B08204000５ 高念慈

HW 4 . Time series , 2023.3.17

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1. $Z_t$ is a sequence of independent rv's defined as

$$\begin{cases} X_t \sim Z_{t-1}, & t: \text{odd} \\ Y_t \sim P(Y_t = 1) = P(Y_t = -1) = \frac{1}{2}, & t: \text{even} \end{cases}$$

(a) Is the process first order stationary in distribution?

(b) Is it second order stationary in distribution?

(a) Since $Z_t$ has the same distribution for all $t$, it is first order stationary ※

$( P(Z_t = \pm 1) = P(Z_{t-1} = \pm 1) = \frac{1}{2} , \forall t \in \mathbb{N}$ ※ $)$

(b) No ※ Because correlation change with time ※

時間序列七通常從一開始.

For example, $corr(Z_0, Z_1) = Corr(Z_0, Z_0) = 1$

$corr(Z_1, Z_2) = Corr(Z_0, Z_2) = 0$ (indep.)

Correlation 是從分佈來的，故 Correlation 不一樣分佈也不一樣 ※

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2. $Z_t = U \sin(2\pi w t) + V \cos(2\pi w t)$
$\begin{cases} U \perp\!\!\!\perp V \quad rv's \\ E(U) = E(V) = 0 . \\ Var(U) = Var(V) = 1 \end{cases}$

(a) $Z_t$ strictly stationary?

(b) $Z_t$ covariance stationary?

題目沒給.

~~(a)~~ $w \in \mathbb{Z}$, fixed, $t \in \mathbb{N}$, $k \in \mathbb{N}$ -3

$w = \frac{1}{4}$, $t = 1$, $k = 1$

$P(Z_1 < z) \neq P(Z_2 < z)$.

$U \sim N(0,1)$, $V \sim U(-\sqrt{3}, \sqrt{3})$

Goal: $(Z_{t_1} \cdots Z_{t_n}) \overset{D}{=} (Z_{t_1+k}, \ldots Z_{t_n+k})$

$\because \sin(2\pi w t) = \sin(2\pi w (t+k)) = \sin(2\pi) = 0$
$\cos(2\pi w t) = \cos(2\pi w (t+k)) = \cos(2\pi) = 1$ , $\forall k, t \in \mathbb{N}$

$\therefore Z_1 \cdots Z_n \sim V(0, 1)$ (不獨立，同分佈)

$\Rightarrow (Z_{t_1} \cdots Z_{t_n}) \overset{D}{=} (Z_{t_1+k}, \cdots Z_{t_n+k})$ is strictly stationary ※

(b) Constant Mean & Variance & Covariance 跟 t 無關

I. $E(Z_t) = \sin(2\pi w t) E(U) + \cos(2\pi w t) E(V) = 0$ ※

II. $Cov(Z_t, Z_{t+k}) = E(Z_t \cdot Z_{t+k}) - \underbrace{E(Z_t) \cdot E(Z_{t+k})}_{0 \quad \cdot \quad 0}$

求 $E(Z_t \cdot Z_{t+k}) = E([U \cdot \sin(2\pi w t) + V \cos(2\pi w t)]$
$[U \cdot \sin(2\pi w (t+k)) + V \cos(2\pi w (t+k))])$

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$= E( U^2 \cdot \sin(2\pi w t) \cdot \sin(2\pi w(t+k)) + V^2 \cos(2\pi w t) \cdot \cos(2\pi w(t+k))$
$+ UV \cdot \sin(2\pi w t) \cdot \cos(2\pi w(t+k)) + VU \cos(2\pi w t) \cdot \sin(2\pi w(t+k)) )$

$= \cancel{0 + E(V^2)} + \left.\begin{array}{l} \sin(2\pi w t) \cdot \cos(2\pi w(t+k)) E(UV) \\ + \cos(2\pi w t) \cdot \sin(2\pi w(t+k)) E(VU) \end{array}\right\} \to 0$

$E(V^2) = Var(V) + (E(V))^2 = 1$, $E(VU) = E(UV) \overset{ind.}{=} E(U)E(V) = 0$

$= \cancel{1}$ (與 t 無關) $\Rightarrow$ Covariance is free of t ※

$$\boxed{\begin{array}{l}\big( \sin(2\pi w t) \cdot \sin(2\pi w(t+k)) + \cos(2\pi w t) \cdot \cos(2\pi w(t+k)) \\ = \cos(2\pi w t - 2\pi w(t+k)) \big) = \cos(2\pi w k) \text{ 與 t 無關}, w \in \mathbb{R}, k \in \mathbb{N} \big)\end{array}}$$

Thus, $Z_t$ is Covariance stationary ※  -1  X.

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3. Verify (a) $\rho_0 = 1$ (b) $|\rho_k| \le 1$ (c) $\rho_k = \rho_{-k}$ of a stationary process

(a) $\rho_k = \dfrac{Cov(Z_t, Z_{t+k})}{\sqrt{Var(Z_t)}\sqrt{Var(Z_{t+k})}} = \dfrac{\gamma_k}{\gamma_0}$, when $k=0$, $\rho_0 = \dfrac{\gamma_0}{\gamma_0} = 1$ ※

(b) By Cauchy - Schwarz inequality $( |E(XY)| \le \sqrt{E(X^2)}\sqrt{E(Y^2)} )$

$\Rightarrow |E((Z_t-\mu)(Z_{t+k}-\mu))| \le \sqrt{E((Z_t-\mu)^2)}\sqrt{E[(Z_{t+k}-\mu)^2]}$

$= \sqrt{Var(Z_t)} \cdot \sqrt{Var(Z_{t+k})} = \gamma_0$

$\Rightarrow |Cov(Z_t, Z_{t+k})| = |\gamma_k| \le \gamma_0$

$\Rightarrow \dfrac{|\gamma_k|}{\gamma_0} = |\rho_k| \le 1$ ※

(c) $\rho_k = \dfrac{Cov(Z_t, Z_{t+k})}{\sqrt{Var(Z_t)}\sqrt{Var(Z_{t+k})}} = \dfrac{Cov(Z_{t+k}, Z_t)}{\sqrt{Var(Z_t)}\sqrt{Var(Z_{t+k})}} = \rho_{-k}$ ※

4. Prove or disprove the processes are covariance stationary

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(a) $Z_t = A \cdot \sin(2\pi w t + \theta)$, A is constant, $\theta$ : r.v. $\sim U(0, 2\pi)$

(b) $Z_t = A \cdot \sin(2\pi w t + \theta)$, $\theta$ is constant, A : r.v. $\sim A(0, \text{unit Var})$

(c) $Z_t = (-1)^t \cdot A$, A : r.v. $\sim A(0, \underbrace{\text{unit Var}}_{1})$

Double A

(a) $E(Z_t) = A \cdot E(\sin(2\pi wt + \theta))$

$= A \cdot \int_0^{2\pi} \sin(2\pi wt + \theta) \cdot \frac{1}{2\pi} \, d\theta$

$= \frac{A}{2\pi} \cdot (-\cos(2\pi wt + \theta))\Big|_0^{2\pi}$

$= \frac{A}{2\pi} \cdot (-\cos(2\pi wt) + \cos(2\pi wt)) = 0$

$Cov(Z_t, Z_{t+k}) = E(Z_t \cdot Z_{t+k}) - E(Z_t) \cdot E(Z_{t+k})$

$= E(A \cdot \sin(2\pi wt + \theta) \cdot A \cdot \sin(2\pi w(t+k) + \theta)) - 0$

$= A^2 E(\sin(2\pi wt + \theta) \cdot \sin(2\pi w(t+k) + \theta))$

$= A^2 E[\frac{1}{2}(\cos(2\pi wk) - \cos(2\pi w(2t+k) + 2\theta))]$

$= A^2 \frac{1}{2}\{\cos(2\pi wk) - \underbrace{E[\cos(2\pi w(2t+k) + 2\theta)]}_{I}\}$

$I = \int_0^{2\pi} \cos(2\pi w(2t+k) + 2\theta) \cdot \frac{1}{2\pi} \, d\theta = \frac{1}{2\pi} \cdot \sin(2\pi w(2t+k) + 2\theta) \cdot \frac{1}{2} \Big|_0^{2\pi}$

$= \frac{1}{4\pi} \cdot (\sin(2\pi w(2t+k)) - \sin(2\pi w(2t+k))) = 0$

$= A^2 \cdot \frac{1}{2} \cdot \cos(2\pi wk)$ free of $t$

$\Rightarrow$ covariance stationary #

(b) $E(Z_t) = \sin(2\pi wt + \theta) \, E(A) = 0$

$Cov(Z_t, Z_{t+k}) = E(Z_t \cdot Z_{t+k}) - E(Z_t) \cdot E(Z_{t+k})$

$= E(A \cdot \sin(2\pi wt + \theta) \cdot A \cdot \sin(2\pi w(t+k) + \theta)) - 0$

$= \sin(2\pi wt + \theta) \cdot \sin(2\pi w(t+k) + \theta) \cdot E(A^2)$

$E(A^2) = Var(A) + (E(A))^2 = Var(A) = 1$

$\Rightarrow$ 跟 $t$ 有関，故 Not covariance stationary #

(c) $E(Z_t) = (-1)^t E(A) = 0$

$Cov(Z_t, Z_{t+k}) = E(Z_t \cdot Z_{t+k}) - E(Z_t) \cdot E(Z_{t+k})$

$= E((-1)^t \cdot A \cdot (-1)^{t+k} \cdot A) - 0$

$= (-1)^{2t+k} \cdot E(A^2) = (-1)^{2t+k} = (-1)^k$

$E(A^2) = Var(A) + (E(A))^2 = 1$

$\Rightarrow$ 跟 $t$ 無関，故 Covariance stationary #