Annotated slides from Tuesday

MA313 : Linear Algebra I

Week 2: Subspaces and Spans

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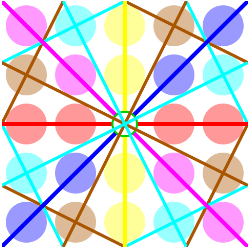
13 and 16 September, 2022



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These slides are based on ones by Tobias Rossmann.

Outline

Assignment 1

- The first assignment has opened. Deadline is Monday, 19 September.
- It contributes 3% to the final grade for MA313.
- Topics: addition of vectors, matrix-vector multiplication, matrix-matrix multiplication, solving linear systems by row-reduction.
- System still has a few glitches. We are working to fix them. Don't worry about time-out errors.

Communications skills

DRAFT: will be edited later

1. Later this week: list of topics will be posted.
2. End of Week 3: confirm your topic.
3. End of Week 7: progress report due. Will include scope, outline of structure, and major sources.
4. Week 12: submission of essay and slides and presentations.

Tutorials start in Week 3. When:
https://forms.office.com/r/0ya9Bp8qBU

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12					
12 – 1				??	Lecture
1 – 2		Lecture			
2 – 3					
3 – 4					
4 – 5					

Everyone who attended Friday's class was available

- ~~Wednesday at 11.00~~
- ~~Friday at 11.00~~

Mostly likely Thursday at 12. Perhaps Tues at 10 too.

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Week 2: Subspaces and Spans

Start of ...

PART 1: Definition of a Vector Space

See Section 4.1 of the text-book:
https://search.library.nuigalway.ie/permalink/f/1pmb9lf/353GAL_ALMA_DS51920676300003626

Definition of a vector space (1/2)

A **vector space** consists of

- a (non-empty!) set V, whose elements we call **vectors**,
- an operation called **addition** which assigns a vector

$$u + v \in V$$

  to any two vectors $u, v \in V$, and
- an operation called **scalar multiplication** which assigns a vector

$$cu \in V$$

  to each scalar $c \in \mathbb{R}$ and vector $u \in V$

such that the axioms on the following slides are satisfied.

Definition of a vector space (2/2)

We require that the following conditions **V1–V8** are satisfied for all vectors $u, v, w \in V$ and scalars $c, d \in \mathbb{R}$:

V1. $u + v = v + u$ (**commutativity** of addition)

V2. $(u + v) + w = u + (v + w)$ (**associativity** of addition)

V3. There exists $\mathbf{0} \in V$, called the **zero vector** such that $u + \mathbf{0} = u$ for all $u \in V$,

V4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$

V5. $c(u + v) = cu + cv$ (**distributivity I**) ✓

V6. $(c + d)u = cu + du$ (**distributivity II**)

V7. $c(du) = (cd)u$ ✓

V8. $1u = u$

Example ($\mathbb{R}^n$ is a vector space.)

We define

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}$$

with addition defined as $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$ and

scalar multiplication defined as $c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$.

Then $\mathbb{R}^n$ is a vector space. The proof is a quite tedious, but quite easy.

It would take too long to show that $\mathbb{R}^n$ satisfies each of the 8 axioms. So we'll just verify the three of them.

V1. $u + v = v + u$ (**commutativity** of addition)

We'll do this for $\mathbb{R}^3$. Write u as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \text{ Then}$$

Scalar addition.

$$u + v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ v_3+u_3 \end{bmatrix}$$

vector addition

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = v + u.$$

V3. There exists $\mathbf{0}$, called the **zero vector**, such that $u + \mathbf{0} = u$ for all $u \in V$.

we'll again do this for $\mathbb{R}^3$.

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{then,}$$

$$u + \vec{0} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 + 0 \\ u_2 + 0 \\ u_3 + 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u$$

V4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$.

(These notes were added after class.)

Let's write the vector u as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

And the zero vector is

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The we can see there is a vector, -u, with

$$-u = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$$

The $u + (-u) = \begin{bmatrix} u_1 - u_1 \\ u_2 - u_2 \\ \vdots \\ u_n - u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

For an integer $n \geqslant 0$, let $\mathbb{P}_n$ consist of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

of degree at most n, where $a_0, \ldots, a_n \in \mathbb{R}$.

We can add polynomials in $\mathbb{P}_n$ in the usual way:

$$\begin{aligned}(a_0 + a_1 t + \cdots + a_n t^n)& \\ + (b_0 + b_1 t + \cdots + b_n t^n)& \\ = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n&.\end{aligned}$$

i

Also,

$$cp(t) = ca_0 + ca_1 t + \cdots + ca_n t^n,$$

where $c \in \mathbb{R}$.

Claim: These operations turn $\mathbb{P}_n$ into a vector space.

The reasoning again just boils down to to properties of real numbers.

Example

Function spaces Let $\mathbb{D}$ be an arbitrary set.
Let V be the set of **all** functions $f : \mathbb{D} \to \mathbb{R}$.
Given $f, g \in V$ and $c \in \mathbb{R}$, we define $f + g \in V$ and $cf \in V$ via

$$(f + g)(x) := f(x) + g(x)$$

and

$$(cf)(x) := cf(x)$$

for $x \in \mathbb{D}$.
Claim: These operations turn V into a vector space.

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Week 2: Subspaces and Spans

Start of ...

PART 2: Not everything is a vector space

Part 2: Not everything is a vector space

So far, all of the examples we have looked at correspond to vector spaces. But not every set equipped with addition and scalar multiplication is a vector space.

Here are a few examples of things that are not vector spaces.

>0

1. The set of vectors in $\mathbb{R}^2$ with strictly positive entries.

Eg $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \pi \\ e \end{bmatrix}$, but not $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

This set does not include the zero vector. Not a vector space

2. The set of vectors in $\mathbb{R}^2$ with non-negative entries.

≥ 0

If $u \in V$, then $-u \notin V$.

Eg $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$, but $\begin{bmatrix} -1 \\ 0 \end{bmatrix} \notin V$.

(fails V4)

Part 2: Not everything is a vector space

3. The set of polynomials of degree **exactly** 3.

Eg V includes $1+2x+3x^2+4x^3$

and $1+\frac{1}{8}x^3$.

But not $1+52x^3-8x^4$ ← too high.

or $1+\frac{1}{3}x^2$ ← not degree 3

Can't include the zero vector. But also because addition is not closed.

Eg If $r(x) = 1+2x+3x^2+4x^3$

$q(x) = 1-2x+3x^2-4x^3$

$r+q = 2+6x^2$: not cubic

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Start of ...

PART 3: Subspaces

Finished here Tuesday