Sum

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Notation

- $1 + 2 + 3 + \cdot \cdot \cdot \cdot + (n - 1) + n$
- Where '....' means to complete the pattern established by the surrounding terms
- Moreover 3 and (n-1) terms are also redundant
- $1 + 2 + \cdot \cdot \cdot + n$
- BUT if we ignore 2 then
- $1 + \cdot \cdot \cdot + n$

Notation

- $1 + 2 + 3 + \cdots + (n - 1) + n$
- $a_1 + a_2 + \cdots + a_n$
- Where a_k is called a term of the sum and will be a function of k
- $2^1 + 2^2 + \cdots + 2^n$
- Where $a_k = 2^k$

Notation

- $a_1 + a_2 + \cdot \cdot \cdot + a_n$
- The Sigma-notation form

$$\sum_{1 \leqslant k \leqslant n} a_k$$

- The delimited form

$$\sum_{k=1}^{n} a_k$$

Notation

- The squares of all odd positive integers below 100
- The Sigma-notation form

$$\sum_{\substack{1 \leqslant k < 100 \\ k \text{ odd}}} k^2$$

- The delimited form

$$\sum_{k=0}^{49} (2k+1)^2$$

Notation

- The sum of reciprocals of all prime numbers between 1 and N
- The Sigma-notation form

$$\sum_{\substack{p \leqslant N \\ p \text{ prime}}} \frac{1}{p}$$

- The delimited form

$$\sum_{k=1}^{\pi(N)} \frac{1}{p_k}$$

where p_k denotes the kth prime and $\pi(N)$ is the number of primes $\leqslant N$.

Notation

- The biggest advantage of general Sigma-notation is that we can manipulate it more easily than the delimited form.
- The Sigma-notation form

$$\sum_{1\leqslant k\leqslant n} a_k = \sum_{1\leqslant k+1\leqslant n} a_{k+1}$$

- The delimited form

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1}$$

Notation

- The general form of Sigma-notation

$$\sum_{P(k)} a_k$$

- Where a "property P(k)" is any statement about k that can be either true or false.

Iverson's convention

$$[P(k)] = \begin{cases} 1, & if\ P(k) is\ True \\ 0, & if\ P(k) is\ False \end{cases}$$

$$[\text{p prime}] = \begin{cases} 1, & \text{if p is a prime number;} \\ 0, & \text{if p is not a prime number.} \end{cases}$$

Notation

- The general form of Sigma-notation

$$\sum_{P(k)} a_k$$

- The general form of Sigma-notation in Iverson's convention

$$\sum_{k} a_k [P(k)]$$

Notation

- The sum of reciprocals of all prime numbers between 1 and N
- The Sigma-notation form

$$\sum_{\substack{p \leqslant N \\ p \text{ prime}}} \frac{1}{p}$$

- The of Sigma-notation form in Iverson's convention

$$\sum_{p} [p \text{ prime}][p \leqslant N]/p$$

SUMS AND RECURRENCES

- The delimited form Sum:

$$S_n = \sum_{k=0}^{n} a_k$$

- Equivalent recurrence form:

$$S_0 = a_0 ;$$
$$S_n = S_{n-1} + a_n , \qquad \text{for } n > 0.$$

SUMS AND RECURRENCES

- Equivalent recurrence form:

$$S_0 = a_0;$$
$$S_n = S_{n-1} + a_n, \qquad \text{for } n > 0.$$

- If $a_n = \beta + \gamma n$
- Then,

$$R_0 = \alpha;$$
$$R_n = R_{n-1} + \beta + \gamma n, \qquad \text{for } n > 0.$$

SUMS AND RECURRENCES

- If a_n = $\beta + \gamma n$
- recurrence form

$$R_0 = \alpha;$$
$$R_n = R_{n-1} + \beta + \gamma n, \qquad \text{for } n > 0.$$

- R1 = ?

$$R_1 = \alpha + \beta + \gamma$$

- R2 = ?

$$R_2 = \alpha + 2\beta + 3\gamma;$$

SUMS AND RECURRENCES

$$R_0 = \alpha$$

$$R_1 = \alpha + \beta + \gamma$$

$$R_2 = \alpha + 2\beta + 3\gamma;$$

- .
- .
- .
- .

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

where $A(n)$, $B(n)$, and $C(n)$ are the coefficients of dependence on the general parameters α, β, and γ.

SUMS AND RECURRENCES

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

- If R_n = 1, then $\alpha = 1,\ \beta = 0,\ \gamma = 0$

  A(n) = 1

- If R_n = n, then $\alpha = 0,\ \beta = 1,\ \gamma = 0;$

  B(n) = n

- If R_n = n^2, then $\alpha = 0,\ \beta = -1,\ \gamma = 2$

  2C(n) – B(n) = n^2

  2C(n) = n^2+B(n)

  2C(n) = n^2+n

  C(n) = (n^2+n)/2

SUMS AND RECURRENCES

- A(n) = 1
- B(n) = n
- C(n) = (n^2+n)/2

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$R_0 = \alpha\,;$$
$$R_n = R_{n-1} + \beta + \gamma n\,, \qquad \text{for } n > 0.$$

$$S_0 = a_0\,;$$
$$S_n = S_{n-1} + a_n\,, \qquad \text{for } n > 0.$$

$$S_n = \sum_{k=0}^{n} a_k$$

Example Problem for Recurrences

$$\sum_{k=0}^{n} (a + bk)$$

- Ro = a
- Rn = R(n-1) + a+b*n

$$R_0 = \alpha;$$
$$R_n = R_{n-1} + \beta + \gamma n, \qquad \text{for } n > 0.$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

- A(n) = 1
- B(n) = n
- C(n) = (n^2+n)/2
- Rn = $a(n+1) + b(n+1)n/2$

$$\alpha = \beta = a, \gamma = b,$$

Recursive equation to sum

- Recursive Equation:

$$\begin{aligned} T_0 &= 0; \\ T_n &= 2T_{n-1} + 1 \end{aligned}$$

- If we divide each side with 2^n:

$$\begin{aligned} T_0/2^0 &= 0; \\ T_n/2^n &= T_{n-1}/2^{n-1} + 1/2^n \end{aligned}$$

Now we can set $S_n = T_n/2^n$, and we have

$$\begin{aligned} S_0 &= 0; \\ S_n &= S_{n-1} + 2^{-n}, \qquad \text{for } n > 0. \end{aligned}$$

Recursive equation to sum

- Recursive Equation:

$$
\begin{aligned}
S_0 &= 0; \\
S_n &= S_{n-1} + 2^{-n}, \qquad \text{for } n > 0.
\end{aligned}
$$

- Sum Form:

$$S_n = \sum_{k=1}^{n} 2^{-k}.$$

$$= 2^{-1} + 2^{-2} + \cdots + 2^{-n} = (\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^n$$

$$= 1 - (\tfrac{1}{2})^n$$

$$T_n = 2^n S_n = 2^n - 1.$$

Generalization

- Previous problem

$$T_0 = 0;$$
$$T_n = 2T_{n-1} + 1, \qquad \text{for } n > 0.$$

- Generalized:

$$a_n T_n = b_n T_{n-1} + c_n$$

multiply both sides by a *summation factor*, s_n:

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n$$

This factor s_n is cleverly chosen to make

$$s_n b_n = s_{n-1} a_{n-1}$$

Generalization

Then if we write $S_n = s_n a_n T_n$ we have a sum-recurrence,

$$S_n = S_{n-1} + s_n c_n$$

$$\begin{aligned} S_n &= s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k \\ &= s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \end{aligned}$$

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

Generalization

But how can we be clever enough to find the right s_n?

$$s_n = s_{n-1} a_{n-1} / b_n$$

$$s_n = \frac{a_{n-1} a_{n-2} \ldots a_1}{b_n b_{n-1} \ldots b_2}$$

$$T_n = 2T_{n-1} + 1 \qquad a_n T_n = b_n T_{n-1} + c_n$$

$a_n = 1$ and $b_n = 2$;

Generalization

$$s_n = \frac{a_{n-1}a_{n-2}\dots a_1}{b_n b_{n-1}\dots b_2}$$

$$T_n = 2T_{n-1} + 1 \qquad a_n T_n = b_n T_{n-1} + c_n$$

$a_n = 1$ and $b_n = 2$;

$s_n = 2^{-n}$

Recursive Equation of Quicksort

$$C_0 = 0;$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k\,, \qquad \text{for } n > 0.$$

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k\,, \qquad \text{for } n > 0$$

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k\,, \qquad \text{for } n-1 > 0$$

$$nC_n - (n-1)C_{n-1} = 2n + 2C_{n-1}\,, \qquad \text{for } n > 1$$

$$nC_n = (n+1)C_{n-1} + 2n\,, \qquad \text{for } n > 0.$$

Recursive Equation of Quicksort

$$C_0 = 0;$$
$$nC_n = (n+1)C_{n-1} + 2n, \qquad \text{for } n > 0$$

$$a_n T_n = b_n T_{n-1} + c_n$$
$$a_n = n,\ b_n = n+1, \text{ and } c_n = 2n.$$

$$s_n = \frac{a_{n-1}a_{n-2}\ldots a_1}{b_n b_{n-1}\ldots b_2} = \frac{(n-1)\cdot(n-2)\cdot\ldots\cdot 1}{(n+1)\cdot n\cdot\ldots\cdot 3} = \frac{2}{(n+1)n}$$

Recursive Equation of Quicksort

$$
\begin{aligned}
C_0 &= 0; \\
nC_n &= (n+1)C_{n-1} + 2n, \qquad \text{for } n > 0
\end{aligned}
$$

$$a_n T_n = b_n T_{n-1} + c_n$$

$$s_n = \frac{a_{n-1}a_{n-2}\ldots a_1}{b_n b_{n-1}\ldots b_2} = \frac{(n-1)\cdot(n-2)\cdot\ldots\cdot 1}{(n+1)\cdot n\cdot\ldots\cdot 3} = \frac{2}{(n+1)n}$$

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

$$C_n = 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1}.$$

Recursive Equation of Quicksort

$$\sum_{k=1}^{n} \frac{1}{k+1} = \sum_{1 \leqslant k \leqslant n} \frac{1}{k+1}$$

$$\begin{aligned}
\sum_{1 \leqslant k \leqslant n} \frac{1}{k+1} &= \sum_{1 \leqslant k-1 \leqslant n} \frac{1}{k} \\
&= \sum_{2 \leqslant k \leqslant n+1} \frac{1}{k} \\
&= \left(\sum_{1 \leqslant k \leqslant n} \frac{1}{k} \right) - \frac{1}{1} + \frac{1}{n+1} = H_n - \frac{n}{n+1}
\end{aligned}$$

Recursive Equation of Quicksort

$$C_n = 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1}.$$

$$\sum_{k=1}^{n}\frac{1}{k+1} = H_n - \frac{n}{n+1}$$

$$C_n = 2(n+1)H_n - 2n$$

Manipulation of Sums

$$\sum_{k\in K} ca_k = c\sum_{k\in K} a_k\,; \qquad \text{(distributive law)}$$

$$\sum_{k\in K} (a_k + b_k) = \sum_{k\in K} a_k + \sum_{k\in K} b_k\,; \qquad \text{(associative law)}$$

$$\sum_{k\in K} a_k = \sum_{p(k)\in K} a_{p(k)}\,. \qquad \text{(commutative law)}$$

Distributive Law

$$\sum_{k \in K} c a_k = c \sum_{k \in K} a_k ; \qquad \text{(distributive law)}$$

- cao + ca1 + ca2 = c(ao+a1+a2)

Associative Law

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k \,; \qquad \text{(associative law)}$$

- (ao+bo)+(a1+b1)+(a2+b2) = (ao+a1+a2)+(bo+b1+b2)

Commutative Law

- Let p(k) = (k+1)%3
- ao+a1+a2 = a1+a2+a0

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)} . \qquad \text{(commutative law)}$$

Applications

$$S = \sum_{0 \leqslant k \leqslant n} (a + bk)$$

$$S = \sum_{0 \leqslant n-k \leqslant n} \big(a + b(n - k)\big) = \sum_{0 \leqslant k \leqslant n} (a + bn - bk)$$

$$2S = \sum_{0 \leqslant k \leqslant n} \big((a + bk) + (a + bn - bk)\big) = \sum_{0 \leqslant k \leqslant n} (2a + bn)$$

$$2S = (2a + bn) \sum 1 = (2a + bn)(n + 1)$$

$$\sum_{k=0}^{n} (a + bk) = (a + \tfrac{1}{2}bn)(n + 1)$$

Perturbation Technique

$$\sum_{0 \leqslant k \leqslant n} a_k = a_0 + \sum_{1 \leqslant k \leqslant n} a_k$$

$$S_n = \sum_{0 \leqslant k \leqslant n} a_k$$

$$\begin{aligned} S_n + a_{n+1} = \sum_{0 \leqslant k \leqslant n+1} a_k &= a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k \\ &= a_0 + \sum_{1 \leqslant k+1 \leqslant n+1} a_{k+1} \\ &= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} \, . \end{aligned}$$

Perturbation Technique

$$S_n = \sum_{0 \leqslant k \leqslant n} ax^k$$

$$S_n + ax^{n+1} = ax^0 + \sum_{0 \leqslant k \leqslant n} ax^{k+1}$$

$$x \sum\nolimits_{0 \leqslant k \leqslant n} ax^k = xS_n$$

perturbation technique

$$S_n + ax^{n+1} = a + xS_n$$

$$S_n = \sum_{0 \leqslant k \leqslant n} k\,2^k$$

$$\sum_{k=0}^{n} ax^k = \frac{a - ax^{n+1}}{1 - x}, \qquad \text{for } x \neq 1.$$

Multiple Sums

$$\sum_{1 \leqslant j,k \leqslant 3} a_j b_k = a_1 b_1 + a_1 b_2 + a_1 b_3 \\ + a_2 b_1 + a_2 b_2 + a_2 b_3 \\ + a_3 b_1 + a_3 b_2 + a_3 b_3 .$$

$$\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} \, [P(j,k)]$$

$$\sum_j \sum_k a_{j,k} \, [P(j,k)] \qquad \sum_j \left(\sum_k a_{j,k} \, [P(j,k)] \right)$$

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} [P(j,k)]$$

$$\sum_{\substack{j\in J\\ k\in K}} a_j b_k = \Big(\sum_{j\in J} a_j\Big)\Big(\sum_{k\in K} b_k\Big)$$

$$\begin{aligned}
\sum_{1\leqslant j,k\leqslant 3} a_j b_k &= \sum_{j,k} a_j b_k[1\leqslant j,k\leqslant 3] \\
&= \sum_{j,k} a_j b_k [1\leqslant j\leqslant 3][1\leqslant k\leqslant 3] \\
&= \sum_{j}\sum_{k} a_j b_k [1\leqslant j\leqslant 3][1\leqslant k\leqslant 3] \quad = \sum_j a_j[1\leqslant j\leqslant 3] \sum_k b_k [1\leqslant k\leqslant 3] \\
&= \sum_j a_j[1\leqslant j\leqslant 3]\Big(\sum_k b_k[1\leqslant k\leqslant 3]\Big) \quad = \Big(\sum_j a_j[1\leqslant j\leqslant 3]\Big)\Big(\sum_k b_k[1\leqslant k\leqslant 3]\Big) \\
&= \Big(\sum_{j=1}^{3} a_j\Big)\Big(\sum_{k=1}^{3} b_k\Big)
\end{aligned}$$

General Methods

$$\Box_n = \sum_{0 \leqslant k \leqslant n} k^2, \qquad \text{for } n \geqslant 0.$$

Method 0: You could look it up.

$$\Box_n = \frac{n(n+1)(2n+1)}{6}, \qquad \text{for } n \geqslant 0.$$

General Methods

Method 1: Guess the answer, prove it by induction.

$$\square_n = \frac{n(n+\frac{1}{2})(n+1)}{3}, \qquad \text{for } n \geqslant 0;$$

$$\square_n = \square_{n-1} + n^2$$

$$\begin{aligned} 3\square_n &= (n-1)(n-\tfrac{1}{2})(n) \; + \; 3n^2 \\ &= (n^3 - \tfrac{3}{2}n^2 + \tfrac{1}{2}n) \; + \; 3n^2 \\ &= (n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n) \\ &= n(n+\tfrac{1}{2})(n+1). \end{aligned}$$

General Methods

Method 2: Perturb the sum.

$$\square_n = \sum_{0 \leqslant k \leqslant n} k^2, \qquad \text{for } n \geqslant 0.$$

$$\begin{aligned}
\square_n + (n+1)^2 &= \sum_{0 \leqslant k \leqslant n} (k+1)^2 = \sum_{0 \leqslant k \leqslant n} (k^2 + 2k + 1) \\
&= \sum_{0 \leqslant k \leqslant n} k^2 + 2 \sum_{0 \leqslant k \leqslant n} k + \sum_{0 \leqslant k \leqslant n} 1 \\
&= \square_n + 2 \sum_{0 \leqslant k \leqslant n} k + (n+1).
\end{aligned}$$

$$2 \sum_{0 \leqslant k \leqslant n} k = (n+1)^2 - (n+1)$$

General Methods

Method 3: Build a repertoire.

$$R_0 = \alpha\,;$$
$$R_n = R_{n-1} + \beta + \gamma n + \delta n^2\,, \qquad \text{for } n > 0,$$

$$R_n \;=\; A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

$A(n)$, $B(n)$, and $C(n)$ when $\delta = 0$.

$R_n = n^3$; $\alpha = 0,\ \beta = 1,\ \gamma = -3,\ \delta = 3$

$$3D(n) - 3C(n) + B(n) \;=\; n^3\,;$$

$$3D(n) \;=\; n^3 + 3C(n) - B(n) \;=\; n^3 + 3\frac{(n+1)n}{2} - n \;=\; n(n+\tfrac{1}{2})(n+1)$$

General Methods

Method 4: Replace sums by integrals.

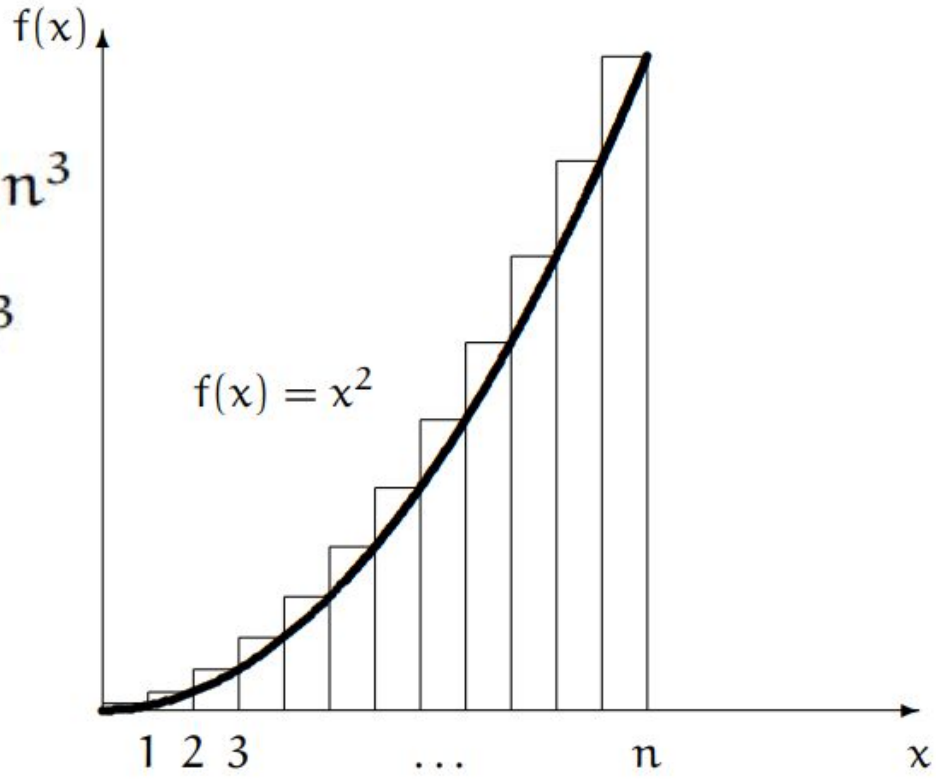
$\int_0^n x^2\,dx = n^3/3$

$$E_n = \square_n - \tfrac{1}{3}n^3 = \square_{n-1} + n^2 - \tfrac{1}{3}n^3$$

$$= E_{n-1} + \tfrac{1}{3}(n-1)^3 + n^2 - \tfrac{1}{3}n^3$$

$$E_n = E_{n-1} + n - \tfrac{1}{3}.$$

we could find E_n and then $\square_n$



General Methods

Method 5: Expand and contract.

$$\begin{aligned}
\square_n &= \sum_{1\leqslant k\leqslant n} k^2 = \sum_{1\leqslant j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n} \sum_{j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n} \left(\frac{j+n}{2}\right)(n-j+1) \\
&= \tfrac{1}{2} \sum_{1\leqslant j\leqslant n} \left(n(n+1)+j-j^2\right) \\
&= \tfrac{1}{2}n^2(n+1) + \tfrac{1}{4}n(n+1) - \tfrac{1}{2}\square_n = \tfrac{1}{2}n(n+\tfrac{1}{2})(n+1) - \tfrac{1}{2}\square_n\,.
\end{aligned}$$

Thanks

- $T_n = 2*T(n-1) + 3$
- $T_n = 2*(2*T(n-2)+3)+3$
- $T_n = 2^2*T(n-2)+ 2*3 + 3$
- $T_n = 2^2*(2* T(n-3) + 3) + 2*3 + 3$
- $T_n = 2^3*T(n-3) + 2^2*3 + 2*3 + 3$
- …..
- $T_n = 2^{(n-1)}*3+\ldots\ldots.+2^2*3+2*3+3$
- $T_n = 3*\{2^{(n-1)}+\ldots\ldots+2^2+2+1\}$
- $T_n = 3*\{2^0+2^1+2^2+\ldots\ldots\ldots..+2^{(n-1)}\}$

- $T_n = 2*T(n-1)+1$
- $T_n + 1 = 2*T(n-1) +2$
- $T_n + 1 = 2*\{T(n-1)+1\}$
- $U_n = T_n + 1$ $U(0) = T(0) + 1 = 1$
- $U(n-1) = T(n-1) +1$
- $U_n = 2*U(n-1)$ $U(n-1) = 2*U(n-2)$
- $U_n = 2^2*U(n-2)$
- …..
- $U_n = 2^n * U(0) = 2^n$
- $T_n +1 = 2^n$
- $T_n = 2^n -1$