Efficient Steady-state Simulation of High-dimensional Reflected Brownian Motions

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Road map

Reflected Brownian Motion (RBM)

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- RBM is the solution of a Skorokhod problem with Brownian input.
- Skorokhod problem:

$$0 \le \mathbf{Y}(t) = \mathbf{Y}(0) + \mathbf{X}(t) + R\mathbf{L}(t), \ \mathbf{L}(0) = 0 \tag{1}$$

  where the i-th entry of $\mathbf{L}(\cdot)$ is non-decreasing and $\int_0^t Y_i(s)\, dL_i(s) = 0$.
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- Multi-dimensional Brownian motion $\mathbf{X} \to$ RBM $\mathbf{Y}$.
- Goal: Find an efficient simulation algorithm to estimate the steady-state expectation of certain functions $f(\cdot)$ of a general multi-dimension RBM for arbitrary dimension d.

Assumptions

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- **Uniform contraction:** let $R = I - Q^T$, where Q is substochastic and satisfies

$$\left\|\mathbf{1}^T Q^n\right\|_\infty \le \kappa_0(1-\beta_0)^n, \; n \ge 1.$$

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- **Lipschitz functions:** The function to be estimated $f(\cdot)$ is Lipschitz continuous in l_∞ norm, i.e. $|f(\mathbf{y}) - f(\mathbf{y}')| \leq \mathcal{L}\|\mathbf{y} - \mathbf{y}'\|_\infty$ for $\mathcal{L} > 0$ independent of d.

Multilevel Monte Carlo Algorithm: Discretization

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- Parameters: discretization granularity $\gamma \in (0, 1)$; simulation horizon $T > 0$; the total number of levels L.

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$$B_i^m(t) = B_i\left(t_m^-\right) + \left(t - t_m^-\right)\frac{B_i\left(t_m^+\right) - B_i\left(t_m^-\right)}{t_m^+ - t_m^-}, \text{ for } i = 1, 2, ..., d.$$

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- $\mathbf{X}^m(t) = \boldsymbol{\mu} t + C\mathbf{B}^m(t)$.
- RBMs driven by $\mathbf{X}_{s:t}$ $(\mathbf{X}_{s:t}^m)$ for $\mathbf{X}_{s:t}(u) = \mathbf{X}(s+u) - \mathbf{X}(s)$:

$$\begin{aligned} \mathbf{Y}\left(t+s; \mathbf{y}, \mathbf{X}_{0:s+t}\right) &= \mathbf{Y}\left(t; \mathbf{Y}\left(s; \mathbf{y}, \mathbf{X}_{0:s}\right), \mathbf{X}_{s:s+t}\right), \\ \mathbf{Y}^m\left(t+s; \mathbf{y}, \mathbf{X}_{0:s+t}^m\right) &= \mathbf{Y}^m\left(t; \mathbf{Y}^m\left(s; \mathbf{y}, \mathbf{X}_{0:s}^m\right), \mathbf{X}_{s:s+t}^m\right). \end{aligned} \tag{2}$$

Multilevel Monte Carlo Algorithm: Estimator

Our estimator:

$$Z = \frac{1}{p(M)} \left(f\left(\mathbf{Y}^{M+1}\left(MT; \mathbf{Y}^{M+1}\left(T; \mathbf{y}_0, \mathbf{X}_{0:T}^{M+1} \right), \mathbf{X}_{T:(M+1)T}^{M+1} \right) \right) \right.$$
$$\left. - f\left(\mathbf{Y}^{M}\left(MT; \mathbf{y}_0, \mathbf{X}_{T:(M+1)T}^{M} \right) \right) \right) + f\left(\mathbf{y}_0\right).$$

for a random variable M following probability distribution

$$P(M=m) = p\left(m\right) = \gamma^m \left(1-\gamma\right)/(1-\gamma^L) \triangleq K(\gamma)\gamma^m, \text{ for } 0 \leq m < L.$$

Multilevel Monte Carlo Algorithm: Estimator

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$$
\begin{aligned}
E[Z] &= E\left[E\left[Z|M\right]\right] \\
&= \sum_{m=0}^{L-1}\left(E\left[f\left(\mathbf{Y}^{m+1}\left(mT;\mathbf{Y}^{m+1}(T;\mathbf{y}_0,\mathbf{X}_{0:T}^{m+1}),\mathbf{X}_{T:(m+1)T}^{m+1}\right)\right)\right]\right. \\
&\quad \left.-E\left[f\left(\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}_{T:(m+1)T}^{m}\right)\right)\right]\right)+f\left(\mathbf{y}_0\right) \\
&= \sum_{m=0}^{L-1}\left(E\left[f\left(\mathbf{Y}^{m+1}\left((m+1)T;\mathbf{y}_0,\mathbf{X}_{0:(m+1)T}^{m+1}\right)\right)\right]\right. \\
&\quad \left.-E[f\left(\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}_{0:mT}^{m}\right)\right)]\right)+f\left(\mathbf{y}_0\right) \\
&= E\left[f\left(\mathbf{Y}^{L}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}^{L}\right)\right)\right].
\end{aligned}
$$

As $L \to \infty$,

$$
E\left[f\left(\mathbf{Y}^{L}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}^{L}\right)\right)\right] \to E[f(\mathbf{Y}(\infty))].
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Error Analysis

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$$
\begin{aligned}
& E\left[f\left(\mathbf{Y}^L\left(TL; \mathbf{y}_0, \mathbf{X}^L_{0:LT}\right)\right)\right] - E\left[f(\mathbf{Y}(\infty))\right] \\
= & \left(E\left[f\left(\mathbf{Y}^L\left(TL; \mathbf{y}_0, \mathbf{X}^L_{0:LT}\right)\right)\right] - E\left[f\left(\mathbf{Y}\left(TL; \mathbf{y}_0, \mathbf{X}_{0:LT}\right)\right)\right]\right) \\
& + \left(E\left[f\left(\mathbf{Y}\left(TL; \mathbf{y}_0, \mathbf{X}_{0:LT}\right)\right)\right] - E\left[f(\mathbf{Y}(\infty))\right]\right) \\
= & \text{ Discretization Error} + \text{Non-stationarity Error.}
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Error Bound

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Parameter specification:

- Step size: we recommend γ around 0.05;
- Path length: $T = O(\log(d)^2)$;
- Number of levels: $L = \lceil (\log(\log(d)) + 2\log(1/\varepsilon) + k_1) \,/\, \log(1/\gamma) \rceil$;
- Number of sample paths: $N = \lceil (1-\gamma^L)(1-\gamma)^{-1}\gamma^{-L}L \rceil = O(\varepsilon^2 \log(d) \log(\log(d)))$.

Theorem

Suppose $\mathbf{Y}$ *(indexed by the number of dimensions* d*) is a sequence of RBMs satisfying Assumptions 1-4. Then, the total expected cost, in terms of* **the number of scalar Gaussian random variables**, *for the Multilevel Monte Carlo Algorithm to produce an estimator of* $E[f(\mathbf{Y}(\infty))]$ *with mean square error (MSE)* ε^2 *is*

$$O\left(\varepsilon^{-2} d \log(d)^3 (\log(\log(d)) + \log(1/\varepsilon))^3\right).$$

Numerical Experiments: Setup

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- Symmetric RBMs: $\boldsymbol{\mu} = -[1, 1, \ldots, 1]^T$

$$\Sigma = \begin{bmatrix} 1 & \rho_\sigma & \ldots & \rho_\sigma \\ \rho_\sigma & 1 & \ldots & \rho_\sigma \\ \vdots & & 1 & \vdots \\ \rho_\sigma & \ldots & \rho_\sigma & 1 \end{bmatrix}, \; R = \begin{bmatrix} 1 & -r & \ldots & -r \\ -r & 1 & \ldots & -r \\ \vdots & & 1 & \vdots \\ -r & \ldots & -r & 1 \end{bmatrix}.$$

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- Pick $\rho_\sigma = -\frac{1-\beta}{d-1}$ and $r = \frac{1-\beta}{d-1}$, and $f(Y(\infty)) = Y_1(\infty)$.

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$$E[Y_1(\infty)] = \frac{1-(d-2)r+(d-1)r\rho_\sigma}{2(1+r)} = \frac{\beta}{2}.$$

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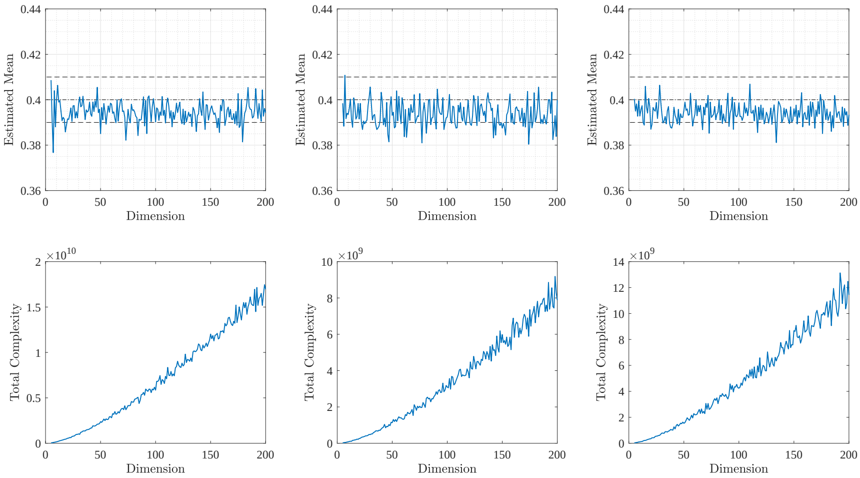
- Pick $\beta = 0.8$ and $E[Y_1(\infty)] = 0.4$.



(a) $\gamma = 0.01$ (b) $\gamma = 0.05$ (c) $\gamma = 0.1$

Figure 1: Simulation results for symmetric RBMs at target error level $\epsilon = 0.01$.

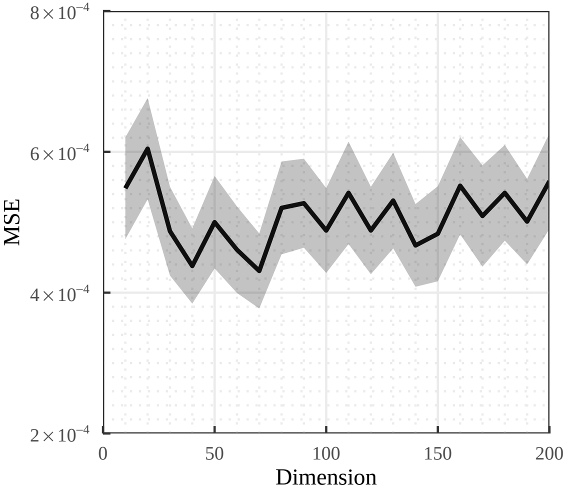


Figure 2: Mean square error of the estimators at target error level $\epsilon = 0.05$ for $\gamma = 0.05$. The shaded area represents 95% confidence band for the MSE.

Reference

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Blanchent, Jose, Xinyun Chen, Peter Glynn, and **Nian Si**. "Efficient Steady-state Simulation of High-dimensional Stochastic Networks." arXiv preprint arXiv:2001.08384 (2020).

Thanks!