

## MA385 Part 1: Solving nonlinear equations

**1.5: Newton's Method**

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Sir Isaac Newton, 1643 - 1727, England. Easily one of the greatest scientist of all time. The method we are studying appeared in his celebrated *Principia Mathematica* in 1687, but it is believed he had used it as early as 1669.

# 0. Outline

- 1 Motivation
- 2 Newton's Method
- 3 Newton Error Formula
- 4 Applying the Newton Error Formula
- 5 Convergence of Newton's Method
- 6 Exercises
  - Some solutions

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For more details, see Section 1.4 (Relaxation and Newton's method) of [Süli and Mayers, \*An Introduction to Numerical Analysis\*](#)

Also, Chapter 3 of Epperson:

[https://search.library.nuigalway.ie/permalink/f/3b1kce/TN\\_cdi\\_askewsholts\\_vlebooks\\_9781118730966](https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9781118730966)

# 1. Motivation

Secant method can be written as

$$x_{k+1} = x_k - f(x_k)\phi(x_k, x_{k-1}),$$

where the function  $\phi$  is chosen so that  $x_{k+1}$  is the root of the secant line joining the points  $(x_{k-1}, f(x_{k-1}))$  and  $(x_k, f(x_k))$ .

A closely related idea leads to **Newton's Method**: set  $x_{k+1} = x_k - f(x_k)\lambda(x_k)$ , where we choose  $\lambda$  so that  $x_{k+1}$  is the zero of the tangent to  $f$  at  $(x_k, f(x_k))$ .

## 2. Newton's Method

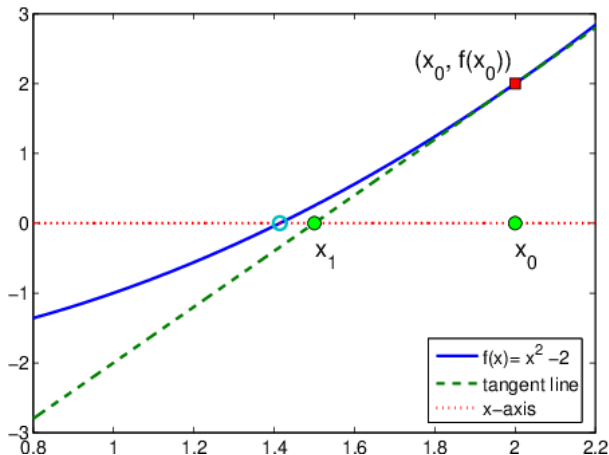


Figure 1: Estimating  $\sqrt{2}$  by solving  $x^2 - 2 = 0$  using Newton's Method

## 2. Newton's Method

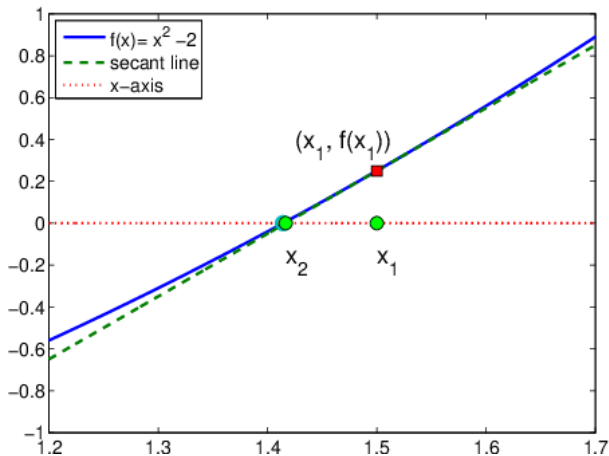


Figure 1: Estimating  $\sqrt{2}$  by solving  $x^2 - 2 = 0$  using Newton's Method

## 2. Newton's Method

The formula for Newton's method may be deduced writing down the equation for the line at  $(x_k, f(x_k))$  with slope  $f'(x_k)$ , and setting  $x_{k+1}$  to be its zero; see Exercise 1.5.1.

### Newton's Method

1. Choose any  $x_0$  in  $[a, b]$ ,
2. For  $k = 0, 1, \dots$ , set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \quad (1)$$

## 2. Newton's Method

### Example 1.5.1

*Use bisection, secant, and Newton's Method to solve  $x^2 - 2 = 0$  in  $[0, 2]$ .*

For this case, Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k},$$

which simplifies as

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

Taking  $x_0 = 2$ , we get  $x_1 = 3/2$ .

Then  $x_2 = 2x_1 + 1/x_1 = 17/12 = 1.46667$ .

Then  $x_3 = 1.4142$ , etc.

## 2. Newton's Method

Iter	Bisection	Secant	Newton
$k$	$ x_k - \tau $	$ x_k - \tau $	$ x_k - \tau $
0	1.41	1.41	5.86e-01
1	5.86e-01	5.86e-01	8.58e-02
2	4.14e-01	4.14e-01	2.45e-03
3	8.58e-02	8.09e-02	2.12e-06
4	1.64e-01	1.44e-02	1.59e-12
5	3.92e-02	4.20e-04	2.34e-16
6	2.33e-02	2.12e-06	—
7	7.96e-03	3.16e-10	—
8	7.66e-03	4.44e-16	—
9	1.51e-04	—	—
10	3.76e-03	—	—
11	1.80e-03	—	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	5.72e-07	—	—

## 2. Newton's Method

Deriving Newton's method geometrically certainly has an intuitive appeal. However, to analyse the method, we need a more abstract derivation based on a **Truncated Taylor Series**.

$$\begin{aligned} f(x) = & f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(x_k) + \dots \\ & + \frac{(x - x_k)^n}{n!}f^{(n)}(x_k) + \frac{(x - x_k)^{n+1}}{(n+1)!}f^{(n+1)}(\eta_k) \end{aligned}$$

where  $\eta_k \in (x, x_k)$ . Truncate at the second term (i.e., take  $n = 1$ )...

## 2. Newton's Method

### Deriving Newton's Method

Take  $n = 1$ , and then

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(\eta_2)$$

### 3. Newton Error Formula

We now want to show that Newton's converges **quadratically**, that is, with (at least) **order  $q = 2$** . To do this, we need to

1. Write down a recursive formula for the error.
2. Show that it converges.
3. Then find the limit of  $\frac{|\tau - x_{k+1}|}{|\tau - x_k|^2}$ .

Step 2 is usually the crucial part.

There are two parts to the proof. The first involves deriving the so-called “**Newton Error formula**”.

We'll assume that the functions  $f$ ,  $f'$  and  $f''$  are defined and continuous on the an interval  $I_\delta = [\tau - \delta, \tau + \delta]$  around the root  $\tau$ .

The following proof is essentially the same as the above derivation (see also Section 3.6 of Epperson:

<https://search.library.nuigalway.ie/permalink/f/>

### 3. Newton Error Formula

#### Theorem 1.5.1 (Newton Error Formula)

If  $f(\tau) = 0$  and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point  $\eta_k$  between  $\tau$  and  $x_k$  such that

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2} \frac{f''(\eta_k)}{f'(x_k)}, \quad (2)$$

## 4. Applying the Newton Error Formula

The Newton Error Formula is important in theory (for proving that Newton's Method converges) and practice (to estimate the error when it is applied to specific problems).

In practical applications, we can use the (2) as follows. So suppose we are applying Newton's Method to solving  $f(x) = 0$  on  $[a, b]$ .

Denote the error at Step  $k$  by  $\varepsilon_k = |\tau - x_k|$ . Then we can deduce that

$$\varepsilon_{k+1} \leq \varepsilon_k^2 \frac{\max_{a \leq x \leq b} |f''(x)|}{2|f'(x_k)|}. \quad (3)$$

Then, using that  $\varepsilon_0 \leq |b - a|$ , (3) can be used repeatedly to bound  $\varepsilon_1$ ,  $\varepsilon_2$ , etc.

## 5. Convergence of Newton's Method

We'll now complete our analysis of this section by proving the convergence of Newton's method.

### Theorem 1.5.2

Let us suppose that  $f$  is a function such that

- ▶  $f$  is continuous and real-valued, with continuous  $f''$ , defined on some closed interval  $I_\delta = [\tau - \delta, \tau + \delta]$ ,
- ▶  $f(\tau) = 0$  and  $f''(\tau) \neq 0$ ,
- ▶ there is some positive constant  $A$  such that

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all } x, y \in I_\delta.$$

Let  $h = \min\{\delta, 1/A\}$ . If  $|\tau - x_0| \leq h$  then Newton's Method converges quadratically.

## 5. Convergence of Newton's Method

## 6. Exercises

### Exercise 1.5.1

Write down the equation of the line that is tangential to the function  $f$  at the point  $x_k$ . Give an expression for its zero. Hence show how to derive Newton's method.

### Exercise 1.5.2

- (i) Is it possible to construct a problem for which the bisection method will work, but Newton's method will fail? If so, give an example.
- (ii) Is it possible to construct a problem for which Newton's method will work, but bisection will fail? If so, give an example.

## 6. Exercises

### Exercise 1.5.3

- (i) Let  $q$  be your student ID number. Find  $k$  and  $m$  where  $k - 2$  is the remainder on dividing  $q$  by 4, and  $m - 2$  is the remainder on dividing  $q$  by 6.
- (ii) Show how Newton's method can be applied to estimate the positive real number  $m^{1/k}$ . That is, state the nonlinear equation you would solve, and give the formula for Newton's method, simplified as much as possible.
- (iii) Do three iterations by hand of Newton's Method for this problem, taking  $x_0 = 1$ .

## 6. Exercises

### Exercise 1.5.4

Suppose we want apply to Newton's method to solving  $f(x) = 0$  where  $f$  is such that  $|f''(x)| \leq 10$  and  $|f'(x)| \geq 2$  for all  $x$ . How close must  $x_0$  be to  $\tau$  for the method to converge?

## 6. Exercises

### Exercise 1.5.5

Here is (yet) another scheme called *Steffenson's Method*: Choose  $x_0 \in [a, b]$  and set

$$x_{k+1} = x_k - \frac{(f(x_k))^2}{f(x_k + f(x_k)) - f(x_k)} \text{ for } k = 0, 1, 2, \dots$$

It is remarkable because its convergence is quadratic, like Newton's, but does not require derivatives of  $f$ .

Show how the method can be derived from Newton's Method, using the formal definition of the derivative.

## 6. Exercises

### Exercise 1.5.6

(This is Exercise 1.6 from Süli and Mayers) The proof of the convergence of Newton's method given in Theorem 13 uses that  $f'(\tau) \neq 0$ . Suppose that it is the case that  $f'(\tau) = 0$ .

- (i) Starting from the Newton Error formula, show that

$$\tau - x_{k+1} = \frac{(\tau - x_k)}{2} \frac{f''(\eta_k)}{f''(\mu_k)},$$

for some  $\mu_k$  between  $\tau$  and  $x_k$ . (*Hint: try using the MVT*).

- (ii) What does the above error formula tell us about the convergence of Newton's method in this case?

**Exer 1.5.2** This question is really intended to prompt discussion, e.g., in a tutorial. This is because we need a careful understanding regarding what constitutes a solution.

**Exer 1.5.2 Part(i):** If we take the most limited definition of the problem: “*there is a continuous function  $f$ , and real numbers  $a$  and  $b$  such that  $f(a)f(b) \leq 0$* ”, then we can answer (i) as “yes”. In this situation, bisection is proven to work. However, Newton’s method requires that, for any  $x_i$ , we have  $f'(x_i) \neq 0$ .

For example, take  $f(x) = x^2 - 4x + 2$ , with  $a = 0$ ,  $b = 2$ . Then  $f(0) = 2$  and  $f(2) = -2$ , so there is a solution on  $[0, 2]$ . (In fact, it is  $\tau = \sqrt{2} \approx 0.5858$ ).

We know, from theory that Bisection works (if you try by hand, with  $x_0 = 0$ , and  $x_1 = 2$ , you get the sequence  $\{0, 2, 1, 0.5, 0.75, 0.675, 0.5625, \dots\}$ ).

However, if we try Newton’s method with  $x_0 = 2$ , we get  $f'(x_0) = 0$ , so the method will fail.

**Exer 1.5.2 (ii):** Very pedantically, if we insist that the problem is defined a function,  $f$  and two points  $a$  and  $b$ , such that  $f(a)f(b) \leq 0$ , (we say “ $a$  and  $b$  bracket the solution”) then bisection cannot fail. However, suppose we just

state that there is a function  $f$  and two points  $a$  and  $b$  such that there is  $\tau \in [a, b]$  such that  $f(\tau) = 0$ , then bisection can fail.

This could happen if, for example, there were two solutions in  $[a, b]$  so  $f(a)f(b) > 0$ . Take, for example,  $f(x) = x^3 - 4x^2 + 2x$  and  $a = -1$ ,  $b = 1$ .