2. The secant method formula for finding the square root of a real number R from the equation $x^2 - R = 0$ is

(A) $\dfrac{x_i x_{i-1} + R}{x_i + x_{i-1}}$

(B) $\dfrac{x_i x_{i-1}}{x_i + x_{i-1}}$

(C) $\dfrac{1}{2}\left(x_i + \dfrac{R}{x_i}\right)$

(D) $\dfrac{2x_i^2 + x_i x_{i-1} - R}{x_i + x_{i-1}}$

Solution

The correct solution is (A).

The secant method formula for finding the root of $f(x) = 0$ is

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

where

$$f(x) = x^2 - R$$

Thus

$$x_{i+1} = x_i - \frac{(x_i^2 - R)(x_i - x_{i-1})}{(x_i^2 - R) - (x_{i-1}^2 - R)}$$

$$= x_i - \frac{(x_i^2 - R)(x_i - x_{i-1})}{x_i^2 - x_{i-1}^2}$$

$$= x_i - \frac{(x_i^2 - R)(x_i - x_{i-1})}{(x_i - x_{i-1})(x_i + x_{i-1})}$$

$$= x_i - \frac{x_i^2 - R}{x_i + x_{i-1}}$$

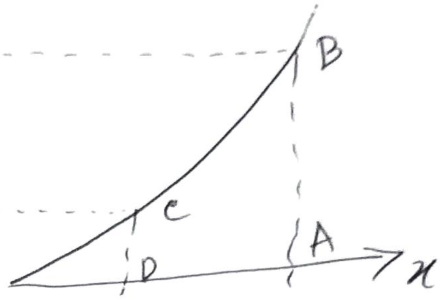
$$= \frac{x_i(x_i + x_{i-1}) - (x_i^2 - R)}{x_i + x_{i-1}}$$

$$= \frac{x_i^2 + x_i x_{i-1} - x_i^2 + R}{x_i + x_{i-1}}$$

$$= \frac{x_i x_{i-1} + R}{x_i + x_{i-1}}$$

3. The next iterative value of the root of $x^2 - 4 = 0$ using secant method, if the initial guesses are 3 and 4, is

(A) 2.2857
(B) 2.5000
(C) 5.5000
(D) 5.7143



Solution

The correct solution is (A).

$x_{i-1} \to 3 \to x_{-1}$ $\quad$ $x_i \to 4 \to x_0$

The first iteration is

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

for $i = 0,\ x_0 = 4,\ x_{-1} = 3$

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= x_0 - \frac{(x_0^2 - 4)(x_0 - x_{-1})}{(x_0^2 - 4) - (x_{-1}^2 - 4)}$$

$$= 4 - \frac{(4^2 - 4)(4 - 3)}{(4^2 - 4) - (3^2 - 4)}$$

$$= 4 - \frac{(12)(1)}{(12) - (5)}$$

$$= 4 - \frac{12}{7}$$

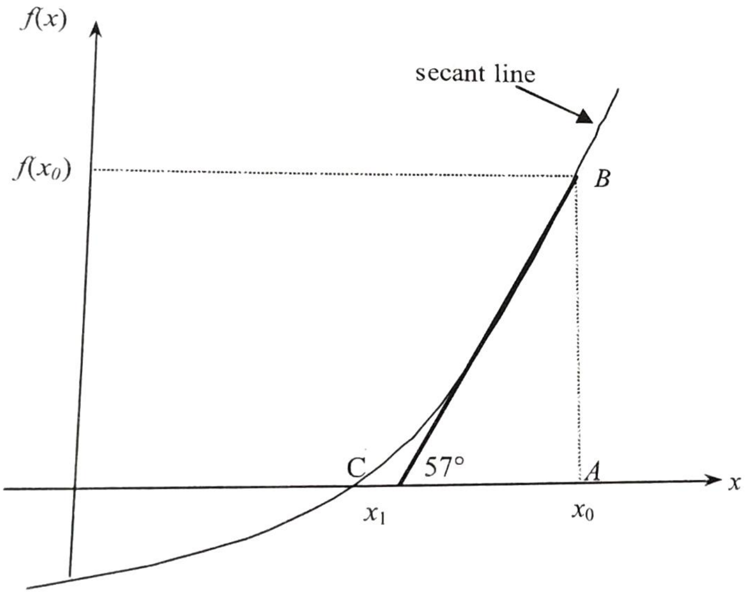
$$= 2.2857$$

4. The root of the equation $f(x)=0$ is found by using the secant method. Given one of the initial estimates is $x_0 = 3$, $f(3)=5$, and the angle the secant line makes with the x-axis is $57°$, the next estimate of the root, x_1, is

(A) −3.2470
(B) −0.24704
(C) 3.247
(D) 6.2470

Solution

The correct answer is (B).



$$\tan(\theta) = \frac{f(x_0)}{x_0 - x_1}$$

$$\tan(57) = \frac{f(3)}{3 - x_1}$$

$$3 - x_1 = \frac{f(3)}{\tan(57)}$$

$$x_1 = -\frac{5}{1.5399} + 3$$

$$= -3.2470 + 3$$

$$= -0.24704$$

$$\tan(\theta) = \frac{f(3)}{x_0 - x_1}$$

$$\tan(57) = \frac{5}{3 - x_1}$$

$$3 - x_1 = \frac{5}{\tan(57)}$$

$$x_1 = 3 - \frac{5}{\tan(57)}$$

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats (Figure 2) for ABC commodes. The floating ball has a specific gravity of 0.6 and a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.
The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993\times 10^{-4} = 0$$

Use the secant method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error and the number of significant digits at least correct at the end of each iteration.

Solution

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0 = 0.05$.

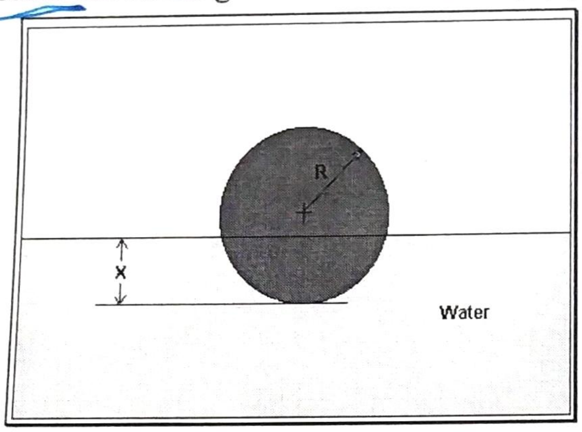


Figure 2 Floating ball problem.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= x_0 - \frac{\left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right)\times\left(x_0 - x_{-1}\right)}{\left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right) - \left(x_{-1}^3 - 0.165x_{-1}^2 + 3.993\times10^{-4}\right)}$$

$$= 0.05 - \frac{\left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right]\times\left[0.05 - 0.02\right]}{\left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right] - \left[0.02^3 - 0.165(0.02)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06461$$

The absolute relative approximate error $|\epsilon_a|$ at the end of Iteration 1 is

$$|\epsilon_a| = \left|\frac{x_1 - x_0}{x_1}\right|\times100$$

$$= \left|\frac{0.06461 - 0.05}{0.06461}\right|\times100$$

$$= 22.62\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for one significant digit to be correct in your result.

Iteration 2

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= x_1 - \frac{\left(x_1^3 - 0.165x_1^2 + 3.993\times10^{-4}\right)\times\left(x_1 - x_0\right)}{\left(x_1^3 - 0.165x_1^2 + 3.993\times10^{-4}\right) - \left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06461 - \frac{\left[0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right]\times(0.06461-0.05)}{\left[0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right] - \left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06241$$

The absolute relative approximate error $|\epsilon_a|$ at the end of Iteration 2 is

$$|\epsilon_a| = \left|\frac{x_2 - x_1}{x_2}\right|\times 100$$

$$= \left|\frac{0.06241 - 0.06461}{0.06241}\right|\times 100$$

$$= 3.525\%$$

The number of significant digits at least correct is 1, as you need an absolute relative approximate error of 5% or less.

<u>Iteration 3</u>

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= x_2 - \frac{(x_2^3 - 0.165x_2^2 + 3.993\times10^{-4})\times(x_2 - x_1)}{(x_2^3 - 0.165x_2^2 + 3.993\times10^{-4}) - (x_1^3 - 0.165x_1^2 + 3.993\times10^{-4})}$$

$$= 0.06241 - \frac{\left[0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right]\times(0.06241-0.06461)}{\left[0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right] - \left[0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06238$$

The absolute relative approximate error $|\epsilon_a|$ at the end of Iteration 3 is

$$|\epsilon_a| = \left|\frac{x_3 - x_2}{x_3}\right|\times 100$$

$$= \left|\frac{0.06238 - 0.06241}{0.06238}\right|\times 100$$

$$= 0.0595\%$$

The number of significant digits at least correct is 2, as you need an absolute relative approximate error of 0.5% or less. Table 1 shows the secant method calculations for the results from the above problem.

Table 1 Secant method results as a function of iterations.

Iteration Number, i	x_{i-1}	x_i	x_{i+1}	$\lvert\epsilon_a\rvert\%$	$f(x_{i+1})$
1	0.02	0.05	0.06461	22.62	-1.9812×10^{-5}
2	0.05	0.06461	0.06241	3.525	-3.2852×10^{-7}
3	0.06461	0.06241	0.06238	0.0595	2.0252×10^{-9}
4	0.06241	0.06238	0.06238	-3.64×10^{-4}	-1.8576×10^{-13}