2425-MA140 Engineering Calculus

Week 2, Lecture 1
Partial Fractions

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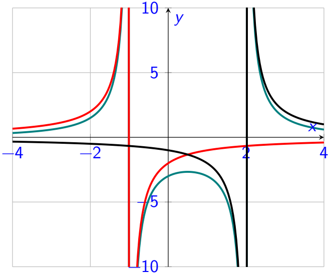

This version of the slides are by Niall Madden, but are adapted from original notes by Dr Kirsten Pfeiffer.

Outline

For more, see Section 2.5 (Rational Functions) of *Modern Engineering Mathematics*:
https://search.library.nuigalway.ie/permalink/f/3b1kce/TN_cdi_askewsholts_vlebooks_9780273742517

Tutorials start **this** week. The schedule is:

- Teams 1+2: Tuesday 15:00 ENG-**2003**
- Teams 3+4: Tuesday 15:00 ENG-**2034**
- Teams 9+10: Thursday 11:00 ENG-**2002**
- Teams 11+12: Thursday 11:00 ENG-**3035**
- Teams 5+6: Friday 13:00 Eng-**2002**
- Teams 7+8: Friday 13:00 Eng-**2035**

If you are interested to taking a tutorial through Irish, please complete this survey: http://tinyurl.com/suirbhe1

- There is currently a "practice" assignment open. See https://universityofgalway.instructure.com/courses/35693/assignments/94873
- During tutorials, the tutor will solve some similar questions. You can access the tutorial sheet at https://universityofgalway.instructure.com/courses/35693/files/2023552?module_item_id=650912
- A new assignment will open by tomorrow...

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In addition, each week I'll post a set of exercises related to the material covered. You don't have to submit your work for these, but you should try them: they are similar in style and standard to exam questions.

Partial Fractions

Rational Functions have the general form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

An (proper) rational function can often be written as a sum of simpler ones: **partial fractions.**

For example

$$\frac{8x - 12}{x^2 - 2x - 3}$$

can be written as

$$\frac{3}{x-3} + \frac{5}{x+1}$$

In order to do this, we try to **factorize** the denominator.

Partial Fractions

Note: Any polynomial (with real coefficients) can be factorised fully into the product of

- linear
- and irreducible quadratic factors.

We get different combinations of factors in the denominator. Let's look at **four cases**, and how to find the partial fractions in each case.

The four cases

1. Linear factors to the power of 1 in the denominator.
2. Linear factors to the power greater than 1 in the denominator, (i.e repeated linear factors).
3. Irreducible quadratic factors.
4. Irreducible quadratic factors to power greater than 1.

1: Eg $f(x) = \frac{p(x)}{q(x)}$ $q(x) = (x-2)(x-4)$.

2. $q(x) = (x+2)(x^2+2x+1) = (x+2)(x+1)^2$

3. $q(x) = x^2+1$

4 $q(x) = (x^2+1)^2$

(1) Linear factors to the power of 1 in the denominator.

Example

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Find A and B.

Method 1: Compare coef (ie of $1 = x^0$ and of x) by matching powers of x. Multiply the terms on the Right, above + below, to get $(x-1)(x+2)$ in the denominator:

$$\frac{3x}{(x-1)(x+2)} = \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x-1)(x+2)}.$$

There are **two methods** for finding A and B.

Method 1: Comparing coefficients (continued).

$$\frac{3x+0}{(x-1)(x+2)} = \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x-1)(x+2)}.$$

$$= \frac{Ax + 2A + Bx - B}{(x-1)(x+2)}.$$

$$= \frac{(A+B)x + (2A-B)}{(x-1)(x+2)}.$$

So
$$\begin{array}{rl} A + B &= 3 \\ 2A - B &= 0 \\ \hline 3A &= 3 \end{array} \Rightarrow A = 1, \quad B = 2$$

Method 2: Substituting specific values for x.

Recall

$$\frac{3x}{(x-1)(x+2)} = \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x-1)(x+2)}.$$

So $3x = A(x+2) + B(x-1)$. for all x.

ⓐ Pick $x = 1$. $\Rightarrow$ $3(1) = A((1)+2) \Rightarrow 3 = 3A$

$\boxed{A = 1}$

ⓑ Pick $x = -2$ $\Rightarrow$ $3(-2) = A(0) + B(-2-1)$

$\Rightarrow -6 = -3B$ $\boxed{B = 2}$

So $$\frac{3x}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{2}{x+2}.$$

Example

Write $\dfrac{8x-12}{x^2-2x-3}$ as sum of partial fractions.

Step 1: factorise x^2-2x-3 as $(x-3)(x+1)$.

$$\frac{8x-12}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}.$$

So $8x-12 = A(x+1) + B(x-3)$

ⓐ $x=3$ $\quad 8(3)-12 = A(4) \Rightarrow 12 = 4A \Rightarrow A=3$

ⓑ $x=-1$ $\quad 8(-1)-12) = B(-4) \Rightarrow -20 = -4B \Rightarrow B=5$

Ans $\dfrac{8x-12}{x^2-2x-3} = \dfrac{3}{x-3} + \dfrac{5}{x+1}$ ✓

Exercise 2.1

Find the constants A, B and C, so that

$$\frac{2x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

Week 2, Exer 1.

(2) Linear factors to the power greater than 1 in the denominator, (i.e repeated linear factors).

If $(x-\alpha)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\dots+\frac{A_k}{(x-\alpha)^k}$$

Example

Find A, B and C such that

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

(Note: we'll find that $A = 5/9$, $B = 4/3$ and $C = -5/9$).

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2)}{(x-1)(x-1)(x+2)} + \frac{B(x+2)}{(x-1)^2(x+2)} + \frac{C(x-1)^2}{(x+2)(x-1)^2}$$

So we see

$$3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

$$3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

(a) $x=1$ $\quad 3(1)+1 = A(0)(3) + B(3) + C(0)^2$

$$\Rightarrow 4 = 3B \qquad B = 4/3.$$

(b) $x=-2$ $\Rightarrow$ $3(-2)+1 = A(-3)(0) + B(0) + C(-3)^2$

$$\Rightarrow -5 = 9C \qquad C = -5/9.$$

So now we have

$$3x+1 = A(x^2+x-2) + (4/3)(x+2) + (-5/9)(x-1)^2$$

Now we match the powers of x and get $x^2(A - 5/9) = 0 \Rightarrow A = 5/9$

(3) Irreducible quadratic factors.

Irreducible quadratic factors can not be factorised using real numbers, e.g. $x^2 + x + 1$.

An irreducible quadratic factor $ax^2 + bx + c$ gives rise to partial fractions of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example 2.34 from textbook

If one writes

irreducible.

$$\frac{5x}{\underbrace{(x^2 + x + 1)}(x - 2)} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 2}$$

then we find $A - 10/7$, $B = 5/7$ and $C = 10/7$.

(4) Irreducible quadratic factors to power greater than 1.

Each repeated irreducible quadratic factor $(ax^2 + bx + c)^k$ in the denominator will give rise to

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$

These can be done in a similar way to the previous case. But the calculations are pretty messy, so we won't even try!

Exercise 2.2

Express the following as partial fractions.

1. $\dfrac{6}{x^2 - x - 2}$
2. $\dfrac{2x - 1}{x^2 - x - 2}$
3. $\dfrac{x - 1}{(x + 1)(x^2 - x - 2)}$
4. $\dfrac{x}{x^2 + 2x + 1}$
5. $\dfrac{1}{x^3 - 1}$

Finished here!