MA378 Chapter 1: Interpolation

§1.4 Hermite Interpolation

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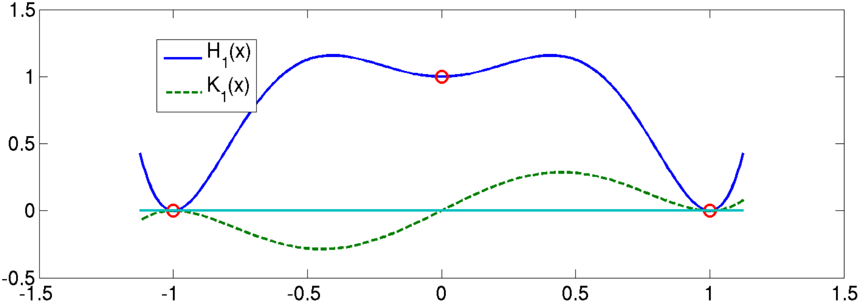

Charles Hermite

Charles Hermite, France, 1822–1901. Apart from this form of interpolation, his contributions to mathematics included the first proof that e is transcendental.

His methods were later used to show that π is transcendental.

4.0 Topics in this section:

Hermite interpolation is a variant on the standard Polynomial Interpolation Problem: we seek a polynomial that not only agrees with a given function f at the interpolation points, but its first derivative also matches f' at those points.

We are not that interested in this problem for its own sake, but the idea recurs again in the sections in piecewise polynomial interpolation and Gaussian quadrature.

Formally, the problem is

The Hermite Polynomial Interpolation Problem (HPIP) *Given a set of interpolation points* $x_0 < x_1 < \cdots < x_n$ *and a continuous, differentiable function* f, *find* $p_{2n+1} \in \mathcal{P}_{2n+1}$ *such that*

$$p_{2n+1}(x_i) = f(x_i) \quad \text{and} \quad p'_{2n+1}(x_i) = f'(x_i), \quad \text{for } i = 0, 1, \dots, n.$$

One can prove that if there is a solution to this problem, then it is unique (see Exercise 4.1).

4.2 Construction

It is possible to solve this problem using an extension of the Lagrange Polynomial approach.

The Hermite basis functions

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Here the L_i are the usual Lagrange polynomials:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

$$\Rightarrow \quad L_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases}$$

$$\boxed{H_i(x) = [L_i(x)]^2 \left(1 - 2L_i'(x_i)(x - x_i)\right)}$$

We can show that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0 \ \forall k$$

First check that $H_i \in \mathbb{P}_{2n+1}$.

Note that $L_i(x) \in \mathbb{P}_n$ so

$(L_i(x))^2 \in \mathbb{P}_{2n}$ and

$(1 - 2L_i'(x_i))(x - x_i) \in \mathbb{P}_1$. So $H_i(x) \in \mathbb{P}_{2n+1}$

4.2 Construction

$$\boxed{H_i(\cancel{x}) = [L_i(x)]^2\left(1 - 2L_i'(x_i)(\cancel{x} - x_i)\right)}$$

We can show that, for $i, k = 0, 1, \ldots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad\qquad H_i'(x_k) = 0\ \forall k$$

next, we show that

$$H_i(x_i) = (\underbrace{L_i(x_i)}_{1})^2\left(1 - 2L_i'(x_i)(\underbrace{x_i - x_i}_{0})\right)$$

So $H_i(x_i) = 1$

Also $H_i(x_k) = (\underbrace{L_i(x_k)}_{0})^2\left(1 - 2L_i'(x_i)(x_k - x_i)\right)$

$\Rightarrow H_i(x_k) = 0$ ✓

4.2 Construction

$(uv)' = u'v + uv'$ with

$u = (L_i(x))^2 \quad v = (1 - 2L'(x_i)(x-x_i))$

$$\boxed{H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big)}$$

We can show that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0 \ \forall k$$

Now we show $H_i'(x_k) = 0$ for all k.

Differentiate $H_i(x)$ to get

$$H_i'(x) = 2L_i(x)L_i'(x)\big(1 - 2L_i'(x_i)(x - x_i)\big) + (L_i(x))^2\big(-2L_i'(x_i)\big)$$

4.2 Construction

$$\boxed{H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big)}$$

We can show that, for $i, k = 0, 1, \ldots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0\ \forall k$$

$$H_i'(x) = 2L_i(x)L_i'(x)\big(1 - 2L_i'(x_i)(x - x_i)\big) + (L_i(x))^2\big(-2L_i'(x_i)\big)$$

$k \neq i$

$$H_i'(x_k) = 2\underbrace{L_i(x_k)}_{0}L'(x_k)(1 - \cdots) - 2\big(\underbrace{L_i(x_k)}_{0}\big)^2 L_i'(x_i)$$

$= 0$. Also (see board) $H_i'(x_i) = 0$

4.2 Construction

$$\boxed{K_i(x) = [L_i(x)]^2(x - x_i)}$$

Also, for $i, k = 0, 1, \dots n$,

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

This part is left to Exercise 4.4(a).

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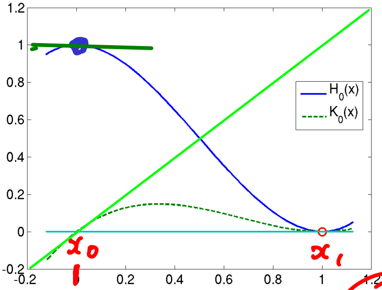


Hermite bases functions H_0 and K_0 for $n = 1$, $x_0 = 0$ and $x_1 = 1$

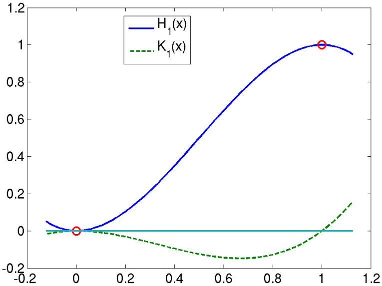
Note $H_0(0) = 1$ $H_0(1) = 0$

$H_0'(0) = H_0'(1) = 0$

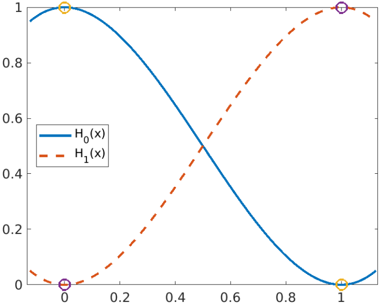
$K_0(0) = K_0(1) = 0$

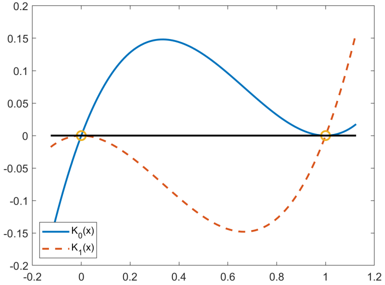
$K_0'(0) = 1$

$K_0'(1) = 0$



Hermite bases functions H_1 and K_1 for $n = 1$, $x_0 = 0$ and $x_1 = 1$





Hermite bases functions H_0, H_1 (left) and K_0, K_1 (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

One can now show that the solution to the HPIP exists and is

$$\boxed{p_{2n+1}(x) = \sum_{i=0}^{n} \left(f(x_i)H_i(x) + f'(x_i)K_i(x) \right)}$$

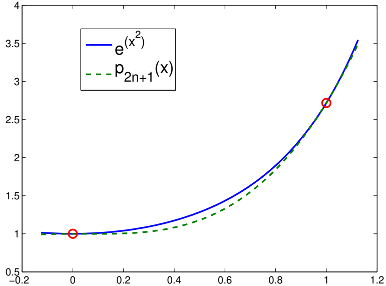
This part is left to Exercise 4.4(b).

Note :

1. verify that $p_{2n+1}(x) \in \mathbb{P}_{2n+1}$
2. $p_{2n+1}(x_k) = f(x_k) \quad k=0,1,\dots,n$
3. $p'_{2n+1}(x_k) = f'(x_k) \quad k=0,1,\dots,n.$

Example 4.1

Find the polynomial of degree 3 that interpolates $\exp(x^2)$, and its first derivative, at $x_0 = 0$ and $x_1 = 1$. (See below).



Write down $H_0(x)$, $H_1(x)$ $K_0(x)$, $K_1(x)$:

$L_0(x) = -(x-1)$ $L_1(x) = x$

$H_0(x) = (x-1)^2(1+2x)$

$H_1(x) = x^2(1-2(x-1))$

We also get $K_0(x) = (x-1)^2(x)$ $K_1(x) = (x-1)x^2$.

Then (eventually) get

$p_3(x) = 2x^3 + (e-3)x^2 + 1.$

Theorem 4.2

Let be a real-valued function that is continuous and defined on $[a, b]$, such that the derivatives of f of order $2n+2$ exist and are continuous on $[a, b]$. Let p_{2n+1} be the Hermite interpolant to f. Then, for any $x \in [a, b]$ there is an $\tau \in (a, b)$ such that

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!}[\pi_{n+1}(x)]^2.$$

We won't do a proof of this in class. However, later in this course we'll be interested in the particular example of finding p_3 the cubic Hermite Polynomial Interpolant to a function f at the points x_0 and x_1.

Exercise 4.1 (*)

Show that there is at most one solution to the Hermite Polynomial Interpolation Problem (HPIP)

Exercise 4.2

For *just* the case $n = 1$, state and prove an appropriate version of Theorem 4.2 (i.e., error in the Hermite interpolant). Use this to find a bound for

$$\max_{x_0 \le x \le x_1} |f(x) - p_3(x)|$$

in terms of f and $h = x_1 - x_0$.

Exercise 4.3

Let $n = 2$ and $x_0 = -1$, $x_0 = 0$ and $x_1 = 1$. Write out the formulae for H_i and K_i for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 4.4

Let L_0, L_1, ..., L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \dots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2\big(1 - 2L_i'(x_i)(x - x_i)\big), \quad \text{and } K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad\qquad H_i'(x_k) = 0.$$

(a) Show that $K_i(x_k) = 0$, for $k = 0, 1, \dots n$, and $K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

(b) Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$

4.4 Exercises

Exercise 4.5

Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$.