3.0 Annotated slides.

# MA378 Chapter 1: Interpolation

## §1.3 Interpolation Error Estimates

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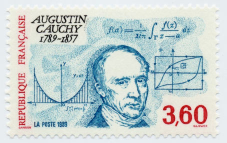


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Augustin-Louis Cauchy (1789–1857), Paris, France. He was a pioneer of analysis, in particular in introducing rigour into calculus proofs. He founded the fields of complex analysis and the study of permutation groups.

**Important:** This section is based on Section 6.2 of the text-book. You can access the book from the Reading List on canvas. I have also posted Sections 6.1 and 6.2 to Canvas.

Text book: Suli & Mayers:

An Intro to Numerical Analysis.

Exer: Read that Section!!

# 3.1 Introduction

In our last example, we wrote down the polynomial of degree $n = 2$ interpolating $f(x) = e^x$ at $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

We now want to investigate how, in general, error in polynomial interpolation depends on

(i) the function (and its derivatives)

(ii) the number of points used (or, equivalently, degree of the polynomial used).

By "Error" we mean

$$f(x) - p_n(x)$$

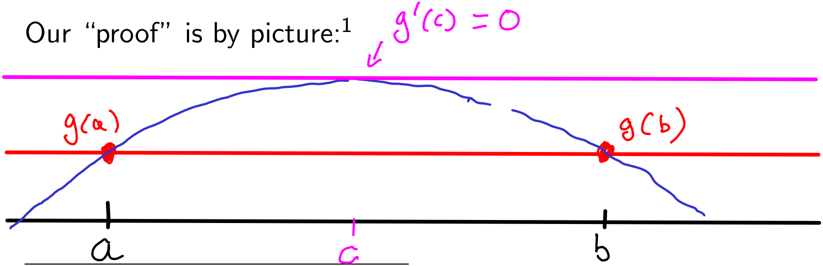
# 3.1 Introduction

The main ingredient we need to the following theorem.

**Theorem 3.1 (Rolle's Theorem)**

*Let $g$ be a function that is continuous and differentiable on the interval $[a, b]$. If $g(a) = g(b)$, then there is at least one point $c$ in $(a, b)$ where $g'(c) = 0$.*

Our "proof" is by picture:[1]




[1] One can easily deduce Rolle's Theorem from the Mean Value Theorem (MVT). But since the standard proof of the MVT uses Rolle's Theorem, that would be cheating.

# 3.2 Error estimate for n=0

The simplest case is when $n = 0$, so the interpolant is a constant, i.e., it is $p_0$ interpolating a function $f$ at a point $x_0$. Here is one way we can deduce the *interpolation error*.

$\$$ i.e. $p_0(x)$ is a constant, and $p_0(x_0) = f(x_0)$

So $p_0(x) = f(x_0)$ for all $x$. Take any fixed $x \neq x_0$. For that $x$ define the auxillary function

$$\boxed{g(t) = f(t) - p_0(t) - \left[\frac{f(x) - p_0(x)}{x - x_0}\right](t - x_0)}$$

Note that $g(x_0) = \underbrace{f(x_0) - p(x_0)}_{0} - \left[\frac{f(x) - p_0(x)}{x - x_0}\right]\underbrace{(x_0 - x_0)}_{0}$

So $g(x_0) = 0$

Also $g(x) = f(x) - p_0(x) - \frac{f(x) - p_0(x)}{\cancel{x - x_0}}\cancel{(x - x_0)} = 0$

## 3.2 Error estimate for n=0

The simplest case is when $n = 0$, so the interpolant is a constant, i.e., it is $p_0$ interpolating a function $f$ at a point $x_0$. Here is one way we can deduce the *interpolation error*.

So we know that: $g(x_0) = g(x) = 0$

(ie $g$ is zero at two distinct points).

By Rolle's Thm, there is a point $c$ between $x_0$ and $x$, such that $g'(c) = 0$.

Since $g(t) = f(t) - p_0(t) - \left[\frac{f(x) - p_0(x)}{x - x_0}\right](t - x_0)$

So $g'(t) = f'(t) - p_0'(t) - \left[\frac{f(x) - p_0(x)}{x - x_0}\right]$

Thus $0 = g'(c) = f'(c) - \frac{f(x) - p_0(x)}{x - x_0}$ [since $p_0'(t) \equiv 0$]

That gives $\boxed{f(x) - p_0(x) = f'(c)(x - x_0)}$.

# 3.2 Error estimate for n=0

It is important to understand what this formula is telling us:

We now know that

$$f(x) - p_0(x) = f'(c)\,(x - x_0)$$

for some $c \in (x_0, x)$. $c$ is unknown, but bounded between $x_0$ & $x$. Moreover

- Error depends on $|x - x_0|$. The further $x$ is from $x_0$, the larger the error.

- The larger $|f'(c)|$ is, the larger the error.

# 3.3 Error estimates for $n$ (general)

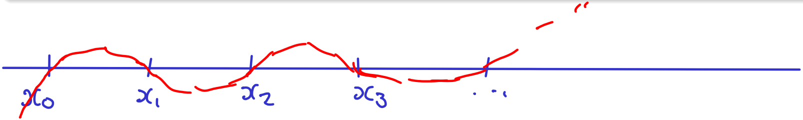
The following is the most important theorem of NA2; it is used repeatedly through-out the semester. It's often called the *Polynomial Interpolation Error Theorem*, or *Cauchy's Theorem*.

First, we need to define an important polynomial.

**Definition 3.2 (Nodal Polynomial)**

The **Nodal Polynomial** $\pi_{n+1}$ associated with the interpolation points that $a = x_0 < x_1 < \cdots < x_n = b$ is

$$\pi_{n+1}(x) := (x - x_0)(x - x_1)\ldots(x - x_n) = \prod_{i=0}^{n}(x - x_i).$$



**Theorem 3.3 (Cauchy, 1840)**

*Suppose that $n \geq 0$ and $f$ is a real-valued function that is continuous and defined on $[a, b]$, such that the derivative of $f$ of order $n+1$ exists and is continuous on $[a, b]$. Let $p_n$ be the polynomial of degree $n$ that interpolates $f$ at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that*

$$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x).} \tag{1}$$

[Note: we denoted "$\tau$" as "c" for the case $n=0$. sorry!]

Notation $f^{(n+1)}(\tau)$ is $\dfrac{d^{n+1}}{dx^{n+1}} f(\tau)$

# 3.3 Error estimates for $n$ (general)

**Proof.**

We won't do the proof in class. It follows the reasoning for the case of $n = 0$, and is given in full detail in Theorem 6.2 of Suli and Mayers. However, you are expected to

- review the proof in the textbook, and make sure you understand it;
- be able to reproduce it for an exam (yes, it could be asked in the final exam).

$\square$

Finished here 19/01/2024

**Example 3.4**

In an earlier example, we wrote down the Lagrange form of the polynomial, $p_2$, that interpolates $f(x) = e^x$ at the points $\{-1, 0, 1\}$. Give a formula for $e^x - p_2(x)$.

Here $n=2$, $x_0=-1$, $x_1=0$ and $x_2=1$.

Since $n=2$, $f(x) - p_2(x) = \dfrac{f^{(3)}(\tau)}{3!}(x-x_0)(x-x_1)(x-x_2)$

Since $f(x)=e^x$, so $f'(x) = f''(x) = f'''(x) = e^x$.

So $f(x) - p_2(x) = \dfrac{e^\tau}{6}(x+1)(x)(x-1)$ for some $\tau \in (-1,1)$

$= \frac{1}{6} e^\tau (x^3 - x)$

In addition

$\max_{-1 \le x \le 1} |f(x) - p_2(x)| \le \frac{1}{6}(e)|x^3 - x|$, since $e^\tau < e \ \forall \tau$

# 3.3 Error estimates for $n$ (general)

Usually (and as in the above example), we can't calculate $f(x) - p_n(x)$ exactly from Formula (1), because we have no way of finding $\tau$. However, we are typically not so interested in what the error is at some given point, but what is the maximum error over the whole interval $[x_0, x_n]$. That is given by:

**Corollary 3.5**

*Define*

$$M_{n+1} = \max_{x_0 \le \sigma \le x_n} |f^{(n+1)}(\sigma)|.$$

*Then*, for any $x$,

$$|f(x) - p_n(x)| \le \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|. \quad (2)$$

Furthermore

$$\max_{x_0 \le x \le x_n} |f(x) - p_n(x)| \le \frac{M_{n+1}}{(n+1)!} \max_{x_0 \le x \le x_n} |\pi_{n+1}(x)|.$$

# 3.3 Error estimates for $n$ (general)

**Example 3.6**

Let $p_1$ be the polynomial of degree 1 that interpolates a function $f$ at distinct points $x_0$ and $x_1$. Letting $h = x_1 - x_0$, show that

$$E = \max_{x_0 \le x \le x_1} |f(x) - p_1(x)| \le \frac{1}{8} h^2 M_2.$$

We know that

$$E \doteq \max_{x_0 \le x \le x_1} |f(x) - p_1(x)| \le \frac{|f''(\tau)|}{2!} \cdot \max_{x_0 \le x \le x} |\pi_2(x)|.$$

$$\le \frac{M_2}{2} \max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)|.$$

But (see white board!) $|(x - x_0)(x - x_1)| \le \frac{h^2}{4}$.

So $E \le \frac{M_2}{8} h^2$

### Example 3.6

Let $p_1$ be the polynomial of degree 1 that interpolates a function $f$ at distinct points $x_0$ and $x_1$. Letting $h = x_1 - x_0$, show that

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

Now that we know this

- The error depends on $(x_1 - x_0)^2$
- As $h \to 0$ so too does the Error!
- Error depends on $M_2$. If $M_2 = 0$ Error is zero.

# 3.4 Exercises

## Exercise 3.1

Read Section 6.2 of An Introduction to Numerical Analysis (Süli and Mayers). Pay particular attention to the proof of Thm 6.2 at `https://ebookcentral.proquest.com/lib/nuig/reader.action?docID=221072&ppg=192`.

## Exercise 3.2

Let $p_2$ be the polynomial of degree 2 that interpolates a function $f$ at the points $x_0$, $x_1$ and $x_2$. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0 \leq x \leq x_2} |f(x) - p_2(x)| \leq \frac{1}{6} \frac{2}{3\sqrt{3}} h^3 M_3 = \frac{1}{9\sqrt{3}} h^3 M_3.$$

*Hint: simplify the calculations by taking* $t = x - x_1$, *writing* $(x - x_0)(x - x_1)(x - x_2)$ *in terms of* $h$ *and* $t$.