

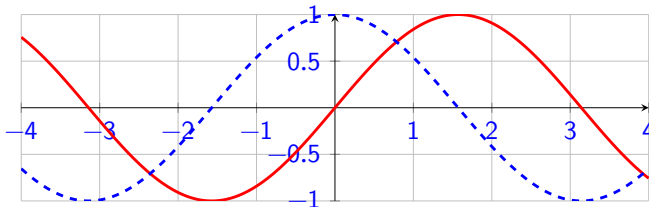
2526-MA140: Week 03, Lecture 1 (L07)

2526-MA140: Week 03, Lecture 1 (L07) The Squeeze Theorem & one-sided limits

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Outline

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- 3 $\sin(\theta)/\theta$
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For more, see Section 2.2 (Limit of a Function) from **Calculus** by Gil Strang and Jed Herman, published by the non-profit OpenStax. See <https://openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function>

Reminder

- ▶ **Assignment 1** has a deadline of 5pm, Monday 6 October. You can access it on Canvas... **2526-MA140...** Assignments. (Or directly, **at this link**).
- ▶ The **Tutorial Sheet** is available at <https://universityofgalway.instructure.com/courses/46734/files/2883465?wrap=1>
- ▶ Assignment 2 is also open, with a deadline of 5pm, 13 Oct.

The first (of two) class tests will take place 2 weeks from now:
Tuesday, 14th October.

- ▶ You will have 40 minutes to complete the test, which will be in the form of a Multiple Choice Test.
- ▶ Test will take place in one of ENG-G017 or ENG-G018.
- ▶ I need to gather information on Reasonable Accommodations for tests. If you want to avail of such, please complete this form: <https://forms.office.com/e/HaAsrzaE3D> by **10am Thursday 2nd Oct.**



Recall... the Squeeze Theorem

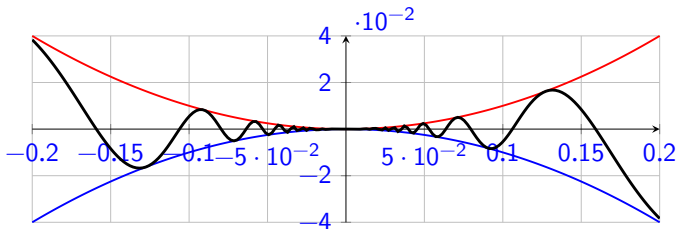
Last Thursday, we finished with...

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Suppose that for functions f , g and h in a given interval I :

$$g(x) \leq f(x) \leq h(x) \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then $\lim_{x \rightarrow c} f(x) = L.$



We'll use the Squeeze Theorem to explain that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

First, we review some facts about trigonometric functions.

► **In this module, we only every use radians** (never degrees).

► Given the triangle drawn below, $\sin \theta = \frac{b}{h}$, $\cos \theta = \frac{a}{h}$,

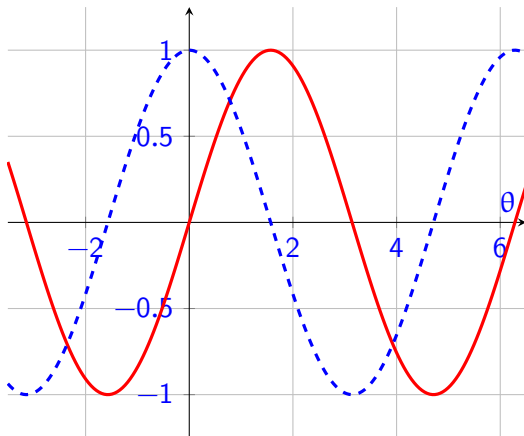
$$\tan \theta = \frac{b}{a} = \frac{\sin \theta}{\cos \theta}$$

$$\sin(\theta)/\theta$$

- ▶ Area of a sector of a circle is $\frac{1}{2}r^2\theta$ where r is the radius of the circle, and θ is the angle subtended by the sector.

$$\sin(\theta)/\theta$$

- The \sin (red) and \cos (blue) functions look like this:

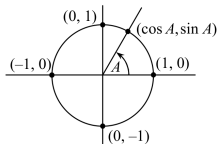


Various other facts are summarised in the State Examination Commission's Tables:

Triantánacht

$$\tan A = \frac{\sin A}{\cos A} \quad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \quad \operatorname{cosec} A = \frac{1}{\sin A}$$



Trigonometry

$$\cos^2 A + \sin^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$

$$\sin(-A) = -\sin A$$

$$\tan(-A) = -\tan A$$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
 $\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

| A (céimeanna) | 0° | 90° | 180° | 270° | 30° | 45° | 60° | A (degrees) |
|-----------------|-----------|-----------------|-------------|------------------|----------------------|----------------------|----------------------|---------------|
| A (raidian) | 0 | $\frac{\pi}{2}$ | π | $\frac{3\pi}{2}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | A (radians) |
| $\cos A$ | 1 | 0 | -1 | 0 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $\cos A$ |
| $\sin A$ | 0 | 1 | 0 | -1 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $\sin A$ |
| $\tan A$ | 0 | - | 0 | - | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\tan A$ |

$$1 \text{ rad.} \approx 57.296^\circ$$

$$1^\circ \approx 0.01745 \text{ rad.}$$

Foirmlí uillinneacha comhshuite

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Compound angle formulae

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Foirmlí uillinneacha dúbailte

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Double angle formulae

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

Iolraigh a thiontú ina suimeanna agus ina ndifríochtaí**Products to sums and differences**

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

Suimeanna agus difríochtaí a thiontú ina n-iolraigh**Sums and differences to products**

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

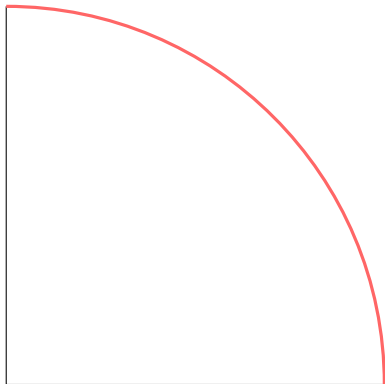
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\sin(\theta)/\theta$$

To show that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$



Example

Evaluate $\lim_{x \rightarrow 0} \frac{\tan(3x)}{\sin(2x)}$

Example

Evaluate $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$

Infinite Limits

So far, we've had lots of examples that are a little like:

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2 - x + 1}{(x - 1)^2} = 2.$$

(Check that this is correct).

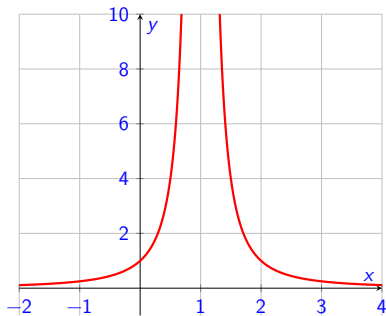
But what about

$$\lim_{x \rightarrow 1} \frac{1}{(x - 1)^2} = ???$$

Let's plot it and see:

Infinite Limits

$$f(x) = \frac{1}{(x-1)^2}$$



As x get closer and closer to 1, the value of $f(x)$ gets larger and larger. In fact, it becomes infinite.

For this we write

$$\lim_{x \rightarrow 1} f(x) = \infty.$$

Digression: How fast can an object travel

- ▶ Q: Is there any limit to the speed at which an object can travel?
- ▶ A: Yes! (Assuming you believe Einstein)

Thanks to Einstein ($E = mc^2$), Lorenz and others, it is known that the mass of a moving charged particle behaves like

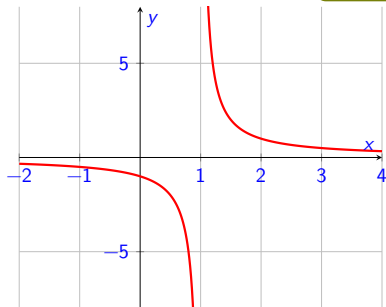
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is its mass at rest, c is the speed of light, and v is the particles current speed. What happens as $v \rightarrow c$?

One-sided Limits

Let's consider a motivating example, very similar to the one where we introduced ∞ .

$$f(x) = \frac{1}{x-1}$$



As x get closer and closer to 1, then $f(x) \rightarrow -\infty$ or $f(x) \rightarrow \infty$, depending on whether x approaches 1 from the left or right.

To express this, we need the concept of a **one-sided limit**

$\lim_{x \rightarrow a^-} f(x)$ is: **limit of f as x approaches a from the left**

$\lim_{x \rightarrow a^+} f(x)$ is: **limit of f as x approaches a from the right**

In the previous example, with $f(x) = \frac{1}{x-1}$, we have

► $\lim_{x \rightarrow 1^-} f(x) = -\infty$

► $\lim_{x \rightarrow 1^+} f(x) = \infty$

In many important examples, we encounter functions that have different definitions in different regions. The most classic example is the **absolute value function**:

$$|x| = \begin{cases} -x & x < 0 \\ x & x > 0. \end{cases}$$

Care has to be taken when evaluating the limits of such functions....

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x \leq 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$$

Find $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$.

Empty and Full Circle Notation:

In the previous sketch, we use the convention that

- ▶ If the end point of a line segment is **not** included in its definition, it terminates with an **open circle**, \circ
- ▶ If the end point of a line segment **is** included in its definition, it terminates with an **closed circle** \bullet .

Existence of a limit

$\lim_{x \rightarrow a} f(x)$ **exists** if and only if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

So if $\lim_{x \rightarrow a} f(x) = L$ exists, we have

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

though it is not necessary that $f(a) = L$

Example

Sketch the function

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 4, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Determine if $\lim_{x \rightarrow 2} f(x)$ exists.

Exercise 3.1.1 (from 2023/24 Q1(b))

Evaluate

$$\lim_{\theta \rightarrow 0} \frac{2 \sin(\theta)}{\theta + 3 \tan(\theta)}$$