MA385: Tutorial 4 ANS with some solutions

These exercises are for Tutorial 4 (Week 10). You do not have to submit solutions to these questions.

Q1. For the following functions show that they satisfy a Lipschitz condition on the corresponding domain, and give an upper-bound for L:

(i) $f(t, y) = 2yt^{-4}$ for $t \in [1, \infty)$,

(ii) $f(t, y) = 1 + t\sin(ty)$ for $0 \leqslant t \leqslant 2$.

Answer: We need to show that a positive real number L exists, such that,

$$|f(t, u) - f(t, v)| \leqslant L|u - v|,$$

for all (t, u) and (t, v) on the domain.

(i) $f(t, y) = 2yt^{-4}$ for $t \in [1, \infty)$,

$$|f(t, u) - f(t, v)| = 2|ut^{-4} - vt^{-4}| = 2t^{-4}|u - v|$$

where we have used that $t > 0$. Since $t^{-4} \leqslant 1$ for all $t \geqslant 1$, we get $|f(t, u) - f(t, v)| \leqslant 2|u - v|$, for all u and v. So f is Lipschitz, and $\mathbf{2}$ is an upper bound for L.

(ii) $f(t, y) = 1 + t\sin(ty)$ for $0 \leqslant t \leqslant 2$

$$|f(t, u) - f(t, v)| = |1 + t\sin(tu) - 1 - t\sin(tv)| = t|\sin(tu) - t\sin(tv)| \leqslant 2|\sin(tu) - t\sin(tv)|,$$

Now apply the Mean Value Theorem:

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{for some } c \in (a, b),$$

with $f(u) = \sin(tu)$. We get that, for some $\eta \in (u, v)$,

$$\frac{|\sin(tu) - \sin(tv)|}{|u - v|} = |t\cos(t\eta)| \leqslant 2,$$

again using that $t \leqslant 2$, and also that $|\cos(x)| \leqslant 1$ for any x. So now we have $|\sin(tu) - \sin(tv)| \leqslant 2|u - v|$. Thus, $|f(t, u) - f(t, v)| \leqslant 4|u - v|$. The function is Lipschitz and the upper bound for $L = 4$.

Q2. Suppose we use Euler's method to find an approximation for $y(2)$, where y solves

$$y(1) = 1, \qquad y' = (t - 1)\sin(y).$$

(i) Give an upper bound for the global error taking $n = 4$ (i.e., $h = 1/4$).

Answer: We wish to find an upper bound for $\mathcal{E}_n := y(t_n) - y_n|$ with $n = 4$. We know that

$$|\mathcal{E}_n| = \frac{T}{L}\left(e^{L(t_n - t_0)} - 1\right).$$

We have $t_0 = 1$ and $t_n = 2$. So we need L and T. We need to find L, such that,

$$\frac{|f(t,u) - f(t,v)|}{|u - v|} \leqslant L.$$

$$\frac{|(t-1)\sin(u) - (t-1)\sin(v)|}{|u-v|} \leqslant L$$

$$|t-1|\frac{|\sin(u) - \sin(v)|}{|u-v|} \leqslant L$$

From the Mean Value Theorem we know

$$|f'(\eta)| = \frac{|f(u) - f(v)|}{|u-v|} \quad \text{for some } \eta \in (u,v),$$

so

$$|\cos(\eta)| = \frac{|\sin(u) - \sin(v)|}{|u-v|},$$

and

$$|t-1||\cos(\eta)| \leqslant L.$$

$\max|t-1| = 1$, $\max|\cos(\eta)| = 1$ so $1 \leqslant L$. The function satisfies a Lipschitz condition on the domain with the upper bound for $L = 1$. For Euler's method

$$T \leqslant \max_{t_0 \leqslant t \leqslant t_n} \frac{h}{2}|f''(y)|.$$

$$y'(t) = (t-1)\sin(y),$$

$$y''(t) = (t-1)\cos(y)\frac{dy}{dt} + \sin(y),$$

$$= (t-1)^2\cos(y)\sin(y) + \sin(y),$$

$\max|t-1| = 1$. Let $f(y) = \cos(y)\sin(y) + \sin(y)$, find $\max|f(y)|$. The max occurs when $f' = 0$ and $f'' < 0$.

$$f'(y) = \cos^2(y) - \underbrace{\sin^2(y)}_{1-\cos^2(y)} + \cos(y)$$

Local maximum occurs when

$$2\cos^2(y) + \cos(y) - 1 = 0,$$

$$(2\cos(y) - 1)(\cos(y) + 1) = 0,$$

local max at $\cos(y) = 1/2$ or $\cos(y) = -1$.

$$f''(y) = -4\cos(y)\sin(y) - \sin(y).$$

Evaluate $f''(y)$ at $\cos(y) = 1/2$, $\sin(y) = \sqrt{3}/2$.

$$f''\left(\frac{1}{2}\right) = -4\frac{1}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2},$$

therefore, local max at $\cos(y) = 1/2$ ($f''(-1) = 0$).

$$\max|y''(t)| = 1\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}.$$

(ii) What n should you take to ensure that the global error is no more that 10^{-3}?

Q3. Here is the tableau for a three stage Runge-Kutta method:

$$\begin{array}{c||ccc} \alpha_1 & & & \\ \alpha_2 & \beta_{21} & & \\ \alpha_3 & \beta_{31} & \beta_{32} & \\ \hline & b_1 & b_2 & b_3 \end{array} = \begin{array}{c||ccc} 0 & & & \\ \alpha_2 & 1/2 & & \\ 1 & \beta_{31} & 2 & \\ \hline & 1/6 & b_2 & 1/6 \end{array}$$

(i) Use that the method is consistent to determine b_2.

(ii) The method is exact when used to compute the solution to

$$y(0) = 0, \quad y'(t) = 2t, \ t > 0.$$

Use this to determine α_2.

(iii) The method should agree with an appropriate Taylor series for the solution to $y'(t) = \lambda y(t)$, up to terms that are $\mathcal{O}(h^3)$. Use this to determine β_{31}.