \tau$ as $k \to \infty$.

In this section, we'll consider one step methods, which, like Newton's method, compute x_{k+1} just from x_k.

The Method is called **Fixed Point Iteration** (FPI):

- Choose a function g such that, if $f(\tau) = 0$, then τ is a fixed point of g.
- Choose x_0, and then iterate with $x_{k+1} = g(x_k)$.

Bisection & Secant are two-step

Methods : $x_{k+1} = \tilde{g}(x_k, x_{k-1})$

Example 1.6.1

Suppose $f(x) = e^x - 2x - 1$ and we are trying to find a solution to $f(x) = 0$ in $[1, 2]$. Then we can take $g(x) = \ln(2x + 1)$.

If we take $x_0 = 1$, then we get the following sequence:

$e^x - 2x - 1 = 0$

$\Rightarrow e^x = 2x + 1$

$\Rightarrow \ln(e^x) = \ln(2x+1)$

$\Rightarrow x = \ln(2x+1)$

k	x_k	$\|f(x_k)\|$	$\|\tau - x_k\|$
0	1.0000	0.2817	2.5643e-01
1	1.0986	0.1972	1.5782e-01
2	1.1623	0.1273	9.4148e-02
3	1.2013	0.0781	5.5092e-02
4	1.2246	0.0464	3.1868e-02
5	1.2381	0.0271	1.8310e-02
6	1.2460	0.0157	1.0479e-02
⋮	⋮	⋮	
10	1.2553	0.0017	1.1079e-03

2. How not to choose g

We have to be quite careful with this method: **not every choice is g is suitable**.

For example, suppose we want the solution to $f(x) = x^2 - 2 = 0$ in $[1, 2]$. We could choose $g(x) = x^2 + x - 2$. Then, if take $x_0 = 1$ we get the sequence:

$x_0 = 1 \qquad x_1 = g(x_0) = 1 + 1 - 2 = 0.$

$x_2 = g(x_1) = 0^2 + 0 - 2 = -2$

$x_3 = g(-2) = (-2)^2 - 2 - 2 = 0$

$x_4 = g(0) = -2$

$x_5 = 0$

etc.

Before we do that in a formal way, consider the following...

Example 1.6.2

Use the Mean Value Theorem to show that the fixed point method $x_{k+1} = g(x_k)$ converges if $|g'(x)| < 1$ for all x near the fixed point.

By the MVT: given a, b there is a point $c \in [a, b]$ such that

$$\frac{g(b) - g(a)}{b - a} = g'(c)$$

$\Rightarrow \quad g(b) - g(a) = g'(c)(b - a).$

Let $a = x_k$ and $b = \tau$. Then $g(\tau) - g(x_k) = g'(c)(\tau - x_k)$

$\Rightarrow \tau - x_{k+1} = g'(c)(\tau - x_k) \Rightarrow |\tau - x_{k+1}| = |g'(c)|\,|\tau - x_k|$

So $|\tau - x_{k+1}| < |\tau - x_k|$

2. How not to choose g

This previous example is useful in two ways:

1. It introduces the tricks of using that $g(\tau) = \tau$ & $g(x_k) = x_{k+1}$.
2. It leads us towards the **contraction mapping theorem**.

Definition: fixed point

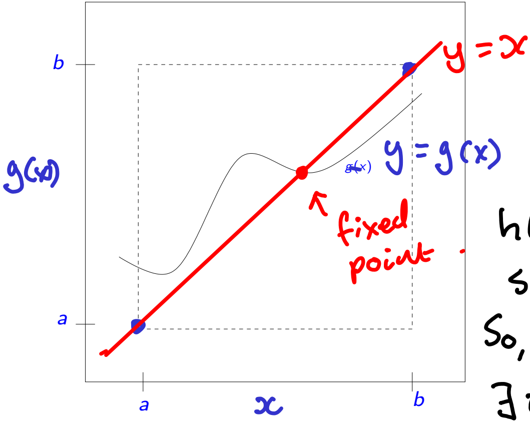
A function $g : \mathbb{R} \to \mathbb{R}$ is said to have a **fixed** point at $x = \tau$ if $g(\tau) = \tau$

Eg $g(x) = x + 1$ does not have a fixed point

Eg If $g(x)$ has a fixed point then $f(x) = g(x) - x$ has a zero.

Theorem 1.6.1 (Fixed Point Theorem)

Suppose the function g is cont's on $[a, b]$, and $a \leq g(x) \leq b$ for all $x \in [a, b]$. Then $g(x)$ has a *fixed point* in $[a, b]$.



Proof: $h(x) = g(x) - x$.

Then

$h(a) = g(a) - a \geq 0$

since $g(a) \geq a$

$h(b) = g(b) - b \leq 0$

since $g(b) \leq b$.

So, by the IVT

$\exists\, \tau \in [a, b]$ s.t. $h(\tau) = 0$ ✓

Next suppose that g is a *contraction*. That is, $g(x)$ is continuous and defined on $[a, b]$ and there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b]. \tag{1}$$

Theorem 1.6.2 (Contraction Mapping Theorem)

Suppose that the function g is a real-valued, defined, continuous, and

(a) maps every point in $[a, b]$ to some point in $[a, b]$, and

(b) is a contraction on $[a, b]$,

then

(i) $g(x)$ has a fixed point $\tau \in [a, b]$,

(ii) the fixed point is unique,

(iii) the sequence $\{x_k\}_{k=0}^{\infty}$ defined by $x_0 \in [a, b]$ and $x_k = g(x_{k-1})$ for $k = 1, 2, \ldots$ converges (at least linearly) to τ.

(i) is true because of Thm 1.6.1 (since $a \le g(x) \le b \quad \forall x \in [a,b]$)

Proof of (ii) (that τ is unique).

Suppose that g has two fixed points τ_1 and τ_2, and $\tau_1 \neq \tau_2$

So $g(\tau_1) = \tau_1$ $\quad g(\tau_2) = \tau_2$

Then $|\tau_1 - \tau_2| = |g(\tau_1) - g(\tau_2)|$

$$\leq L|\tau_1 - \tau_2|$$

But $0 < L < 1$ so

$|\tau_1 - \tau_2| < |\tau_1 - \tau_2|$

which is not possible.

(iii) Show that $x_k \to \tau$ as $k \to \infty$

$$|\tau - x_k| = |g(\tau) - g(x_{k-1})|$$
$$\leq L|\tau - x_{k-1}|$$

So

$$|\tau - x_1| \leq L|\tau - x_0|$$
$$|\tau - x_2| \leq L|\tau - x_1| \leq L^2|\tau - x_0|$$

And, in general $|\tau - x_k| \leq L^k|\tau - x_0|$

Since $0 < L < 1$ so $L^k \to 0$ as $k \to \infty$

So $|\tau - x_k| \to 0$ as $k \to \infty$. $\boxed{}$

4. How many iterations?

The algorithm generates as sequence $\{x_0, x_1, \ldots, x_k\}$. Eventually we must stop. Suppose we want the solution to be accurate to say 10^{-6}, how many steps are needed? That is, how big do we need to take k so that

$$|x_k - \tau| \leq 10^{-6}?$$

The answer is obtained by first showing that

$$|\tau - x_k| \leq \frac{L^k}{1-L}|x_1 - x_0|. \qquad (2)$$

This is because $|\tau - x_0| = |\tau - x_1 + x_1 - x_0|$

So $|\tau - x_0| \leq |\tau - x_1| + |x_1 - x_0|$

$\Rightarrow |\tau - x_0| \leq |g(\tau) - g(x_0)| + |x_1 - x_0|$

So $|\tau - x_0| \leq L\,|\tau - x_0| + |x_1 - x_0|$

$\Rightarrow |\tau - x_0| \leq \frac{1}{L-1}|x_1 - x_0|$

Example 1.6.3

Suppose we are using FPI to find the fixed point $\tau \in [1,2]$ of $g(x) = \ln(2x+1)$ with $x_0 = 1$, and we want $|x_k - \tau| \leq 10^{-6}$, then we can use (2) to determine the number of iterations required.

Here $g'(x) = \dfrac{2}{2x+1}$. So, on $[1,2]$ $|g'(x)| \leq g(1) = \dfrac{2}{3}$

So, from the Mean Value theorem,

$$|g(\tau) - g(x_k)| \leq |g'(\eta)|\,|\tau - x_k|$$

$$\Rightarrow |\tau - x_{k+1}| \leq \left(\tfrac{2}{3}\right)|\tau - x_k|$$

So, take $L = \frac{2}{3}$.

So $|\tau - x_k| \leq \dfrac{L^k}{1-L}|x_1 - x_0| = 3\left(\tfrac{2}{3}\right)^k |x_1 - x_0|$

Then (Check) need $k \geq 36.783\ldots$ $k \geq 37$.

5. Newton's method as a FPI

Newton's method can be thought of as an example of a fixed point method, where we take

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

However, we know that, when Newton's Method converges it does so quadratically, whereas FPI converges (at least linearly).

5. Newton's method as a FPI

Let's remind ourselves of the definition:

- We have a sequence of numbers ε_0, ε_1, ..., such that $\lim_{k\to\infty} \varepsilon_k = 0$.
- These bound the errors: $|\tau - x_k| \leq \varepsilon_k$ Let $\tau = \lim_{k\to\infty} x_k$.
- We know that $\lim_{k\to\infty} x_k = \tau$
- Then we say that the sequence $\{x_k\}_{k=0}^{\infty}$ converges *with at least order* q if

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu,$$

  for some constant μ.

For $q = 1$ we get linear convergence. If $q = 2$, we say it is *quadratic*.

5. Newton's method as a FPI

Suppose that we have a convergent Fixed Point Method, $x_{k+1} = g(x_k)$, but with the additional property that $g'(\tau) = 0$. Then, in fact, FPI converges (at least) quadratically):

Use a truncated Taylor Series

$$g(x) = g(\tau) + (x-\tau)\,g'(\tau) + \tfrac{1}{2}(x-\tau)^2 g''(\eta)$$

for some $\eta \in [x, \tau]$.

Then $$g(x_k) - g(\tau) = (x_k - \tau)\,g'(\tau) + (x_k - \tau)^2 \frac{g''(\eta)}{2}.$$

Since $x_{k+1} = g(x_k)$ and $\tau = g(\tau)$, and $g'(\tau) = 0$

we get

$$x_{k+1} - \tau = (x_k - \tau)^2 \frac{g''(\eta)}{2}$$

5. Newton's method as a FPI

Suppose that we have a convergent Fixed Point Method, $x_{k+1} = g(x_k)$, but with the additional property that $g'(\tau) = 0$. Then, in fact, FPI converges (at least) quadratically):

Since the method converges, so

$$\lim_{k\to\infty} x_k = \tau.$$

So $\lim_{k\to\infty} g''(\eta) = g''(\tau)$ which is a constant.

Therefor

$$\lim_{k\to\infty} \frac{|\tau - x_{k+1}|}{|\tau - x_k|^2} = \mu$$

where $\mu = \frac{1}{2} g''(\tau)$ ∎

5. Newton's method as a FPI

Finally, we show that, in the FPI setting, Newton converges quadratically:

We need to show that, if we write Newton's Method as a FPI method, then $g'(\tau) = 0$.

Newton: $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

So take $g(x) = x - \dfrac{f(x)}{f'(x)}$

Then $g'(x) = 1 - \dfrac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \dfrac{f(x)f''(x)}{(f'(x))^2}$

So $g'(\tau) = 0$ since $f(\tau) = 0$ $\boxed{\text{M}}$

Exercise 1.6.1

Is it possible for g to be a contraction on $[a, b]$ but not have a fixed point in $[a, b]$? Give an example to support your answer.

Exercise 1.6.2

Show that $g(x) = \ln(2x + 1)$ is a contraction on $[1, 2]$. Give an estimate for L. (Hint: Use the Mean Value Theorem).

Exercise 1.6.3

Suppose we wish to numerically estimate the famous *golden ratio*, $\tau = (1+\sqrt{5})/2$, which is the positive solution to $x^2 - x - 1$. We could attempt to do this by applying fixed point iteration to the functions $g_1(x) = x^2 - 1$ or $g_2(x) = 1 + 1/x$ on the region $[3/2, 2]$.

(i) Show that g_1 is *not* a contraction on $[3/2, 2]$.

(ii) Show that g_2 *is* a contraction on $[3/2, 2]$, and give an upper bound for L.

Exercise 1.6.4

In class we saw that if $g(\tau) = \tau$, and the fixed point method given by

$$x_{k+1} = g(x_k),$$

converges to the point τ (where $g(\tau) = \tau$), and if $g'(\tau) = 0$, then the method converges quadraticially.
Show that, in fact if

$$g'(\tau) = g''(\tau) = \cdots = g^{(p-1)}(\tau) = 0,$$

then it converges with order p.