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35/40

No.: HW1

Subject: Time Series

Date: 2023/4/7

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1. Find the autocovariance generating function

(a) $Z_t = \mu + a_t - 0.5a_{t-1} + 2a_{t-2} + 3a_{t-3}$, $a_t \sim WN(0,1)$

(b) $Z_t = 0.5Z_{t-1} + a_t$, $a_t \sim WN(0, \sigma_a^2)$

(a) MA(3), $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$

$= 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3$

$r(B) = \sum_{k=-\infty}^{\infty} r_k B^k = \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} \psi_j \psi_{j+k} \sigma_a^2 B^k$

$= \psi(B) \cdot \psi(B^{-1}) \sigma_a^2$

$\Rightarrow r(B) = (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3)(1 + \psi_1 B^{-1} + \psi_2 B^{-2} + \psi_3 B^{-3}) \cdot 1^2$

$= (1 - 0.5B + 2B^2 + 3B^3)(1 - 0.5B^{-1} + 2B^{-2} + 3B^{-3}) \cdot 1$

$= 3B^3 + 0.5B^2 + 4.5B + 14.25 + 4.5B^{-1} + 0.5B^{-2} + 3B^{-3}$ #

(b) AR(1), $\pi(B)\dot{Z}_t = a_t \Rightarrow \dot{Z}_t = \frac{1}{\pi(B)} a_t$, $\mu = 0$

$\Rightarrow (1 - 0.5B) Z_t = a_t$

?泰勒展開?

$Z_t = \frac{1}{1-0.5B} a_t = \sum_{i=0}^{\infty} (0.5B)^i \cdot a_t$

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$\Rightarrow r(B) = (\sum_{i=0}^{\infty} (0.5B)^i)(\sum_{i=0}^{\infty} (0.5B)^{-i}) \cdot \sigma_a^2$ # ✗ −5

2. AR(2) models : (i) $Z_t - 0.6Z_{t-1} - 0.3Z_{t-2} = a_t$

(ii) $Z_t - 0.8Z_{t-1} + 0.5Z_{t-2} = a_t$

(a) Find general expression for ρ_k

(b) Plot the ρ_k, for $k = 0, 1, \dots 10$

(c) Calculate σ_z^2 by assuming $\sigma_a^2 = 1$

(i) $\underbrace{(1 - 0.6B - 0.3B^2)}_{\text{characteristic equation}} Z_t = a_t$, $\rho_0 = 1$, $\rho_1 = \frac{\phi_1}{1-\phi_2}$, $\phi_1 = 0.6$, $\phi_2 = 0.3$

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$, $k \geq 2$

(a) $1 - 0.6B - 0.3B^2 = 0$

$0.36 + 4 \cdot 0.3 = 1.56 > 0$

$\rho_k = a\left(\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}\right)^k + b\left(\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}\right)^k$

$= a\left(\frac{3 + \sqrt{39}}{10}\right)^k + b\left(\frac{3 - \sqrt{39}}{10}\right)^k$, $\rho_1 = \frac{0.6}{0.7} = \frac{6}{7}$

$\Rightarrow \begin{cases} 1 = a + b \\ \frac{6}{7} = a\left(\frac{3+\sqrt{39}}{10}\right) + b\left(\frac{3-\sqrt{39}}{10}\right) \end{cases} \Rightarrow \begin{cases} b = \frac{7-\sqrt{39}}{14}, \quad a = \frac{7+\sqrt{39}}{14} \end{cases}$

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$$\Rightarrow \rho_k = \left(\frac{7+\sqrt{39}}{14}\right)\left(\frac{3+\sqrt{39}}{10}\right)^k + \left(\frac{7-\sqrt{39}}{14}\right)\left(\frac{3-\sqrt{39}}{10}\right)^k, \quad k=0,1\ldots \quad \#$$

(c) $Var(Z_t) = \gamma_0 = \sigma_Z^2, \quad \phi_1 = 0.6, \quad \phi_2 = 0.3$

$$\begin{cases} \gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + \sigma_a^2 + 2\phi_1\phi_2\gamma_1 \\ \gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_0 = 0.36\gamma_0 + 0.09\gamma_0 + 1 + 0.36\gamma_1 \\ \gamma_1 = 0.6\gamma_0 + 0.3\gamma_1 \end{cases}$$

$$\Rightarrow \begin{cases} (1-0.45)\gamma_0 = 1 + 0.36\gamma_1 \\ \gamma_1 = \frac{0.6\gamma_0}{0.7} \end{cases}, \quad \gamma_1 = \frac{0.55\gamma_0 - 1}{0.36}, \quad \gamma_0 = \frac{1+0.36\gamma_1}{0.55}$$

$$\Rightarrow \frac{0.55\gamma_0 - 1}{0.36} = \frac{0.6\gamma_0}{0.7} \quad \Rightarrow \gamma_0 = \frac{700}{169} \doteqdot 4.1420 \quad \#$$

(ii) $(1-0.8B+0.5B^2)Z_t = a_t, \quad \phi_1 = 0.8, \quad \phi_2 = -0.5$

$$\rho_0 = 1, \quad \rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{8}{15}$$

(a) characteristic equation : $(1-0.8B+0.5B^2) = 0$

$$0.64 - 4\cdot 0.5 = -1.36 < 0$$

$$\Rightarrow \rho_k = c\cdot\alpha^k\cos(k\theta) + d\,\alpha^k\sin(k\theta), \quad \alpha = \sqrt{-\phi_2} = \frac{\sqrt{2}}{2}$$

$$\cos\theta = \frac{\phi_1}{2\sqrt{-\phi_2}} = \frac{2\sqrt{2}}{5}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2\sqrt{2}}{5}\right) \doteqdot 0.9695321 \qquad \sin\theta = \frac{\sqrt{17}}{5}$$

$$\Rightarrow \begin{cases} 1 = c \\ \frac{8}{15} = \frac{\sqrt{2}}{2}\cdot\frac{2\sqrt{2}}{5} + d\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{17}}{5} \end{cases}, \quad d = \frac{2\sqrt{34}}{51}$$

$$\Rightarrow \rho_k = \left(\frac{\sqrt{2}}{2}\right)^k\cos\left(k\cdot\cos^{-1}\left(\frac{2\sqrt{2}}{5}\right)\right) + \frac{2\sqrt{34}}{51}\cdot\left(\frac{\sqrt{2}}{2}\right)^k\sin\left(k\cdot\cos^{-1}\left(\frac{2\sqrt{2}}{5}\right)\right), \quad k=0,1\ldots \quad ※$$

(c) $$\begin{cases} \gamma_0 = 0.64\gamma_0 + 0.25\gamma_0 + 1 + (-0.4)\gamma_1\cdot 2 \\ \gamma_1 = 0.8\gamma_0 + (-0.5)\gamma_1 \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_1 = \frac{0.11\gamma_0 - 1}{-0.8} \\ \gamma_1 = \frac{0.8\gamma_0}{1.5} \end{cases}, \quad \frac{0.11\gamma_0 - 1}{-0.8} = \frac{0.8\gamma_0}{1.5}, \quad \gamma_0 = \frac{1500}{805} \doteqdot 1.86335 \quad ※$$

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3. Given AR(2) : $Z_t = Z_{t-1} - 0.25 Z_{t-2} + a_t$

(a) Calculate ρ_1

(b) Use $\rho_0 . \rho_1$ as starting values and the difference equation to obtain ρ_k

(c) Calculate the value ρ_k for $k = 1.2 \dots 10$

$(1 - B + 0.25 B^2) = 0$, $\rho_0 = 1$, $\rho_1 = \frac{\phi_1}{1-\phi_2}$, $\phi_1 = 1$, $\phi_2 = -0.25$

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$, $k \geq 2$

(a) $\rho_1 = \frac{1}{1.25} = \frac{4}{5}$ ※

(b) $1 - 4 \cdot 0.25 = 0$

$B^{-1} = \frac{1}{0.5} = 2$ (重根)

$\Rightarrow \rho_k = \frac{1}{2}^k (b_1 + b_2 k)$

$\Rightarrow \begin{cases} 1 = b_1 \\ \frac{4}{5} = \frac{1}{2}(1 + b_2) \end{cases}$, $b_2 = \frac{3}{5}$

$\Rightarrow \rho_k = (\frac{1}{2})^k (1 + \frac{3}{5} k)$, $k = 0.1.2 \dots$ ※

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4. Derive the stationary region of ϕ_1 and ϕ_2 for AR(2) : $(1 - \phi_1 B - \phi_2 B^2) \dot{Z}_t = a_t$

$\{Z_t\}$ stationary $\Leftrightarrow (1 - \phi_1 B - \phi_2 B^2) = 0$ 的根落在單位圓外 (|根| > 1)

$$\begin{cases} B_1 = \dfrac{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \\ B_2 = \dfrac{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \end{cases} \Rightarrow \begin{cases} B_1^{-1} = \dfrac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \\ B_2^{-1} = \dfrac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} \end{cases}$$

Z_t 有平穩解 $\Leftrightarrow |B_i^{-1}| < 1$, $i = 1.2$

考慮 $x^2 - \phi_1 x - \phi_2 = 0$ 的根與係數 $\begin{cases} 1 > |B_1^{-1} \cdot B_2^{-1}| = |\phi_2| \\ 2 > |B_1^{-1} + B_2^{-1}| = |\phi_1| \end{cases}$

故 $\begin{cases} -1 < \phi_2 < 1 \\ -2 < \phi_1 < 2 \end{cases}$ 為平穩的必要條件

(I) 實根時, $\phi_1^2 + 4\phi_2 \geq 0$, 則 $-1 < B_2^{-1} \leq B_1^{-1} < 1 \Rightarrow \begin{cases} \phi_2 + \phi_1 < 1 \\ \phi_2 - \phi_1 < 1 \end{cases}$

$-1 < \dfrac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$ $\Rightarrow \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} > -2$

$\Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < \phi_1 + 2$

$\Rightarrow \phi_2 - \phi_1 < 1$ ※

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1 \quad \Rightarrow \phi_1 + \sqrt{\phi_1^2+4\phi_2} < 2$$

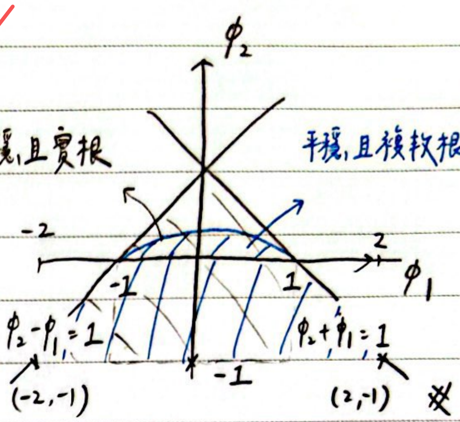
$$\Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

$$\Rightarrow \phi_2 + \phi_1 < 1 \quad ※$$

(II) 虛根時, $\phi_1^2 + 4\phi_2 < 0$

Thus, stationary region of AR(2)

$$\begin{cases} \phi_2 - \phi_1 < 1 \\ \phi_2 + \phi_1 < 1 \\ -1 < \phi_2 < 1 \end{cases} \quad ※$$



Time Series HW7

B082040005 高念慈

2023-04-07

2. Consider the following AR(2) models:

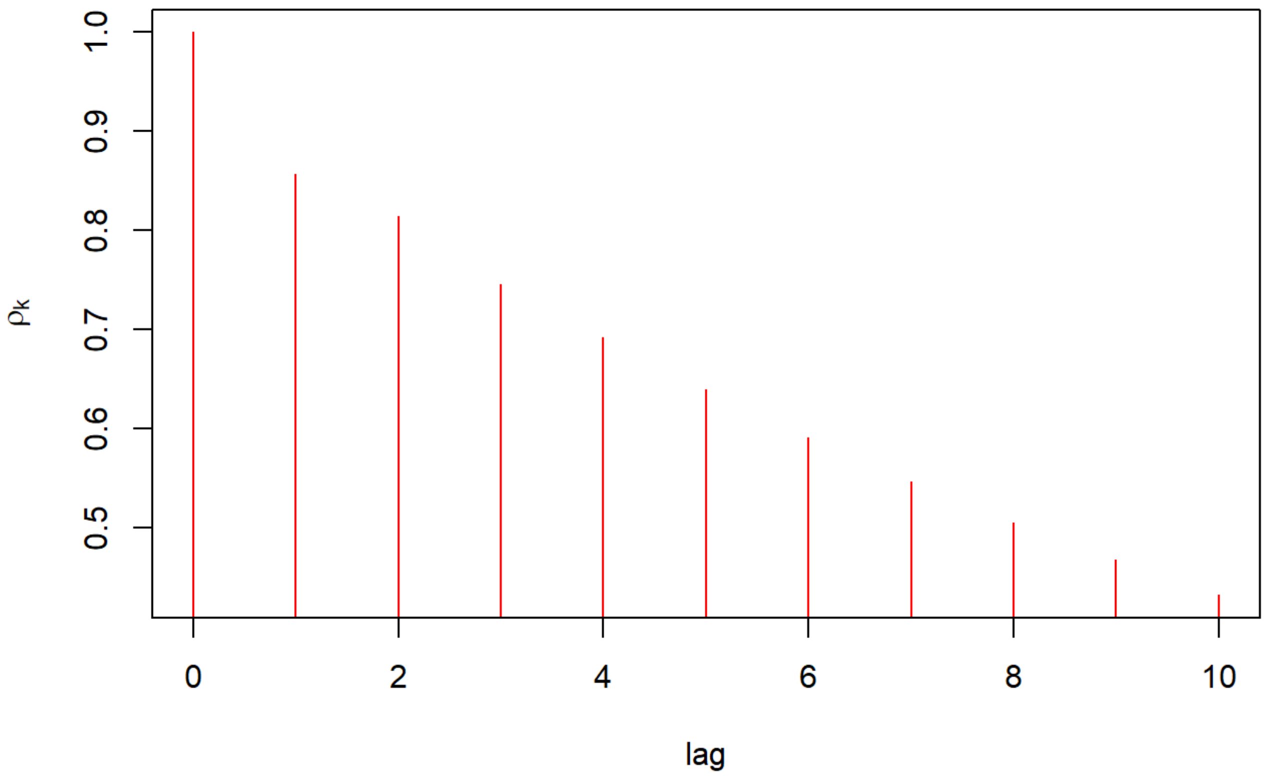
$$(i) Z_t - 0.6Z_{t-1} - 0.3Z_{t-2} = a_t$$

b. Plot the ρ_k for k = 0,1,2,…,10

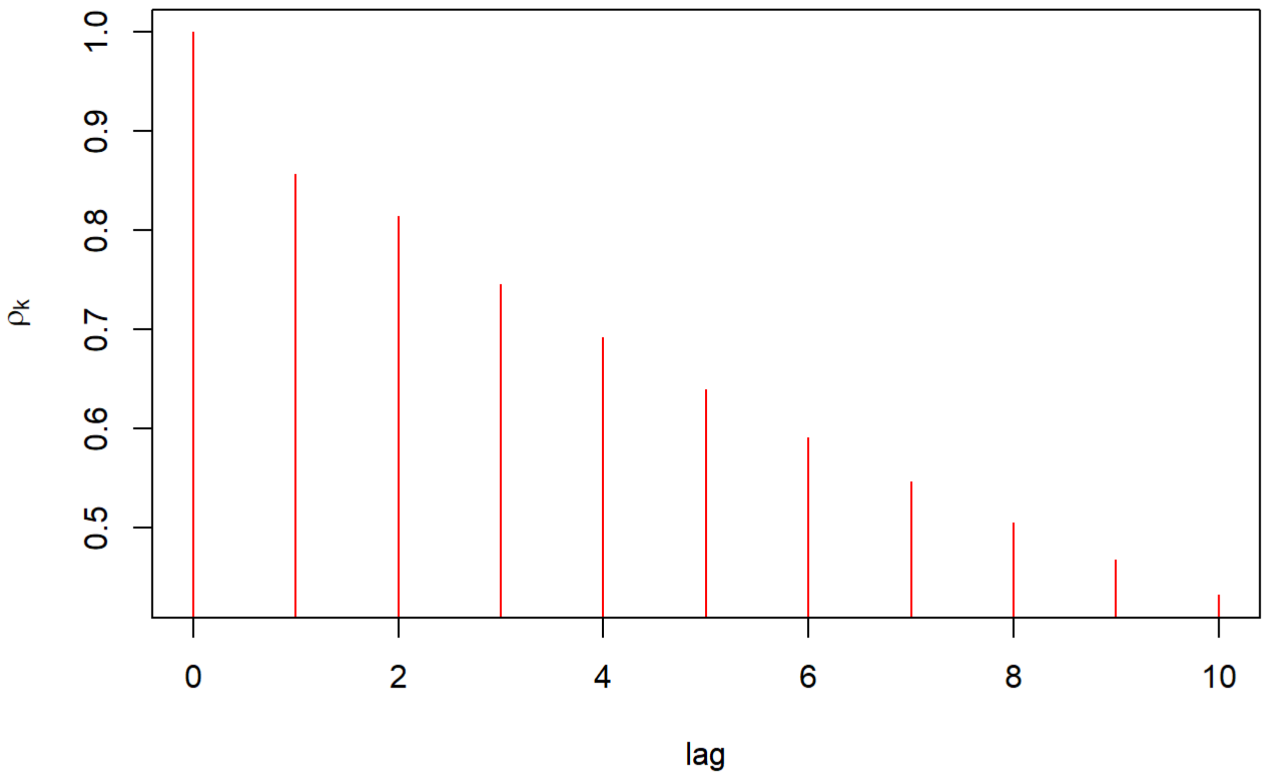
```
rho_k_2i_a <- c()

for (i in 0:10)
  rho_k_2i_a <- c(rho_k_2i_a,(1/2+sqrt(39)/14)*((3+sqrt(39))/10)^i+(1/2-sqrt(39)/14)*((3-sqrt
(39))/10)^i)

cbind(k=seq(0,10,1), ρk=rho_k_2i_a)
```

```
##        k        ρk
##  [1,]  0 1.0000000
##  [2,]  1 0.8571429
##  [3,]  2 0.8142857
##  [4,]  3 0.7457143
##  [5,]  4 0.6917143
##  [6,]  5 0.6387429
##  [7,]  6 0.5907600
##  [8,]  7 0.5460789
##  [9,]  8 0.5048753
## [10,]  9 0.4667488
## [11,] 10 0.4315119
```

```
plot(x=seq(0,10,1),
     y=rho_k_2i_a,
     type ="h",
     col="Red",
     xlab="lag",
     ylab=expression(rho[k]))
```




```
plot(x=seq(0,10,1),
     y=(ARMAacf(c(0.6,0.3),lag.max=10)),
     type ="h",
     col="Red",
     xlab="lag",
     ylab=expression(rho[k]))
```



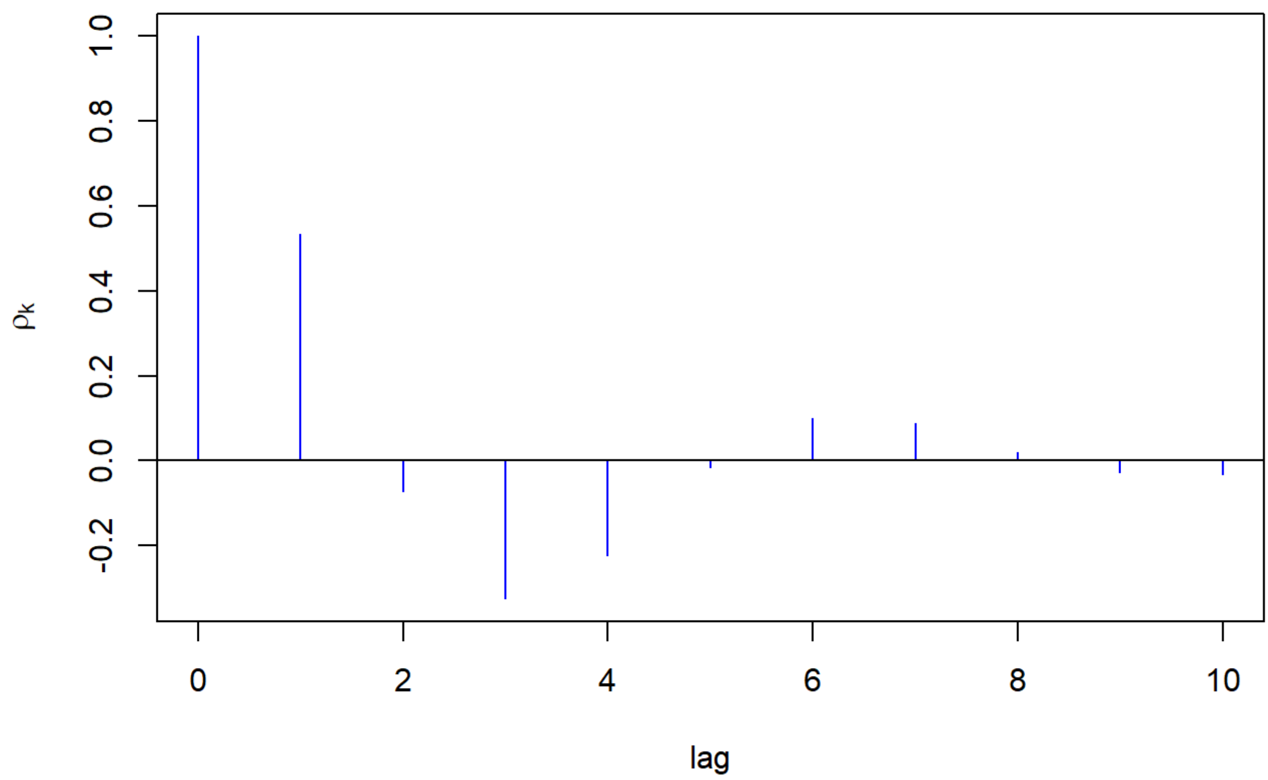
$$(ii) Z_t - 0.8Z_{t-1} + 0.5Z_{t-2} = a_t$$

b. Plot the ρ_k for k = 0,1,2,…,10

```
rho_k_2ii_a <- c()

for (i in 0:10)
  rho_k_2ii_a <- c(rho_k_2ii_a, (sqrt(2)/2)^i*(cos(i*acos(2*sqrt(2)/5)))+(2*sqrt(34)/51)*(sqr
t(2)/2)^i*sin(i*acos(2*sqrt(2)/5)))

cbind(k=seq(0,10,1), ρk=rho_k_2ii_a)
```

```
##         k          ρk
##  [1,]  0  1.00000000
##  [2,]  1  0.53333333
##  [3,]  2 -0.07333333
##  [4,]  3 -0.32533333
##  [5,]  4 -0.22360000
##  [6,]  5 -0.01621333
##  [7,]  6  0.09882933
##  [8,]  7  0.08717013
##  [9,]  8  0.02032144
## [10,]  9 -0.02732791
## [11,] 10 -0.03202305
```

```
plot(x=seq(0,10,1),
     y=(ARMAacf(c(0.8,-0.5),lag.max=10)),
     type="h",
     col="Blue",
     xlab="lag",
     ylab=expression(rho[k]))

abline(h=0)
```



3.

Given the AR(2) process:

$$Z_t = Z_{t-1} - 0.25Z_{t-2} + a_t$$

(c) Calculate the value ρ_k for k = 0,1,2,…,10

```
rho_k_3_c <- c()

for (i in 1:10)
  rho_k_3_c <- c(rho_k_3_c, (1/2)^i*(1+(3/5)*i))

cbind(k=seq(1,10,1), ρk=rho_k_3_c)
```

```
##          k           ρk
##   [1,]   1 0.800000000
##   [2,]   2 0.550000000
##   [3,]   3 0.350000000
##   [4,]   4 0.212500000
##   [5,]   5 0.125000000
##   [6,]   6 0.071875000
##   [7,]   7 0.040625000
##   [8,]   8 0.022656250
##   [9,]   9 0.012500000
##  [10,]  10 0.006835938
```