CS4423: Homework Assignment 2: Part 1 [ANS] with solutions

Q1. Show that, for any network, its Laplacian is singular.

> **Answer:** All the row sums of L are zero. Therefore, if $e = (1,1,\ldots,1)^\mathsf{T}$, then $Le = 0$. Therefore, zero is an eigenvalue, since the "ones" vector is an eigenvector. Consequently, L has no inverse.

Q2. Explain why we cannot use Perron-Frobenious theory to deduce that the eigenvalue of L with largest modulus (i.e., absolute value) is positive.

> **Answer:** PF applies only to non-negative matrices, but L has negative entries.

Q3. Show that (in fact) all the eigenvalues of L are non-negative.

> **Answer:** Since $A = B^\mathsf{T}B$, if $Ax = \lambda x$, and assume $x^\mathsf{T}x = 1$. $\lambda = x^\mathsf{T}Ax = x^\mathsf{T}(B^\mathsf{T}B)x = (Bx)^\mathsf{T}(Bx) = y^\mathsf{T}y \geqslant 0$. However, it is enough to mention that L is diagonally dominant, or to appeal to Gerschgorin's Theorem.

Q4. Give an example of a network with at least five nodes for which traversal by Depth First Search (DFS) and Breath First Search (BFS) will give the nodes in the same order.
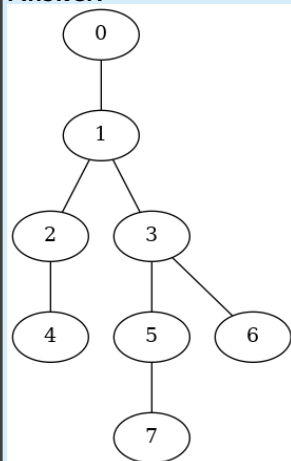
> **Answer:** Path graph on 5 nodes. I can't think of any other examples. If you can, let me know!

Q5. Let G_1 be the tree with Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$
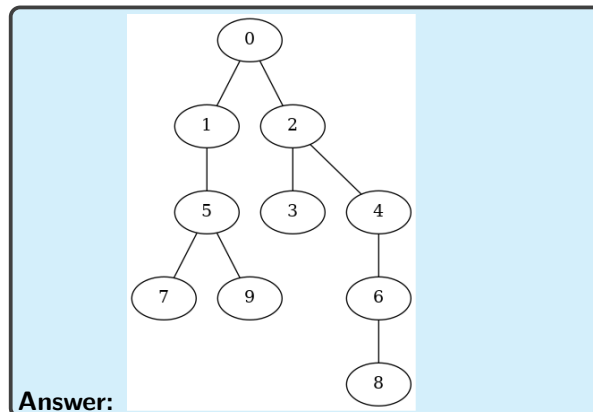
Sketch G_1.

> **Answer:**
>

Q6. Give the Prüfer code for G_1.

> **Answer:** [1, 2, 1, 3, 3, 5]

Q7. Sketch the tree, T_2, on the nodes $\{0, 1, \ldots, 9\}$ that has as its Prüfer code $[2, 5, 6, 4, 2, 0, 1, 5]$.

>
> **Answer:**

Q8. Give the order in which the nodes of T_2 would be visited if it is traversed by **Depth First Searh**.

> **Answer:** [0,2,4,6,8,3,1,5,7,9]. However, there are other acceptable correct answers: so long as a branch is followed until an end, and then you retrace until the last unexplored branch. Also, it should have been explicitly stated that one starts from Node 0. So it is OK to start from a different initial node.

Q9. Give the order in which the nodes of T_2 would be visited if it is traversed by *Breadth First Searh*.

> **Answer:** [0,1,2,5,3,4,7,9,6,8]
> There are other acceptable correct answers: as long as any node that is a distance k away from the initial node, is listed before one that is a distance greater than k it is OK. Also, it should have been explicitly stated that one starts from Node 0. So it is OK to start from a different initial node.