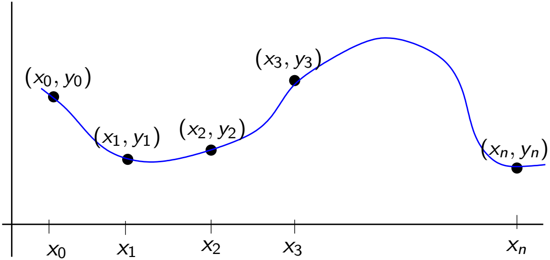
MA378 Chapter 1: Interpolation
§1.1 Introduction to Interpolation

Dr Niall Madden, Start: 14 Jan 2026 (W01.1)



1.1 Introduction

Suppose that we have two sets of $n+1$ real numbers:

- $\{x_i\}_{i=0}^{n}$, which are are *strictly* increasing, meaning that $x_0 < x_1 < x_2 < \cdots < x_n$
- and $\{y_i\}_{i=0}^{n}$.

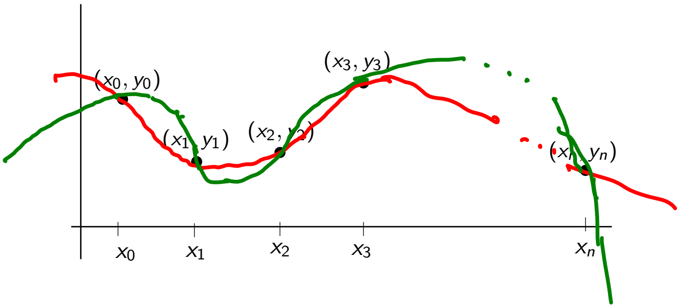
x_i are called "node points", "interpolation" points or "a mesh".

Interpolation problems are of the form: *Find a function, p, that is continuous and defined on* $[x_0, x_n]$, *such that*

$$p(x_k) = y_k, \quad \textit{for} \quad k = 0, 1, \ldots, n.$$

We say: "*p interpolates* the points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n)".

1.1 Introduction



1.2 Why do this?

Why would one like to do this? There are several possibilities, including

- The points belong to an underlying, but unknown function, f. We wish to establish likely values of f at points other than x_0, x_1, ..., x_n. The values of f may have been obtained from physical experiments, or numerical procedures (e.g., Newton's method for initial value problems). Or it may be that some values of the function are easily available. For example $2! = 2$, and $3! = 6$, but what about $2\frac{1}{2}!$ or $\pi!$?
- We may know the function, but prefer to work with an interpolant to it. For example, in order to estimate derivatives or integrals of a function.

Mathematics, from number theory to information theory, and nearly every aspect of numerical analysis, features many interpolation problems.

Elsewhere, the methods are used in fields ranging from aircraft design to computer animation.

The main reference for this section is Chapter 6 of Suli and Mayers, See also Lectures 18–20 of Stewart's *Afternotes on Numerical Analysis*.

Finished here Wed @ 2pm.

Definition 1.1

$\mathcal{P}_n$ is the set of polynomials of degree at most n and real-valued coefficients, i.e., $p_n \in \mathcal{P}_n$ if

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_i \in \mathbb{R}$.

Examples:

Eg $p(x) = 1 + 2x + \frac{1}{2}x^2 \in \mathbb{P}_2$,

Also $q(x) = 1 + 2x \in \mathbb{P}_1$ and $q(x) \in \mathbb{P}_2$

$r(x) = \frac{1}{2}x^2 \in \mathbb{P}_2$ so $q(x) = p(x) - r(x)$.

Definition 1.1

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$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_i \in \mathbb{R}$.

Examples:

$x^3 + 4x^5 - 3x \in \mathbb{P}_5$

Also $\pi x - e \in \mathbb{P}_1$

But $\sqrt{-1}\, x^2 - 3$ is not a polynomial (over the reals). Neither are $x^{1/2}$ or x^{-1}.

1.3 Polynomial Interpolation

In Exercise 1.2 you are asked to verify that $\mathcal{P}_n$ is a vector space.

- It is particularly important to note that if p_n and q_n both belong to $\mathcal{P}_n$, then so too does their sum.

- Also the zero polynomial belongs to $\mathbb{P}_n$.

The Polynomial Interpolation Problem comes in two forms.

The Polynomial Interpolation Problem I (PIP1)

Given is set of points $x_0 < x_1 < \cdots < x_n$, and a set of real numbers $y_0, y_1, \dots, y_n$, find $p_n \in \mathcal{P}_n$ such that

$$p_n(x_k) = y_k, \quad \text{for } k = 0, 1, \dots, n.$$

The Polynomial Interpolation Problem II (PIP2)

Given is set of points $x_0 < x_1 < \cdots < x_n$, and a function $f : [x_0, x_n] \to \mathbb{R}$, find $p_n \in \mathcal{P}_n$ such that

$$p_n(x_k) = f(x_k), \quad \text{for } k = 0, 1, \ldots, n.$$

Clearly PIP2 is just PIP1 with $y_k = f(x_k)$.

We say "p_n interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$". Or "p_n is the polynomial interpolant of f".

The questions that we must ask (and answer) are

(i) Is there a solution to the polynomial interpolation problem?

(ii) Is it unique?

(iii) How do we find it?

(iv) How accurate is it? If f is the underlying function (i.e., $f(x_k) = y_k$), can we find an upper bound for

$$\max_{x_0 \leq x \leq x_n} \{|f(x) - p_n(x)|\}?$$

1.4 Exercises

Exercise 1.1

Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

(d) What is the minimum possible degree of pq?

Exercise 1.2

Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space. Find a basis for $\mathcal{P}_n$. Find another basis for $\mathcal{P}_n$.

1.4 Exercises

Exercise 1.3

(a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

Exercise 1.4

For each of the following interpolation problems, determine (with explanation) if there is no solution, exactly one solution, or more than one solution. In all cases p_n denotes a polynomial of degree (at most) n. You are not required to determine p_n where it exists.

(a) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(b) Find $p_1(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = -2$.

(c) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = i - 1$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(d) Find $p_2(x)$ that interpolates (x_0, y_0), (x_1, y_1), and (x_2, y_2), where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$, $y_2 = 1$.

(e) Find $p_2(x)$ that interpolates (x_0, y_0) and (x_1, y_1) where $x_i = (-1)^{i+1}$ and $y_0 = 0$, $y_1 = -1$.