Introduction to Generalised Linear Models for Ecologists

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https://github.com/niamhmimnagh/GLME01---Introduction-to-Generalised-Linear-Models-for-Ecologists

PR STATS

Count Data

- Count data are non-negative integers that often have a skewed distribution and a variance that increases with the mean.
- Examples:
  - The number of birds observed in a plot
  - The number of disease cases reported per day
  - The number of accidents per year.
- Using standard linear regression for such data can lead to nonsensical predictions (e.g. negative counts), and incorrect inferences because linear regression assumes constant variance and normally distributed errors.
- Poisson regression solves these issues by explicitly modelling the distribution of counts.

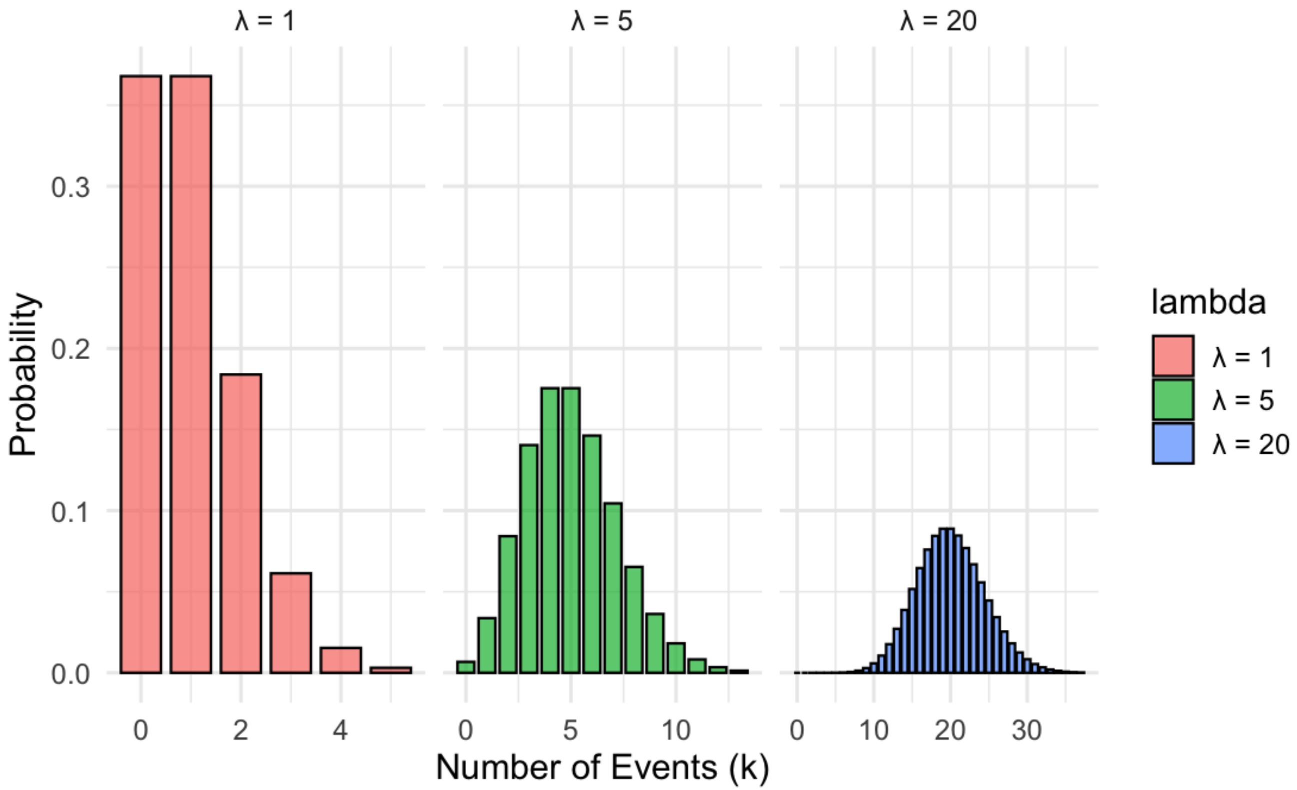
Why Not Use Linear Regression?

- Count data violates the assumptions of linear regression.
- Linear regression assumes that residuals are normally distributed and have constant variance.
- When applied to count data, linear models can predict negative values, which are meaningless for counts.
- Count data typically exhibit heteroscedasticity, where the variance grows with the mean. We need a model that respects the nature of count data.

The Poisson Distribution

- The Poisson distribution gives the probability of observing a count (0, 1, 2, …) of events in a fixed time or space window.
- It assumes events occur independently and at a constant average rate λ (equivalently, counts in non-overlapping intervals are independent).
- A key property of this distribution is equidispersion: the mean and variance are both λ.
- Use it for rare-event counts (e.g., calls per minute, nests per plot).
- If the variance ≫ mean or there are many zeros, consider a negative binomial or zero-inflated model.

The Poisson Distribution



The Poisson Distribution

- When λ is small the Poisson distribution is strongly right-skewed, with most of the probability at zero.
- As λ increases to around five, the distribution becomes less skewed, and counts spread out and cluster around the mean λ.
- For large λ values (about 20 or more), the distribution looks nearly symmetric and is well approximated by a Normal distribution with mean and variance equal to λ.
- In general, skewness decreases as λ grows, and the most likely count is roughly λ rounded down.

Poisson Regression as a GLM

- We model each count Y_i as Poisson with mean λ_i.

$$Y_i \sim Poisson(\lambda_i)$$

- The model links the mean λ_i to a linear predictor through the natural log function:

$$\log(\lambda_i) = \beta_0 + \beta_1 x_{1_i} + \beta_2 x_{2_i} + \ldots \beta_p x_{p_i}$$

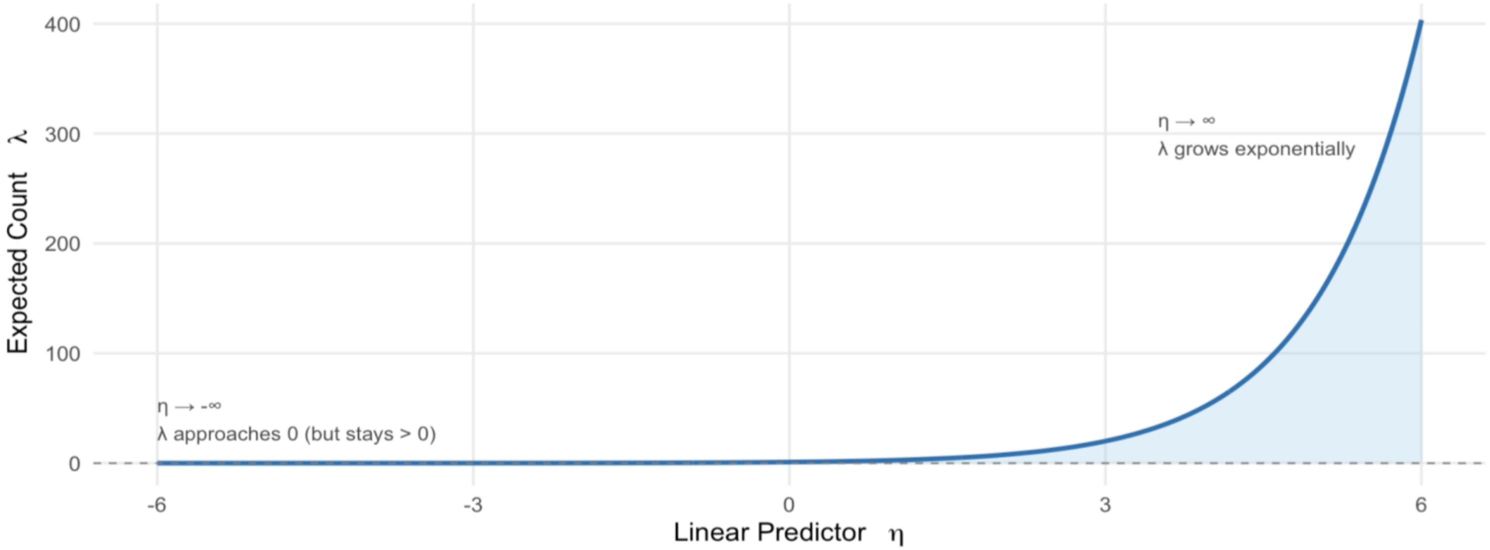
- This log link keeps λ_i positive and treats effects additively on the log scale (multiplicative on the count scale).

The Log Link Function

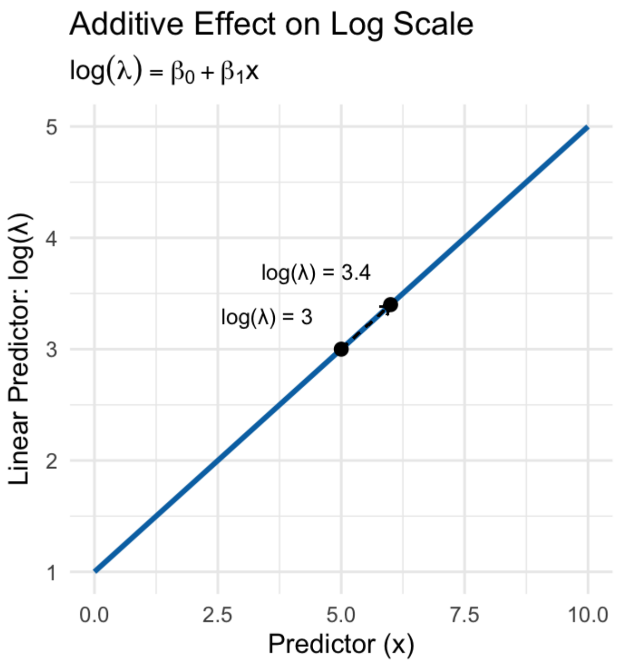
- The log link function transforms the linear predictor into a positive expected count. It does this by exponentiating the linear predictor.

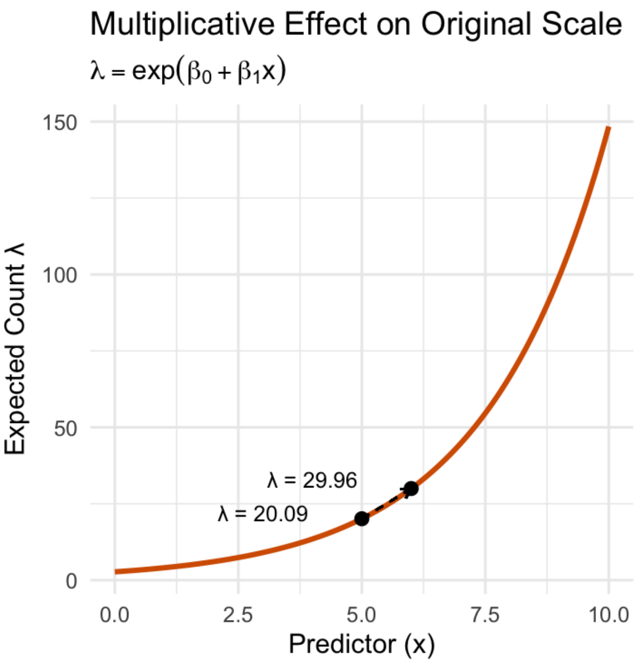
$$\log(\lambda_i) = \beta_0 + \beta_1 x_{1_i} \rightarrow \lambda_i = e^{\beta_0 + \beta_1 x_{1_i}}$$

- For any real number, the predicted mean count is always positive.



The Log Link Function





Adding 0.4 to 3 gives 3.4

Multiplying 20.09 by $e^{0.4}$ gives 29.96

The Log Link Function

- In a Poisson regression with a log link:
$$\log(\lambda_i) = \beta_0 + \beta_1 x_{1i}$$
- On the log scale, a one-unit increase in x_1 simply adds β_1
- On the original scale, taking the exponential gives:
$$\lambda_i = e^{\beta_0 + \beta_1 x_i} = e^{\beta_0} e^{\beta_1 x_i}$$
- So, each 1-unit increase in x multiplies λ by e^{β_1}.
- This impacts the way that we interpret the coefficients of a Poisson model.

Interpreting Coefficients

- β_1 is the slope on the log scale. It is the log rate ratio (sometimes called the log incidence rate ratio). A 1-unit increase in x adds β_1 to log(λ)
- e^{β_1} is called the rate ratio. It tells you how many times larger (or smaller) the expected count is for a one-unit increase in x.
- If $\beta_1 = 0.5$,then the rate ratio is $e^{0.5} \approx 1.65$, meaning there is a 65% increase in the expected count, per unit increase in x
- If $\beta_1 = -0.3$,then $e^{-0.3} \approx 0.74$, meaning that for a one unit increase in x, we multiply the expected count by 0.74. (there is a 26% decrease in the expected count, per unit increase in x)
- β_0 is the intercept on the log scale. It is the expected count when all predictors are 0 (or at average predictor values, if they have been centered).

Example

- Consider an example where the count of bird species depends on habitat area.
- Y_i is the number of bird species observed at site i
- $Area_i$ is the habitat area for site i in hectares.
- The response variable Y_i follows a Poisson distribution

$$Y_i \sim Poisson(\lambda_i)$$
$$\log(\lambda_i) = \beta_0 + \beta_1 Area_i.$$

- Each additional hectare of habitat changes the expected count of bird species by a multiplicative factor of e^{β_1}

Coding Demo

Poisson Regression Assumptions

1. Each observed count is assumed to arise from an independent process.

- Violations:
  - Temporal or spatial autocorrelation
  - Repeated measures on the same unit

2. The expected mean equals the variance (equidispersion):

$$E[Y_i] = \lambda_i,$$
$$Var(Y_i) = \lambda_i$$

- The Poisson distribution assumes variance grows in lockstep with the mean.

Adjusting for Exposure

- In many real-world datasets, not all observations are equally exposed.
- This could be due to:
  - Unequal observation windows (e.g. different follow-up times)
  - Different population at risk (cases per person-year)
  - Variable measurement effort (e.g. number of traps checked)

- For example, bird surveys may be of different lengths, disease counts may be over different population sizes, or accident counts may vary by the number of kilometres travelled.
- If we ignore this, we may draw misleading conclusions about the underlying rates.

Counts vs. Rates

- Sometimes we model rates instead of raw counts.
- Counts are the number of events observed in a fixed space, time or population (e.g., 10 bird species recorded at a certain site). These are modelled as raw counts Y_i
- Rates are counts that are standardised by an exposure variable, such as time or area (e.g., bird species per hectare, or cases per year).

$$Rate_i = \frac{Count_i}{Exposure_i}$$

- We model rates because different observations may have different levels of exposure. If you just model counts, a longer observation time or larger area will naturally have more events, and this can confound the effect of interest.

Offsets

- An offset is a known quantity that we include in the model with a fixed coefficient of one.
- In Poisson regression, the offset is typically the log of the exposure variable. Including it in the linear predictor adjusts the expected counts proportionally to the exposure.
- Using an offset turns the model into a rate model: we explain counts per unit exposure (e.g., per trap-night, per km surveyed, per hour observed).
- A site watched twice as long is expected to have twice the count, all else equal.
- Practical notes: offsets must be positive and entered untransformed except for the log (don't centre/scale them).

How Poisson Regression Handles Rates

- Standard Poisson regression for counts:

$$\log(\lambda_i) = \beta_0 + \beta_1 x_{i1} + \ \dots + \beta_p \, x_{ip}$$

- For rates, we include the log of exposure E_i as an offset:

$$\log(\lambda_i) = \beta_0 + \beta_1 x_{i1} + \ \dots + \beta_p \, x_{ip} + \log(E_i)$$

- Rearranging:

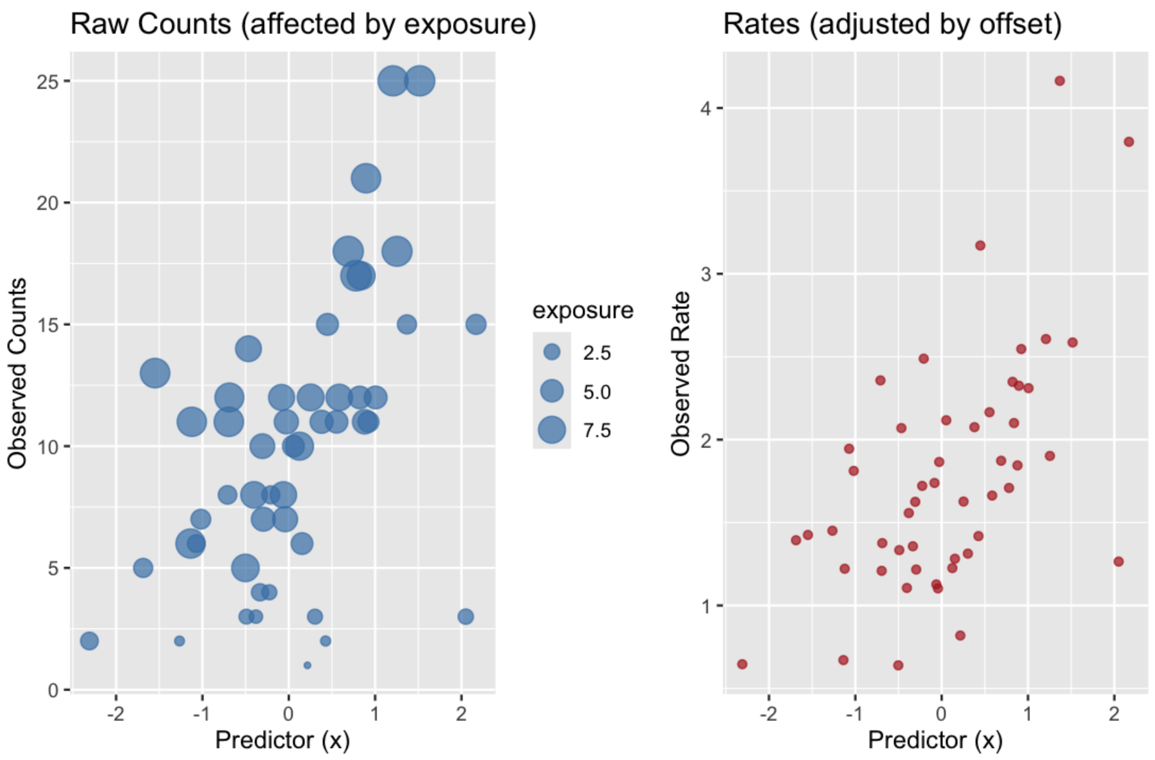
$$log\left(\frac{\lambda_i}{E_i}\right) = \beta_0 + \beta_1 x_{i1} + \ \dots + \beta_p \, x_{ip}$$

- So we're modelling the rate directly:

$$Rate_i = \frac{\lambda_i}{E_i} = e^{\beta_0 + \beta_1 x_{i1} + \ \dots + \beta_p \, x_{ip}}$$

- The offset forces the expected count to scale linearly with exposure

Visualising Offsets



Interpreting Coefficients with an Offset

- Without an offset: coefficients describe a multiplicative effect on the raw counts.
- With an offset: coefficients describe a multiplicative effect on the rate.

- Intercept β_0: baseline log rate per unit exposure at the reference covariate levels.
- e^{β_0} = expected rate (e.g., animals per hour).
- β_1(continuous x_j): a one-unit increase in x_j multiplies the rate by e^{β_1}
- β_1 (factor x_j): e^{β_1} is the rate ratio vs the reference level (per unit exposure).

Example 1

- Suppose you're counting the number of bird species in each site, but site areas differ wildly (1-50 hectares).
- Larger sites will naturally have more species simply due to the greater area. Rather than modelling species counts, you model rate per hectare by including an offset:

$$\log(\lambda_i) = \beta_0 + \beta_1 HabitatQuality_i + \log(Area_i)$$

- Which is equivalent to:

$$\log\left(\frac{\lambda_i}{Area_i}\right) = \beta_0 + \beta_1 HabitatQuality_i$$

- Now, β_1 affects the density of species per hectare, not just raw counts

Example 2

- Imagine you're studying disease across different animal populations. Yi is the number of infected individuals in population i.
- Clearly, larger populations can be expected to have more infected individuals, simply because there are more individuals present to become infected.
- In this case, it makes sense to examine the infection rate per individual, rather than just counts of infected individuals

$$\log(\lambda_i) = \beta_0 + \beta_1 x_{1i} + \log(Pop_i)$$

$$log\left(\frac{\lambda_i}{Pop_i}\right) = \beta_0 + \beta_1 x_{1i}$$

- So the model is actually for the infection rate per individual

When a Single Offset Isn't Enough

- Sometimes more than one exposure factor affects how counts scale
- Example scenarios:
  - Accident counts depend on both time on the road AND distance travelled
  - Wildlife survey counts depend on survey duration AND area covered
  - Hospital infections depend on patient-days AND bed capacity
- We may need to include multiple offset terms to properly standardise rates

Example: Road Accidents

- Modelling the number of road accidents in different regions
- Accident counts scale with:
  - Population size (more people leads to more accidents)
  - Vehicle kilometres travelled (more driving leads to more accidents)
- Ignoring one exposure risks biasing covariate effects
- Solution: include both log(Population) and log(distance travelled) as offsets

Multiple Offsets in the Linear Predictor

- Offsets add linearly inside the log link function:

$$\log(\lambda_\mathrm{i}) = \beta_0 + \beta_1 x_{\mathrm{i}1} + \ldots + \beta_\mathrm{p} x_{\mathrm{ip}} + \log(E_{1\mathrm{i}}) + \log(E_{2\mathrm{i}})$$

By log rules: $\log(E_{1\mathrm{i}}) + \log(E_{2\mathrm{i}}) = \log(E_{1\mathrm{i}} \times E_{2\mathrm{i}})$

So multiple offsets combine multiplicatively: $\lambda_\mathrm{i} \propto E_{1\mathrm{i}} \times E_{2\mathrm{i}}$

- You’re still modelling a rate, but per unit of BOTH exposures

Example: Wildlife Survey

- Imagine we are counting animals in plots of varying size and survey duration.
- Expected count should scale with both area surveyed (we expect to see more animals if we're examining a larger habitat) and survey effort (the longer we survey, the more likely we are to see animals)

$$\log(\lambda_i) = \beta_0 + \beta_1 HabitatQuality_i + \log(Area_i) + \log(Effort_i)$$

$$\log\left(\frac{\lambda_i}{Area_i \times Effort_i}\right) = \beta_0 + \beta_1 HabitatQuality_i$$

- β_1 affects density per hectare per hour, not raw counts

Interpreting Multiple Offset Models

- β_0 is the baseline log density (animals per hectare-hour) at the reference level of habitat quality.
- e^{β_0} is the expected density when habitat quality is at its reference (e.g., 0 for continuous variables or baseline category).
- β_1 interprets the effect on density per hectare per hour.
- If habitat quality is categorical, e^{β_1} is the density (rate) ratio for that category vs the reference category (e.g., density in "High" habitat compared to "Low"), holding other predictors and offsets constant.
- If habitat quality is continuous, A one-unit increase in habitat quality multiplies density by e^{β_1}.

What an Offset Really Does

- An offset is a known adjustment with a fixed coefficient of 1.
- It says: 'If the exposure doubles, the expected count also doubles.'
- There is no uncertainty: the effect is assumed perfectly proportional.

$$\log(\lambda_i) = \beta_0 + \beta_1 x_i + \log(E_i) \rightarrow$$
$$\lambda_i = E_i \times \exp(\beta_0 + \beta_1 x_i)$$

- E_i is NOT estimated. It just scales the expected mean.

What a Predictor Really Does

- A predictor is treated as an unknown effect estimated from data.
- Instead of forcing the coefficient to be 1, you estimate it:

$$\log(\lambda_{\mathrm{i}}) = \beta_0 + \beta_1 x_{\mathrm{i}} + \beta_2 \log(E_{\mathrm{i}})$$

- β_2 can be 1 (proportional) but may be $\beta_2 < 1$ or $\beta_2 > 1$
- This lets you test if exposure has a different-than-proportional effect.

When Should it Be an Offset?

Use an offset if:

- It's a known property of the process that counts scale exactly with exposure
  - More time → proportionally more events
  - Larger population → proportionally more infections
- You're not interested in testing its effect, just standardising for it

Example: Counting accidents per 10,000 km → km is just exposure

When Should it Be a Predictor?

Use a predictor if:

- You want to test whether the relationship is truly proportional
- You suspect nonlinear or non-proportional scaling
- It’s a variable of scientific interest, not just nuisance scaling

Example: Does larger hospital size change infection risk per patient-day?

Examples

Disease counts:

- Offset population if just comparing infection rates per individual
- Predictor population if testing herd immunity effects

Bird surveys:

- Offset survey hours if assuming linear effort → counts
- Predictor if longer surveys yield diminishing returns

Road accidents:

- Offset vehicle km for rates per km
- Predictor if more traffic volume alters risk per km

Model Fitting in R

```
glm(cases ~ covariate, family = poisson,  offset =
log(time) + log(population))
```

Equivalently:

```
glm(cases ~ covariate, family = poisson, offset =
log(time * population))
```

Example: Wildlife Survey

- Imagine we are counting animals in plots of varying size and survey duration.
- Expected count should scale with both area surveyed (we expect to see more animals if we're examining a larger habitat) and survey effort (the longer we survey, the more likely we are to see animals)

$$\log(\lambda_i) = \beta_0 + \beta_1 HabitatQuality_i + \log(Area_i) + \log(Effort_i)$$

$$\log\left(\frac{\lambda_i}{Area_i \times Effort_i}\right) = \beta_0 + \beta_1 HabitatQuality_i$$

- β_1 affects density per hectare per hour, not raw counts

Coding Demo

Workflow for Practical Poisson Modelling

The practical workflow is:

1. Start with raw counts.
2. Identify appropriate exposure.
3. Include log(exposure) as an offset.
4. Fit the model.
5. Visualise and check fit.

This ensures fair comparisons and valid inference.

Common Pitfalls

- Forgetting some exposure components - e.g., including log(area) but forgetting log(time), so you model counts per hectare, not per hectare-hour.
- Treating exposure as a covariate instead of an offset – the model estimates a slope for exposure (not fixed at 1), breaking the rate interpretation and biasing effects.
- Double-counting exposure – don't use an offset and divide the response by exposure, or include the same component twice (e.g., time in days and hours).
- Misinterpreting coefficients as effects on counts - with offsets you're modelling rate. Predicted counts still scale with exposure.

Overdispersion

- If the variance is greater than the mean, this is called overdispersion, and violates the Poisson assumption of equidispersion. This can lead to underestimated standard errors, and inflated Type I errors.
- Possible causes of overdispersion are:
  - Unobserved heterogeneity (missing covariates)
  - Clustering or repeated measures
  - Zero inflation
- We can check for overdispersion by comparing residual deviance to the residual degrees of freedom.

$$\varphi = \frac{Residual\ Deviance}{Residual\ df}$$

  - $\varphi \approx 1 \rightarrow$ No overdispersion
  - $\varphi > 1 \rightarrow$ Overdispersion