MA378 Chapter 4: Finite Element Methods

§4.1 Boundary value problems

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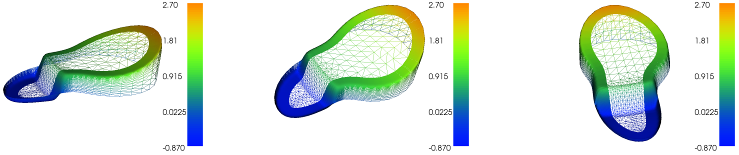

(The above images have nothing to do with this lecture. But I like them. The are solutions to a certain PDE computed using a Python-based finite element system called FEniCS.)

1.1 Introduction

In this final section of MA378 we consider numerical schemes for particular ordinary differential equations called

Boundary Value Problems (BVPs).

BVPs are types of **differential equations**, meaning that

- They are equations (i.e., feature an "$=$" symbol) that we try to solve for some unknown.
- The equation features derivatives of the unknown.

1.1 Introduction

BVPs V IVPs

The main differences BVPs and the initial value problems (IVPs) of MA385 are:

- The equation gives the solution to the DE **two** (boundary) points, not just one (initial) point.
- The second derivative of the solution **always** appears in the equation. The IVPs we studied usually only had first derivatives.
- We usually think of the independent variable, x, as representing **space** rather than time.

1.1 Introduction

For some BVPs it is possible to write down the exact solution (an example is given below). Usually, however, this is not easy, or, indeed, possible. So numerical methods are needed to given *approximate* solutions. Two of the most popular numerical methods are

- **Finite Difference Methods** (FDMs), where one approximates the differential equation, based on Taylor series expansions.
- **Finite Element Methods** (FEMs), where the solution space is approximated, using ideas related to interpolation.

Since **FEMs** are in keeping with the philosophy of this course, we will study them.

More details are in Chapter 14 of the Süli and Mayers.

1.2 Boundary value problems

To formulate a model BVP we first define a differential operator

$$L(u) := -u''(x) + r(x)u(x). \tag{1}$$

The practice of using L for such an operator comes from the fact that it is **linear**, meaning that $L(u+v) = L(u) + L(v)$.

This operator can be applied to any function with a defined second derivative.

Eg $\quad r(x)\big(u+v\big)(x) = r(x)u(x) + r(x)v(x)$

And $\quad (u+v)''(x) = u''(x) + v''(x).$

1.2 Boundary value problems

The general form of a BVP is:

Find a function u, defined on the interval $[a, b]$, such that

$$L(u) = f(x) \quad \text{for } a < x < b, \qquad \text{with } u(a) = 0, u(b) = 0. \quad (2)$$

We make the assumptions that r and f are continuous and have as many continuous derivatives as we would like. **Furthermore we always have that $r(x) > 0$ for all $x \in [a, b]$**.

Some BVPs are easy to solve by hand. That is, we can write down a solution in terms of elementary functions. That is certainly the case if the functions r and f are constant.

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Example 1.1

Verify that, for any constants c_0 and c_1, $u(x) = c_0 e^{2x} + c_1 e^{-2x} + x/4$ is a solution to

$$-u''(x) + 4u(x) = x \quad \text{for} \quad 0 \le x \le 3, \qquad (3)$$

(If we have two boundary conditions, we can uniquely determine c_0 and c_1. See Exercise 1.2).

$$u'(x) = 2c_0 e^{2x} - 2c_1 e^{-2x} + 1/4$$

$$u''(x) = 4c_0 e^{2x} + 4c_1 e^{-2x}$$

Then $-u''(x) + 4u(x) =$

$$\cancel{-4c_0 e^{2x}} \cancel{-4c_1} e^{-2x} + \cancel{4c_0 e^{2x}} + \cancel{4c_1} e^{-2x} + x$$

$$= x$$

1.2 Boundary value problems

Unlike the example above, most boundary value problems are difficult or impossible to solve analytically and so, in general, we need approximations. However, it can often be possible to obtain **qualitative** information about the solutions.

One crucial tool for this is called a **maximum principle**.

Lemma 1.2 (Maximum Principle)

Let L be the differential operator $L(u) := -u''(x) + r(x)u(x)$, where $r(x) > 0$ for all $x \in [a, b]$, and u is a function such that $Lu \geq 0$ on (a, b). If $u(a) \geq 0$, $u(b) \geq 0$. Then $u \geq 0$ for all $x \in [a, b]$.

Proof: Suppose that x^* is such that $u(x^*) = \min_{a \leq x \leq b} u(x)$. Suppose that $u(x^*) < 0$. Clearly $x^* \neq a$ and $x^* \neq b$, so $u(x)$ has a LOCAL MINIMUM at $x = x^*$. Then $u'(x^*) = 0$. Moreover $u''(x^*) \geq 0$

Then $\underbrace{-u''(x^*)}_{\leq 0} + \underbrace{r(x^*)}_{>0}\,\underbrace{u(x^*)}_{<0} < 0$. This is a contradiction!

This lemma is as useful as it is simple (see also Exercise 1.4).

Lemma 1.3

There is at most one solution to the BVP

$$L(u) = f(x) \quad \textit{for } a < x < b, \qquad \textit{with } u(a) = 0, u(b) = 0.$$

Suppose both u and v are solutions to this problem. Then $-u''(x) + r(x)u(x) = f(x)$ (i)

and $-v''(x) + r(x)v(x) = f(x)$. (ii)

Let $w = u - v$. Then, subrating (ii) from (i) gives $-(u-v)''(x) + r(x)(u-v)(x) = 0$.

So $L(w) = 0$. Since $L(w) \geq 0$, we know $w(x) \geq 0$

But this also shows $L(-w) = 0 \Rightarrow L(-w) \geq 0$

$\Rightarrow w(x) \leq 0.$ Thus $w(x) \equiv 0$

1.3 Where the solution lives

The boundary value problem

$$-u''(x) + r(x)u(x) = f(x) \quad a < x < b,$$

and

$$u(a) = 0, u(b) = 0,$$

equates the terms in u and u'' with a function f that is continuous on (a, b), so we must have that u, u' and u'' are all continuous on the interval (a, b). That is, u belongs to the space of functions $C^2(a, b)$.

So we can state the problem more precisely: *find* $u \in C^2(a, b)$ *such that*

$$-u''(x) + r(x)u(x) = f(x) \qquad \textit{for all } x \textit{ in } (a, b),$$

$$\textit{and } u(a) = u(b) = 0.$$

1.4 The variational form

Before we give a numerical method for computing an approximate solution to a boundary value problem, we will rewrite it as an integral equation.

Definition 1.4 ($H_0^1(a, b)$)

Define (again) the inner product: $(u, v) := \int_a^b u(x)v(x)dx$. With this we can defined the norm: $\|u\|_2 := (u, u)^{1/2}$,
Then $H_0^1(a, b)$ is defined to be the space of (absolutely) continuous functions on $[a, b]$ such that if $w \in H_0^1(a, b)$ then

$$w(a) = w(b) = 0 \quad \text{and } \|w'\|_2 < \infty.$$

(We met similar space before in the section on cubic splines.)

1.4 The variational form

Consider the Boundary Value Problem: *find* $u \in C^2(a,b)$ *such that*

$$-u''(x) + r(x)u(x) = f(x) \text{ on } (a,b); \text{ and } u(a) = u(b) = 0. \quad (4)$$

Suppose that we have a solution u to this equation. Then the **variational form** of (4) is derived as on the following slide.

1.4 The variational form

$$\boxed{-u''(x) + r(x)u(x) = f(x) \text{ on } (a,b); \quad \text{and } u(a) = u(b) = 0.}$$

Suppose this equation holds. Then, for any function, v,

$$-u''(x)v(x) + r(x)u(x)v(x) = f(x)v(x).$$

Integrate:

$$\int_a^b -u''(x)v(x) + r(x)u(x)v(x)\,dx = \int_a^b f(x)v(x)\,dx.$$

Using Integration By Parts:

$$\int_a^b u''(x)v(x)\,dx = u'(x)v(x)\Big|_a^b - \int_a^b u'(x)v'(x)\,dx.$$

This holds for <u>any</u> v, including if $v(a) = v(b) = 0$.

$$\boxed{-u''(x) + r(x)u(x) = f(x) \text{ on } (a,b); \quad \text{and } u(a) = u(b) = 0.}$$

So we now have
$$\int_a^b u'(x)\,v'(x)\,dx + \int_a^b r(x)\,u(x)\,v(x)\,dx = \int_a^b f(x)\,v(x)\,dx$$
for any v for which $v(a) = v(b) = 0$.

Recall that $(u,v) = \int_a^b u(x)\,v(x)\,dx$. So now we have
$$(u', v') + (ru, v) = (f, v).$$

Note that here I only need that I can integrate u' and v'.

Definition 1.5 (Variational formulation)

The variational/weak formulation of (4) is: *Find* $u \in H_0^1(a,b)$ *such that*

$$A(u,v) = (f,v) \quad \textit{for all} \quad v \in H_0^1(a,b). \tag{5}$$

where $\mathcal{A}(\cdot,\cdot)$ is the symmetric bilinear functional

$$\mathcal{A}(u,v) := (u',v') + (ru,v).$$

Symmetric : $A(u,v) = A(v,u)$

Bilinear : $A(u, v+w) = A(u,v) + A(u,w)$

$A(cu,v) = c\,A(u,v)$ c a constant

functional : A maps to $\mathbb{R}$.

1.4 The variational form

This $\mathcal{A}$ has several important properties. One of these is that, for any function w,

$$A(w, w) \geq 0,$$

and

$$A(w, w) = 0 \Leftrightarrow w = 0.$$

The following lemma is an immediate consequence of these properties.

Lemma 1.6

The variational problem (5) *has at most one solution.*

Suppose there are two solutions: u, and w.

That is $A(u,v) = (f,v)$ for all $v \in H_0^1(a,b)$

and $A(w,v) = (f,v)$ " " ",

Then $A(u-w, v) = 0$ " " "

Since u, w both in $H_0^1(a,b)$ so too is $u-w$

So we can take $v = u-w$. That gives

$A(u-w, u-w) = 0 \implies u-w=0.$

1.5 Exercises

Exercise 1.1

Suppose, instead of the differential operator defined in (1), we had the more general one:

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x)u(x).$$

Does this L_q also satisfy a maximum principle? If so, provide a proof. If not, give a counter example.

Exercise 1.2

Verify that

$$u(x) = \frac{x}{4} + \frac{3e^6(e^{-2x} - e^{2x})}{4(e^{12} - 1)}$$

is the exact solution to (1.1) with the boundary conditions $u(0) = 0$, $u(3) = 0$,

1.5 Exercises

Exercise 1.3

In this section of the course, we'll always assume homogeneous boundary conditions. That is, that $u(x) = 0$ at the boundaries. Suppose the problem we wish to solve is

$$-u''(x) + r(x)u(x) = f(x) \quad u(0) = \alpha, u(1) = \beta.$$

Show how to find a problem which has the same left-hand side as this one, homogeneous boundary conditions, and with a solution that differs from this one only by a known linear function.

Exercise 1.4

Suppose that u solves $-u''(x) + r(x)u(x) = f(x) \quad$ on $(0, 1)$, and $u(0) = u(1) = 0$. Let ϱ be such $r(x) \geq \varrho > 0$, and define

$$C = \max_{0 \leq x \leq 1} |f(x)|/\varrho.$$

Prove that $u(x) \leq C$. (Hint: Consider $L(C - u)$).

1.5 Exercises

Exercise 1.5

Consider the differential equation:

$$-u''(x) = \exp(x+1), \text{ on } (0,2), \quad \text{and} \quad u(0) = u(2) = 0.$$

(i) State the variational formulation of this differential equation.

(ii) Show that the solution to the variational problem is unique.