

2526-MA140 Engineering Calculus

Week 07, Lecture 3 The Fundamental Theorem of Calculus

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Suimeálaithe

Tá tuairisc na suimeálaí fágtha ar lír.

| $f(x)$ | $\int f(x)dx$ |
|---------------------|-----------------------|
| $x^n \ (n \neq -1)$ | $\frac{x^{n+1}}{n+1}$ |
| $\frac{1}{x}$ | $\ln x $ |
| e^x | e^x |
| e^{ax} | $\frac{1}{a}e^{ax}$ |
| a^x | $\frac{a^x}{\ln a}$ |
| $\cos x$ | $\sin x$ |
| $\sin x$ | $-\cos x$ |
| $\tan x$ | $\ln \sec x $ |

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|------------------------------|--|
| $\cos^2 x$ | $\frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]$ |
| $\sin^2 x$ | $\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]$ |
| $\frac{1}{\sqrt{a^2 - x^2}}$ | $\sin^{-1} \frac{x}{a}$ |
| $\frac{1}{x^2 + a^2}$ | $\frac{1}{a} \tan^{-1} \frac{x}{a}$ |

Suimeáil
na mireanna

$$\int u dv = uv - \int v du$$

Integrals

Constants of integration omitted.

| $f(x)$ | $\int f(x)dx$ |
|-------------------------------|---|
| $\frac{1}{x\sqrt{x^2 - a^2}}$ | $\frac{1}{a} \sec^{-1} \frac{x}{a}$ |
| $\frac{1}{\sqrt{x^2 + a^2}}$ | $\ln \left \frac{x + \sqrt{x^2 + a^2}}{a} \right $ |
| $\frac{1}{a^2 - x^2}$ | $\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ |
| $\frac{1}{\sqrt{x^2 - a^2}}$ | $\ln \left \frac{x + \sqrt{x^2 - a^2}}{a} \right $ |

Integration by parts

Dad/Bad Joke of the Day

Today's joke (with thanks to Julie M).

**Me peeling
potatoes**

$$\sum_{k=1}^n f(x_k) \cdot \Delta x$$

**My mum peeling
potatoes**

$$\int f(x) dx$$

The exciting topics that await us in today:

- 1 Recall from yesterday:
- 2 Fundamental Thm of Calculus: Part 1
- 3 FTC1+Chain Rule
- 4 Antiderivatives
 - Indefinite Integrals
 - Common functions
 - Properties
- 5 The Fundamental Thm of Calculus: Part 2
- 6 Exercises

See also: Sections **4.10** (Antiderivatives) and **5.3** (Fundamental Theorem of Calculus) of **Calculus** by Strang & Herman:

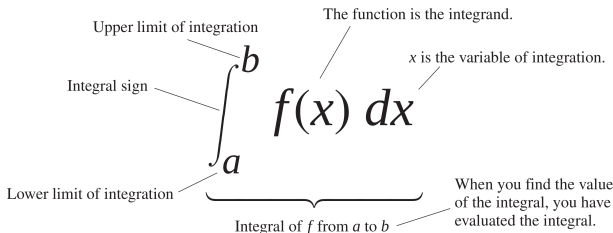
[math.libretexts.org/Bookshelves/Calculus/Calculus_\(OpenStax\)](https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax))

Recall from yesterday:

Let $f(x)$ be function defined on an interval $[a, b]$. The **definite integral** of f from a to b is

$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} hf(x_i),$$

where $h = (b - a)/n$ and $x_i = a + ih$. It is the **area** of the region in space bounded by $y = 0$, $y = f(x)$, $x = a$, and $x = b$.



Recall from yesterday:

Given a function, f , we can define another, F as

$$F(x) = \int_a^x f(t) dt.$$

That is, the variable in F is the upper limit of integration on the right.

Recall from yesterday:

Example

Let $f(t) \equiv 1$, and $F(x) = \int_0^x f(t) dt$. Give a formula for $F(x)$, using the “area” meaning of the definite integral.

Fundamental Thm of Calculus: Part 1

Fundamental Theorem of Calculus: Part 1 (FTC1)

Let $f(x)$ be a continuous function on $[a, b]$. If as

$$F(x) = \int_a^x f(t) dt, \quad \text{then} \quad \frac{dF}{dx}(x) = f(x).$$

I.e., $F'(x) = f(x)$ for $x \in [a, b]$.

Roughly: f is the derivative its own integral. You can find a proof in Section 5.3 of the textbook.

Fundamental Thm of Calculus: Part 1

Example

Let $g(x) = \int_1^x \frac{1}{t^3 + 1} dt$. Find $g'(x)$.

FTC1+Chain Rule

Sometimes the limit of integration is a more complicated function of x . In that case, we can apply the **Chain Rule**, along with the FTC1.

Example

Let $F(x) = \int_1^{\sqrt{x}} \sin(t) dt$. Find $F'(x)$.

Idea: Let $u(x) = \sqrt{x} = x^{1/2}$. So

► $F(u) = \int_1^u \sin(t) dt$, and

► $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$.

Then...

$$\frac{dF}{dx} = \frac{dF}{du} \frac{du}{dx} = \sin(u(x)) \left(\frac{1}{2\sqrt{x}} \right) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

Antiderivatives

Definition: Antiderivative

A function F is an **antiderivative** of f on $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$. Thus,

f is the derivative of $F \Leftrightarrow F$ is an antiderivative of f .

Note: If F is an antiderivative of f , then the most general antiderivative of f is

$$F(x) + C$$

where C is an *arbitrary* constant, called a **constant of integration**.

- ▶ The word “arbitrary” here means that any choice is valid.
- ▶ The derivative of C is zero.

Antiderivatives

Examples:

- ▶ $F(x) = x + C$ is an antiderivative of $f(x) \equiv 1$.
- ▶ $F(x) = x^2 + C$ is an antiderivative of $f(x) = ???$...
- ▶ $F(x) = ???$ is an antiderivative of $f(x) = 3x^2$.

Antiderivatives

Examples

Find all antiderivatives of the following functions

(i) $f(x) = \frac{1}{x}$ for $x > 0$.

(ii) $f(x) = \sin(x)$

(iii) $f(x) = e^x$.

Definition: indefinite integral

Given a function f , the **indefinite integral** of f , denoted

$$\int f(x) \, dx$$

is the general antiderivative of f . That is, if F is an antiderivative of f , then

$$\int f(x) \, dx = F(x) + C.$$

Examples:

- ▶ $\int 2x \, dx = x^2 + C$
- ▶ $\int 3x^2 \, dx = x^3 + C$

- ▶ $\int x \, dx = \frac{1}{2}x^2 + C$
- ▶ $\int x^2 \, dx = \frac{1}{3}x^3 + C.$

Spotting the pattern we can deduce...

Power Rule of Integration

$$\text{If } n \neq -1, \quad \text{then} \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Note: For $n = -1$, we have

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C.$$

Here is a list of the antiderivatives of some common functions.

$$\blacktriangleright \int \frac{1}{x} dx = \ln |x| + C$$

$$\blacktriangleright \int e^x dx = e^x + C$$

$$\blacktriangleright \int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\blacktriangleright \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\blacktriangleright \int \sin(x) dx = -\cos(x) + C$$

$$\blacktriangleright \int \cos(x) dx = \sin(x) + C$$

$$\blacktriangleright \int \tan(x) dx = \ln |\sec(x)| + C$$

$$\blacktriangleright \dots$$

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Suimeáil
na míreanna

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Integration by parts

Properties of Integration

1. If k is a constant, then

$$\int kf(x) dx = k \int f(x) dx.$$

2. Integration is additive:

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

Example

Evaluate the integral

$$\int 2x^2 + 9x^7 \, dx$$

Example

Evaluate the integral

$$\int \frac{4}{1+x^2} dx.$$

The Fundamental Thm of Calculus: Part 2

Now that we know all about antiderivatives, we can see how the link to **definite integrals**

Theorem (The Fundamental Thm of Calculus, Part 2)

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Notation: We can write $F(b) - F(a)$ as $F(x) \Big|_{x=a}^{x=b}$, or, more often,

as $F(x) \Big|_a^b$.

The Fundamental Thm of Calculus: Part 2

Example: Show that $\int_{-1}^1 (x^2 + 2) \, dx = \frac{14}{3}$

The Fundamental Thm of Calculus: Part 2

Example: Show that $\int_{-1}^1 (x^3 + x) \, dx = 0$

Exer 7.3.1

Let $F(x) = \int_x^{2x} t \, dt$. Use the Fundamental Theorem of Calculus to evaluate $F'(x)$.

Hint: we can split this into two integrals:

$$F(x) = \int_x^{2x} t \, dt = \int_x^0 t \, dt + \int_0^{2x} t \, dt = -\int_0^x t \, dt + \int_0^{2x} t \, dt.$$

Now apply the FTC to each term, including the Chain Rule for the second.

Exercises

Exer 7.3.2

Evaluate the following integrals.

1. $\int e^{2x} + \frac{1}{2x} dx$

2. $\int \frac{3}{\sqrt{2-x^2}} dx$

Exer 7.3.3

Evaluate the definite integral $\int_1^e e^{2x} + \frac{1}{2x} dx$

Exer 7.3.4

Find two values of q for which $\int_q^0 2x + x^2 dx = 0$.