CS4423: Networks

Week 6, Part 1: Centrality Measures

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These notes were revised after the lecture on 20 Feb, to correct for a mathematical error in Slide 14; see Slide 15.

These slides include material by Angela Carnevale.

Outline

Today's notes are split between these slides, and a Jupyter Notebook.

Slides are at:

https://www.niallmadden.ie/2425-CS4423

Centrality Measures

What is it that makes a node in a network important?

Key nodes in networks can be identified through **centrality measures**: a way of assigning "scores" to nodes that represents their "importance". However, what it means to be central depends on the context.

Examples

- In a friendship network, who is most popular?
- In a epidemiology network, who is most likely to get infected?
- In a banking, which institution poses the greatest danger to the system should it fail?

Accordingly, in the context of network analysis, a variety of different centrality measures have been developed.

Centrality Measures

Measures of centrality include:

- **Degree Centrality** which is just the degree of the node. It can be important in e.g., transport networks.
- **Eigenvector Centrality**, defined in terms of properties of the network's **adjacency matrix**.
- **Closeness Centrality**, defined in terms of a nodes **distance** to other nodes on the network.
- **Betweenness Centrality**, defined in terms of **shortest paths**.

Degree Centrality

Definition (Degree Centrality)

In a (simple) graph $G = (X, E)$, with $X = \{0, \dots, n-1\}$ and adjacency matrix $A = (a_{ij})$, the degree centrality c_i^D of node $i \in X$ is defined as

$$c_i^D = k_i = \sum_j a_{ij},$$

where k_i is the degree of node i.

Example:

In some cases, this measure can be misleading, since it depends–among other things–on the order of the graph. A better measure is then the following.

Normalized Degree Centrality

The **normalized degree centrality** C_i^D of node $i \in X$ is defined as

$$C_i^D = \frac{k_i}{n-1} = \frac{c_i^D}{n-1} \left(= \frac{\text{degree centrality of node } i}{\text{number of potential neighbors of } i} \right).$$

Note: in a directed graph one distinguishes between the in-degree and the out-degree of a node and defines the in-degree centrality and the out-degree centrality accordingly.

We now recall from important facts from **Linear Algebra**.

Eigenvalues and Eigenvectors

Let A be a square $n \times n$ matrix. An n-dimensional vector, $\boldsymbol{v}$, is called an **eigenvector** of A if

$$A\mathsf{v} = \lambda \mathsf{v}$$

for some scalar (number), λ, which is called an **eigenvalue** of A.

Example:

- When is a real-valued matrix, one usually finds that λ and $\boldsymbol{v}$ are *complex valued*. However, if A is symmetric then they are *real valued*.
- A may have up to n eigenvalues: λ_1, λ_2, …, λ_n.
- The **spectral radius** of A is $\rho(A) := \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$
- If v is an eigenvector associated with the eigenvalue λ, so too is any non-zero multiple of v

Centrality

The basic idea of eigenvector centrality is that **a node's ranking in a network should relate to the rankings of the nodes it is connected to**.

More specifically, up to some scalar λ, the centrality c_i^E of node i should be equal to the sum of the centralities c_j^E of its neighbouring nodes j.

In terms of the adjacency matrix $A = (a_{ij})$, this relationship is expressed as

$$\lambda c_i^E = \sum_j a_{ij} c_j^E,$$

which in turn, in matrix language is

$$\lambda c^E = A c^E,$$

for the vector $c^E = (c_i^E)$, which then is an eigenvector of A.

So c^E is an eigenvector of A! (But which one???)

Centrality

How to find c^E and/or λ?

If the network is small, one could use the usual method (although it is almost never a good idea)

1. Find the *characteristic polynomial* $p_A(x)$ of A, as *determinant* of the matrix $xI - A$, where I is the $n \times n$ identity matrix);
2. Find the *roots* λ of $p_A(x)$ (i.e. scalars λ such that $p_A(\lambda) = 0$);
3. Find a *nontrivial solution* of the linear system $(\lambda I - A)c = 0$ (where 0 stands for an all-0 column vector, and $c = (c_1, \ldots, c_n)$ is a column of *unknowns*).

For large networks, there are much better algorithms, such as the **Power Method** that we'll study later (in the Week 6 – Part 3 Jupyter Notebook).

Presently, we'll learn that the adjacency matrix always has one eigenvalue which is greater than all the others.

First, some definitions:

Irreducible Matrix

A matrix A is called **reducible** if, for some simultaneous permutation of its rows and columns, it has the block form

$$P^T A P = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices, and P is some permutation matrix A is **irreducible** if it is not reducible.

Some important points:

- The use a permutation matrix is not a major technical detail: it is just the same as swapping around some of the node labels.
- **Important:** The adjacency matrix of a simple graph G is irreducible if and only if G is connected.
- When A is reducible, and written in the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

then the (square) matrices A_{11} and A_{22} are the adjacency matrices of components of A.

Non-negative matrix

A matrix $A = (a_{ij})$ is **non-negative** if

$$a_{ij} \geq 0 \qquad \text{for all } i, j.$$

For simplicity, we usually write $A \geq 0$.

Important: Adjacency matrices are examples non-negative matrices.

There are similar concepts of, say, positive matrices (nothing to do with positive definite!!), negative matrices.

Of particular importance are **positive vectors**: $v = (v_i)$ is positive if $v_i > 0$ for all i. We write $v > 0$.

The Theorem

Theorem (Perron-Frobenius Theorem 1907/1912)

Suppose that A is a square, nonnegative, **irreducible** *matrix. Then:*

- *A has a real eigenvalue $\lambda = \rho(A)$ and $\lambda \geq |\lambda'|$ for any other eigenvalue λ' of A. λ is called the* **Perron root** *of A*
- *λ is a simple root of the characteristic polynomial of A (so has just one corresponding eigenvector)*
- *There is an eigenvector, $\boldsymbol{v}$ associated with λ, such that $v > 0$.*

For us this means:

(i) The adjacency matrix of a connected graph has an eigenvalue that positive; no other eigenvalue is greater in magnitude. (*See next page*)

(ii) It has an eigenvector, $\boldsymbol{v}$ that is positive.

(iii) v_i is the Eigenvector Centrality node i.

CORRECTION

The version of Slide 14 shown in class had two errors:

(a) In the statement of the theorem, it claimed that the Perron Root, λ, satisfied $|\lambda| > |\lambda'|$ where $\lambda'|$ is any other eigenvalue of A. It should have read $|\lambda| \geq |\lambda'|$.

(b) This error was repeated in the first statement after the theorem. That has been corrected (see text in red)

Interestingly, the statement that $|\lambda| > |\lambda'|$ would be true if either of the following is true:

1. The matrix A is positive (which is never the case for the adjacency matrix of a simple graph: no loops means that the diagonal entry is always zero.
2. If there is some number k such that $A^k > 0$. That can happen. Will discuss more in class next week.