Forward Difference Approximation of the First Derivative ~~33~~ 34

From differential calculus, we know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite Δx,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The above is the forward divided difference approximation of the first derivative. It is called forward because you are taking a point ahead of x. To find the value of $f'(x)$ at $x = x_i$, we may choose another point Δx ahead as $x = x_{i+1}$. This gives

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

where

$$\Delta x = x_{i+1} - x_i$$

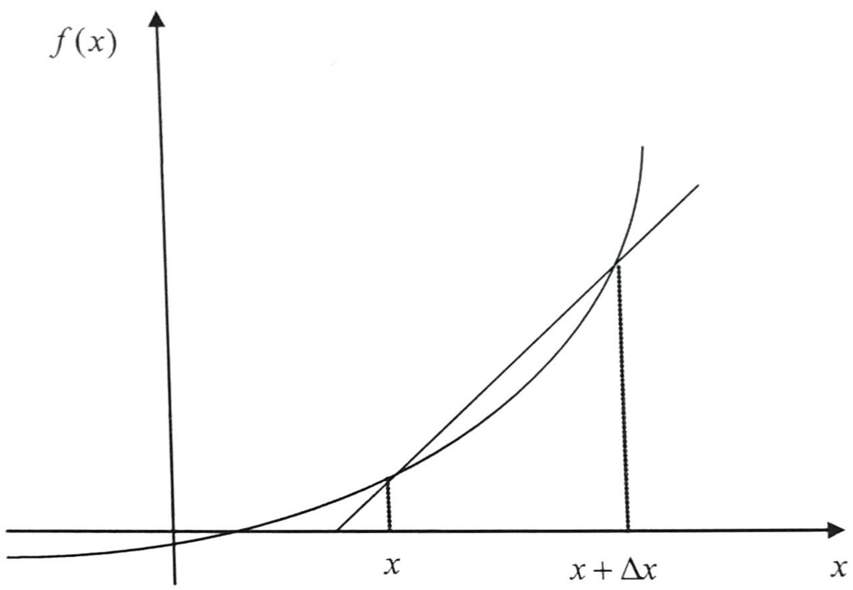

Figure 1 Graphical representation of forward difference approximation of first derivative.

Backward Difference Approximation of the First Derivative

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We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite Δx,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If Δx is chosen as a negative number,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

This is a backward difference approximation as you are taking a point backward from x. To find the value of $f'(x)$ at $x = x_i$, we may choose another point Δx behind as $x = x_{i-1}$. This gives

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

where

$$\Delta x = x_i - x_{i-1}$$

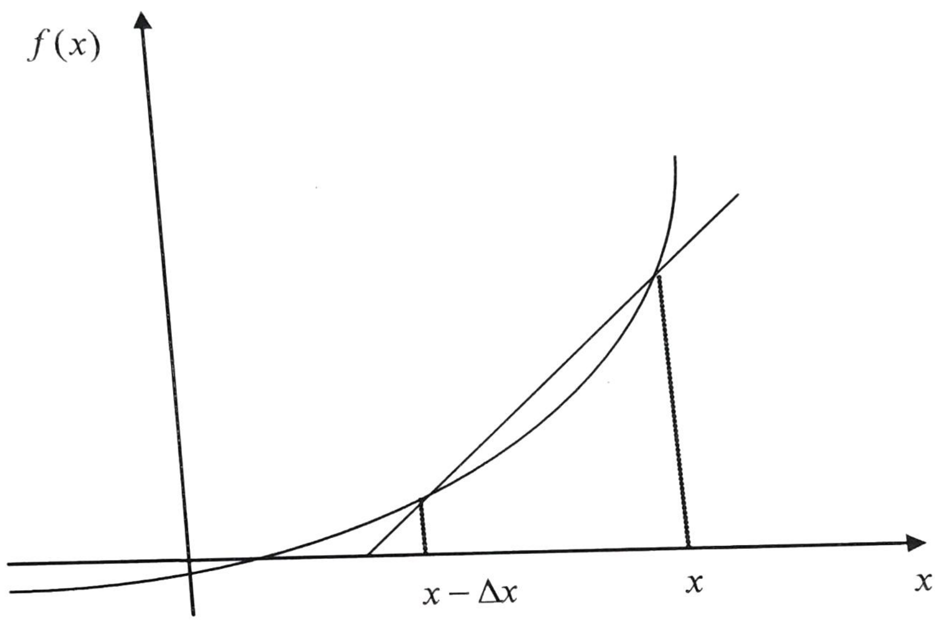

Figure 2 Graphical representation of backward difference approximation of first derivative.

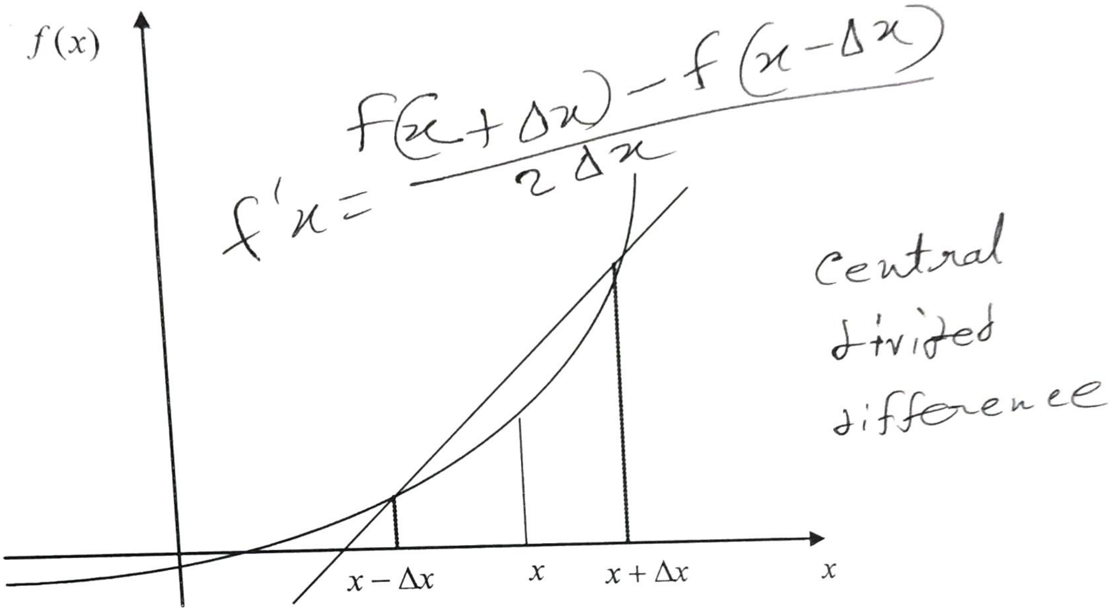

Figure 3 Graphical representation of central difference approximation of first derivative.

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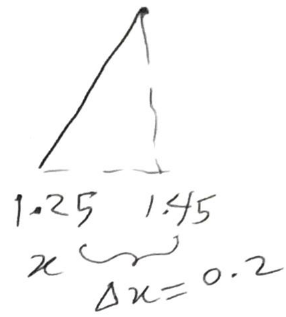
3. Using the forwarded divided difference approximation with a step size of 0.2, the derivative of $f(x)=5e^{2.3x}$ at $x=1.25$ is

(A) 163.4
(B) 203.8
(C) 211.1
(D) 258.8



Solution

The correct answer is (D).

The forward divided difference approximation is

$$f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

where

$$x = 1.25$$

$$\Delta x = 0.2$$

Thus,

$$f'(1.25) \approx \frac{f(1.25+0.2)-f(1.25)}{0.2}$$

$$= \frac{f(1.45)-f(1.25)}{0.2}$$

$$= \frac{5e^{2.3(1.45)}-5e^{2.3(1.25)}}{0.2}$$

$$= 258.8$$

$f(x) = 5e^{2.3x}$

(Ans.)

(41)

$f(x) = e^x$

4. A student finds the numerical value of $\frac{d}{dx}(e^x) = 20.220$ at $x = 3$ using a step size of 0.2. Which of the following methods did the student use to conduct the differentiation?

(A) Backward divided difference
(B) Calculus, that is, exact
(C) Central divided difference
(D) Forward divided difference

$x = 3$ $\delta x = 0.2$

Solution

The correct answer is (C).

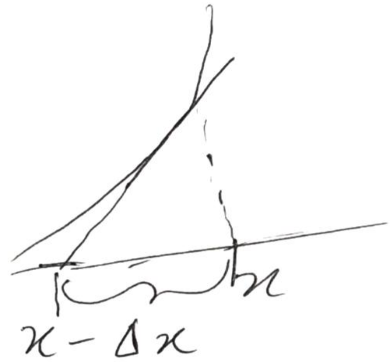
Choice (A)

The backward divided difference approximation is

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

where

$$x = 3$$
$$\Delta x = 0.2$$



Thus,

$$f'(3) \approx \frac{f(3) - f(3 - 0.2)}{(0.2)}$$
$$= \frac{f(3) - f(2.8)}{(0.2)}$$
$$= \frac{e^3 - e^{2.8}}{0.2}$$
$$= 18.204$$

Choice (B)

Using calculus,

$$\frac{d}{dx}(e^x) = e^x$$

Thus,

$$f'(3) = e^3$$
$$= 20.086$$

Choice (C)

The central divided difference approximation is

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where

$$x = 3$$

$$\Delta x = 0.2$$

Thus,

$$f'(3) \approx \frac{f(3+0.2) - f(3-0.2)}{2(0.2)}$$

$$= \frac{f(3.2) - f(2.8)}{2(0.2)}$$

$$= \frac{e^{3.2} - e^{2.8}}{0.4}$$

$$= 20.220$$

Choice (D)

The forward divided difference approximation is

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

where

$$x = 3$$

$$\Delta x = 0.2$$

Thus,

$$f'(3) \approx \frac{f(3+0.2) - f(3)}{(0.2)}$$

$$= \frac{f(3.2) - f(3)}{(0.2)}$$

$$= \frac{e^{3.2} - e^{3}}{0.2}$$

$$= 22.235$$

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$\nearrow f(x) = e^x$

$\nearrow$ True error

5. Using the backward divided difference approximation, $\frac{d}{dx}(e^x) = 4.3715$ at $x = 1.5$ for a step size of 0.05. If you keep halving the step size to find $\frac{d}{dx}(e^x)$ at $x = 1.5$ before two significant digits can be considered to be at least correct in your answer, the step size would be (you cannot use the exact value to determine the answer)

(A) 0.05/2
(B) 0.05/4
(C) 0.05/8 ✓
(D) 0.05/16

Solution

The correct answer is (C).

The equation for the backward difference approximation is

$$f'(x) \approx \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}$$

Half the step size and find the value of

$$\frac{d}{dx}(e^x) \text{ at } x = 1.5$$

$$\Delta x = 0.05/2$$
$$= 0.025$$

$$f'(1.5) = \frac{f(1.5) - f(1.475)}{0.025}$$
$$= \frac{e^{1.5} - e^{1.475}}{0.025}$$
$$= 4.4261$$

present (Approx)

The absolute relative approximate error is

$$|\epsilon_a| = \left|\frac{4.4261 - 4.3715}{4.4261}\right| \times 100$$
$$= 1.2345\%$$

Since $1.2345\% \le 0.5 \times 10^{2-m}$ for a maximum integer value of $m = 1$, there is at least one significant digit correct. But, we are looking for 2 significant digits so we must halve the previous step size and find the backward difference approximation again.

$$\Delta x = 0.05/4$$
$$= 0.0125$$

$$f'(1.5) = \frac{f(1.5) - f(1.4875)}{0.0125}$$
$$= \frac{e^{1.5} - e^{1.4875}}{0.0125}$$
$$= 4.4538$$

The absolute relative approximate error is

$$|\epsilon_a| = \left|\frac{4.4538 - 4.4261}{4.4538}\right| \times 100$$

$$= 0.62111\%$$

Since for $0.62111\% \le 0.5 \times 10^{2-m}$ for a maximum integer value of $m = 1$, again, there is only at least one significant digit correct. We must halve the previous step size and find the backward difference again.

$$\Delta x = 0.05/8$$

$$= 0.00625$$

$$f'(1.5) = \frac{f(1.5) - f(1.49375)}{0.00625}$$

$$= \frac{e^{1.5} - e^{1.49375}}{0.00625}$$

$$= 4.4677$$

The absolute relative approximate error is

$$|\epsilon_a| = \left|\frac{4.4677 - 4.4538}{4.4677}\right| \times 100$$

$$= 0.31153\%$$

Since $0.31153\% \le 0.5 \times 10^{2-m}$ for a maximum integer value of $m = 2$. Now, there are at least two significant digits correct in the iteration. Thus, the answer is

$$\Delta x = 0.05/8$$

Example 1 44

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t,\ 0 \le t \le 30$$

where v is given in m/s and t is given in seconds. At $t = 16\,\text{s}$,

a) use the forward difference approximation of the first derivative of $v(t)$ to calculate the acceleration. Use a step size of $\Delta t = 2\,\text{s}$.

b) find the exact value of the acceleration of the rocket.

c) calculate the absolute relative true error for part (b).

Solution

(a) $$a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

Continuous Differentiation

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$$a(16) \approx \frac{v(18)-v(16)}{2} \quad \text{......} \quad (1)$$

$$v(18) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$
$$= 453.02 \text{ m/s}$$

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$
$$= 392.07 \text{ m/s}$$

Hence

$$a(16) \approx \frac{v(18)-v(16)}{2} \quad \text{......} \quad (1)$$
$$= \frac{453.02 - 392.07}{2}$$
$$= 30.474 \text{ m/s}^2$$

(b) The exact value of $a(16)$ can be calculated by differentiating

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t$$

as

$$a(t) = \frac{d}{dt}[v(t)]$$

Knowing that

$$\boxed{\frac{d}{dt}[\ln(t)] = \frac{1}{t} \text{ and } \frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}}$$

$$\frac{d}{dt}[v(t)] = a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{(14\times10^4 - 2100t)^2}\right)(-2100) - 9.8$$

differentiatig

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

$$a(16) = \frac{-4040 - 29.4(16)}{-200 + 3(16)}$$
$$= 29.674 \text{ m/s}^2$$

(c) The absolute relative true error is

$$|\epsilon_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{29.674 - 30.474}{29.674}\right| \times 100$$

$$= 2.6967\%$$

Example 2 46

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

(a) Use the backward difference approximation of the first derivative of $v(t)$ to calculate the acceleration at $t = 16\text{s}$. Use a step size of $\Delta t = 2\text{ s}$.

(b) Find the absolute relative true error for part (a).

Solution

$$a(t) \approx \frac{v(t_i) - v(t_{i-1})}{\Delta t}$$ उत्तर

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i-1} &= t_i - \Delta t \\ &= 16 - 2 \\ &= 14 \end{aligned}$$

$$a(16) \approx \frac{v(16) - v(14)}{2}$$

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$

$$= 392.07\text{ m/s}$$

$$v(14) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(14)}\right] - 9.8(14)$$

$$= 334.24 \text{ m/s}$$

$$a(16) \approx \frac{v(16) - v(14)}{2}$$

$$= \frac{392.07 - 334.24}{2}$$

$$= 28.915 \text{ m/s}^2$$

(b) The exact value of the acceleration at $t = 16\text{s}$ from Example 1 is

$\frac{d}{d} v(t) = a(t)$
Ex 1 P2/15

$$a(16) = 29.674 \text{ m/s}^2$$

The absolute relative true error for the answer in part (a) is

$$|\epsilon_t| = \left| \frac{29.674 - 28.915}{29.674} \right| \times 100$$

$$= 2.5584\%$$

Example 3 48

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30.$$

(a) Use the central difference approximation of the first derivative of $v(t)$ to calculate the acceleration at $t = 16\,\text{s}$. Use a step size of $\Delta t = 2\,\text{s}$.

(b) Find the absolute relative true error for part (a).

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_{i-1})}{2\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

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$$t_{i+1} = t_i + \Delta t$$
$$= 16 + 2$$
$$= 18$$
$$t_{i-1} = t_i - \Delta t$$
$$= 16 - 2$$
$$= 14$$

$$a(16) \approx \frac{v(18) - v(14)}{2(2)}$$
$$= \frac{v(18) - v(14)}{4}$$

$$v(18) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$
$$= 453.02 \text{ m/s}$$

$$v(14) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100(14)}\right] - 9.8(14)$$
$$= 334.24 \text{ m/s}$$

$$a(16) \approx \frac{v(18) - v(14)}{4}$$
$$= \frac{453.02 - 334.24}{4}$$
$$= 29.694 \text{ m/s}^2$$

(b) The exact value of the acceleration at $t = 16\,\text{s}$ from Example 1 is Ex2.05

$$a(16) = 29.674 \text{ m/s}^2$$

The absolute relative true error for the answer in part (a) is

$$|\epsilon_t| = \left|\frac{29.674 - 29.694}{29.674}\right| \times 100$$
$$= 0.069157\%$$

The results from the three difference approximations are given in Table 1.

Table 1 Summary of $a(16)$ using different difference approximations Summary

Type of difference approximation	$a(16)$ (m/s^2)	$\lvert\epsilon_t\rvert\%$
Forward	30.475	2.6967
Backward	28.915	2.5584
Central	29.695	0.069157

Clearly, the central difference scheme is giving more accurate results because the order of accuracy is proportional to the square of the step size. In real life, one would not