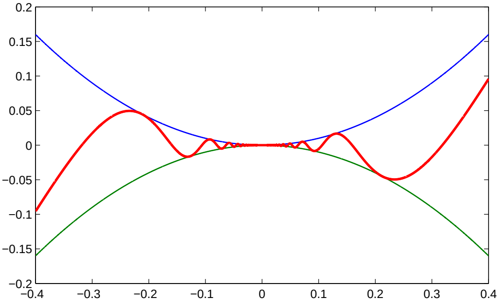
MA211

Lecture 4: Limits and Derivatives

Wednesday 17 September 2008



Outline

Problem Solving Sessions

Reminder: Problem Solving Sessions (tutorials) will start next week. There will be **two** per week. Attend whichever one you like

- Tuesday, 3pm, AC202
- Wednesday, 5pm, QA003 (Physiology lecture room)

Recall... Limits

When we write

$$\lim_{x \to c} f(x) = L$$

or say "*The limit of f as x approaches c is L*" we mean that we can make f as close to L as we would like by taking x as close to c as is needed.

Definition (Limit)

If for any $\varepsilon > 0$, no matter how small, we can find $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{when} \quad |x - c| < \delta.$$

then we can say

$$\lim_{x \to c} f(x) = L.$$

Recall... Limits

As the end of *Lecture 3* we introduced problems of the following form:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{where} \quad \lim_{x \to c} g(x) = 0$$

We'll look at two techniques for solving such problems:

- The *The Squeeze Theorem* (today).
- *l'Hospital's Rule*

Theorem (The Squeeze Theorem)

Let f, g and h be functions such that

$$f(x) \leq g(x) \leq h(x) \quad \text{for } x \text{ near } c.$$

If

$$\lim_{x \longrightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \longrightarrow c} h(x) = L,$$

then

$$\lim_{x \longrightarrow c} g(x) = L.$$

Example

Show that $\lim_{x \longrightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Exercise (4.1)

Use the Squeeze theorem to answer the following questions.

(i) Find $\lim_{x\to 0} f(x)$ if f is a function such that

$$2 - x^2 \leq f(x) \leq 2\cos(x),$$

(ii) If $\lim_{x\to 0} |f(x)| = 0$, show that $\lim_{x\to 0} f(x) = 0$.

(iii) What is the largest possible domain of $f(x) = x^4 \cos(2/x)$? Show that $\lim_{x\to 0} f(x) = 0$.

The rule of l'Hospital is

$$\frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{\lim_{x \to c} f'(x)}{\lim_{x \to c} g'(x)}$$

where here, for example, $f'(x)$ is the derivative of f with respect to x.

And explaining what that means is one of the real reasons we've introduced the idea of a limit.

We'll return to l'Hospital's rule toward the end of next Monday's lecture.

Derivatives

How can you calculate the slope of the tangent to a function f at a given point x?

One approach is to compute the slope of the line that intersects the function at x and some near by point $x + h$. (This is called a *secant* line)...

And then get the limit of the slope of the secant lines as h tends to zero.

This gives us the definition

$$f'(x) := \frac{d}{dx} f(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example

Find the derivative of $f(x) = x^2$ using the above definition. (i.e, "*differentiate* $f(x) = x^2$ *from first principles.*").

Example

Find the derivative of $f(x) = 1/x$ from first principles.

Exercise (4.2)

From first principles, find the derivative of

(a) $f(x) = \frac{1}{3}x^3$.

(b) $f(x) = x^n$ for any $n = 1, 2, 3, ...$

(c) $f(x) = x^{-n}$

For a hint for Part (ii), use the *binomial theorem* (see Slides 19 and 20).

Derivatives

In the majority of cases, you don't have to use first principles to calculate derivatives. The answer for the most common functions are given in the *Mathematical ("Log") Tables*.

We then use these, often in conjunction with some so-called rules, such as the *Product Rule* and *Quotient Rule*, which are also given in the tables.

Elementary properties

- $\big(f+g\big)'(x) = f'(x) + g'(x)$.
- $\big(f-g\big)'(x) = f'(x) - g'(x)$.
- $\big(Cf\big)'(x) = Cf'(x)$, for a constant C.

The derivative of the product of 2 functions

Let $f(x) = u(x)v(x)$. Then

$$\frac{d}{dx}f(x) = (u \cdot v)'(x) = u(x)v'(x) + u'(x)v(x).$$

Example

What is the derivative of

$$f(t) = (t^2 + 1)(t^3 + 2t)?$$

$\left(u/v\right)'(t)$

The derivative of the ratio of 2 functions

Let $f(x) = u(x)/v(x)$. Then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{\left(v(x)\right)^2}.$$

We'll postpone a proof of this for a while.

$\left(u/v\right)'(t)$

Example

Calculate the derivative of

$$f(t) = \frac{\sqrt{t}}{2t-1}$$

Extra: Binomial Expansions

Exercise 4.2 (b) required us to find the derivative (with respect to x) of $f(x) = x^n$ for any $n = 1, 2, 3, \ldots$

This involves working with the expression

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

So we need to expand the expression $(x+h)^n$, using the *Binomial Theorem*

Extra: Binomial Expansions

Recall *Binomial Expansion*

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + b^n$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

Here the *Binomial Coefficient* $\binom{n}{k}$ ("n choose k") is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

and $n!$ ("n factorial") is $n! = n \cdot (n-1) \ldots 3 \cdot 2 \cdot 1$.