2425-MA140 Engineering Calculus

Week 04, Lecture 1
Introduction to Derivatives

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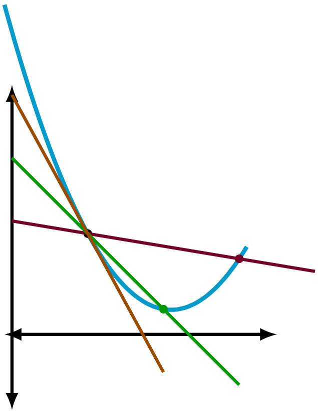


What we'll study today

Further reading:

- Sections 3.1 and 3.2 of **Calculus** by Strang & Herman: `https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative`
- Nice animation: `https://www.geogebra.org/m/MeMdCUEm`

Remember:

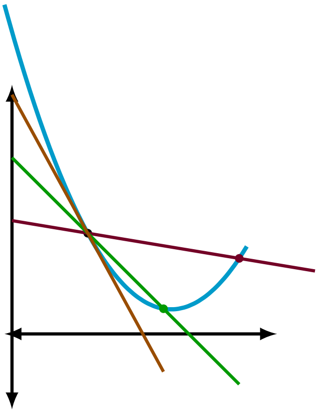
Reminders

- **Assignment 1** finished yesterday at 5pm. Grades are available on Canvas
- Assignment 2 is open; deadline is 5pm, 13 Oct. You can access it at `https://universityofgalway.instructure.com/courses/46734/assignments/129715`. (Or: go to Canvas, click on `Assignments` ... `Problem Set 2` ... the bottom of the page, click `Load Problem Set 2 in a new window`
- This week's **Tutorial Sheet** is available at `https://universityofgalway.instructure.com/courses/46734/files/2883465?module_item_id=943734`
- The first (of two) class tests will take place next Tuesday, 14th October. I'll be in touch about accommodations for those who completed the request form.

Derivative at a point

At the end of the last class, we visualised a function, f, with a line intersecting it at the points $(a, f(x)$ and $(b, f(b))$, where $b = a + h$

We moved the point b closer and closer to a (by taking smaller and smaller h, until the intersecting line eventually became the tangent to f at $x = a$.



Conclude: the slope of the tangent to f at $x = a$ is the limit:

$$\lim_{h\to 0} \frac{f(x+h) - f(a)}{h}.$$

The slope of the curve $y = f(x)$ at the point $P = (a, f(a))$ is given by the number (if it exists)

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, it is called the **derivative of f at $x = a$** and we denote it by $f'(a)$.

Definition: derivative at a point

Let $f(x)$ be a function that has $x = a$ in its domain. The **derivative** of the function $f(x)$ at a, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

Some terminology

$f'(a)$ exists then we say that function f is **differentiable at $x = a$**.

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the "limit" formula, we are doing "**differentiation from first principles"**.

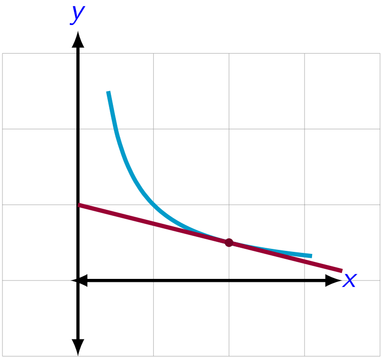
Example

Use the limit definition of a derivative to compute the slope of the tangent to $f(x) = x^2$ at $x = 3$.

Example

Use the limit definition of a derivative to find the equation of the tangent to $f(x) = 1/x$ at $x = 2$.

$$\boxed{f(x) = \frac{1}{x}} \text{ and } \boxed{y = 1 - \frac{x}{4}}$$



Derivative as a function

We've seen how to compute $f'(a)$: the derivative of the function f at a given point, $x = a$.

But if $f'(a)$ has a value for all $x = a$ (in the domain of $f(x)$), we can think $f'(x)$ as a function itself!

Definition: derivative as a function

Let f be a function. The derivative function, denoted f'. is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative as a function

Terminology and notation

- We usually refer to f' simply as the derivative of $f(x)$.
- Where $y = f(x)$, we often we write f' as $\dfrac{dy}{dx}$, or y', or $\dfrac{d}{dx}(f)(x)$.

Derivative as a function

Example

Use the above definition to find the derivative of $f(x) = x^2$.

Solution

The derivative is defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Here $f(x+h) = (x+h)^2 = x^2 + h^2 + 2hx,$ so we get:

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x^2 + h^2 + 2hx) - x^2}{h} \\ &= \lim_{h \to 0} \frac{h(h+2x)}{h} = \lim_{h \to 0} (h + 2x) = 2x \end{aligned}$$

Derivative as a function

Example

Use the "limit" definition to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

Derivative as a function

Consider the absolute value function $f(x) = |x|$. What is its derivative at (i) $x = 2$, (ii) $x = -3$, or (iii) $x = 0$?

Derivative as a function

Show that $\dfrac{d}{dx}(\sin x) = \cos x$.

Solution: We need to evaluate

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h},$$

where $f(x) = \sin(x)$. From p5 of the "log" tables, we have that
$\sin(A) - \sin(B) = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$.
Here $A = x + h$, and $B = x$, so
$\sin(x+h) - \sin(x) = 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$.
So now we evaluate

$$f'(x) = \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \lim_{h \to 0} \frac{2}{h}\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right).$$

Derivative as a function

But

$$\lim_{h\to 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right) = \left(\lim_{h\to 0} \frac{2}{h} \sin\left(\frac{h}{2}\right)\right)\left(\lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right)\right).$$

We learned last week that,

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Taking $\theta = h/2$, we get that

$$\lim_{h\to 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) = 1.$$

And finally,

$$\lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right) = \cos(x).$$

and we are done!

Differentiation by rule

We've seen we can compute derivatives of some functions using the "limit" definition (i.e., **differentiation from first principles**). However, that approach is tedious, and unnecessary in many case.

Instead we can use a set of "**rules**" which makes the process much more efficient. These rules are themselves derived from the "limit" definition – but we don't have to use that every time.

The Constant Rule

If f is a constant function, i.e. $f(x) = c$ for all x, then:

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Why:

We've already deduced that

- The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- The derivative of $f(x) = x^{1/2}$ is $f'(x) = \frac{1}{2}x^{-1/2}$

These are particular examples of the **Power Rules**

The Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Examples Calculate the derivatives of the following functions

1. $f(x) = x^6$
2. $f(x) = \sqrt[3]{x}$

The constant multiple rule

Let $f(x)$ be any differentiable function, and let k be constant, then

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big).$$

Example: Find the derivative of $f(x) = 5x^4$.

The Sum and Difference Rules

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}\big(u(x)+v(x)\big)=\frac{d}{dx}\big(u(x)\big)+\frac{d}{dx}\big(v(x)\big).$$

Similarly, $\frac{d}{dx}\big(u(x)-v(x)\big)=\frac{d}{dx}\big(u(x)\big)-\frac{d}{dx}\big(v(x)\big).$

Example: Find the derivative of $f(x)=1+x+x^2$.

Actually, the "**Difference Rule**", which states that

$$\boxed{\frac{d}{dx}\big(u(x) - v(x)\big) = \frac{d}{dx}\big(u(x)\big) - \frac{d}{dx}\big(v(x)\big).}$$

can be combined by combining the **Sum Rule** and the **Constant Multiple Rule**.

Example

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$;

(b) The slope of the tangent line at $x = 2$;

(c) The equation of the tangent at $x = 2$.

(a) $f'(x) = -15x^2 + 6x - 9$

(b) The slope of the tangent line at $x = 2$ is $f'(2)$:

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57.$$

(c) The equation of the line with slope M and passing through a point (x_1, y_1) is

$$y - y_1 = M(x - x_1)$$

The y coordinate at $x = 2$ is

$$\begin{aligned} f(2) &= -5(2)^3 + 3(2)^2 - 9(2) + 7 \\ &= -5(8) + 3(4) - 18 + 7 \\ &= -40 + 12 - 18 + 7 \\ &= -39. \end{aligned}$$

So the tangent line passes through the point $(2, -39)$ and the slope of the line is -57.

Therefore, the equation of this line is $y + 39 = -57(x - 2)$

Ans: The equation of the tangent line is $x = 2$ is $y = 75 - 57x$.

Tomorrow's Rules

Tomorrow we'll focus on two more rules, with important applications:

- The **Product Rule** for computing the the derivative of the **product** of two functions.
- **The Quotient Rule** for differentiating the **ratio** of two functions.

Exercises

Exercises 4.1.1 (Based on Q2(a), 2019/2020)

Use the (limit) definition of a derivative to differentiate the function $f(x) = x^2 + 2$.

Exercise 4.1.2

Use the (limit) definition of a derivative to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.