(V) Taylor Series,

General Taylor series:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots\dots$$

Some examples of Taylor series,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$$

(Maclaurin series)

Drivation of taylor series

Derive series for $\exp(x)$ (e^x)

Drive: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Slv: Now, General form Taylor:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

Since $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$

$\therefore f''(0) = e^0 = 1$

Now drive the Maclaurin series:

$f(x) = e^x$ $\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$

$f''(x) = e^x$ $\quad$ $f''(0) = 1$

$f'''(x) = e^x$ $\quad$ $f'''(0) = 1$

Now,

$$f(0+h) = f(0) + f'(0)h + \frac{f''(0)}{2!}h^2 + \frac{f'''(0)}{3!}h^3 + \cdots$$

$$\Rightarrow f(h) = e^0 + (e^0)h + \frac{e^0}{2!}h^2 + \frac{(e^0)}{3!}h^3$$

$$\Rightarrow f(h) = 1 + h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 + \cdots$$

So if we change h to x, because h is just a dummy variable in this series:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ (Ans)}$$

Example 3

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Derive the Maclaurin series of $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Solution

In the previous example, we wrote the Taylor series for $\sin(x)$ around the point $x = \frac{\pi}{2}$.

Maclaurin series is simply a Taylor series for the point $x = 0$.

$$f(x) = \sin(x), \quad f(0) = 0$$

$$f'(x) = \cos(x),\ f'(0) = 1$$
$$f''(x) = -\sin(x),\ f''(0) = 0$$
$$f'''(x) = -\cos(x),\ f'''(0) = -1$$
$$f''''(x) = \sin(x),\ f''''(0) = 0$$
$$f'''''(x) = \cos(x),\ f'''''(0) = 1$$

Using the Taylor series now,

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4} + f'''''(x)\frac{h^5}{5} + \cdots$$

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

$$f(h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

$$= 0 + 1(h) - 0\frac{h^2}{2!} - 1\frac{h^3}{3!} + 0\frac{h^4}{4} + 1\frac{h^5}{5} + \cdots$$

$$= h - \frac{h^3}{3!} + \frac{h^5}{5!} + \cdots$$

So

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Example Math:

Example 4 Taylor

Find the value of $f(6)$ given that $f(4)=125$, $f'(4)=74$, $f''(4)=30$, $f'''(4)=6$ and all other higher derivatives of $f(x)$ at $x=4$ are zero.

Solution

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+f'''(x)\frac{h^3}{3!}+\cdots$$

$$x=4$$
$$\begin{aligned} h &= 6-4 \\ &= 2 \end{aligned}$$

Since fourth and higher derivatives of $f(x)$ are zero at $x=4$.

$$f(4+2)=f(4)+f'(4)2+f''(4)\frac{2^2}{2!}+f'''(4)\frac{2^3}{3!}$$

$$\begin{aligned} f(6) &= 125+74(2)+30\left(\frac{2^2}{2!}\right)+6\left(\frac{2^3}{3!}\right) \\ &= 125+148+60+8 \\ &= 341 \end{aligned}$$

1. The coefficient of the x^5 term in the Maclaurin polynomial for $\sin(2x)$ is 30

(A) 0
(B) 0.0083333
(C) 0.016667
(D) 0.26667

Solution

The correct answer is (D).

sin(2x)

The Maclaurin series for $\sin(2x)$ is

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots$$

$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots$$

$$= 2x - 1.3333x^3 + 0.26667x^5 + \cdots$$

Hence, the coefficient of the x^5 term is 0.26667.

2. Given $f(3)=6$, $f'(3)=8$, $f''(3)=11$, and all other higher order derivatives of $f(x)$ are zero at $x=3$, and assuming the function and all its derivatives exist and are continuous between $x=3$ and $x=7$, the value of $f(7)$ is

(A) 38.000
(B) 79.500
(C) 126.00
(D) 331.50

Solution

The correct answer is (C).

The Taylor series is given by

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2!}h^2+\frac{f'''(x)}{3!}h^3+\cdots$$

$$x=3,\ h=7-3=4$$

$$f(3+4)=f(3)+f'(3)4+\frac{f''(3)}{2!}4^2+\frac{f'''(3)}{3!}4^3+\cdots$$

$$f(7)=f(3)+f'(3)4+\frac{f''(3)}{2!}4^2+\frac{f'''(3)}{3!}4^3+\ldots$$

Since all the derivatives higher than second are zero,

$$f(7)=f(3)+f'(3)4+\frac{f''(3)}{2!}4^2$$

$$=6+8\times 4+\frac{11}{2!}4^2$$

$$=126$$

3. Given that $y(x)$ is the solution to $\frac{dy}{dx} = y^3 + 2$, $y(0) = 3$ the value of $y(0.2)$ from a second order Taylor polynomial around x=0 is

(A) 4.400
(B) 8.800
(C) 24.46
(D) 29.00

Solution

The correct answer is (C).

The second order Taylor polynomial is

$$y(x+h) = y(x) + y'(x)h + \frac{y''(x)}{2!}h^2$$

$$x = 0,\ h = 0.2 - 0 = 0.2$$

$$y(0+0.2) = y(0) + y'(0)\times 0.2 + \frac{y''(0)}{2!}0.2^2$$

$$y(0.2) = y(0) + y'(0)\times 0.2 + y''(0)\times 0.02$$

$$y(0) = 3$$

$$y'(x) = y^3 + 2$$

$$y'(0) = 3^3 + 2$$
$$= 29$$

$$y''(x) = 3y^2\frac{dy}{dx}$$
$$= 3y^2\left(y^3 + 2\right)$$

$$y''(0) = 3(3)^2\left(3^3 + 2\right)$$
$$= 783$$

$$y(0.2) = 3 + 29\times 0.2 + 783\times 0.02$$
$$= 24.46$$

Error in Taylor Series

As you have noticed, the Taylor series has infinite terms. Only in special cases such as a finite polynomial does it have a finite number of terms. So whenever you are using a Taylor series to calculate the value of a function, it is being calculated approximately.

The Taylor polynomial of order n of a function $f(x)$ with $(n+1)$ continuous derivatives in the domain $[x, x+h]$ is given by

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \cdots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x+h)$$

where the remainder is given by

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

where

$$x < c < x + h$$

that is, c is some point in the domain $(x, x+h)$.