① Introduction

Solving an engineering problem requires four steps :

1. Formulate the problem
2. Mathematically model the problem
3. Solve the mathematical model
4. Implement the results in

Q. One of the roots of the equation x^3-3x^2+x-3 is,

$$x^3-3x^2+x-3=0$$

$$x^2(x-3)+1(x-3)=0$$

$$(x^2+1)(x-3)=0$$

$$\therefore x=3$$

Ans.

Q. The exact integral of $\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx$ is most nearly,

$$\int_0^{\pi/4} 2\cos 2x\,dx$$

$$= \left[2\,\frac{\sin(2x)}{2}\right]_0^{\pi/4}$$

$$= \left[\sin(2x)\right]_0^{\pi/4}$$

$$= \sin \pi/2 - \sin(0)$$

$$= 1 - 0 = 1 \text{ (Ans.)}$$

Q. The value of $\frac{dy}{dx}(1.0)$, given $y = 2\sin(3x)$ most nearly is,

$$y = 2\sin(3x)$$

$$\frac{dy}{dx} = 2(3\cos(3x)) = 6\cos(3x)$$

$$\frac{dy}{dx}(1.0) = 6\cos(3(1.0))$$

$$= 6(-0.98999)$$

$$= -5.9399$$

Q. The form of the exact solution of the ordinary differential equation, $2\frac{dy}{dx}+3y=5e^{-x}$, $y(0)=5$ is,

$$2\frac{dy}{dx}+3y=5e^{-x}, \quad y(0)=5$$

The characteristic equation for the homogeneous part of the solution is

$$2m^1+3m^0=0$$

$$2m+3=0$$

$$m=-1.5$$

The homogeneous part of the solution hence is

$$y_H=Be^{-x}$$

So the form of the solution to the ordinary differential equation is,

$$y=y_H+y_p$$

$$=Ae^{-1.5x}+Be^{-x}$$

(II) Measuring Errors

Q. Why measure errors?

Ans:

1. To determine the accuracy of numerical results.
2. To develop stopping criteria for iterative algorithms.

True error:

True error is the difference between the true value and the approximate value.

it denoted by,

True Error = True Value - Approximate Value

Q. The derivative of a function f(x) at a particular value of x can be approximately calculated by

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

$\boxed{h = \Delta x}$

If $f(x) = 7e^{0.5x}$ and $h = 0.3$

Now, (a) find the approximate value of $f'(2)$ x

(b) the true value of $f'(2)$

(c) the true error for part (a)

Solve: a) $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ [Approximate]

For $x=2$ and $h=0.3$

x

$$\therefore f'(2) \approx \frac{f(2+0.3)-f(2)}{h}$$

$$\approx \frac{f(2.3)-f(2)}{h}$$

$$\approx \frac{7e^{0.5*2.3}-7e^{0.5*2}}{0.3}$$

$$\approx \frac{22.107-19.027}{0.3}$$

$$\approx 10.267$$

b) The exact value of $f'(2)$ can be calculaed,

$f(x) = 7e^{0.5x}$ [exact value / true value]

$f'(x) = 7 \times 0.5 e^{0.5x}$ $\left[\frac{d}{dx} e^{mx} = me^{mx}\right]$

$\therefore f'(2) = 3.5e^{0.5(2)}$

$= 9.5140$

c) True error,

E_t = True value − Approximate value

$= 9.5140 - 10.265$

$= -0.75061$

D) Q. if said to calculate,

Relative true error, represent as persentage %

Absolute relative true error $|\epsilon_t|$

D) Slv:

Relative true error,

$$\epsilon_t = \frac{\text{True Error}}{\text{True Value}}$$

$$= \frac{-0.75061}{9.5140}$$

$$= -0.078895$$

or, -7.58895%

Absolute relative true errors may,

$$|\epsilon_t| = |-0.075888|$$

$$= 0.07588$$

$$= 0.07588\%$$

Q. What is relative true error?

Ans:

Relative true error is the retio between the true error and the true value.

$$\text{Relative true error } (E_t) = \frac{\text{True Error}}{\text{True value}}$$

Approximate error:

Approximate error is denoted by Ea and is defined as the difference between the present approximation and previous approximation.

$$\underset{E_a}{\text{Approximate error}} = \overset{\text{current}}{\text{Present}} \text{ Approximation} - \text{previous Approximation}$$

Example: Approximate Error

Q. The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$f(x)$

for $f(x) = 7e^{0.5x}$ and at $x=2$, find the following

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

c) approximate error for the value of $f'(2)$ Part (b)

Ans:

a) The approximate expression for the derivative of a function is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

previous Approximate value

$x=2,\ h=0.3$

$$\therefore f'(2) = \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3} = 10.265$$

8

b) Now with, $h = 0.15$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(2) \approx \frac{f(2+0.15) - f(2)}{0.15}$$

$$\approx \frac{f(2.15) - f(2)}{0.15}$$

$$\approx \frac{7e^{0.5(2.15)} - 7e^{0.5(2)}}{0.15}$$

$$\approx 9.8799$$ [present Approx]

c) So the approximate error, E_a is

E_a = present Approx − Previous Approx

$= 9.8799 - 10.265$

$= -0.38474$

The relative approximate errors are also presented as percentages.

$$E_a = \frac{\text{Approximate Error}}{\text{Present Approx}}$$

$$= \frac{-0.38474}{\text{present Approx}}$$

$$= \frac{-0.38474}{9.8799}$$

$$= -0.038942$$

or, -3.8942%

Absoulute relative approximate errors,

$$|\epsilon_a| = |-0.038942|$$

$$= 0.038942$$

or, 3.8942%

Q. What is relative approximate error?

Ans :

Relative approximate error is denoted by ϵ_a and is defined as the ratio between the approximate error and the present approximation

$$\text{Relative Approximate Error} = \frac{\text{Approximate Error}}{\text{present Approx}}$$

ϵ_a

12

Q: While solving a mathematical model using numerical methods, how can we use relative approximate errors to minimize the error?

A: In a numerical method that uses iterative methods, a user can calculate relative approximate error $\in_a$ at the end of each iteration. The user may pre-specify a minimum acceptable tolerance called the pre-specified tolerance, $\in_s$. If the absolute relative approximate error $\in_a$ is less than or equal to the pre-specified tolerance $\in_s$, that is, $|\in_a| \leq \in_s$, then the acceptable error has been reached and no more iterations would be required.

Alternatively, one may pre-specify how many significant digits they would like to be correct in their answer. In that case, if one wants at least m significant digits to be correct in the answer, then you would need to have the absolute relative approximate error, $|\in_a| \leq 0.5 \times 10^{2-m}\,\%$.

13

Example 5

If one chooses 6 terms of the Maclaurin series for e^x to calculate $e^{0.7}$, how many significant digits can you trust in the solution? Find your answer without knowing or using the exact answer.

Solution

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots\ldots$$

Using 6 terms, we get the current approximation as

$$e^{0.7} \cong 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \frac{0.7^4}{4!} + \frac{0.7^5}{5!}$$

$$= 2.0136$$

Using 5 terms, we get the previous approximation as

$$e^{0.7} \cong 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \frac{0.7^4}{4!}$$

$$= 2.0122$$

The percentage absolute relative approximate error is

$$|\in_a| = \left|\frac{2.0136 - 2.0122}{2.0136}\right| \times 100$$

$$= 0.069527\%$$

Since $|\in_a| \le 0.5 \times 10^{2-2}\%$, at least 2 significant digits are correct in the answer of

$$e^{0.7} \cong 2.0136$$