3. Assuming an initial bracket of $[1,5]$, the second (at the end of 2 iterations) iterative value of the root of $te^{-t} - 0.3 = 0$ using the bisection method is

→bisection

(A) 0

(B) 1.5

(C) 2

(D) 3

Solution

The correct answer is (C).

$$f(t) = te^{-t} - 0.3$$

If the initial bracket is [1,5] then

$$t_u = 5$$

$$t_\ell = 1$$

Check to see if the function changes sign between t_ℓ and t_u

$$f(t_u) = 5e^{-5} - 0.3$$
$$= -0.2663$$

$$f(t_\ell) = 1e^{-1} - 0.3$$
$$= 0.0679$$

Hence,

$$f(t_u)f(t_\ell) = f(5)f(1)$$
$$= (-0.2663)(0.0679)$$
$$= -0.0181 < 0$$

→ এর মধ্যে আছে

So there is at least one root between t_ℓ and t_u.

Iteration 1

The estimate of the root is

$$t_m = \frac{t_\ell + t_u}{2}$$
$$= \frac{1+5}{2}$$
$$= 3$$

$$f(t_m) = 3e^{-3} - 0.3$$
$$= -0.1506$$

Thus,

$$f(t_\ell)f(t_m) = f(1)f(3)$$
$$= (0.0679)(-0.1506)$$
$$= -0.0102 < 0$$

→ এর মধ্যে আছে তাহলে tu এর সাথে check করার দরকার নাই

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The root lies between t_ℓ and t_m, so the new upper and lower guesses for the root are

$$t_\ell = t_\ell = 1$$

$$t_u = t_m = 3$$

Iteration 2

The estimate of the root is

$$t_m = \frac{t_\ell + t_u}{2}$$

$$= \frac{1+3}{2}$$

$$= 2$$ value after Iteration 2

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4. To find the root of $f(x)=0$, a scientist is using the bisection method. At the beginning of an iteration, the lower and upper guesses of the root are x_l and x_u. At the end of the iteration, the absolute relative approximate error in the estimated value of the root would be

(A) $\left|\dfrac{x_u}{x_u+x_\ell}\right|$

(B) $\left|\dfrac{x_\ell}{x_u+x_\ell}\right|$

(C) $\left|\dfrac{x_u-x_\ell}{x_u+x_\ell}\right|$

(D) $\left|\dfrac{x_u+x_\ell}{x_u-x_\ell}\right|$

Solution

The correct answer is (C).

The absolute relative approximate error is

$$|\epsilon_a|=\left|\frac{x_m^{new}-x_m^{old}}{x_m^{new}}\right|$$

where

$$x_m^{new}=\frac{x_\ell+x_u}{2}$$

If

$$x_m^{old}=x_\ell$$

$$|\epsilon_a|=\left|\frac{\dfrac{x_\ell+x_u}{2}-x_\ell}{\dfrac{x_\ell+x_u}{2}}\right|$$

$$=\left|\frac{(x_\ell+x_u)-2x_\ell}{x_\ell+x_u}\right|$$

$$=\left|\frac{x_u-x_\ell}{x_u+x_\ell}\right|$$

If

$$x_m^{old}=x_u$$

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$$|\in_a| = \left| \frac{\frac{x_\ell + x_u}{2} - x_u}{\frac{x_\ell + x_u}{2}} \right|$$

$$= \left| \frac{(x_\ell + x_u) - 2x_u}{x_\ell + x_u} \right|$$

$$= \left| \frac{x_l - x_u}{x_u + x_\ell} \right|$$

The answer is the same whether $x_m^{old} = x_\ell$ *or* x_u as x_m is exactly in the middle of x_ℓ and x_u.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water. R

R=0.055 m

The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the bisection method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the end of each iteration.

Solution

From the physics of the problem, the ball would be submerged between $x = 0$ and $x = 2R$, where

$R =$ radius of the ball,

that is

$$0 \le x \le 2R$$
$$0 \le x \le 2(0.055)$$
$$0 \le x \le 0.11$$

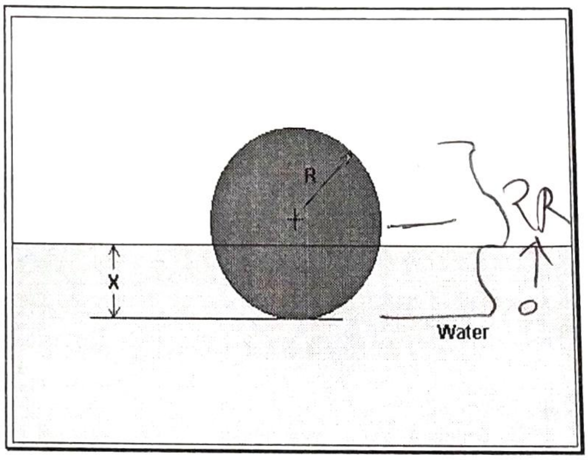
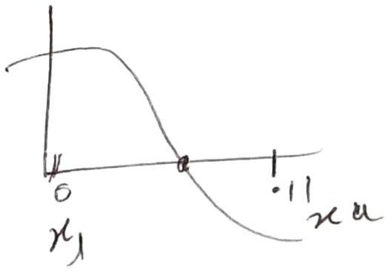


Figure 5 Floating ball problem.



Lets us assume

$$x_\ell = 0, x_u = 0.11$$

Check if the function changes sign between x_ℓ and x_u. x_l, x_u

$$f(x_\ell) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_\ell)f(x_u) = f(0)f(0.11) = (3.993\times10^{-4})(-2.662\times10^{-4}) < 0$$

So there is at least one root between x_ℓ and x_u, that is between 0 and 0.11.

Iteration 1

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{0 + 0.11}{2}$$
$$= 0.055$$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

$$f(x_\ell)f(x_m) = f(0)f(0.055) = (3.993\times10^{-4})(6.655\times10^{-4}) > 0$$

Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limit of the new bracket is

$$x_\ell = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$

$$= \frac{0.055 + 0.11}{2}$$

$$= 0.0825$$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993 \times 10^{-4} = -1.622 \times 10^{-4}$$

$$f(x_\ell)f(x_m) = f(0.055)f(0.0825) = \left(6.655 \times 10^{-5}\right) \times \left(-1.622 \times 10^{-4}\right) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.0825. So the lower and upper limit of the new bracket is

$$x_\ell = 0.055,\ x_u = 0.0825$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100$$

$$= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100$$

$$= 33.33\%$$

None of the significant digits are at least correct in the estimated root of $x_m = 0.0825$ because the absolute relative approximate error is greater than 5%.

Iteration 3

$$x_m = \frac{x_\ell + x_u}{2}$$

$$= \frac{0.055 + 0.0825}{2}$$

$$= 0.06875$$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993 \times 10^{-4} = -5.563 \times 10^{-5}$$

$$f(x_\ell)f(x_m) = f(0.055)f(0.06875) = (6.655 \times 10^{5}) \times (-5.563 \times 10^{-5}) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.06875. So the lower and upper limit of the new bracket is

$$x_\ell = 0.055,\ x_u = 0.06875$$

The absolute relative approximate error $|\in_a|$ at the ends of Iteration 3 is

$$|\epsilon_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.06875 - 0.0825}{0.06875}\right| \times 100$$

$$= 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

Seven more iterations were conducted and these iterations are shown in Table 1.