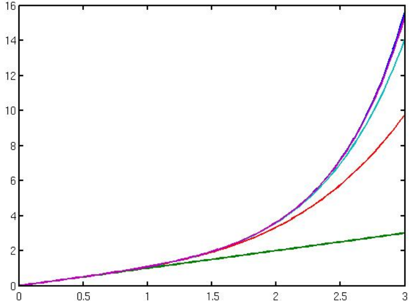
MA211

Lecture 15: Nonhomogeneous 2^{nd}-order DEs (final part). Series Solutions

Wed 29^{th} October 2008



Today...

1. f is the product of two functions
   - $f(x) = P(x)e^{Tx}$

2. $f(x) = P(x)\sin(x)$ or $P(x)\cos(x)$

3. Power Series

For further details and examples, look at the section on *Nonhomogeneous Linear Equations*, Section 17.2 of Stewart *Calculus: early transcendentals*.

So far we have studied how to find general solutions to the following types of problems:

$$ay'' + by' + cy = 1 + 2x^2 + 3x^4.$$
$$ay'' + by' + cy = e^{-x/2}.$$
$$ay'' + by' + cy = \sin(3x).$$

Then we moved onto ones of the form:

$$ay'' + by' + cy = P(x) + e^{Tx} + \cos(\omega x),$$

proceeding by solving each of

$$ah'' + bh' + ch = 0;$$
$$au'' + bu' + cu = P(x);$$
$$av'' + bv' + cv = e^{Tx}.$$
$$aw'' + bw' + cw = \cos(\omega x).$$

Then the general solution will be

$$y(x) = h(x) + u(x) + v(x) + w(x).$$

f is the product of two functions

Finally we will consider how to solve problems with the form:

$$ay'' + by' + cy = P(x)e^{Tx},$$
$$av'' + bv' + cv = P(x)\sin(Tx).$$

where T is some real number and $P(x)$ is a polynomial of degree x.

f is the product of two functions $\quad f(x) = P(x)e^{Tx}$

To find the solution to

$$ay'' + by' + cy = P(x)e^{Tx}$$

where P is a polynomial of degree n.

- Let h be the general solution to $ah'' + bh' + ch = 0$.
- Let u be one of
  - $(q_0 + q_1 x + \cdots + q_n x^n)e^{Tx}$, if T is not a solution to the auxiliary equation.
  - $(q_0 + q_1 x + \cdots + q_n x^n)xe^{Tx}$, if the auxiliary equation has two solutions, one of which is T.
  - $(q_0 + q_1 x + \cdots + q_n x^n)x^2e^{Tx}$, if T is the only solution to the auxiliary equation.
- Substitute u into

$$au'' + bu' + cu' = P(x)e^{Tx},$$

divide by e^{Tx} and solve for q_n, q_{n-1}, ..., q_0.

Example

Find the general solution to the non-homogeneous problem:

$$y'' - 4y' + 4y = x^2 e^x.$$

To find the solution to

$$ay'' + by' + cy = P(x)\cos(Tx)$$

where P is a polynomial of degree n.

- Let h be the general solution to
$$ah'' + bh' + ch = 0.$$
- Let $u = (A_0 + A_1 x \cdots + A_n x^n)\cos(Tx)$
$$+(B_0 + B_1 x \cdots + B_n x^n)\sin(Tx).$$
- Substitute u into the DE. Extract the equations for $\cos(Tx)$ and $\sin(Tx)$.
- Solve for A_n, A_{n-1}, ..., A_0; B_n, B_{n-1}, ..., B_0;

$f(x) = P(x)\sin(x)$ or $P(x)\cos(x)$

Example (Q4, Autumn, 06/07)

Find the general solution to the non-homogeneous problem:

$$y'' + 2y = x\cos(x).$$

Exercise (Q15.1)

Find general solutions to the following differential equations:

1. $y'' - 2y' + y = x + 1 + \sin(x)$
2. $y'' - 4y = x\sin(2x)$.
3. $y'' + 4y = 5xe^{-x}$.

Power Series

We conclude this section with a note about **Power Series**.

We have a way of solving explicitly for problems with *constant coefficients*:

$$ay'' + by' + cy = 0.$$

But for more general problems, such as,

$$y'' + 2xy' + y = 0,$$

we have no such method.

However, we can find a very good *approximation* by using a **Power Series**.

Power Series

Power Series

The key idea is that we suppose that we can write y as

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

The general solution will always have arbitrary constants, so we let these be c_0 and c_1.

Then we substitute the power series is into the differential equation, and get equations for c_2, c_3, c_4, ...

The more terms we take, the more accurate the solution is.

Power Series

Example

Find a Series Solution to

$$y'' + y = 0,$$

where $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$.

This last example is not very typical. Usually we want a formula for *all* of the coefficients c_2, c_3, c_4, ...

Typically, for a second order problem, we get a formula for c_k in terms of $c_k - 2$ that is called *recurrence relation*.

Power Series

Example

Find a recurrence relation for the coefficients of $c_0, c_1, c_2, \ldots$, of the series solution $y = \sum_{n=0}^{\infty} c_n x^n$ to

$$y'' + y = 0.$$