MA313 : Linear Algebra I

Week 4: Spanning sets and column spaces

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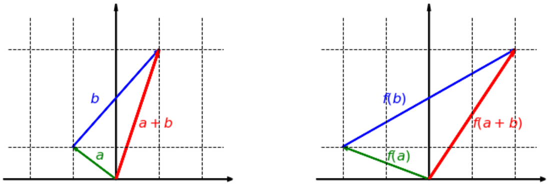
27 and 30 September, 2022



Adapted from https://commons.wikimedia.org/wiki/File:Streckung_der_Summe_zweier_Vektoren.gif

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details, see Section 4.2 of the text-book:
https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=5174425

Assignment 2

- Opened last Monday (19 Sep 2022).
- **Deadline:** 5pm, Friday 30 Sep 2022.
- It contributes 5% to the final grade for MA313.
- Tutorials continue Thursdays at 12 in IT206.

Communication Skills

1. Topics and Info posted on Blackboard and at https://www.niallmadden.ie/teaching/2223-MA313/22_23_Communication_Skills.pdf
2. Confirm your topic by 5pm, 26 September (Monday of Week 4). To that by first emailing Niall with your choice and, if agreed, entering in on Blackboard.

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Week 4: Spanning sets and column spaces

Start of ...

PART 1: Recall from last week

Part 1: Recall from last week

Linear combinations

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form $c_1 u_1 + \cdots + c_p u_p$ for scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

Span

The **span** of a set of vectors is the set of all possible liner combinations of them. That is, given vectors $u_1, \ldots, u_p$ in some vector space V, their **span** is

$$\mathrm{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

Part 1: Recall from last week

Subspaces

Given any set of a vectors in a vector space V, their span is a **subspace** of V.

Null space

Given a $m \times n$ matrix, A, its **null space** is the set of all vectors for which $Ax = 0$. That is:

$$\mathrm{Nul}\, A = \{x \in \mathbb{R}^n : Ax = 0\}.$$

- For some matrices, the only vector in the null space is the zero vector.
- The null space of an $m \times n$ matrix is itself a vector space (and so a subspace of $\mathbb{R}^N$).

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Start of ...

PART 2: Spanning Sets

Part 2: Spanning Sets

Definition (Spanning Set)

A **spanning set** of a vector space V is a collection of vectors in V whose span is all of V.

Equivalently, the set of vectors $\{v_1, \ldots, v_p\}$ in V form a spanning set if and only if every vector in V can be written as a linear combination of $v_1, \ldots, v_p$.

Example (A spanning set for $\mathbb{R}^2$)

The vectors

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

form a spanning set of $\mathbb{R}^2$.

Example (A spanning set for $\mathbb{R}^n$)

In the same way, for each $n \geqslant 1$, the vectors

$$\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \in \mathbb{R}^n$$

form a spanning set of $\mathbb{R}^n$.

Recall that $\mathbb{P}_n$ is the vector space of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

of degree n or less.

Example

$\mathbb{P}_n = \mathrm{span}\{1, t, \ldots, t^n\}$.

Recall: $M_{m\times n}$ is the vector space of all $m \times n$ matrices.

Example

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Important: Spanning sets are (in general) not unique.

Example (Another spanning set of $M_{2\times 2}$)

We also, for example,

$$M_{2\times 2} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

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Start of ...

PART 3: Column spaces

Part 3: Column spaces

Definition (COLUMN SPACE)

Let $A = [a_1 \cdots a_n]$ be an $m \times n$ matrix, where $a_1, \ldots, a_n \in \mathbb{R}^m$. That is, a_i is the ith column of A.
The **column space** of A is

$$\mathrm{Col}\, A := \mathrm{span}\{a_1, \ldots, a_n\}.$$

Note that $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$.

Part 3: Column spaces

Example

Let I_n be the $n \times n$ identity matrix.
Then $\mathbb{R}^n = \operatorname{Col} I_n$.

Part 3: Column spaces

Here is another way of thinking about the column space: we have already seen that Ax is a linear combination of the columns of A. So, ...

$$\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\}$$

and

$$\operatorname{Col} A = \{b \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \colon b = Ax\}.$$

Given a matrix A, we can construct two vector spaces:

$\operatorname{Nul} A$

- Easy to test membership: does $x \in \mathbb{R}^n$ belong to $\operatorname{Nul} A$?
- Not as easy to produce a (finite) spanning set.

$\operatorname{Col} A$

- Very easy to give a spanning set: it is how the space is defined!
- Not as easy to check to test membership.

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Start of ...

PART 4: Spanning sets of null spaces

Part 4: Spanning sets of $\operatorname{Nul} A$

Question

Given an $m \times n$ matrix A, can we find a finite spanning set of $\operatorname{Nul} A$?

That is, can we find vectors $v_1, \ldots, v_p \in \mathbb{R}^n$ such that those vectors $x \in \mathbb{R}^n$ with $Ax = 0$ are precisely the linear combinations

$$c_1 v_1 + \cdots + c_p v_p,$$

where $c_1, \ldots, c_p \in \mathbb{R}$?

To see the answer, we'll recall that the $Ax = b$ is just another way of writing a linear system of equations.

When we write

$$Ax = b$$

where A is an $n \times n$ matrix, and $x, b \in \mathbb{R}^n$, we mean

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This is the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Remember that we used solve such systems using "row reduction" (a.k.a., Gaussian Elimination): we rearrange the equations to get

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
\hat{a}_{22}x_2 + \hat{a}_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
\hat{a}_{33}x_3 + \cdots + \hat{a}_{3n}x_n &= b_2\\
&\vdots\\
\hat{a}_{nn}x_n &= b_n
\end{aligned}$$

This is done by so-called *elementary row operations*. And we do this because it is easy to solve this version.

Elementary row operations

Performing an **elementary row operation** on a matrix means:

- Multiply some row by a non-zero scalar.
- Add a scalar multiple of some row to another row.
- Interchange (i.e., swap) two rows.

Fact!

Let A' be obtained from A by performing an **elementary row operation**. The

$$\operatorname{Nul} A = \operatorname{Nul} A'.$$

Definition (Row Echelon Form)

A matrix is in **row echelon form** if

- all non-zero rows are above all zero rows and
- the **leading entry** (or "*pivot*") in a row is in a column to the right of the leading entry in the row above it.
- All entries in a column below a leading entry are zero.

Definition (Reduced Row Echelon Form)

A matrix is in **reduced row echelon form** if it is in row echelon form, and also

- Each leading entry is one;
- If a column contains a leading entry, all its other entries are zero.

Theorem and Definition

Using elementary row operations, *every* matrix A can be row reduced to obtain a **unique** matrix A' in reduced row echelon form. We call A' **the reduced row echelon form** of A.

It turns out that we can read off a spanning set of $\mathrm{Nul}\, A$ from the reduced row echelon form of A.

Example

Find a spanning set of $\operatorname{Nul} A$, where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

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Start of ...

PART 5: Checking column space

Part 5: Checking column space

Question

Given an $m \times n$ matrix A and $b \in \mathbb{R}^m$, how can we decide if $b \in \operatorname{Col} A$?

Since $\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\}$, this problem is equivalent to deciding whether there exists a solution $x \in \mathbb{R}^n$ to the system of linear equations

$$Ax = b.$$

Again, **row reduction** (a.k.a. **Gaussian elimination**) can be used for this purpose.

Part 5: Checking column space

Example

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$. Is $b \in \mathrm{Col}\, A$?

Part 5: Checking column space

Example (From 2018/19 exam paper)

Decide (with justification) if

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ belongs to the column space of } A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix}.$$

Answer: No!. Why? The RREF of

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ -1 & 3 & 5 & 4 & 2 \\ 2 & 1 & -3 & -1 & -1 \end{array}\right] \text{ is } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

So...

So now we know that, given an $m \times n$ matrix A, we can use **row reduction** to perform the following tasks:

- Construct a finite spanning set of $\mathrm{Nul}\, A$.
- Decide, for a given $b \in \mathbb{R}^m$, whether $b \in \mathrm{Col}\, A$.

But what has this to do with **vector spaces**?
Do these matrix computations (row reduction) and concepts (null spaces, column spaces) have analogues for general vector spaces?

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Start of ...

PART 6: Linear Transformations

Part 6: Linear Transformations

Definition (LINEAR TRANSFORMATIONS)

Let V and W be vector spaces. A **linear transformation** from V to W is a function $T\colon V \to W$ (i.e., a "rule" which assigns a unique $T(u) \in W$ to each $u \in V$) such that

- $T(u+v) = T(u) + T(v)$ for all $u, v \in V$ and
- $T(cu) = cT(u)$ for all $u \in V$ and $c \in \mathbb{R}$.

That is, a linear transformations is a function which "respects" (or "is compatible with") the vector space structures.

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Example

Determine if the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a *linear transformation*.

$$T_3\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Example

The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

defines a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

An important fact

Linear transformations preserve linear combinations: if $T\colon V \to W$ is a linear transformation, then

$$T(c_1 v_1 + \cdots + c_p v_p) = c_1 T(v_1) + \cdots + c_p T(v_p)$$

for all $v_1, \ldots, v_p \in V$ and $c_1, \ldots, c_p \in \mathbb{R}$.

Example (Matrices)

Let A be an $m \times n$ matrix. Define $T\colon \mathbb{R}^n \to \mathbb{R}^m$ via

$$T(x) = Ax \qquad (x \in \mathbb{R}^n).$$

Then T is a linear transformation.

Question

Are there any other linear transformations $\mathbb{R}^n \to \mathbb{R}^m$?

Answer: No! Linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ and $m \times n$ matrices are essentially the "same thing". What we mean is,

- Every $m \times n$ matrix defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can find a matrix that defines it.

The matrix of a linear transformation

Let e_i be the usual vector in $\mathbb{R}^n$ with 1 is row i, and zero everywhere else. Then the matrix for a given linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$A := [T(e_1) \cdots T(e_n)].$$

Why?

Since linear transformations are generalizations of matrices, we need the analogous idea of **null spaces** and **column spaces**.

Definition (KERNEL and RANGE of a linear transformation)

Let $T: V \to W$ be a linear transformation.

- The **kernel** of T is $\mathrm{Ker}\, T = \{u \in V : T(u) = 0\}$.
- The **range** (or *image*) of T is $\mathrm{Ran}\, T = \{T(u) : u \in V\}$.

Example

Let A be an $m \times n$ matrix. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Ax$ be the associated linear transformation. Then:

- $\mathrm{Ker}\, T = \{x \in \mathbb{R}^n : T(x) = Ax = 0\} = \mathrm{Nul}\, A$.
- $\mathrm{Ran}\, T = \{T(x) = Ax : x \in \mathbb{R}^n\} = \mathrm{Col}\, A$.

Theorem

Let $T: V \to W$ be a linear transformation. Then:

- $\mathrm{Ker}\, T$ *is a subspace of* V.
- $\mathrm{Ran}\, T$ *is a subspace of* W.

Here is another result, though the importance might not be clear yet.

Theorem

Let V be a vector space and let $H \subseteq V$ be a subspace.
Then there are vector spaces U and W and linear transformations $S\colon U \to V$ and $T\colon V \to W$ such that

$$\operatorname{Ran} S = H = \operatorname{Ker} T.$$

We essentially get S for free...

But some new ideas would be required to produce T.

Exercises

Q1. Construct a finite spanning set of each of the null space of each of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Exercises

Q2. Let

$$w = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

Determine whether w belongs to $\mathrm{Nul}\,A$ and whether w belongs to $\mathrm{Col}\,A$.

Q3. Let

$$w = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix}.$$

Determine whether w belongs to $\mathrm{Nul}\,A$ and whether w belongs to $\mathrm{Col}\,A$.

Q4. *4.2.30* Let $T\colon V \to W$ be a linear transformation from a vector space V to a vector space W.

Q1..1 Show that the kernel $\mathrm{Ker}\,T$ of T is a subspace of V.

Q2..2 Show that the range $\mathrm{Ran}\,T$ of T is a subspace of W.

Exercises

Q5. *4.2.31* Recall that $\mathbb{P}_n$ is the vector space of polynomials of the form $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ for $a_0, \ldots, a_n \in \mathbb{R}$. Define $T \colon \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

Q1..1 Show that T is a linear transformation.

Q2..2 Find a polynomial $p(t) \in \mathbb{P}_2$ with $\mathrm{Ker}\, T = \mathrm{span}\{p(t)\}$.

Q3..3 What is the range of T?

Q6. *4.2.32* Define $T \colon \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}.$$

Q1..1 Show that T is a linear transformation.

Q2..2 Find polynomials $p_1(t), p_2(t) \in \mathbb{P}_2$ with $\mathrm{Ker}\, T = \mathrm{span}\{p_1(t), p_2(t)\}$.

Q3..3 What is the range of T?

Exercises

Q7. *4.2.33* Recall that $M_{m\times n}$ denotes the vector space of $m \times n$ matrices with real entries. Further recall that $A^\top$ denotes the *transpose* of a matrix A. Define $T\colon M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^\top$.

- Q1..1 Show that T is a linear transformation.
- Q2..2 Show that the range of T consists precisely of those matrices $B \in M_{2\times 2}$ with $B = B^\top$. (Such matrices are called *symmetric*.)
- Q3..3 Describe the kernel of T.

Q8. *4.2.34* Recall that $C([a, b])$ denotes the vector space of all continuous functions $[a, b] \to \mathbb{R}$. Define $T\colon C([0, 1]) \to C([0, 1])$ as follows: for $f \in C([0, 1])$, let $T(f)$ be the antiderivative F of f with $F(0) = 0$. Show that T is a linear transformation.