Example 4

2nd derivatives

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

Use the forward difference approximation of the second derivative of $v(t)$ to calculate the jerk at $t = 16\text{ s}$. Use a step size of $\Delta t = 2\text{ s}$.

Solution

$$j(t_i) \approx \frac{v(t_{i+2}) - 2v(t_{i+1}) + v(t_i)}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i+2} &= t_i + 2(\Delta t) \\ &= 16 + 2(2) \\ &= 20 \end{aligned}$$

$$j(16) \approx \frac{v(20) - 2v(18) + v(16)}{(2)^2}$$

$$\begin{aligned} v(20) &= 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(20)}\right] - 9.8(20) \\ &= 517.35\text{ m/s} \end{aligned}$$

$$\begin{aligned} v(18) &= 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18) \\ &= 453.02\text{ m/s} \end{aligned}$$

$$\begin{aligned} v(16) &= 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16) \\ &= 392.07\text{ m/s} \end{aligned}$$

$$\begin{aligned} j(16) &\approx \frac{517.35 - 2(453.02) + 392.07}{4} \\ &= 0.84515\text{ m/s}^3 \end{aligned}$$

The exact value of $j(16)$ can be calculated by differentiating

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t$$

twice as

$$a(t) = \frac{d}{dt}[v(t)] \text{ and}$$

$$j(t) = \frac{d}{dt}[a(t)]$$

Knowing that

$$\frac{d}{dt}[\ln(t)] = \frac{1}{t} \text{ and}$$

$$\frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$\frac{d}{dt}\left[v(t)\right] = a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{(14\times10^4 - 2100t)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

Similarly it can be shown that

$$\frac{d}{dt}\left[a(t)\right] = j(t) = \frac{d}{dt}[a(t)]$$

$$= \frac{18000}{(-200 + 3t)^2}$$

$$j(16) = \frac{18000}{[-200 + 3(16)]^2}$$

$$= 0.77909 \text{ m/s}^3$$

The absolute relative true error is

$$|\epsilon_t| = \left|\frac{0.77909 - 0.84515}{0.77909}\right| \times 100$$

$$= 8.4797\%$$

Example 5

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t,\ 0 \le t \le 30,$$

(a) Use the central difference approximation of the second derivative of $v(t)$ to calculate the jerk at $t = 16\,\text{s}$. Use a step size of $\Delta t = 2\,\text{s}$.

Solution

The second derivative of velocity with respect to time is called jerk. The second order approximation of jerk then is

$$j(t_i) \approx \frac{v(t_{i+1}) - 2v(t_i) + v(t_{i-1})}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$t_{i+2} = t_i - \Delta t$$

$$= 16 - 2$$

$$= 14$$

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$

$$v(18) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$
$$= 453.02\ \text{m/s}$$

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$
$$= 392.07\ \text{m/s}$$

$$v(14) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(14)}\right] - 9.8(14)$$
$$= 334.24\ \text{m/s}$$

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$
$$= \frac{453.02 - 2(392.07) + 334.24}{4}$$
$$= 0.77969\ \text{m/s}^3$$

The absolute relative true error is

$$|\epsilon_t| = \left|\frac{0.77908 - 0.77969}{0.77908}\right| \times 100$$
$$= 0.077992\%$$

exact value
Example 4