Integer Functions

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FLOORS AND CEILINGS

- Floor Function

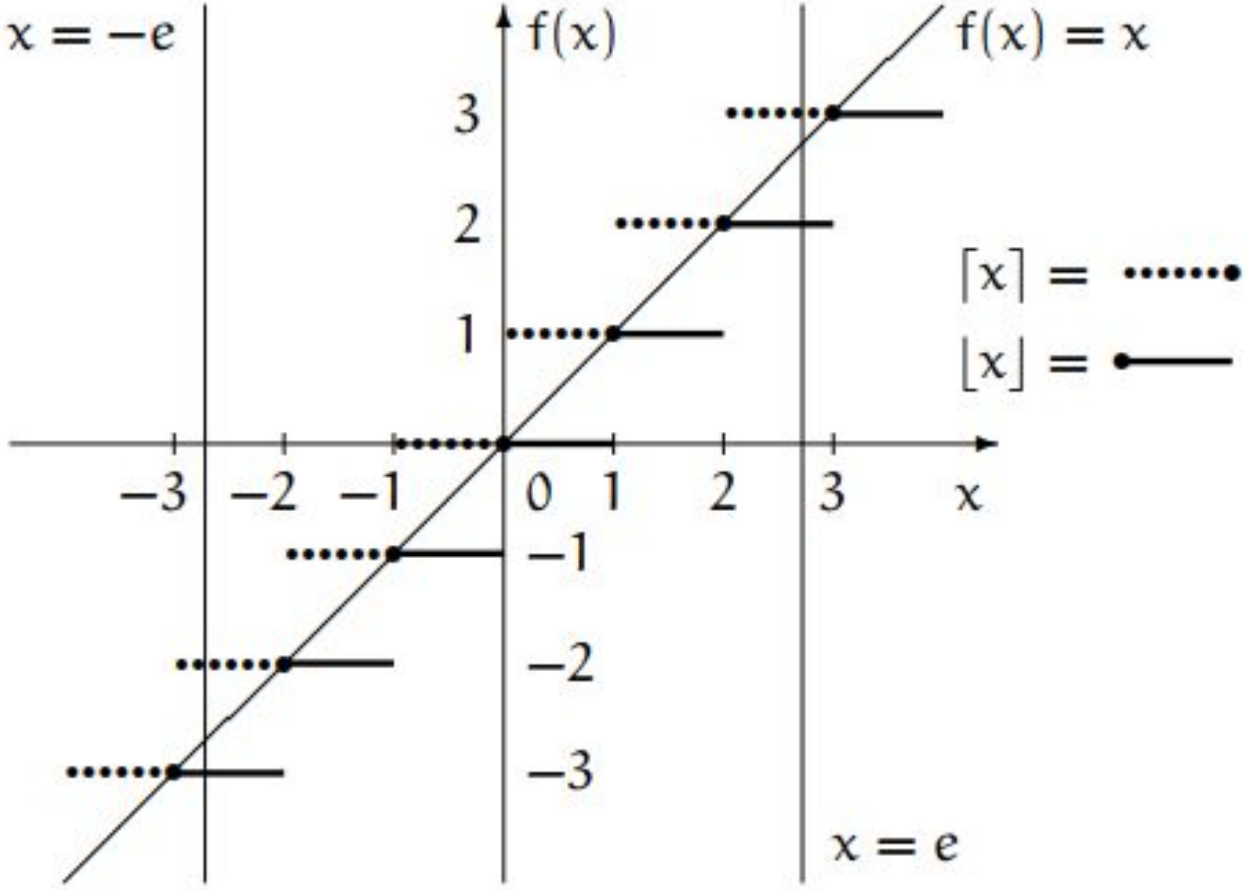
$\lfloor x \rfloor$ = the greatest integer less than or equal to x

- Ceiling Function

$\lceil x \rceil$ = the least integer greater than or equal to x.

- Kenneth E. Iverson introduced this notation.
- His notation has become sufficiently popular that floor and ceiling brackets can now be used in a technical paper without an explanation of what they mean.

FLOORS AND CEILINGS GRAPH



Properties of FLOOR AND CEILING Function

$$\lfloor x \rfloor \leqslant x$$

$$\lceil x \rceil \geqslant x$$

$$\lfloor x \rfloor = x \quad \Longleftrightarrow \quad x \text{ is an integer} \quad \Longleftrightarrow \quad \lceil x \rceil = x$$

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]$$

$$x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$$

$$\lfloor -x \rfloor = -\lceil x \rceil ; \qquad \lceil -x \rceil = -\lfloor x \rfloor .$$

Properties of FLOOR AND CEILING Function

$$\lfloor x \rfloor \leqslant x$$

Properties of FLOOR AND CEILING Function

$$\lceil x \rceil \geq x$$

Properties of FLOOR AND CEILING Function

$$\lfloor x \rfloor = x \quad \Longleftrightarrow \quad x \text{ is an integer} \quad \Longleftrightarrow \quad \lceil x \rceil = x$$

Properties of FLOOR AND CEILING Function

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]$$

Properties of FLOOR AND CEILING Function

$$x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$$

Properties of FLOOR AND CEILING Function

$$\lfloor -x \rfloor = -\lceil x \rceil ; \qquad \lceil -x \rceil = -\lfloor x \rfloor .$$

Rules for FLOOR AND CEILING Function

$$\lfloor x \rfloor = n \iff n \leqslant x < n+1$$
$$\lfloor x \rfloor = n \iff x-1 < n \leqslant x$$
$$\lceil x \rceil = n \iff n-1 < x \leqslant n$$
$$\lceil x \rceil = n \iff x \leqslant n < x+1$$

Rules for FLOOR AND CEILING Function

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n, \qquad \text{integer } n.$$

$$\lfloor nx \rfloor \neq n\lfloor x \rfloor$$

Redundant Use of FLOOR AND CEILING Function

$$x < n \iff \lfloor x \rfloor < n$$

$$n < x \iff n < \lceil x \rceil$$

$$x \leqslant n \iff \lceil x \rceil \leqslant n$$

$$n \leqslant x \iff n \leqslant \lfloor x \rfloor$$

Fractional Part

The difference between x and $\lfloor x \rfloor$ is called the *fractional part* of x,

$$\{x\} = x - \lfloor x \rfloor$$

$$x = \lfloor x \rfloor + \{x\}.$$

$$x = n + \theta$$

$n = \lfloor x \rfloor$ and $\theta = \{x\}$.

$$0 \leqslant \theta < 1$$

Fractional Part

$\lfloor x + y \rfloor$

$x = \lfloor x \rfloor + \{x\}$

$y = \lfloor y \rfloor + \{y\}$

$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$

$0 \leqslant \{x\} + \{y\} < 2$

we find that sometimes $\lfloor x + y \rfloor$ is $\lfloor x \rfloor + \lfloor y \rfloor$,

otherwise it's $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

FLOOR/CEILING APPLICATIONS

- How many bits are needed to represent a number n in binary representation?
- Let the number n has m bits in its binary representation.
- Question?
- Range of Number that can be representation in binary by using m bit?

$$2^{m-1} \leqslant n < 2^m$$

$$m - 1 = \lfloor \lg n \rfloor$$

$$m = \lfloor \lg n \rfloor + 1.$$

$$\lfloor x \rfloor = n \iff n \leqslant x < n + 1$$

$$\lfloor x \rfloor = n \iff x - 1 < n \leqslant x$$

$$\lceil x \rceil = n \iff n - 1 < x \leqslant n$$

$$\lceil x \rceil = n \iff x \leqslant n < x + 1.$$

FLOOR/CEILING APPLICATIONS

$m = \lfloor \lg n \rfloor + 1.$

$\lceil \lg(n+1) \rceil$; this formula holds for $n = 0$

FLOOR/CEILING APPLICATIONS

$$\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor, \qquad \text{real } x \geqslant 0.$$

$$m = \lfloor\sqrt{\lfloor x\rfloor}\rfloor$$

$$m \leqslant \sqrt{\lfloor x\rfloor} < m+1$$

$$m^2 \leqslant \lfloor x\rfloor < (m+1)^2.$$

$$m^2 \leqslant x < (m+1)^2$$

$$m \leqslant \sqrt{x} < m+1$$

$$m = \lfloor\sqrt{x}\rfloor$$

$$\lfloor\sqrt{\lfloor x\rfloor}\rfloor = m = \lfloor\sqrt{x}\rfloor;$$

$$\lfloor x\rfloor = n \iff n \leqslant x < n+1$$

$$x < n \iff \lfloor x\rfloor < n, \qquad (a)$$

$$n < x \iff n < \lceil x\rceil, \qquad (b)$$

$$x \leqslant n \iff \lceil x\rceil \leqslant n, \qquad (c)$$

$$n \leqslant x \iff n \leqslant \lfloor x\rfloor. \qquad (d)$$

$$\lfloor x\rfloor = n \iff n \leqslant x < n+1$$

FLOOR/CEILING APPLICATIONS

$$\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil, \qquad \text{real } x \geqslant 0$$

FLOOR/CEILING APPLICATIONS

- Let f(x) be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \implies x = \text{integer}$$

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \quad \text{and} \quad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

FLOOR/CEILING APPLICATIONS

$$\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

FLOOR/CEILING APPLICATIONS

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \qquad \text{and} \qquad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

$$\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \qquad \text{and} \qquad \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$$

if m and n are integers and the denominator n is positive.

$$\lfloor \lfloor \lfloor x/10 \rfloor /10 \rfloor /10 \rfloor = \lfloor x/1000 \rfloor$$

FLOOR/CEILING APPLICATIONS

- Close Interval $[\alpha \,..\, \beta]$ $\alpha \leqslant x \leqslant \beta$
- Open Interval $(\alpha \,..\, \beta)$ $\alpha < x < \beta$
- Half Open Interval $[\alpha \,..\, \beta)$

  $(\alpha \,..\, \beta]$

- How many integers are contained in such intervals?

FLOOR/CEILING APPLICATIONS

- How many integers are contained in such intervals?

The answer is easy if α and β are integers

$[\alpha \,..\, \beta)$ contains the $\beta - \alpha$ integers α, $\alpha + 1$, $\ldots$, $\beta - 1$, assuming that $\alpha \leqslant \beta$

$(\alpha \,..\, \beta]$

But our problem is harder, because α and β are arbitrary reals

FLOOR/CEILING APPLICATIONS

$[\alpha\,..\,\beta)$

$\alpha \leqslant n < \beta \iff \lceil\alpha\rceil \leqslant n < \lceil\beta\rceil$

$x < n \iff \lfloor x\rfloor < n$

$n < x \iff n < \lceil x\rceil$

$x \leqslant n \iff \lceil x\rceil \leqslant n$

$n \leqslant x \iff n \leqslant \lfloor x\rfloor$

$[\alpha\,..\,\beta) \qquad \lceil\beta\rceil - \lceil\alpha\rceil$

FLOOR/CEILING APPLICATIONS

$[\alpha \,..\, \beta)$	$\lceil\beta\rceil - \lceil\alpha\rceil$
$(\alpha \,..\, \beta]$	$\lfloor\beta\rfloor - \lfloor\alpha\rfloor$

FLOOR/CEILING APPLICATIONS

interval	integers contained	restrictions
$[\alpha\,..\,\beta]$	$\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$	$\alpha \leqslant \beta\,,$
$[\alpha\,..\,\beta)$	$\lceil\beta\rceil - \lceil\alpha\rceil$	$\alpha \leqslant \beta\,,$
$(\alpha\,..\,\beta]$	$\lfloor\beta\rfloor - \lfloor\alpha\rfloor$	$\alpha \leqslant \beta\,,$
$(\alpha\,..\,\beta)$	$\lceil\beta\rceil - \lfloor\alpha\rfloor - 1$	$\alpha < \beta\,.$

Roulette Wheel Problem

- There's a roulette wheel with one thousand slots, numbered 1 to 1000.

$$\lfloor \sqrt[3]{n} \rfloor \setminus n$$

The notation $a \setminus b$, read "a divides b,"

- How many integers n, where 1 <= n <= 1000, satisfy the relation $\lfloor \sqrt[3]{n} \rfloor \setminus n$?

Roulette Wheel Problem

- Then it's a winner and the house pays us \$5; otherwise it's a loser and we must pay \$1.
- Let the number of winners W, then the number L = 1000 - W of losers.

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}$$

- How can we count the number of winners among 1 through 1000?
- It's not hard to spot a pattern.
- The numbers from 1 through 2^3 - 1 = 7 are all winners because $\lfloor \sqrt[3]{n} \rfloor = 1$ for each.
- Among the numbers 2^3 = 8 through 3^3 - 1 = 26, only the even numbers are winners.
- And among 3^3 = 27 through 4^3 - 1 = 63, only those divisible by 3 are. And so on.

Roulette Wheel Problem

$$
\begin{aligned}
W &= \sum_{n=1}^{1000} [n \text{ is a winner}] \\
&= \sum_{1 \leqslant n \leqslant 1000} \left[\lfloor \sqrt[3]{n} \rfloor \backslash n \right] \\
&= \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor \right] [k \backslash n] [1 \leqslant n \leqslant 1000] \\
&= \sum_{k,m,n} \left[k^3 \leqslant n < (k+1)^3 \right] [n = km] [1 \leqslant n \leqslant 1000] \\
&= 1 + \sum_{k,m} \left[k^3 \leqslant km < (k+1)^3 \right] [1 \leqslant k < 10] \\
&= 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots (k+1)^3/k \right) \right] [1 \leqslant k < 10]
\end{aligned}
$$

Roulette Wheel Problem

$$= 1 + \sum_{1 \leqslant k < 10} \left(\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil \right)$$

$$= 1 + \sum_{1 \leqslant k < 10} (3k + 4)$$

$$= 1 + \frac{7 + 31}{2} \cdot 9$$

$$= 172$$

- (6*172 - 1000)/1000 dollars, which is 3.2 cents

The General Solution of Roulette Wheel Problem

$$K = \lfloor \sqrt[3]{N} \rfloor$$

$$\begin{aligned} W &= \sum_{1 \leqslant k < K} (3k+4) + \sum_m [K^3 \leqslant Km \leqslant N] \\ &= \tfrac{1}{2}(7 + 3K + 1)(K-1) + \sum_m \left[m \in [K^2 \,..\, N/K] \right] \\ &= \tfrac{3}{2}K^2 + \tfrac{5}{2}K - 4 + \sum_m \left[m \in [K^2 \,..\, N/K] \right] \end{aligned}$$

$$\lfloor N/K \rfloor - \lceil K^2 \rceil + 1 = \lfloor N/K \rfloor - K^2 + 1$$

$$W = \lfloor N/K \rfloor + \tfrac{1}{2}K^2 + \tfrac{5}{2}K - 3, \qquad K = \lfloor \sqrt[3]{N} \rfloor$$

The Approximate Solution of Roulette Wheel Problem

$$W = \lfloor N/K \rfloor + \tfrac{1}{2}K^2 + \tfrac{5}{2}K - 3, \qquad K = \lfloor \sqrt[3]{N} \rfloor$$

$$W = \tfrac{3}{2}N^{2/3} + O(N^{1/3})$$

N	$\frac{3}{2}N^{2/3}$	W	% error
1,000	150.0	172	12.791
10,000	696.2	746	6.670
100,000	3231.7	3343	3.331
1,000,000	15000.0	15247	1.620
10,000,000	69623.8	70158	0.761
100,000,000	323165.2	324322	0.357
1,000,000,000	1500000.0	1502496	0.166

The General Solution of Josephus problem

$$J(1) = 1;$$
$$J(2n) = 2J(n) - 1, \qquad \text{for } n \geqslant 1$$
$$J(2n+1) = 2J(n) + 1, \qquad \text{for } n \geqslant 1$$

$$J(1) = 1;$$
$$J(n) = 2J(\lfloor n/2 \rfloor) - (-1)^n, \qquad \text{for } n > 1$$

Josephus problem in which every third person is eliminated, instead of every second.

$$J_3(n) = \left\lceil \tfrac{3}{2} J_3\left(\lfloor \tfrac{2}{3} n \rfloor\right) + a_n \right\rceil \bmod n + 1$$

$$a_n = -2,\ +1,\ \text{or } -\tfrac{1}{2} \text{ according as } n \bmod 3 = 0,\ 1,\ \text{or } 2$$

The General Solution of Josephus problem

- There's another approach to the Josephus problem that gives a much better setup.
- Whenever a person is passed over, we can assign a new number.
- Thus, 1 and 2 become $n + 1$ and $n + 2$
- then 3 is executed
- 4 and 5 become $n + 3$ and $n + 4$
- then 6 is executed
- $3k + 1$ and $3k + 2$ become $n + 2k + 1$ and $n + 2k + 2$
- then $3k + 3$ is executed
- then $3n$ is executed (or left to survive)

The General Solution of Josephus problem

- For example, when n = 10 the numbers are

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

$$N := 3n;$$
$$\textbf{while } N > n \textbf{ do } N := \left\lfloor \frac{N-n-1}{2} \right\rfloor + N - n$$
$$J_3(n) := N.$$

- The kth person eliminated ends up with number 3k. So we can figure out who the survivor is if we can figure out the original number of person number 3n.

$$N := 3n;$$

$$\textbf{while } N > n \textbf{ do } N := \left\lfloor \frac{N-n-1}{2} \right\rfloor + N - n$$

$$J_3(n) := N.$$

$D = 3n + 1 - N$ in place of N

$$\begin{aligned} D &:= 3n+1-\left(\left\lfloor \frac{(3n+1-D)-n-1}{2} \right\rfloor + (3n+1-D)-n\right) \\ &= n+D-\left\lfloor \frac{2n-D}{2} \right\rfloor = D-\left\lfloor \frac{-D}{2} \right\rfloor = D+\left\lceil \frac{D}{2} \right\rceil = \left\lceil \tfrac{3}{2}D \right\rceil \end{aligned}$$

$$D := 1;$$

$$\textbf{while } D \leqslant 2n \textbf{ do } D := \left\lceil \tfrac{3}{2}D \right\rceil$$

$$J_3(n) := 3n+1-D.$$

$$
\begin{aligned}
&D := 1\,;\\
&\textbf{while}\ \ D \leqslant (q-1)n\ \ \textbf{do}\ \ D := \left\lceil \tfrac{q}{q-1} D \right\rceil\\
&J_q(n) := qn + 1 - D\,.
\end{aligned}
$$

In the case $q = 2$ that we know so well, this makes D grow to 2^{m+1} when $n = 2^m + l$; hence $J_2(n) = 2(2^m + l) + 1 - 2^{m+1} = 2l + 1$. Good.

$$
\begin{aligned}
&D_0^{(q)} = 1\,;\\
&D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil \qquad \text{for } n > 0.
\end{aligned}
$$