

Chap. 2: Initial Value Problems

§2.1: Introduction

MA385/530 – Numerical Analysis 1

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Emile Picard: his fundamental work on differential equations was only one of his many contributions to mathematics



Olga Ladyzhenskaya: her extensive achievements include providing the first proof of the convergence of finite difference methods for the Navier-Stokes equations

(See Chap 6 of Epperson for the introduction, and Chapter 12 of Süli and Mayers for the rest).

Motivation

The growth of some tumours can be modelled as

$$R'(t) = -\frac{1}{3}S_i R(t) + \frac{2\lambda\sigma}{\mu R + \sqrt{\mu^2 R^2 + 4\sigma}},$$

subject to the initial condition $R(t_0) = a$, where R is the radius of the tumour at time t .

Clearly, it would be useful to know the value of R at certain times in the future. Though it's essentially impossible to solve for R exactly, we can accurately estimate it. In this section, we'll study techniques for this.

Initial Value Problems (IVPs)

Initial Value Problems (IVPs) are differential equations of the form: *Find $y(t)$ such that*

$$\frac{dy}{dt} = f(t, y) \text{ for } t > t_0, \quad \text{and } y(t_0) = y_0. \quad (1)$$

Here $y' = f(t, y)$ is the *differential equation* and $y(t_0) = y_0$ is the *initial value*.

Some IVPs are easy to solve. For example:

$$y' = t^2 \quad \text{with } y(1) = 1.$$

Most problems are much harder, and some don't have solutions at all. In many cases, it is possible to determine that a given problem does indeed have a solution, even if we can't write it down. The idea is that the function f should be "Lipschitz", a notion closely related to that of a *contraction*.

Definition 2.1 (Lipschitz Condition)

A function f satisfies a **Lipschitz Condition** (with respect to its second argument) in the rectangular region D if there is a positive real number L such that

$$|f(t, u) - f(t, v)| \leq L|u - v| \quad (2)$$

for all $(t, u) \in D$ and $(t, v) \in D$.

Example 2.2

For each of the following functions f , show that it satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L .

- (i) $f(t, y) = y/(1 + t)^2$ for $0 \leq t \leq \infty$.
- (ii) $f(t, y) = 4y - e^{-t}$ for all t .
- (iii) $f(t, y) = -(1 + t^2)y + \sin(t)$ for $1 \leq t \leq 2$.

Theorem 2.3 (Picard's)

Suppose that the real-valued function $f(t, y)$ is continuous for $t \in [t_0, t_M]$ and $y \in [y_0 - C, y_0 + C]$; that $|f(t, y_0)| \leq K$ for $t_0 \leq t \leq t_M$; and that f satisfies the *Lipschitz condition* (2). If

$$C \geq \frac{K}{L} \left(e^{L(t_M - t_0)} - 1 \right),$$

then (1) has a unique solution on $[t_0, t_M]$. Furthermore

$$|y(t) - y(t_0)| \leq C \quad t_0 \leq t \leq t_M.$$

You are not required to know this theorem for this course. However, it's important to be able to determine when a given f satisfies a Lipschitz condition.

Exercise 2.1

For the following functions show that they satisfy a Lipschitz condition on the corresponding domain, and give an upper-bound for L :

- (i) $f(t, y) = 2yt^{-4}$ for $t \in [1, \infty)$,
- (ii) $f(t, y) = 1 + t \sin(ty)$ for $0 \leq t \leq 2$.

Exercise 2.2

Many text books, instead of giving the version of the Lipschitz condition we use, give the following: *There is a finite, positive, real number L such that*

$$\left| \frac{\partial}{\partial y} f(t, y) \right| \leq L \quad \text{for all } (t, y) \in D.$$

Is this statement *stronger than* (i.e., more restrictive than), *equivalent to* or *weaker than* (i.e., less restrictive than) the usual Lipschitz condition? Justify your answer.

Hint: the Wikipedia article on [Lipschitz continuity](#) is very informative.