Annotated slides from Tuesday

MA313 : Linear Algebra I

Week 4: Spanning sets and column spaces

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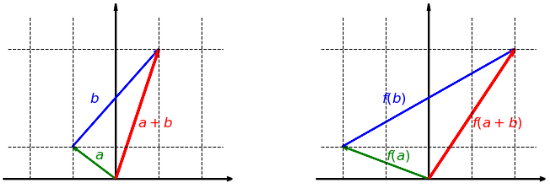
27 and 30 September, 2022



Adapted from https://commons.wikimedia.org/wiki/File:Streckung_der_Summe_zweier_Vektoren.gif

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details, see Section 4.2 of the text-book:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=5174425

Assignment 2

- Opened last Monday (19 Sep 2022).
- **Deadline:** 5pm, Friday 30 Sep 2022.
- It contributes 5% to the final grade for MA313.
- Tutorials continue Thursdays at 12 in IT206.

Communication Skills

1. Topics and Info posted on Blackboard and at https://www.niallmadden.ie/teaching/2223-MA313/22_23_Communication_Skills.pdf
2. Confirm your topic by 5pm, 26 September (Monday of Week 4). To that by first emailing Niall with your choice and, if agreed, entering in on Blackboard.

MA313
Week 4: Spanning sets and column spaces

Start of ...

PART 1: Recall from last week

Part 1: Recall from last week

Linear combinations

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form $c_1 u_1 + \cdots + c_p u_p$ for scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

Span

The **span** of a set of vectors is the set of all possible liner combinations of them. That is, given vectors $u_1, \ldots, u_p$ in some vector space V, their **span** is

$$\mathrm{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

Part 1: Recall from last week

Subspaces

Given any set of a vectors in a vector space V, their span is a **subspace** of V.

Null space

Given a $m \times n$ matrix, A, its **null space** is the set of all vectors for which $Ax = 0$. That is:

$$\mathrm{Nul}\, A = \{x \in \mathbb{R}^n : Ax = 0\}.$$

- For some matrices, the only vector in the null space is the zero vector.
- The null space of an $m \times n$ matrix is itself a vector space (and so a subspace of $\mathbb{R}^N$).

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Week 4: Spanning sets and column spaces

Start of ...

PART 2: Spanning Sets

Part 2: Spanning Sets

Definition (Spanning Set)

A **spanning set** of a vector space V is a collection of vectors in V whose span is all of V.

Equivalently, the set of vectors $\{v_1, \ldots, v_p\}$ in V form a spanning set if and only if every vector in V can be written as a linear combination of $v_1, \ldots, v_p$.

Example (A spanning set for $\mathbb{R}^2$)

The vectors

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

form a spanning set of $\mathbb{R}^2$.

$\mathbb{R}^2$ is the space of all vectors of the form $\begin{bmatrix}a\\b\end{bmatrix}$ where $a, b \in \mathbb{R}$.

These can be written as $a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

So they are a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

Example (A spanning set for $\mathbb{R}^n$)

In the same way, for each $n \geqslant 1$, the vectors

$$\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \in \mathbb{R}^n$$

form a spanning set of $\mathbb{R}^n$.

Eg $n=3$: set is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix},$

v

Recall that $\mathbb{P}_n$ is the vector space of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

of degree n or less.

Example

$\mathbb{P}_n = \mathrm{span}\{1, t, \ldots, t^n\}$.

Recall: $M_{m\times n}$ is the vector space of all $m \times n$ matrices.

Example

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right\}.$$

Any matrix in $M_{2\times 2}$ is of the form $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ $a,b,c,d \in \mathbb{R}$.

And $\begin{bmatrix}a & b\\c & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$.

So this is a spanning set for $M_{2\times 2}$.

Important: Spanning sets are (in general) not unique.

Example (Another spanning set of $M_{2\times 2}$)

We also, have for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \underset{u_1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{u_4}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \right\}.$$

We can write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$a\left(\tfrac{1}{2}u_1 + \tfrac{1}{2}u_4\right) + b u_2 + c u_3 + d\left(\tfrac{1}{2}u_1 - \tfrac{1}{2}u_4\right).$$

Note, eg, $\tfrac{1}{2}u_1 + \tfrac{1}{2}u_4 = \tfrac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

Important: Spanning sets are (in general) not unique.

Example (Another spanning set of $M_{2\times 2}$)

We also, for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Note also

① $M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

② But $M_{2\times 2} \neq \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$

Important: Spanning sets are (in general) not unique.

Example (Another spanning set of $M_{2\times 2}$)

We also, for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Note also $V = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \neq M_{2\times 2}$,

since any matrix in V is of the form $a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

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Week 4: Spanning sets and column spaces

Start of ...

PART 3: Column spaces

Part 3: Column spaces

Definition (COLUMN SPACE)

Let $A = [a_1 \cdots a_n]$ be an $m \times n$ matrix, where $a_1, \ldots, a_n \in \mathbb{R}^m$. That is, a_i is the ith column of A.

The **column space** of A is

$$\mathrm{Col}\, A := \mathrm{span}\{a_1, \ldots, a_n\}.$$

Note that $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$.

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \end{bmatrix}$. then

$$\mathrm{Col}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right).$$

Eg $\begin{bmatrix} 0 \\ -1 \end{bmatrix} \in \mathrm{col}(A)$ since $\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Example

Let I_n be the $n \times n$ identity matrix.
Then $\mathbb{R}^n = \mathrm{Col}\, I_n$.

Eg $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

And $\mathbb{R}^3 = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \mathrm{Col}\, I_3$

Part 3: Column spaces

Here is another way of thinking about the column space: we have already seen that Ax is a linear combination of the columns of A. So, ...

$$\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\}$$

and

$$\operatorname{Col} A = \{b \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \colon\ b = Ax\}.$$

$$I_3 x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Given a matrix A, we can construct two vector spaces:

$\operatorname{Nul} A$

- Easy to test membership: does $x \in \mathbb{R}^n$ belong to $\operatorname{Nul} A$?
- Not as easy to produce a (finite) spanning set.

$\operatorname{Col} A$

- Very easy to give a spanning set: it is how the space is defined!
- Not as easy to check to test membership.

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Week 4: Spanning sets and column spaces

Start of ...

PART 4: Spanning sets of null spaces

Part 4: Spanning sets of $\mathrm{Nul}\, A$

Question

Given an $m \times n$ matrix A, can we find a finite spanning set of $\mathrm{Nul}\, A$?

That is, can we find vectors $v_1, \ldots, v_p \in \mathbb{R}^n$ such that those vectors $x \in \mathbb{R}^n$ with $Ax = 0$ are precisely the linear combinations

$$c_1 v_1 + \cdots + c_p v_p,$$

where $c_1, \ldots, c_p \in \mathbb{R}$?

To see the answer, we'll recall that the $Ax = b$ is just another way of writing a linear system of equations.

"precisely" - includes all such vectors,
but not anything else.

When we write

$$Ax = b$$

where A is an $n \times n$ matrix, and $x, b \in \mathbb{R}^n$, we mean

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This is the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Remember that we used solve such systems using "row reduction" (a.k.a., Gaussian Elimination): we rearrange the equations to get

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
\hat{a}_{22}x_2 + \hat{a}_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
\hat{a}_{33}x_3 + \cdots + \hat{a}_{3n}x_n &= b_2\\
&\vdots\\
\hat{a}_{nn}x_n &= b_n
\end{aligned}$$

This is done by so-called *elementary row operations*. And we do this because it is easy to solve this version.

Elementary row operations

Performing an **elementary row operation** on a matrix means:

- Multiply some row by a non-zero scalar.
- Add a scalar multiple of some row to another row.
- Interchange (i.e., swap) two rows.

Fact!

Let A' be obtained from A by performing an **elementary row operation**. The

$$\operatorname{Nul} A = \operatorname{Nul} A'.$$

Definition (Row Echelon Form)

A matrix is in **row echelon form** if

- all non-zero rows are above all zero rows and
- the **leading entry** (or "*pivot*") in a row is in a column to the right of the leading entry in the row above it.
- All entries in a column below a leading entry are zero.

Eg $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

But not $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 7 & 0 & 8 \end{bmatrix}$

Finished here Tuesday

Definition (Reduced Row Echelon Form)

A matrix is in **reduced row echelon form** if it is in row echelon form, and also

- Each leading entry is one;
- If a column contains a leading entry, all its other entries are zero.

Theorem and Definition

Using elementary row operations, *every* matrix A can be row reduced to obtain a **unique** matrix A' in reduced row echelon form. We call A' **the reduced row echelon form** of A.

It turns out that we can read off a spanning set of $\operatorname{Nul} A$ from the reduced row echelon form of A.

Example

Find a spanning set of $\operatorname{Nul} A$, where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

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Week 4: Spanning sets and column spaces

Start of ...

PART 5: Checking column space

Part 5: Checking column space

Question

Given an $m \times n$ matrix A and $b \in \mathbb{R}^m$, how can we decide if $b \in \operatorname{Col} A$?

Since $\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\}$, this problem is equivalent to deciding whether there exists a solution $x \in \mathbb{R}^n$ to the system of linear equations

$$Ax = b.$$

Again, **row reduction** (a.k.a. **Gaussian elimination**) can be used for this purpose.

Part 5: Checking column space

Example

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$. Is $b \in \operatorname{Col} A$?

Details on board.... we see
x4=1/17
x3 is free, and then x2=-2 + 5x3 + 4/17.
Take x3=0, to get x2=-30/17.

Similarly, x1=5.

Part 5: Checking column space

Example (From 2018/19 exam paper)

Decide (with justification) if

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ belongs to the column space of } A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix}.$$

Answer: No!. Why? The RREF of

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ -1 & 3 & 5 & 4 & 2 \\ 2 & 1 & -3 & -1 & -1 \end{array}\right] \text{ is } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

So...

So now we know that, given an $m \times n$ matrix A, we can use **row reduction** to perform the following tasks:

- Construct a finite spanning set of $\operatorname{Nul} A$.
- Decide, for a given $b \in \mathbb{R}^m$, whether $b \in \operatorname{Col} A$.

But what has this to do with **vector spaces**?
Do these matrix computations (row reduction) and concepts (null spaces, column spaces) have analogues for general vector spaces?

Finished here Friday.

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Week 4: Spanning sets and column spaces

Start of ...

PART 6: Linear Transformations

Part 6: Linear Transformations

Definition (LINEAR TRANSFORMATIONS)

Let V and W be vector spaces. A **linear transformation** from V to W is a function $T \colon V \to W$ (i.e., a "rule" which assigns a unique $T(u) \in W$ to each $u \in V$) such that

- $T(u + v) = T(u) + T(v)$ for all $u, v \in V$ and
- $T(cu) = cT(u)$ for all $u \in V$ and $c \in \mathbb{R}$.

That is, a linear transformations is a function which "respects" (or "is compatible with") the vector space structures.

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_2\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_1^2-x_2^2\\x_1^2+x_2^2\end{bmatrix}$$

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Example

The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

defines a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

An important fact

Linear transformations preserve linear combinations: if $T\colon V \to W$ is a linear transformation, then

$$T(c_1 v_1 + \cdots + c_p v_p) = c_1 T(v_1) + \cdots + c_p T(v_p)$$

for all $v_1, \ldots, v_p \in V$ and $c_1, \ldots, c_p \in \mathbb{R}$.

Example (Matrices)

Let A be an $m \times n$ matrix. Define $T\colon \mathbb{R}^n \to \mathbb{R}^m$ via

$$T(x) = Ax \qquad (x \in \mathbb{R}^n).$$

Then T is a linear transformation.

Question

Are there any other linear transformations $\mathbb{R}^n \to \mathbb{R}^m$?

Answer: No! Linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ and $m \times n$ matrices are essentially the "same thing". What we mean is,

- Every $m \times n$ matrix defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can find a matrix that defines it.

The matrix of a linear transformation

Let e_i be the usual vector in $\mathbb{R}^n$ with 1 is row i, and zero everywhere else. Then the matrix for a given linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$A := [T(e_1) \cdots T(e_n)].$$

Why?

Since linear transformations are generalizations of matrices, we need the analogous idea of **null spaces** and **column spaces**.

Definition (KERNEL and RANGE of a linear transformation)

Let $T: V \to W$ be a linear transformation.

- The **kernel** of T is $\mathrm{Ker}\, T = \{u \in V : T(u) = 0\}$.
- The **range** (or *image*) of T is $\mathrm{Ran}\, T = \{\, T(u) : \; u \in V\}$.

Example

Let A be an $m \times n$ matrix. Let $T \colon \mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Ax$ be the associated linear transformation. Then:

- $\operatorname{Ker} T = \{x \in \mathbb{R}^n : T(x) = Ax = 0\} = \operatorname{Nul} A$.
- $\operatorname{Ran} T = \{T(x) = Ax : x \in \mathbb{R}^n\} = \operatorname{Col} A$.

Theorem

Let $T\colon V \to W$ be a linear transformation. Then:

- $\mathrm{Ker}\, T$ *is a subspace of* V.
- $\mathrm{Ran}\, T$ *is a subspace of* W.

Here is another result, though the importance might not be clear yet.

Theorem

Let V be a vector space and let $H \subseteq V$ be a subspace.
Then there are vector spaces U and W and linear transformations $S\colon U \to V$ and $T\colon V \to W$ such that

$$\operatorname{Ran} S = H = \operatorname{Ker} T.$$

We essentially get S for free...

But some new ideas would be required to produce T.

Exercises

Q1. Construct a finite spanning set of each of the null space of each of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Exercises

Q2. Let

$$w = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

Determine whether w belongs to $\mathrm{Nul}\, A$ and whether w belongs to $\mathrm{Col}\, A$.

Q3. Let

$$w = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix}.$$

Determine whether w belongs to $\mathrm{Nul}\, A$ and whether w belongs to $\mathrm{Col}\, A$.

Q4. *4.2.30* Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W.

Q1..1 Show that the kernel $\mathrm{Ker}\, T$ of T is a subspace of V.

Q2..2 Show that the range $\mathrm{Ran}\, T$ of T is a subspace of W.

Exercises

Q5. *4.2.31* Recall that $\mathbb{P}_n$ is the vector space of polynomials of the form $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ for $a_0, \ldots, a_n \in \mathbb{R}$. Define $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

- Q1..1 Show that T is a linear transformation.
- Q2..2 Find a polynomial $p(t) \in \mathbb{P}_2$ with $\mathrm{Ker}\ T = \mathrm{span}\{p(t)\}$.
- Q3..3 What is the range of T?

Q6. *4.2.32* Define $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}.$$

- Q1..1 Show that T is a linear transformation.
- Q2..2 Find polynomials $p_1(t), p_2(t) \in \mathbb{P}_2$ with $\mathrm{Ker}\ T = \mathrm{span}\{p_1(t), p_2(t)\}$.
- Q3..3 What is the range of T?

Exercises

Q7. *4.2.33* Recall that $M_{m\times n}$ denotes the vector space of $m \times n$ matrices with real entries. Further recall that $A^\top$ denotes the *transpose* of a matrix A. Define $T\colon M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^\top$.

- Q1..1 Show that T is a linear transformation.
- Q2..2 Show that the range of T consists precisely of those matrices $B \in M_{2\times 2}$ with $B = B^\top$. (Such matrices are called *symmetric*.)
- Q3..3 Describe the kernel of T.

Q8. *4.2.34* Recall that $C([a,b])$ denotes the vector space of all continuous functions $[a,b] \to \mathbb{R}$. Define $T\colon C([0,1]) \to C([0,1])$ as follows: for $f \in C([0,1])$, let $T(f)$ be the antiderivative F of f with $F(0) = 0$. Show that T is a linear transformation.