26/34

Subject: Time Series , 高念慈 B082040005 No.: HW6 Date: 2023/3/31

10

1. (i) Show ACF ρ_k for the AR(1) process satisfies the difference equation

$$\rho_k - \phi_1 \rho_{k-1} = 0 \quad , \text{ for } k \geq 1$$

(ii) Find the general expression for ρ_k

(i) AR(1) : $\dot{z}_t = \phi_1 \dot{z}_{t-1} + a_t$, $\dot{z}_t = z_t - \mu$

$a_t \sim WN(0, \sigma_a^2)$, $a_t \perp \dot{z}_{t-i}$, $i > 0$

同乘 $\dot{z}_{t-k}$ $\Rightarrow$ $\dot{z}_t \dot{z}_{t-k} = \phi_1 \dot{z}_{t-1} \dot{z}_{t-k} + a_t \cdot \dot{z}_{t-k}$

取期望值 $\Rightarrow$ $E(\dot{z}_t \dot{z}_{t-k}) = \phi_1 E(\dot{z}_{t-1} \dot{z}_{t-k}) + \underbrace{E(a_t)E(\dot{z}_{t-k})}_{0}$ (By indep.)

$\Rightarrow$ $E(z_t - \mu)(z_{t-k} - \mu) = \phi_1 E(z_{t-1} - \mu)(z_{t-k} - \mu) + 0$

$\Rightarrow$ $\gamma_k = \phi_1 \gamma_{k-1}$

同除 γ_0 $\Rightarrow$ $\rho_k = \phi_1 \rho_{k-1}$, $k \geq 1$ ※

(ii) $\rho_k = \phi_1 \rho_{k-1}$

$= \phi_1 (\phi_1 \rho_{k-2}) = \phi_1^2 \rho_{k-2}$

$= \phi_1^3 \rho_{k-3} \cdots = \phi_1^k \rho_0 = \phi_1^k$, $|\phi_1| < 1$ ※

☆ Bartlett's approximation (lecture 2 p.34 p.35)

$Cov(\hat{\rho}_k, \hat{\rho}_{k+j}) \approx \frac{1}{n} \sum_{i=-\infty}^{\infty} (\rho_i \rho_{i+j} + \rho_{i+k+j} \rho_{i-k} - 2\rho_k \rho_i \rho_{i-k-j}$

$k > 0$, $k+j > 0$ $\qquad - 2\rho_{k+j} \rho_i \rho_{i-k} + 2\rho_k \rho_{k+j} \rho_i^2)$

$Var(\hat{\rho}_k) \approx \frac{1}{n} \sum_{-\infty}^{\infty} (\rho_i^2 + \rho_{i+k} \rho_{i-k} - 4\rho_k \rho_i \rho_{i-k} + 2\rho_k^2 \rho_i^2)$

In this case, $\rho_k = \phi_1^k$, $k = 0.1.2 \ldots$

$\Rightarrow Var(\hat{\rho}_k) \approx \frac{1}{n} \sum_{i=-\infty}^{\infty} (\phi_1^{2i} + \phi_1^{2i} - 4\phi_1^{2i} + 2\phi_1^{2k+2i})$ X

$= \frac{1}{n} \sum_{i=-\infty}^{\infty} (+2\phi_1^{2i}(\phi_1^{2k} - 1))$

$= \frac{2}{n} (\phi_1^{2k} - 1) \sum_{i=-\infty}^{\infty} \phi_1^{2i}$

$= \frac{2}{n} (\phi_1^{2k} - 1) \cdot (2 \cdot \sum_{i=0}^{\infty} \phi_1^{2i} - 1)$

$= \frac{2}{n} (\phi_1^{2k} - 1) (\frac{\phi_1^2}{1-\phi_1^2})$ ※

No. :
Date : ..../........./..........
Subject : ....

3. (a) $z_t - 0.5 z_{t-1} = a_t$

$z_t = 0.5 z_{t-1} + a_t$, $\phi_0 = 0$, $\phi_1 = 0.5$, $\mu = \dfrac{\phi_0}{1-\phi_1} = 0$

$\Rightarrow z_t - 0 = 0.5(z_{t-1} - 0) + a_t$

$\Rightarrow \dot{z}_t = 0.5\, \dot{z}_{t-1} + a_t$ ※

(b) $z_t + 0.98 z_{t-1} = a_t$

$z_t = -0.98 z_{t-1} + a_t$, $\phi_0 = 0$, $\phi_1 = -0.98$, $\mu = 0$

$\Rightarrow (z_t - 0) = -0.98(z_t - 0) + a_t$

$\Rightarrow \dot{z}_t = -0.98(\dot{z}_{t-1}) + a_t$ ※

(c) $z_t - 1.3 z_{t-1} + 0.4 z_{t-2} = a_t$

$z_t = 1.3 z_{t-1} - 0.4 z_{t-2} + a_t$, $\phi_1 = 1.3$, $\phi_2 = -0.4$

$(1 - 1.3B + 0.4B^2) = 0$ $\qquad \mu = \dfrac{\phi_0}{1-\phi_1-\phi_2} = 0$

$(-1.3)^2 - 4 \cdot 0.4 = 0.09 > 0$

$$\rho_k = a\left(\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}\right)^k + b\left(\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}\right)^k$$

$$= a \cdot \left(\frac{4}{5}\right)^k + b\left(\frac{1}{2}\right)^k$$

$\rho_0 = 1$, $\rho_1 = \dfrac{\phi_1}{1-\phi_2} = \dfrac{13}{14}$

$$\begin{cases} 1 = a + b \\ \frac{13}{14} = \frac{4}{5}a + \frac{1}{2}b \end{cases} \Rightarrow \begin{cases} a = \frac{10}{7} \\ b = -\frac{3}{7} \end{cases}$$

$\Rightarrow \hat{\rho}_k = \dfrac{10}{7}\left(\dfrac{4}{5}\right)^k + \left(-\dfrac{3}{7}\right)\left(\dfrac{1}{2}\right)^k$, for $k = 0, 1, \cdots, 20$ ※

(d) $z_t - 1.2 z_{t-1} + 0.8 z_{t-2} = a_t$

$z_t = 1.2 z_{t-1} - 0.8 z_{t-2} + a_t$, $\phi_0 = 0$, $\phi_1 = 1.2$, $\phi_2 = -0.8$

$(1 - 1.2B + 0.8B^2) = 0$

$(-1.2)^2 - 4 \cdot (0.8) = -1.76 < 0$

$\Rightarrow \rho_k = c\,\alpha^k \cos k\theta + d\,\alpha^k \sin k\theta$, $\alpha = \sqrt{-\phi_2} = \sqrt{0.8}$

$\cos\theta = \dfrac{\phi_1}{2\sqrt{-\phi_2}} = \dfrac{1.2}{2\sqrt{0.8}} = \dfrac{3\sqrt{5}}{10}$

$$\theta = \cos^{-1}\left(\frac{3\sqrt{5}}{10}\right) \doteq 0.8355$$

$$\rho_0 = 1 , \quad \rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{2}{3}$$

$$\begin{cases} 1 = c , \\ \frac{2}{3} = 1 \cdot \sqrt{\frac{4}{5}} \cos\left(\cos^{-1}\left(\frac{3\sqrt{5}}{10}\right)\right) + d\sqrt{\frac{4}{5}} \sin\left(\cos^{-1}\frac{3\sqrt{5}}{10}\right) \end{cases}$$

$$\Rightarrow \left(\frac{2}{3} - \frac{3}{5}\right) = d \cdot 0.663325 , \quad d \doteq 0.05527708$$

$$\Rightarrow \rho_k = \left(\sqrt{\frac{4}{5}}\right)^k \cdot \cos\left(k \cdot \cos^{-1}\left(\frac{3\sqrt{5}}{10}\right)\right) + 0.05527708 \left(\frac{2}{\sqrt{5}}\right)^k \sin\left(k \cdot \cos^{-1}\left(\frac{3\sqrt{5}}{10}\right)\right)$$

for $k = 0.1 \ldots 20$ #

Time Series HW6

B082040005 高念慈

2023-03-31

2.

8/12

i. Simulate 100 observations from the following AR(1) process:

$$Z_t = 0.5 - 0.5Z_{t-1} + \epsilon_t$$

Where $Z_0 = 0$ and $\epsilon_t\ i.i.d.\ N(0, 0.3)$

Compute the sample ACF $\hat{\rho}_k$ for k=0,1,2,…,10 and PACF $\hat{\phi}_{kk}$ for k=0,1,2,…,10.

- $Z_t = 0.5 - 0.5Z_{t-1} + \epsilon_t$
- $\mu = \frac{\phi 0}{1 - \phi_1} = \frac{0.5}{1 + 0.5} = \frac{1}{3}$
- $Z_t - \frac{1}{3} = -0.5(Z_{t-1} - \frac{1}{3}) + \epsilon_t$
- $\dot{Z}_t = -0.5\dot{Z}_{t-1} + \epsilon_t$
- R中常數項為μ
- $Z_t = \dot{Z}_t + \mu$

模擬

- R 中 Arima.sim() 模型的輸入值是多少？(https://stackoverflow.com/questions/51296915/whats-the-input-value-for-model-in-arima-sim-in-r)
- 使用指定的非零均值和 AR 係數在 R 中模擬 AR(1) 過程 (https://stats.stackexchange.com/questions/305224/simulate-ar1-process-in-r-with-specified-nonzero-mean-and-ar-coefficient)

```
set.seed(20230331)
yt = arima.sim(list(order=c(1,0,0), ar=-0.5, sq=sqrt(0.3)), n=100)  # c(1,0,0):AR模型
ar1_100 = yt + 1/3
head(ar1_100)
```

```
## [1]  0.8794098 -1.9086424  0.7830030  2.4863706 -0.4684484 -0.7201929
```

Compute the sample ACF and PACF

- $\rho_k = (\phi_1)^k$
- $\hat{\phi}_{kk}: \phi_{11} = \rho_1 = \phi_1 = -0.5, other = 0$

```
rho_k = c()

for (i in 0:10)
 rho_k = c(rho_k,(-0.5)^i)

rho_k                          # ACF
```

```
##  [1]  1.0000000000 -0.5000000000  0.2500000000 -0.1250000000  0.0625000000
##  [6] -0.0312500000  0.0156250000 -0.0078125000  0.0039062500 -0.0019531250
## [11]  0.0009765625
```

Compute the sample ACF and PACF

```
acf(ar1_100, type = "correlation", plot = FALSE,lag.max = 10)  # ACF
```

```
##
## Autocorrelations of series 'ar1_100', by lag
##
##      0      1      2      3      4      5      6      7      8      9     10
##  1.000 -0.402  0.137 -0.038 -0.062  0.059 -0.017 -0.042  0.207 -0.131  0.062
```

```
acf(ar1_100, type = "partial", plot = FALSE, lag.max = 10)     # PACF
```

```
##
## Partial autocorrelations of series 'ar1_100', by lag
##
##      1      2      3      4      5      6      7      8      9     10
## -0.402 -0.030  0.008 -0.085  0.002  0.020 -0.055  0.198  0.040 -0.002
```

ii. Repeating the procedure(程序) of (i) for 1000 times, find the values of the mean, variance and covariance of $\hat{\rho}_1, \hat{\rho}_2$ and $\hat{\rho}_3$

mean, variance and covariance

```
set.seed(20220328)

M = 1000

rho_1 = rep(0,M);rho_2 = rep(0,M);rho_3 = rep(0,M)

for (i in 1:M){
  yt_1000 = arima.sim(list(order=c(1,0,0), ar=-0.5, sq=sqrt(0.3)), n=100)  # c(1,0,0):AR模型
  ar1_1000 = yt_1000 + 1/3

  ar100_acf = acf(ar1_1000, type = "correlation", plot = FALSE,lag.max = 3)[[1]]  # ACF
  rho_1[i] = ar100_acf[2]
  rho_2[i] = ar100_acf[3]
  rho_3[i] = ar100_acf[4]
}

cbind(mean_rho = c(mean(rho_1),mean(rho_2),mean(rho_3)),
      var_rho = c(var(rho_1),var(rho_2),var(rho_3)))
```

```
##         mean_rho     var_rho
## [1,] -0.4937733 0.007009579
## [2,]  0.2347892 0.012109305
## [3,] -0.1179250 0.013892444
```

covariance

```
rbind(co_rho12 = cov(rho_1,rho_2),
      co_rho23 = cov(rho_2,rho_3),
      co_rho13 = cov(rho_1,rho_3))
```

```
##                   [,1]
## co_rho12 -0.006991068
## co_rho23 -0.010039551
## co_rho13  0.005001310
```

iii. Compare the result of (ii) with the Bartlett's approximation.

from HW5

- $Var(\hat{\rho}_k) \approx \frac{2}{n}(\phi_1^{2k}-1)(\frac{\phi_1^2}{1-\phi_1^2})$

```
phi1 = -0.5
n = 1000

var_approx = function(k){
  return((2/n)*(phi1^(2*k)-1)*(phi1^2/(1-phi1^2)))
}

rbind(var_rho_1=var_approx(1),
      var_rho_2=var_approx(2),
      var_rho_3=var_approx(3))
```

```
##                    [,1]
## var_rho_1 -0.00050000
## var_rho_2 -0.00062500
## var_rho_3 -0.00065625
```

- 隨著 n 越大應該是要越接近的
- mean_rho1: -0.4937733 var_rho1: 0.007009579
- mean_rho2: 0.2347892 var_rho2: 0.012109305
- mean_rho3: -0.1179250 var_rho3: 0.013892444
- co_rho12: -0.006991068
- co_rho23: -0.010039551
- co_rho13: 0.005001310

8/12

3.

Simulate a series of 100 observations from each of the following models where the a_t is a Gaussian white noise process with $E(a_t) = 0$ and $Var(a_t) = 1$:

$$(a) Z_t - 0.5Z_{t-1} = a_t$$

$$(b) Z_t + 0.98Z_{t-1} = a_t$$

$$(c) Z_t - 1.3Z_{t-1} + 0.4Z_{t-2} = a_t$$

$$(d) Z_t - 1.2Z_{t-1} + 0.8Z_{t-2} = a_t$$

For each case, plot the simulated series, and calculate and study its sample ACF $\hat{\rho}_k$ for k=0,1,2,…,20 and PACF $\hat{\phi_{kk}}$ for k=0,1,2,…,20.

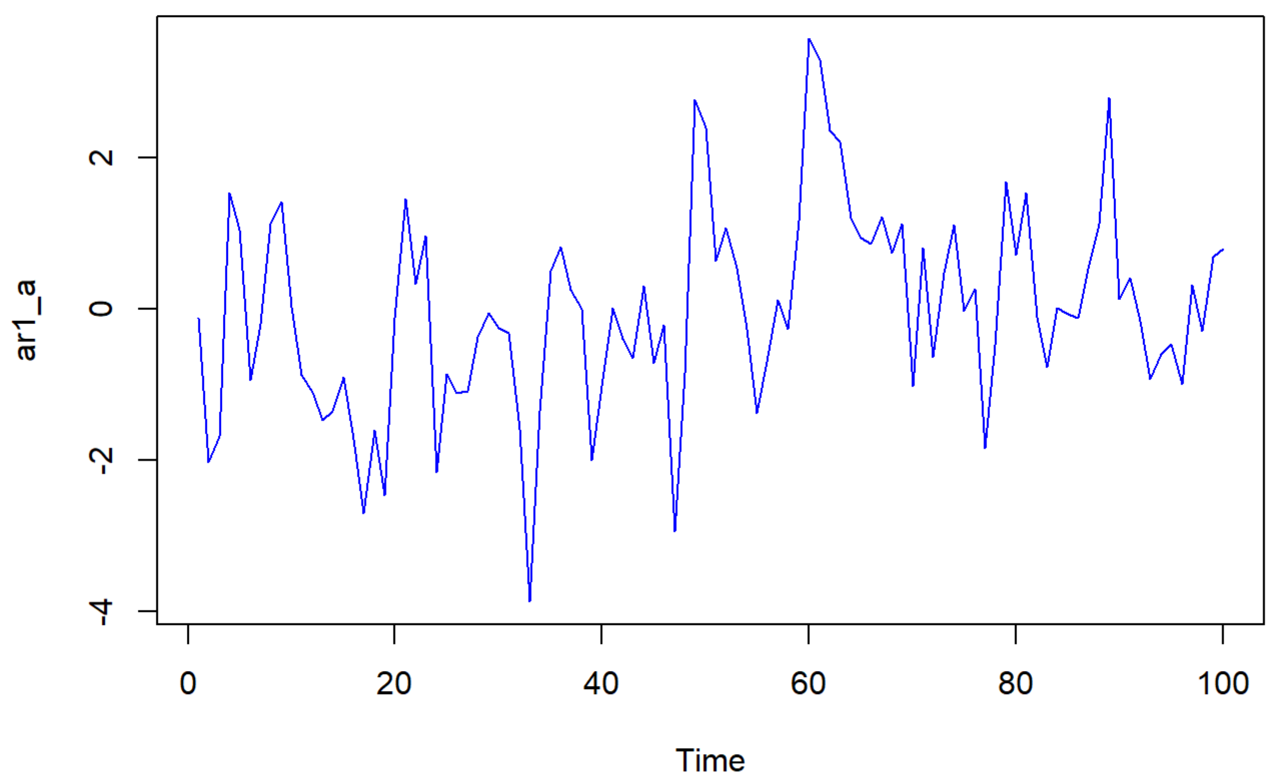
模擬(a)

```
set.seed(20230331)
yta = arima.sim(list(order=c(1,0,0), ar=0.5, sq=1), n=100)  # c(1,0,0):AR模型
ar1_a = yta + 0
head(ar1_a)
```

```
## [1] -0.1147034 -2.0262893 -1.6844628  1.5356407  1.0425573 -0.9331384
```

plot

```
ts.plot(ar1_a,col="blue")
```



Compute the sample ACF and PACF

```
acf(ar1_a, type = "correlation", plot = FALSE,lag.max = 20)  # ACF
```

```
##
## Autocorrelations of series 'ar1_a', by lag
##
##      0      1      2      3      4      5      6      7      8      9     10
##  1.000  0.534  0.210  0.073  0.005  0.058  0.069  0.139  0.215  0.112  0.106
##     11     12     13     14     15     16     17     18     19     20
##  0.106  0.087  0.167  0.234  0.184  0.072 -0.013  0.003  0.003  0.019
```

```
acf(ar1_a, type = "partial", plot = FALSE, lag.max = 20)     # PACF
```

```
##
## Partial autocorrelations of series 'ar1_a', by lag
##
##      1      2      3      4      5      6      7      8      9     10     11
##  0.534 -0.104  0.007 -0.027  0.103 -0.005  0.130  0.114 -0.094  0.105  0.033
##     12     13     14     15     16     17     18     19     20
##  0.017  0.133  0.130 -0.050 -0.070 -0.002  0.009 -0.043  0.017
```

結論?

- $\rho_k = (\phi_1)^k$
- $\hat{\phi_{kk}} : \phi_{11} = \rho_1 = \phi_1 = 0.5, other = 0$

```
rho_k_a = c()

for (i in 0:20)
 rho_k_a = c(rho_k_a,(0.5)^i)

rho_k_a                          # ACF
```

```
##  [1] 1.000000e+00 5.000000e-01 2.500000e-01 1.250000e-01 6.250000e-02
##  [6] 3.125000e-02 1.562500e-02 7.812500e-03 3.906250e-03 1.953125e-03
## [11] 9.765625e-04 4.882812e-04 2.441406e-04 1.220703e-04 6.103516e-05
## [16] 3.051758e-05 1.525879e-05 7.629395e-06 3.814697e-06 1.907349e-06
## [21] 9.536743e-07
```

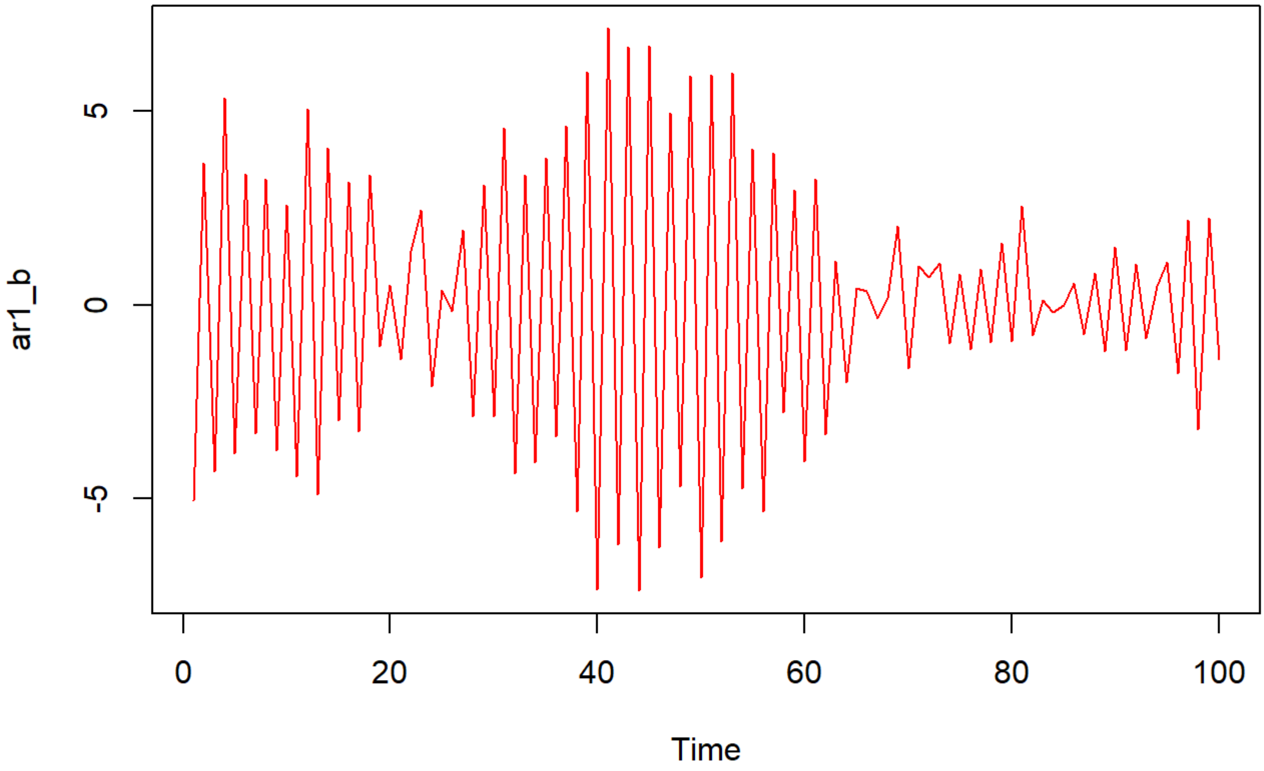
模擬(b)

```
set.seed(20230331)
ytb = arima.sim(list(order=c(1,0,0), ar=-0.98, sq=1), n=100)  # c(1,0,0):AR模型
ar1_b = ytb + 0
head(ar1_b)
```

```
## [1] -5.043674  3.661535 -4.293398  5.334082 -3.833109  3.386770
```

plot

```
ts.plot(ar1_b,col="red")
```



Compute the sample ACF and PACF

```
acf(ar1_b, type = "correlation", plot = FALSE,lag.max = 20)  # ACF
```

```
##
## Autocorrelations of series 'ar1_b', by lag
##
##      0      1      2      3      4      5      6      7      8      9     10
##  1.000 -0.949  0.908 -0.881  0.852 -0.812  0.773 -0.743  0.718 -0.682  0.643
##     11     12     13     14     15     16     17     18     19     20
## -0.597  0.552 -0.502  0.453 -0.407  0.365 -0.329  0.276 -0.230  0.190
```

```
acf(ar1_b, type = "partial", plot = FALSE, lag.max = 20)     # PACF
```

```
##
## Partial autocorrelations of series 'ar1_b', by lag
##
##      1      2      3      4      5      6      7      8      9     10     11
## -0.949  0.069 -0.137 -0.014  0.105 -0.037 -0.046  0.049  0.092 -0.053  0.143
##     12     13     14     15     16     17     18     19     20
## -0.056  0.085 -0.023  0.012  0.004 -0.053 -0.193 -0.022 -0.026
```

- $\rho_k = (\phi_1)^k$
- $\hat{\phi}_{kk} : \phi_{11} = \rho_1 = \phi_1 = -0.98, other = 0$

结論?

```
rho_k_b = c()

for (i in 0:20)
 rho_k_b = c(rho_k_b,(-0.98)^i)

rho_k_b                         # ACF
```

```
##  [1]  1.0000000 -0.9800000  0.9604000 -0.9411920  0.9223682 -0.9039208
##  [7]  0.8858424 -0.8681255  0.8507630 -0.8337478  0.8170728 -0.8007314
## [13]  0.7847167 -0.7690224  0.7536419 -0.7385691  0.7237977 -0.7093218
## [19]  0.6951353 -0.6812326  0.6676080
```

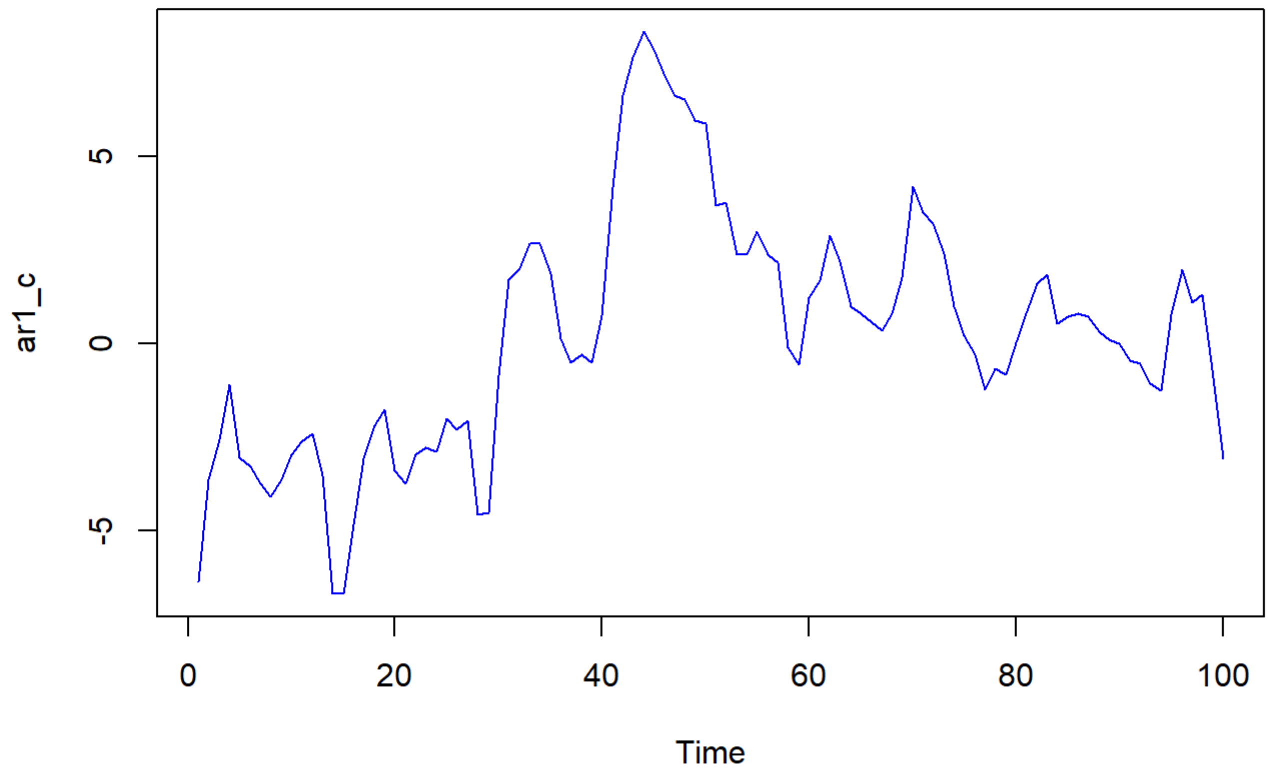
模擬(c)

```
set.seed(20230331)
ytc = arima.sim(list(order=c(2,0,0), ar=c(1.3,-0.4), sq=1), n=100)  # c(1,0,0):AR模型
ar1_c = ytc + 0
head(ar1_c)
```

```
## [1] -6.345467 -3.619559 -2.569870 -1.089690 -3.033029 -3.286991
```

plot

```
ts.plot(ar1_c,col="blue")
```



Compute the sample ACF and PACF

```
acf(ar1_c, type = "correlation", plot = FALSE,lag.max = 20)  # ACF
```

```
##
## Autocorrelations of series 'ar1_c', by lag
##
##     0     1     2     3     4     5     6     7     8     9    10    11    12
## 1.000 0.904 0.782 0.675 0.604 0.564 0.545 0.528 0.504 0.465 0.427 0.393 0.369
##    13    14    15    16    17    18    19    20
## 0.345 0.288 0.214 0.134 0.074 0.047 0.033 0.012
```

```
acf(ar1_c, type = "partial", plot = FALSE, lag.max = 20)     # PACF
```

```
##
## Partial autocorrelations of series 'ar1_c', by lag
##
##      1      2      3      4      5      6      7      8      9     10     11
##  0.904 -0.187  0.026  0.119  0.087  0.079  0.008  0.003 -0.046  0.030  0.000
##     12     13     14     15     16     17     18     19     20
##  0.020 -0.031 -0.201 -0.083 -0.083  0.012  0.049 -0.041 -0.076
```

- $AR(2): Z_t = \phi_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$
- $\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}, k \geq 2$
- $\rho_0 = 1, \rho_1 = \frac{\phi_1}{1-\phi_2}$
- $\hat{\phi_{kk}} : \phi_{11} = \phi_1 = 1.3, \phi_{22} = \phi_2 = -0.4, other = 0$
- $\hat{\rho}_k = \frac{10}{7}(\frac{4}{5})^k - \frac{3}{7}(\frac{1}{2})^k, k = 0, 1, \ldots$

```
rho_k_c = c()

for (i in 0:20)
 rho_k_c =  c(rho_k_c,(10/7)*(0.8)^i+(-3/7)*(0.5)^i)

rho_k_c                         # ACF
```

```
##  [1] 1.00000000 0.92857143 0.80714286 0.67785714 0.55835714 0.45472143
##  [7] 0.36779500 0.29624493 0.23800041 0.19090256 0.15297316 0.12250409
## [13] 0.09806605 0.07848423 0.06280308 0.05025031 0.04020417 0.03216530
## [19] 0.02573322 0.02058707 0.01646990
```

結論？

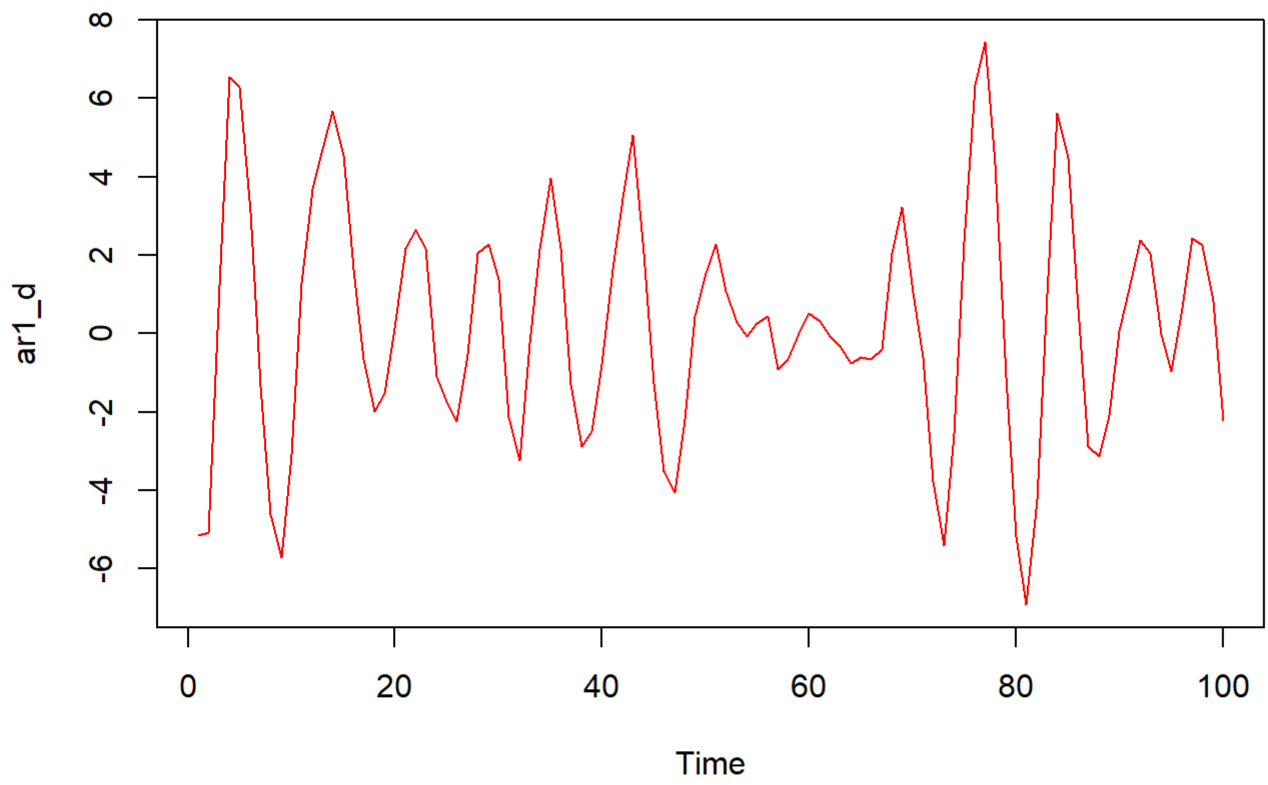
模擬(d)

```
set.seed(20230331)
ytd = arima.sim(list(order=c(2,0,0), ar=c(1.2,-0.8), sq=1), n=100)  # c(1,0,0):AR模型
ar1_d = ytd + 0
head(ar1_d)
```

```
## [1] -5.142741 -5.060436  1.225517  6.537260  6.299656  3.091521
```

plot

```
ts.plot(ar1_d,col="red")
```



Compute the sample ACF and PACF

```
acf(ar1_d, type = "correlation", plot = FALSE,lag.max = 20)  # ACF
```

```
##
## Autocorrelations of series 'ar1_d', by lag
##
##      0      1      2      3      4      5      6      7      8      9     10
##  1.000  0.667 -0.024 -0.609 -0.769 -0.476  0.033  0.442  0.535  0.304 -0.047
##     11     12     13     14     15     16     17     18     19     20
## -0.290 -0.313 -0.155  0.037  0.150  0.131  0.040 -0.044 -0.073 -0.044
```

```
acf(ar1_d, type = "partial", plot = FALSE, lag.max = 20)     # PACF
```

```
##
## Partial autocorrelations of series 'ar1_d', by lag
##
##      1      2      3      4      5      6      7      8      9     10     11
##  0.667 -0.844 -0.105 -0.129 -0.091 -0.079 -0.039 -0.072 -0.069  0.108  0.009
##     12     13     14     15     16     17     18     19     20
##  0.025  0.012 -0.072  0.027 -0.102  0.043 -0.087 -0.036 -0.024
```

- $AR(2) : Z_t = \phi_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$
- $\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}, k \geq 2$
- $\rho_0 = 1, \rho_1 = \frac{\phi_1}{1-\phi_2}$
- $\hat{\phi_{kk}} : \phi_{11} = \phi_1 = 1.2, \phi_{22} = \phi_2 = -0.8, other = 0$

```
rho_k_d <- c()

d = (2/3-3/5)/sqrt(4/5)*sin(acos(3*sqrt(5)/10))

for (i in 0:20){
   rho_k_d <- c(rho_k_d,(2*sqrt(5)/5)^i*(cos(i*acos(3*sqrt(5)/10)))+d*(2*sqrt(5)/5)
^i*sin(i*acos(3*sqrt(5)/10)))
}

rho_k_d
```

```
##  [1]  1.000000000  0.636666667 -0.036000000 -0.552533333 -0.634240000
##  [6] -0.319061333  0.124518400  0.404671147  0.385990656  0.139451870
## [11] -0.141450281 -0.281301833 -0.224401975 -0.044240903  0.126432496
## [16]  0.187111718  0.123388065 -0.001623697 -0.100658888 -0.119491708
## [21] -0.062862939
```

結論？