MA385 Class Test

When:	15:00 Thursday, 30 October 2025.
Duration:	40 minutes.
Instructions:	Answer all questions in the answer book provided
Don't forget	to write your name and ID number on the answer book.
Rules	No notes or books allowed. No calculators allowed.
	Don't sit beside anybody else.

1. Suppose we have a function g and points a and b, such that $a \leq g(x) \leq b$ for all $x \in [a, b]$.

   (a) What does it mean for g to be a *contraction* on $[a, b]$?

   Answer: It means that $|g(\alpha) - g(\beta)| < |\alpha - \beta|$ for any pair of points α and β in $[a, b]$.

   (b) We know that, if g is a contraction on $[a, b]$, then it has a fixed point in $[a, b]$. Show that that fixed point is unique.

   Answer: Suppose that g has two fixed points, τ_1 and τ_2, and that $\tau_1 \neq \tau_2$. Then

$$\begin{aligned} |\tau_1 - \tau_2| &= |g(\tau_1) - g(\tau_2)| \qquad \text{(using that they are fixed points)} \\ &< |\tau_1 - \tau_2| \quad \text{(since } g \text{ is a contraction.)} \end{aligned}$$

   So $|\tau_1 - \tau_2| < |\tau_1 - \tau_2|$, which is not possible.

   (c) Suppose we know that a particular *Fixed Point* iteration, $x_{k+1} = g(x_k)$, converges at least linearly to the fixed point $\tau = g(\tau)$. Use a Taylor Series to show that, if $g'(\tau) = 0$, then, in fact, it converges at least quadratically.

   Answer: A method converges with at least order 2 (i.e., quadratically) if there is a constant $\mu \geq 0$ such that $\lim_{k\to\infty} \frac{|\tau - x_{k+1}|}{|\tau - x_k|^2} = \mu$. So, we need to find μ such that $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^2} \to \mu$. Let's write out a Taylor series for $g(x_k)$ about τ:

$$g(x_k) = g(\tau) + (x_k - \tau)g'(\tau) + \frac{1}{2}(x_k - \tau)^2 g''(\eta),$$

   for some $\eta \in [x_k, \tau]$. Since $g'(\tau) = 0$, and using that $g(\tau) = \tau$ and $g(x_k) = x_{k+1}$, this simplifies to $x_{k+1} - \tau = \dfrac{1}{2}(x_k - \tau)^2 g''(\eta)$. Rearranging:

$$\frac{|x_{k+1} - \tau|}{|x_k - \tau|^2} = \frac{1}{2}|g''(\eta)|$$

   To finish, since we know the method converges, $x_k \to \tau$. So, since $\eta \in [x_k, \tau]$, it must be that $\eta \to \tau$, as $k \to \infty$. Thus $g''(\eta_k) \to g''(\tau)$, which is a constant.

$$\frac{|\tau - x_{k+1}|}{|\tau - x_k|^2} \to \mu \qquad \text{where } \mu = \frac{|g''(\tau)|}{2}.$$

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2. (a) State *Newton's method* for solving the nonlinear equation $f(x) = 0$ *for some* $x \in [a, b]$.

Answer: Choose $x_0 \in [a, b]$ and then set $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ for $k = 0, 1, 2, \ldots$.

(b) Explain how Newton's Method can thought of as *Fixed Point* iteration $x_{k+1} = g(x_k)$. That is, *what is* g?

Answer: Set $g(x) = x - f(x)/f'(x)$.

(c) Assuming Newton's method converges, show that it does so at least quadratically.

Answer: We need to show that $g'(\tau) = 0$. In this case, $g'(x) = 1 - \frac{f'(x)f'(x)-f(x)f''(x)}{(f'(x))^2}$. Using that $f(\tau) = 0$, we get $g'(\tau) = 1 - \dfrac{\big(f'(\tau)\big)^2}{\big(f'(\tau)\big)^2} = 1 - 1 = 0$. Now apply the result from Q1(c).

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3. (a) State *Euler's method* for solving initial value problems:

$$y(t_0) = y_0 \qquad \text{and } y'(t) = f(t, y) \text{ for } t > t_0.$$

Answer: Euler's method is $y_{i+1} = y_i + hf(t_i, y_i)$, for $i = 0, 1, \ldots, n-1$.

(b) Show how Euler's method can be motivated by using a Taylor Series Expansion.

Answer: Taylor series for $y(t_{i+1})$ about t_i:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\eta_i),$$

for some $\eta_i \in [t_i, t_{i+1}]$. Using $h = t_{i+1} - t_i$, and that, from the IVP, $y'(t_i) = f\big(t_i, y(t_i)\big)$, giving

$$y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} y''(\eta_i).$$

Neglecting the $\mathcal{O}(h^2)$ term, and denoting the resulting approximation for $y(t_i)$ as y_i, we get the method.

(c) Give an example of an Initial Value problem for which Euler's method would give the exact solution (i.e., with no error). Justify your answer with the Taylor Series you used to motivate Euler's method.

Answer: We need a problem for which the solution has $y''(t) \equiv 0$ for all t. Take, for example $y(t) = t$, so $y'(t) = 1$, and $y''(t) = 0$. Now we just need a problem for which this is a solution. Acceptable options include taking $f(t, y) = 1$, or (equivalently) $f(t, y) = y/t$, along with the initial condition (for example) $y_0 = 1$ at $t_0 = 1$.