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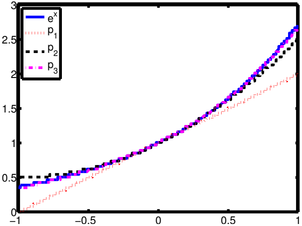
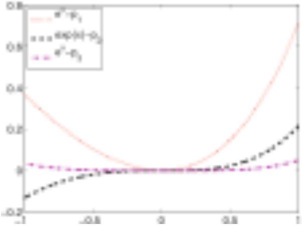
§0. Introduction to MA385; Taylor's Theorem

§0.2 Taylor's Theorem

MA385/530 – Numerical Analysis 1

September 2019





Taylor's Theorem is perhaps the most important mathematical tool in Numerical Analysis. Providing we can evaluate the derivatives of a given function at some point, it gives us a way of approximating the function by a polynomial.

Working with polynomials, particularly ones of degree 3 or less, is much easier than working with arbitrary functions. For example, polynomials are easy to differentiate and integrate. Most importantly for the next section of this course, their zeros are easy to find.

Our starting point is the classic mean value theorem.

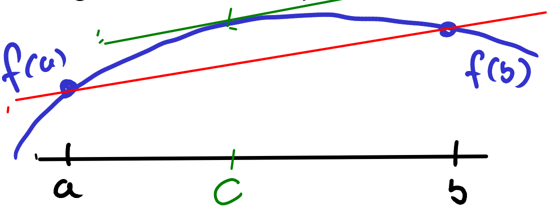
Brook Taylor, 1865 – 1731, England. He (re)discovered this polynomial approximation in 1712, though its full importance was not realised for another 50 years.

Theorem 1 (Mean Value Theorem)

If f is function that is continuous and differentiable for all $a \le x \le b$, then there is a point $c \in [a, b]$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). = \frac{df}{dx}(c).$$

This is just a consequence of Rolle's Theorem, and has few different interpretations. One is that the slope of the line that intersects f at the points a and b is equal to the slope of the tangent to f at some point between a and b.



Right now, we interested in the fact that the MVT tells us that we can approximate the value of a function by a near-by value, with accuracy that depends on f':

$\exists c \in [a,b]$ such that $f(b) - f(a) = f'(c)(b-a)$

So, e.g., $f(b) = f(a) + f'(c)(b-a)$.

Or we can think of it as approximating f by a line: Since b can be any point, just denote it x. Then $f(x) = f(a) + (x-a)f'(c)$. Of course, we don't know c. But, if x is "close" to a, c is close to a. So $f'(c) \approx f'(a)$. So $f(x) \approx f'(a)x + f(a) - af'(a)$.

What if we want a better approximation? We could replace our function with a quadratic polynomial. For example, let

$$p_2(x) = b_0 + b_1(x-a) + b_2(x-a)^2,$$

and solve for the coefficients b_0, b_1 and b_2 so that

$$p_2(a) = f(a), \quad p_2'(a) = f'(a), \quad p_2''(a) = f''(a).$$

If $p_2(a) = f(a)$ then $b_0 + b_1 \underbrace{(a-a)}_{\to 0} + b_2 \underbrace{(a-a)^2}_{\to 0} = f(a)$

That is, $\boxed{b_0 = f(a)}$

If $p_2'(a) = f'(a)$, then, $p_2'(x) = b_1 + 2b_2(x-a)$, so

$p_2'(a) = b_1 = f'(a)$. So $\boxed{b_1 = f'(a)}$.

Finally, $p_2''(x) = 2b_2$. So $b_2 = \frac{1}{2} f''(a)$.

Next, if we try to construct an approximating cubic of the form

$$\begin{aligned} p_3(x) &= b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3, \\ &= \sum_{k=0}^{3} b_k (x-a)^k, \end{aligned}$$

with the property that

$$\begin{aligned} p_3(a) &= f(a), \qquad & p_3'(a) &= f'(a), \\ p_3''(a) &= f''(a), \qquad & p_3'''(a) &= f'''(a). \end{aligned} \tag{1}$$

$f^{(k)} = \dfrac{d^k f}{dx^k}.$

Again we find that

$$b_k = \frac{f^{(k)}(a)}{k!} \quad \text{for} \quad k = 0, 1, 2, 3.$$

note $0! = 1$. So $b_0 = f(a)$, $b_1 = f'(a)$, $b_2 = \dfrac{f''(a)}{2}$, $b_3 = \dfrac{f'''(a)}{6}$

Definition 2 (Taylor Polynomial)

The *Taylor Polynomial* of degree k (also called the *Truncated Taylor Series*) that approximates the function f about the point $x = a$ is

$$p_k(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^k}{k!}f^{(k)}(a).$$

Finished here Thu 12/9/2019.

Example 3

Write down the Taylor polynomial of degree k that approximates $f(x) = e^x$ about the point $x = 0$, that is, take $a = 0$.

$$P_1 = f(a) + (x-a)f'(a)$$
$$= e^0 + x e^0 = 1 + x$$

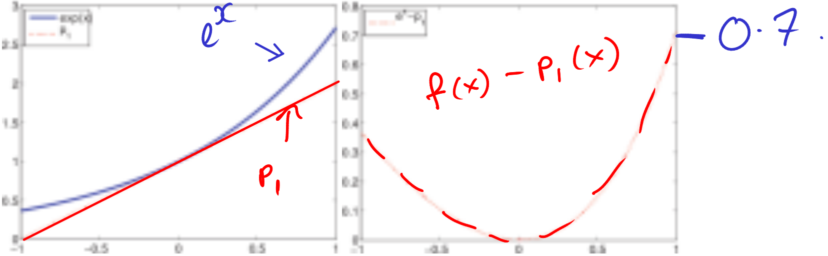
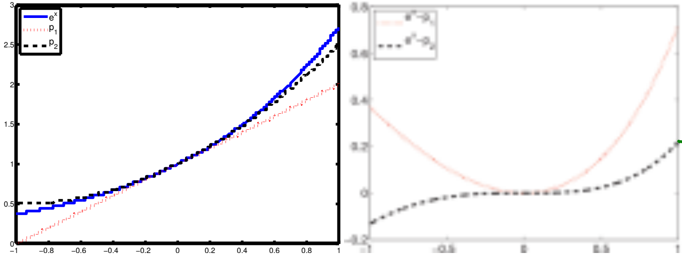


Figure: Taylor polys for $f(x) = e^x$ about $x = 0$ (left), and errors (right)

Example 3

Write down the Taylor polynomial of degree k that approximates $f(x) = e^x$ about the point $x = 0$.

$$P_2(x) = 1 + x + \frac{x^2}{2}.$$



0.2

Figure: Taylor polys for $f(x) = e^x$ about $x = 0$ (left), and errors (right)

Example 3

Write down the Taylor polynomial of degree k that approximates $f(x) = e^x$ about the point $x = 0$.

In general

$$p_k(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{k!}x^k.$$

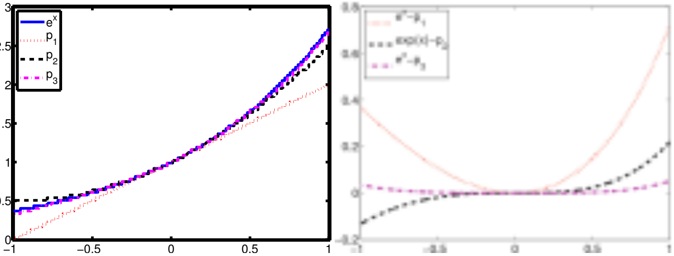


Figure: Taylor polys for $f(x) = e^x$ about $x = 0$ (left), and errors (right)

We now want to examine the *accuracy* of the Taylor polynomial as an approximation. In particular, we would like to find a formula for the *remainder* or *error*:

$$R_k(x) := f(x) - p_k(x).$$

we dont know σ in practice.

With a little bit of effort one can prove that:

$$R_k(x) := \frac{(x-a)^{k+1}}{(k+1)!} f^{(k+1)}(\sigma), \text{ for some } \sigma \in [x, a].$$

We won't prove this in class. But for the sake of completeness, a proof is included in the notes.

Example 4

With $f(x) = e^x$ and $a = 0$, we get that

$$R_k(x) = \frac{x^{k+1}}{(k+1)!} e^{\sigma}, \text{ some } \sigma \in [0, x].$$

Example 5

How many terms are required in the Taylor Polynomial for e^x about $x = 0$ to ensure that the error at $x = 1$ is

- no more than 10^{-1}? Ans: 4.
- no more than 10^{-2}? Ans: 5.
- no more than 10^{-6}? Ans: 10
- no more than 10^{-10}? Ans: 14.

note: $e^x \leq e$ for all $x \in [0,1]$. So $|R_K(x)| \leq \frac{x^{K+1}}{(K+1)!} e$

The reasons for emphasising Taylor's theorem so early in this course are that

- It introduces us to the concept of approximation, and error estimation, but in a very simple setting;
- It is the basis for deriving methods for solving both nonlinear equations, and initial value ordinary differential equations.

With the last point in mind, we'll now outline how to derive Newton's method for nonlinear equations.

Suppose we want to solve $f(x)=0$.
We know

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(\sigma)$$

for some σ in $[a,x]$.

$([1,0]=[0,1])$

We want to find x so that $f(x)=0$.
If x is "close" to a, then $|x-a|$ is "small" and $(x-a)^2$ is smaller still, so neglect that term.
So $f(x) \stackrel{?}{=} f(a) + (x-a)f'(a)$. Set $f(x)=0$ & solve to get

$$x = a - \frac{f(a)}{f'(a)}$$

Exercise 0.1

Write down the formula for the Taylor Polynomial for

(i) $f(x) = 3x^2 + 3x - 12$

(ii) $f(x) = \sqrt{1+x}$ about the point $a = 0$,

(iii) $f(x) = \log(x)$ about the point $a = 1$.

Exercise 0.2

Write out the Taylor polynomial at x, about $a = 0$, of degree 7 for $f(x) = \sin(x)$. How does its derivative compare to the corresponding Taylor polynomial for $f(x) = \cos(x)$?

The purpose of the next exercise is to demonstrate that, usually, the closer x is to a, the better the Taylor polynomial approximates that function's value.

Exercise 0.3

Write out the Taylor Polynomial about $a = 1$ of degree 4 and corresponding remainder for $f(x) = \ln(x)$. Give an upper bound for this remainder when $x = 2$, $x = 1.1$ and $x = 1.01$.

The purpose of the next exercise is to demonstrate that some functions do not have sensible Taylor polynomials.

Exercise 0.4

Write out the Taylor polynomial about $a = 0$, of degree 4, for $f(x) = e^{-1/x^2}$.
Hint: $\lim_{x\to 0} e^{-1/x^2} x^{-p} = 0$ for any positive, finite p.

Exercise 0.5

Prove the *Integral Mean Value Theorem*: there exists a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$