MA211

# Lecture 3: Limits

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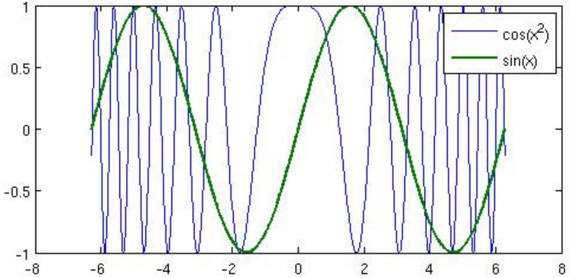


# Outline

## Problem Solving Sessions

Problem Solving sessions (tutorials) will start next week. There will be three per week. Attend whichever one you like

- Tuesday, 3pm, AC202
- Wednesday, 5pm, QA003 (Physiology lecture room)
- ????? (Thursday, 6pm, IT207??)

A a **function** $f$ from a set $X$ to a set $Y$ is a rule of correspondences that associates every element of $X$ with some (single) element of $Y$. We write:

$$f : X \to Y.$$

- $X$ is called the **Domain** of $f$, and
- $Y$ is called the *Codomain*.
- the *Range* of $f$ the subset of $Y$ that contains all the elements of $Y$ that are the image under $f$ of some element of $X$.

A function from $X$ to $Y$ is *One-to-One* if no two elements of $X$ are mapped to the same element of $Y$. (In some books this is called *injective*).

**Definition (One-to-one)**

The function $f : X \rightarrow Y$ is one-to-one if whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$.

**Example**

Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is one-to-one? Why?

Find other sets $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ such that $f : X \to Y$ is one-to-one.

A function from $X$ to $Y$ is *onto* if every element of $Y$ is the image of some element of $X$. (In some books this is called *surjective*).

**Definition (Onto)**

The function $f : X \to Y$ is onto if for each $y \in B$ there exists $x \in A$ such that $y$ is the image of $x$. That is

$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y.$$

A third way of expressing this is by saying the *range* of $f$ is equal to its *codomain*.

**Example**

Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not **onto**.

Find other sets $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ such that $f : X \to Y$ is onto.

**Exercise (3.1)**

Give an example of a function:

(i) $f : \mathbb{Z} \to \mathbb{N}$ that is onto but *not* one-to-one.

(ii) $f : \mathbb{N} \to \mathbb{N}$ that is one-to-one, but not onto.

# Inverse functions

When a function is both one-to-one and onto is has an **Inverse**.

**Definition (Inverse)**

If the function $g$ is the inverse of $f$ then

$$\text{when } f(x) = y, \text{ we get that } g(y) = x.$$

Usually we write $g = f^{-1}$.

*Examples:*

1. $y = f(x) \iff x = f^{-1}(y)$

2. The domain of $f^{-1}$ is the range of $f$
3. The range of $f^{-1}$ is the domain of $f$

4. $f^{-1}\big(f(x)\big) = x$
5. $f\big(f^{-1}(x)\big) = x$

6. $\big(f^{-1}\big)^{-1} = f$.

7. The graph of $f^{-1}$ is the reflection of the graph of $f$ in the line $x = y$.

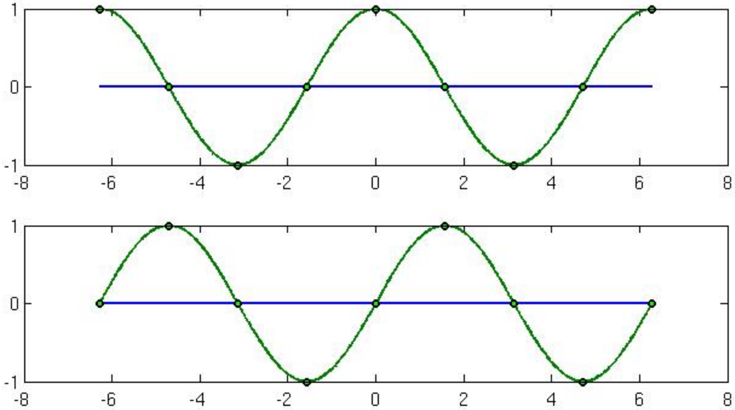
# The trigonometric functions

You'll remember the definition of cos and sin in terms of a right-angle triangle.

Here is another, equivalent definition.

- Take the usual coordinate axes centred on $(0,0)$ and draw the unit circle,
- Mark the point $(1,0)$.
- Trace an arc of length $t$ *anti-clockwise* from $(1,0)$ along the circle. Call the point at the end of that arc $P$.
- The $x$-coordinate of $P$ is $\cos(t)$ and the $y$-coordinate is $\sin(t)$.

## **The** cos **(top) and** sin **(bottom) functions**



## Exercise (3.2)

Find subsets $X$ and $Y$ of the real numbers such that the functions $f : X \to Y$ are *invertible* (i.e., both *one-to-one* and *onto*) for

(i) $f(x) = \sin(x)$

(ii) $f(x) = \cos(x^2)$

# Limits

When we write

$$\lim_{x \longrightarrow c} f(x) = L$$

or say "*The limit of* $f$ *as* $x$ *approaches* $c$ *is* $L$" we mean that we can make $f$ as close to $L$ as we would like by taking $x$ as close to $c$ as is needed.

**Definition (Limit)**

If for any $\varepsilon > 0$, no matter how small, we can find $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{when} \quad |x - c| < \delta.$$

then we can say

$$\lim_{x \longrightarrow c} f(x) = L.$$

# Limits

**Example**

Show that

$$\lim_{x \longrightarrow 3} (2x + 1) = 7.$$

# Limits

**Exercise (3.3)**

Show *carefully* that

(i) $\lim_{x \to 4} 3x - 7 = 5$.

(ii) $\displaystyle\lim_{x \to 2} \left(\frac{x}{2} + 3\right)$ is 4

# Limits

Calculating the limit of a given function as it approaches a certain point is a fairly standard task.

**Example**

Find the limit of

$$f(x) = \frac{2x^2 - 3x - 2}{x - 2}$$

as $x$ approaches 2.

Let $n$ be an integer, $k$ a constant real number, and $f$ and $g$ be functions that have a limit at $c$. Then

1. $\lim_{x \to c} k = k$;
2. $\lim_{x \to c} x = c$;
3. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$;
4. $\lim_{x \to c} \big(f(x) + g(x)\big) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.
5. $\lim_{x \to c} \big(f(x) - g(x)\big) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$.

7. $\lim_{x \to c} \big(f(x) \cdot g(x)\big) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$.

8. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$. *providing that* $\lim_{x \to c} g(x) \neq 0$.

9. $\lim_{x \to c} \left(f(x)\right)^n = \left(\lim_{x \to c} f(x)\right)^n$.

10. $\lim_{x \to c} \left(f(x)\right)^{(1/n)} = \left(\lim_{x \to c} f(x)\right)^{(1/n)}$.

There are many neat tricks for computing limits, particularly those of the form

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}. \quad \text{where} \quad \lim_{x \to c} g(x) = 0$$

One of these is *The Squeeze Theorem*, but even better is l'Hopital's Rule

We'll cover these on Wednesday.