2425-MA140 Engineering Calculus

Week 04, Lecture 1
Introduction to Derivatives

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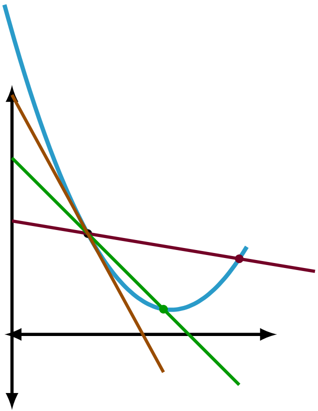


What we'll study today

Further reading:

- Sections 3.1 and 3.2 of **Calculus** by Strang & Herman: `https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative`
- Nice animation: `https://www.geogebra.org/m/MeMdCUEm`

Remember:

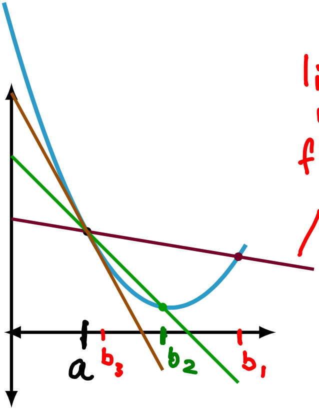
Reminders

- **Assignment 1** finished yesterday at 5pm. Grades are available on Canvas
- Assignment 2 is open; deadline is 5pm, 13 Oct. You can access it at `https://universityofgalway.instructure.com/courses/46734/assignments/129715`. (Or: go to Canvas, click on `Assignments` ... `Problem Set 2` ... the bottom of the page, click `Load Problem Set 2 in a new window` ✓
- This week's **Tutorial Sheet** is available at `https://universityofgalway.instructure.com/courses/46734/files/2883465?module_item_id=943734`
- The first (of two) class tests will take place next Tuesday, 14th October. I'll be in touch about accommodations for those who completed the request form.

- Assignment 3: start tomorrow.

At the end of the last class, we visualised a function, f, with a line intersecting it at the points $(a, f(x)$ and $(b, f(b))$, where $b = a + h$

We moved the point b closer and closer to a (by taking smaller and smaller h, until the intersecting line eventually became the tangent to f at $x = a$.



line intersecting f at $x=a$ & $x=b_1$

Conclude: the slope of the tangent to f at $x = a$ is the limit:

$$\lim_{h \to 0} \frac{f(\cancel{x}\,a + h) - f(a)}{h}.$$

$h = (a+h) - a$
$= h$

The slope of the curve $y = f(x)$ at the point $P = (a, f(a))$ is given by the number (if it exists)

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, it is called the **derivative of f at $x = a$** and we denote it by $f'(a)$.

Read as "f prime at a"
or "the derivative of f at a".

Definition: derivative at a point

Let $f(x)$ be a function that has $x = a$ in its domain. The **derivative** of the function $f(x)$ at a, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

if the limit exists.

Some terminology

$f'(a)$ exists then we say that function f is **differentiable at $x = a$**.

Also, we'll soon learn that there are formulae for derivatives of many well-known functions. But when we use the "limit" formula, we are doing "**differentiation from first principles**".

Note, sometimes in Physics

where $f = f(t)$ and t is time,

one writes $\dot{f}(t)$

Example

Use the limit definition of a derivative to compute the slope of the tangent to $f(x) = x^2$ at $x = 3$.

we wont to compute

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$

$$= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h\to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h\to 0} 6 + h = 6$$

Example

Use the limit definition of a derivative to find the equation of the tangent to $f(x) = 1/x$ at $x = 2$.

First, we the slope of the tangent, which is

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{1}{2+h} - \frac{1}{2}\right)$$

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{2}{2(2+h)} - \frac{2+h}{2(2+h)}\right) = \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{4+2h}\right)$$

$$= \lim_{h\to 0} \frac{-1}{4+2h} = -1/4$$

So the slope is $-1/4$.

Example

Use the limit definition of a derivative to find the equation of the tangent to $f(x) = 1/x$ at $x = 2$.

So now we need the equation of the line with slope $m = -1/4$ through $(2, \frac{1}{2})$.

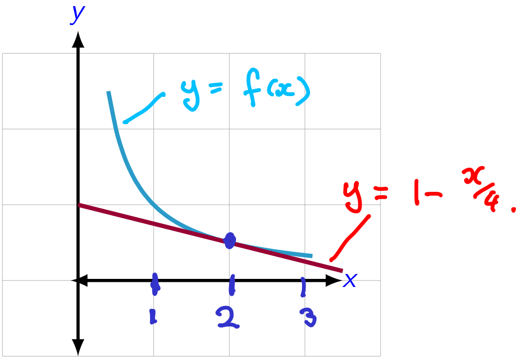
Formula $\quad y - y_1 = m(x - x_1)$

with $\quad y_1 = 1/2, \; m = -1/4, \; x_1 = 2$

$$y - 1/2 = (-1/4)(x - 2)$$

which is $\quad y = 1 - x/4$.

$$\boxed{f(x) = \frac{1}{x}} \text{ and } \boxed{y = 1 - \frac{x}{4}}$$



Derivative as a function

We've seen how to compute $f'(a)$: the derivative of the function f at a given point, $x = a$.

But if $f'(a)$ has a value for all $x = a$ (in the domain of $f(x)$), we can think $f'(x)$ as a function itself!

Definition: derivative as a function

Let f be a function. The derivative function, denoted f'. is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative as a function

Terminology and notation

- We usually refer to f' simply as the derivative of $f(x)$.
- Where $y = f(x)$, we often we write f' as $\frac{dy}{dx}$, or y', or $\frac{d}{dx}(f)(x)$.

or $\frac{df}{dx}$

Derivative as a function

Example

Use the above definition to find the derivative of $f(x) = x^2$.

Solution

The derivative is defined as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Here $f(x+h) = (x+h)^2 = \underline{x^2 + h^2 + 2hx}$, so we get:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(\cancel{x^2} + h^2 + 2hx) - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+2x)}{\cancel{h}} = \lim_{h\to 0} (h+2x) = 2x$$

Derivative as a function

Example

Use the "limit" definition to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) = \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

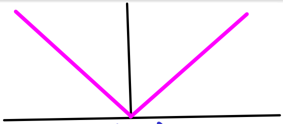
$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}}$$

So $\quad f'(x) = \dfrac{1}{2\sqrt{x}}.$

Derivative as a function

Consider the absolute value function $f(x) = |x|$. What is its derivative at (i) $x = 2$, (ii) $x = -3$, or (iii) $x = 0$?

Recall $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$



(i) $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{2+h-2}{h} = 1$ ✓

(ii) $f'(-3) = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \to 0} \frac{3-h-(-3)}{h} = -1$

(iii) $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ which does not exist since $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \neq \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

Derivative as a function

Show that $\frac{d}{dx}(\sin x) = \cos x$.

Solution: We need to evaluate

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h},$$

where $f(x) = \sin(x)$. From p5 of the "log" tables, we have that
$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.
Here $A = x + h$, and $B = x$, so
$\sin(x+h) - \sin(x) = 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$.
So now we evaluate

$$f'(x) = \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \lim_{h \to 0} \frac{2}{h} \sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right).$$

Derivative as a function

But

$$\lim_{h\to 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right) = \left(\lim_{h\to 0} \underbrace{\frac{2}{h} \sin\left(\frac{h}{2}\right)}_{1}\right)\left(\lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right)\right).$$

We learned last week that,

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Taking $\theta = h/2$, we get that

$$\lim_{h\to 0} \frac{2}{h} \sin\left(\frac{h}{2}\right) = 1.$$

And finally,

$$\sin'(x) = \lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right) = \cos(x).$$

and we are done!

Differentiation by rule

We've seen we can compute derivatives of some functions using the "limit" definition (i.e., **differentiation from first principles**). However, that approach is tedious, and unnecessary in many case.

Instead we can use a set of "**rules**" which makes the process much more efficient. These rules are themselves derived from the "limit" definition – but we don't have to use that every time.

The Constant Rule

If f is a constant function, i.e. $f(x) = c$ for all x, then:

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Why:

(i) If $f'(x)$ is the rate of change of f at x, but f is <u>not</u> changing (it is constant) then the rate of change is 0.

(ii) Formally: $\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$ ✓

$x^{1/2} = \sqrt{x}$.

We've already deduced that

- The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- The derivative of $f(x) = x^{1/2}$ is $f'(x) = \frac{1}{2}x^{-1/2}$

These are particular examples of the **Power Rules**

The Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

we wont prove this. If $n \in \mathbb{N}$ use binomial thm. With $n \notin \mathbb{N}$ need more...

Check with $n = \frac{1}{2}$: $f(x) = x^{1/2} = \sqrt{x}$.

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{(1/2 - 1)} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Examples Calculate the derivatives of the following functions

1. $f(x) = x^6$
2. $f(x) = \sqrt[3]{x}$

1: $n=6$ $f'(x) = (x^6)' = 6x^5$.

2.: $f(x) = \sqrt[3]{x} = x^{1/3}$. so $n = 1/3$.

Then $f'(x) = \frac{1}{3} \cdot x^{(1/3 - 1)} = \frac{1}{3} \cdot x^{-2/3}$

$= \frac{1}{3x^{2/3}}$.

Finished here Tuesday

The constant multiple rule

Let $f(x)$ be any differentiable function, and let k be constant, then

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big).$$

Example: Find the derivative of $f(x) = 5x^4$.

The Sum and Difference Rules

Let $u(x)$ and $v(x)$ be any differentiable functions. Then

$$\frac{d}{dx}\big(u(x)+v(x)\big)=\frac{d}{dx}\big(u(x)\big)+\frac{d}{dx}\big(v(x)\big).$$

Similarly, $\frac{d}{dx}\big(u(x)-v(x)\big)=\frac{d}{dx}\big(u(x)\big)-\frac{d}{dx}\big(v(x)\big).$

Example: Find the derivative of $f(x)=1+x+x^2$.

Actually, the "**Difference Rule**", which states that

$$\boxed{\frac{d}{dx}\big(u(x) - v(x)\big) = \frac{d}{dx}\big(u(x)\big) - \frac{d}{dx}\big(v(x)\big).}$$

can be combined by combining the **Sum Rule** and the **Constant Multiple Rule**.

Example

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$;

(b) The slope of the tangent line at $x = 2$;

(c) The equation of the tangent at $x = 2$.

(a) $f'(x) = -15x^2 + 6x - 9$

(b) The slope of the tangent line at $x = 2$ is $f'(2)$:

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57.$$

(c) The equation of the line with slope M and passing through a point (x_1, y_1) is

$$y - y_1 = M(x - x_1)$$

The y coordinate at $x = 2$ is

$$\begin{aligned} f(2) &= -5(2)^3 + 3(2)^2 - 9(2) + 7 \\ &= -5(8) + 3(4) - 18 + 7 \\ &= -40 + 12 - 18 + 7 \\ &= -39. \end{aligned}$$

So the tangent line passes through the point $(2, -39)$ and the slope of the line is -57.

Therefore, the equation of this line is $y + 39 = -57(x - 2)$

Ans: The equation of the tangent line is $x = 2$ is $y = 75 - 57x$.

Tomorrow's Rules

Tomorrow we'll focus on two more rules, with important applications:

- The **Product Rule** for computing the the derivative of the **product** of two functions.
- **The Quotient Rule** for differentiating the **ratio** of two functions.

Exercises

Exercises 4.1.1 (Based on Q2(a), 2019/2020)

Use the (limit) definition of a derivative to differentiate the function $f(x) = x^2 + 2$.

Exercise 4.1.2

Use the (limit) definition of a derivative to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.