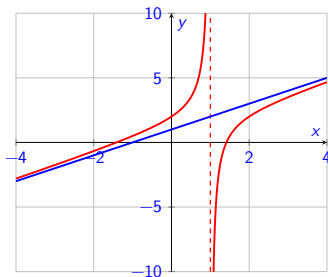


Introduction to Limits

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Slides by Niall Madden, with some material adapted from textbooks, and original notes by Dr Kirsten Pfeiffer.

Outline

- 1 Reminders
- 2 Towards Limits
- 3 Definition of a Limit
- 4 Properties of Limits
 - Evaluating limits
- 5 Limits of rational functions
- 6 More limits
- 7 Exercises

For more, see Chapter 2 (Limits) of Strang and Herman's **Calculus**, especially Sections 2.2 (Limit of a Function) and 2.3 (Limit Laws).

Slides are on canvas, and at niallmadden.ie/2526-MA140



Reminders

- ▶ Tutorials started **this** week.
- ▶ Current assignment (for this week's tutorials) is PS-0. Just for practice. See <https://universityofgalway.instructure.com/courses/46734/assignments/128373>
- ▶ **Assignment 1** (PS-1) due 5pm, Monday 5 October. Will be covered in tutorials next week.
- ▶ Two class tests planned for this module, each worth 10% of the final grade.
 - ▶ Test 1: **Tuesday, 14 October** (Week 5)
 - ▶ Test 2: **Tuesday, 18 November** (Week 10)
 - ▶ Contact Niall if you have any concerns, or wish to avail of alternative arrangements, as provided by LENS reports.

Towards Limits

When we were considering the domain of a function, we looked at those x -values for which the function was not defined.

Example

$$f(x) = \frac{x^2 - 2}{x - 1}$$

$$g(x) = \frac{x^2 - 1}{x - 1}$$

Neither f nor g are defined at $x = 1$.

But what happens if x gets very closed to 1?

x	0.900	0.990	0.999	1	1.001	1.010	1.100
$f(x)$							
$g(x)$							

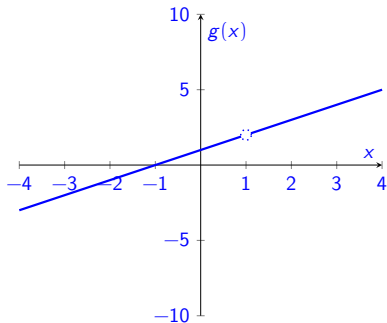
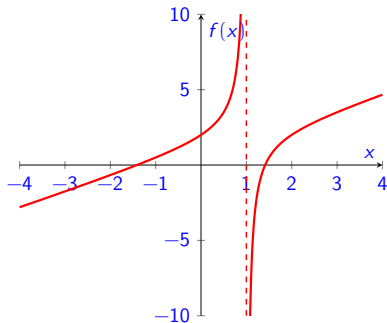
Let's look at the graphs of f and g .

Towards Limits

Example

$$f(x) = \frac{x^2 - 2}{x - 1}$$

$$g(x) = \frac{x^2 - 1}{x - 1}$$



Towards Limits

In the previous example, we saw that, although neither f nor g was defined at $x = 1$, they behaved very differently as x approaches 1.

To discuss this we'll need the concept of a **limit** which, roughly, relates to the value of function as it **approaches** a point (but not actually at that point).

$$\lim_{x \rightarrow a} f(x) = L$$

The concept of a limit is a prerequisite for a proper understanding of calculus, and numerical methods.

Definition of a Limit

Some conventions and terminology we'll use:

- ▶ x is a variable.
- ▶ a is a fixed number.
- ▶ ϵ is a **small** positive number (that we get to choose).
- ▶ δ is another **small** positive number (determined by ϵ).
- ▶ $|x - a| < \delta$ means that the distance between x and a is less than δ , i.e. very small.
- ▶ As x approaches a , so $f(x)$ approaches a number L .

When we write

$$\lim_{x \rightarrow a} f(x) = L,$$

we read

"The limit of f , as x goes to a , is L ".

Definition of a Limit

LIMIT: formal definition

$$\lim_{x \rightarrow a} f(x) = L,$$

means that, for every number $\epsilon > 0$, it is possible to find a number $\delta > 0$, such that

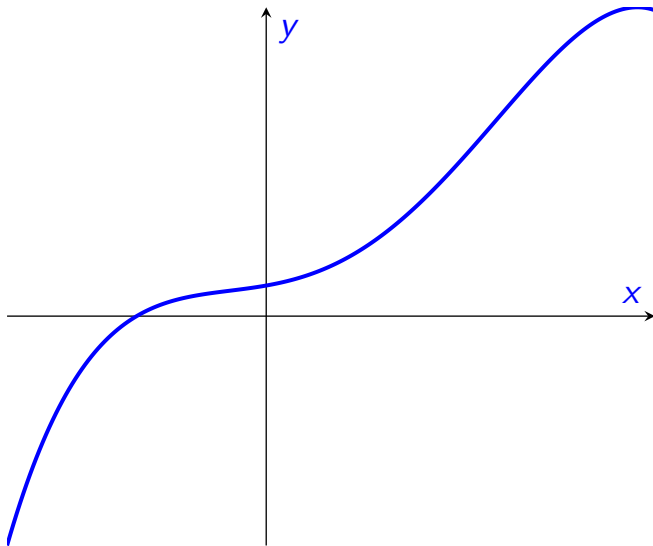
$$|f(x) - L| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

LIMIT: Informal explanation

$$\lim_{x \rightarrow a} f(x) = L,$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

Definition of a Limit



Definition of a Limit

Example

Prove formally that $\lim_{x \rightarrow 3} (4x - 5) = 7$.

That is, for arbitrary ϵ , find a δ such that

$$|(4x - 5) - 7| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta.$$

Definition of a Limit

The approach we just used is technically correct, but not very practical in many cases.

Fortunately, there are other methods that can be used

- ▶ to show that a limit exists;
- ▶ find the limit of a function, $f(x)$ as $x \rightarrow a$.

Properties of Limits

See also...

... Section 2.3 of the textbook: **Limit Laws**

Suppose that $\lim_{x \rightarrow a} f_1(x) = L_1$, and $\lim_{x \rightarrow a} f_2(x) = L_2$ and $c \in \mathbb{R}$ is any constant. Then,

$$(1) \lim_{x \rightarrow a} c = c, \quad c \in \mathbb{R}$$

$$(2) \lim_{x \rightarrow a} x = a$$

Properties of Limits

$$(3) \lim_{x \rightarrow a} [cf_1(x)] = cL_1$$

$$(4) \lim_{x \rightarrow a} [f_1(x) + f_2(x)] = L_1 + L_2 \text{ and} \\ \lim_{x \rightarrow a} [f_1(x) - f_2(x)] = L_1 - L_2$$

Properties of Limits

$$(5) \lim_{x \rightarrow a} (f_1(x)f_2(x)) = L_1L_2$$

$$(6) \lim_{x \rightarrow a} ((f_1(x))^n) = (L_1)^n$$

Properties of Limits

$$(7) \lim_{x \rightarrow a} \left(\frac{f_1(x)}{f_2(x)} \right) = \frac{L_1}{L_2}, \quad \text{providing } L_2 \neq 0.$$

$$(8) \lim_{x \rightarrow a} \sqrt[n]{f_1(x)} = \sqrt[n]{L_1}$$

Note: we can combine these properties as needed. For example, (5) and (8) together give that

$$\lim_{x \rightarrow a} x^n = a^n$$

Example

Evaluate the limit $\lim_{x \rightarrow 1} (x^3 + 4x^2 - 3)$

Example

Evaluate $\lim_{x \rightarrow 1} \frac{x^4 + x^2 - 1}{x^2 + 5}$ using the Properties of Limits.

Limits of rational functions

In many cases it's more complicated. In particular, we'll consider numerous examples where we want to evaluate $\lim_{x \rightarrow a} f(x)$ where a is not in the domain of f .

A typical example of this is when we evaluate a rational function:

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$$

where **both** $p(a) = 0$ and $q(a) = 0$.

Idea: Since we care about the value of p and q **near** $x = a$, but not actually at $x = a$, it is safe to factor out an $(x - a)$ term from both.

Limits of rational functions

Three examples

Evaluate the limits:

$$(a) \lim_{x \rightarrow 0} \frac{x}{x} \qquad (b) \lim_{x \rightarrow 0} \frac{x^2}{x} \qquad (c) \lim_{x \rightarrow 0} \frac{x}{x^2}$$

Limits of rational functions

Example

Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$$

Limits of rational functions

In that last example, we found that

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x}$$

But these are different functions:

Limits of rational functions

Evaluate the limit

$$\lim_{x \rightarrow 2} \left(\frac{\frac{1}{2} - \frac{1}{x}}{x - 2} \right)$$

More limits

Very often, we'll evaluate limits of the form:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where f and g are not polynomials. Some of the same ideas still apply.

Example

Evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2}$$

Exercises

Exercise 2.2.1

Evaluate the following limits

$$(a) \lim_{x \rightarrow \frac{1}{2}} \frac{x - \frac{1}{2}}{x^2 - \frac{1}{4}}$$

$$(b) \lim_{x \rightarrow -4} \frac{x^2 + 3x - 4}{x^2 + x - 12}$$

Exercise 2.2.2 (from 2425-MA140 exam)

Let $f(x) = \frac{x^2 - 2x - 15}{3x^3 - 6x^2 - 45x}$. For each of the following, evaluate the limit, or determine that it does not exist.

$$(i) \lim_{x \rightarrow -3} f(x)$$

$$(ii) \lim_{x \rightarrow 0} f(x)$$

$$(iii) \lim_{x \rightarrow 5} f(x)$$