MA378 Chapter 5: Review

§5.1 Review (and Exam Preview)

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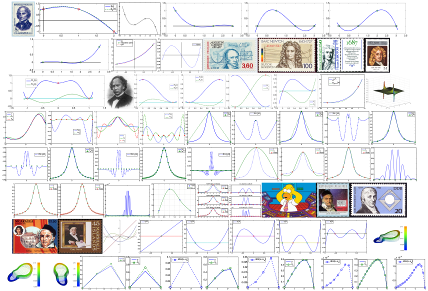


This module has four chapters:

1. Polynomial Interpolation
2. Piecewise polynomial interpolation
3. Numerical Integration (Quadrature)
4. Finite element methods

We also studied implementation of methods in Octave/MATLAB.

We now review these — primarily to provide you with a check-list of topics.

1.1 The polynomial interpolation problem (PIP) in 2 forms;

1.2 Uniqueness (*if* $p_n \in \mathcal{P}_n$ *has* $n+1$ *zeros, then* $p_n \equiv 0$)

1.2 **Lagrange Polynomials and Lagrange Interpolation**

1.3 Rolle's Theorem

1.3 **Cauchy's theorem**: $f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x)$.

1.4 Hermite Interpolation: construction and error estimates.

1.5 Convergence & Runge's Example

2.1 Linear Interpolating Splines:
- construction,
- error analysis,
- best approximation, and
- the Minimum Energy property.

2.2 (Natural) Cubic Splines: construction (error analysis stated without proof)

2.3 PCHIP: Piecewise Cubic Hermite Interpolating Polynomial.
- construction,
- error analysis.

§3 Numerical Integration ("Quadrature")

- 3.1 Newton-Cotes methods, and the Trapezium Rule
- 3.1 Undetermined Coefficients
- 3.2 Simpson's Rule and (non-sharp) error estimates
- 3.3 Precision
- 3.3 Composite rules
- 3.4 Gaussian Quadrature (and *Undetermined Coefficients* again)
- 3.5 A sequence of Orthogonal Monic Polynomials (and vector spaces and inner products); properties of orthogonal monic polynomials; their roots as quadrature points.
- 3.6 Precision and convergence of Gaussian Quadrature methods.

1.2 §4: Finite Element Methods

4.1 Boundary value problems, and Maximum Principles.

4.1 The **variational** (weak) **formulation** of the problem, and the uniqueness of its solution.

4.2 The **FEM**, including finite and infinite dimensional spaces.

4.2 **FE implementation**, including Galerkin basis functions ("hat functions"), how to construct the linear system, and why the system matrix is tridiagonal and symmetric.

4.3 Analysis, **Cea's Lemma**.

Numerical Analysis is an ancient field of mathematics.

The origins of the topics we've studied have roots in astronomy: **interpolation** techniques were used in astronomy. Earliest known examples date from use of linear interpolation in 200 BC (Babylon; modern Iraq), polynomial interpolation in 600 AD (China), and 650 AD (India).

Quadrature (numerical integration) has been studied for about as long – and long before the invention/discovery of calculus. The Trapezium Rule was known to the Babylonians, and was used by the ancient Greeks for tasks such as estimating π.

Finite Elements have a more recent history, and have been developed since the 1950, starting with the work of Richard Courant.

All three area feature in modern mathematical research.

- Since about 2010 there has been reviewed interest in robust methods for very high-order polynomial interpolation, and Chebychev methods in particular;
- Modern research in quadrature focuses on fast methods for high-dimensional problems;
- 1,000s of papers are published every year on finite element methods, with much effort aimed at understanding how to construct meshes automatically, use of discontinuous basis functions, and applications to non-linear problems.

So: in spite of the long history, what you have studied is very relevant to modern mathematics.

FYI: The monic polynomials we studied in Chapter 3 are also known as Legendre Polynomials, named after Adrien-Marie Legendre.
This is the only known "portrait" of him, which was (re)discovered in 2008.

1.3 Assessment

MA378 is assessed by

- Labs (10%). These have been graded.
- Two Assignments (20%). (Assignment 2 still not graded; sorry!)
- Class test (10%)
- Written two-hour exam (60%)

The exam will have **4 questions**: answer all 4.

Q1 (28 marks) has 4 parts, all concerning theorems and proofs. **Answer any two**. You'll be given the statements of the theorems, and asked to provide the proofs. You won't be ask to apply those results in Q1 (applications of theorems will be in other questions).

For the **Semester 2 2023/2024** exam, the theorems in question are selected from the following:

- **Chapter 1**:
  - 2.3+2.7 (There exists a unique solution to the PIP);
  - 3.3 (Cauchy's Theorem).
- **Chapter 2**:
  - 1.7 (Error is no worse than twice the best possible);
  - 1.9 (Minimum Energy)

- **Chapter 3**:
  - 3.4: If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k+1$ points, then it has precision $2k+1$.
  - 5.15 (Zeros of the monic polys are (i) Simple; (ii) real and in $[a, b]$).
- **Chapter 4**:
  - 1.2+1.3 (Max Principle, and unique solution to the BVP),
  - Lemma 3.2 (Cea's Lemma - both parts)

On the **Semester 2 2023/2024** exam, the remaining 3 questions all are worth 24 marks.

Q2: Chapter 1 (Polynomial interpolation)

Q3: Chapter 2 (Piecewise polynomial interpolation)

Q4: Chapter 3 (i.e., numerical integration/quadrature of any type) and Chapter 4: (i.e., BVPs+FEMs)

For each of these, the emphasis is on derivation of methods, and demonstration of understanding, rather than proving theorems.

The marks for each sub-section are indicated on the paper. Pay attention to those, since they indicate, roughly, the effort involved.
Octave/MATLAB will *not* feature on the exam.

Important: the above guide is only for the **Semester 2 2023/2024** exam. If there is an Autumn exam, the theorems on Q1 could be different, as could the mapping of chapters to Questions 2, 3 and 4.

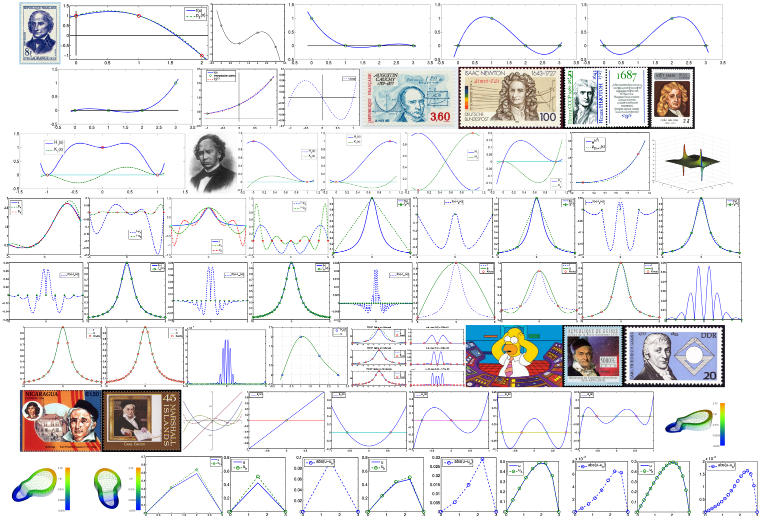
1.4 Thanks

Thanks!

Thanks for taking this module. I hope you have found it interesting, and that some of the topics covered are useful in your future mathematical and computational efforts.

Feedback from graduates suggests its one of the most useful "MA" modules. If you are interested in graduate studies in this field, in Ireland or elsewhere, please talk to me.

1.4 Thanks


AUGUSTIN CAUCHY
RÉPUBLIQUE FRANÇAISE
3,60
ISAAC NEWTON 1643-1727
DEUTSCHE BUNDESPOST
100
1687
REPUBLIQUE DE GUINEE
DDR
20
NICARAGUA
45 MARSHALL ISLANDS
CARL GAUSS