Solving nonlinear equations

§1.4: Fixed Point Iteration

MA385 – Numerical Analysis 1

September 2019

Newton's method can be considered to be a special case of a very general approach called *Fixed Point Iteration* or *Simple Iteration*.

The basic idea is:

> *If we want to solve* $f(x) = 0$ *in* $[a, b]$, *find a function* $g(x)$ *such that, if* τ *is such that* $f(\tau) = 0$, *then* $g(\tau) = \tau$. *Choose* x_0 *and set* $x_{k+1} = g(x_k)$ *for* $k = 0, 1, 2, \ldots$.

ANNOTATED SLIDES from Sep 30th + Oct 3rd

Example 4

Suppose that $f(x) = e^x - 2x - 1$ and we are trying to find a solution to $f(x) = 0$ in $[1, 2]$. Then we can take $g(x) = \ln(2x + 1)$.

If we take $x_0 = 1$, then we get the following sequence:

If
$e^x - 2x - 1 = 0$
Then
$e^x = 2x + 1$, so
$\log(e^x) = \log(2x+1)$
$\Rightarrow x = \log(2x+1)$

k	x_k	$\lvert\tau - x_k\rvert$
0	1.0000	2.564e-1
1	1.0986	1.578e-1
2	1.1623	9.415e-2
3	1.2013	5.509e-2
4	1.2246	3.187e-2
5	1.2381	1.831e-2
$\vdots$	$\vdots$	$\vdots$
10	1.2558	6.310e-4

what if we try
$2x = e^x - 1$
$\Rightarrow x = \frac{1}{2}(e^x - 1)$
??

We have to be quite careful with this method: **not every choice is g is suitable**.

For example, suppose we want the solution to $f(x) = x^2 - 2 = 0$ in $[1, 2]$. We could choose $g(x) = x^2 + x - 2$. Then, if take $x_0 = 1$ we get the sequence:

$x_0 = 1$

$x_1 = (1^2) + 1 - 2 = 0,$

$x_2 = 0^2 + 0^2 - 2 = -2$

$x_3 = (-2)^2 + (-2) - 2 = +4 - 4 = 0$

$x_4 = -2$ $\quad x_5 = 0$ $\quad x_6 = -2$

etc.

We need to refine the method that ensure that it will converge.

Before we do that in a formal way, consider the following...

Example 5

Use the Mean Value Theorem to show that the fixed point method $x_{k+1} = g(x_k)$ converges if $|g'(x)| < 1$ for all x near the fixed point.

By the MVT, for any a, b, there is $c \in [a, b]$ such that $\dfrac{g(b) - g(a)}{b - a} = g'(c)$. So

$g(b) - g(a) = g'(c)(b - a)$. Take $a = x_k$ and $b = \tau$

so $g(\tau) - g(x_k) = g'(c)(\tau - x_k)$. But $g(\tau) = \tau$, $g(x_k) = x_{k+1}$

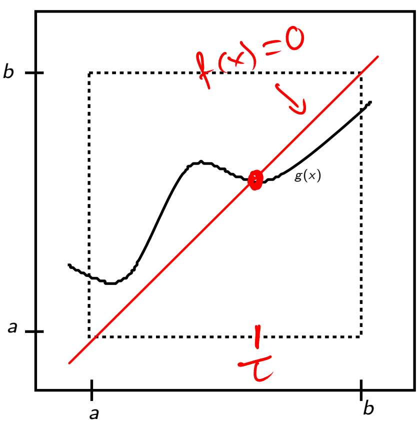
So $|\tau - x_{k+1}| = \underbrace{|g'(c)|}_{<1} |\tau - x_k| < |\tau - x_k|$

This example:

- introduces the tricks of using that $g(\tau) = \tau$ & $g(x_k) = x_{k+1}$.
- Leads us towards the ***contraction mapping theorem***.

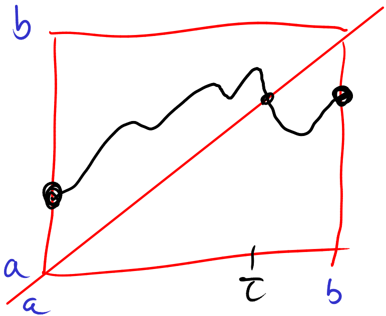
Theorem 6 (Fixed Point Theorem)

Suppose that $g(x)$ is defined and continuous on $[a, b]$, and that $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there exists $\tau \in [a, b]$ such that $g(\tau) = \tau$. That is, $g(x)$ has a fixed point in $[a, b]$.



$g(x) \in [a,b]$,

$\Leftrightarrow \quad a \le g(x) \le b$.



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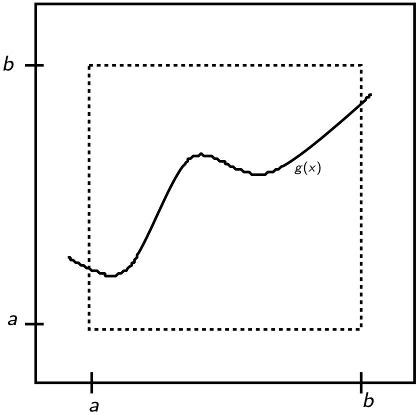

Proof. Let

$h(x) = g(x) - x$.

So $h(a) = g(a) - a \geq 0$

and $h(b) = g(b) - b \leq 0$.

So, by the intermediate value theorem, $h(\tau) = 0$ for some $\tau \in [a, b]$.

So $g(\tau) = \tau$.

Next suppose that g is a *contraction*. That is, $g(x)$ is continuous and defined on $[a, b]$ and there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b]. \tag{8}$$

Theorem 7 (Contraction Mapping Theorem)

Suppose that the function g is a real-valued, defined, continuous, and

(a) *maps every point in $[a, b]$ to some point in $[a, b]$, and*

(b) *is a contraction on $[a, b]$,*

then

Previous Theorem

(i) *$g(x)$ has a fixed point $\tau \in [a, b]$,*

(ii) *the fixed point is unique,*

(iii) *the sequence $\{x_k\}_{k=0}^{\infty}$ defined by $x_0 \in [a, b]$ and $x_k = g(x_{k-1})$ for $k = 1, 2, \ldots$ converges to τ.*

(ii) The fixed point of g is unique.

Suppose g has two fixed points: τ_1, τ_2.

i.e., $g(\tau_1) = \tau_1$, $g(\tau_2) = \tau_2$

Then $|\tau_1 - \tau_2| = |g(\tau_1) - g(\tau_2)|$

$\leq L|\tau_1 - \tau_2|$

So $|\tau_1 - \tau_2|(1 - L) \leq 0$.

But $0 < L < 1$, So $1 - L > 0$. Thus $|\tau_1 - \tau_2| \leq 0$.

So it must be that $|\tau_1 - \tau_2| = 0$, ie $\tau_1 = \tau_2$.

(iii) Show that $g(x_k) \to \tau$ as $k \to \infty$.

Proof:

$$|\tau - x_k| = |g(\tau) - g(x_{k-1})|$$
$$\leq L\,|\tau - x_{k-1}| \quad \text{for all } k.$$

In particular, $|\tau - x_1| \leq L\,|\tau - x_0|$

$$|\tau - x_2| \leq L\,|\tau - x_1| \leq L^2\,|\tau - x_0|.$$

In general $|\tau - x_k| \leq L^k\,|\tau - x_0|$.

Since $0 < L < 1$, $\lim_{k\to\infty} L^k = 0$. So $x_k \to \tau$.

The algorithm generates as sequence $\{x_0, x_1, \ldots, x_k\}$. Eventually we must stop. Suppose we want the solution to be accurate to say 10^{-6}, how many steps are needed? That is, how big do we need to take k so that

$$|x_k - \tau| \leq 10^{-6}?$$

The answer is obtained by first showing that

$$|\tau - x_k| \leq \frac{L^k}{1-L}|x_1 - x_0|. \tag{9}$$

This is because $|\tau - x_0| = |\tau - x_1 + x_1 - x_0|$

$$\leq |\tau - x_1| + |x_1 - x_0|$$

$$\leq L|\tau - x_0| + |x_1 - x_0|$$

$$\Rightarrow \quad |\tau - x_0| \leq \frac{1}{1-L}|x_1 - x_0|.$$

Example 8

Suppose we are using FPI to find the fixed point $\tau \in [1, 2]$ of $g(x) = \ln(2x + 1)$ with $x_0 = 1$, and we want $|x_k - \tau| \leq 10^{-6}$, then we can use (9) to determine the number of iterations required.

Here $g'(x) = \frac{2}{2x+1}$. So $|g'(x)| \leq 2/3$ (ie at $x=1$).

So, from the MVT,

$$|g(\tau) - g(x_k)| \leq |f'(c)|\,|\tau - x_k|$$
$$\leq 2/3\,|\tau - x_k|.$$

So take $L = 2/3$. So $|x_k - \tau| \leq \frac{L^k}{1-L}|x_1 - x_0|$

$$\leq (3)\left(\tfrac{2}{3}\right)^k |x_1 - x_0|.$$

So (check!) need $k \geq 36.783$. So $k = 37$.

Exercise 1.14

Is it possible for g to be a contraction on $[a, b]$ but not have a fixed point in $[a, b]$? Give an example to support your answer.

Exercise 1.15 ($\star$ Homework problem)

Show that $g(x) = \ln(2x + 1)$ is a contraction on $[1, 2]$. Give an estimate for L. (Hint: Use the Mean Value Theorem).

Exercise 1.16

Suppose we wish to numerically estimate the famous *golden ratio*, $\tau = (1+\sqrt{5})/2$, which is the positive solution to $x^2 - x - 1$. We could attempt to do this by applying fixed point iteration to the functions $g_1(x) = x^2 - 1$ or $g_2(x) = 1 + 1/x$ on the region $[3/2, 2]$.

(i) Show that g_1 is *not* a contraction on $[3/2, 2]$.

(ii) Show that g_2 *is* a contraction on $[3/2, 2]$, and give an upper bound for L.

Exercise 1.17

Consider the function $g(x) = x^2/4 + 5x/4 - 1/2$.

(i) It has two fixed points – what are they?

(ii) For each of these, find the largest region around them such that g is a contraction on that region.

Exercise 1.18

(i) Prove that if $g(\tau) = \tau$, and the fixed point method given by

$$x_{k+1} = g(x_k),$$

converges to the point τ (where $g(\tau) = \tau$), and

$$g'(\tau) = g''(\tau) = \cdots = g^{(p-1)}(\tau) = 0,$$

then it converges with order p. (Hint: you don't have to prove that the method converges; you can assume that. Also, use a Taylor Series).

(ii) We can think of Newton's Method for the problem $f(x) = 0$ as fixed point iteration with $g(x) = x - f(x)/f'(x)$. Use this, and Part (i), to show that, if Newton's method converges, it does so with order 2, providing that $f'(\tau) \neq 0$.