OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

School of Mathematical and Statistical Sciences
University of Galway

The 21st Workshop on Numerical Methods for Problems with Layer Phenomena

24th and 25th of April 2025

Book of Abstracts DRAFT

Workshop Organisers:

- Niall Madden (Chair)
- Nanda Poddar
- Jekaterina Mosalska
- Sean Tobin

We gratefully acknowledge the support of the the Irish Mathematical Society, and the University of Galway School of Mathematical and Statistical Sciences.

An Evolve-Filter-Relax Regularized Reduced Order Model for Buoyancy-Driven Flows

Medine Demir
Weierstrass Institute for Applied Analysis and Stochastic
10117 Berlin, Germany

demir@wias-berlin.de

The a priori error analysis of reduced order models (ROMs) for buoyancy-driven flows is relatively scarce. In this study, we take a step in this direction and conduct numerical analysis of the evolve-filter-relax ROM (EFR-ROM), which uses spatial filtering to stabilize ROMs for convection-dominated flows. This study extends the EFR-ROM model of [1] for the Navier-Stokes equations to the Boussinesq equations with the spectral element discretization framework. Specifically, we prove stability, and an a priori error bound for the EFR-ROM. Our numerical investigation shows that the theoretical convergence rates are recovered numerically. In addition, we show that EFR-ROM yields more accurate solutions and quantity of interest than the Galerkin-ROM (G-ROM) in two test problems.

This is joint work with Ping-Hsuan Tsai (VA, USA).

References

[1] Strazzullo, M., Girfoglio, M., Ballarin, F., Iliescu, T., Rozza, G.: Consistency of the full and reduced order models for evolve-filter-relax regularization of convection-dominated, marginally-resolved flows, INT J NUMER MODEL EL, **123**(14), 3148–3178 (2022)

On a posteriori estimation in the energy norm for convection-diffusion problems

Sebastian Franz
University of Technology Dresden, Germany
sebastian.franz@tu-dresden.de

We consider a singularly perturbed convection-diffusion problem

$$Lu = -\varepsilon\Delta u + b \cdot \nabla u + cu = f$$

in a domain $\Omega \subset \mathbb{R}^2$ with Dirichlet and Neumann boundary conditions. A result by Verfürth showed that the classical residual and jump estimators

$$\begin{aligned}\eta(T) &= \sqrt{\eta_{Vol}(T)^2 + \eta_{jump}(T)^2},\\ \eta_{Vol}(T) &= \alpha_T \|f - Lu_h\|_{L_2(T)},\\ \eta_{jump}(T) &= \sqrt{\beta_T\varepsilon}\|[\![u_h]\!]\|_{L_2(\partial T)}\end{aligned}$$

can be used as a posteriori error estimator. However, the associated norm contains a dual norm which is not computable.

We present a different norm that bounds this estimator, is efficient and computable. A mesh adaptation algorithm using this estimator can be applied to problems with boundary and interior layers to solve them and produce reliable results.

This is joint work with Natalia Kopteva (Limerick).

Pointwise-in-time error bounds for a fractional-derivative parabolic problem on quasi-graded meshes.

Seán Kelly
University of Limerick
sean.f.kelly@ul.ie

An initial-boundary value, subdiffusion problem involving a Caputo time derivative of fractional order $\alpha \in (0, 1)$ is considered. The solutions of which typically exhibit a singular behaviour at initial time. We propose an extension to the approach, by Kopteva and Meng [1], used to analyse the error of L1-type discretizations on both graded and uniform temporal meshes. We broaden the assumption on the regularity of the solution to incorporate more general solution behaviour, such that $|\delta_t^l u(\cdot, t)| \lesssim 1 + t^{\sigma - l}$ for some $\sigma \in (0, 1) \cup (1, 2)$ and any $l = 0, 1, 2$. Under this more general assumption on the solution, we give sharp pointwise-in-time error bounds on quasi-graded temporal meshes with arbitrary degree of grading (including uniform meshes, also considered by Li, Qin, and Zhang [2]). Extensions to the semilinar case will also be considered.

This is joint work with Professor Natalia Kopteva (University of Limerick).
Supported by the Science Foundation Ireland under Grant number 18/CRT/6049.

References

[1] N. Kopteva and X. Meng: Error Analysis for a fractional-derivative parabolic problem on quasi-graded meshes using barrier functions. SIAM J. Numer. Anal., 58, (2020)

[2] D. Li, H. Qin, and J. Zhang: Sharp pointwise-in-time error estimate of L1 scheme for non-linear subdiffusion equations J. Comput. Math., 42 (2024)

The Bound Preserving Method applied to the 2D Induction Heating Problem

Katherine MacKenzie
University of Strathclyde

katherine.mackenzie@strath.ac.uk

Induction heating is a process widely used in the metallurgical manufacturing industry to heat conductive materials. Using an alternating current with a very high frequency, a magnetic field generates a current in the material, which produces heat due to the Joule heating process. This current is concentrated in a very thin layer near the boundary of the material, and as such there is a boundary layer in the magnetic field. This creates a highly irregular source term ($f \in L^1(\Omega)$) in the heat equation, and in the time-dependent case, generates a boundary layer in temperature near $t = 0$.

In this talk, I will describe the application of the Nodally Bound Preserving Method [1] to the induction heating equations. This method is designed to satisfy given bounds on the solution and guarantees stability for meshes for which standard methods do not guarantee bound preservation. The main technical result shows that when imposing non-physical bounds on the discrete solution, the method converges to the best approximation in the infinite-dimensional constrained convex set. As a result, for the induction heating problem, where the bounds are not explicit, this leads to a method that converges without imposing a restriction on the mesh.

This is joint work with Gabriel R. Barrenechea.

Supported by a University of Strathclyde SEA scholarship with EPSRC and additional support by Bifrangi UK Ltd.

References

[1] Barrenechea, G. R., Georgoulis, E. H., Pryer, T., Veeser, A.,: A nodally bound-preserving finite element method. IMA J. Numer. Anal. **44**(4) 2198–2219 (2024)

A Stabilized Scheme for an Optimal Control Problem Governed by Convection–Diffusion–Reaction Equation

Christos Pervolianakis
Friedrich-Schiller-Universität Jena

christos.pervolianakis@uni-jena.de

It is well-known that convection-diffusion equations may exhibit layers, which can render standard finite element methods inadequate for accurately approximating the exact solution. These layers can cause issues such as spurious oscillations, violating physical properties of the solution. To address this, nonlinear discretizations have been developed that preserve the maximum principle of the solution and accurately capture the position of these layers.

In this talk, we will consider an optimal control problem on a bounded domain $\Omega \subset \mathbb{R}^2$, governed by a time–dependent convection–diffusion–reaction equation with pointwise control constraints. Following the optimize–then–discretize approach, the resulting optimality conditions yield a coupled system of two time-dependent convection–diffusion–reaction equations.

To stabilize the fully–discrete scheme derived from the optimality conditions, we employ the algebraic flux correction method. Additionally, we discuss the well-posedness of the resulting fully–discrete scheme and present a priori and residual-type a posteriori error estimates.

References

[1] C. Pervolianakis, "Numerical analysis of a stabilized scheme for an optimal control problem governed by a parabolic convection–diffusion equation," *arXiv preprint*, 2024. Available: https://arxiv.org/pdf/2412.21070.

Nodally bound-preserving discontinuous Galerkin methods for charge transport

Alex Trenam
Heriot–Watt University
A.Trenam@hw.ac.uk

Preserving the positivity of charge density variables is often critical to ensuring the well-posedness of models describing the transport of charged particles. A prototypical example is the coupled nonlinear *Poisson–Nernst–Planck (PNP)* equations, or *drift-diffusion* equations, which provide a continuum description for the two-way interaction between charged particle densities and an associated electric field. Motivated by the PNP system, and its extension to fluidic media through the *Navier–Stokes–PNP* model, in this talk I will present recent work [2] adopting the nodally bound-preserving method first introduced in [1] to the context of discontinuous Galerkin methods for charge transport.

This is joint work with Tristan Pryer (University of Bath) and Gabriel R. Barrenechea (University of Strathclyde).

Supported by the EPSRC grants EP/S023364/1, EP/X030067/1, EP/W026899/1 and the Leverhulme Trust Research Project Grant RPG-2021-238.

References

[1] Gabriel R. Barrenechea, Emmanuil H. Georgoulis, Tristan Pryer, Andreas Veeser: A nodally bound-preserving finite element method. IMA Journal of Numerical Analysis **44(4)**, 2198–2219 (2024)

[2] Gabriel R. Barrenechea, Tristan Pryer, Alex Trenam: A nodally bound-preserving discontinuous Galerkin method for the drift-diffusion equation. Accepted for publication in Journal of Computational and Applied Mathematics, preprint available at arXiv:2410.05040.

On the decomposition of the solution to reaction-diffusion two-point boundary value problems with data of finite regularity

Christos Xenophontos
University of Cyprus

xenophontos@ucy.ac.cy

We consider reaction-diffusion two-point boundary value problems with data of finite regularity, i.e. H^2. It is well known that the solution may be decomposed into a smooth part, two boundary layers at the endpoints, and a remainder. We provide a proof of the regularity of each term in the decomposition that *does not use the maximum principle*, but rather utilizes *exponentially weighted* spaces. Even though the end result is known, our method of proof may be extended to problems for which the maximum principle does not hold, e.g. fourth order problems, Reissner-Mindlin plate model, etc. Using our result, we show how the h version of the Finite Element Method (with piece-wise linears) on the exponentially graded (eXp) mesh from [2], converges uniformly at the optimal rate. The results presented in this talk appear in [1].

This is joint work with Ch. Schwab (ETH, Zürich).

References

[1] Schwab, Ch. and Xenophontos, C.: Neural Networks for Singular Perturbations II: Finite Regularity, in preparation (2025).

[2] Xenophontos, C.: Optimal mesh design for the finite element approximation of reaction-diffusion problems, Int. J. Numer. Meth. Eng., Vol. 53, No 4, pp. 929–943 (2002).

Gradient-robust finite element - finite volume scheme for the compressible Stokes equations

Marwa Zainelabdeen
WIAS Berlin

marwa@wias-berlin.de

We consider a steady compressible Stokes problem on a domain $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$, in primitive variables velocity, pressure and non-constant density $(\boldsymbol{u}, p, \varrho)$. A *barotropic flow* is assumed, where the pressure depends solely on the density under an exponential *equation of state* $p = c_M \varrho^\gamma$ for $\gamma \geq 1$.

A finite element scheme for the momentum balance, coupled to a finite volume discretization for the continuity equation $\nabla \cdot (\varrho \boldsymbol{u}) = 0$, was proposed in [1] for a linear equation of state $(\gamma = 1)$. In this talk, we present an extension of the scheme to the nonlinear equation of state $(\gamma > 1)$. The scheme satisfies several desired structural properties, namely stability, convergence, the preservation of non-negativity and mass constraints for the density, and gradient-robustness. The latter property is related to the locking phenomenon observed in incompressible flow at high Reynolds number regimes, which carries over to the compressible setting. To achieve gradient-robustness, we employed the reconstruction operator proposed in [2]. The structural properties of the scheme were tested using various numerical benchmark problems.

This is joint work with Volker John and Christian Merdon (Berlin).

References

[1] Akbas, M., Gallouët, T., Gassmann, A., Linke, A., Merdon, C.: A gradient-robust well-balanced scheme for the compressible isothermal Stokes problem In: Computer Methods in Applied Mechanics and Engineering **74**(367), 2024-06-20 (2020). https://doi.org/10.1016/j.cma.2020.113069

[2] Alexander Linke.: On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime In: Computer Methods in Applied Mechanics and Engineering **268**(782–800), 0045-7825 (2014). https://doi.org/10.1016/j.cma.2013.10.011