Some notes from the tutorial on Monday 22/Jan/2024

MA378 Chapter 1: Interpolation

Exercises (from 1.1, 1.2, and 1.3)

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1.0

*Submit carefully written solutions to Exercises 1.4$\star$, 2.3$\star$, 2.5$\star$, and **??**$\star$.*

Deadline: 5pm, Friday 9 February.
Your solutions must be clearly written, and neatly presented. You can submit an electronic copy, through Canvas, or a hard copy (ideally at the lecture on the 9th). Make sure pages of the hard copy are stapled together. Marks will be given for quality and clarity of exposition. Collaboration is encouraged; policy will be discussed in class.

Exercise 1.1

Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$. (with $m, n \geq 0$)

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

P_m is the space (or set) of polynomials of degree at most m. Eg, P_2 is set of all constant, linear & quadratic polys.

(a) answer: $\max\{\deg(p), \deg(q)\}$

Eg if $p = 1 + 2x$, $q = 3 + 4x - 5x^3$, $p + q = 4 + 6x - 5x^3$.

Both q and $p+q$ are cubic.

Exercise 1.1

Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

b Ans : 0 since we could have $p = q$

c Ans: deg(p) + deg(q). Eg If

$p = 1 + 2x$, $q = 3 + 4x^2$

Then $pq = (1+2x)(3+4x^2) = (3 + 4x^2 + 6x + 8x^3)$

Exercise 1.1

Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

d : Ans: Same! (?) if $\deg(p), \deg(q) \geq 1$.

But if, say, p is the zero polynomial, then so too is pq.

In which case $\deg(pq) = 0$.

Exercise 1.2

Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

Note: P_n is a vector space over the real numbers. Roughly this means that

if $p, q \in P_n$

Then $ap + bq \in P_n$ for any $a, b \in \mathbb{R}$.

Basis for P_n: $\{1, x, x^2, x^3, \dots, x^n\}$, since any P_n can be written as a linear combination of these. This is the "canonical" basis.

Exercise 1.2

Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

We know $\{b_0, b_1, b_2, \ldots, b_n\} = \{1, x, x^2, \ldots, x^n\}$ is a basis because any poly

$$p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

can be written as

$$p = a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

What about $\{b_0, b_1, \ldots, b_n\} = \{2, x, \ldots, x^n\}$?

Yes: $p = \frac{1}{2} a_0 b_0 + a_1 b_1 + \cdots + a_n b_n$.

Or $\{b_0, b_1, \ldots, b_n\} = \{1+x, 1-x, x^2, \ldots, x^n\}$

since $\frac{1}{2}(b_0 + b_1) = 1$ $\quad \frac{1}{2}(b_0 - b_1) = x$.

Exercise 1.3

(a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

Yes, it is always possible.

Eg $p(x) = y_0 + (x - x_0)$ is a poly of degree 1

and $p(x_0) = y_0$ ✓

But there are infinitely many others!

Any $p(x) = y_0 + c\,(x - x_0)$

for $c \in \mathbb{R}$ will do.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

If $x_0 = x_1$ and $y_0 = y_1$ this is the same as part (a).

If $x_0 = x_1$ and $y_0 \neq y_1$ then there is <u>no</u> solution since the polynomial would need to take 2 different values at the same point.

Otherwise, ie if $x_0 \neq x_1$, there is always exactly one solution:

$$p_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

Suppose $x_0 \neq x_1$, $x_1 \neq x_2$, $x_0 \neq x_2$. Can it be done?

Eg $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

And $y_0 = 0$ $y_1 = 1$ $y_2 = 2$.

Eg $p_1(x) = x$!

Exercise 1.4 ($\star$)

For each of the following interpolation problems, determine (with explanation) if there is no solution, exactly one solution, or more than one solution. In all cases p_n denotes a polynomial of degree (at most) n. You are not required to determine p_n where it exists.

(a) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(b) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = -2$.

(c) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(d) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(e) Find $p_2(x)$ that interpolates (x_0, y_0) and (x_1, y_1) where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$.

Exercise 2.1

The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

$$\text{Its determinant is : } \det(V_n) = \prod_{0 \le i < j \le n} (x_j - x_i). \tag{1}$$

Verify (1) for the 2×2 and 3×3 cases.

(Note that from Formula (1) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \ldots, x_n$ are all distinct.)

Exercise 2.2

Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute

$$\max_{0 \leq x \leq a} |f(x) - p_1(x)|.$$

Source: Chapter 6 of Süli and Mayers.

Exercise 2.3 ($\star$)

Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \quad \text{for all } x.$$

Exercise 2.4

Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0, 3)$, $(1, 2)$ and $(2, 4)$.

Exercise 2.5 ($\star$)

Show that all the following represent the same polynomial (usually called the "Chebyshev Polynomial of Degree 3"),
$T_3(x) = 4x^3 - 3x$.

(a) Horner form: $\big((4x + 0)x - 3\big)x + 0$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3}\left(\prod_{j=0,j\neq k}^{3}\frac{x-x_j}{x_k-x_j}\right)(-1)^{k+1}$, where $x_0=-1, x_1=-1/2, x_2=1/2, x_3=1$.

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and
$T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

Exercise 3.1

Read Section 6.2 of An Introduction to Numerical Analysis (Süli and Mayers). Pay particular attention to the proof of Thm 6.2 at https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=192.

Exercise 3.2

Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0 \leq x \leq x_2} |f(x) - p_2(x)| \leq \frac{1}{6}\frac{2}{3\sqrt{3}}h^3 M_3 = \frac{1}{9\sqrt{3}}h^3 M_3.$$

Hint: simplify the calculations by taking $t = x - x_1$, writing $(x - x_0)(x - x_1)(x - x_2)$ in terms of h and t.