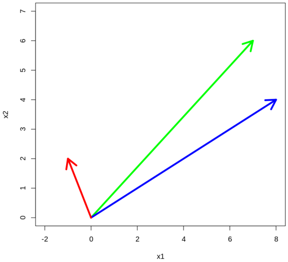
Week 10: Orthogonal Everything

Dr Niall Madden

8 and 11 November, 2022



R code

```
v <- c(7,6)
w <- c(8,4)
z <- c(-1,2)
plot(NULL, xlim=c(-2,8), ylim=c(0,7)),
   xlab="x1", ylab="x2")
arrows(0,0, v[1], v[2],lwd=4,col="green")
arrows(0,0, w[1], w[2],lwd=4,col="blue")
arrows(0,0, z[1], z[2],lwd=4,col="red")
```

These slides are adapted (slightly) from ones by Tobias Rossmann.

Outline

For more details,

- Section 6.1 (Inner Product, Length and Orthogonality) of the Lay et al text-book https://nuigalway-primo.hosted.exlibrisgroup.com/permalink/f/1pmb9lf/353GAL_ALMA_DS5192067630003626
- Chapters 6 and 9 of *Linear Algebra for Data Science* https://shainarace.github.io/LinearAlgebra/norms.html and https://shainarace.github.io/LinearAlgebra/orthog.html

MA313
Week 10: Orthogonal Everything

Start of ...

PART 1: Announcements and Preview of Week 10

Assignment 5

Assignment 5 opened on Thursday 10 Nov). Deadline is 5pm, Friday, 25th of November.

Communication Skills : Next steps...

- Instructions at https://www.niallmadden.ie/2223-MA313/22_23_Communication_Skills.pdf have been updated.
- Deadline is 5pm Friday, 18 November.
- Presentations will be during the week 21–25 November:
  - Monday at 12.00 in AC204 (i.e., MA335 class time)
  - Tuesday at 13.00 in Ac202 (i.e., MA313 class time)
  - Some other time ... (probably Thursday at 12).

Theorem: Unique representation/Orthogonal decomposition

Let W be a subspace of $\mathbb{R}^n$. Then:

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- If $W = \operatorname{span}\{w_1, \ldots, w_r\}$, then $W^\perp = \{z \in \mathbb{R}^n : z \perp w_1, \ldots, z \perp w_r\}$.
- Every vector $v \in \mathbb{R}^n$ has a **unique representation**

$$v = \hat{v} + z \qquad \text{for } \hat{v} \in W, \qquad \text{and } z \in W^\perp.$$

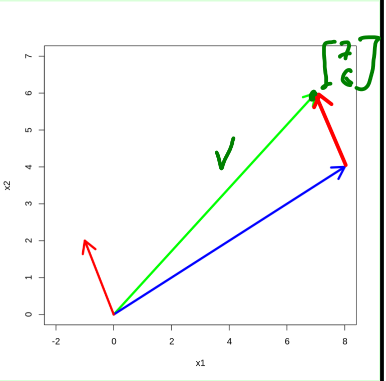
- The function $\operatorname{proj}_W : \mathbb{R}^n \to W, \quad v \mapsto \hat{v}$ is a linear transformation, called the **orthogonal projection** of $\mathbb{R}^n$ onto W.
- $W \cap W^\perp = \{0\}$.
- $\dim W^\perp = n - \dim W$.

Example

Let $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $W = \mathrm{span}\left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$. Then

The orthogonal projection of v onto W is $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$.

The component of v orthogonal to W is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



Proposition

Let $W = \operatorname{span}\{u\}$ be a subspace of $\mathbb{R}^n$, where $0 \neq u \in \mathbb{R}^n$. (That is, W is a line through the origin.)
Then the orthogonal projection $\hat{v} = \operatorname{proj}_W(v)$ of $v \in \mathbb{R}^n$ on W is

$$\hat{v} = \frac{v \cdot u}{u \cdot u} u.$$

To see this, we have to show that

- $\hat{v} \in W$
- If $z := v - \hat{v}$, then $z \perp u$.

Note: $\frac{v \cdot u}{u \cdot u}$ is a scalar. So $\hat{v}$ is some multiple of u.

Proposition

Let $W = \operatorname{span}\{u\}$ be a subspace of $\mathbb{R}^n$, where $0 \neq u \in \mathbb{R}^n$. (That is, W is a line through the origin.)
Then the orthogonal projection $\hat{v} = \operatorname{proj}_W(v)$ of $v \in \mathbb{R}^n$ on W is

$$\hat{v} = \frac{v \cdot u}{u \cdot u}\, u.$$

To see this, we have to show that

- $\hat{v} \in W$
- If $z := v - \hat{v}$, then $z \perp u$.

To see that $z \cdot u = 0 \ldots$

$$z \cdot u = (v - \hat{v}) \cdot u = \left(v - \frac{v \cdot u}{u \cdot u} u\right) \cdot u$$
$$= v \cdot u - \frac{v \cdot u}{\cancel{u \cdot u}} \cancel{u \cdot u} = v \cdot u - v \cdot u = 0.$$

Example

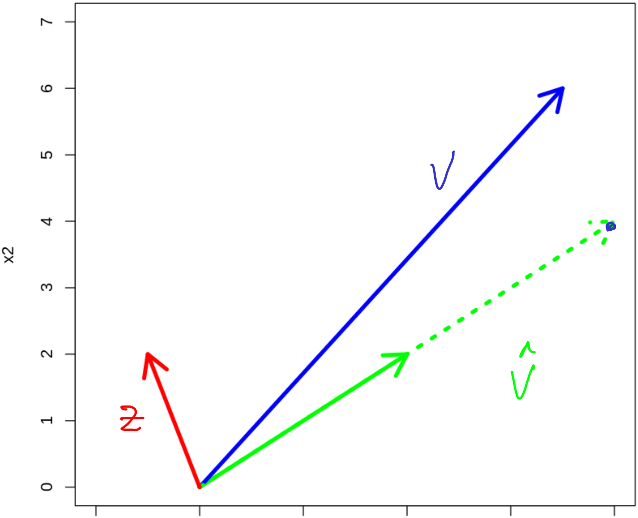
Let $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$. Compute

- the orthogonal projection, $\hat{v}$, of v onto $\mathrm{span}\{u\}$;
- $z = v - \hat{v}$.
- Verify that $z \perp u$.

$$\hat{v} = \frac{v \cdot u}{u \cdot u} u = \frac{\begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{(28) + (12)}{16 + \cancel{2}4} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

$$z = v - \hat{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$


x2
0
1
2
3
4
5
6
7
v
z
$\hat{v}$

In case you are interested, here is how to do this in R.

Column vector.

Solution in R

inner (dot) product

```
u <- c(4,2)
v <- c(7,6)
vhat = c((v %*% u)/(u %*% u))*u
z = v - vhat

plot(NULL, xlim=c(-2,8), ylim=c(0,7),
   xlab="x1", ylab="x2")
arrows(0,0, u[1], u[2],lwd=4,col="green")
arrows(0,0, v[1], v[2],lwd=4,col="blue")
arrows(0,0, vhat[1], vhat[2],lwd=4, lty=3, col="green")
arrows(0,0, z[1], z[2],lwd=4, col="red")
```

MA313
Week 10: Orthogonal Everything

Start of ...

PART 3: Orthogonal Bases

Part 3: Orthogonal Bases

In Part 2, we saw how to compute the projection of a vector orthogonal to a one-dimensional space.

Questions

Let W be an arbitrary subspace of $\mathbb{R}^n$, with $\dim W \geq 1$. How can we compute proj_W?
That is, given $v \in \mathbb{R}^n$, how can we find $\hat{v} = \mathrm{proj}_W(v)$?
Also, why bother?

Part 3: Orthogonal Bases

Definition (ORTHOGONAL BASIS)

- A sequence of vectors $u_1, \ldots, u_p \in \mathbb{R}^n$ is **orthogonal** if $u_i \perp u_j$ for all $i \neq j$.
- An **orthogonal basis** of a subspace W of $\mathbb{R}^n$ is a basis of W which is orthogonal.

For $\mathbb{R}^n$, the standard basis is an example of an orthogonal basis. But there are others.

$u_1 \cdot u_j = 0$ if $j \neq 1$.

Proposition

If $u_1, \ldots, u_p$ is an orthogonal sequence of non-zero vectors, then these vectors are linearly independent.

Suppose $u_1, u_2, \ldots, u_p$ are orthogonal but dependent. Then there are $c_1, c_2, \ldots, c_p$ which are not all zero, and

$$c_1 u_1 + c_2 u_2 + \cdots + c_p u_p = 0.$$

Then
$$\begin{aligned} 0 &= (c_1 u_1 + c_2 u_2 + \cdots + c_p u_p) \cdot u_j \\ &= c_1 u_1 \cdot u_j + c_2 u_2 \cdot u_j + \cdots + c_j u_j \cdot u_j \\ &\quad + c_p u_p \cdot u_j = c_j u_j \cdot u_j \end{aligned}$$

But $u_j \cdot u_j \neq 0$. So $c_j = 0$.

Part 3: Orthogonal Bases

Now we can generalise the idea on Part 2, Slide 17 which was for one-dimensional spaces.

Theorem

Let $(u_1, \ldots, u_p)$ be an orthogonal basis of a subspace W of $\mathbb{R}^n$. Then the orthogonal projection of $v \in \mathbb{R}^n$ onto W is given by

$$\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{v \cdot u_p}{u_p \cdot u_p} u_p.$$

Note: each $\dfrac{v \cdot u_j}{u_j \cdot u_j} \in \mathbb{R}$. So

$\hat{v} \in \text{Span}\{u_1, u_2, \ldots, u_p\} = W$

Also, $z = v - \hat{v} \in W^\perp$. (Check this! Show $z \perp u_j \ \forall j$)

Example

Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $W = \operatorname{span}\{u_1, u_2\}$.
What is $\hat{v} = \operatorname{proj}_W(v)$?

First: check $u_1 \perp u_2$.

$$\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = (-4) + (5) + (-1) = 0 \quad \checkmark$$

So (u_1, u_2) is an orthogonal basis for W.

Thus

$$\hat{v} = \frac{u_1 \cdot v}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot v}{u_2 \cdot u_2} u_2 = \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} -12 \\ 60 \\ 6 \end{bmatrix}$$

MA313
Week 10: Orthogonal Everything

Start of ...

PART 4: Gram-Schmidt Process

 Finished here Friday.

Question(s)

- Does every subspace of $\mathbb{R}^n$ have an orthogonal basis?
- If so, how do we construct it?

If $W = \text{Span}\{u_1\}$ this is easy.
(ie $\dim W = 1$).

If $W = \text{Span}\{u_1, u_2\}$

Idea: take u_1 as one basis vector.
Then make a second, $\hat{u}_2$, which is a linear combination of u_1 & u_2 but $u_1 \perp \hat{u}_2$.

Part 4: Gram-Schmidt Process

Theorem: "Gram-Schmidt process"

Let $(v_1, \ldots, v_p)$ be a basis of a subspace W of $\mathbb{R}^n$.
Define vectors $u_1, \ldots, u_p$ via

- $u_1 := v_1$,
- $u_2 := v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$,
- $u_3 := v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$,
- $\vdots$
- $u_p := v_p - \frac{v_p \cdot u_1}{u_1 \cdot u_1} u_1 - \cdots - \frac{v_p \cdot u_{p-1}}{u_{p-1} \cdot u_{p-1}} u_{p-1}$.

Then $(u_1, \ldots, u_p)$ is an orthogonal basis of W.

Part 4: Gram-Schmidt Process

Example

Let $W = \operatorname{span}\{v_1, v_2\}$ for $v_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Construct an orthogonal basis of W.

First, check if $v_1 \perp v_2$. Since $v_1 \cdot v_2 = (3)(1) + (6)(2) + (0)(2) = 15$ we can say v_1 & v_2 are not orthogonal.

To make an orthogonal basis, set

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} \cdot u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Note that $u_1 \perp u_2$.

So $\left(\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right)$ is an orthogonal basis for W.

MA313
Week 10: Orthogonal Everything

Start of ...

PART 5: Orthogonal Matrices

Finished here.