MA385 Part 2: Initial Value Problems

2.5: Runge-Kutta 4 (RK4)

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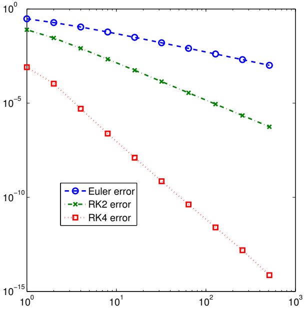

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For more details, see Chapter 6 of Süli and Mayers, *An Introduction to Numerical Analysis*. In particular, see Section 12.4 (Runge-Kutta methods)

1. A higher order method

It is possible to construct methods that have higher orders of accuracy than **RK2** methods. Of these, the most used are probably those that belong to the **Runge-Kutta 4 (RK4)** family, and have the property that

$$|y(t_n) - y_n| \leq Ch^4.$$

However, even writing down the general form of the RK4 method, and then deriving conditions on the parameters is rather complicated. Therefore, we'll focus on just one RK4 method, and use examples, rather than theory, to demonstrate that it is 4th-order.

"The RK4 Method"

$$\begin{cases} k_1 = f(t_i, y_i), \\ k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1), \\ k_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2), \\ k_4 = f(t_i + h, y_i + hk_3), \\ y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{cases}$$

ie $\Phi(t_i, y_i; h) = \frac{1}{6}(k_1 + k_2 + k_3 + k_4).$

The RK4 method can be interpreted as follows :

k_1 – comes from Euler's Method, and gives an $\mathcal{O}(h)$ approximation to y_{i+1}

k_2 - approximates the solution using an RK2 method based at $t_i + h/2$ which is $\mathcal{O}(h^2)$

k_3 – same again, but using k_2 $\left(\mathcal{O}(h^3)\right)$

k_4 – uses k_3 to estimate y_{i+1}.

Then: Take a weighted average.

1. A higher order method

As the following example shows, RK4 can be much more accurate than the Euler or RK2 methods for small h (i.e., large n). For the RK4, doubling n reduces the error by a factor of 8 (compared with 2 and 4 for the Euler and RK2 methods, respectively).

Example 2.5.1

Compare Euler, Modified Euler, and RK4 for approximating $y(1)$ *where:* $y(0) = 1, \quad y'(t) = y\log(1+t^2)$.

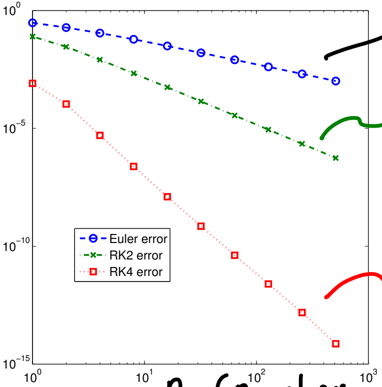
Error: $|y(t_n) - y_n|$

n	Euler	Modified	RK4
1	3.02e-01	7.89e-02	8.14e-04
2	1.90e-01	2.90e-02	1.08e-04
4	1.11e-01	8.20e-03	5.07e-06
8	6.02e-02	2.16e-03	2.44e-07
16	3.14e-02	5.55e-04	1.27e-08
32	1.61e-02	1.40e-04	7.11e-10
64	8.13e-03	3.53e-05	4.18e-11
128	4.09e-03	8.84e-06	2.53e-12
256	2.05e-03	2.21e-06	1.54e-13
512	1.03e-03	5.54e-07	7.33e-15

Example 2.5.2

Compare Euler, Modified Euler, and RK4 for approximating $y(1)$ *where:* $y(0) = 1, \quad y'(t) = y\log(1+t^2)$.

Error



slope -1

slope of -2

slope of -4

n (number of steps)

2. Consistency and convergence of RK4

Although we won't do a detailed analysis of RK4, we can do a little. In particular, we would like to show it is

(i) consistent,

(ii) convergent and fourth-order, at least for some examples.

Example 2.5.3

It is easy to see that RK4 is consistent:

Recall: a method, $y_{i+1} = y_i + h\,\Phi(t_i; y_i; h)$

is consistent if $\Phi(t_i; y_i; 0) = f(t_i, y_i)$

$k_1 = f(t_i, y_i)$

$k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) = f(t_i, y_i)$ if $h=0$

Similarly $k_3 = k_4 = f(t_i, y_i)$

So $\Phi(t_i, y_i; 0) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = f(t_i, y_i)$

Example 2.5.4

In general, showing the rate of convergence is tricky. Instead, we'll demonstrate how the method relates to a Taylor Series expansion for the problem $y' = \lambda y$ where λ is a constant.

Our ODE is $y'(t) = \lambda y$.

We can solve this $y(t) = e^{\lambda t} + C$,

for some constant C.

So then $y'(t) = \lambda e^{\lambda t} = \lambda y(t)$

Also $y''(t) = \lambda^2 e^{\lambda t} = \lambda^2 y(t)$

$y'''(t) = \lambda^3 y(t)$, $y^{(n)}(t) = \lambda^n y(t)$.

Now write out a Taylor series for $y(t_{i+1})$ about t_i

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{3!} y'''(t_i) + \frac{h^4}{4!} y^{(4)}(t_i) + \frac{h^5}{5!} y^{(5)}(\eta)$$

for some $\eta \in [t_i, t_{i+1}]$ and using $t_{i+1} - t_i = h$.

Then, using $y^{(n)}(t_i) = \lambda^n y(t_i)$, we get

$$y(t_{i+1}) = y(t_i)\left[1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24}\right] + \mathcal{O}(h^5)$$

Next, write out the RK4 method for this problem:

$$\boxed{f(t,y) = \lambda y(t)}$$

$$k_1 = f(t_i, y_i) = \lambda y_i$$

$$k_2 = f\left(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2}\lambda y_i\right)$$
$$= \lambda\left(y_i + \tfrac{h}{2}\lambda y_i\right) = y_i\left(\lambda + \frac{h\lambda^2}{2}\right)$$

$$k_3 = f\left(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2}k_2\right) = \lambda\left(y_i + \tfrac{h}{2}y_i\left(\lambda + \frac{h\lambda^2}{2}\right)\right)$$
$$= y_i\left(\lambda + \tfrac{1}{2}h\lambda^2 + \tfrac{1}{4}h^2\lambda^3\right)$$

And (check!)

$$k_4 = y_i\left(\lambda + h\lambda^2 + \tfrac{1}{2}h^2\lambda^3 + \tfrac{1}{4}h^3\lambda^4\right)$$

Many (seemingly different) RK have been proposed and studied. A unified approach of representing them was developed by John Butcher: write an s-stage method as

Each stage gives a "k". So RK4, which has k_1, k_2, k_3 k_4 is 4-stage.

$$\Phi(t_i, y_i; h) = \sum_{j=1}^{s} b_j k_j, \quad \text{where}$$

$$\begin{aligned}
k_1 &= f(t_i + \alpha_1 h, y_i),\\
k_2 &= f(t_i + \alpha_2 h, y_i + \beta_{21} h k_1),\\
k_3 &= f(t_i + \alpha_3 h, y_i + \beta_{31} h k_1 + \beta_{32} h k_2),\\
&\vdots\\
k_s &= f(t_i + \alpha_s h, y_i + \beta_{s1} h k_1 + \dots \beta_{s,s-1} h k_{s-1}),
\end{aligned}$$

The most convenient way to represent the coefficients is in a tableau:

$$
\begin{array}{c||ccccc}
\alpha_1 & 0 & 0 & 0 & 0 & \\
\alpha_2 & \beta_{21} & 0 & 0 & 0 & \\
\alpha_3 & \beta_{31} & \beta_{32} & 0 & 0 & \\
\vdots & & & & 0 & \\
\alpha_s & \beta_{s1} & \beta_{s2} & \cdots & \beta_{s,s-1} & \\
\hline\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

The tableaux for basic Euler, Modified Euler, and RK4 are:

$$
\begin{array}{c||c}
0 & \\
\hline\hline
 & 1
\end{array}
\qquad
\begin{array}{c||cc}
0 & & \\
1/2 & 1/2 & \\
\hline\hline
 & 0 & 1
\end{array}
\qquad
\begin{array}{c||cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline\hline
 & 1/6 & 2/6 & 2/6 & 1/6
\end{array}
$$

4. Even higher-order methods?

A Runge Kutta method has s stages if it involves s evaluations of the function f. (That it, its formula features $k_1, k_2, \ldots, k_s$).

We've seen a 1-stage method that is 1^{st}-order.

We studied 2-stage methods that are 2^{nd}-order.

In an exercise, you'll construct a 3-stage method that is 3rd order.

And, of course, we have just considered a four-stage method that is 4^{th}-order.

4. Even higher-order methods?

It is tempting to think that we could we can get an Order s method by using s stages for any s.
However, it can be shown that, for example,

- for a method to have order $p = 5$, we need $s = 6$ stages (Kutta, 1901)
- for order $p = 6$, we need at least $s = 7$ stages (Butcher, 1964).
- for order $p = 7$, we need at $s = 9$ stages (Butcher, 1968).
- for order $p = 8$, we need at $s = 11$ stages (Curtis, 1970).
- for order $p = 10$, we need at $s = 17$ stages (Hairer, 1978).

The theory involved is both intricate and intriguing, and involves aspects of group theory, graph theory, and differential equations.
See https://doi.org/10.1016/0168-9274(95)00108-5

Finished here at the end of the lecture Thu 16 Oct (W6)

Exercise 2.5.1

We claim that, for $RK4$:

$$|\mathcal{E}_N| = |y(t_N) - y_N| \leq Kh^4.$$

for some constant K. How could you verify that the statement is true using the data of Table 2.3, at least for test problem in Example 2.4.2? Give an estimate for K.

Exercise 2.5.2

Recall the problem in Example 2.2.2: *Estimate* $y(2)$ *given that*

$$y(1) = 1, \qquad y' = f(t, y) := 1 + t + \frac{y}{t},$$

(i) Show that $f(t, y)$ satisfies a Lipschitz condition and give an upper bound for L.

(ii) Use Euler's method with $h = 1/4$ to estimate $y(2)$. Using the true solution, calculate the error.

(iii) Repeat this for the RK2 method of your choice (with $a \neq 0$) taking $h = 1/2$.

(iv) Use $RK4$ with $h = 1$ to estimate $y(2)$.

Exercise 2.5.3

Here is the tableau for a three stage Runge-Kutta method:

$$\begin{array}{c||ccc}
0 & & & \\
\alpha_2 & 1/2 & & \\
1 & \beta_{31} & 2 & \\
\hline\hline
 & 1/6 & b_2 & 1/6
\end{array}$$

(i) Use that the method is consistent to determine b_2.

(ii) The method is exact when used to compute the solution to

$$y(0) = 0, \quad y'(t) = 2t, \ t > 0.$$

Use this to determine α_2.

(iii) The method should agree with an appropriate Taylor series for the solution to $y'(t) = \lambda y(t)$, up to terms that are $\mathcal{O}(h^3)$. Use this to determine β_{31}.