Inner products. Given a vector space, V, a real (or complex) *inner product* is a function that maps pairs of elements of V to a real (or complex) number. For us, the most important vector space is the usual "dot"-product. If $u, v \in \mathbb{C}^n$, then their inner product is

$$(u, v) := \sum_{i=1}^{n} \bar{u}_i v_i = u^\star v.$$

You should check that, for any pair of vectors we have

- *Conjugate symmetry:* $(u, v) = \overline{(v, u)}$.
- *Linearity:* $(au + bv, y) = a(u, y) + b(v, y)$ where a, b are scalars, and u, v, y are vectors.
- *Positive-definiteness:* $(u, u) > 0$.

We've already encountered the idea of *orthogonal* vectors: $u \perp v \iff (u, v) = 0$.
For real matrices, we can define matrix products in terms of inner products: $(AB)_{ij} = ((A^\top)_i, b_j)$, where b_j is column j of B, and $(A^\top)_i$ is column i of $A^\top$ (equivalently, the transpose of the row i of A).

Vector norm. A *norm* is a function that maps an element of a vector space to a non-negative real number. Norms are used to quantify the "magnitude" of a vector, or how much two vectors differ from each other. For vectors in $\mathbb{C}^n$, there is the family of p-norms:

$$\|u\|_p = \begin{cases} \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p} & 1 \leqslant p < \infty \\ \max_i |u_i| & p = \infty. \end{cases}$$

In practice, the interesting "values" of $p = 1$, $p = 2$ and $p = \infty$. And of those, the most crucial is $p = 2$, which gives the "Euclidean" or "2"-norm: $\|u\|_2 := \sqrt{(u, u)}$.
A norm is required to satisfy the following properties:

- $\|u\| \geqslant 0$ for all u, with $\|u\| = 0$ exactly when u is the zero vector.
- $\|\lambda u\| = |\lambda|\|u\|$, where λ is any scalar from the field where u is defined.
- $\|u + v\| \leqslant \|u\| + \|v\|$; the triangle inequality.

Norms are important in *approximation*. For example, if we wanted to solve the linear system of equations, $Ax = b$, and computed a sequence of approximations $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$, we could try to quantify of these are approaching the true x by computing $\|b - Ax^{(i)}\|$.
It should be noted that there are other ways of thinking about how "similar" two vectors are; in data science, one often looks at the "angle" between then: $\dfrac{(u, v)}{\|u\|\|v\|}$.

Unitary matrices.

Definition 1. *A matrix $U \in M_n(\mathbb{C})$ is* unitary *if its Hermitian transpose is equal to its inverse, i.e. if $U^* U = I_n$. If $U \in M_n(\mathbb{R})$ has this property, U is called an* orthogonal *matrix. This means that the (ordinary) transpose of U is equal to the inverse of U, i.e. $U^T U = I_n$.*

If $U = (u_1 | u_2 | \cdots | u_n)$ is unitary then $u_i^\star u_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.
It is important to note that, for the 2-norm:

$$\|Ux\| = \sqrt{(Ux)^\star Ux} = \sqrt{x^\star U^\star Ux} = \sqrt{x^\star x} = \|x\|.$$