# Week 09, Lecture 1 (L25)
# Introduction to Volumes

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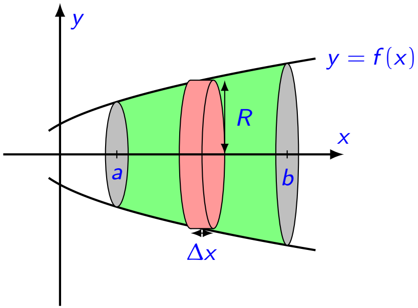
**Tuesday, 12 November, 2024**



# Assignments (sorry!)

So... thanks to a blunder while setting up Assignment 7, some progress on Assignment 6 was lost. Your grades, up to Sunday, were backed up, but actual answers lost.

1. Everyone should (re)do Q8 in **Assignment-6-Q8**. That is, reenter the answer you had already provided (or a new one, if you prefer). It is tedious, but I've given you an extra week to do it.
2. On Canvas, `Assignment-6-Q1-Q7` records your score for Assignment 6 as it was before my error. A score of 89% means you had a perfect score for Questions 1-7; so you only need to complete the ungraded Q8.
3. If you have a score of less than 89%, you can choose to keep it, or redo **Assignment-6**. Tedious, but you have a whole extra week to improve your grade!

# Today's class revolves around:

See also: Section **6.2** (Determining Volumes by Slicing) **Calculus** by Strang & Herman: math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)

# Computing Volumes

Last week, we used definite integrals to compute **areas**. Now we'll compute **volumes**. We already know how to compute the volumes of certain simple objects:

- Volume of a **cube** with size of length $a$:

- Volume of a **rectangular solid**, length $a$, width $b$, and height $c$:

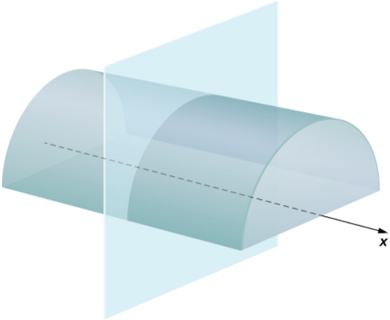
## Computing Volumes

We also know formulae (e.g., from P10 of the Formulae and Tables booklet) for the volumes of a cylinder ($\pi r^2 h$), cone ($\pi r^2 h/3$), sphere ($\frac{4}{3}\pi r^3$), pyramid ($Ah/3$), etc.

We'll now see how these can be derived using integration.

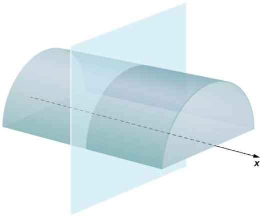
Usually, we think of a **cylinder** as something with a circular top and bottom, with the same radius. Furthermore, every cross section (parallel to top and bottom) are also circles of the same radius.

In mathematics, the term “cylinder” includes any object for which all cross-sections (in the same place) are the same.




Three-dimensional cylinder

Two-dimensional cross section




Three-dimensional cylinder

Two-dimensional cross section

If the cross-sections all have area $A$, and the cylinder has length $h$, then the volume is $V = Ah$.

But we can go further, and study objects for cross-sections all have the same shape, but different areas (but we have a formula for the areas).

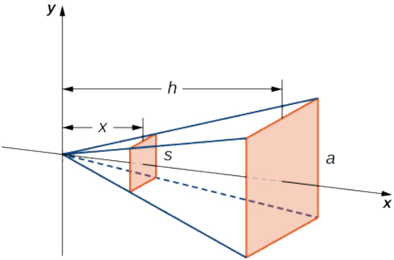
**Volume of Cylinder**

Suppose that we have an object for which every cross-section, perpendicular to the $x$-axis, through a given $x$ has area $A(x)$.

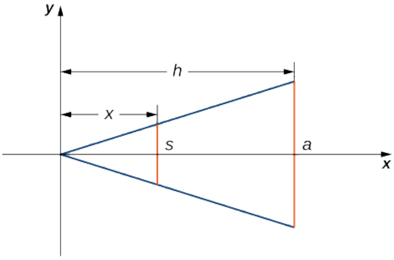
Then the volume is $V = \int_a^b A(x)dx$.

For our first example, we'll derive the formula for volume of a square-based pyramid.

Consider a square pyramid, with height $h$, and base with sides of length $a$. We need to determine the length of the side of the cross-section which is a distance $x$ from the vertex.




(a)




(b)

Reasoning from the side view in (b), we can see that

$$\frac{s}{x} = \frac{a}{h}.$$

# Slicing

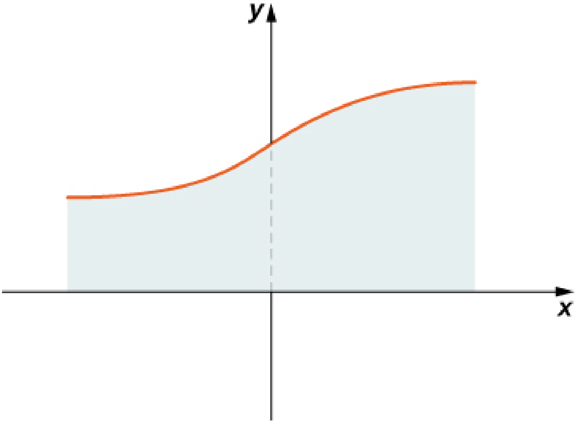
The method we just used is called **slicing**.

Next, we'll use it to calculate the volumes of **solids of revolution**.

# Introducing "Solids of Revolution"

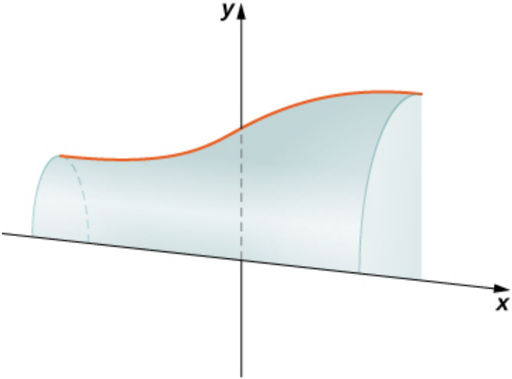
If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**. Here is the idea...

**1. Start with a region in the $xy$-plane.**



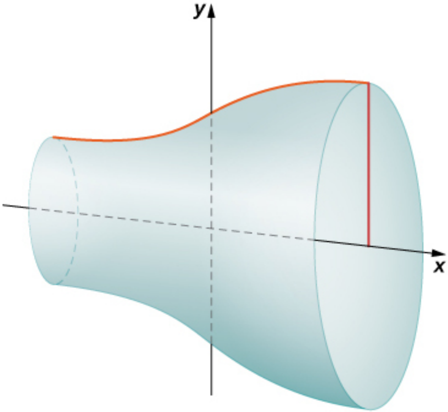
# Introducing "Solids of Revolution"

## 2. Revolve the region abount the $x$-axis



# Introducing "Solids of Revolution"

**3. Continue until you have produced a "solid of revolution"**



# Introducing "Solids of Revolution"

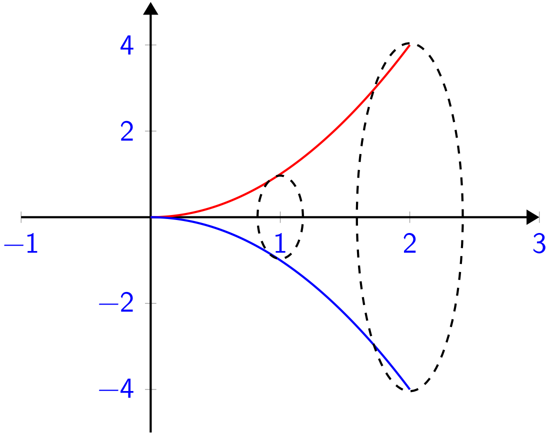
## Examples

1. What is the solid of revolution of a triangle with vertices $(0, 0)$, $(h, 0)$ and $(h, r)$?
2. What is the solid of revolution of a semicircle with radius $r$?

# Volumes of Solids of Revolution: slicing

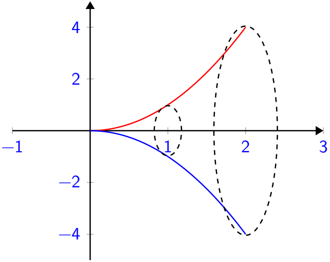
**Example**

Find the volume of the solid of revolution that is bounded by the graphs of $f(x) = x^2$, $x = 0$ and $x = 2$



# Volumes of Solids of Revolution: slicing

So, with $a = 0$, $b = 2$, and $f(x) = x^2$, the volume is...



# Solids of revolution: disk method

Since, for solids of revolution, each "slice" is actually a disk, it is often called the **disk method**. Furthermore, since, at a given $x$ the disk has radius $f(x)$, and so area, $A(x) = \pi\big(f(x)\big)^2$, we can directly compute the volume

**Solids of revolution: disk method**

Let $f(x)$ be continuous and nonnegative. The volume of region formed by revolving the region between $f(x)$ and the $x$-axis, and between $x = a$ and $x = b$, about the $x$-axis is

$$V = \int_a^b \pi\big(f(x)\big)^2 dx.$$

# Solids of revolution: disk method

Note: the following example is taken from the textbook, which has a nice animation of the process. Also try this link.

**Example**

Find the volume of the the solid of revolution generated by revolving the region between the graph of the function $f(x) = x^2 - 2x + 2$ and the $x$-axis over the interval $[-1, 3]$.

# Solids of revolution: disk method

**Example**

Use the disk method to verify that the volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$.

# Exercises

## Exer 9.1.1

Use the "slicing" method to derive the formula for the volume of a circular cone, of height $h$ and base with radius $r$.

## Exer 9.1.2

Use the "disk" method to derive the formula for the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$, $x = 1$ and $x = 2$.