MA385 Part 2: Initial Value Problems

2.4: Runge-Kutta 2 (RK2)

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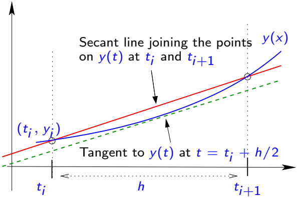

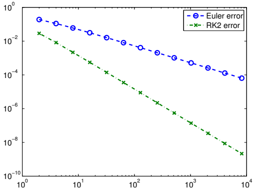

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For more details, see Chapter 6 of Süli and Mayers, *An Introduction to Numerical Analysis*. In particular:

- Section 12.3 deals with Consistency; pay attention to the discussion leading to (12.21).
- Section 12.4 (Runge-Kutta methods)

Recall our original motivation of Euler's method: use the slope of the tangent to y at t_i as an approximation for the slope of the secant line joining the points $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$.
One could argue, given the diagram on the next slide, that the slope of the tangent to y at $t = (t_i + t_{i+1})/2 = t_i + h/2$ would be a better approximation. This would give

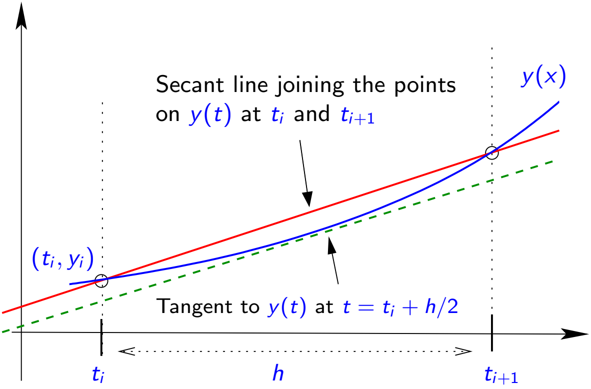
$$y(t_{i+1}) \approx y_i + hf\big(t_i + \frac{h}{2}, y(t_i + \frac{h}{2})\big). \tag{1}$$

However, we don't know $y(t_i + h/2)$, but can approximate it using Euler's Method: $y(t_i + h/2) \approx y_i + (h/2)f(t_i, y_i)$.

1. Modified Euler Method

Modified (Midpoint) Euler's Method

$$y_{i+1} = y_i + hf\Big(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)\Big). \tag{2}$$



Example 2.4.1

Use the Modified Euler Method to approximate $y(1)$ *where*

$$y(0) = 1, \qquad y'(t) = y\log(1+t^2).$$

This has the solution $y(t) = (1+t^2)^t \exp(-2t + 2\tan^{-1} t)$.

1. Modified Euler Method

	Euler		Modified	
n	$\mathcal{E}_n$	$\mathcal{E}_n/\mathcal{E}_{n-1}$	$\mathcal{E}_n$	$\mathcal{E}_n/\mathcal{E}_{n-1}$
1	3.02e-01		7.89e-02	
2	1.90e-01	1.59	2.90e-02	2.72
4	1.11e-01	1.72	8.20e-03	3.54
8	6.02e-02	1.84	2.16e-03	3.79
16	3.14e-02	1.91	5.55e-04	3.90
32	1.61e-02	1.95	1.40e-04	3.95
64	8.13e-03	1.98	3.53e-05	3.98
128	4.09e-03	1.99	8.84e-06	3.99

Clearly we get a much more accurate result using the Modified Euler Method. Even more importantly, we get a higher *order of accuracy*: if h is reduced by a factor of **two**, the error in the Modified method is reduced by a factor of **four**.

We can also make a direct comparison of the two methods by using a log-log plot of the errors.

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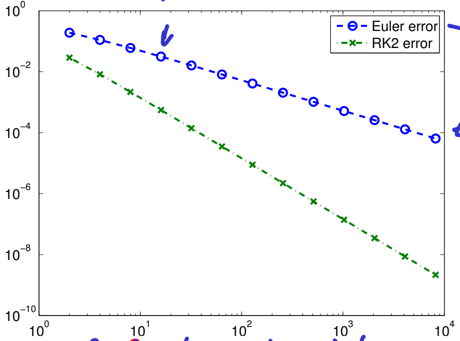
Error



n = 1024.

n, number of point

Figure 1: Log-log plot of the errors when Euler's and Modified Euler's methods are applied to the problem in Example 2.4.1

2. General RK2

The "*Modified Euler Method*" is an example of one of the (large) family of 2^{nd}-*order Runge-Kutta* (RK2). Recall that that one-step methods are written as $y_{i+1} = y_i + h\Phi(t_i, y_i; h)$

The general RK2 method is

$$k_1 = f(t_i, y_i) \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1).$$
$$\Phi(t_i, y_i; h) = (ak_1 + bk_2) \tag{3}$$

Example: take $a = 1, b = 0$.

Then $y_{i+1} = y_i + h\,(k_1 + 0 \cdot k_2)$

$= y_i + h\, f(t_i, y_i)$, ie Euler' method

[Boring!]

The general RK2 method is

$$k_1 = f(t_i, y_i) \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1).$$
$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

Example 2: take $\alpha = \beta = 1/2, a = 0, b = 1$.

$k_1 = f(t_i, y_i) \qquad k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i))$

So the method is

$$y_{i+1} = y_i + h f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right)$$

which is the Modified (Midpoint Method).

Our aim now is to deduce general rules for choosing a, b, α and β. We'll see that if we pick any one of these four parameters, then the requirement that the method be consistent and second-order determines the other three.

As well now see, by demanding that RK2 be **consistent** we get that $a + b = 1$:

Recall, a one step method

$$y_{i+1} = y_i + h\,\Phi(t_i, y_i; h)$$

is consistent if

$$\Phi(t_i, y_i; 0) = f(t_i, y_i)$$

For RK2,

$$\Phi(t_i; y_i; h) = a f(t_i, y_i) + b f(t_i + \alpha h, y_i + \beta h f(t_i, y_i))$$

so $\Phi(t_i, y_i; 0) = (a+b) f(t_i, y_i)$. So $\boxed{a+b=1}$

Next we need to know how to choose α and β. The formal way is to use a two-dimensional Taylor series expansion. However, it is quite technical, involving long calculations.

So, instead, we'll take an approach based on applying the method to a simple, but representative problem.

Because we expect that, for a second order accurate method, $|\mathcal{E}_n| \leq Kh^2$ where K depends on $y'''(t)$, if we choose a problem for which $y'''(t) \equiv 0$, we expect no error...

We'll take $y(t) = t^2$. So $y'(t) = 2t$, $y''(t) = 2$ and $y'''(t) \equiv 0$. An equation that has this as a solution is

$$y'(t) = 2t \quad (\text{ie } f(t,y) = 2t) \quad \text{and} \quad y(1) = 1$$

The method should give the true solution for any h (we say it is "exact")

So we'll take $h=1$ for simplicity. Then

$y_0 = 1$ $\quad t_1 = t_0 + h = 2$, so $y(t_1) = (t_1)^2 = 4$.

Next $\quad k_1 = f(t_0, y_0) = f(1, 1) = 2$

since $f(t,y) = 2t$.

Because we expect that, for a second order accurate method, $|\mathcal{E}_n| \leq Kh^2$ where K depends on $y'''(t)$, if we choose a problem for which $y'''(t) \equiv 0$, we expect no error...

$(f(t,y) = 2$

$$k_2 = f(t_0 + \alpha h, \; t_0 + \beta h k_1)$$
$$= 2(t_0 + \alpha h) = 2 + 2\alpha h.$$

Then
$$y_1 = y_0 + h(a k_1 + b k_2)$$
$$= 1 + a(2) + b(2 + 2\alpha)$$

If $y(t_1) = y_1$, ie $y(2) = 2$ we get

$$4 = 1 + 2(\underbrace{1-b}_{a}) + 2b + 2b\alpha.$$

So this gives $\alpha = \dfrac{1}{2b}$ (check!)

In the above example, the right-hand side of the differential equation, $f(t, y)$, depended only on t. Now we'll try the same trick: using a problem with a simple known solution (and zero error), but for which f depends explicitly on y.

Consider the DE $y(1) = 1, y'(t) = y(t)/t$. It has a simple solution: $y(t) = t$. We now use that any RK2 method should be exact for this problem to deduce that $\alpha = \beta$.

we have $y'(t) = \dfrac{y(t)}{t} = \dfrac{t}{t} = 1.$ So $f(t,y) = \dfrac{y}{t}.$

Again we take $h = 1$, and use that the method is exact at $t_1 = t_0 + 1 = 2$.

$$k_1 = f(t_0, y_0) = \frac{y_0}{t_0} = 1$$

$$k_2 = f(t_0 + \alpha h, y_0 + \beta h k_1) = \frac{1+\beta}{1+\alpha}$$

So now we have

$$k_1 = 1 \qquad k_2 = \frac{1+\beta}{1+\alpha}$$

Then, using $y(2) = 2$ we get

$$y_1 = y_0 + a k_1 + b k_2$$

$$\Rightarrow \quad 2 = 1 + a + b\left(\frac{1+\beta}{1+\alpha}\right)$$

$$\Rightarrow \quad 1 = (1-b) + b\left(\frac{1+\beta}{1+\alpha}\right)$$

So we get $\dfrac{1+\beta}{1+\alpha} = 1 \Rightarrow \alpha = \beta.$

Now we collect the above results all together and show that the second-order Runge-Kutta (RK2) methods are:

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(t_i, y_i), \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1),$$

where we choose any $b \neq 0$ and then set

$$a = 1 - b, \qquad \alpha = \frac{1}{2b}, \qquad \beta = \alpha.$$

It is easy to verify that the Modified method satisfies these criteria.

Exercise 2.4.1

A popular RK2 method, called the *Improved Euler Method*, is obtained by choosing $\alpha = 1$.

(i) Use the Improved Euler Method to find an approximation for $y(4)$ when

$$y(0) = 1, \qquad y' = y/(1+t^2),$$

taking $n = 2$. (If you wish, use Python.)

(ii) Using a diagram similar to the one used to motivate the Modified Euler Method, justify the assertion that the Improved Euler Method is more accurate than the basic Euler Method.

(iii) Show that the method is consistent.

(iv) Write out what this method would be for the problem: $y'(t) = \lambda y$ for a constant λ. How does this relate to the Taylor series expansion for $y(t_{i+1})$ about the point t_i?

Exercise 2.4.2 (Assignment!)

In his seminal paper of 1901, Carl Runge gave an example of what we now call a *Runge-Kutta 2 method*, where

$$\Phi(t_i, y_i; h) = \frac{1}{4} f(t_i, y_i) + \frac{3}{4} f\big(t_i + \frac{2}{3} h, y_i + \frac{2}{3} h f(t_i, y_i)\big).$$

(i) Show that it is consistent.

(ii) Show how this method fits into the general framework of RK2 methods. That is, what are a, b, α, and β? Do they satisfy the conditions

$$\beta = \alpha, \qquad b = \frac{1}{2\alpha}, \qquad a = 1 - b?$$

(iii) Use it to estimate the solution at the point $t = 2$ to $y(1) = 1$, $y' = 1 + t + y/t$ taking $n = 2$ time steps.

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