2223-MA378: Sample Exercises for Class Test, Week 7: **Solutions**

1. *Let p_n be the polynomial of degree n that interpolates the function f at the distinct points $\{x_0, x_1, \ldots, x_N\}$. State Cauchy's Theorem for $f(x) - p_n(x)$. (You do* not *have to prove it).*
   Answer: Suppose that $n \geq 0$ and f is a real-valued function that is continuous and defined on $[a, b]$, such that the derivative of f of order $n+1$ exists and is continuous on $[a, b]$. Let p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that
   $$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x). \tag{1}$$

2. *Suppose that S is a natural cubic spline on $[0, 2]$ with*
   $$S(x) = \begin{cases} -x + 2(1-x) + a(1-x)^3 + \frac{2}{3}x^3, & \text{for } 0 \leq x < 1, \\ b(2-x) + c(2-x)^3 + d(x-1)^3, & \text{for } 1 \leq x \leq 2. \end{cases}$$
   Find a, b, c, and d.
   Solution: Differentiate to get
   $$S'(x) = \begin{cases} -3 - 3a(1-x)^2 + 2x^2, & \text{for } 0 \leq x < 1, \\ -b - 3c(2-x)^2 + 3d(x-1)^2, & \text{for } 1 \leq x \leq 2, \end{cases}$$
   and
   $$S''(x) = \begin{cases} 6a(1-x) + 4x, & \text{for } 0 \leq x < 1, \\ 6c(2-x) + 6d(x-1), & \text{for } 1 \leq x \leq 2. \end{cases}$$
   Since S is a *natural* spline,
   - $S''(0) = 0$, which gives that $a = 0$, and
   - $S''(2) = 0$, which gives that $d = 0$.

   To find b and c use any two of
   - S is continuous at $x = 1$, which gives $b + c = -1/3$,
   - S' is continuous at $x = 1$, which gives $b + 3c = 1$, and
   - S'' is continuous at $x = 1$, which gives $6c = 4$.

   That will give that $b = -1$, and $c = 2/3$.

3. *Suppose that S is the cubic spline interpolant to $f(x) = xe^{-x}$ on the $N+1$ equally spaced points $\{x_0 = 0 < x_1 < \cdots < x_N = 2\}$. We know that*
   $$\|f - S\| := \max_{0 \leq x \leq 2} |f - S| \leq \frac{5h^4}{384} \max_{0 \leq x \leq 2} |f^{(4)}(x)|,$$
   where $h = 2/N$.
   What value of N should one take to ensure that $\|f - S\|$ is no more than 10^{-8}.
   Solution: We have to find h such that
   $$\frac{5h^4}{384} \max_{0 \leq x \leq 2} |f^{(4)}(x)| \leq 10^{-8}.$$
   For this problem, $f^{(4)} = d^4 f/dx^4 = (x-4)e^{-x}$. This is negative, but increasing on $[0, 2]$, so
   $$\max_{0 \leq x \leq 2} |f^{(4)}(x)| = |f^{(4)}(0)| = 4.$$
   So we have to choose h such that
   $$h^4 \leq 10^{-8} \times \frac{384}{20} \approx 1.92 \times 10^{-7}.$$
   This gives $h \leq 0.02093$. Since $N = 2/h$, we get $N \geq 95.544$. So take $\mathbf{N = 96}$.

4. *Suppose that S is the natural cubic spline interpolant to a function g on $[-1, 1]$. If*
   $$\max_{-1 \leq x \leq 1} |g(x) - S(x)| = 0,$$
   what can we say about g? **Answer:** g must be a cubic polynomial, with $g''(-1) = g''(1) = 0$. (That is sufficient for a correct answer. With a little more work, you can show this means that, in fact, g is a polynomial of degree 1).