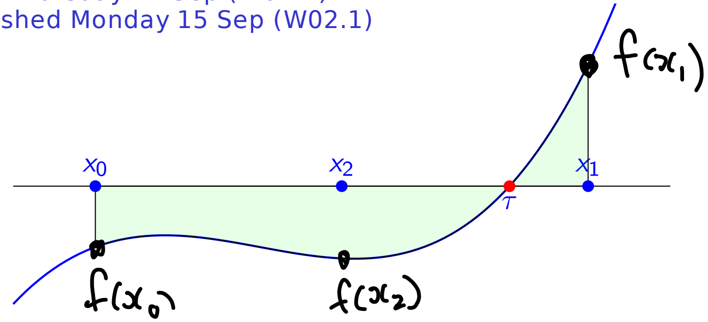
Solving nonlinear equations

1.2: Interval Bisection

MA385 – Numerical Analysis

September 2025

Started Thursday 11 Sep (W01.2)
and finished Monday 15 Sep (W02.1)



0. Outline

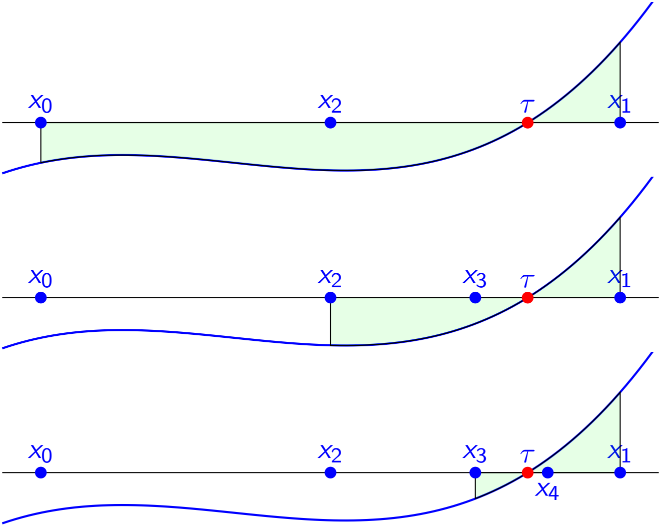
For more details, see Section 1.6 (The Bisection Method) of Süli and Mayers, *An Introduction to Numerical Analysis*

1. Bisection

The most elementary algorithm is the "*Bisection Method*" (also known as "Interval Bisection"). Suppose that we know that f changes sign on the interval $[a, b] = [x_0, x_1]$ and, thus, $f(x) = 0$ has a solution, τ, in $[a, b]$. Proceed as follows

1. Set x_2 to be the midpoint of the interval $[x_0, x_1]$.
2. Choose one of the sub-intervals $[x_0, x_2]$ and $[x_2, x_1]$ where f change sign;
3. Repeat Steps 1–2 on that sub-interval, until f is sufficiently small at the end points of the interval.

1. Bisection



1. Bisection

This may be expressed more precisely using some *pseudocode*.

The Bisection Algorithm

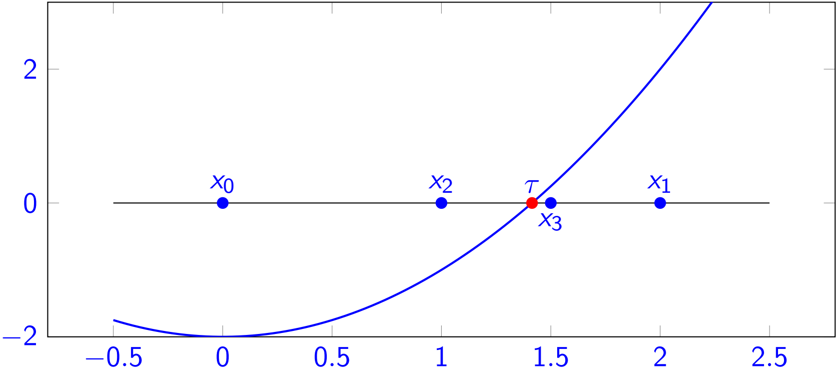
Set eps to be the stopping criterion.
If $|f(a)| \leq eps$, return a. Exit.
If $|f(b)| \leq eps$, return b. Exit.
Set $x_L = a$ and $x_R = b$.
Set $k = 1$
while($|f(x_k)| > eps$)
    $x_{k+1} = (x_L + x_R)/2$;
    if $(f(x_L)f(x_{k+1}) < 0)$
        $x_R = x_{k+1}$;
    else
        $x_L = x_{k+1}$
    end if;
    $k = k + 1$
end while;

1. Bisection

Example 1

Find an estimate for $\sqrt{2}$ that is correct to 6 decimal places.

Solution: Use bisection to solve $f(x) := x^2 - 2 = 0$ on the interval $[0, 2]$.



Example 1

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k	x_k	$\lvert f(x_k)\rvert$	$\lvert \tau - x_k\rvert$
0	0.000000	2.00e+00	1.41e+00
1	2.000000	2.00e+00	5.86e-01
2	1.000000	1.00e+00	4.14e-01
3	1.500000	2.50e-01	8.58e-02
4	1.250000	4.38e-01	1.64e-01
5	1.375000	1.09e-01	3.92e-02
6	1.437500	6.64e-02	2.33e-02
7	1.406250	2.25e-02	7.96e-03
8	1.421875	2.17e-02	7.66e-03
9	1.414062	4.27e-04	1.51e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	1.414214	1.62e-06	5.72e-07
23	1.414214	2.69e-07	9.50e-08

One of the main advantages of the Bisection method is that it will always work, providing only that f is continuous on $[a, b]$, and that the solution exists. Furthermore, we can prove...

Theorem 2.1

Let x_k be the kth iteration generated by the Bisection Method. Then

$$|\tau - x_k| \leq \Big(\frac{1}{2}\Big)^{k-1}|b - a|, \qquad \text{for } k = 2, 3, 4, \ldots$$

Proof: Since $a \leq \tau \leq b$, so $x_0 \leq \tau \leq x_1$

So $|\tau - x_1| \leq |x_1 - x_0| = |b - a|$.

Since x_2 is the midpoint of x_0 & x_1, so

$|\tau - x_2| \leq \frac{1}{2}|b - a|$.

Similarly

$$|\tau - x_3| \le \tfrac{1}{2}|x_1 - x_2|$$
$$\le \tfrac{1}{2}\left(\tfrac{1}{2}\right)|b-a| = \left(\tfrac{1}{2}\right)^2|b-a|$$

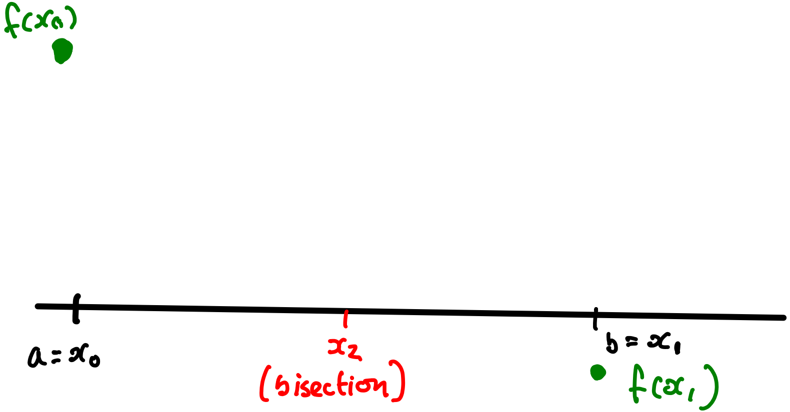
And (by induction)

$$|\tau - x_k| \le \left(\tfrac{1}{2}\right)^{k-1}|b-a|.$$

Note: as $k \to \infty$, so $\left(\tfrac{1}{2}\right)^k \to 0$

So $|\tau - x_k| \to 0$ i.e. $x_k \to \tau$.

A disadvantage of bisection is that it is not particularly efficient. So our next goal will be to derive better methods, particularly the **Secant Method** and **Newton's method**. We also have to come up with some way of expressing what we mean by "**better**".



Exercise 1.1

Suppose we want to find $\tau \in [a, b]$ *such that* $f(\tau) = 0$ *for some given* f, a *and* b. *Write down an estimate for the number of iterations* K *required by the bisection method to ensure that, for a given* ε, *we know* $|x_k - \tau| \leq \varepsilon$ *for all* $k \geq K$. *In particular, how does this estimate depend on* f, a *and* b*?*

Exercise 1.2

How many (decimal) digits of accuracy are gained at each step of the bisection method? (If you prefer, how many steps are needed to gain a single (decimal) digit of accuracy?)

Exercise 1.3

Let $f(x) = e^x - 2x - 2$*. Show that there is a solution to the problem: find* $\tau \in [0, 2]$ *such that* $f(\tau) = 0$*.*
Taking $x_0 = 0$ *and* $x_1 = 2$*, use 6 steps of the bisection method to estimate* τ*. You may use a computer program to do this, but please note that in your solution.*
Give an upper bound for the error $|\tau - x_6|$*.*

Exercise 1.4

We wish to estimate $\tau = \sqrt[3]{4}$ *numerically by solving* $f(x) = 0$ *in* $[a, b]$ *for some suitably chosen* f, a *and* b.

(i) *Suggest suitable choices of* f, a, *and* b *for this problem.*

(ii) *Show that* f *has a zero in* $[a, b]$.

(iii) *Use 6 steps of the bisection method to estimate* $\sqrt[3]{4}$. *You may use a computer program to do this, but please note that in your solution.*

(iv) *Use Theorem 2.1 to give an upper bound for the error* $|\tau - x_6|$.