2425-MA140 Engineering Calculus

Week 07, Lecture 3
The Fundamental Theorem of Calculus

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Thursday, 30 October, 2025

Suimeálaithe | **Integrals**

Tá tairisigh na suimeála fágtha ar lár. | Constants of integration omitted.

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$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
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e^x	e^x
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Suimeáil na míreanna $\quad \int u\,dv = uv - \int v\,du \quad$ **Integration by parts**

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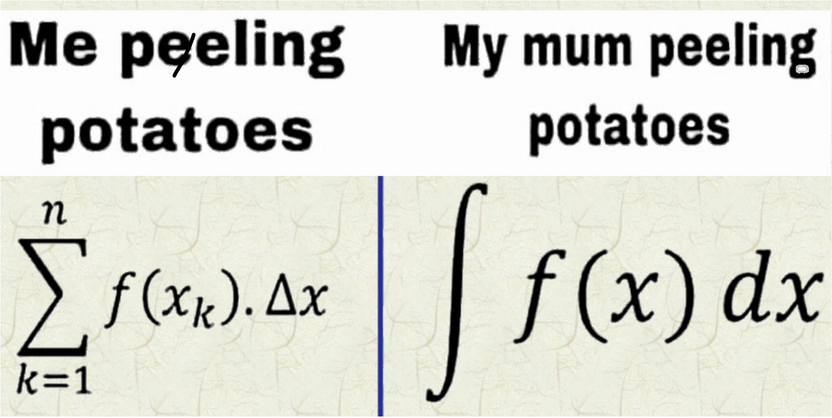
Today's joke (with thanks to Julie M).



The exciting topics that await us in today:

1. Recall from yesterday:
2. Fundamental Thm of Calculus: Part 1
3. FTC1+Chain Rule
4. Antiderivatives
   - Indefinite Integrals
   - Common functions
   - Properties
5. The Fundamental Thm of Calculus: Part 2
6. Exercises

See also: Sections **4.10** (Antiderivatives) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman:
math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

Recall from yesterday:

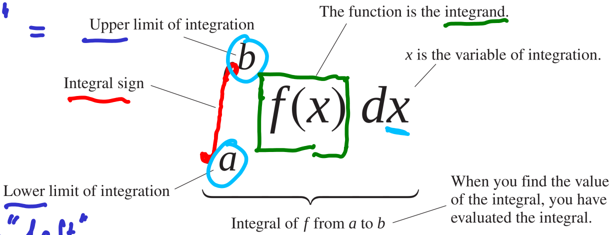
Let $f(x)$ be function defined on an interval $[a, b]$. The **definite integral** of f from a to b is

$$\boxed{\int_a^b f(x)dx} := \lim_{n\to\infty} \sum_{i=0}^{n-1} hf(x_i),$$

"dx" = "delta x" = "change in x" = "h"

where $h = (b-a)/n$ and $x_i = a + ih$. It is the **area** of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$, and $x = b$.

"Right" =



= "left"

Recall from yesterday:

Given a function, f, we can define another, F as

$$F(x) = \int_a^x f(t)dt.$$

That is, the variable in F is the upper limit of integration on the right.

x,

Recall from yesterday:

Example

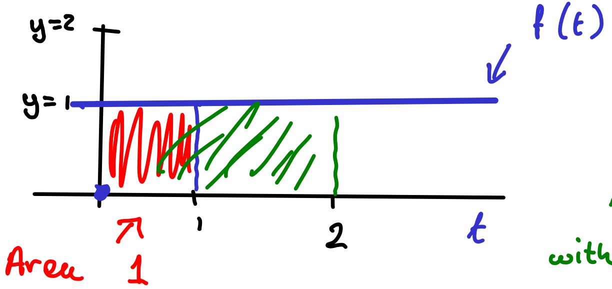
Let $f(t) \equiv 1$, and $F(x) = \int_0^x f(t)\,dt$. Give a formula for $F(x)$, using the "area" meaning of the definite integral.

Check some values:

$F(0) = \int_0^0 1\,dt = 0$

$F(1) = \int_0^1 1\,dt = 1$

$F(2) = \int_0^2 1\,dt = 2$



$F(x) = \int_0^x 1\,dt = x$

"area of rectangle with base x, height 1"

$= x$

Fundamental Thm of Calculus: Part 1

Fundamental Theorem of Calculus: Part 1 (FTC1)

Let $f(x)$ be a continuous function on $[a, b]$. If as

$$F(x) = \int_a^x f(t)dt, \qquad \text{then} \qquad \frac{dF}{dx}(x) = f(x).$$

I.e., $F'(x) = f(x)$ for $x \in [a, b]$.

Roughly: $\boxed{f \text{ is the derivative its own integral}}$. You can find a proof in Section 5.3 of the textbook.

Fundamental Thm of Calculus: Part 1

Example

Let $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$. Find $g'(x)$.

By the FTC 1:

$$g'(x) = \frac{1}{x^3+1}$$

[Note: the correct answer is $\frac{1}{x^3+1}$
and not $\frac{1}{t^3+1}$]

FTC1+Chain Rule

Sometimes the limit of integration is a more complicated function of x. In that case, we can apply the **Chain Rule**, along with the FTC1.

Example

Let $F(x) = \int_1^{\sqrt{x}} \sin(t)dt$. Find $F'(x)$.

Idea: Let $u(x) = \sqrt{x} = x^{1/2}$. So

- $F(u) = \int_1^u \sin(t)dt$, and
- $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$.

By FTC1, $\frac{dF}{du} = \sin(u)$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

Chain Rule.

Then...

$$\boxed{\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx}} = \sin(u(x))\left(\frac{1}{2\sqrt{x}}\right) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

Antiderivatives

Definition: Antiderivative

A function F is an **antiderivative** of f on $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$. Thus,

$$f \text{ is the derivative of } F \Leftrightarrow F \text{ is an antiderivative of } f.$$

Note: If F is an antiderivative of f, then the most general antiderivative of f is

$$\boxed{F(x) + C}$$

where C is an *arbitrary* constant, called a **constant of integration**.

- The word "arbitrary" here means that any choice is valid.
- The derivative of C is zero.

Eg $F(x) = 3x^2 + 1$ $\quad f(x) = F'(x) = 6x$

$F(x) = 3x^2 - 52$ $\quad f(x) = F'(x) = 6x.$

Examples:

- $F(x) = x + C$ is an antiderivative of $f(x) \equiv 1$.

  Since $F'(x) = \frac{d}{dx}(x+C) = \frac{d}{dx}(x) + \frac{d}{dx}(C)$
  $= 1 + 0 = 1.$

- $F(x) = x^2 + C$ is an antiderivative of $f(x) = ???$ ...

  Differentiate: $F'(x) = \frac{d}{dx}(x^2+C) = 2x + 0 = \underline{\underline{2x}}$
  So $f(x) = 2x$

- $F(x) = ???$ is an antiderivative of $f(x) = 3x^2$.

  $\boxed{F(x) = x^3 + C}$ then $F'(x) = 3x^2 + 0$ ✓

Examples

Find all antiderivatives of the following functions

(i) $f(x) = \frac{1}{x}$ for $x > 0$.

(ii) $f(x) = \sin(x)$

(iii) $f(x) = e^x$.

(i) $F(x) = \ln(x) + C$ "ln(x) is the Natural Log of x"

$F'(x) = \frac{1}{x}$.

(ii) $f(x) = \sin(x)$. Recall $\frac{d}{dx}\cos(x) = -\sin(x)$

So take $F(x) = -\cos(x) + C$.

(iii) $F(x) = e^x + C$

Definition: indefinite integral

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,\mathrm{d}x$$

is the general antiderivative of f. That is, if F is an antiderivative of f, then

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

Examples:

- $\int 2x\,dx = x^2 + C$
- $\int 3x^2\,dx = x^3 + C$

- $\int x \, dx = \frac{1}{2}x^2 + C$
- $\int x^2 \, dx = \frac{1}{3}x^3 + C$.

Spotting the pattern we can deduce...

Power Rule of Integration

If $n \neq -1$, then $$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$

If $n = -1$, note that $\frac{x^{n+1}}{n+1} = \frac{x^0}{0}$?? . But $\int \frac{1}{x} dx = \ln(x) + C$.

Note: For $n = -1$, we have

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C \, .$$

Here is a list of the antiderivatives of some common functions.

- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$ Note $\ln(e) = 1$.
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \tan(x)\,dx = \ln|\sec(x)| + C$
- ...

Suimeálaithe
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Suimeáil na míreanna $\int udv = uv - \int vdu$ **Integration by parts**

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Properties of Integration

1. If k is a constant, then

$$\int kf(x)\,dx = k\int f(x)\,dx\,.$$

2. Integration is additive:

$$\int (f(x) \pm g(x))\ dx = \int f(x)\,dx \pm \int g(x)\,dx\,.$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx\,.$$

Example

Evaluate the integral

$$\int 2x^2 + 9x^7 \, dx$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int 2x^2 + 9x^7 \, dx = \int 2x^2 \, dx + \int 9x^7 \, dx \quad (\text{Additive})$$

$$= 2\int x^2 \, dx + 9\int x^7 \, dx$$

$$= 2\,\frac{x^3}{3} + C_1 + 9\,\frac{x^8}{8} + C_2$$

$$= \frac{2}{3}x^3 + \frac{9}{8}x^8 + C \quad (C = C_1 + C_2).$$

Example

Evaluate the integral

$$\int \frac{4}{1+x^2}\,\mathrm{d}x.$$

$$\int \frac{4}{1+x^2}\,dx = 4\int \frac{1}{x^2+1}\,dx$$

$$= 4\int \frac{a}{x^2+a}\,dx \qquad a=1$$

$\rightarrow$ Check table!

$$= 4\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= 4\tan^{-1}(x). + C$$

The Fundamental Thm of Calculus: Part 2

Now that we know all about antiderivatves, we can see how the link to **definite integrals**

Theorem (The Fundamental Thm of Calculus, Part 2)

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

"Evaluation Theorem".

Notation: We can write $F(b) - F(a)$ as $F(x)\Big|_{x=a}^{x=b}$, or, more often, as $F(x)\Big|_a^b$.

So $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b$

The Fundamental Thm of Calculus: Part 2

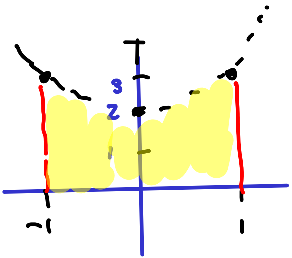
Example: Show that $\displaystyle\int_{-1}^{1}(x^2+2)\,\mathrm{d}x = \frac{14}{3}$

$f(x) = x^2 + 2$

$f(-1) = 1+2 = 3$

$f(0) = 2$

$f(1) = 3$



$$\int_{-1}^{1} x^2+2\,dx = \int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} dx$$

$$= \frac{1}{3}x^3\Big|_{-1}^{1} + 2\,\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) + 2 + 2$$

$$= 14/3 \ (!)$$

The Fundamental Thm of Calculus: Part 2

Example: Show that $\int_{-1}^{1} (x^3 + x)\, \mathrm{d}x = 0$

(These notes were added after class)

$$\int_{-1}^{1} x^3 + x \, dx = \int_{-1}^{1} x^3 \, dx + \int_{-1}^{1} x \, dx$$

$$= \frac{1}{4} x^4 \Big|_{-1}^{1} + \frac{1}{2} x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{4}(1)^4 - \frac{1}{4}(-1)^4 + \frac{1}{2}(1)^2 - \frac{1}{2}(-1)^2$$

$$= \frac{1}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = 0.$$

Exercises

Exer 7.3.1

Let $F(x) = \int_x^{2x} t \, \mathrm{d}t$. Use the Fundamental Theorem of Calculus to evaluate $F'(x)$.

Hint: we can split this into two integrals:

$$F(x) = \int_x^{2x} t \, \mathrm{d}t = \int_x^{0} t \, \mathrm{d}t + \int_0^{2x} t \, \mathrm{d}t = -\int_0^{x} t \mathrm{d}t + \int_0^{2x} t \mathrm{d}t.$$

Now apply the FTC to each term, including the Chain Rule for the second.

Exercises

Exer 7.3.2

Evaluate the following integrals.

1. $\displaystyle\int e^{2x} + \frac{1}{2x}\,\mathrm{d}x$
2. $\displaystyle\int \frac{3}{\sqrt{2-x^2}}\,\mathrm{d}x$

Exer 7.3.3

Evaluate the definite integral $\displaystyle\int_1^e e^{2x} + \frac{1}{2x}\,\mathrm{d}x$

Exer 7.3.4

Find two values of q for which $\displaystyle\int_q^0 2x + x^2\,\mathrm{d}x = 0.$

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