Stochastic Boolean Satisfiability

Decision Procedures, Generalization, and Applications

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Outline

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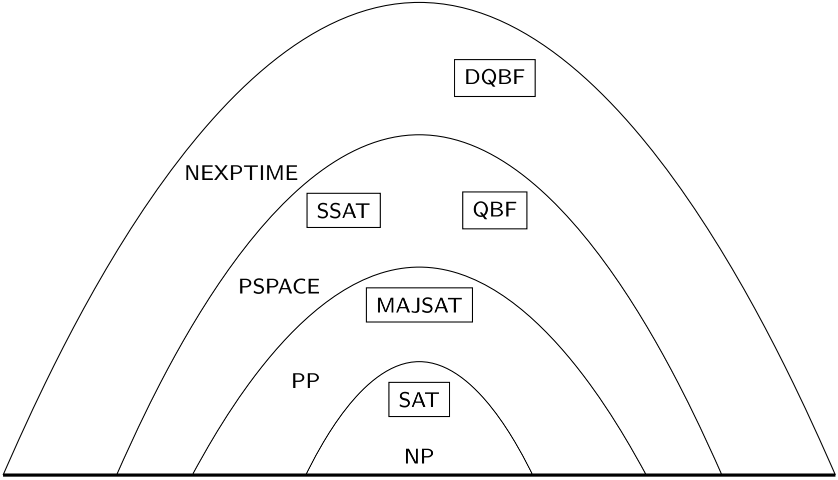
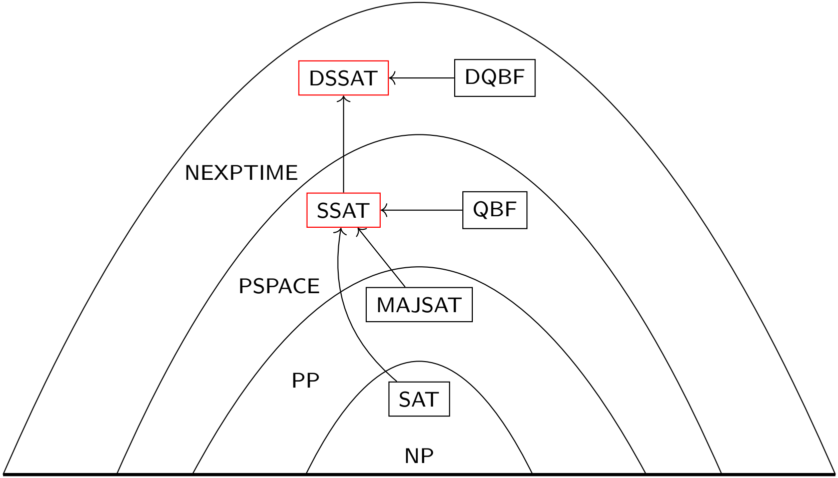
Satisfiability Solving: A Success Story

- Satisfiability solvers [6] have succeeded in various fields
  - Artificial intelligence [50, 55]
  - Electronic design automation [43, 65]
  - Formal verification [4, 27]

Satisfiability beyond Propositional Logic



This Dissertation in a Nutshell



Decision Making under Uncertainty

- Stochastic Boolean satisfiability (SSAT)
  - *Games against nature* [51]
  - Randomized quantifier $\reflectbox{R}^{p} x$: $\Pr[x = \textsc{true}] = p$
  - Logical formalism for problems with uncertainty
    - Probabilistic planning [40–42] and POMDP [56]

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  - Logical formalism for problems with uncertainty
    - Probabilistic planning [40–42] and POMDP [56]
- Application to VLSI systems?
  - Conventionally: error detection [15] or correction [46]
  - Post-Moore: probabilistic behavior of devices [12]

Accepting Device Imperfection

- New computational paradigms
  - Approximate design: deterministic deviation
    - E.g., neural-network deployment to edge devices
    - Circuit architectures [28, 29, 66]
    - Performance analysis [38, 64]
    - Automatic synthesis [44, 45, 48, 53, 63]
  - Probabilistic design: nondeterministic deviation
    - E.g., low-power video decoding
    - Energy consumption vs. correct switching of probabilistic CMOS [12]

Analyzing Probabilistic Design

- Circuit reliability analysis
  - Permanent defects or transient faults
  - Error probability at primary outputs
  - Monte Carlo simulation [47] or statistical methods [3, 31, 54]
  - Inadequate to analyze probabilistic design
    - Single-gate failure
    - Average error rate

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- Research need: a framework to analyze probabilistic design
  - Design space exploration
  - Fault-tolerant applications
  - Intrinsically probabilistic systems
  - SSAT is a suitable logical formalism

State-of-the-Art SSAT Solving

- DPLL search [19]
  - MAXPLAN [41]: pure variables and unit propagation
  - ZANDER [42]: threshold-pruning heuristics and memorization
  - DC-SSAT [40]: divide-and-conquer (structure of planning problems)

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  - PSPACE-complete [61]: the same as QBF
- Research need: novel algorithms for SSAT solving
  - Leverage advancements of other formalisms

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    - E.g., decentralized POMDP (Dec-POMDP) [5]
    - Difficult to obtain succinct encodings using SSAT

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  - E.g., decentralized POMDP (Dec-POMDP) [5]
  - Difficult to obtain succinct encodings using SSAT
- Research need: modeling NEXPTIME problems with uncertainty
  - Extend DQBF to stochastic domain

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  - Open-source solvers and formula instances are barely available
- Research need: public SSAT solvers and instances
  - Convenient comparison of different algorithms

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Contributing to the Research Needs

- Probabilistic-design analysis: probabilistic property evaluation
  - Random-exist/Exist-random SSAT for average-/worst-case analysis

[1] https://github.com/NTU-ALComLab/ssatABC

[2] https://github.com/NTU-ALComLab/ssat-benchmarks

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- Novel algorithms: random-exist (RE) and exist-random (ER) SSAT
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  - Approximate SSAT solving

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  - Applications to probabilistic/approximate design and Dec-POMDP
- Algorithm evaluation: open-source solver[1] and benchmark set[2]
  - Benchmark set will become public after necessary licenses are added

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Dissertation Structure

- The dissertation is based on the following publications
  - Chapter 4: probabilistic property evaluation
    - Published at ICCAD '14 [33] and in Trans. Computers '18 [34]
  - Chapter 5: random-exist quantified SSAT solving
    - Published at IJCAI '17 [36]
  - Chapter 6: exist-random quantified SSAT solving
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- Repeat the experiments with `BenchExec`[3]
  - Precise measurement: reproducibility
  - Data visualization

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Propositional Logic

Symbol	Description
τ	An assignment (a mapping from a variable set to $\mathbb{B}$)
ϕ	A quantifier-free formula
$\tau \models \phi$	τ satisfies ϕ
$\phi\|_x, \phi\|_{\neg x}$	Positive and negative cofactors of ϕ w.r.t. x
$\phi\|_\tau$	The resultant formula after cofactoring ϕ with τ

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Stochastic Boolean Satisfiability

Definition: SSAT

Given a quantified formula $\Phi = Q_1 x_1, \ldots, Q_n x_n.\phi$:

- $Q_1 x_1, \ldots, Q_n x_n$: quantification structure, $Q_i \in \{\text{Я}^p, \exists\}$ (*prefix*)
- ϕ: quantifier-free formula over $\{x_1, \ldots, x_n\}$ (*matrix*)

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Definition: Satisfying Probability of SSAT

Given an SSAT formula Φ, $\Pr[\Phi]$ is computed by:

1. $\Pr[\top] = 1$
2. $\Pr[\bot] = 0$
3. $\Pr[\Phi] = \max\{\Pr[\Phi|_{\neg x}], \Pr[\Phi|_x]\}$, if x is quantified by $\exists$
4. $\Pr[\Phi] = (1-p)\Pr[\Phi|_{\neg x}] + p\Pr[\Phi|_x]$, if x is quantified by Ɐ^p

Example: Satisfying Probability of SSAT

$$\Phi = \reflectbox{R}^{0.5} x_1, \exists y_1, \reflectbox{R}^{0.5} x_2, \exists y_2 . \phi$$

$$\phi = (x_1 \vee \neg y_1)(\neg x_1 \vee y_1)(\neg x_1 \vee \neg x_2 \vee y_2)(x_1 \vee \neg y_2)(x_2 \vee \neg y_2)$$

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$$\boxed{\text{Я}^{0.5} x_1}$$

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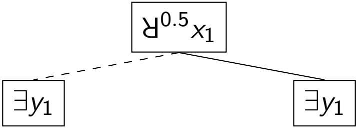

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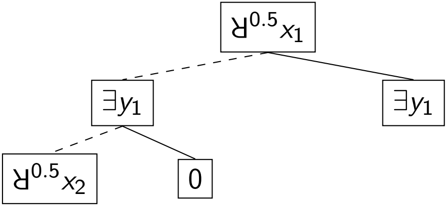

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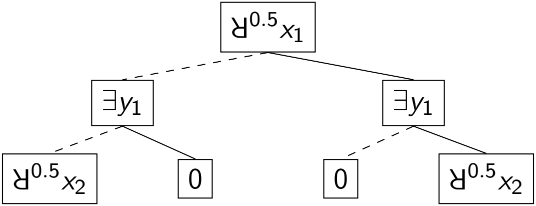

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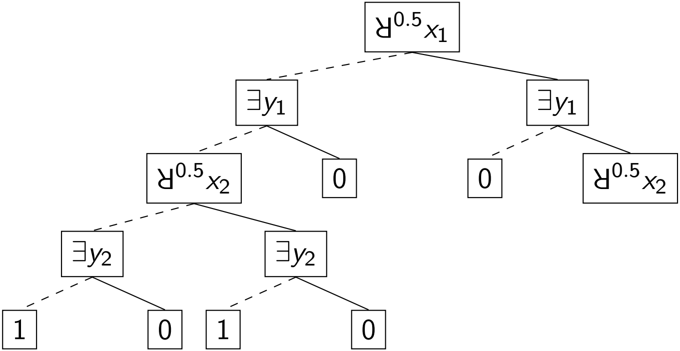

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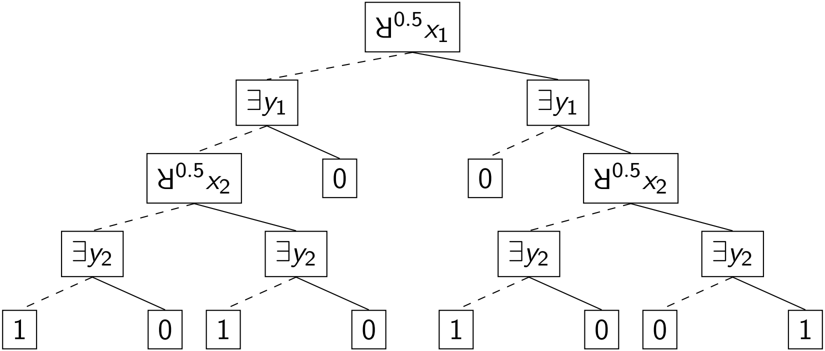

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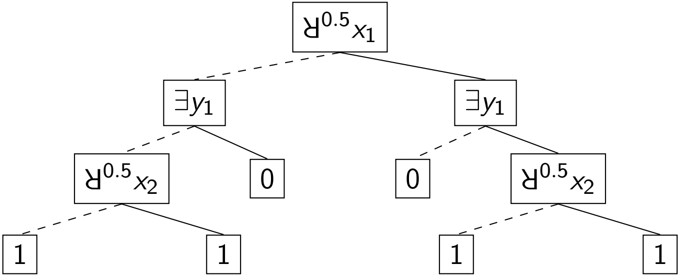

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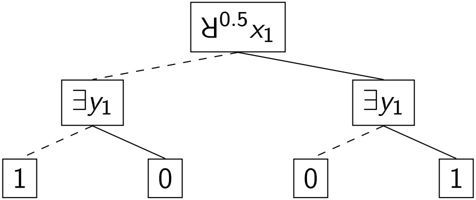

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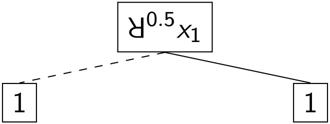

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$$\boxed{\Pr[\Phi] = 1}$$

Game-Theoretical Interpretations of SSAT

- $\Phi = Q_1 x_1, \ldots, Q_n x_n.\phi, Q_i \in \{\reflectbox{R}^p, \exists\}$
  - $\reflectbox{R}^p$: nondeterministic factors
  - $\exists$: an agent who plays under uncertainty
  - ϕ: game matrix
  - $\Pr[\Phi]$: the maximum winning probability of the agent
  - Skolem functions: a strategy of the agent
  - Optimal Skolem functions: maximize the winning probability

Game-Theoretical Interpretations of SSAT

Example: Optimal Skolem Functions

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- Variable y_1: $f_1(x_1) = x_1$; variable y_2: $f_2(x_1, x_2) = x_1 \wedge x_2$

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Model Counting

Definition: Unweighted Model Counting

Given a Boolean formula ϕ:

- Exact: find $\#\phi$
- Approximate: $\Pr[(1+\epsilon)^{-1}\#\phi \leq A \leq (1+\epsilon)\#\phi] \geq 1-\delta$

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Definition: Weighted Model Counting

Given ϕ and $\omega : \mathtt{vars}(\phi) \mapsto [0,1]$:

- Weight of ϕ: sum of the weights of the satisfying assignments
  - Weight of x: $\omega(x)$
  - Weight of $\neg x$: $1-\omega(x)$
  - Weight of τ: product of the weights of the individual literals

Model Counting

- Exact
  - `Cachet` [57, 58]: DPLL search plus subformula caching
  - `c2d` [16, 17]: CNF-to-d-DNNF compilation
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- Variants: expressible by SSAT
  - Weighted model counting [13, 59]
  - Projected model counting [2]
  - Maximum model counting [23]
  - Weighted projected model counting
    - `ProCount` [21]: ordering of projected and non-projected variables

Express Model-Counting Variants with SSAT

Variant	SSAT encoding
Unweighted	$\reflectbox{R}^{0.5} x_1, \ldots, \reflectbox{R}^{0.5} x_n . \phi$
Weighted	$\reflectbox{R}^{p_1} x_1, \ldots, \reflectbox{R}^{p_n} x_n . \phi$
Projected	$\reflectbox{R}^{0.5} x_1, \ldots, \reflectbox{R}^{0.5} x_n, \exists y_1, \ldots, \exists y_m . \phi$
Maximum	$\exists x_1, \ldots, \exists x_n, \reflectbox{R}^{0.5} y_1, \ldots, \reflectbox{R}^{0.5} y_m . \phi$
Weighted projected	$\reflectbox{R}^{p_1} x_1, \ldots, \reflectbox{R}^{p_n} x_n, \exists y_1, \ldots, \exists y_m . \phi$
Maximum weighted	$\exists x_1, \ldots, \exists x_n, \reflectbox{R}^{p_1} y_1, \ldots, \reflectbox{R}^{p_m} y_m . \phi$

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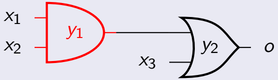
Modeling Probabilistic Design

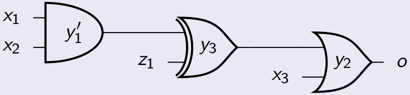
Example: Probabilistic Boolean Network and Standardization

A PBN with $V_I = \{x_1, x_2, x_3\}$ and $V_O = \{o\}$:



- $p_{x_1} = p_{x_2} = p_{x_3} = 0.5; p_{y_1} = 0.25; p_{y_2} = p_o = 0$

After standardization:



- $p_{z_1} = 0.25; p_{y_1'} = p_{y_3} = 0$

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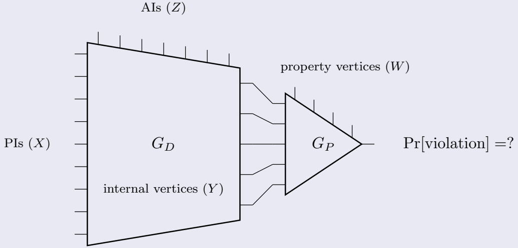
Probabilistic Property Evaluation

Definition: Property Violation Probability



- PPE: the average violation probability under $\pi : X \mapsto [0,1]$

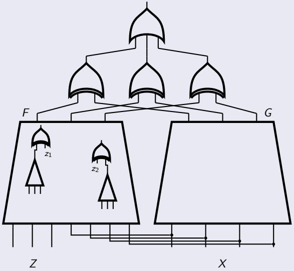
$$\reflectbox{R}^{\pi(x_1)} x_1, \ldots, \reflectbox{R}^{\pi(x_n)} x_n, \reflectbox{R}^{p_{z_1}} z_1, \ldots, \reflectbox{R}^{p_{z_l}} z_l, \reflectbox{R}^{p_{w_1}} w_1, \ldots, \reflectbox{R}^{p_{w_q}} w_q, \exists y_1, \ldots, \exists y_m . \phi_M$$

- MPPE: the maximum violation probability

$$\exists x_1, \ldots, \exists x_n, \reflectbox{R}^{p_{z_1}} z_1, \ldots, \reflectbox{R}^{p_{z_l}} z_l, \reflectbox{R}^{p_{w_1}} w_1, \ldots, \reflectbox{R}^{p_{w_q}} w_q, \exists y_1, \ldots, \exists y_m . \phi_M$$

Probabilistic Property Evaluation

Example: Probabilistic Equivalence Checking



- PEC: the average difference probability under $\pi : X \mapsto [0,1]$
$$\forall X, \forall Z, \exists Y.(F(X,Z) \neq G(X))$$
- MPEC: the maximum difference probability
$$\exists X, \forall Z, \exists Y.(F(X,Z) \neq G(X))$$

Solving MPPE and PPE

- MPPE: SSAT
  - CNF-based
  - BDD-based: graph traversal

Solving MPPE and PPE

- MPPE: SSAT
  - CNF-based
  - BDD-based: graph traversal
- PPE: model counting
  - Weighted model counting
  - Unweighted model counting with formula rewriting
    - Approximate model counting mostly focuses on unweighted instances
    - Express a weight of the form $\frac{k}{2^n}$ with additional variables and clauses

Outline

Evaluation

- Compared approaches
  - SSAT formulation (MPPE and PPE)
    - `BDDsp`: `C` language using `CUDD` [60] inside `ABC` [7]
    - `BDDsp-nr`: `BDDsp` without variable reordering
    - `DC-SSAT`: state-of-the-art CNF-based solver
  - Model-counting formulation (PPE)
    - `Cachet`: exact weighted model counter
    - `ApproxMC-4.0.1`: epsilon=0.99 and delta=0.01

Evaluation

- Compared approaches
  - SSAT formulation (MPPE and PPE)
    - `BDDsp`: `C` language using `CUDD` [60] inside `ABC` [7]
    - `BDDsp-nr`: `BDDsp` without variable reordering
    - `DC-SSAT`: state-of-the-art CNF-based solver
  - Model-counting formulation (PPE)
    - `Cachet`: exact weighted model counter
    - `ApproxMC`-4.0.1: epsilon=0.99 and delta=0.01
- Experimental setup
  - A machine with one 2.2 GHz CPU (Intel Xeon Silver 4210) with 40 processing units and 134 616 MB of RAM
  - Ubuntu 20.04 (64 bit), running Linux 5.4
  - CPU time: 15 min; memory: 15 GB

Benchmark Set

- Probabilistic equivalence checking
  - Average case: PEC
  - Worst case: MPEC

Benchmark Set

- Probabilistic equivalence checking
  - Average case: PEC
  - Worst case: MPEC
- ISCAS '85 [8] and EPFL [1] benchmark suites
  - And-inverter graphs (AIGs)
  - 30 circuits with sizes from 100 to 100K gates
  - Error rate: $\epsilon = 0.125$
  - Defect rate: $\delta = 0.01$ and 0.1

- BDDsp performs the best for small- and medium-sized circuits

Implications from the Results

- `BDDsp` performs the best for small- and medium-sized circuits
- `ApproxMC` uniquely solves large instances

Implications from the Results

- `BDDsp` performs the best for small- and medium-sized circuits
- `ApproxMC` uniquely solves large instances
- `Cachet` and `DC-SSAT` do not scale well

Results for PEC ($\delta = 0.01$)

Circuit	BDDsp T (s)	BDDsp Pr	DC-SSAT T (s)	DC-SSAT Pr	Cachet T (s)	Cachet Pr	ApproxMC T (s)	ApproxMC Pr
adder	3.71e−1	7.28e−1	–	–	–	–	6.70e+2	7.19e−1
bar	7.04e+0	9.85e−1	–	–	–	–	5.84e+2	1.00e+0
c1355	5.20e+0	4.32e−1	–	–	–	–	2.95e+1	4.30e−1
c1908	4.93e−1	6.25e−2	–	–	–	–	1.35e+1	6.05e−2
c2670	2.67e−1	3.10e−1	–	–	–	–	4.75e+2	3.13e−1
c3540	8.26e+0	2.28e−1	–	–	–	–	4.39e+1	2.30e−1
c432	6.19e−2	3.15e−2	–	–	2.87e−1	3.15e−2	1.17e+1	3.13e−2
c499	2.14e+0	2.62e−1	–	–	–	–	1.98e+1	2.66e−1
c5315	6.57e+1	6.53e−1	–	–	–	–	4.52e+2	6.56e−1
c6288	–	–	–	–	–	–	6.93e+1	9.06e−1
c7552	–	–	–	–	–	–	6.54e+2	7.03e−1
c880	6.31e−1	1.23e−1	–	–	1.10e+1	1.23e−1	3.17e+1	1.25e−1
cavlc	4.86e−2	4.96e−2	1.51e−1	4.96e−2	1.06e−1	4.96e−2	1.76e+1	4.98e−2
ctrl	4.44e−2	1.87e−1	9.67e−3	1.87e−1	3.52e−2	1.87e−1	1.05e+0	1.88e−1
dec	4.32e−2	6.56e−1	6.66e−3	6.56e−1	3.74e−2	6.56e−1	6.03e+0	6.56e−1
i2c	8.01e−2	4.33e−1	–	–	7.42e+2	4.33e−1	3.12e+2	4.22e−1
int2float	4.20e−2	6.39e−3	1.12e−2	6.39e−3	3.95e−2	6.39e−3	1.04e+0	6.47e−3
priority	1.16e−1	3.93e−1	–	–	–	–	1.70e+2	3.91e−1
router	5.06e−2	9.40e−4	–	–	6.84e+0	9.40e−4	3.69e+1	9.16e−4
sin	–	–	–	–	–	–	4.90e+2	1.00e+0

Results for MPEC ($\delta = 0.01$)

Circuit	BDDsp T (s)	BDDsp Pr	BDDsp-nr T (s)	BDDsp-nr Pr	DC-SSAT T (s)	DC-SSAT Pr
adder	$3.65e+0$	$7.99e-1$	–	–	–	–
c1355	$2.23e+1$	$6.56e-1$	$1.58e+0$	$6.56e-1$	–	–
c1908	$9.23e-1$	$4.14e-1$	$2.32e-1$	$4.14e-1$	$4.78e+1$	$4.14e-1$
c2670	$3.15e-1$	$5.51e-1$	–	–	–	–
c3540	$2.82e+1$	$6.07e-1$	$1.92e+1$	$6.07e-1$	–	–
c432	$6.40e-2$	$2.34e-1$	$4.99e-2$	$2.34e-1$	–	–
c499	$3.32e+0$	$4.14e-1$	$3.91e-1$	$4.14e-1$	–	–
c880	$6.71e-1$	$3.30e-1$	$9.85e-1$	$3.30e-1$	–	–
cavlc	$1.51e+0$	$5.42e-1$	$4.58e-2$	$5.42e-1$	$1.46e-1$	$5.42e-1$
ctrl	$4.53e-2$	$2.34e-1$	$4.45e-2$	$2.34e-1$	$7.27e-3$	$2.34e-1$
dec	$4.55e-2$	$6.56e-1$	$4.55e-2$	$6.56e-1$	$6.06e-3$	$6.56e-1$
i2c	–	–	$3.65e-1$	$8.57e-1$	–	–
int2float	$4.63e-2$	$2.34e-1$	$4.14e-2$	$2.34e-1$	$1.31e-2$	$2.34e-1$
priority	$1.78e+0$	$6.34e-1$	$6.86e-2$	$6.34e-1$	–	–
router	$5.39e-2$	$5.42e-1$	$4.84e-2$	$5.42e-1$	–	–

Outline

Outline

Definition: RE-SSAT

$\Phi = \text{Я} X, \exists Y. \phi(X, Y)$:

- X and Y: two disjoint sets of Boolean variables
- $\phi(X, Y)$: a CNF formula

Preliminaries

Definition: RE-SSAT

$\Phi = \reflectbox{R}X, \exists Y.\phi(X, Y)$:

- X and Y: two disjoint sets of Boolean variables
- $\phi(X, Y)$: a CNF formula

Definition: SAT/UNSAT Minterms and Cubes

Given $\Phi = \reflectbox{R}X, \exists Y.\phi(X, Y)$ and an assignment τ over X:

- $\phi(X, Y)|_\tau$ is satisfiable: τ is a SAT minterm of ϕ over X
- $\phi(X, Y)|_\tau$ is unsatisfiable: τ is an UNSAT minterm of ϕ over X
- τ is a partial assignment: τ is a SAT or an UNSAT cube

Generalization of SAT and UNSAT Minterms

- Given $\Phi = \reflectbox{R} X, \exists Y.\phi(X, Y)$
  - SAT minterm τ over X
    - Find a minimum subset of literals from τ that satisfy all clauses
    - Also known as *minimum hitting set*
  - UNSAT minterm τ over X
    - Modern SAT solvers: conflict analysis
    - $\phi(X, Y)|_\tau$ is UNSAT: a subset of literals from τ
    - Also known as *minimum UNSAT core*

Outline

Decision Procedure

- Given $\Phi = \reflectbox{R}X, \exists Y.\phi(X, Y)$, $\Pr[\Phi]$ equals
  - Sum of weights of all SAT minterms over X
  - One minus sum of weights of all UNSAT minterms over X

Decision Procedure

- Given $\Phi = \text{Ɐ}X, \exists Y.\phi(X, Y)$, $\Pr[\Phi]$ equals
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- Avoid exhaustive enumeration by minterm generalization
  - Overlapping cubes: weights cannot be summed up directly
  - Handle overlap by weighted model counting

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- Avoid exhaustive enumeration by minterm generalization
  - Overlapping cubes: weights cannot be summed up directly
  - Handle overlap by weighted model counting
- The collected cubes reflect bounds of $\Pr[\Phi]$
  - SAT cubes: lower bound
  - UNSAT cubes: upper bound

Decision Procedure

Example: RE-SSAT Solving

Consider $\text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3, \exists e_1, \exists e_2, \exists e_3 . \phi$ with

$C_1 : (r_1 \vee r_2 \vee e_1);\ C_2 : (r_1 \vee \neg r_3 \vee e_2);\ C_3 : (r_2 \vee \neg r_3 \vee \neg e_1 \vee \neg e_2)$

$C_4 : (r_3 \vee e_3);\ C_5 : (r_3 \vee \neg e_3)$

Decision Procedure

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Assignment	Minterm Type	Generalization	UB	LB
$\tau_1 = \neg r_1 \neg r_2 \neg r_3$	UNSAT	$\tau_1^+ = \neg r_3$	0.5	0

Decision Procedure

Example: RE-SSAT Solving

Consider $\text{Ɐ}^{0.5} r_1, \text{Ɐ}^{0.5} r_2, \text{Ɐ}^{0.5} r_3, \exists e_1, \exists e_2, \exists e_3 . \phi$ with

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$\tau_1 = \neg r_1 \neg r_2 \neg r_3$	UNSAT	$\tau_1^+ = \neg r_3$	0.5	0
$\tau_2 = \neg r_1 \neg r_2 r_3$	UNSAT	$\tau_2^+ = \neg r_1 \neg r_2$	0.375	0

Decision Procedure

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Consider $\text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3, \exists e_1, \exists e_2, \exists e_3 . \phi$ with

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$\tau_2 = \neg r_1 \neg r_2 r_3$	UNSAT	$\tau_2^+ = \neg r_1 \neg r_2$	0.375	0
$\tau_3 = \neg r_1 r_2 r_3$	SAT	$\tau_3^+ = r_2 r_3$	0.375	0.25

Decision Procedure

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Consider $\text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3, \exists e_1, \exists e_2, \exists e_3 . \phi$ with

$$C_1 : (r_1 \vee r_2 \vee e_1);\ C_2 : (r_1 \vee \neg r_3 \vee e_2);\ C_3 : (r_2 \vee \neg r_3 \vee \neg e_1 \vee \neg e_2)$$

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$\tau_3 = \neg r_1 r_2 r_3$	SAT	$\tau_3^+ = r_2 r_3$	0.375	0.25
$\tau_4 = r_1 \neg r_2 r_3$	SAT	$\tau_4^+ = r_1 r_3$	0.375	0.375

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$\tau_4 = r_1 \neg r_2 r_3$	SAT	$\tau_4^+ = r_1 r_3$	0.375	0.375

- $C_\perp = \{\tau_1^+, \tau_2^+\} = \{\neg r_3, \neg r_1 \neg r_2\}$
- $C_\top = \{\tau_3^+, \tau_4^+\} = \{r_2 r_3, r_1 r_3\}$
- $\Pr[\Phi] = 0.375$

Outline

Evaluation

- Compared solvers
  - `reSSAT`: `C++` language inside `ABC` environment [7]
    - SAT solver: `MiniSat-2.2` [22]
    - Weighted model counter: `Cachet` [57]
  - `reSSAT-b`: `reSSAT` without minterm generalization
  - `DC-SSAT`: state-of-the-art DPLL-based SSAT solver

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  - Ubuntu 20.04 (64 bit), running Linux 5.4
  - CPU time: 15 min; memory: 15 GB

Benchmark Set

- Random k-CNF formulas (by `CNFgen` [32])
  - $k \in \{3, \ldots, 9\}, n \in \{10, 20, \ldots, 50\}, \frac{m}{n} = \{k-1, k, k+1, k+2\}$
  - Quantify half variables randomly and others existentially
  - 5 samples per configuration: 700 formulas

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  - Forall-exist QBF: decide whether a company is *strategic*
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- PEC formulas: 60 formulas

Implications from the Results

- `reSSAT` outperforms `DC-SSAT` on random and strategic formulas

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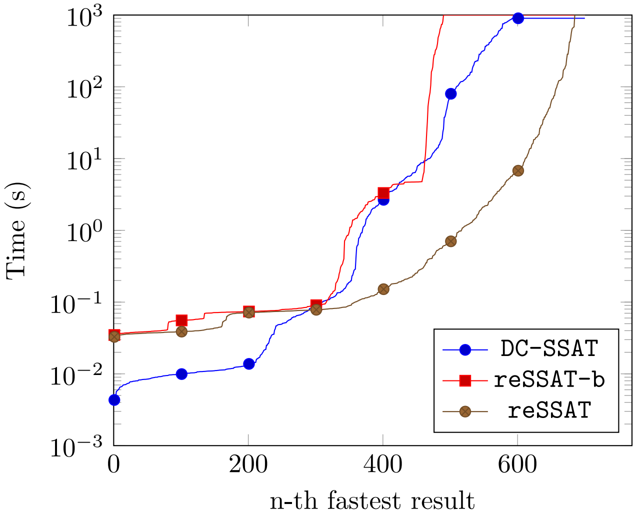
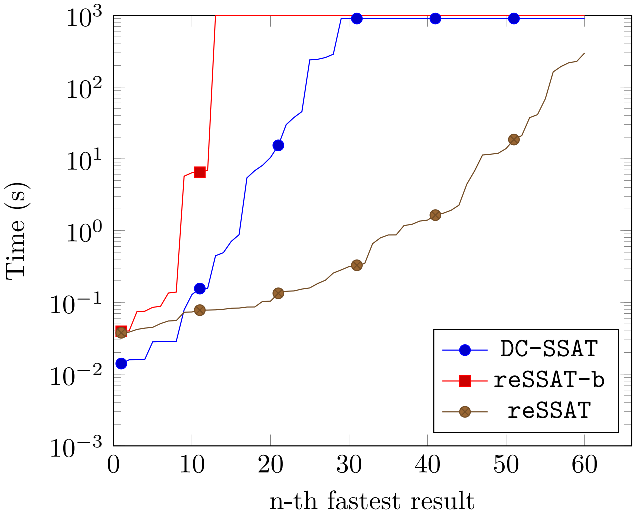
- `reSSAT` outperforms `DC-SSAT` on random and strategic formulas
- `reSSAT` is able to derive non-trivial bounds when `DC-SSAT` timeouts
- Minterm generalization is crucial to the performance of `reSSAT`

Quantile Plot for Random Formulas



Quantile Plot for Strategic-Company Formulas



Results for PEC Formulas ($\delta = 0.01$)

Formula	DC-SSAT T (s)	DC-SSAT Pr	reSSAT T (s)	reSSAT Pr	reSSAT UB	reSSAT-b T (s)	reSSAT-b Pr	reSSAT-b UB
adder	–	–	–	–	$7.98e{-1}$	–	–	–
bar	–	–	–	–	$9.98e{-1}$	–	–	–
c1908	–	–	–	–	–	–	–	$2.65e{-1}$
c2670	–	–	–	–	$5.03e{-1}$	–	–	$1.00e{+0}$
c3540	–	–	–	–	$2.98e{-1}$	–	–	$9.96e{-1}$
c432	–	–	–	–	$3.15e{-2}$	–	–	$5.92e{-1}$
c5315	–	–	–	–	$9.66e{-1}$	–	–	–
c6288	–	–	–	–	$1.00e{+0}$	–	–	$1.00e{+0}$
c880	–	–	–	–	$1.74e{-1}$	–	–	$8.83e{-1}$
cavlc	$1.60e{-1}$	$4.96e{-2}$	–	–	$4.96e{-2}$	–	–	$4.96e{-2}$
ctrl	$1.20e{-2}$	$1.87e{-1}$	$7.65e{-2}$	$1.87e{-1}$	–	$7.94e{-2}$	$1.87e{-1}$	–
dec	$9.19e{-3}$	$6.56e{-1}$	$7.46e{-2}$	$6.56e{-1}$	–	$1.00e{+1}$	$6.56e{-1}$	–
i2c	–	–	–	–	$8.18e{-1}$	–	–	$8.07e{-1}$
int2float	$1.84e{-2}$	$6.39e{-3}$	$8.49e{-2}$	$6.39e{-3}$	–	$1.08e{-1}$	$6.39e{-3}$	–
max	–	–	–	–	$9.74e{-1}$	–	–	–
priority	–	–	–	–	$7.61e{-1}$	–	–	–
router	–	–	–	–	$1.41e{-3}$	–	–	$2.41e{-1}$

Outline

Outline

Preliminaries

Definition: ER-SSAT (E-MAJSAT)

$\Phi = \exists X, \rotatebox[origin=c]{180}{R} Y.\phi(X, Y)$:

- X and Y: two disjoint sets of Boolean variables
- $\phi(X, Y)$: a CNF formula

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$\Phi = \exists X, \text{Ⅎ} Y.\phi(X, Y)$:

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Definition: Clause Selection

Given a CNF formula $\phi(X, Y)$:

- $C = C^X \vee C^Y$
- An assignment τ over X *selects* C if τ falsifies every literal in C^X
- Selection variable: $s_C \equiv \neg C^X$
- Selection relation: $\psi(X, S) = \bigwedge_{C \in \phi} (s_C \equiv \neg C^X)$

Outline

Decision Procedure

- Given $\Phi = \exists X, \text{Ɐ} Y.\phi(X, Y)$, $\Pr[\Phi]$ equals
  - The maximum conditional satisfying probability $\Pr[\Phi|_{\tau^*}]$

Decision Procedure

- Given $\Phi = \exists X, \reflectbox{R} Y.\phi(X, Y)$, Pr[$\Phi$] equals
  - The maximum conditional satisfying probability $\Pr[\Phi|_{\tau^*}]$
- Clause-containment learning
  - Prune assignments that select a superset of the selected clauses
    - An assignments τ_1 selects a set of clauses $\phi|_{\tau_1}$
    - $\phi|_{\tau_1} \subseteq \phi|_{\tau_2} \implies (\phi|_{\tau_2} \rightarrow \phi|_{\tau_1}) \implies \Pr[\Phi|_{\tau_2}] \leq \Pr[\Phi|_{\tau_1}]$
    - Learnt clause: $\bigvee_{C \in \phi|_{\tau_1}} \neg s_C$

Decision Procedure

Example: E-MAJSAT Solving

Consider $\Phi = \exists e_1, \exists e_2, \exists e_3, \text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3.\phi$ with

$C_1 : (e_1 \vee r_1 \vee r_2)$ $C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$

$C_3 : (\neg e_2 \vee \neg e_3 \vee r_2 \vee \neg r_3)$ $C_4 : (\neg e_1 \vee e_3 \vee r_3)$

Decision Procedure

Example: E-MAJSAT Solving

Consider $\Phi = \exists e_1, \exists e_2, \exists e_3, \text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3.\phi$ with

$$C_1 : (e_1 \vee r_1 \vee r_2) \; C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$$

$$C_3 : (\neg e_2 \vee \neg e_3 \vee r_2 \vee \neg r_3) \; C_4 : (\neg e_1 \vee e_3 \vee r_3)$$

The selection relation $\psi(X, S)$ of $\phi(X, Y)$ equals

$$(s_1 \equiv \neg e_1) \wedge (s_2 \equiv \neg e_1 \wedge \neg e_2) \wedge (s_3 \equiv e_2 \wedge e_3) \wedge (s_4 \equiv e_1 \wedge \neg e_3)$$

Decision Procedure

Example: E-MAJSAT Solving

Consider $\Phi = \exists e_1, \exists e_2, \exists e_3, \text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3 . \phi$ with

$$C_1 : (e_1 \vee r_1 \vee r_2) \;\; C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$$

$$C_3 : (\neg e_2 \vee \neg e_3 \vee r_2 \vee \neg r_3) \;\; C_4 : (\neg e_1 \vee e_3 \vee r_3)$$

The selection relation $\psi(X, S)$ of $\phi(X, Y)$ equals

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Assignment	Selected Clauses	$\Pr[\Phi\|_\tau]$	Learnt Clause	LB
$\tau_1 = \neg e_1 \neg e_2 \neg e_3$	$\{C_1, C_2\}$	0.75	$(\neg s_1 \vee \neg s_2)$	0.75

Example: E-MAJSAT Solving

Consider $\Phi = \exists e_1, \exists e_2, \exists e_3, \text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3 . \phi$ with

$$C_1 : (e_1 \vee r_1 \vee r_2) \; C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$$

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The selection relation $\psi(X, S)$ of $\phi(X, Y)$ equals

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Assignment	Selected Clauses	$\Pr[\Phi\|_\tau]$	Learnt Clause	LB
$\tau_1 = \neg e_1 \neg e_2 \neg e_3$	$\{C_1, C_2\}$	0.75	$(\neg s_1 \vee \neg s_2)$	0.75
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Example: E-MAJSAT Solving

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Decision Procedure

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Heuristics to Strengthen Learnt Clauses

- A shorter learnt clause prunes more search space
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Outline

Evaluation

- Compared solvers
  - erSSAT: C++ language inside ABC [7]
    - SAT solver: MiniSat-2.2 [22]
    - Weighted model counter: Cachet [57] and CUDD [18]
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  - DC-SSAT: state-of-the-art DPLL-based SSAT solver
- Experimental setup
  - A machine with one 2.2 GHz CPU (Intel Xeon Silver 4210) with 40 processing units and 134 616 MB of RAM
  - Ubuntu 20.04 (64 bit), running Linux 5.4
  - CPU time: 15 min; memory: 15 GB

Benchmark Set

- Random k-CNF formulas (by `CNFgen` [32])
  - $k \in \{3, \ldots, 9\}, n \in \{10, 20, \ldots, 50\}, \frac{m}{n} = \{k-1, k, k+1, k+2\}$
  - Quantify half variables existentially and others randomly
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- Application formulas: 212 formulas

Family	Description	Number
Toilet-A	Adapted from exist-forall QBFs [49]	77
Conformant	Adapted from exist-forall QBFs [49]	24
Sand-Castle	A probabilistic planning problem [41]	25
Max-Count	Adapted from maximum model counting [23]	26
MPEC	Maximum probabilistic equivalence checking	60

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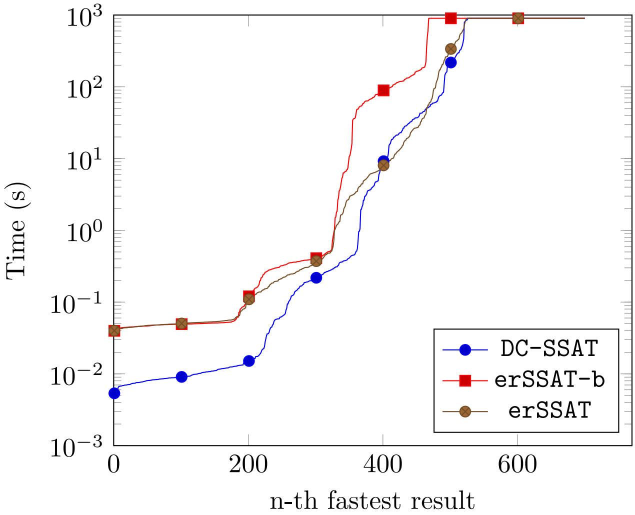
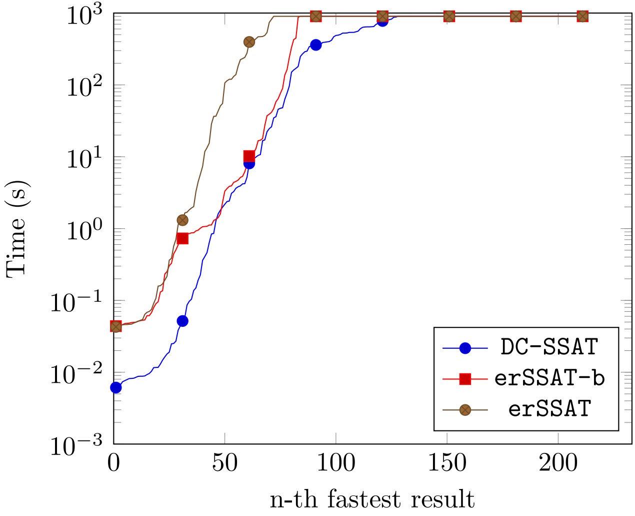
Quantile Plot for Random Formulas



Quantile Plot for Application Formulas



Summary of Results for Application Formulas

Algorithm	`DC-SSAT`	`erSSAT`	`erSSAT-b`
Solved formulas	78	59	65
Toilet-A	44	38	46
Conformant	1	2	1
Sand-Castle	22	13	14
Max-Count	3	3	1
MPEC	8	3	3
Timeouts	85	141	129
Out of memory	38	0	0
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- *Sand-Castle* favors `DC-SSAT`: same subformulas across the stages

Approximate E-MAJSAT Solving: erSSAT on *MPEC*

	DC-SSAT		erSSAT			
Formula	T (s)	Pr	T (s)	Pr	LB	T-LB (s)
c1355-0.01	–	–	–	–	4.14e−1	6.76e+2
c1908-0.01	4.80e+1	4.14e−1	–	–	3.18e−1	5.87e+2
c2670-0.01	–	–	–	–	4.87e−1	1.46e+2
c3540-0.01	–	–	–	–	–	–
c432-0.01	–	–	–	–	2.34e−1	1.41e+2
c499-0.01	–	–	–	–	4.14e−1	3.33e+1
c880-0.01	–	–	–	–	3.30e−1	4.14e+0
cavlc-0.01	1.49e−1	5.42e−1	–	–	–	–
ctrl-0.01	8.87e−3	2.34e−1	7.01e−2	2.34e−1	–	–
ctrl-0.10	5.77e−2	8.65e−1	–	–	–	–
dec-0.01	8.50e−3	6.56e−1	4.28e−2	6.56e−1	–	–
dec-0.10	1.81e+2	9.88e−1	–	–	–	–
i2c-0.01	–	–	–	–	–	–
int2float-0.01	1.17e−2	2.34e−1	1.31e+0	2.34e−1	–	–
int2float-0.10	4.22e+0	9.01e−1	–	–	–	–
priority-0.01	–	–	–	–	4.45e−1	1.40e+2
router-0.01	–	–	–	–	1.25e−1	3.60e−1

Outline

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Preliminaries

Definition: Dependency QBF (DQBF)

$\Phi = \forall x_1, \ldots, \forall x_n, \exists y_1(D_{y_1}), \ldots, \exists y_m(D_{y_m}).\phi$

- $D_{y_j} \subseteq \{x_1, \ldots, x_n\}$: the dependency set of y_j
- Φ is satisfiable if
  1. A Boolean function f_j over variables in D_{y_j} exists for each y_j
  2. ϕ becomes a tautology over $\{x_1, \ldots, x_n\}$ after substituting y_j with f_j

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Example: DQBF

$$\Phi = \forall x_1, \forall x_2, \exists y_1(\{x_1\}), \exists y_2(\{x_1, x_2\}).\phi$$
$$\phi = (x_1 \vee \neg y_1)(\neg x_1 \vee y_1)(\neg x_1 \vee \neg x_2 \vee y_2)(x_1 \vee \neg y_2)(x_2 \vee \neg y_2)$$

- Φ is satisfiable: $f_1(x_1) = x_1; f_2(x_1, x_2) = x_1 \wedge x_2$

Outline

Formulation

Definition: Dependency SSAT (DSSAT)

$\Phi = \reflectbox{R}^{p_1} x_1, \ldots, \reflectbox{R}^{p_n} x_n, \exists y_1(D_{y_1}), \ldots, \exists y_m(D_{y_m}).\phi$

- Satisfying probability of Φ with respect to $\mathcal{F} = \{f_1, \ldots, f_m\}$

$$\Pr[\Phi|_{\mathcal{F}}] = \Pr[\reflectbox{R}^{p_1} x_1, \ldots, \reflectbox{R}^{p_n} x_n.\phi|_{\mathcal{F}}]$$

- Decision version: given Φ and θ decide if $\Pr[\Phi|_{\mathcal{F}}] \geq \theta$ for some $\mathcal{F}$
- Optimization version: find $\mathcal{F}$ to maximize $\Pr[\Phi|_{\mathcal{F}}]$

Formulation

Example: DSSAT

$$\Phi = \reflectbox{R}^{0.5} x_1, \reflectbox{R}^{0.5} x_2, \exists y_1(\{x_1\}), \exists y_2(\{x_2\}).\phi$$

$$\phi = (x_1 \vee \neg y_1)(\neg x_1 \vee y_1)(\neg x_1 \vee \neg x_2 \vee y_2)(x_1 \vee \neg y_2)(x_2 \vee \neg y_2)$$

$$\mathcal{F} = \{f_1(x_1) = x_1, f_2(x_2) = x_2\}$$

- $\Pr[\Phi|_{\mathcal{F}}] = \Pr[\reflectbox{R}^{0.5} x_1, \reflectbox{R}^{0.5} x_2.(x_1 \vee \neg x_2)] = 0.75$

Complexity

Theorem: NEXPTIME-Completeness of DSSAT

The decision version of DSSAT is NEXPTIME-complete

- NEXPTIME: guess $\mathcal{F}$ in exponential time
- NEXPTIME-hard: DQBF $\leq_P$ DSSAT

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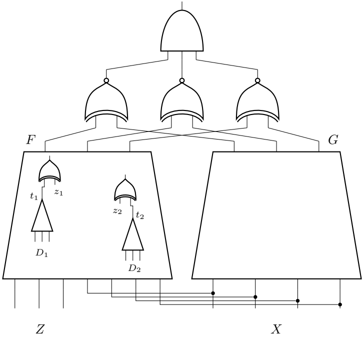
- Stochastic vs. Deterministic
  - DSSAT: natural encoding for NEXPTIME problems with uncertainty
  - DQBF: less straightforward for problems with probabilities

Outline

Analyzing Probabilistic/Approximate Partial Design

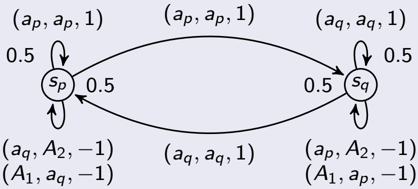
- Partial design problem [25] of probabilistic design
  - Synthesize black-box outputs T to realize specification
  - Constraints on inputs to black boxes: $D_i \subseteq X \cup Y$



$$\rotatebox[origin=c]{180}{R} X, \rotatebox[origin=c]{180}{R} Z, \forall Y, \exists T(D).(Y \equiv E(X)) \rightarrow (F(X, Z, T) \equiv G(X))$$

Modeling Decentralized POMDP

Example: Dec-POMDP



$$\mathcal{M} = (\{1, 2\}, \{s_p, s_q\}, \{a_p, a_q\}, T, \rho, \{o_p, o_q\}, \Omega, \Delta_0, h)$$

- $T(s_p, a_p, a_p, s_p) = T(s_p, a_p, a_p, s_q) = 0.5$
- $\rho(s_p, a_p, a_p) = 1; \rho(s_p, a_q, A_2) = \rho(s_p, A_1, a_q) = -1$
- $\Omega(s_p, o_p) = \Pr[o_p | s_p]$

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- Optimal *joint policy* to maximize the expected total reward
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- Policy selection for an individual agent [56]
  - Agent 1: $\exists x_a^{1,0}, \rotatebox[origin=c]{180}{R} x_o^{1,0}, \exists x_a^{1,1}, \rotatebox[origin=c]{180}{R} x_o^{1,1}, \exists x_a^{1,2}, \ldots$
  - Agent 2: $\exists x_a^{2,0}, \rotatebox[origin=c]{180}{R} x_o^{2,0}, \exists x_a^{2,1}, \rotatebox[origin=c]{180}{R} x_o^{2,1}, \exists x_a^{2,2}, \ldots$
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- Explicitly specify dependency sets in DSSAT
  - $\reflectbox{R} x_o^{1,0}, \reflectbox{R} x_o^{2,0}, \reflectbox{R} x_o^{1,1}, \reflectbox{R} x_o^{2,1}, \exists x_a^{1,0}, \exists x_a^{1,1}(\{x_o^{1,0}\}), \exists x_a^{1,2}(\{x_o^{1,0}, x_o^{1,1}\}), \ldots$

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  - Verify fairness of supervised learning [24]

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Thanks for your attention!
Questions?

BDD-Based SSAT Solving

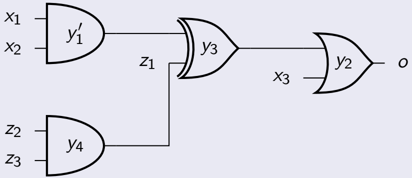
Input: An ROBDD node N and a prefix Q
Output: $\Pr[N = \top]$ under Q

```
if (N is a terminal node) then
    return N.sp
end if
if (N.visited = FALSE) then
    if (Q(N.var) = Ⴘ^p) then
        N.sp := (1 − p) · BddSsatRecur(N.else, Q) + p · BddSsatRecur(N.then, Q)
    else
        N.sp := max{BddSsatRecur(N.else, Q), BddSsatRecur(N.then, Q)}
    end if
    N.visited := TRUE
end if
return N.sp
```

Rewriting WMC into Unweighted MC

Example: WMC Rewriting

Express $p_{z_1} = 0.25$ with $z_1 \equiv z_2 \wedge z_3$:



- Unweighted: $p_{x_1} = p_{x_2} = p_{x_3} = p_{z_2} = p_{z_3} = 0.5$

WMC Rewriting Procedure

Input: A formula ϕ, a base set X_d for ϕ, a wt. func. ω s.t. $\forall x \in X_d . \omega(x) = \frac{k}{2^n}$

Output: A formula ϕ', a base set X'_d for ϕ', a wt. func. ω' s.t. $\forall x \in X'_d . \omega'(x) = \frac{1}{2}$

$\phi' := \phi, X'_d := X_d$
for all $(x \in X_d)$ **do**
    $var := x, wt := \omega(x)$
    while $(wt \neq \frac{1}{2})$ **do**
        $inv := \bot$
        if $(wt > \frac{1}{2})$ **then**
            $wt := 1 - wt, inv := \top$
        end if
        $\phi' := \phi' \wedge ((inv \oplus var) \equiv (y_{var} \wedge z_{var}))$
        $X'_d := X'_d \setminus \{var\} \cup \{y_{var}\}$
        $\omega'(y_{var}) = \frac{1}{2}$
        $var = z_{var}, wt = 2 \cdot wt$
    end while
    $X'_d := X'_d \cup \{var\}, \omega'(var) = \frac{1}{2}$
end for
return (ϕ', X'_d, ω')

Results for PEC ($\delta = 0.1$)

	BDDsp		DC-SSAT		Cachet		ApproxMC	
CIRCUIT	T (s)	Pr	T (s)	Pr	T (s)	Pr	T (s)	Pr
adder	1.77e+0	1.00e+0	–	–	–	–	–	–
c1355	–	–	–	–	–	–	2.92e+2	9.22e−1
c1908	–	–	–	–	–	–	2.49e+2	9.22e−1
c432	1.12e+1	4.99e−1	–	–	–	–	7.13e+1	4.84e−1
c499	–	–	–	–	–	–	1.89e+2	8.13e−1
c880	–	–	–	–	–	–	1.85e+2	8.75e−1
cavlc	2.58e−1	6.89e−1	–	–	1.77e+0	6.89e−1	3.32e+2	6.72e−1
ctrl	4.57e−2	8.22e−1	4.88e−2	8.22e−1	5.01e−2	8.22e−1	3.33e+1	8.28e−1
dec	5.38e−2	9.87e−1	1.84e+2	9.87e−1	1.34e+0	9.87e−1	8.17e+1	1.00e+0
int2float	5.70e−2	4.32e−1	4.30e+0	4.32e−1	3.55e−1	4.32e−1	7.27e+1	4.30e−1
router	3.60e−1	1.76e−1	–	–	–	–	1.56e+2	1.76e−1

Results for MPEC ($\delta = 0.1$)

	BDDsp		BDDsp-nr		DC-SSAT	
Circuit	T (s)	Pr	T (s)	Pr	T (s)	Pr
cavlc	–	–	$5.62e{-}2$	$9.78e{-}1$	–	–
ctrl	$4.87e{-}2$	$8.65e{-}1$	$4.61e{-}2$	$8.65e{-}1$	$5.83e{-}2$	$8.65e{-}1$
dec	$6.10e{-}2$	$9.88e{-}1$	$4.85e{-}2$	$9.88e{-}1$	$1.81e{+}2$	$9.88e{-}1$
int2float	–	–	$4.70e{-}2$	$9.01e{-}1$	$4.21e{+}0$	$9.01e{-}1$
router	–	–	$2.75e{+}0$	$8.96e{-}1$	–	–

Generalization of SAT Minterms

Example: SAT Generalization

Consider $\text{Ʀ}^{0.5} r_1, \text{Ʀ}^{0.5} r_2, \text{Ʀ}^{0.5} r_3, \exists e_1, \exists e_2, \exists e_3.\phi$ with ϕ consisting of the following clauses, and $\tau = \neg r_1 r_2 r_3$:

$$C_1 : (r_1 \vee r_2 \vee e_1);\ C_2 : (r_1 \vee \neg r_3 \vee e_2);\ C_3 : (r_2 \vee \neg r_3 \vee \neg e_1 \vee \neg e_2)$$
$$C_4 : (r_3 \vee e_3);\ C_5 : (r_3 \vee \neg e_3)$$

- τ is a SAT minterm: ϕ is satisfiable with τ and $\mu = \neg e_1 e_2 \neg e_3$
- Generalize τ to a SAT cube $\tau^+ = r_2 r_3$

Generalization of UNSAT Minterms

Example: UNSAT Generalization

Consider $\reflectbox{R}^{0.5} r_1, \reflectbox{R}^{0.5} r_2, \reflectbox{R}^{0.5} r_3, \exists e_1, \exists e_2, \exists e_3 . \phi$ with ϕ consisting of the following clauses, and $\tau = \neg r_1 \neg r_2 \neg r_3$:

$$C_1 : (r_1 \vee r_2 \vee e_1);\ C_2 : (r_1 \vee \neg r_3 \vee e_2);\ C_3 : (r_2 \vee \neg r_3 \vee \neg e_1 \vee \neg e_2)$$

$$C_4 : (r_3 \vee e_3);\ C_5 : (r_3 \vee \neg e_3)$$

- τ is an UNSAT minterm: C_4 and C_5 conflict
- Generalize τ to an UNSAT cube $\tau^+ = \neg r_3$

RE-SSAT Solving

Input: $\Phi = \text{\reflectbox{R}} X, \exists Y.\phi(X, Y)$ and a run-time limit $\texttt{TO}$
Output: Lower and upper bounds (P_L, P_U) of $\Pr[\Phi]$

$\psi(X) := \top$
$C_\top := \emptyset$
$C_\bot := \emptyset$
while $(\texttt{SAT}(\psi)$ **and** $\texttt{run-time} < \texttt{TO})$ **do**
    $\tau := \psi.\text{model}$
    if $(\texttt{SAT}(\phi|_\tau))$ **then**
        $\tau^+ := \texttt{MinimalSatisfying}(\phi, \tau)$
        $C_\top := C_\top \cup \{\tau^+\}$
    else
        $\tau^+ := \texttt{MinimalConflicting}(\phi, \tau)$
        $C_\bot := C_\bot \cup \{\tau^+\}$
    end if
    $\psi := \psi \wedge \neg\tau^+$
end while
return $(\texttt{ComputeWeight}(C_\top), 1 - \texttt{ComputeWeight}(C_\bot))$

Results for PEC Formulas ($\delta = 0.1$)

Formula	DC-SSAT T (s)	DC-SSAT Pr	reSSAT T (s)	reSSAT Pr	reSSAT UB	reSSAT-b T (s)	reSSAT-b Pr	reSSAT-b UB
c1908	–	–	–	–	$1.00e{+}0$	–	–	–
c3540	–	–	–	–	$1.00e{+}0$	–	–	–
c432	–	–	–	–	$7.81e{-}1$	–	–	$9.76e{-}1$
c880	–	–	–	–	$9.98e{-}1$	–	–	–
cavlc	–	–	–	–	$7.66e{-}1$	–	–	$8.88e{-}1$
ctrl	$5.27e{-}2$	$8.22e{-}1$	–	–	$8.49e{-}1$	–	–	$8.73e{-}1$
dec	$1.85e{+}2$	$9.87e{-}1$	–	–	$9.88e{-}1$	–	–	$9.99e{-}1$
i2c	–	–	–	–	$9.98e{-}1$	–	–	–
int2float	$4.33e{+}0$	$4.32e{-}1$	–	–	$5.22e{-}1$	–	–	$6.46e{-}1$
max	–	–	–	–	$1.00e{+}0$	–	–	–
priority	–	–	–	–	$1.00e{+}0$	–	–	–
router	–	–	–	–	$2.41e{-}1$	–	–	$3.10e{-}1$

Minimal Clause Selection

- Iteratively solve $\psi(X, S)$ to select a minimal set of clauses
- Recall $\Phi = \exists e_1, \exists e_2, \exists e_3, \text{Я}^{0.5} r_1, \text{Я}^{0.5} r_2, \text{Я}^{0.5} r_3 . \phi$ with

  $C_1 : (e_1 \vee r_1 \vee r_2)$ $C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$
  $C_3 : (\neg e_2 \vee \neg e_3 \vee r_2 \vee \neg r_3)$ $C_4 : (\neg e_1 \vee e_3 \vee r_3)$
- $\tau_1 = \neg e_1 \neg e_2 \neg e_3$ selects $\{C_1, C_2\}$
- Solve $\psi \wedge (\neg s_1 \vee \neg s_2)$ under assumptions $s_3 \wedge s_4$
  - $\tau_2 = \neg e_1 e_2 \neg e_3$, which only selects $\{C_1\}$
  - Replace an expensive model-counting call with a SAT call

Induced Clause Subsumption

- C_1 subsumes C_2 if $C_1 \subseteq C_2$ (treat C_1 and C_2 as sets of literals)
- Subsumed clauses can be removed from a CNF formula
- Recall $\Phi = \exists e_1, \exists e_2, \exists e_3, \text{Ⅎ}^{0.5} r_1, \text{Ⅎ}^{0.5} r_2, \text{Ⅎ}^{0.5} r_3.\phi$ with

  $C_1 : (e_1 \vee r_1 \vee r_2)$ $C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$
  $C_3 : (\neg e_2 \vee \neg e_3 \vee r_2 \vee \neg r_3)$ $C_4 : (\neg e_1 \vee e_3 \vee r_3)$
- $\tau_1 = \neg e_1 \neg e_2 \neg e_3$ selects $\{C_1, C_2\}$
- C_1^Y subsumes C_2^Y: remove C_2 from the set of selected clauses
  - Strengthen the learnt clause to $(\neg s_1)$
- Subsumption among C^Y clauses is *induced* by clause selection

Partial Assignment Pruning

- $C_L = \bigvee_{C \in \phi|_\tau} \neg s_C = \bigvee_{C \in \phi|_\tau} C^X$
- Discard literals from C_L by model counting
- Recall $\Phi = \exists e_1, \exists e_2, \exists e_3, \rotatebox[origin=c]{180}{R}^{0.5} r_1, \rotatebox[origin=c]{180}{R}^{0.5} r_2, \rotatebox[origin=c]{180}{R}^{0.5} r_3 . \phi$ with

  $C_1 : (e_1 \vee r_1 \vee r_2)$ $C_2 : (e_1 \vee e_2 \vee r_1 \vee r_2 \vee \neg r_3)$
  $C_3 : (\neg e_2 \vee \neg e_3 \vee r_2 \vee \neg r_3)$ $C_4 : (\neg e_1 \vee e_3 \vee r_3)$
- $\tau_1 = \neg e_1 \neg e_2 \neg e_3$: $C_L = (\neg s_1 \vee \neg s_2) = (e_1 \vee e_2), \Pr[\Phi|_{\tau_1}] = 0.75$
  - Can we discard e_2 from C_L?
  - (e_1) blocks τ that selects C_1: $\Pr[\Phi|_\tau] \leq \Pr[C_1] = 0.75 \leq \Pr[\Phi|_{\tau_1}]$
- Count weight of selected clauses and compare to current maximum

ER-SSAT Solving

Input: $\Phi = \exists X, \reflectbox{R} Y.\phi(X,Y)$
Output: $\Pr[\Phi]$

$\psi(X,S) := \bigwedge_{C \in \phi} (s_C \equiv \neg C^X) \wedge \bigwedge_{\text{pure } l : \texttt{var}(l) \in X} l$
$prob := 0$
$\texttt{s-table} := \texttt{BuildSubsumptionTable}(\phi)$
while $(\texttt{SAT}(\psi))$ **do**
  $\tau := \psi.\mathrm{model}$ (discarding the selection variables)
  if $(\texttt{SAT}(\phi|_\tau))$ **then**
    $\tau' := \texttt{SelectMinimalClauses}(\phi, \psi)$
    $\varphi := \texttt{RemoveSubsumedClauses}(\phi|_{\tau'}, \texttt{s-table})$
    $prob := \max\{prob, \texttt{ComputeWeight}(\reflectbox{R} Y.\varphi)\}$
    $C_S := \bigvee_{C \in \varphi} \neg s_C$
    $C_L := \texttt{DiscardLiterals}(\phi, C_S, prob)$
  else
    $C_L := \texttt{MinimalConflicting}(\phi, \tau)$
  end if
  $\psi := \psi \wedge C_L$
end while
return $prob$

Approximate E-MAJSAT Solving: erSSAT-b on *MPEC*

	DC-SSAT		erSSAT-b			
FORMULA	T (s)	Pr	T (s)	Pr	LB	T-LB (s)
c1355-0.01	–	–	–	–	$6.56e{-1}$	$4.10e{+1}$
c1908-0.01	$4.80e{+1}$	$4.14e{-1}$	–	–	$4.14e{-1}$	$2.23e{+1}$
c2670-0.01	–	–	–	–	$5.51e{-1}$	$5.74e{+0}$
c3540-0.01	–	–	–	–	$5.51e{-1}$	$7.20e{+1}$
c432-0.01	–	–	–	–	$2.34e{-1}$	$2.91e{+0}$
c499-0.01	–	–	–	–	$4.14e{-1}$	$3.80e{-1}$
c880-0.01	–	–	–	–	$3.30e{-1}$	$6.17e{+0}$
cavlc-0.01	$1.49e{-1}$	$5.42e{-1}$	–	–	–	–
ctrl-0.01	$8.87e{-3}$	$2.34e{-1}$	$8.98e{-2}$	$2.34e{-1}$	–	–
ctrl-0.10	$5.77e{-2}$	$8.65e{-1}$	–	–	–	–
dec-0.01	$8.50e{-3}$	$6.56e{-1}$	$4.58e{-2}$	$6.56e{-1}$	–	–
dec-0.10	$1.81e{+2}$	$9.88e{-1}$	–	–	–	–
i2c-0.01	–	–	–	–	$7.21e{-1}$	$1.75e{+2}$
int2float-0.01	$1.17e{-2}$	$2.34e{-1}$	$4.45e{-1}$	$2.34e{-1}$	–	–
int2float-0.10	$4.22e{+0}$	$9.01e{-1}$	–	–	–	–
priority-0.01	–	–	–	–	$5.89e{-1}$	$5.57e{+2}$
router-0.01	–	–	–	–	$1.25e{-1}$	$3.80e{-1}$

Computational Complexity of DSSAT

Theorem: NEXPTIME-Completeness of DSSAT

The decision version of DSSAT is NEXPTIME-complete

- DSSAT is NEXPTIME
- DSSAT is NEXPTIME-hard

Proof Sketch

- NEXPTIME
  1. nondeterministically construct a set of Skolem functions $\mathcal{F}$
  2. compute $\Pr[\Phi|_{\mathcal{F}}]$ and compare it with the threshold θ
- NEXPTIME-hard: DQBF $\leq_P$ DSSAT

$$\Phi_Q = \forall x_1, \ldots, \forall x_n, \exists y_1(D_{y_1}), \ldots, \exists y_m(D_{y_m}).\phi$$

$$\Phi_S = \rotatebox[origin=c]{180}{R}^{0.5} x_1, \ldots, \rotatebox[origin=c]{180}{R}^{0.5} x_n, \exists y_1(D_{y_1}), \ldots, \exists y_m(D_{y_m}).\phi$$

Claim: Φ_Q is satisfiable if and only if $\Pr[\Phi_S|_{\mathcal{F}}] \geq 1$ for some $\mathcal{F}$

Modeling Decentralized POMDP

$$\bigwedge_{0\leq t\leq h-2} [x_p^t \equiv \bot \rightarrow \bigwedge_{i\in I} x_o^{i,t} \equiv 0 \wedge x_s^{t+1} \equiv 0 \wedge x_p^{t+1} \equiv \bot]$$

$$x_p^{h-1} \equiv \bot$$

$$\bigwedge_{s\in S} \bigwedge_{\vec{a}\in\vec{A}} [x_p^0 \equiv \bot \wedge x_s^0 \equiv s \wedge \bigwedge_{i\in I} x_a^{i,0} \equiv a_i \rightarrow x_r^0 \equiv N_r(s,\vec{a})]$$

$$\bigwedge_{1\leq t\leq h-1} \bigwedge_{s\in S} \bigwedge_{\vec{a}\in\vec{A}} [x_p^{t-1} \equiv \top \wedge x_p^t \equiv \bot \wedge x_s^t \equiv s \wedge \bigwedge_{i\in I} x_a^{i,t} \equiv a_i \rightarrow x_r^t \equiv N_r(s,\vec{a})]$$

$$\bigwedge_{0\leq t\leq h-2} \bigwedge_{s\in S} \bigwedge_{\vec{a}\in\vec{A}} \bigwedge_{s'\in S} [x_p^t \equiv \top \wedge x_s^t \equiv s \wedge \bigwedge_{i\in I} x_a^{i,t} \equiv a_i \wedge x_s^{t+1} \equiv s' \rightarrow x_{T_{s,\vec{a}}}^t \equiv s']$$

$$\bigwedge_{0\leq t\leq h-2} \bigwedge_{s'\in S} \bigwedge_{\vec{a}\in\vec{A}} \bigwedge_{\vec{o}\in\vec{O}} [x_p^t \equiv \top \wedge x_s^{t+1} \equiv s' \wedge \bigwedge_{i\in I} x_a^{i,t} \equiv a_i \wedge \bigwedge_{i\in I} x_o^{i,t} \equiv o_i \rightarrow x_{\Omega_{s',\vec{a}}}^t \equiv N_\Omega(\vec{o})]$$