Piecewise Polynomials

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MA378 Chapter 2: Splines

§2.1 Linear Interpolating Splines

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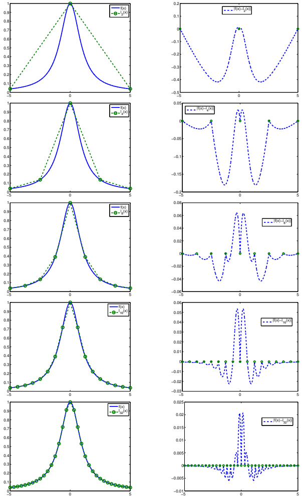

1.1 Introduction

In Section 1.5 (Convergence and Runge's Example), we learned that it is not always a good idea to interpolate functions by a high-order polynomials at equally spaced points. However, it is possible to obtain very good approximations using a very simple method. The trick is to use a **spline**: a ***piecewise polynomial interpolating function***.

We'll consider three important example of splines:

1. ***linear splines***
2. ***(natural) cubic splines***.
3. ***Hermite*** piecewise cubics.

For more details about splines, have a look at Chap. 11 of Süli and Mayers, and Lectures 10 and 11 Stewart's "Afternotes goes to Grad School".

1.1 Introduction

In this section, we always have N equally spaced ~~points~~ intervals: let $h = (b - a)/N$, then (and so $N+1$ points).

$$a = x_0, \quad b = x_N \quad \text{and} \quad x_i = x_0 + ih \ \text{ for } \ i = 0, 1, \ldots, N.$$

Often these are referred to as *knots points* (or simply as *knots*), and denote the set of knot points by $\omega^N := \{x_i\}_{i=0}^{N}$.

$$x_0 = x_0 + 0(h) = x_0 = a$$

$$x_1 = x_0 + h$$

$$\ldots$$

$$x_N = x_0 + N(h) = x_0 + N\left(\frac{x_N - x_0}{N}\right)$$

$$= x_N = b$$

1.2 Linear Interpolating Splines

We first study the ***piecewise linear interpolant***, also called a ***linear spline***. We will see that they have important properties, including

(a) they are easy to construct and analyse;

(b) the bound on the error decreases as the number of interpolation points increases;

(c) the error we get using a linear spline is no more than twice the error using the best possible (piecewise linear) approximation; amd

(d) of all the interpolants to f at a given set of points, the linear spline is the one with the $\underbrace{\textbf{smallest first derivative}}_{\text{"Minimum Energy".}}$.

1.3 Construction on linear splines

Definition 1.1

Let f be a function that is continuous on $[a, b]$. The *linear spline interpolant* to f is the continuous function, l, such that

(i) $l(x_i) = f(x_i)$ for each $i = 0, 1, \ldots, N$, "Interpolates f"

(ii) l is a linear function l_i on each interval $[x_{i-1}, x_i]$. That is,

$$l(x) = \begin{cases} l_1(x) & x_0 \leq x \leq x_1 \\ l_2(x) & x_1 \leq x \leq x_2 \\ \ldots & \\ l_N(x) & x_{N-1} \leq x \leq x_N \end{cases}$$

it is piece-wise linear.

1.3 Construction on linear splines

It is easy to write down a formula for the l_i, based on Lagrange polynomials:

- Set $h = (b - a)/N$.
- For each $i = 1, 2, \dots, N$, define

$$l_i(x) = f(x_{i-1})\frac{x_i - x}{h} + f(x_i)\frac{x - x_{i-1}}{h}, \quad x \in [x_{i-1}, x_i]. \quad (1)$$

Note that each l_i is a linear polynomial.

(ie highest power of x is 1)

Also, for example,

$$l_i(x_i) = f(x_{i-1})\underbrace{\frac{x_i - x_i}{h}}_{0} + f(x_i)\underbrace{\frac{x_i - x_{i-1}}{h}}_{1} = f(x_i)$$

similarly $l_i(x_{i-1}) = f(x_{i-1})$. So l interpolates f.

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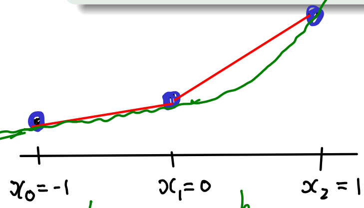
Example 1.2

Write down the linear spline interpolant to $f(x) = e^x$ at the knot points $\{-1, 0, 1\}$.



h $\quad$ h

$$l(x) = \begin{cases} l_1(x) & -1 \le x < 0 \\ l_2(x) & 0 \le x \le 1 \end{cases}$$

Here $l_1(x) = f(x_0)\dfrac{x_0 - x}{h} + f(x_1)\dfrac{x - x_1}{h}$

$$= e^{-1}(-1 - x) + (1)x$$

And $l_2(x) = (1 - x) + e x.$

We know that if p_N is the polynomial of degree N that interpolates f at n equally spaced points, it does **not** follow that $p_N \to f$ as $N \to \infty$. But as we will see, the piecewise linear interpolant to f converges to f, albeit slowly.

This is verified in the following theorem, which is a direct consequence of Cauchy's theorem.

That is, we dont have that

$$\lim_{N\to\infty} \|f - p_N\|_\infty = 0.$$

But,
$$\lim_{N\to\infty} \|f - \ell\|_\infty = 0 \quad !!$$

Theorem 1.3

Suppose that f, f' and f'' are all continuous and defined on the interval $[a, b]$. Let l be the linear spline interpolant to f on the $N + 1$ equally spaced points $a = x_0 < x_1 \cdots < x_N = b$ with $h = x_i - x_{i-1} = (b - a)/N$. Then

$$\|f - l\|_\infty \leq \frac{h^2}{8}\|f''\|_\infty,$$

(Here $\|g\|_\infty := \max_{a \leq x \leq b} |g(x)|$.) . Proof: Cauchy's theorem gives that, on any interval $[x_{i-1}, x_i]$

$$|f(x) - p_1(x)| \leq \frac{|f''(\tau_i)|}{2} |(x - x_{i-1})(x - x_i)|$$

where $\tau_i \in [x_{i-1}, x_i]$

Since $|f''(\tau_i)| \leq \|f''\|_\infty$ for any i,

and $l_i = p_1$ on $[x_{i-1}, x_i]$ we get

$$|f(x) - l_i(x)| \leq \frac{\|f''\|_\infty}{2} |(x - x_i)(x - x_{i-1})|$$

Furthermore $\displaystyle\max_{x_{i-1} \leq x \leq x_i} |(x - x_i)(x - x_{i-1})| \leq \frac{h^2}{4}$

(Did this last week!)

Then, for any i, $|f(x) - l_i(x)| \leq \frac{h^2}{8}\|f''\|_\infty$

So, in fact

$$|f(x) - l(x)| \leq \frac{h^2}{8}\|f''\|_\infty$$

It follows directly from this theorem that

$h = \frac{b-a}{N}$

$$\lim_{N\to\infty} \|f - l\|_\infty = 0.$$

This is because, for any N,

$$\|f - l\|_\infty \leq \frac{h^2}{8}\|f''\|_\infty = \frac{1}{N^2}\underbrace{\left[\frac{(b-a)^2}{8}\|f''\|_\infty\right]}_{C}$$

So

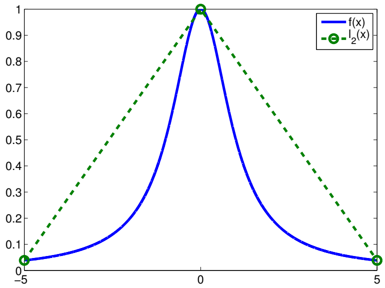
$$\lim_{N\to\infty} \|f - l\|_\infty = \lim_{N\to\infty} \frac{C}{N^2} = 0$$

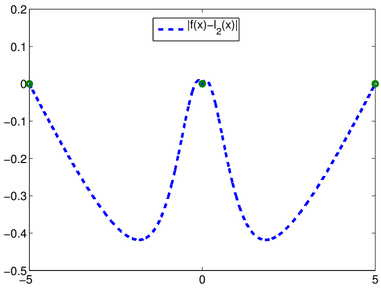
Example 1.4

The figure below shows linear spline interpolations of Runge's example:

$$f(x) = \frac{1}{1+x^2} \text{ on } [-5,5].$$

These diagrams appear to support our assertion that the error tends to zero as $N \to \infty$.





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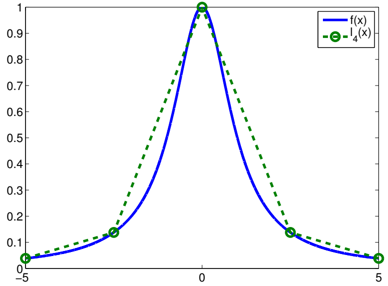

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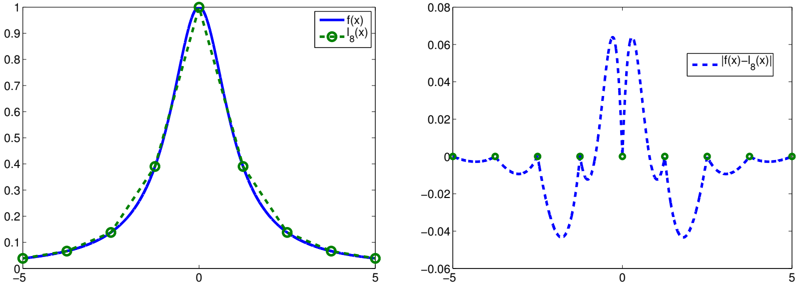

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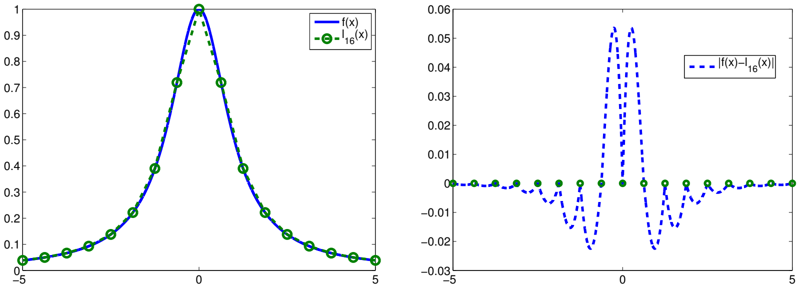

1.4 Analysis

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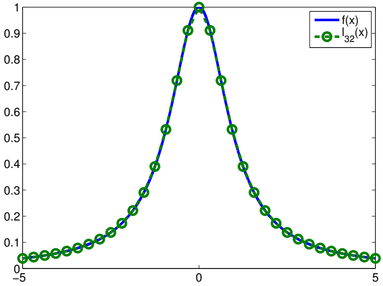
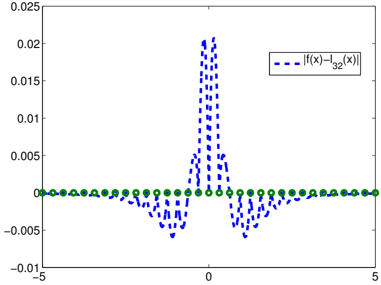




Example 1.5

Suppose you are interpolating $f(x) = e^x$ on n equally spaced intervals between $x_0 = -1$ and $x_N = 1$. What value of N would you have to take to ensure that the maximum error is less than 10^{-2}?

N

we want $\|f - l\|_\infty \le 10^{-2}$, so we choose N (or h) such that

$$\|f - l\|_\infty \le \frac{h^2}{8}\|f''\|_\infty \le 10^{-2}$$

Since $f(x) = e^x$, so $f''(x) = e^x$. Then on $[-1, 1]$

$\|f''\|_\infty = e \sim 2.7813$. So we require h so that

$$\frac{h^2}{8} e \le 10^{-2} \Leftrightarrow h^2 \le \frac{8}{e} 10^{-2} \Rightarrow h \le \sqrt{\frac{8}{e} 10^{-2}}$$

So $h \le 0.17155$. $N = \frac{(b-a)}{h} \Rightarrow N \geq 11.655$. So $N = 12$

For the next part of the analysis it will help to think of piecewise linear interpolation as an *operator*. Then we can compare the linear spline to all the other piecewise linear approximations.

First, observe that one can define an infinite number of piecewise linear functions on a given set of $N + 1$ knot points, denoted ω^N. We'll call the set of these functions $\mathcal{L}$.

An operator maps functions to functions.
Eg differentiation can be thought of as an operator

1.5 Best approximation

Definition 1.6

For a fixed set of knot points ω^N, let L be the operator that maps the continuous function f to its linear spline interpolant $l \in \mathcal{L}$.

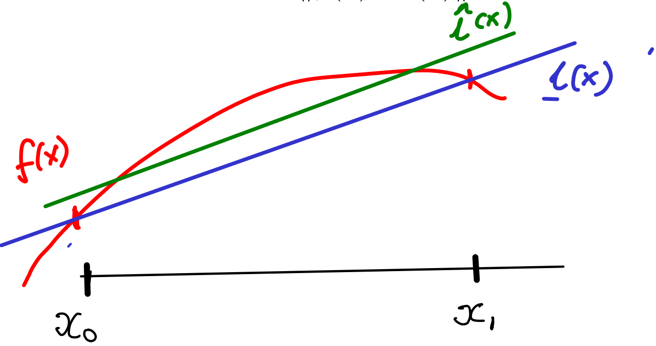
Now suppose that $g \in \mathcal{L}$. Then $\mathrm{L}(g) = g$. That is L is a *projection*: $\mathrm{L}(\mathrm{L}(f)) = \mathrm{L}(f)$.

1.5 Best approximation

It is not hard to see that one could find a different function $\hat{l} \in \mathcal{L}$ that is a better approximation of f in sense that

$$\max_{x_0 \le x \le x_n} |f(x) - \hat{l}(x)| < \max_{x_0 \le x \le x_n} |f(x) - l(x)|.$$

However, l is very easy to find, and the associated error is no worse than twice $\|f(x) - \hat{l}(x)\|_\infty$.



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Theorem 1.7 (Stewart's "Afternotes goes to grad school", Lecture 10)

Let $l = \mathrm{L}(f)$. *For all* $\hat{l} \in \mathcal{L}$,

$\|\cdot\| = \|\cdot\|_\infty$

$$\|f - l\|_\infty \leq 2\|f - \hat{l}\|_\infty.$$

(That L is a projection is key to the proof.)

Proof $\|f - l\| = \|f - \hat{l} + \hat{l} - l\|$

$\leq \|f - \hat{l}\| + \|\hat{l} - l\|$ (Triangle ineq),

$= \|f - \hat{l}\| + \|L(\hat{l}) - L(f)\|$

$= \|f - \hat{l}\| + \|L(\hat{l} - f)\|$

$\leq \|f - \hat{l}\| + \|\hat{l} - f\|$ Since $\|L(g)\| \leq \|g(x)\|$

1.6 Minimum Energy

The final interesting property of l that we will study is called the *minimum energy property*.

Definition 1.8

Let u be a function that is continuous and defined on the interval $[a, b]$ except, maybe, at the (countable set) ω^N of knot points[a] Then the 2-norm of u is

$$\|u\|_{2,[a,b]} := \Big(\int_a^b u^2(x)dx \Big)^{1/2}.$$

Usually we just write this as $\|u\|_2$.

[a] More precisely, we should say "everywhere, except on a set of measure zero". However, since not everyone is familiar with the terminology, we'll skip the details.

1.6 Minimum Energy

Let H^1 be the set of all functions u that are continuous on $[a, b]$ and have $\|u'\|_2 < \infty$. Note that $l \in H^1$, even though we have not properly defined l' at the mesh points ω^N.

Theorem 1.9 (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l we the linear spline interpolant of f. Then

$$\|l'\|_2 \leq \|w'\|_2.$$

For any $w \in H^1$, $\quad \|w'\|_2^2 := \int_a^b (w')^2\,dx = \int_a^b \big((w'-l')+l'\big)^2\,dx.$

$$= \int_a^b \underbrace{(w'-l')^2}_{\|w'-l'\|_2^2 \geq 0} + 2(w'-l')l' + \underbrace{(l')^2\,dx}_{=\|l'\|_2^2} \geq \|l'\|_2^2$$

which will follow if we can show that

$$2\int_a^b (w'-l')l'\,dx = 0.$$

Theorem 1.9 (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l we the linear spline interpolant of f. Then

$$\|l'\|_2 \leq \|w'\|_2.$$

To see this we use that

$$\int_a^b (w-l)'l'\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (w-l)'l'\,dx .$$

Integrating by part, we ge

$$\int_{x_{i-1}}^{x_i} (w-l)'l'\,dx = l'(w-l)\Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (w-l)l''\,dx$$

But both w & l interpolate f at x_i, so $(w-l)(x_i)=0$ for all i, so $l'(w-l)\Big|_{x_{i-1}}^{x_i} = 0$.

Theorem 1.9 (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l we the linear spline interpolant of f. Then

$$\|l'\|_2 \leq \|w'\|_2.$$

Also, since $l(x)$ is piecewise linear,

so $l'(x)$ is piecewise constant, and $l''(x) \equiv 0$.

Thus $\int_{x_{i-1}}^{x_i} (w-l)(x) l''(x)\,dx = 0$.

So we know $\int_a^b (w-l)' \, l' \, dx = 0$, finishing the proof.

1.7 Looking ahead

Piecewise linear interpolation is one of the most standard tools in computational science. Very likely, every time you have zoomed into an image on your phone, you have used it.

It's major drawback is that it can't represent the *curvature* of the function it is interpolating. In the next section we'll investigate how to do that using *cubic* splines.

1.8 Exercises

Exercise 1.1

Page 28 of the Department of Education's old Mathematics Tables ("The *Log Tables*") reports that $\ln(1) = 0$, $\ln(1.5) = 0.4055$ and $\ln(2) = 0.6931$.

(i) Write down the linear spline l that interpolates $f(x) = \ln(x)$ at the points $x_0 = 1$, $x_1 = 1.5$ and $x_2 = 2$.

(ii) Use this to estimate $\ln(x)$ at $x = 1.2$. How does this compare to the value in the tables? (0.1823)

(iii) Give an estimate for the maximum error:

$$\max_{1 \leq x \leq 2} |f(x) - l(x)|.$$

(iv) What value of n would you choose to ensure that $|f(x) - l(x)| \leq 0.001$ for all $x \in [1, 2]$.

Exercise 1.2

As an alternative to (1), one can define the linear spline interpolant to a function is as a linear combination of a set of piecewise linear basis functions $\{\psi_i\}_{i=0}^{N}$:

$$\psi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(i) Write down a formula for the $\psi_i(x)$;

(ii) derive a formula for $l(x)$ in terms of the ψ_i.

This exercise is useful: we'll use these basis functions (called "hat" functions) in the final section of the course.

1.8 Exercises

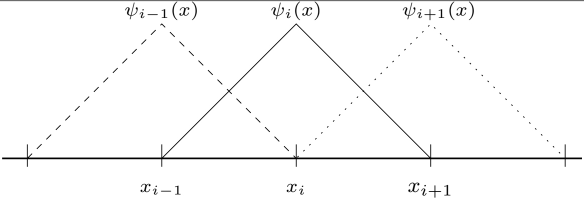

Figure: Some hat functions