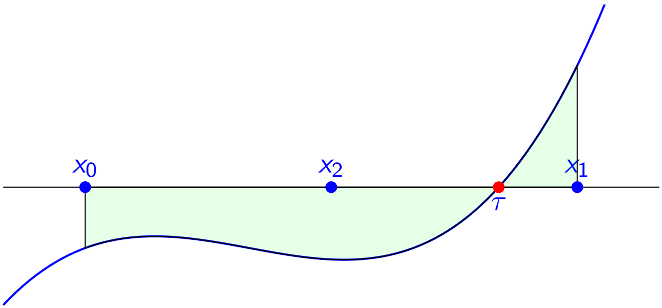
Solving nonlinear equations

1.2: Interval Bisection

MA385 – Numerical Analysis

September 2025



0. Outline

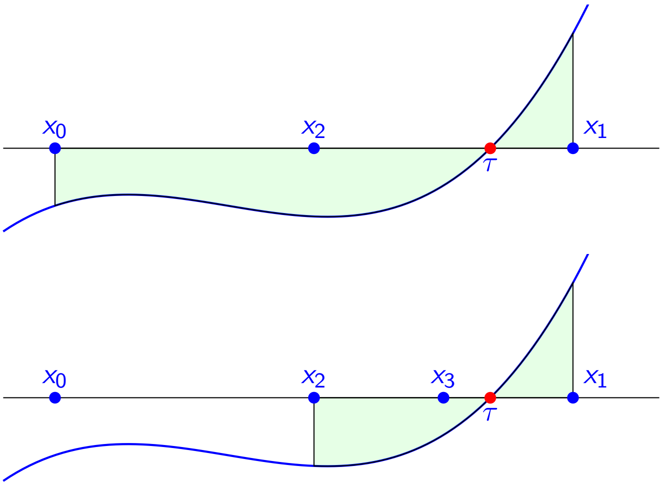
For more details, see Section 1.6 (The Bisection Method) of Süli and Mayers, *An Introduction to Numerical Analysis*

1. Bisection

The most elementary algorithm is the "*Bisection Method*" (also known as "Interval Bisection"). Suppose that we know that f changes sign on the interval $[a, b] = [x_0, x_1]$ and, thus, $f(x) = 0$ has a solution, τ, in $[a, b]$. Proceed as follows

1. Set x_2 to be the midpoint of the interval $[x_0, x_1]$.
2. Choose one of the sub-intervals $[x_0, x_2]$ and $[x_2, x_1]$ where f change sign;
3. Repeat Steps 1–2 on that sub-interval, until f is sufficiently small at the end points of the interval.

1. Bisection



1. Bisection

This may be expressed more precisely using some *pseudocode*.

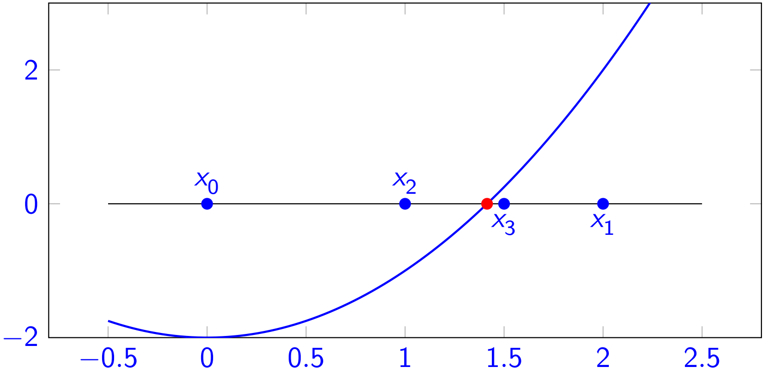
The Bisection Algorithm

Set eps to be the stopping criterion.
If $|f(a)| \leq eps$, return a. Exit.
If $|f(b)| \leq eps$, return b. Exit.
Set $x_L = a$ and $x_R = b$.
Set $k = 1$
while($|f(x_k)| > eps$)
    $x_{k+1} = (x_L + x_R)/2$;
    if $(f(x_L)f(x_{k+1}) < eps)$
        $x_R = x_{k+1}$;
    else
        $x_L = x_{k+1}$
    end if;
    $k = k + 1$
end while;

Example 1

Find an estimate for $\sqrt{2}$ that is correct to 6 decimal places.

Idea: Use bisection to solve $x^2 - 2 = 0$ on the interval $[0, 2]$.



Example 1

Find an estimate for $\sqrt{2}$ that is correct to 6 decimal places.
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k	x_k	$\lvert x_k - \tau\rvert$	$\lvert x_k - x_{k-1}\rvert$
0	0.000000	1.41	
1	2.000000	5.86e-01	
2	1.000000	4.14e-01	1.00
3	1.500000	8.58e-02	5.00e-01
4	1.250000	1.64e-01	2.50e-01
5	1.375000	3.92e-02	1.25e-01
6	1.437500	2.33e-02	6.25e-02
7	1.406250	7.96e-03	3.12e-02
8	1.421875	7.66e-03	1.56e-02
9	1.414062	1.51e-04	7.81e-03
10	1.417969	3.76e-03	3.91e-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	1.414214	5.72e-07	9.54e-07

2. The bisection method works

The main advantages of the Bisection method are

- It will always work, providing only that f is continuous on $[a, b]$, and that the solution exists.
- After k steps we know that

Theorem 1.1

$$|\tau - x_k| \leq \left(\frac{1}{2}\right)^{k-1}|b-a|, \qquad \text{for } k = 2, 3, 4, \ldots$$

2. The bisection method works

3. Improving upon bisection

A disadvantage of bisection is that it is not particularly efficient. So our next goal will be to derive better methods, particularly the **Secant Method** and **Newton's method**. We also have to come up with some way of expressing what we mean by "**better**".

4. Exercises

Exercise 1.1

Suppose we want to find $\tau \in [a, b]$ such that $f(\tau) = 0$ for some given f, a and b. Write down an estimate for the number of iterations K required by the bisection method to ensure that, for a given ε, we know $|x_k - \tau| \leq \varepsilon$ for all $k \geq K$. In particular, how does this estimate depend on f, a and b?

Exercise 1.2

How many (decimal) digits of accuracy are gained at each step of the bisection method? (If you prefer, how many steps are needed to gain a single (decimal) digit of accuracy?)

Exercise 1.3

Let $f(x) = e^x - 2x - 2$. Show that there is a solution to the problem: *find $\tau \in [0, 2]$ such that $f(\tau) = 0$.*

Taking $x_0 = 0$ and $x_1 = 2$, use 6 steps of the bisection method to estimate τ. You may use a computer program to do this, but please note that in your solution.

Give an upper bound for the error $|\tau - x_6|$.

Exercise 1.4

We wish to estimate $\tau = \sqrt[3]{4}$ numerically by solving $f(x) = 0$ in $[a, b]$ for some suitably chosen f, a and b.

(i) Suggest suitable choices of f, a, and b for this problem.

(ii) Show that f has a zero in $[a, b]$.

(iii) Use 6 steps of the bisection method to estimate $\sqrt[3]{4}$. You may use a computer program to do this, but please note that in your solution.

(iv) Use Theorem 1.1 to give an upper bound for the error $|\tau - x_6|$.