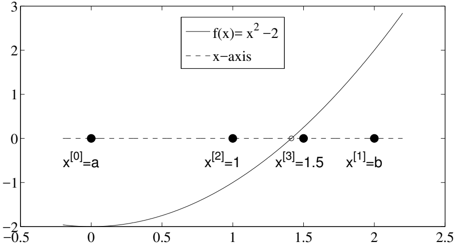
Solving nonlinear equations

§1.1: The bisection method

MA385/530 – Numerical Analysis

September 2019



Linear equations are of the form:

$$\textit{find } x \textit{ such that } ax + b = 0$$

and are easy to solve. Some nonlinear problems are also easy to solve, e.g.,

$$\textit{find } x \textit{ such that } ax^2 + bx + c = 0.$$

Similarly, there are formulae for all cubic and quartic polynomial equations. But most equations do not have simple formulae for their solutions, so numerical methods are needed.

References

- Chap. 1 of Süli and Mayers (Introduction to Numerical Analysis). We'll follow this pretty closely in lectures, though we will do the sections in reverse order!
- Stewart (*Afternotes ...*), Lectures 1–5. A well-presented introduction, with lots of diagrams to give an intuitive introduction.
- Chapter 4 of Moler's "Numerical Computing with MATLAB". Gives a brief introduction to the methods we study, and description of MATLAB functions for solving these problems.
- The proof of the convergence of Newton's Method is based on the presentation in Thm 3.2 of Epperson.

Our generic problem is:

> *Let f be a continuous function on the interval $[a, b]$.*
> *Find $\tau \in [a, b]$ such that $f(\tau) = 0$.*

Here f is some specified function, and τ is the ***solution*** to $f(x) = 0$.

This leads to two natural questions:

(1) How do we know there is a solution?

(2) How do we find it?

The following gives *sufficient* conditions for the existence of a solution:

Theorem 1.1

Let f be a real-valued function that is defined and continuous on a bounded closed interval $[a, b] \subset \mathbb{R}$. Suppose that $f(a)f(b) \leq 0$. Then there exists $\tau \in [a, b]$ such that $f(\tau) = 0$.

So now we know there is a solution τ to $f(x) = 0$, but how to we actually solve it? **Usually we don't!** Instead we construct a sequence of estimates $\{x_0, x_1, x_2, x_3, \dots\}$ that ***converge*** to the true solution. So now we have to answer these questions:

(1) How can we construct the sequence $x_0, x_1, \dots$?

(2) How do we show that $\lim_{k\to\infty} x_k = \tau$?

There are some subtleties here, particularly with part (2). What we would like to say is that at each step the error is getting smaller. That is

$$|\tau - x_k| < |\tau - x_{k-1}| \qquad \text{for } k = 1, 2, 3, \ldots.$$

But we can't. Usually all we can say is that the ***bounds*** on the error is getting smaller. That is: ***let ε_k be a bound on the error at step k***

$$|\tau - x_k| < \varepsilon_k,$$

then $\varepsilon_{k+1} < \mu\varepsilon_k$ for some number $\mu \in (0, 1)$. It is easiest to explain this in terms of an example, so we'll study the simplest method: ***Bisection***.

The most elementary algorithm is the "*Bisection Method*" (also known as "Interval Bisection"). Suppose that we know that f changes sign on the interval $[a, b] = [x_0, x_1]$ and, thus, $f(x) = 0$ has a solution, τ, in $[a, b]$. Proceed as follows

1. Set x_2 to be the midpoint of the interval $[x_0, x_1]$.
2. Choose one of the sub-intervals $[x_0, x_2]$ and $[x_2, x_1]$ where f change sign;
3. Repeat Steps 1–2 on that sub-interval, until f is sufficiently small at the end points of the interval.

This may be expressed more precisely using some *pseudocode*.

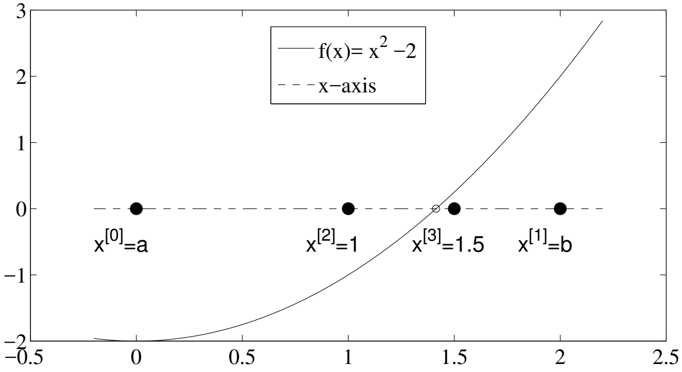
The Bisection Algorithm

Set eps to be the stopping criterion.
If $|f(a)| \leq eps$, return a. Exit.
If $|f(b)| \leq eps$, return b. Exit.
Set $x_L = a$ and $x_R = b$.
Set $k = 1$
while($|f(x_k)| > eps$)
    $x_{k+1} = (x_L + x_R)/2$;
    if $(f(x_L)f(x_{k+1}) < eps)$
        $x_R = x_{k+1}$;
    else
        $x_L = x_{k+1}$
    end if;
    $k = k + 1$
end while;

Example 1.2

Find an estimate for $\sqrt{2}$ that is correct to 6 decimal places.

Solution: Use bisection to solve $f(x) := x^2 - 2 = 0$ on the interval $[0, 2]$.



Find an estimate for $\sqrt{2}$ that is correct to 6 decimal places.
Solution: Use bisection to solve $f(x) := x^2 - 2 = 0$ on the interval $[0, 2]$.

k	x_k	$\lvert x_k - \tau \rvert$	$\lvert x_k - x_{k-1} \rvert$
0	0.000000	1.41	
1	2.000000	5.86e-01	
2	1.000000	4.14e-01	1.00
3	1.500000	8.58e-02	5.00e-01
4	1.250000	1.64e-01	2.50e-01
5	1.375000	3.92e-02	1.25e-01
6	1.437500	2.33e-02	6.25e-02
7	1.406250	7.96e-03	3.12e-02
8	1.421875	7.66e-03	1.56e-02
9	1.414062	1.51e-04	7.81e-03
10	1.417969	3.76e-03	3.91e-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	1.414214	5.72e-07	9.54e-07

The main advantages of the Bisection method are

- It will always work.
- After k steps we know that

Theorem 1.3

$$|\tau - x_k| \leq \left(\frac{1}{2}\right)^{k-1}|b-a|, \quad \text{for } k = 2, 3, 4, \ldots$$

A disadvantage of bisection is that it is not particularly efficient. So our next goal will be to derive better methods, particularly the ***Secant Method*** and ***Newton's method***. We also have to come up with some way of expressing what we mean by "better"; and we'll have to use Taylor's theorem in our analyses.

Exercise 1.1

Does Proposition 1.1.1 mean that, if there is a solution to $f(x) = 0$ in $[a, b]$ then $f(a)f(b) \leq 0$? That is, is $f(a)f(b) \leq 0$ a *necessary* condition for their being a solution to $f(x) = 0$? Give an example that supports your answer.

Exercise 1.2

Suppose we want to find $\tau \in [a, b]$ such that $f(\tau) = 0$ for some given f, a and b. Write down an estimate for the number of iterations K required by the bisection method to ensure that, for a given ε, we know $|x_k - \tau| \leq \varepsilon$ for all $k \geq K$. In particular, how does this estimate depend on f, a and b?

Exercise 1.3

How many (decimal) digits of accuracy are gained at each step of the bisection method? (If you prefer, how many steps are needed to gain a single (decimal) digit of accuracy?)

Exercise 1.4

Let $f(x) = e^x - 2x - 2$. Show that there is a solution to the problem: *find* $\tau \in [0, 2]$ *such that* $f(\tau) = 0$.
Taking $x_0 = 0$ and $x_1 = 2$, use 6 steps of the bisection method to estimate τ. You may use a computer program to do this, but please note that in your solution.
Give an upper bound for the error $|\tau - x_6|$.

Exercise 1.5

We wish to estimate $\tau = \sqrt[3]{4}$ numerically by solving $f(x) = 0$ in $[a, b]$ for some suitably chosen f, a and b.

(i) Suggest suitable choices of f, a, and b for this problem.

(ii) Show that f has a zero in $[a, b]$.

(iii) Use 6 steps of the bisection method to estimate $\sqrt[3]{4}$. You may use a computer program to do this, but please note that in your solution.

(iv) Use Theorem 1.3 to give an upper bound for the error $|\tau - x_6|$.

§1.2: The secant method

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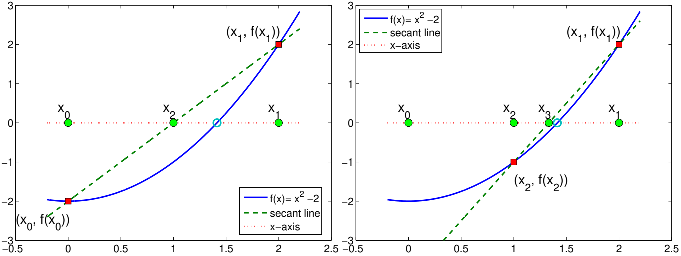
Idea:

- Choose two points, x_0 and x_1.
- Take x_2 to be the zero of the line joining $(x_0, f(x_0))$ to $(x_1, f(x_1))$.
- Take x_3 to be the zero of the line joining $(x_1, f(x_1))$ to $(x_2, f(x_2))$.
- Etc.



The Secant Method

Choose x_0 and x_1 so that there is a solution in $[x_0, x_1]$. Then define

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}. \tag{1}$$

Example 1.4

Use the Secant Method to solve $x^2 - 2 = 0$ in $[0, 2]$. Results are shown below. We see that, not only does the method appear to converge to the true solution, it seem to do so *much* more efficiently than Bisection. We'll return to why this is later.

	Secant		Bisection	
k	x_k	$\|x_k - \tau\|$	x_k	$\|x_k - \tau\|$
0	0.000000	1.41	0.000000	1.41
1	2.000000	5.86e-01	2.000000	5.86e-01
2	1.000000	4.14e-01	1.000000	4.14e-01
3	1.333333	8.09e-02	1.500000	8.58e-02
4	1.428571	1.44e-02	1.250000	1.64e-01
5	1.413793	4.20e-04	1.375000	3.92e-02
6	1.414211	2.12e-06	1.437500	2.33e-02
7	1.414214	3.16e-10	1.406250	7.96e-03
8	1.414214	4.44e-16	1.421875	7.66e-03

To compare different methods, we need the following concept.

Definition 1.5 (Linear Convergence)

Suppose that $\tau = \lim_{k\to\infty} x_k$. *Then we say that the sequence* $\{x_k\}_{k=0}^{\infty}$ *converges to* τ ***at least linearly*** *if there is a sequence of positive numbers* $\{\varepsilon_k\}_{k=0}^{\infty}$, *and* $\mu \in (0,1)$, *such that*

$$\lim_{k\to\infty} \varepsilon_k = 0, \tag{2a}$$

and

$$|\tau - x_k| \le \varepsilon_k \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{2b}$$

and

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu. \tag{2c}$$

For Example 1.4, the bisection method converges at least linearly.

As we have seen, there are methods that converge more quickly than bisection. Now we'll give a more precise description of what "more quickly" means.

Definition 1.6 (Order of Convergence)

Let $\tau = \lim_{k\to\infty} x_k$. Suppose there exists $\mu > 0$ and a sequence of positive numbers $\{\varepsilon_k\}_{k=0}^{\infty}$ such that (2a) and and (2b) both hold. Then we say that the sequence $\{x_k\}_{k=0}^{\infty}$ converges with at least order q if

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \mu.$$

Two particular values of q are important to us:

(i) If $q = 1$, and we have that $0 < \mu < 1$, then the rate is ***linear***.

(ii) If $q = 2$, the rate is ***quadratic*** for any $\mu > 0$.

Theorem 1.7

Suppose that f and f' are real-valued functions, continuous and defined in an interval $I = [\tau - h, \tau + h]$ for some $h > 0$. If $f(\tau) = 0$ and $f'(\tau) \neq 0$, then the sequence (1) converges at least linearly to τ.

- We wish to show that $|\tau - x_{k+1}| < |\tau - x_k|$.
- From the (MVT), there is a point $w_k \in [x_{k-1}, x_k]$ s.t.

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(w_k). \tag{3}$$

- Also by the MVT, there is a point $z_k \in [x_k, \tau]$ such that

$$\frac{f(x_k) - f(\tau)}{x_k - \tau} = \frac{f(x_k)}{x_k - \tau} = f'(z_k). \tag{4}$$

Therefore $\quad f(x_k) = (x_k - \tau) f'(z_k).$

- Using (3) and (4), we can show that

$$\tau - x_{k+1} = (\tau - x_k)\Big(1 - f'(z_k)/f'(w_k)\Big).$$

  Therefore

$$\frac{|\tau - x_{k+1}|}{|\tau - x_k|} = \Big|1 - \frac{f'(z_k)}{f'(w_k)}\Big|.$$

- Suppose that $f'(\tau) > 0$. (If $f'(\tau) < 0$ just tweak the arguments accordingly). Saying that f' *is continuous in the region* $[\tau - h, \tau + h]$ means that, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f'(x) - f'(\tau)| < \varepsilon \text{ for any } x \in [\tau - \delta, \tau + \delta].$$

  Take $\varepsilon = f'(\tau)/4$. Then $|f'(x) - f'(\tau)| < f'(\tau)/4$. Thus

$$\frac{3}{4}f'(\tau) \leq f'(x) \leq \frac{5}{4}f'(\tau) \quad \text{for any } x \in [\tau - \delta, \tau + \delta].$$

Then, so long as w_k and z_k are both in $[\tau - \delta, \tau + \delta]$

$$\frac{f'(z_k)}{f'(w_k)} \leq \frac{5}{3}.$$

Given enough time and effort we *could* show that the Secant Method converges faster that linearly. In particular, that the order of convergence is

$$q = (1+\sqrt{5})/2 \approx 1.618.$$

This number arises as the only positive root of $q^2 - q - 1$. It is called the ***Golden Mean***, and arises in many areas of Mathematics, including finding an explicit expression for the Fibonacci Sequence:

$$\begin{aligned} f_0 &= 1, \\ f_1 &= 1, \\ f_{k+1} &= f_k + f_{k-1} \text{ for } k = 2, 3, \ldots. \end{aligned}$$

That gives, $f_0 = 1$, $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, $\ldots$.

The connection here is that it turns out that $\varepsilon_{k+1} \leq C\varepsilon_k\varepsilon_{k-1}$. Repeatedly using this we get:

- Let $r = |x_1 - x_0|$ so that $\varepsilon_0 \leq r$ and $\varepsilon_1 \leq r$,
- Then $\varepsilon_2 \leq C\varepsilon_1\varepsilon_0 \leq Cr^2$
- Then $\varepsilon_3 \leq C\varepsilon_2\varepsilon_1 \leq C(Cr^2)r = C^2r^3$.
- Then $\varepsilon_4 \leq C\varepsilon_3\varepsilon_2 \leq C(C^2r^3)(Cr^2) = C^4r^5$.
- Then $\varepsilon_5 \leq C\varepsilon_4\varepsilon_3 \leq C(C^4r^5)(C^2r^3) = C^7r^8$.
- And in general, $\varepsilon_k = C^{f_k-1}r^{f_k}$.

Exercise 1.6

Suppose we define the Secant Method as follows.

> *Choose any two points x_0 and x_1.*
> *For $k = 1, 2, \dots,$ set x_{k+1} to be the point where the line through $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ that intersects the x-axis.*

Show how to derive the formula for the secant method.

Exercise 1.7

(i) Is it possible to construct a problem for which the bisection method will work, but the secant method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which the secant method will work, but bisection will fail? If so, give an example.

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§1.3: Newton's Method

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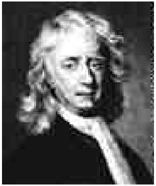

Sir Isaac Newton, 1643 - 1727, England. Easily one of the greatest scientist of all time. The method we are studying appeared in his celebrated *Principia Mathematica* in 1687, but it is believed he had used it as early as 1669.

Secant method can be written as

$$x_{k+1} = x_k - f(x_k)\phi(x_k, x_{k-1}),$$

where the function ϕ is chosen so that x_{k+1} is the root of the secant line joining the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$.

A closely related idea leads to **Newton's Method**: set $x_{k+1} = x_k - f(x_k)\lambda(x_k)$, where we choose λ so that x_{k+1} is the zero of the tangent to f at $(x_k, f(x_k))$.

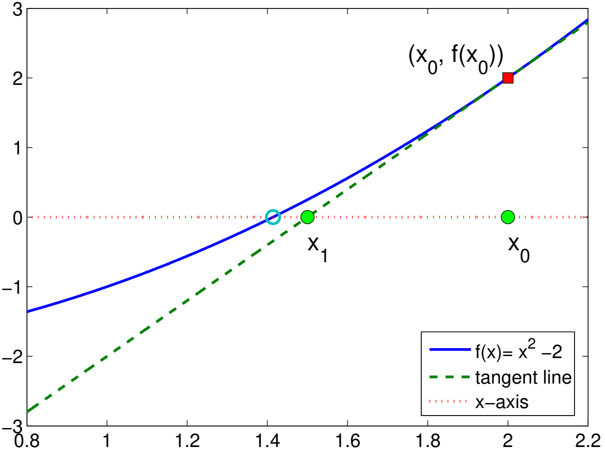

Figure: Estimating $\sqrt{2}$ by solving $x^2 - 2 = 0$ using Newton's Method

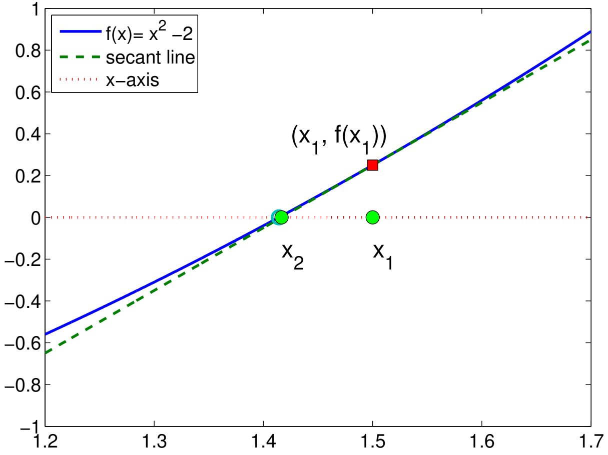

Figure: Estimating $\sqrt{2}$ by solving $x^2 - 2 = 0$ using Newton's Method

The formula for Newton's method may be deduced writing down the equation for the line at $\big(x_k, f(x_k)\big)$ with slope $f'(x_k)$, and setting x_{k+1} to be its zero; see Exercise 1.8-(i)).

Newton's Method

1. Choose any x_0 in $[a, b]$,
2. For $k = 0, 1, \ldots,$ set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{5}$$

Example 1.8

Use bisection, secant, and Newton's Method to solve $x^2 - 2 = 0$ in $[0, 2]$.

For this case, Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k},$$

which simplifies as

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

Taking $x_0 = 2$, we get $x_1 = 3/2$.
Then $x_2 = 2x_1 + 1/x_1 = 17/12 = 1.46667$.
Then $x_3 = 1.4142$, etc.

Iter	Bisection	Secant	Newton
k	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$
0	1.41	1.41	5.86e-01
1	5.86e-01	5.86e-01	8.58e-02
2	4.14e-01	4.14e-01	2.45e-03
3	8.58e-02	8.09e-02	2.12e-06
4	1.64e-01	1.44e-02	1.59e-12
5	3.92e-02	4.20e-04	2.34e-16
6	2.33e-02	2.12e-06	—
7	7.96e-03	3.16e-10	—
8	7.66e-03	4.44e-16	—
9	1.51e-04	—	—
10	3.76e-03	—	—
11	1.80e-03	—	—
⋮	⋮	⋮	⋮
22	5.72e-07	—	—

Deriving Newton's method geometrically certainly has an intuitive appeal. However, to analyse the method, we need a more abstract derivation based on a **Truncated Taylor Series**.

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(x_k) + \dots \\ + \frac{(x - x_k)^n}{n!}f^{(n)}(x_k) + \frac{(x - x_k)^{n+1}}{(n+1)!}f^{(n+1)}(\eta_k)$$

where $\eta_k \in (x, x_k)$. Now truncate at the second term (i.e., take $n = 1$):

We now want to show that Newton's converges ***quadratically***, that is, with *at least order* $q = 2$. To do this, we need to

1. Write down a recursive formula for the error.
2. Show that it converges.
3. Then find the limit of $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^2}$.

Step 2 is usually the crucial part.

There are two parts to the proof. The first involves deriving the so-called "***Newton Error formula***".

We'll assume that the functions f, f' and f'' are defined and continuous on the an interval $I_\delta = [\tau - \delta, \tau + \delta]$ around the root τ.

The following proof is essentially the same as the above derivation (see also Theorem 3.2 in Epperson).

Theorem 1.9 (Newton Error Formula)

If $f(\tau) = 0$ and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point η_k between τ and x_k such that

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2} \frac{f''(\eta_k)}{f'(x_k)}, \tag{6}$$

The Newton Error Formula is important in theory (for proving that Newton's Method converges) and practice (to estimate the error when it is applied to specific problems).

In practical applications, we can use the (6) as follows. So suppose we are applying Newton's Method to solving $f(x) = 0$ on $[a, b]$. Denote the error at Step k by $\varepsilon_k = |\tau - x_k|$. Then we can deduce that

$$\varepsilon_{k+1} \leq \varepsilon_k^2 \frac{\max\limits_{a \leq x \leq b} |f''(x)|}{2|f'(x_k)|}. \tag{7}$$

Then, using that $\varepsilon_0 \leq |b - a|$, (7) can be used repeatedly to bound ε_1, ε_2, etc.

We'll now complete our analysis of this section by proving the convergence of Newton's method.

Theorem 1.10

Let us suppose that f is a function such that

- f is continuous and real-valued, with continuous f'', defined on some closed interval $I_\delta = [\tau - \delta, \tau + \delta]$,
- $f(\tau) = 0$ and $f''(\tau) \neq 0$,
- there is some positive constant A such that

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all} \quad x, y \in I_\delta.$$

Let $h = \min\{\delta, 1/A\}$. If $|\tau - x_0| \leq h$ then Newton's Method converges quadratically.

Exercise 1.8 ($\star$ Homework problem)

Write down the equation of the line that is tangential to the function f at the point x_k. Give an expression for its zero. Hence show how to derive Newton's method.

Exercise 1.9

(i) Is it possible to construct a problem for which the bisection method will work, but Newton's method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which Newton's method will work, but bisection will fail? If so, give an example.

Exercise 1.10 ($\star$ Homework problem)

(i) Let q be your student ID number. Find k and m where $k-2$ is the remainder on dividing q by 4, and $m-2$ is the remainder on dividing q by 6.

(ii) Show how Newton's method can be applied to estimate the postive real number $\sqrt[k]{m}$. That is, state the nonlinear equation you would solve, and give the formula for Newton's method, simplified as much as possible.

(iii) Do three iterations by hand of Newton's Method for this problem.

Exercise 1.11 ($\star$ Homework problem)

Suppose we want apply to Newton's method to solving $f(x) = 0$ where f is such that $|f''(x)| \leq 10$ and $|f'(x)| \geq 2$ for all x. How close must x_0 be to τ for the method to converge?

Exercise 1.12

Here is (yet) another scheme called *Steffenson's Method*: Choose $x_0 \in [a, b]$ and set

$$x_{k+1} = x_k - \frac{\big(f(x_k)\big)^2}{f\big(x_k + f(x_k)\big) - f(x_k)} \text{ for } k = 0, 1, 2, \dots.$$

It is remarkable because its convergence is quadratic, like Newton's, but does not require derivatives of f.
Show how the method can be derived from Newton's Method, using the formal definition of the derivative.

Exercise 1.13

(This is Exercise 1.6 from Süli and Mayers) The proof of the convergence of Newton's method given in Theorem 1.10 uses that $f'(\tau) \neq 0$. Suppose that it is the case that $f'(\tau) = 0$.

(i) Starting from the Newton Error formula, show that

$$\tau - x_{k+1} = \frac{(\tau - x_k)}{2} \frac{f''(\eta_k)}{f''(\mu_k)},$$

for some μ_k between τ and x_k. (*Hint: try using the MVT*).

(ii) What does the above error formula tell us about the convergence of Newton's method in this case?

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§1.4: Fixed Point Iteration

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Newton's method can be considered to be a special case of a very general approach called *Fixed Point Iteration* or *Simple Iteration*.

The basic idea is:

> *If we want to solve* $f(x) = 0$ *in* $[a, b]$*, find a function* $g(x)$ *such that, if* τ *is such that* $f(\tau) = 0$*, then* $g(\tau) = \tau$*.*
> *Choose* x_0 *and set* $x_{k+1} = g(x_k)$ *for* $k = 0, 1, 2, \dots$ *.*

Example 1.11

Suppose that $f(x) = e^x - 2x - 1$ and we are trying to find a solution to $f(x) = 0$ in $[1, 2]$. Then we can take $g(x) = \ln(2x + 1)$.

If we take $x_0 = 1$, then we get the following sequence:

k	x_k	$\lvert\tau - x_k\rvert$
0	1.0000	2.564e-1
1	1.0986	1.578e-1
2	1.1623	9.415e-2
3	1.2013	5.509e-2
4	1.2246	3.187e-2
5	1.2381	1.831e-2
⋮	⋮	⋮
10	1.2558	6.310e-4

We have to be quite careful with this method: **not every choice is g is suitable**.

For example, suppose we want the solution to $f(x) = x^2 - 2 = 0$ in $[1, 2]$. We could choose $g(x) = x^2 + x - 2$. Then, if take $x_0 = 1$ we get the sequence:

We need to refine the method that ensure that it will converge.

Before we do that in a formal way, consider the following...

Example 1.12

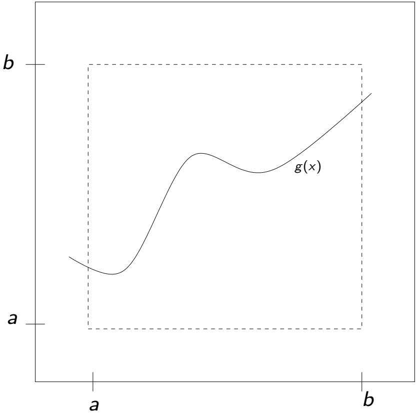
Use the Mean Value Theorem to show that the fixed point method $x_{k+1} = g(x_k)$ converges if $|g'(x)| < 1$ for all x near the fixed point.

This example:

- introduces the tricks of using that $g(\tau) = \tau$ & $g(x_k) = x_{k+1}$.
- Leads us towards the ***contraction mapping theorem***.

Theorem 1.13 (Fixed Point Theorem)

Suppose that $g(x)$ is defined and continuous on $[a, b]$, and that $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there exists $\tau \in [a, b]$ such that $g(\tau) = \tau$. That is, $g(x)$ has a *fixed point* in $[a, b]$.



Next suppose that g is a *contraction*. That is, $g(x)$ is continuous and defined on $[a, b]$ and there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b]. \tag{8}$$

Theorem 1.14 (Contraction Mapping Theorem)

Suppose that the function g is a real-valued, defined, continuous, and

(a) maps every point in $[a, b]$ to some point in $[a, b]$, and

(b) is a contraction on $[a, b]$,

then

(i) $g(x)$ has a fixed point $\tau \in [a, b]$,

(ii) the fixed point is unique,

(iii) the sequence $\{x_k\}_{k=0}^{\infty}$ defined by $x_0 \in [a, b]$ and $x_k = g(x_{k-1})$ for $k = 1, 2, \ldots$ converges to τ.

The algorithm generates as sequence $\{x_0, x_1, \ldots, x_k\}$. Eventually we must stop. Suppose we want the solution to be accurate to say 10^{-6}, how many steps are needed? That is, how big do we need to take k so that

$$|x_k - \tau| \leq 10^{-6}?$$

The answer is obtained by first showing that

$$|\tau - x_k| \leq \frac{L^k}{1 - L}|x_1 - x_0|. \tag{9}$$

Example 1.15

Suppose we are using FPI to find the fixed point $\tau \in [1, 2]$ of $g(x) = \ln(2x + 1)$ with $x_0 = 1$, and we want $|x_k - \tau| \leq 10^{-6}$, then we can use (9) to determine the number of iterations required.

Exercise 1.14

Is it possible for g to be a contraction on $[a, b]$ but not have a fixed point in $[a, b]$? Give an example to support your answer.

Exercise 1.15 ($\star$ Homework problem)

Show that $g(x) = \ln(2x + 1)$ is a contraction on $[1, 2]$. Give an estimate for L. (Hint: Use the Mean Value Theorem).

Exercise 1.16

Suppose we wish to numerically estimate the famous *golden ratio*, $\tau = (1+\sqrt{5})/2$, which is the positive solution to $x^2 - x - 1$. We could attempt to do this by applying fixed point iteration to the functions $g_1(x) = x^2 - 1$ or $g_2(x) = 1 + 1/x$ on the region $[3/2, 2]$.

(i) Show that g_1 is *not* a contraction on $[3/2, 2]$.

(ii) Show that g_2 *is* a contraction on $[3/2, 2]$, and give an upper bound for L.

Exercise 1.17

Consider the function $g(x) = x^2/4 + 5x/4 - 1/2$.

(i) It has two fixed points – what are they?

(ii) For each of these, find the largest region around them such that g is a contraction on that region.

Exercise 1.18

(i) Prove that if $g(\tau) = \tau$, and the fixed point method given by

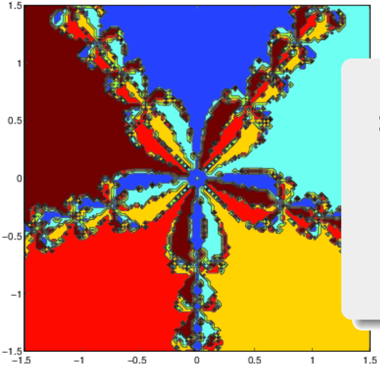
$$x_{k+1} = g(x_k),$$

converges to the point τ (where $g(\tau) = \tau$), and

$$g'(\tau) = g''(\tau) = \cdots = g^{(p-1)}(\tau) = 0,$$

then it converges with order p. (Hint: you don't have to prove that the method converges; you can assume that. Also, use a Taylor Series).

(ii) We can think of Newton's Method for the problem $f(x) = 0$ as fixed point iteration with $g(x) = x - f(x)/f'(x)$. Use this, and Part (i), to show that, if Newton's method converges, it does so with order 2, providing that $f'(\tau) \neq 0$.



§1 Solving nonlinear equations

§1.5: Wrap up: what has Newton's method ever done for me?

MA385/530 – Numerical Analysis 1

October 2019

When studying a numerical method (or any piece of Mathematics) you should ask *why* you are doing this. For example, it might be

- because it will help you can understand other topics later;
- because it is interesting/beautiful in its own right; or
- (most commonly) because it is useful.

Here are some instances of each of these:

1. The analyses we have used in this section allowed us to consider some important ideas in a simple setting.

Examples include

- **Convergence**, including *rates of convergence*
- **Fixed-point theory**, and contractions. We'll be seeing analogous ideas in the next section (Lipschitz conditions).
- **The approximation of functions by polynomials** (Taylor's Theorem). This point will reoccur in the next section, and all through-out next semester.

2. Applications come from lots of areas of science and engineering. In financial mathematics, the **Black-Scholes equation** for pricing a put option can be written as

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0.$$

- $V(S,t)$ is the current value of the right (but not the obligation) to buy or sell ("put" or "call") an asset at a future time T;
- S is the current value of the underlying asset;
- r is the current interest rate (because the value of the option has to be compared with what we would have gained by investing the money we paid for it)
- σ is the volatility of the asset's price.

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0.$$

Often one knows S, T and r, but not σ. The method of *implied volatility* is when we take data from the market and then find the value of σ which, if used in the Black-Scholes equation, would match this data. This is a nonlinear problem and so Newton's method can be used. See Chapters 13 and 14 of Higham's "An Introduction to Financial Option Valuation" for more details.

(We will return to the Black-Scholes problem again at the end of the next section).

3. Some of these ideas are interesting *and* beautiful. The ***complex*** n^{th} ***roots of unity*** is the set of numbers $\{z_0, z_1, \ldots, z_{n-1}\}$ who's n^{th} roots are 1. They can be expressed as

$$z_k = e^{i\theta} \quad \text{where } \theta = \frac{2k\pi}{n}$$

for $k \in \{0, 1, 2 \ldots, n-1\}$ and $i = \sqrt{-1}$.

But suppose we wanted to estimate these numbers using Newton's method. We could try to solve $f(z) = 0$ with $f(z) = z^n - 1$. The iteration is:

$$z_{k+1} = z_k - \frac{(z_k)^n - 1}{n(z_k)^{n-1}}.$$

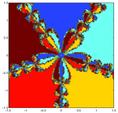
The iteration is

$$z_{k+1} = z_k - \frac{(z_k)^n - 1}{n(z_k)^{n-1}}.$$

However, there are n possible solutions to

$$z^n - 1 = 0.$$

Given a particular starting point, which root with the method converge to? If we take a number of points in a region of space, iterate on each of them, and then colour the points to indicate the ones that converge to the same root, we get the famous Julia set, an example of a fractal.



A contour plot of a Julia set with $n = 5$, generated by the MATLAB script Julia.m, available from the MA385 website.

(Gaston Julia, French mathematician 1893–1978. The famous paper which introduced these ideas was published in 1918. Interest later waned until the 1970s when Mandelbrot's computer experiments reinvigorated