MA378: §1 Interpolation

§3 Interpolation Error Estimates

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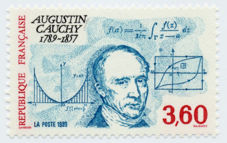
January 2023



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Augustin-Louis Cauchy (1789–1857), Paris, France. He was a pioneer of analysis, in particular in introducing rigour into calculus proofs. He founded the fields of complex analysis and the study of permutation groups.

3.1 Introduction

In our last example, we wrote down the polynomial of degree $n = 2$ interpolating $f(x) = e^x$ at $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

We now want to investigate how, in general, error in polynomial interpolation depends on

(i) the function (and its derivatives)

(ii) the number of points used (or, equivalently, degree of the polynomial used).

3.1 Introduction

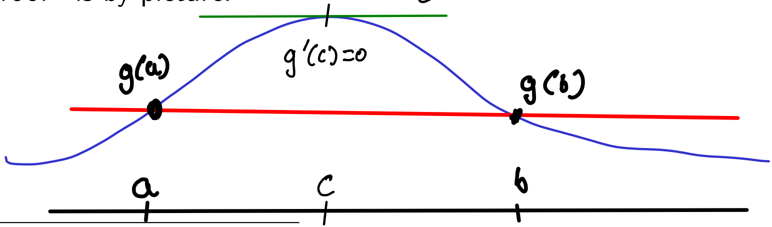
The main ingredient we need to the following theorem.

Theorem 3.1 (Rolle's Theorem)

Let g be a function that is continuous and differentiable on the interval $[a, b]$. If $g(a) = g(b)$, then there is at least one point c in (a, b) where $g'(c) = 0$.

Tangent to g has slope at c.

Our "proof" is by picture:[1]



[1] One can easily deduce Rolle's Theorem from the Mean Value Theorem (MVT). But since the standard proof of the MVT uses Rolle's Theorem, that would be cheating.

3.2 Error estimate for n=0

The simplest case is when $n = 0$, so the interpolant is a constant, i.e., it is p_0 interpolating a function f at a point x_0. Here is one way we can deduce the *interpolation error*.

See notes written on the board in class.

3.2 Error estimate for n=0

It is important to understand what this formula is telling us:

First, we don't know τ, but it is in $[x_0, x]$.

If f is constant, the error is zero (as it should be!) because $f'(x)=0$ $\forall x$.

The larger f' is, the larger the error.

The larger $|x-x_0|$, the larger the error.

Finally, although we assumed $x \neq x_0$, the formula holds in this case.

3.3 Error estimates for $n \geq 1$

The following is the most important theorem of NA2; it is used repeatedly through-out the semester. It's often called the *Polynomial Interpolation Error Theorem*, or *Cauchy's Theorem*.

First, we need to define an important polynomial.

Definition 3.2 (Nodal Polynomial)

The **Nodal Polynomial** π_{n+1} associated with the interpolation points that $a = x_0 < x_1 < \cdots < x_n = b$ is

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\ldots(x - x_n) = \prod_{i=0}^{n}(x - x_i).$$

3.3 Error estimates for $n \geq 1$

Theorem 3.3 (Cauchy, 1840)

Suppose that $n \geq 0$ and f is a real-valued function that is continuous and defined on $[a, b]$, such that the derivative of f of order $n+1$ exists and is continuous on $[a, b]$. Let p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x). \tag{1}$$

$f^{(k)} = \frac{d^k f}{dx^k}$ $\qquad f^{(0)} = f$

3.3 Error estimates for $n \geq 1$

Proof. First, suppose that $x = x_i$ for some $i = 0, 1, \ldots, n$.
Then $f(x_i) = p_n(x_i)$ so $f(x_i) - p_n(x_i) = 0$.
Also $\pi_{n+1}(x_i) = 0$. So the formula holds.

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3.3 Error estimates for $n \geq 1$

Next, take any $x \neq x_0$ and define the auxillary function g

$$g(t) = f(t) - p_n(t) - \left[\frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \right] \pi_{n+1}(t).$$

Then $g(x_i) = \underbrace{f(x_i) - p_n(x_i)}_{=0} - \left[\frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \right] \underbrace{\pi_{n+1}(x_i)}_{=0}$

$= 0$. So g has $n+1$ zeros.

Also, $g(x) = f(x) - p_n(x) - \left[f(x) - p_n(x) \right] \dfrac{\pi_{n+1}(x)}{\pi_{n+1}(x)} = 0$

So, in fact, g has $n+2$ zeros:

$\{x_0, x_1, \ldots, x_n, x\}$.

3.3 Error estimates for $n \geq 1$

Now by Rolles Theorem, between every pair of adjacent zeros, g' has a zero. There are $n+1$ of these.

By repeated application of Rolle's Theorem,

g'' has n distint zeros,

g''' has $n-1$ " ",

$\vdots$

$g^{(n+1)}$ has at least one zero. We'll call this point τ.

That is, there is a point $\tau \in [a, b]$ such that $g^{(n+1)}(\tau) = 0$. Thus,

$$f^{(n+1)}(\tau) - p_n^{(n+1)}(\tau) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}^{(n+1)}(\tau) = 0.$$

But p_n is a poly of degree n, so $p_n^{(n+1)}(x) = 0$ for all x. Furthermore

$\pi_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_{n+1})$. This is can be written as $\pi_{n+1}(x) = x^{n+1} + \sum_{i=0}^{n} c_i x^i$

So $\pi_{n+1}^{(n+1)}(x) = (n+1)!$ So now we have

$$f^{(n+1)}(\tau) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n+1)! = 0.$$

Reorrange to finish.

Example 3.4

In an earlier example, we wrote down the Lagrange form of the polynomial, p_2, that interpolates $f(x) = e^x$ at the points $\{-1, 0, 1\}$. Give a formula for $e^x - p_2(x)$.

$f(x) = e^x$. $\quad x_0 = -1, \quad x_1 = 0, \quad x_2 = 1, \quad n = 2.$

Note $\quad f^{(3)}(x) = e^x.$

Also $\quad \pi_3(x) = (x - x_0)(x - x_1)(x - x_2)$

$\quad = (x+1)(x)(x-1) = x^3 - x.$

So $\quad e^x - p_2(x) = \dfrac{e^\tau}{6}(x^3 - x) \quad$ for some $\tau \in [-1, 1]$.

Note: $\quad |e^x - p_2(x)| \leq \dfrac{e}{6}|x^3 - x|.$

3.3 Error estimates for $n \geq 1$

Usually (and as in the above example), we can't calculate $f(x) - p_n(x)$ exactly from Formula (1), because we have no way of finding τ. However, we are typically not so interested in what the error is at some given point, but what is the maximum error over the whole interval $[x_0, x_n]$. That is given by:

Corollary 3.5

Define

$$\underbrace{M_{n+1} = \max_{x_0 \leq \sigma \leq x_n} |f^{(n+1)}(\sigma)|.}\quad \geq \quad |f^{(n+1)}(\tau)|$$

Then

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|. \tag{2}$$

Example 3.6

Let p_1 be the polynomial of degree 1 that interpolates a function f at distinct points x_0 and x_1. Letting $h = x_1 - x_0$, show that

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

We know $|f(x) - p_1(x)| \leq \dfrac{M_2}{2} |\pi_2(x)|$

So $\displaystyle\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{M_2}{2} \max_{x_0 \leq x \leq x_1} |\pi_2(x)|.$

Here $\pi_2(x) = (x - x_0)(x - x_1) = x^2 - x(x_0 + x_1) + x_0 x_1$

$\max |\pi_2(x)|$ occurs where $\pi_2'(x) = 0$.

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max $|\pi_2(x)|$ occurs where $\pi_2'(x) = 0$.

$\pi_2'(x) = 2x - (x_0 + x_1)$. So

$\pi_2'(x) = 0$ when $x = \frac{x_0 + x_1}{2}$ (ie mid-point!)

Then $\pi_2\left(\frac{x_0+x_1}{2}\right) = \left(\frac{x_0+x_1}{2} - x_0\right)\left(\frac{x_0+x_1}{2} - x_1\right) = \left(\frac{x_1 - x_0}{2}\right)\left(\frac{x_0 - x_1}{2}\right)$

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$= \left(\frac{h}{2}\right)\left(-\frac{h}{2}\right).$

So $|\pi_2(x)| \leq \frac{h^2}{4}.$

So $\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{M_2}{8} h^2$