2425-MA140 Engineering Calculus

Week 2, Lectures 3
The Squeeze Theorem

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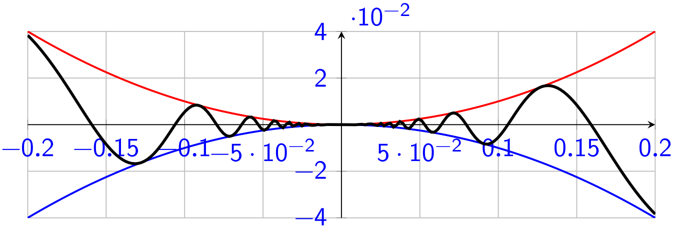
Thursday, 26 September, 2024



This version of the slides are by Niall Madden, but are adapted from original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Section 7.8.1 (Limit of a function of a real number) in *Modern Engineering Mathematics*:
https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Assignment 1

- **Assignment 1** has started! You can access it on Canvas... 2425-MA140... Assignments.
- Deadline: 5pm, Friday 4 Oct 2024. (Note: that's just the deadline, you can actually start before then!)
- The **Tutorial Sheet** is available at https://universityofgalway.instructure.com/files/2040359/download?download_frd=1

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Tutorials started **this** week. The schedule is on the Canvas "Course Information" page: https://universityofgalway.instructure.com/courses/35693/pages/2425-ma140-information

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Support is also available at **SUMS**...

Recall... Limits

Yesterday, we learned that

$$“\lim_{x \to a} f(x) = L,”$$

means that we can make $f(x)$ as close to L as we like, by taking x as close to a as needed.

Crucially, we are usually interested in finding the limit of $f(x)$ as $x \to a$, when a is not in the domain of f.

A typical example of this is when we evaluate a rational function:

$$\boxed{\lim_{x \to a} \frac{p(x)}{q(x)}}$$

where **both** $p(a) = 0$ and $q(a) = 0$. **Idea:** Since we care about the value of p and q **near** $x = a$, but not actually at $x = a$, it is safe to factor out and $(x - a)$ term from both.

Example

Evaluate Consider

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{p(x)}{q(x)}$$

$p(x) = x^2+x-2$ $\quad q(x) = x^2 - x$.

Check $p(1) = 1^2+1-2=0$ $\quad q(1) = 1-1 = 0$.

So we factorise $p(x) = x^2+x-2 = (x+2)(x-1)$.

$q(x) = x^2-x = x(x-1)$.

So $\dfrac{x^2+x-2}{x^2-x} = \dfrac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} = \dfrac{x+2}{x}$ f.. $c \neq 1$

$$\lim_{x\to 1} \frac{p(x)}{q(x)} = \lim_{x\to 1} \frac{x+2}{x} = \frac{3}{1} = 3 \quad \checkmark$$

Limits of rational functions

In that last example, we found that

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x}$$

But these are different functions:

$x=1$ is not in the domain of $\dfrac{x^2+x-2}{x^2-x}$ but is in the domain of $\dfrac{x+2}{x}$.

Evaluate the limit

$$\lim_{x \to 2} \left(\frac{\frac{1}{2} - \frac{1}{x}}{x - 2} \right) = \lim_{x \to 2} \frac{p(x)}{q(x)} \quad \text{s.t.}$$

where $p(x) = \frac{1}{2} - \frac{1}{x}$ $\qquad q(x) = x - 2$

Note $p(2) = \frac{1}{2} - \frac{1}{2} = 0$ $\qquad q(2) = 2 - 2 = 0$

Note, however, $\frac{1}{2} - \frac{1}{x} = \frac{x}{2x} - \frac{2}{2x} = \frac{x-2}{2x}$.

Then $\left(\frac{1}{2} - \frac{1}{x}\right) \cdot \frac{1}{x-2} = \left(\frac{x-2}{2x}\right)\left(\frac{1}{x-2}\right)$

If $x \neq 2$, this is $\frac{1}{2x}$.

So $\lim_{x \to 2} \left(\frac{\frac{1}{2} - \frac{1}{x}}{x - 2} \right) = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$.

More limits

Very often, we'll evaluate limits of the form:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

where f and g are not polynomials. Some of the same ideas still apply.

Example

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

Idea: simplify by completing the square in the numerator.

More limits

$$\lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

(Note if $f(x) = \frac{\sqrt{1+x^2}-1}{x^2}$, then $f(0) = \frac{0}{0}$. Not defined!)

However

$$\frac{\sqrt{1+x^2}-1}{x^2} = \frac{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)}{x^2(\sqrt{1+x^2}+1)}$$

$$= \frac{(\sqrt{1+x^2})^2-1}{x^2(\sqrt{1+x^2}+1)} = \frac{1+x^2-1}{x^2(\sqrt{1+x^2}+1}$$

$$= \frac{x^2}{x^2(\sqrt{1+x^2}+1)} = \frac{1}{\sqrt{1+x^2}+1} \text{ for } x \neq 0.$$

So $\lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{1+x^2}+1} = \frac{1}{2}$

Exercise 2.4

(From 2023/2024 MA140 exam, Q1(a)) Evaluate the limit

$$\lim_{x\to 4} \frac{x-4}{(\sqrt{x}-2)(x+9)}$$

The Squeeze Theorem

There are various approaches to evaluating limits. One significant one is...

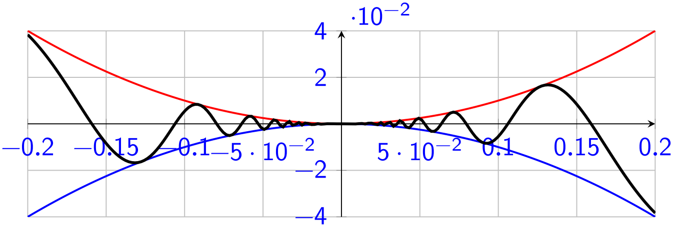
The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that for functions f, g and h in a given interval I:

$$g(x) \leqslant f(x) \leqslant h(x) \quad \underline{\text{and}} \quad \lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

So, if f(x) is between g & h and g, h have same limit, so too does f.

Then $\boxed{\lim_{x\to c} f(x) = L.}$



The Squeeze Theorem

Example

Suppose $f(x)$ is a function such that

$$\underbrace{1 - \frac{x^2}{4}}_{g(x)} \leqslant f(x) \leqslant \underbrace{1 + \frac{x^2}{2}}_{h(x)}, \ \forall x \neq 0$$

Find $\lim_{x \to 0} f(x)$.

Since $\lim_{x \to 0} g(x) = \lim_{x \to 0} 1 - \frac{x^2}{4} = 1$.

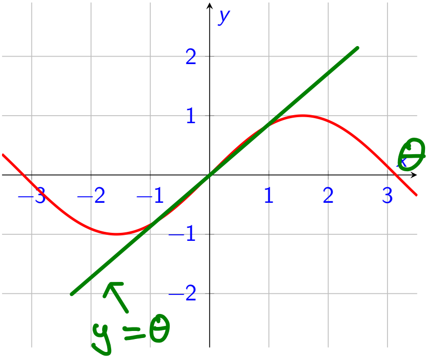
Similarly $\lim_{x \to 0} h(x) = 1$ So

$\lim_{x \to 0} f(x) = 1$.

We use the Squeese Theorem to explain an important limit:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$$

Before we show this is true, let's convince ourselves:



Note $\sin(0)=0$.

So $\dfrac{\sin(\theta)}{\theta}=\dfrac{0}{0}$ at $\theta=0$.

Finish here Thursday

Before we ues the Squeeze Theorem, we need a few facts about trigonometric functions.

- **In this module, we only every use radians** (never, ever degrees).
- Given the triangle drawn below, $\sin\theta = \dfrac{b}{h}$, $\cos\theta = \dfrac{a}{h}$, $\tan\theta = \dfrac{b}{a} = \dfrac{\sin\theta}{\cos\theta}$
- Area of a sector of a circle is $\frac{1}{2}r^2\theta$ where r is the radius of the circle, and θ is the angle subtended by the sector.

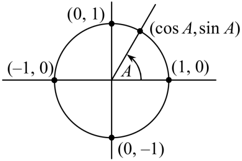
Various other facts are summarised in the State Examination Commission's Tables:

The Squeeze Theorem — $\sin(\theta)/\theta$

Triantánacht — Trigonometry

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	$0°$	$90°$	$180°$	$270°$	$30°$	$45°$	$60°$	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. $\approx 57.296°$ $\qquad$ $1° \approx 0.01745$ rad.

- 13 -

Foirmlí uillinneacha comhshuite **Compound angle formulae**

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Foirmlí uillinneacha dúbailte **Double angle formulae**

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

- 14 -

Iolraigh a thiontú ina suimeanna agus ina ndifríochtaí **Products to sums and differences**

$$2\cos A\cos B = \cos(A+B)+\cos(A-B)$$

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

$$2\cos A\sin B = \sin(A+B)-\sin(A-B)$$

Suimeanna agus difríochtaí a thiontú ina n-iolraigh **Sums and differences to products**

$$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

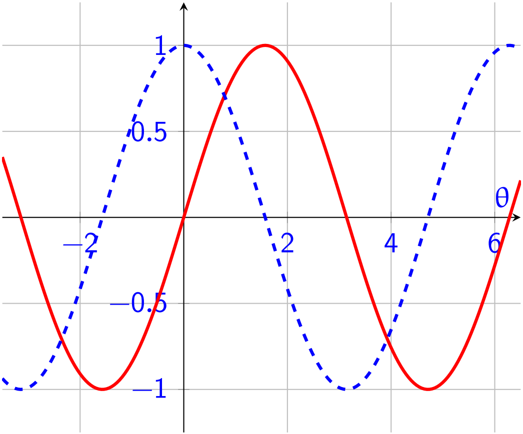
$$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

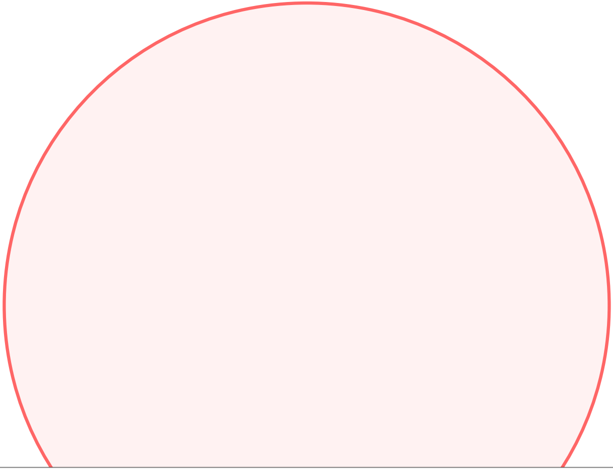
- 15 -

Here are plots of $\sin\theta$ (**red**) and $\cos\theta$ (**blue**).



$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Now let's reason more carefully:



Example

Evaluate $\lim_{x \to 0} \dfrac{\tan 3x}{\sin 2x}$

Example

Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$