MA385 Part 4: Linear Algebra 2

4.1: Vector Norms

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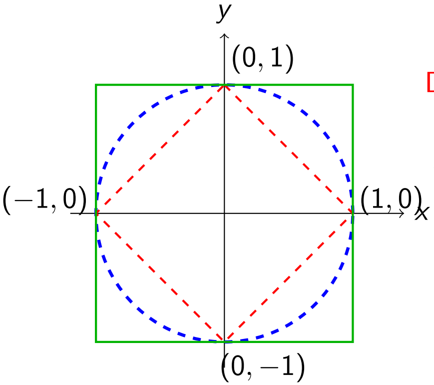
November 2025



Diamond: $\|x\|_1 \leq 1$

Circle: $\|x\|_2 \leq 1$

Square: $\|x\|_\infty \leq 1$

1. Outline Section 4.1

For more, see Section 2.7 of Suli and Mayers:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=51&c=UERG

2. Introduction

This is the final section of MA385. It is kinda a direct continuation of Section 3 – and much of the material is from the same chapter as Section 3 in the text-book (though we'll also take some material from Chapter 5).

At its heart, is the task of bounding the eigenvalues and singular values of a matrix. Our motivation comes from doing an error analysis for LU-factorization. However, the applications are far more general than that.

2. Introduction

But for now, we'll just note that all computer implementations of algorithms that involve floating-point numbers (roughly, finite decimal approximations of real numbers) contain errors due to round-off error.

It transpires that computer implementations of LU-factorization, and related methods, lead to these round-off errors being greatly magnified: and we want to understand why.

2. Introduction

You might remember from earlier sections of the course that we had to assume functions where well-behaved in the sense that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L,$$

for some number L, so that our numerical schemes (e.g., fixed point iteration, Euler's method, etc) would work. If a function *doesn't* satisfy a condition like this, we say it is "ill-conditioned". One of the consequences is that a small error in the inputs gives a large error in the outputs.

We'd like to be able to express similar ideas about matrices: that $A(u - v) = Au - Av$ is not too "large" compared to $u - v$. To do this we used the notion of a "norm" to describing the relatives sizes of the vectors u and Au.

3. Three vector norms

When we want to consider the size of a real number, without regard to sign, we use the *absolute value*. Important properties of this function are:

1. $|x| \geq 0$ for all x.
2. $|x| = 0$ if and only if $x = 0$.
3. $|\lambda x| = |\lambda||x|$.
4. $|x + y| \leq |x| + |y|$ (triangle inequality).

This notion can be extended to vectors and matrices.

3. Three vector norms

Definition 4.1.1

Let $\mathbb{R}^n$ be all the vectors of length n of real numbers. The function $\|\cdot\|$ is called a **norm** of $\mathbb{R}^n$ if, for all $u, v \in \mathbb{R}^n$

1. $\|v\| \geq 0$,
2. $\|v\| = 0$ if and only if $v = 0$.
3. $\|\lambda v\| = |\lambda|\|v\|$ for any $\lambda \in \mathbb{R}$,
4. $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).

Norms on vectors in $\mathbb{R}^n$ quantify the *size* of the vector. But there are different ways of doing this...

Definition 4.1.2

Let $\mathrm{v} \in \mathbb{R}^n$: $\mathrm{v} = (v_1, v_2, \dots, v_{n-1}, v_n)^T$.

(i) The 1-*norm* (a.k.a. the *Taxi cab* norm) is
$$\|v\|_1 = \sum_{i=1}^{n} |v_i|.$$

(ii) The 2-*norm* (a.k.a. the *Euclidean norm*)
$$\|v\|_2 = \Big(\sum_{i=1}^{n} v_i^2\Big)^{1/2}.$$
Note, if v is a vector in $\mathbb{R}^n$, then
$$\mathrm{v}^T\mathrm{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathrm{v}\|_2^2.$$

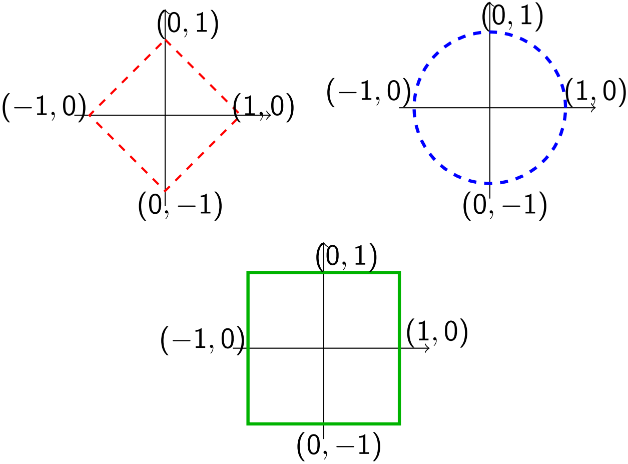
(iii) The ∞-*norm* (a.k.a. the max-*norm*) $\|v\|_\infty = \max_{i=1}^{n} |v_i|$.

3. Three vector norms

Example: Compute the 1-, 2- and ∞-norm of $v = (-2, 4, -4)$

3. Three vector norms

The unit balls in $\mathbb{R}^2$ given by $\|\cdot\|_1$ (top left), $\|x\|_2 = \sqrt{x_1^2 + x_2^2} = 1$ (top right), and $\|\cdot\|_\infty$.



3. Three vector norms

It is easy to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms (see next slide).

And it is not hard to show that $\|\cdot\|_2$ satisfies conditions (1), (2) and (3) of Definition 4.1.1.
It takes a little bit of effort to show that $\|\cdot\|_2$ satisfies the triangle inequality; so we'll do that with care.

4. $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$

As mentioned, it takes a little effort to show that $\|\cdot\|_2$ is indeed a norm on $\mathbb{R}^2$; in particular to show that it satisfies the triangle inequality, we need the Cauchy-Schwarz inequality.

Lemma (Cauchy-Schwarz)

$$|\sum_{i=1}^{n} u_i v_i| \leq \|\mathrm{u}\|_2 \|\mathrm{v}\|_2, \qquad \forall \mathrm{u}, \mathrm{v} \in \mathbb{R}^n.$$

Idea: $0 \leq \|\lambda u + v\|_2^2$.

Example: Pick two vectors in $\mathbb{R}^3$ and convince yourself they satisfy the Cauchy-Schwarz Inequality.

Now can now apply Cauchy-Schwatz to show that

$$\|\mathrm{u} + \mathrm{v}\|_2 \leq \|\mathrm{u}\|_2 + \|\mathrm{v}\|_2.$$

This is because

$$\begin{aligned}
\|u + v\|_2^2 &= (u + v)^T (u + v) \\
&= u^T u + 2u^T v + v^T v \\
&\leq u^T u + 2|u^T v| + v^T v \quad \text{(by the triangle-inequality)} \\
&\leq u^T u + 2\|u\|\|v\| + v^T v \quad \text{(by Cauchy-Schwarz)} \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

It follows directly that

Corollary

$\|\cdot\|_2$ *is a norm.*

6. Exercises

Exercise 4.1.1

Show that, for any vector $x \in \mathbb{R}^n$, $\|x\|_\infty \le \|x\|_2$ and $\|x\|_2^2 \le \|x\|_1 \|x\|_\infty$. For each of these inequalities, give an example for which the equality holds. Deduce that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.

Exercise 4.1.2

Show that if $x \in \mathbb{R}^n$, then $\|x\|_1 \le n\|x\|_\infty$ and that $\|x\|_2 \le \sqrt{n}\|x\|_\infty$.