

Theory & Computation of Singularly Perturbed Differential Equations

IIT (BHU) Varanasi, Dec 2017

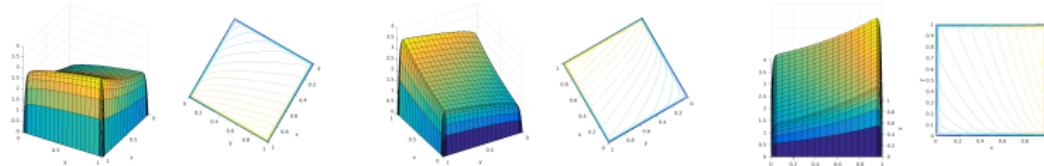
<https://skumarmath.wordpress.com/gian-17/singular-perturbation-problems/>

<http://www.maths.nuigalway.ie/~niall/TCSPDEs2017>

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§8 Singularly perturbed elliptic PDEs

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Presentation version: not for printing

Outline

Monday, 4 December		
09:30 – 10:30	Registration and Inauguration	NM
10:45 – 11:45	1. Introduction to singularly perturbed problems	NM
12:00 – 13:00	2. Numerical methods and uniform convergence	NM
14:30 – 15:30	Tutorial (Convection diffusion problems)	NM
15:30 – 16:30	Lab 1 (Simple FEMs in MATLAB)	NM
Tuesday, 5 December		
09:30 – 10:30	3. Finite difference methods and their analyses	NM
10:45 – 11:45	4. Coupled systems of SPPDEs	NM
14:00 – 16:00	Lab 2 (Fitted mesh methods for ODEs)	NM
Thursday, 7 December		
09:00 – 10:00	8. Singularly perturbed elliptic PDEs	NM
10:15 – 11:15	9. Finite Elements in two and three dimensions	NM
01:15 – 15:15	Lab 4 (Singularly perturbed PDEs)	NM
Friday, 8 December		
09:00 – 10:00	10. Preconditioning for SPPs	NM

§8. Singular Perturbed Elliptic Problems

(≈ 1 hour)

In this section we will study the robust solution, by a **finite difference method**, of PDEs of the form

$$-\varepsilon^2 \Delta u + bu = f \quad \text{on } \Omega := (0, 1)^d.$$

The focus is on $d = 2$, but many of the ideas for $d = 3$ are similar, which will be mentioned in the next section.

- 1 A 2D, SP, reaction-diffusion equation
- 2 Solution decomposition
 - The domain
 - Compatibility conditions
 - Extended domain
 - The regular component
 - Edge components
 - Corner components
- 3 Discretization
 - The FEM
 - A piecewise uniform ("Shishkin") mesh
- 4 Analysis (regular part only)
- 5 References

Primary references

The key reference for this presentation is [Clavero et al., 2005]. From that, the most important component is the solution decomposition, which itself was first established by [Shishkin, 1992]. The compatibility conditions provided by [Han and Kellogg, 1990] are also vital.

Extensions to coupled systems can be found in [Kellogg et al., 2008a] and [Kellogg et al., 2008b], and a unified treatment is given in [Linß, 2010, Chap. 9].

The most general treatment is given in [Shishkin and Shishkina, 2009], though it is not the easiest book to read.

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The references above are mentioned only because they are related to the this presentation.

There are, of course, many other important papers on the solution of two-dimensional reaction-diffusion problems...

A 2D, SP, reaction-diffusion equation

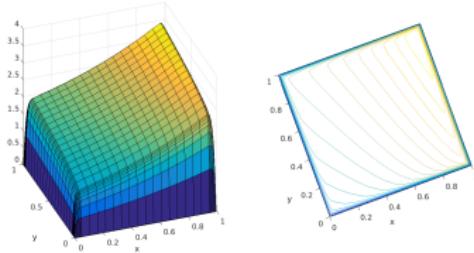
A 2D singularly perturbed problem

$$\begin{aligned} -\varepsilon^2(u_{xx} + u_{yy}) + b(x, y)u &= f(x, y), \text{ on } \Omega := (0, 1)^2 \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

A 2D, SP, reaction-diffusion equation

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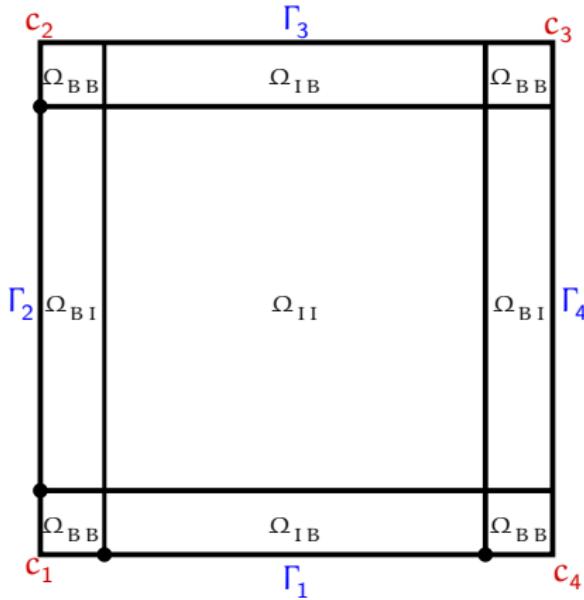


- Typically, on this domain, solutions feature four “edge” layers that behave like $\exp(-x/\varepsilon)$ or $\exp(-y/\varepsilon)$.
- They also have four corner layers, that behave like $\exp(-(x+y)/\varepsilon)$.

Solution decomposition

The domain

- We'll denote the corners of the domain c_1, \dots, c_4 , labelled clockwise from $c_1 = (0, 0)$.
- The edges are $\Gamma_1, \dots, \Gamma_4$, labelled clockwise from $\Gamma_1 = [0, 1]$.
- $u(x, y) = g(x, y)$ on $\partial\Omega$, and g_i is the restriction of g to Γ_i .



From [Han and Kellogg, 1990], we shall assume that $f, b \in C^{2,\alpha}(\bar{\Omega})$, the $g_i \in C^{4,\alpha}([0, 1])$ and that we have compatibility conditions at each corner. For example, at $c_1 = (0, 0)$, these are

$$g_1 = g_2, \quad (2a)$$

$$-\varepsilon^2 \left(\frac{\partial^2}{\partial x^2} g_1 + \frac{\partial^2}{\partial y^2} g_2 \right) + bg_1 = f, \quad ^1 \quad (2b)$$

$$\frac{\partial^2}{\partial x^2} \left(-\varepsilon^2 \frac{\partial^2}{\partial x^2} g_1 + bg_1 - f \right) = \frac{\partial^2}{\partial y^2} \left(-\varepsilon^2 \frac{\partial^2}{\partial y^2} g_2 + bg_2 - f \right). \quad (2c)$$

If u solves (1), and the conditions (2) are satisfied, as well as analogous ones at the other three corners, then $u \in C^{4,\alpha}$.

¹Actually, g_1 and g_2 are functions of a single variable, x and y respectively, but it is notationally convenient to express these ordinary derivatives as partial derivatives, particularly in (2c).

One can show that

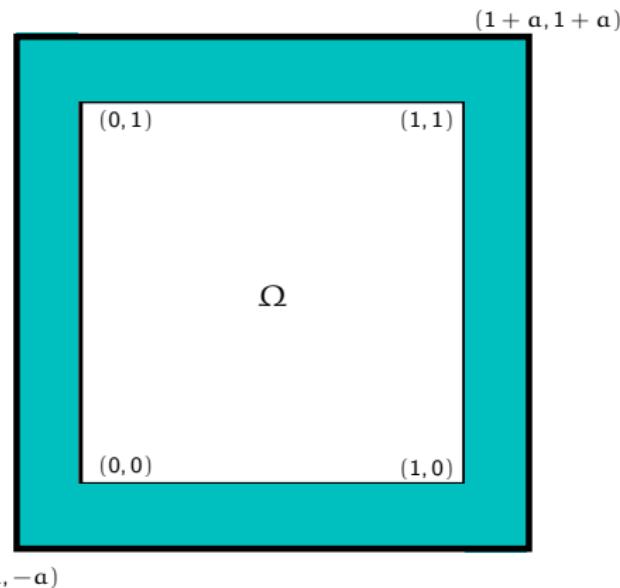
$$\left\| \frac{\partial^{(k+j)}}{\partial x^k \partial y^j} u \right\| \leq C \varepsilon^{-(k+j)}, \quad \text{for } k, j \in \mathbb{N}_0, k + j \leq 4, \quad (3)$$

but finer results are needed.

One of the key ideas in proving the existence of a suitable solution decomposition for this problem is to use an *extended domain*:

$$\Omega^* = (-a, 1+a)^2.$$

Define smooth extensions to b and f to $\bar{\Omega}^*$, denoted b^* and f^* respectively. Similarly the extension of g_i to $[-a, 1+a]$ is g_i^* .



We will let $v^* = v_0^* + \varepsilon v_1^*$, where

- $v_0^* = f^*/b^*$.

- v_1^* solves

$$\mathcal{L}^* v_1^* = \Delta v_0^* \quad \text{on } \Omega^*, \quad v_1^*|_{\partial\Omega^*} = 0.$$

- Then v is taken as the solution to

$$\mathcal{L}v = f \quad \text{on } \Omega^*, \quad v = v^* \text{ on } \partial\Omega.$$

It follows that

$$\left\| \frac{\partial^{(k+j)}}{\partial x^k \partial y^j} u \right\| \leq C(1 + \varepsilon^{-(k+j)}), \quad \text{for } 0 \leq k+j \leq 4. \quad (4)$$

Solution decomposition

Edge components

Next define a function w_1 which is associated with the edge along Γ_1 . That is, we would like to construct w_1 so that $|w_1(x, y)| \leq C e^{-\beta y/\varepsilon}$.

Define a new extended domain,

$$\Omega^{**} = (-a, 1+a) \times (0, 1).$$

Let w^* solve

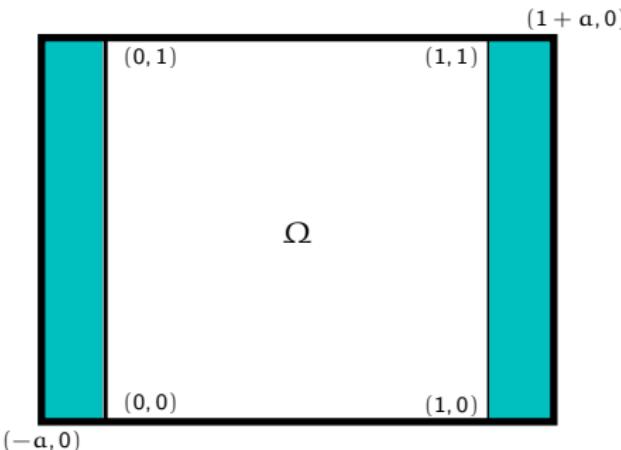
$$\mathcal{L}^{**} w_1 = 0 \quad \text{on } \Omega^{**},$$

$$w_1^* = u - v \quad \text{on } \Gamma_1$$

$$w_1^*(x, 1) = 0 \quad \text{for } x \in [-a, 1+a],$$

$$w_1^*(-a, y) = 0 \quad \text{for } y \in [0, 1],$$

$$w_1^*(1+a, y) = 0 \quad \text{for } y \in [0, 1],$$



and whatever conditions are needed on the remaining regions,

$((-a, 0) \cup (1, 1+a)) \times \{0\}$, to ensure that $w_1 \in C^{4,\alpha}(\bar{\Omega}^{**})$. One can then show that

$$|w_1^*(x, y)| \leq C \left(\frac{a+x}{a} \right) \left(\frac{1+a-x}{1+a} \right) e^{-\beta y/\varepsilon} \quad \text{for } (x, y) \in \bar{\Omega}^{**}.$$

Next, define w_1 as the solution to

$$\begin{aligned} \mathcal{L}w_1 &= 0 && \text{on } \Omega, \\ w_1 &= u - v && \text{on } \Gamma_1 \\ w_1 &= 0 && \text{on } \Gamma_3 \\ w_1 &= w_1^* && \text{on } \{0, 1\} \times [0, 1] \end{aligned}$$

Using the previous bound on w_1^* we get

$$|w_1(x, y)| \leq Ce^{-\beta y/\varepsilon} \quad \text{for } (x, y) \in \bar{\Omega}.$$

So this shows that $w_1(x, y)$ decays rapidly away from Γ_1 , the edge at $y = 1$.

It is possible establish analogous bounds for lower derivatives of w (more about that after coffee...).

Moreover, analogous bounds are possible for:

$$\begin{aligned} |w_2(x, y)| &\leq Ce^{-\beta x/\varepsilon} & |w_3(x, y)| &\leq Ce^{-\beta(1-y)/\varepsilon} \\ |w_4(x, y)| &\leq Ce^{-\beta(1-x)/\varepsilon} \end{aligned}$$

Finally, define z_1 (the component associated with the corner $c_1 = (0, 0)$), as the solution to

$$\begin{aligned}\mathcal{L}z_1 &= 0 && \text{on } \Omega, \\ z_1 &= -w_2 && \text{on } \Gamma_1 \\ z_1 &= -w_1 && \text{on } \Gamma_2 \\ z_1 &= 0 && \text{on } \Gamma_3 \cup \Gamma_4\end{aligned}$$

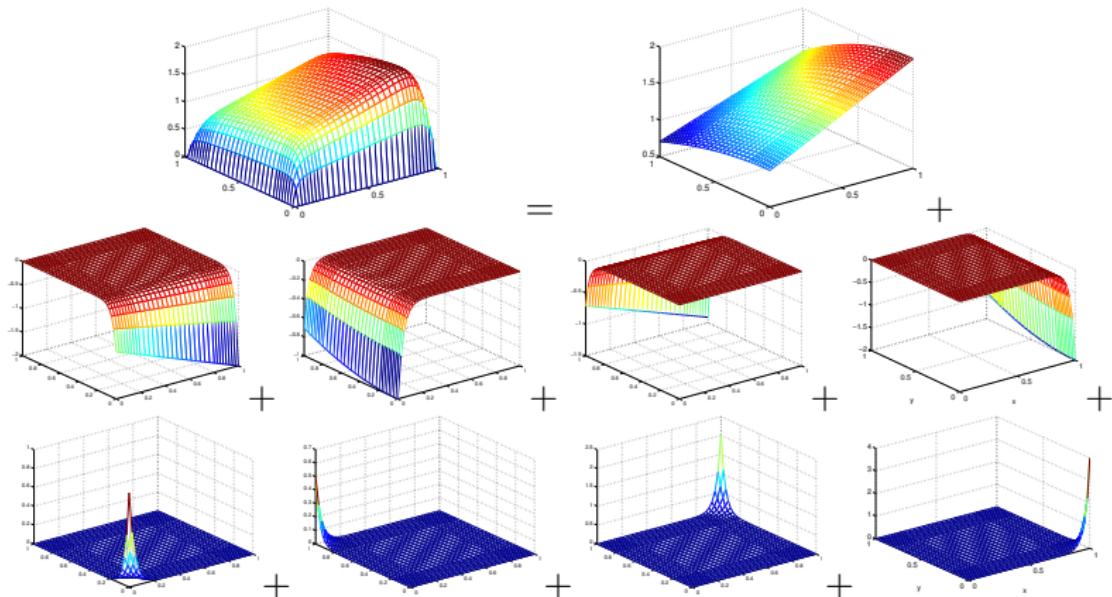
Since we have suitable compatibility conditions, $z_1 \in \mathcal{C}^{4,\alpha}$. A comparison principle then gives

$$|z_1(x, y)| \leq C e^{-\beta(x+y)/\varepsilon}.$$

There are analogous functions, z_2 , z_3 and z_4 associated with the other corners.

The decomposition is

$$u = v + \sum_{i=1}^4 w_i + \sum_{i=1}^4 z_i.$$



We re-use the finite difference method that we employed for 1D problems, extended in the obvious way.

Let $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$ be arbitrary meshes with N intervals on $[0, 1]$.

Set $\bar{\Omega}^N = \{(x_i, y_j)\}_{i,j=0}^N$ to be the Cartesian product of $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$.

Set $h_i = x_i - x_{i-1}$ and $k_i = y_i - y_{i-1}$ for each i .

Define the standard second-order central difference operators

$$\delta_x^2 v_{i,j} := \frac{1}{h_i} \left(\frac{v_{i+1,j} - v_{i,j}}{h_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{h_i} \right)$$

$$\delta_y^2 v_{i,j} := \frac{1}{k_i} \left(\frac{v_{i,j+1} - v_{i,j}}{k_{i+1}} - \frac{v_{i,j} - v_{i,j-1}}{k_i} \right)$$

Define $\Delta^N v_{i,j} := (\delta_x^N + \delta_y^N) v_{i,j}$.

Then the difference operator is

$$(L^N U)_{i,j} = -\epsilon^2 \Delta^N U_{i,j} + b(x_i, y_j) U_{i,j}, \quad i, j = 1, \dots, N-1.$$

To generate a numerical approximation of the solution to (1) solve the system of $(N + 1)^2$ linear equations

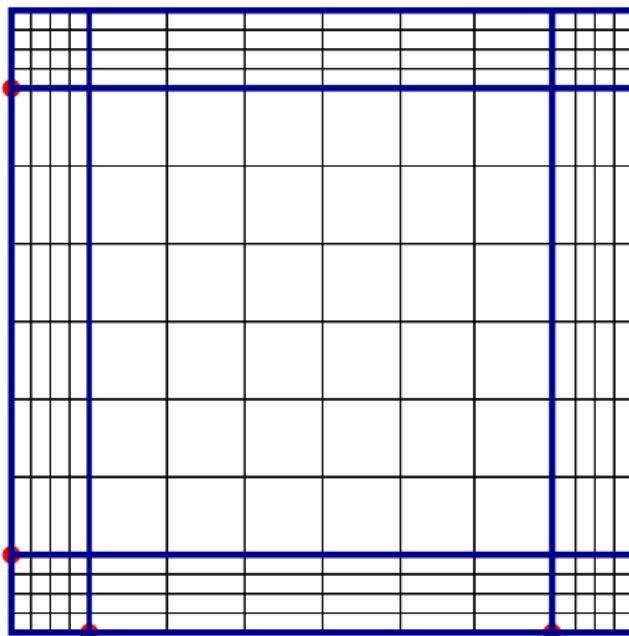
$$\begin{aligned}(L^N U)_{i,j} &= f(x_i, y_j) && \text{for } (x_i, y_j) \in \Omega^N, \\ U_{i,j} &= g(x_i, y_i) && \text{for } (x_i, y_j) \in \partial\Omega^N.\end{aligned}\tag{5}$$

Solving such linear systems is interesting and challenging. It will be the topic of Lecture 10 tomorrow.

Discretization

A piecewise uniform ("Shishkin") mesh

Define $\tau_\varepsilon = \min \left\{ \frac{1}{4}, 2 \frac{\varepsilon}{\beta} \ln N \right\}$, and construct $\bar{\Omega}_x^N$ and $\bar{\Omega}_y^N$ to be Shishkin meshes as before.



For the method and mesh, we would like to prove that

$$\|u - U\|_{\Omega^N} \leq C(N^{-1} \ln N)^2.$$

However, we shall show some restraint, and prove the easiest part of this: for the regular part.

But we will at least focus on how, without greatly complicating the analysis, we may show *almost second-order convergence*, compared to the first-order convergence we obtained for the scalar problem.

That is, assume there exists a decomposition of the discrete solution U :

$$U = V + \sum_{i=1}^4 W_i + \sum_{i=1}^4 Z_i.$$

We will just estimate $\|v - V\|_{\bar{\Omega}^N}$. The idea used is originally from [Miller et al., 1998], though the version given here is exactly from [Clavero et al., 2005].

Analysis (regular part only)

We need only a bound for the truncation error. Standard arguments give

$$|L^N(u-u)(x_i, y_j)| \leq \begin{cases} C\varepsilon^2 (\bar{h}_i \|\frac{\partial^3}{\partial x^3} u\| + \bar{k}_j \|\frac{\partial^3}{\partial y^3} u\|) & x_i, y_j \in \{\tau_\varepsilon, 1-\tau_\varepsilon\} \\ C\varepsilon^2 (\bar{h}_i^2 \|\frac{\partial^4}{\partial x^4} u\| + \bar{k}_j^2 \|\frac{\partial^4}{\partial y^4} u\|) & \text{otherwise.} \end{cases}$$

From this

$$|L^N(V-v)(x_i, y_j)| \leq \begin{cases} C\varepsilon N^{-1} & x_i, y_j \in \{\tau_\varepsilon, 1-\tau_\varepsilon\} \\ CN^{-2}. & \end{cases}$$

Define the barrier function

$$\Phi(x_i, y_j) = C \frac{(\tau_\varepsilon)^2}{\varepsilon^2} N^{-2} (\Theta(x_i) + \Theta(y_j)) + CN^{-2},$$

where Θ is the piecewise linear function interpolating the points

$$\left\{ (0, 0), (\tau_\varepsilon, 1), (1 - \tau_\varepsilon, 1), (1, 0) \right\}.$$

Analysis (regular part only)

Define the barrier function

$$\Phi(x_i, y_j) = C \frac{(\tau_\varepsilon)^2}{\varepsilon^2} N^{-2} (\Theta(x_i) + \Theta(y_j)) + CN^{-2},$$

where Θ is the piecewise linear function interpolating the points

$$\left\{ (0, 0), (\tau_\varepsilon, 1), (1 - \tau_\varepsilon, 1), (1, 0) \right\}.$$

Then, for example,

$$\delta_x^2 \Theta(x) = \begin{cases} -N/\tau_\varepsilon & x \in \{\tau_\varepsilon, 1 - \tau_\varepsilon\} \\ 0 & \text{otherwise.} \end{cases}$$

Analysis (regular part only)

It follows directly that

$$0 \leq \Phi(x_i, y_i) \leq CN^{-2} \ln^2 N,$$

and

$$|L^N \Phi(x_i, y_j)| \leq \begin{cases} C\tau_\varepsilon N^{-1} + (b\Phi)(x_i, y_j) & x_i, y_j \in \{\tau_\varepsilon, 1 - \tau\} \\ (b\Phi)(x_i, y_j) & \text{otherwise.} \end{cases}$$

Application of a maximum principle gives

$$\|v - V\|_{\bar{\Omega}^N} \leq CN^{-2} \ln^2 N.$$

The remaining analysis for $\|w_i - W_i\|_{\bar{\Omega}^N}$ and $\|z_i - Z_i\|_{\bar{\Omega}^N}$ is quite involved, and the details are not presented here.

However, in the next section of this short course, we'll look at the analysis of such terms when studying a *finite element method*.

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