

2425-MA140 Engineering Calculus

Week 06, Lecture 3
Optimization and L'Hôpital's Rule

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- ▶ **Assignment 3:** Reminder...
 1. If you think your correct grade is not showing, send me an email with your **results summary**.
 2. If you like, you reattempt. (Deadline Friday).
- ▶ **Assignment 4:** this “optional” assignment has started. Deadline 5pm, Tuesday 29th October.

This fine autumn morning, we'll learn about...

1 Optimization

- Introduction
- Strategy
- Examples

2 L'Hôpital's Rule

- The Rule (Part I)
- Repeated application
- The Rule (Part II)
- Extra: why it works

3 Exercises

See also: Sections **4.7** (Applied Optimization Problems) and **4.8** (L'Hôpital's Rule) in **Calculus** by Strang & Herman:

[math.libretexts.org/Bookshelves/Calculus/Calculus_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

Now that we know how to find maxima and minima of functions, we can solve **optimization** problems. Here is a classic example:

Example

What is the largest rectangular field we could enclose with 40m of fencing?

We can “solve” this problem by checking a few cases.

Now use calculus:

Here is a more general approach:

1. Write down a function, f , describing the quantity to be minimized/maximized.
2. If f is in more than one variable, use other information, linking the variables, to reduce it to a function of one variable.
3. Differentiate f , and find its critical points. Determine which correspond to maxima and minima.

Example:

A stretch of land is bordered by a (remarkably straight) river. What is the largest field we could enclose with 40m of fencing, if we don't have fence along by the river?

Sometimes, we are given the formula of the quantity to be optimised explicitly.

Example

Suppose that if a particular vehicle is been driven at a speed of x km/hr then its fuel usage, measured, in L/100km is given by

$$y = \frac{x^2}{1000} - \frac{1}{10}x + 10,$$

1. What speed should you drive at in order to minimise your fuel usage?
2. What is the fuel usage (in L/100km) at that speed?

L'Hôpital's Rule

Now that we've learned some differential calculus, we'll use a powerful tool for computing limits of the quotient of two functions: that is, something like

$$\lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)}.$$

We know (from Week 2, Lecture 2), that, if $\lim_{x \rightarrow a} f_1(x) = L_1$, and $\lim_{x \rightarrow a} f_2(x) = L_2 \neq 0$, then

$$\lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}.$$

But what happens in both $L_1 = 0$ and $L_2 = 0$? This is called an **indeterminate form**, and some other methods are needed, e.g., if $f_1(x)$ and $f_2(x)$ are polynomials, we could factorise them.

But now we'll learn a powerful, more general approach...

L'Hôpital's Rule: the $\frac{0}{0}$ case

Suppose that f and g are both differentiable everywhere on an open interval containing a (except possibly at a). If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

providing the limit on the right exists, or is $\pm\infty$. This is true also for one-sided limits, or if $a = \pm\infty$.

Note about the spelling: L'Hôpital's Rule (1696) is named after Guillaume de l'Hospital, who spelled L'Hôpital as L'Hospital. But since then, French spelling has changed. Also, L'Hôpital's Rule was discovered (invented?) by Johann Bernoulli in 1694. Confused?

Example

Evaluate the limit

$$\lim_{x \rightarrow 1} \frac{\ln(x)}{x - 1}$$

One may apply L'Hôpital's Rule multiple times.

Example

Evaluate the limit $\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3}$

L'Hôpital's Rule: the $\frac{\infty}{\infty}$ case

Suppose that f and g are both differentiable everywhere on an open interval containing a (except possibly at a). If $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

providing the limit on the right exists, or is $\pm\infty$. This is true also for one-sided limits, or if $a = \pm\infty$.

Example

Evaluate the limit $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$

In case you are wondering *why* L'Hôpital's Rule works. Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof: Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\begin{aligned} \frac{f'(a)}{g'(a)} &= \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \rightarrow a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \end{aligned}$$

Exercises

Exer 6.3.1

Use L'Hôpital's Rule to evaluate the following:

1. $\lim_{x \rightarrow 1} \frac{(x-1)^2}{\ln(x)}.$
2. $\lim_{x \rightarrow \infty} \frac{(x-1)^2}{\ln(x)}.$

Exer 6.3.2

Use L'Hôpital's Rule to evaluate $\lim_{x \rightarrow 0^+} x \ln(x)$. (*Hint: write $\ln(x)$ as $\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ tend to either $-\infty$ or ∞ as $x \rightarrow 0$.*)