Annotated slides

Chap. 2: Initial Value Problems

§2.1: Introduction

MA385/530 – Numerical Analysis 1

October 2019

Emile Picard: his fundamental work on differential equations was only one of his many contributions to mathematics

Olga Ladyzhenskaya: her extensive achievements include providing the first proof of the convergence of finite difference methods for the Navier-Stokes equations

(See Chap 6 of Epperson for the introduction, and Chapter 12 of Süli and Mayers for the rest).

Motivation

The growth of some tumours can be modelled as

$$\frac{dR}{dt} = R'(t) = -\frac{1}{3}S_i R(t) + \frac{2\lambda\sigma}{\mu R + \sqrt{\mu^2 R^2 + 4\sigma}},$$

subject to the initial condition $R(t_0) = a$, where R is the radius of the tumour at time t.

Clearly, it would be useful to know the value of R as certain times in the future. Though it's essentially impossible to solve for R exactly, we can accurately estimate it. In this section, we'll study techniques for this.

Initial Value Problems (IVPs)

Initial Value Problems (IVPs) are differential equations of the form: *Find $y(t)$ such that*

Initial value

D.E. →

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = f(t, y) \text{ for } t > t_0, \qquad \text{and } y(t_0) = y_0. \tag{1}$$

Here $y' = f(t, y)$ is the *differential equation* and $y(t_0) = y_0$ is the *initial value*.

Some IVPs are easy to solve. For example:

$$y' = t^2 \quad \text{with } y(1) = 1.$$

We can solve this by integrating:

$\frac{dy}{dt} = t^2$, so $y(t) = \frac{1}{3}t^3 + C$.

using $t=1 \Rightarrow y=1$ gives $C = 2/3$.

So $y(t) = \frac{1}{3}t^3 + 2/3$

Most problems are much harder, and some don't have solutions at all. In many cases, it is possible to determine that a giving problem does indeed have a solution, even if we can't write it down. The idea is that the function f should be "Lipschitz", a notion closely related to that of a ***contraction***.

Definition 2.1 (Lipschitz Condition)

A function f satisfies a ***Lipschitz Condition*** (with respect to its second argument) in the rectangular region D if there is a positive real number L such that

$$|f(t,u) - f(t,v)| \le L|u - v| \tag{2}$$

for all $(t,u) \in D$ and $(t,v) \in D$.

Compare with contraction

$$|g(\alpha) - g(\beta)| \le L|\alpha - \beta|$$

with $L \in (0,1)$.

Example 2.2

For each of the following functions f, show that is satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L.

(i) $f(t,y) = y/(1+t)^2$ for $0 \leq t \leq \infty$.

(ii) $f(t,y) = 4y - e^{-t}$ for all t.

(iii) $f(t,y) = -(1+t^2)y + \sin(t)$ for $1 \leq t \leq 2$.

Eg (i) $f(t,y) = \dfrac{y}{(1+t)^2}$ $\quad t \in [0, \infty)$

$|f(t,u) - f(t,v)| = \dfrac{1}{(1+t)^2} |u - v|$

So this is Lipschitz, with $L = 1$

since $\dfrac{1}{(1+t)^2} \leq 1$ for all $t \geq 1$.

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(ii) $f(t,y) = 4y - e^{-t}$.

$$|f(t,u) - f(t,v)| = |4u - e^{-t} - 4v + e^{-t}|$$
$$= 4|u-v|$$

So this is Lipschitz, with $L = 4$.

Example 2.2

For each of the following functions f, show that is satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L.

(i) $f(t, y) = y/(1+t)^2$ for $0 \le t \le \infty$.

(ii) $f(t, y) = 4y - e^{-t}$ for all t.

(iii) $f(t, y) = -(1+t^2)y + \sin(t)$ for $1 \le t \le 2$.

(iii) Here

$$|f(t,u) - f(t,v)| = \left| -(1+t^2)u + \cancel{\sin(t)} + (1+t^2)v - \cancel{\sin(t)} \right|$$

$$= (1+t^2)\,|u-v|.$$

So this is Lipschitz, with $L = 5$.

Theorem 2.3 (Picard's)

Suppose that the real-valued function $f(t, y)$ is continuous for $t \in [t_0, t_M]$ and $y \in [y_0 - C, y_0 + C]$; that $|f(t, y_0)| \leq K$ for $t_0 \leq t \leq t_M$; and that f satisfies the *Lipschitz condition* (2). If

$$C \geq \frac{K}{L}\left(\mathrm{e}^{L(t_M - t_0)} - 1\right),$$

then (1) has a unique solution on $[t_0, t_M]$. Furthermore

$$|y(t) - y(t_0)| \leq C \qquad t_0 \leq t \leq t_M.$$

You are not required to know this theorem for this course. However, it's important to be able to determine when a given f satisfies a Lipschitz condition.

Exercise 2.1

For the following functions show that they satisfy a Lipschitz condition on the corresponding domain, and give an upper-bound for L:

(i) $f(t, y) = 2yt^{-4}$ for $t \in [1, \infty)$,

(ii) $f(t, y) = 1 + t\sin(ty)$ for $0 \leq t \leq 2$.

Exercise 2.2

Many text books, instead of giving the version of the Lipschitz condition we use, give the following: *There is a finite, positive, real number L such that*

$$|\frac{\partial}{\partial y} f(t, y)| \leq L \qquad \text{for all } (t, y) \in D.$$

Is this statement *stronger than* (i.e., more restrictive then), *equivalent to* or *weaker than* (i.e., less restrictive than) the usual Lipschitz condition? Justify your answer.
Hint: the Wikipedia article on Lipschitz continuity is very informative.

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