1. 随机过程

$X(t)$ $\quad X(\omega, t)$ (ω: $\omega\Omega$) Sample path

$R_X(t,s) = E(X(t)\overline{X(s)})$ (Auto) Correlation Function

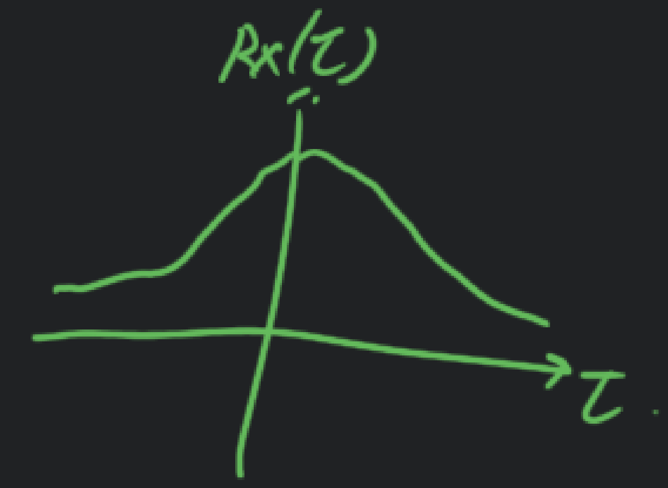
① $R_X(t,s) = R_X(s,t)$ ② $R_X(t,t) \geq 0$ ③ $|R_X(t,s)| \leq (R_X(t,t)R_X(s,s))^{1/2}$ $\left(|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}\right)$.

Stationary: wide sense (w.w.s. 宽平稳).

(Invariance) ① $E(X(t)) = \text{Const}$. (不本质) ② $R_X(t+T, s+T) = R_X(t,s)$ $\forall T$ 自相关函数是一元函数.

$= R_X(t-s)$

$$\begin{cases} R_X(-\tau) = R_X(\tau) \\ R_X(0) \geq 0 \\ |R_X(\tau)| \leq R_X(0) \end{cases}$$

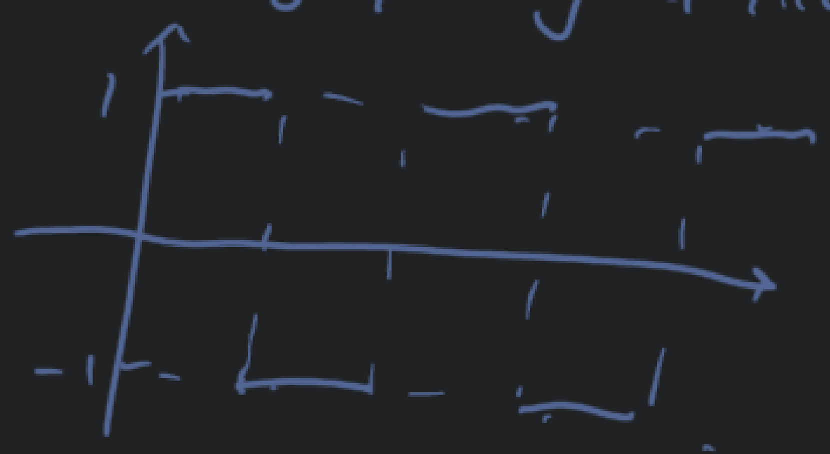


e.g. Modulated signal $X(t) = A(t)\cos(2\pi f t + \theta)$ $A(t)$, θ random independent $A(t)$ w.w.s $\theta \sim U(0, 2\pi)$.

$E(X(t)) = E(A(t))\, E(\cos(2\pi f t + \theta)) = 0$ $\quad E(X(t)X(s)) = E(A(t)A(s))\, E(\cos(2\pi f t + \theta)\cos(2\pi f s + \theta))$.

$= \frac{1}{2} R_A(t-s)\cos(2\pi f(t-s))$ 仍然 w.w.s.

e.g. Random Telegraph signal $X(t)$ 时间段s内跳变次数 $N(s) \sim P(\lambda s)$ $\quad P(N(s)=k) = \frac{(\lambda s)^k}{k!} \exp(-\lambda s)$.



$X(0) \sim \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix}$ 伯努利分布

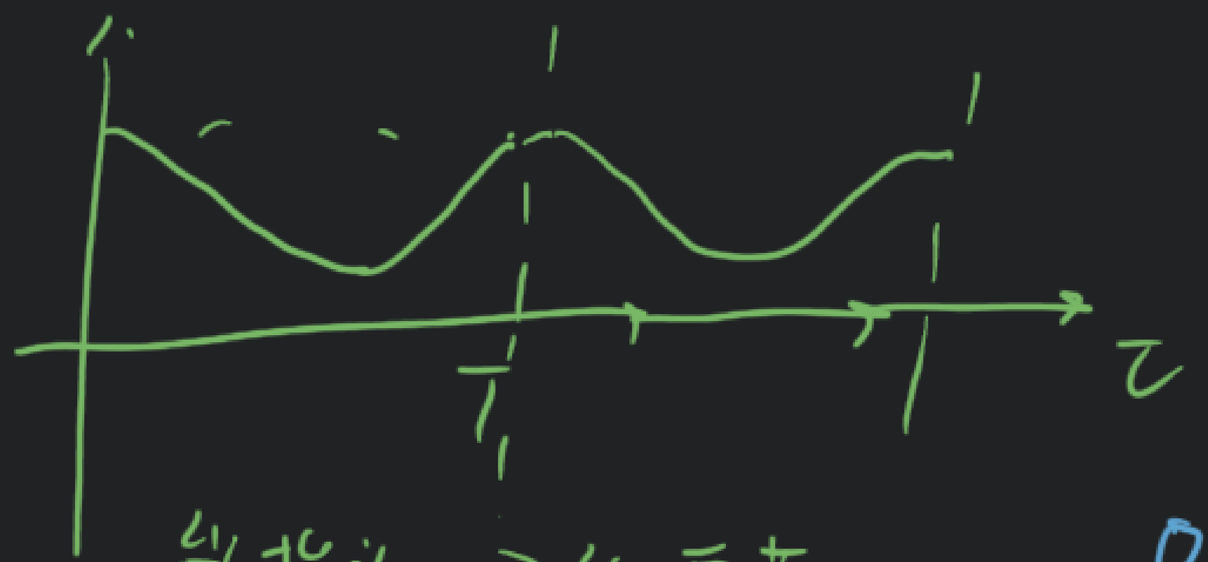
$E(X(t)) = 1 \cdot P(X(t)=1) + (-1) \cdot P(X(t)=-1)$ $\quad P(X(t)=1) = \frac{1}{2}\sum_{k,\text{even}} P(N(t)=k) + \frac{1}{2}\sum_{k,\text{odd}} P(N(t)=k) = \frac{1}{2}$

$P(X(t) = -1) = \frac{1}{2}$ $\Rightarrow E(X(t)) = 0$

$E(X(t)X(s)) = 1\cdot P(X(t)X(s)=1) + (-1)P(X(t)X(s)=-1)$.

$= 1\cdot\frac{1}{2}(1+\exp(-2\lambda|t-s|)) + (-1)\cdot\frac{1}{2}(1-\exp(-2\lambda|t-s|)) = \exp(-2\lambda|t-s|)$. w.w.s.

$X(t)$ w.s.s. if $\exists T>0$, $R_X(0)=R_X(T) \Leftrightarrow R_X(T+\tau)=R_X(\tau)$.



能指头 ⇒ 能震荡.

prove: $R_X(0)=R_X(T) \Rightarrow E(|X(t)-X(t+T)|^2) = E(X^2(t)) + E(X^2(t+T)) - 2E(X(t)X(t+T))$

$= 2R_X(0) - 2R_X(T) = 0$.

$0 \le |R_X(\tau)-R_X(T+\tau)| = |E(X(0)X(\tau)) - E(X(0)X(T+\tau))|$

$= |E(X(0)(X(\tau)-X(T+\tau)))| \le E(|X(0)|\,|X(\tau)-X(\tau+T)|)$

$\le [E(X^2(0))\, E(|X(\tau)-X(\tau+T)|^2)]^{1/2} = 0$.

$|E(X)| \le E(|X|)$.

$X(t)$ w.s.s. $R_X(\tau)$ continuous at 0 $\Rightarrow$ $R_X(\tau)$ continuous at everywhere.

2. Positive Functions (函数的正定性 p.d.)

$f(x)$ p.d. $\Leftrightarrow \forall n\in\mathbb{N}, \forall x_1,\cdots,x_n\in\mathbb{R}$ matrix $(f(x_i-x_j))_{ij}$ is positive

$R_X(i,j) = R_X(t_i-t_j) = E(X(t_i)\overline{X(t_j)})$. $X=(X(t_1),\cdots,X(t_n))^T$. $R_X = E(XX^T) \ge 0$.

Correlation Function $\Longleftrightarrow$ Positive Definite Function (assume w.w.s.)

Bochner: $f(x)$ is p.d. $\Leftrightarrow \int_{-\infty}^{+\infty} f(x)\exp(-j\omega x)\,dx \ge 0$. (函数正定 ⇔ 傅里叶变换是正的)

$\Leftarrow$: $F(\omega) = \int_{-\infty}^{+\infty} f(x)\exp(-j\omega x)\,dx \ge 0$ $f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\exp(j\omega x)\,d\omega$.

① $\exp(j\omega x)$ is p.d. $\forall n, \forall x_1,\cdots x_n \in \mathbb{R}$ $\quad R = (\exp(j\omega(x_i - x_j)))_{ij}$.

$\forall z = (z_1,\cdots,z_n)^T \in \mathbb{C}^n$. $\quad z^+ R z = \sum_{ij} R_{ij} z_i^* z_j = \sum_{ij} \exp(j\omega(x_i-x_j)) z_i^* z_j = [\sum_i \exp(-j\omega x_i)^* z_i^*][\sum_j \exp(-j\omega x_j) z_j] \geq 0$.

② $\forall f_1,\cdots,f_n \geq 0$ $\quad \sum_{i=1}^{n} f_i \exp(j\omega_i x)$ is p.d.

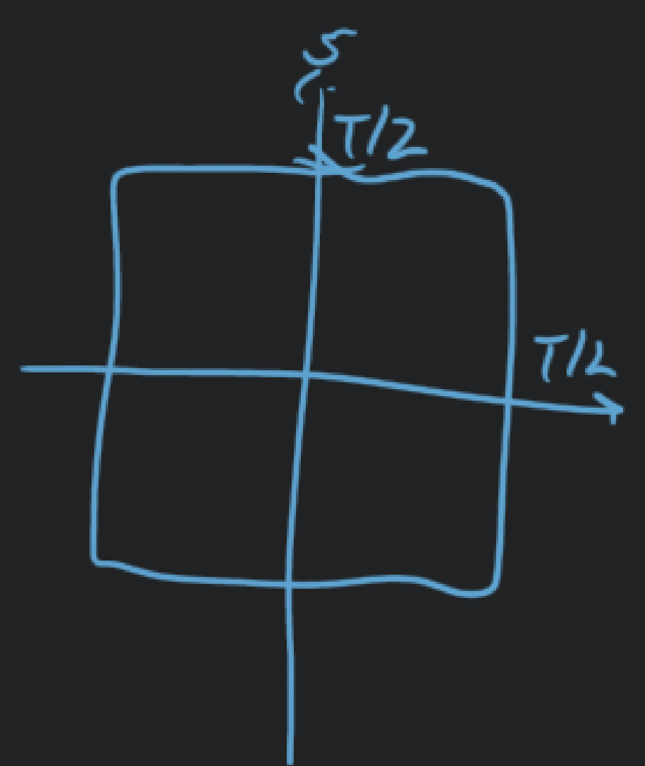
③ continuation

$\int_{-\infty}^{+\infty} F(\omega)\exp(j\omega x)\,d\omega$ is p.d.

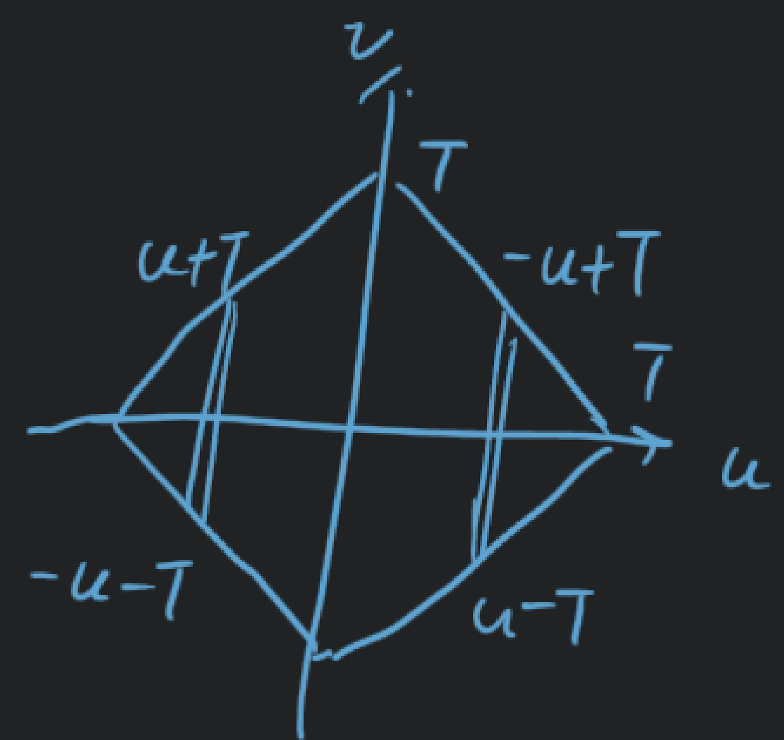
$\Rightarrow$: Fix n. choose $x_1,\cdots,x_n$ $(f(x_i-x_j))_{ij}$ is positive $\quad$ Choose $z = (\exp(j\omega x_1),\cdots,\exp(j\omega x_n))^T$.

$$0 \leq z^+ R z = \sum_{ij} f(x_i - x_j)\exp(j\omega(x_i-x_j)) \to \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds\, f(t-s)\exp(j\omega(t-s))$$

$$\begin{cases} u = t-s \\ v = t+s \end{cases} \quad 0 \leq \frac{1}{T}\iint_{\Diamond} f(u)\exp(j\omega u)\frac{1}{2}\,du\,dv = \frac{1}{T}\left(\int_{-T}^{0} du \int_{-u-T}^{u+T} dv + \int_0^T du \int_{u-T}^{-u+T} dv\right) f(u)\exp(j\omega u)\frac{1}{2}$$



$\Rightarrow$



$$= \frac{1}{T}\int_{-T}^{T} du \int_{|u|-T}^{-|u|+T} dv\, f(u)\exp(j\omega u)\frac{1}{2} = \frac{1}{T}\int_{-T}^{T} du\,(T-|u|)\,f(u)\exp(j\omega u)$$

$$= \int_{-T}^{T}\left(1 - \frac{|u|}{T}\right) du\, f(u)\exp(j\omega u) \xrightarrow{T\to+\infty} \int_{-\infty}^{+\infty} du\, f(u)\exp(j\omega u) \geq 0$$

W.W.S. $\Rightarrow$ 没有衰减,不绝对可积 $\Rightarrow$ 无法取 Fourier 变换.

取代:

$$\lim_{T\to\infty}\frac{1}{T}E\left|\int_{-T/2}^{T/2} X(t)\exp(-j\omega t)\,dt\right|^2$$ (short time) physical.

$$= \lim_{T\to+\infty}\frac{1}{T}E\left(\int_{-T/2}^{T/2} X(t)\exp(-j\omega t)\,dt\right)\left(\overline{\int_{-T/2}^{T/2} X(s)\exp(-j\omega s)\,ds}\right) \to \text{取共轭}$$

$$= \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds \; E(X(t)\overline{X(s)}) \exp(-j\omega(t-s)) = \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau \geq 0$$

$$= S_X(\omega)$$

物理含义：功率谱 $\frac{\text{功率}}{\text{频率}}$

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega) e^{j\omega\tau} d\omega \Rightarrow \int_{-\infty}^{+\infty} S_X(\omega)\,d\omega = 2\pi R_X(0) = 2\pi E(|X(t)|^2) \sim \text{功率量纲}$$

性质：$S_Z(-\omega) = S_Z(\omega)$

verify: $S_Z(-\omega) = \int_{-\infty}^{+\infty} R_X(\tau)\exp(j\omega\tau)d\tau = \int_{-\infty}^{+\infty} R_X(\tau)[\cos(\omega\tau) + j\sin(\omega\tau)]d\tau = \int_{-\infty}^{+\infty} R_X(\tau)\cos\omega\tau\, d\tau$

$= S_Z(\omega)$.

线性变换 $\overset{\text{w.w.s.}}{X(t)} \longrightarrow \boxed{H} \longrightarrow Y(t)$ $\qquad Y(t) = \int_{-\infty}^{+\infty} h(t-s)X(s)ds$

$$R_Y(t,s) = E(Y(t)Y(s)) = E\left[\int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau \int_{-\infty}^{+\infty} h(s-r)X(r)dr\right] = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dr\, h(t-\tau)h(s-r)R_X(\tau-r)$$

卷积的实质：所有积分函数的变元相加与积分变元无关（是个常数）

$\Rightarrow R_Y(t,s) = (h * \tilde{h} * R_X)(t-s)$ $\qquad$ def $\tilde{h}(t) = h(-t)$

线性变换保持宽平稳性 $\qquad S_Y(\omega) = H(\omega)\tilde{H}(\omega)S_X(\omega) = |H(\omega)|^2 S_X(\omega)$

3. Non-Stationary 非平稳

① Cyclostationary 周期平稳 $\quad \exists T \quad R_X(t,s) = R_X(t+T, s+T)$

与w.s.s.关系 $\quad$ 设 $X(t)$ Cyclostationary let $\theta \sim U(0,T)$ independent of $X(t)$ $\quad Y(t) = X(t+\theta)$ is w.s.s.

$$R_Y(t,s) = E(Y(t)Y(s)) = E(X(t+\theta)X(s+\theta)) = E_\theta\left(E_X(X(t+\theta)X(s+\theta)|\theta)\right) = E_\theta(R_X(t+\theta, s+\theta)) = \frac{1}{T}\int_0^T R_X(t+\theta, s+\theta)d\theta$$

def $\theta' = s+\theta$ $\quad R_Y(t,s) = \frac{1}{T}\int_s^{s+T} R_X(t-s+\theta', \theta')\,d\theta' = \frac{1}{T}\int_0^T R_X(t-s+\theta', \theta')\,d\theta'$ (apply cyclostationary)

补充: 条件期望 $E(Y|X)$ ① is a random variable ② $E(E(g(X,Y)|Y)) = E(g(X,Y))$. 其中内层E对X求期望 外层E对Y求期望

(逐次求期望)

$$\int_{-\infty}^{+\infty} E(g(X,Y)|X) f_X(X)dX = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} g(X,Y) f_{Y|X}(Y,X)dY\right) f_X(X)dX = \int_{-\infty}^{+\infty}dX\int_{-\infty}^{+\infty}dY\, g(X,Y)\, f_{X,Y}(X,Y) = E(g(X,Y))$$

$f_{X,Y}$ 联合p.d.f.

e.g. $X_1, \cdots, X_n$ i.i.d. N r.v. independent of $\{X_k\}$. $E(X_1+\cdots+X_N) = E(E(X_1+\cdots+X_N|N)) = E(NE(X_1)) = E(N)E(X_1)$

③ $E(Yg(X)|X) = g(X)E(Y|X)$ (被条件住的变量暂时丧失随机性)

e.g. Mean Square Estimation (均方估计) Y, X solve $g(X) \Rightarrow \min_g E((Y-g(X))^2)$

$$E((Y-g(X))^2) = E((Y-E(Y|X)+E(Y|X)-g(X))^2) = E((Y-E(Y|X))^2) + E((E(Y|X)-g(X))^2) + \underline{2E((Y-E(Y|X))(E(Y|X)-g(X)))}$$

$$E_X\left(E_Y\left((Y-E(Y|X))(E(Y|X)-g(X))|X\right)\right) = E_X\left[(E(Y|X)-g(X))E(Y-E(Y|X)|X)\right]$$

其中 $E(Y-E(Y|X)|X) = E(Y|X) - E(E(Y|X)|X) = 0$!!

Application: Pulse Amplitude Modulation (PAM)

$$X(t) = \sum_{k=-\infty}^{+\infty} \alpha_k \phi(t-kT)$$

α_k: 随机 $\quad \phi$: 基带波形固定

assume α_k w.s.s. $\quad R_X(t,s) = E(X(t)X(s)) = E\left(\sum_{k,n}\alpha_k\alpha_n\phi(t-kT)\phi(s-nT)\right)$

$$= \sum_{k,n} R_\alpha(k-n)\phi(t-kT)\phi(s-nT)$$

$X(t)$ Cyclostationary but not w.w.s.

$$R_X(t+T, s+T) = \sum_{k',n'} R_\alpha(k'-n')\phi(t-k'T)(s-n'T) \quad k'=k-1 \quad n'=n-1.$$

$$= R_X(t,s).$$

② Orthogonal Increment $X(t)$ $X(0)=0$ $\forall t_1<t_2<t_3<t_4$ $X(t_4)-X(t_3) \perp X(t_2)-X(t_1)$ (相关系数为0, $E(\Delta_{43}\Delta_{21})=0$)

正交增量 不是w.w.s.

$\forall t>s$, $R_X(t,s) = E(X(t)X(s)) = E\big((X(t)-X(s)+X(s))X(s)\big) = E\big((X(t)-X(s)+X(s))(X(s)-X(0))\big) = E(X^2(s)) = E(X^2(\min(t,s)))$

反过来也成立, Assume $R_X(t,s) = g(\min(t,s))$ $\quad E\big((X(t_4)-X(t_3))(X(t_2)-X(t_1))\big) = R_X(t_4,t_2)+R_X(t_3,t_1)-R_X(t_3,t_2)-R_X(t_4,t_1)$

$= g(t_2)+g(t_1)-g(t_2)-g(t_1) = 0.$

$X(t)$ 是正交增量过程 $\Longleftrightarrow$ $R_X(t,s)$ 只依赖 t 与 s 的最小值.

Application: Brown Motion 布朗运动 $B(t)$ def ① $B(0)=0$ ② Orthogonal Increment ③ 增量 $B(t)-B(s) \sim N(0,\sigma^2(t-s))$ $(t>s)$. σ^2是常数

$R_B(t,s) = E(B^2(\min(t,s))) = Var(B(\min(t,s))) = \sigma^2 \min(t,s)$ 即依赖于之前时刻 $\min(t,s)$

$Y(t) = \frac{d}{dt}B(t)$ is w.w.s. $\quad R_Y(t,s) = E(Y(t)Y(s)) = E\left(\frac{d}{dt}B(t)\frac{d}{ds}B(s)\right) = \frac{\partial^2}{\partial t\partial s}E(B(t)B(s)) = \frac{\partial^2}{\partial t\partial s}R_B(t,s).$

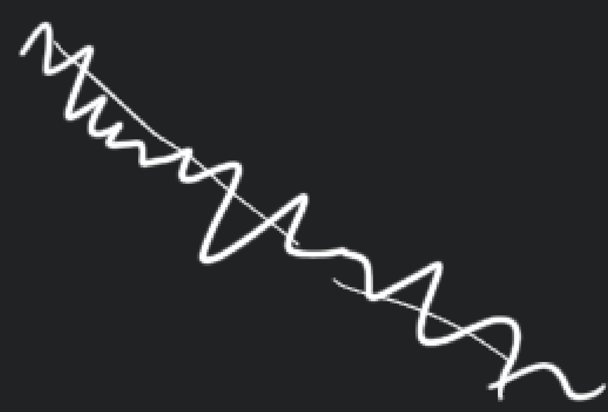
$R_Y(t,s) \sim \delta(t-s)$ $\quad = \frac{\partial^2}{\partial t\partial s}\sigma^2\min(t,s)$

use $\min(t,s) = -\frac{1}{2}(t+s-|t-s|)$.

$R_Y(t,s) = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial t\partial s}|t-s| = -\frac{\sigma^2}{2}\frac{d}{ds}\mathrm{sgn}|t-s| = \sigma^2\delta(t-s)$ $\rightarrow$ white noise

理解: 求导是高通滤波

stationary $\neq$ smooth



trend: smooth

Brown derivate: stationary

4. Multirariate Correlation 多元相关

$X = (X_1,\cdots,X_n)^T$ $\quad R_X = E(XX^T)$ Correlation Matrix $\quad R_X(i,j) = E(X_iX_j)$ $\quad R_X = R_X^T$ (对称矩阵)

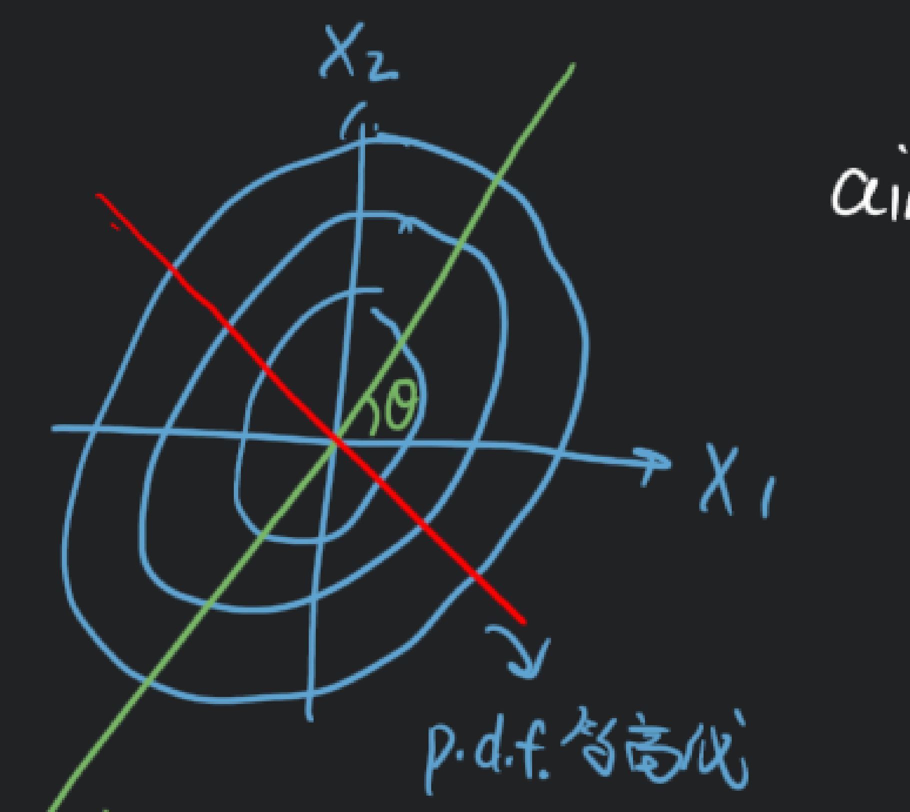
① Decorrelation (Whitening) 去相关(白化) Aim to find matrix $A\in\mathbb{R}^{n\times n}$ $\quad Y = AX \in \mathbb{R}^n$ i.e. $E(Y_iY_j) = 0\ (i\neq j)$.

$R_Y = E[AX(AX)^T] = AE(XX^T)A^T = AR_XA^T$ $\quad R_X$对称 $\Rightarrow$ 一定可以完成对角化.

$R_X = \sum_{k=1}^{n}\lambda_k U_kU_k^T = U\Lambda U^T$ $\quad U=(U_1,\cdots,U_n)$ $\quad U\cdot U^T = U^TU = I$ $\quad \Lambda = \mathrm{diag}(\lambda_1,\cdots,\lambda_n)$ $\quad \lambda_k \geq 0\ \forall k.$

Choose $A = U^T$ $Y = U^T X$ $R_Y = \Lambda$

② Principal Component Analysis (PCA)



← Aim to find this axis
(Dimensional Reduction
Compression)

aim to find $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$ i.e. $E\left[(\underset{\text{投影}}{\mathrm{Proj}_\alpha X})^2\right]$ max   (沿主方向投影方差最大)

$$E|\mathrm{Proj}_\alpha X|^2 = E|(\alpha^T X)\alpha|^2 = E|\alpha^T X|^2 |\alpha|^2 = E[(\alpha^T X)^2]$$
$$= E(\alpha^T X X^T \alpha) = \alpha^T R_X \alpha$$

拉格朗日: $L(\alpha,\lambda) = \alpha^T R_X \alpha - \lambda(|\alpha|^2 - 1)$

$\nabla_\alpha L(\alpha,\lambda) = 2R_X\alpha - 2\lambda\alpha = 0$ $\Rightarrow R_X\alpha = \lambda\alpha$ $\alpha^T R_X \alpha = \lambda\alpha^T\alpha = \lambda$

$\Rightarrow$ Eigenvector w.r.t. Largest Eigenvalue   应用: 主成分 降噪

e.g. $E(X_1) = E(X_2) = 0$ $E(X_1^2) = E(X_2^2) = 1$ $E(X_1X_2) = \rho$ $X = (X_1, X_2)^T$

$R_X = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = 1+\rho & U_1 = (1,1)^T \\ \lambda_2 = 1-\rho & U_2 = (1,-1)^T \end{matrix}$

$\Rightarrow$ 主方向角度θ和相关无关, 和 X_1, X_2 方差有关   相关决定纺锤的胖瘦 $\begin{cases} \rho = 1 \\ \rho = 0 \end{cases}$

③ Expansion $Y = U^T X \Rightarrow X = UY = \sum_{k=1}^{n} U_k Y_k$ $U_i^T U_j = 0 \ (i \neq j)$ orthogonal $E(Y_i Y_j) = 0 \ (i \neq j)$

Karhunen - Loeve Expansion (K-L展开) $\begin{cases} \text{基正交 (几何意义上)} \\ \text{系数正交 (概率意义上)} \end{cases}$ 双正交分解 Biorthogonal Expansion

Continuous $X(t) = \sum_{k=-\infty}^{\infty} \underset{\text{随机系数}}{\alpha_k} \underset{\text{基}}{\phi_k(t)}$ Assume $\{\phi_k\}$ orthogonal $\int_I \phi_i(t)\phi_j(t)\,dt = \delta_{ij}$

$\alpha_k = \int_I X(t)\phi_k(t)\,dt \Rightarrow E(\alpha_i\alpha_j) = \int_I dt \int_I ds \, R_X(t,s)\phi_i(t)\phi_j(s)$

类比：应有 $\int_I ds\, R_X(t,s)\phi_i(s) = \lambda_i \phi_i(t) \quad \forall i$ 即取 $R_X(t,s)$ 的特征函数 $\phi_i(t)$ 作为基，可证 $\{\phi_i(t)\}$ 的正交归一性

(Mercer Theorem，K-L展开对连续随机变量的推广)

assume $X(t)$ w.w.s $I = [-T/2, T/2]$ Solve equation $\int_{-T/2}^{T/2} ds\, R_X(t-s)\phi_k(s) = \lambda_k \phi_k(t) \Rightarrow \phi_k(t) = \exp\left(j\frac{2k\pi}{T}t\right)$

$$\int_{-T/2}^{T/2} R_X(t-s)\exp\left(j\frac{2k\pi}{T}s\right)ds \overset{s'=t-s}{=\!=} \left[\int_{t-T/2}^{t+T/2} R_X(s')\exp\left(-j\frac{2k\pi}{T}s'\right)ds'\right]\exp\left(j\frac{2k\pi}{T}t\right)$$

Need to assume $R_X(\tau) = R_X(\tau+T) \iff R_X(0) = R_X(T)$

$X(t)$ w.w.s. $R_X(\tau) = R_X(T+\tau) \Rightarrow X(t) = \sum_{k=-\infty}^{+\infty} \alpha_k \exp\left(j\frac{2k\pi}{T}t\right)$ (Fourier Expansion is Bi-Orthogonal)

$T\to\infty$? Stielfies Integration $\sum_k f(x_k)\Delta g(x_k) \Rightarrow \int f(x)\,dg(x)$ $\Delta g(x_k) = g(x_k) - g(x_{k-1})$

$X(t) = \int_{-\infty}^{+\infty} \exp(j\omega t)\, dF_X(\omega)$ w.w.s.随机过程的谱表示 $F_X(\omega)$: Spectral Function 形式称 ($\alpha(\omega)$可能发散)

双正交：$E\left(dF_X(\omega_1)\overline{dF_X(\omega_2)}\right) = 0 \quad (\omega_1 = \omega_2)$ $E\left(|dF_X(\omega)|^2\right) = \frac{1}{2\pi}S_X(\omega)d\omega$ 谱函数和功率谱密度的关系

$$R_X(t,s) = E(X(t)X(s)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(j(\omega_1 t - \omega_2 s))\, E\left(dF_X(\omega_1)\overline{dF_X(\omega_2)}\right) = \int_{-\infty}^{+\infty} \exp(j\omega(t-s))\, E\left(|dF_X(\omega)|^2\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(j\omega(t-s))\, S_X(\omega)\, d\omega$$

用谱表示解释线性变换 $Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau = \int_{-\infty}^{+\infty} d\tau\, h(t-\tau)\int_{-\infty}^{+\infty}\exp(j\omega\tau)dF_X(\omega) = \int_{-\infty}^{+\infty} dF_X(\omega)\left[\int_{-\infty}^{+\infty} h(t-\tau)\exp(j\omega\tau)d\tau\right]$

$$\overset{\tau'=t-\tau}{=\!=} \int_{-\infty}^{+\infty} dF_X(\omega)\left[\int_{-\infty}^{+\infty} d\tau'\, h(\tau')\exp(-j\omega\tau')\right]\exp(j\omega t) = \int_{-\infty}^{+\infty}\exp(j\omega t)H(\omega)\,dF_X(\omega) = \int_{-\infty}^{+\infty}\exp(j\omega t)\,dF_Y(\omega)$$

$dF_Y(\omega) = H(\omega)\,dF_X(\omega) \Rightarrow S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$

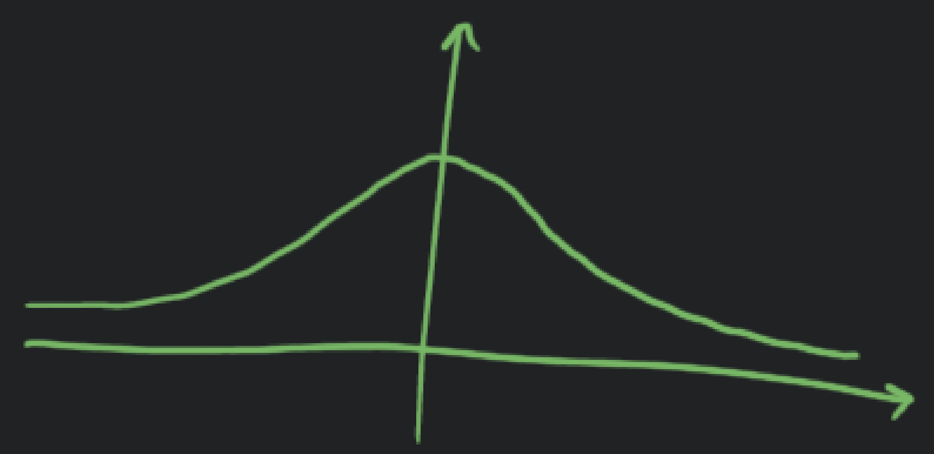
w.w.s. $X(t) \longleftrightarrow \exp(j\omega t)$ Isometry 等距同构 $\|X(t)-X(s)\|^2_{H_1} = \|\exp(j\omega t)-\exp(j\omega s)\|^2_{H_2}$

H_1 H_2 关于t

$$E(|X(t)-X(s)|^2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_x(\omega)\,|\exp(j\omega t)-\exp(j\omega s)|^2 ds$$

5. Gaussian Everywhere

① Physically Diffusion



$f(x,t)$: Density of Certain Particle

$$f(x,t+\tau) = \int_{-\infty}^{+\infty} f(x-y,t)\overset{\text{ratio}}{\rho(\tau,y)}\,dy \quad \text{(Diffusion Integral)}$$

$$f(x,t) + \tau\frac{\partial f}{\partial t} + o(\tau) = \int_{-\infty}^{+\infty}\left(f(x,t) - \frac{\partial f}{\partial x}y + \frac{1}{2}y^2\frac{\partial^2 f}{\partial x^2} + o(y^2)\right)\rho(\tau,y)\,dy$$

$$f(x,t) + \tau\frac{\partial f}{\partial t} = \int_{-\infty}^{+\infty}\left(f(x,t) - y\frac{\partial f}{\partial x} + \frac{1}{2}y^2\frac{\partial^2 f}{\partial x^2}\right)\rho(\tau,y)\,dy$$

$\rho(\tau,y)$ "Probability Density" $\int_{-\infty}^{+\infty}\rho(\tau,y)\,dy = 1$ $\int_{-\infty}^{+\infty} y\rho(\tau,y)\,dy = 0$ $\int_{-\infty}^{+\infty} y^2\rho(\tau,y)\,dy = D^2$

$$\Rightarrow f(x,t) + \tau\frac{\partial f}{\partial t} = f(x,t) + \frac{1}{2}D^2\frac{\partial^2 f}{\partial x^2} \Rightarrow \tau\frac{\partial f}{\partial t} = \frac{D^2}{2}\frac{\partial^2 f}{\partial x^2} \quad \text{(Diffusion Equation)} \quad f(x,0) = \delta(x)$$

$$\Rightarrow f(x,t) = \frac{1}{\sqrt{2\pi Dt}}\exp\left(-\frac{x^2}{2Dt}\right) \sim N(0, Dt)$$ 其中 $D^2/\tau \to D$.

② Information Theory

X r.v. $H(X) = -\int_{-\infty}^{+\infty} f_X(x)\log f_X(x)\,dx$ Maximum Entropy (Randomness)

约定：(1) $(-\infty,\infty)$ $E(X)=0$ $Var(X)=\sigma^2$ (2) $[0,\infty)$ $E(X)=\mu$ (3) $[a,b]$

Solution to (1):

$$\max_f\left(-\int_{-\infty}^{+\infty} f(x)\log f(x)\,dx\right) \text{ s.t. } \int_{-\infty}^{+\infty} xf(x)\,dx = 0 \quad \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \sigma^2 \quad \int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\delta\left[-\int_{-\infty}^{+\infty} f(x)\log f(x)\,dx + \lambda_1\left(\int_{-\infty}^{+\infty} x f(x)\,dx\right) + \lambda_2\left(\int_{-\infty}^{+\infty} x^2 f(x)\,dx - \sigma^2\right) + \lambda_3\left(\int_{-\infty}^{+\infty} f(x)\,dx - 1\right)\right] = 0$$

$$\Rightarrow \log f + 1 - \lambda_1 x - \lambda_2 x^2 - \lambda_3 = 0 \Rightarrow f(x) = \exp(\lambda_2 x^2 + \lambda_1 x + \lambda_3 - 1) \sim C e^{\lambda_2(x-\mu)^2}$$ 高斯分布

Solution to (2) $\log f + 1 - \lambda_1 x - \lambda_2 = 0 \Rightarrow f(x) = \exp(\lambda_1 x + \lambda_2 - 1) \sim Ce^{\lambda_1 x}$ 指数分布

Solution to (3) $\log f + 1 - \lambda = 0 \Rightarrow f(x) = C$ 均匀分布.

③ Probability Central Limit Theorem (CLT, 中心极限定理)

$X_1, X_2, \cdots, X_n$ i.i.d. $E(X_k)=0$ $Var(X_k)=1$ $\forall k$ $\frac{X_1+\cdots+X_n}{\sqrt{n}} \xrightarrow{n\to\infty} N(0,1)$

Characteristic Functions X r.v. $\phi_X(\omega) = E(\exp(j\omega X))$ $\phi_X(\omega) = \int_{-\infty}^{+\infty} \exp(j\omega x) f_X(x)dx$ 即 p.d.f. 的傅里叶变换

e.g. $X_1 \sim f_{X_1}(x)$ $X_2 \sim f_{X_2}(x)$ independent $X = X_1 + X_2 \sim f_{X_1} \circledast f_{X_2}$ 卷积、 $\Phi_X(\omega) = E(\exp(j\omega X)) = \Phi_{X_1}(\omega)\cdot\Phi_{X_2}(\omega) \Rightarrow$ p.d.f. 是卷积

$$\phi_{\frac{X_1+\cdots+X_n}{\sqrt{n}}}(\omega) = E\left(\prod_{k=1}^{n} \exp\left(j\omega \frac{X_k}{\sqrt{n}}\right)\right) \overset{iid}{=} \prod_{k=1}^{n} \phi_{X_k}\left(\frac{\omega}{\sqrt{n}}\right) \overset{iid}{=} \left(\phi_{X_1}\left(\frac{\omega}{\sqrt{n}}\right)\right)^n = \left(1 - \frac{\omega^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \xrightarrow{n\to\infty} \exp\left(-\frac{\omega^2}{2}\right) \xrightarrow{\text{Fourier}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \quad N(0,1) \quad \#$$

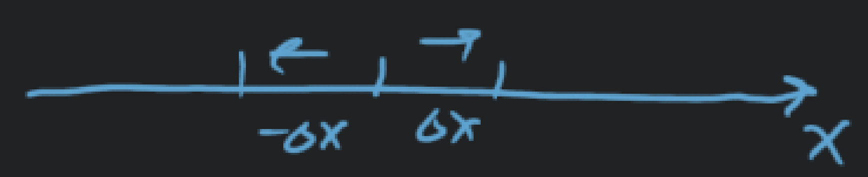
$$\phi_{X_1}\left(\frac{\omega}{\sqrt{n}}\right) = E\left(\exp\left(j\frac{\omega}{\sqrt{n}}X_1\right)\right) = E\left(1 + \frac{j\omega X_1}{\sqrt{n}} - \frac{\omega^2 X_1^2}{2n} + o\left(\frac{1}{n}\right)\right) = 1 - \frac{\omega^2}{2n} + o\left(\frac{1}{n}\right)$$

大数定律 $X_1, \cdots, X_n$ i.i.d. $\frac{X_1+\cdots+X_n}{n} \xrightarrow{n\to\infty} E(X_1)$

$$\phi_{\frac{X_1+\cdots+X_n}{n}}(\omega) = \left(\phi_{X_1}\left(\frac{\omega}{n}\right)\right)^n = \left(1 + j\frac{\omega}{n}E(X_1) + o\left(\frac{1}{n}\right)\right)^n \xrightarrow{n\to\infty} \exp(j\omega E(X_1)) \xrightarrow{\text{Fourier}} \text{Constant}.$$

恰好出现随机性的恰当分母: $\sqrt{n\ln\ln n}$ 重对数律

④ Stochastic Processes One-Dimensional Random Walk Symmetric.



$S_n = \sum_{k=1}^{n} X_n$. $X_n \sim \begin{pmatrix} \Delta x & -\Delta x \\ 1/2 & 1/2 \end{pmatrix}$ $\xrightarrow{\text{Continuous}}$ $X(t) = \left(\sum_{k=1}^{n} \frac{X_k}{\Delta x}\right)\Delta x = \left(\frac{\sum_{k=1}^{n} X_k/\Delta x}{\sqrt{n}}\right)\sqrt{\frac{t}{\Delta t}}\,\Delta x$

let $\Delta t \to 0$, $\Delta x \to 0$, $\frac{(\Delta x)^2}{\Delta t} \to D$ $X(t) \sim N(0,1)\sqrt{Dt} = N(0, Dt)$

6. Joint Gaussian

$X(t)$ Contiuous Time, Continous states. $\forall n\ \forall t_1,\cdots,t_n\ (X(t_1)\cdots X(t_n))^T = X \sim N(\mu,\Sigma)$
$\Rightarrow X(t)$ is a Gaussian Process. Joint Gaussian.

Joint Gaussian : $n=1$ $X\sim N(\mu,\sigma^2)$ $f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.

$n=2$ $(X_1,X_2)\sim N(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ $f_X(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right)\right)$.

$X\in\mathbb{R}^n$ $X\sim N(\mu,\Sigma)$ $f_X(x)=\frac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$

$E(X)=\mu$ $\Sigma=E[(X-\mu)(X-\mu)^T]$ (Covariance Matrix 协方差矩阵，对称，正定).

verify: $\int_{\mathbb{R}^n} f_X(x)dx^n = \frac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)dx^n$

$\Sigma = U^T\Lambda U$ $\Lambda=\mathrm{diag}(\sigma_1^2,\cdots,\sigma_n^2)$ $U^TU=UU^T=I$. $\Sigma^{-1}=U^T\Lambda^{-1}U$ $\Lambda^{-1}=\mathrm{diag}(\sigma_1^{-2},\cdots,\sigma_n^{-2})$

let $\begin{cases} y=U(x-\mu) \\ x=U^Ty+\mu\end{cases}$ $\begin{cases} J=\left(\frac{\partial x_i}{\partial y_j}\right)_{ij}=U^T \\ |J|=1\end{cases}$ $\Rightarrow \int_{\mathbb{R}^n} f_X(x)dx^n = \frac{1}{(2\pi)^{n/2}\prod_{k=1}^n\sigma_k}\prod_{k=1}^n\left[\int_{-\infty}^{+\infty}dy_k\exp\left(-\frac{y_k^2}{2\sigma_k}\right)\right]=1$

Characteristic Functions (n-dim) $\phi_X(\omega)=E(\exp(j\omega^TX))$ $\omega=(\omega_1,\cdots,\omega_n)^T$.

$$\phi_X(\omega) = \frac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}}\int_{\mathbb{R}^n}\exp(j\omega^T X)\exp\left(-\frac{1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu)\right)d^n X$$

1-dim : $-\frac{1}{2\sigma^2}(x-\mu)^2 + j\omega x = -\frac{1}{2\sigma^2}\left[x-(\mu+j\sigma^2\omega)\right]^2 + j\omega\mu - \frac{1}{2}\sigma^2\omega^2$

n-dim : $-\frac{1}{2}(x-\mu-j\Sigma\omega)^T\Sigma^{-1}(x-\mu-j\Sigma\omega) + j\omega^T\mu - \frac{1}{2}\omega^T\Sigma\omega$ #

$\Rightarrow \phi_X(\omega) = \exp\left(j\omega^T\mu - \frac{1}{2}\omega^T\Sigma\omega\right)$

Linearity $X\in\mathbb{R}^n$ $X\sim N(\mu,\Sigma)$ $A\in\mathbb{R}^{m\times n}$ $Y=AX$ $\Rightarrow$ $Y\sim N(A\mu, A\Sigma A^T)$

Prove : $\phi_Y(\omega) = E(\exp(j\omega^T Y)) = E(\exp(j\omega^T AX)) = E(\exp(j(A^T\omega)^T X))$

$= \exp\left(j(\omega^T A)\mu - \frac{1}{2}(\omega^T A)\Sigma(A^T\omega)\right)$ #

抽样

e.g. $(X_1,\cdots,X_n)^T\sim N(\mu,\Sigma)$ $\forall\{n_1,\cdots,n_k\}\subseteq(1,\cdots,n)$. $\Rightarrow$ $(X_{n_1},\cdots,X_{n_k})\sim N$. (高斯的边缘也是高斯)

但边缘是高斯 $\nRightarrow$ 整体是高斯 e.g. $f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)(1+\sin x_1\sin x_2)$

充分条件 : $X_1,\cdots,X_n$ independent $X_k\sim N(\mu_k,\sigma_k^2)$ $\Rightarrow$ $(X_1,\cdots,X_n)\sim N(\mu,\Sigma)$.

$\mu=(\mu_1,\cdots,\mu_n)^T$ $\Sigma=\mathrm{diag}(\sigma_1^2,\cdots,\sigma_n^2)$

充要条件 : $(X_1,\cdots,X_n)^T = X\sim N(\mu,\Sigma) \Leftrightarrow \forall\alpha\in\mathbb{R}^n$ $\alpha^T X\sim N$ (X的任意线性组合服从一维Gauss分布)

"$\Rightarrow$"显然 "$\Leftarrow$" $\forall\alpha$, $\phi_{\alpha^T X}(\omega) = \exp\left(j\omega m_{\alpha^T X} - \frac{1}{2}\omega^2\sigma_{\alpha^T X}^2\right) = E[\exp(j(\alpha^T X)\omega)]$

$$\phi_X(\alpha) = E[\exp(j\alpha^T X)] \xlongequal{\text{let } \omega = 1} \exp\left(j m_{\alpha^T X} - \frac{1}{2}\sigma^2_{\alpha^T X}\right).$$

$$m_{\alpha^T X} = E(\alpha^T X) = \alpha^T \mu. \quad \sigma^2_{\alpha^T X} = E\left[(\alpha^T X - \alpha^T\mu)(\alpha^T X - \alpha^T \mu)^T\right] = \alpha^T E[(X-\mu)(X-\mu)^T]\alpha = \alpha^T \Sigma \alpha$$

$$\Rightarrow \phi_X(\alpha) = \exp\left(j\alpha^T X - \frac{1}{2}\alpha^T \Sigma \alpha\right) \Rightarrow X \sim N(\mu, \Sigma).$$

ω的任意性由 α 可任取保证.

DDPM (2015) Denoise Diffusion Probabilistic Model

Computer Drawing $\Longleftrightarrow$ Random Number Sampling.

X_0

$$X_0 \xrightarrow[\text{加噪声}]{+N_1} X_1 \xrightarrow{+N_2} X_2 \rightarrow \cdots \rightarrow X_T \sim N(0, \sigma^2 I).$$

Direct Diffusion

$$\downarrow$$

$$X_0 \xleftarrow{NN} X_1 \xleftarrow{NN} X_2 \xleftarrow{NN} \cdots \xleftarrow{NN} X_T \sim N(0, \sigma^2 I)$$

神经网络 Reverse Influence

① Diffusion 加噪声: $X_t = \sqrt{\alpha_t}\, X_{t-1} + \sqrt{1-\alpha_t}\,\varepsilon \qquad \varepsilon \sim N(0, I) \qquad \alpha_t$: Parameter.

$$X_t = \sqrt{\alpha_t \alpha_{t-1}}\, X_{t-2} + \sqrt{\alpha_t(1-\alpha_{t-1})}\,\varepsilon_1 + \sqrt{1-\alpha_t}\,\varepsilon_2 = \sqrt{\alpha_t\alpha_{t-1}}\, X_{t-2} + \sqrt{1-\alpha_t\alpha_{t-1}}\,\varepsilon \qquad \varepsilon \sim N(0,1).$$

$$= \cdots = \sqrt{\alpha_1 \cdots \alpha_t}\, X_0 + \sqrt{1-\alpha_1 \cdots \alpha_t}\,\varepsilon. \qquad \bar{\alpha}_t = \prod_{k=1}^{t} \alpha_k \qquad X_t = \sqrt{\bar{\alpha}_t}\, X_0 + \sqrt{1-\bar{\alpha}_t}\,\varepsilon.$$

② Denoise 去噪

$$P(X_{t-1} | X_t, X_0) = \underbrace{P(X_t | X_{t-1}, X_0)}_{N(\sqrt{\alpha_t} X_{t-1},\, 1-\alpha_t)} \frac{P(X_{t-1}|X_0)}{P(X_t | X_0)} \begin{matrix} \sim N(\sqrt{\bar{\alpha}_{t-1}}\, X_0,\, 1-\bar{\alpha}_{t-1}) \\ \sim N(\sqrt{\bar{\alpha}_t}\, X_0,\, 1-\bar{\alpha}_t) \end{matrix}$$

指数项： $-\frac{1}{2}\left[\frac{(X_t-\sqrt{\alpha_t}X_{t-1})^2}{1-\alpha_t}-\frac{(X_t-\sqrt{\bar\alpha_t}X_0)^2}{1-\bar\alpha_t}+\frac{(X_{t-1}-\sqrt{\bar\alpha_{t-1}}X_0)^2}{1-\bar\alpha_{t-1}}\right]$ 配方（X_{t-1}为变量）

与X_{t-1}有关的项： $-\frac{1}{2}\left[\left(\frac{\alpha_t}{1-\alpha_t}+\frac{1}{1-\bar\alpha_{t-1}}\right)X_{t-1}^2-2X_{t-1}\left(\frac{\sqrt{\alpha_t}}{1-\alpha_t}X_t+\frac{\sqrt{\bar\alpha_{t-1}}}{1-\bar\alpha_{t-1}}X_0\right)+C(X_0,X_t)\right]$

$Var[X_{t-1}]=\left(\frac{\alpha_t}{1-\alpha_t}+\frac{1}{1-\bar\alpha_{t-1}}\right)^{-1}=\frac{(1-\alpha_t)(1-\bar\alpha_{t-1})}{1-\bar\alpha_t}$ X_{t-1}仍服从高斯分布

$\mu[X_{t-1}]=\left[\frac{\sqrt{\alpha_t}}{1-\alpha_t}X_t+\frac{\sqrt{\bar\alpha_{t-1}}}{1-\bar\alpha_{t-1}}X_0\right]\frac{(1-\alpha_t)(1-\bar\alpha_{t-1})}{1-\bar\alpha_t}=\frac{\sqrt{\alpha_t}(1-\bar\alpha_{t-1})}{1-\bar\alpha_t}X_t+\frac{\sqrt{\bar\alpha_{t-1}}(1-\alpha_t)}{1-\bar\alpha_t}X_0$

③ $X_0\to\frac{1}{\sqrt{\bar\alpha_t}}(X_t-\sqrt{1-\bar\alpha_t}\,\varepsilon)$

$\mu[X_{t-1}]=\frac{1}{\sqrt{\alpha_t}}X_t-\frac{\sqrt{\bar\alpha_{t-1}}(1-\alpha_t)}{\sqrt{1-\bar\alpha_t}}\varepsilon$

$X_1,\cdots,X_n\overset{i.i.d.}{\sim}N(0,\sigma^2)$ $\bar X=\frac{1}{n}\sum_{k=1}^n X_k$ Sample Mean

$\bar S=\frac{1}{n-1}\sum_{k=1}^n(X_k-\bar X)^2$ Sample Variance

无偏性： $E(\bar X)=\mu$ $E(\bar S)=\sigma^2$ Don't need to assume $X_k\sim N$

$X_k\sim N\Rightarrow\bar X,\bar S$ independent. $\sum_{k=1}^n(X_k-\bar X)^2=\sum_{k=1}^n X_k^2-n(\bar X)^2$ let $Y=AX$ $\sum_{k=1}^n X_k^2=\sum_{k=1}^n Y_k$ $\sum_{k=1}^n\frac{X_k}{\sqrt n}=Y_1$

$A=\begin{pmatrix}1/\sqrt n & \cdots & 1/\sqrt n\\ & \vdots & \\ & \text{构造正交基} & \\ & \vdots & \end{pmatrix}$ i.e. $A^TA=I\Rightarrow\sum_{k=1}^n X_k^2=\sum_{k=1}^n Y_k^2$ $X\sim N(\mu\cdot\mathbb{1},\sigma^2 I)\Rightarrow Y\sim N(A(\mu\mathbb{1}),\underbrace{A\sigma^2IA^T}_{=\sigma^2I})$

$\Rightarrow Y_1,\cdots,Y_n$ independent $Y_1=\frac{1}{\sqrt n}\sum_{k=1}^n X_k$ $(n-1)\bar S=\sum_{k=2}^n Y_k^2\Rightarrow\bar X,\bar S$ independent.

Conditional Distribution of Gaussian

$X=(\overset{n_1}{X_1},\overset{n_2}{X_2})^T\in\mathbb{R}^{m\times(n_1+n_2)}$ $X\sim N(\mu,\Sigma)$ $X_1|X_2\sim N$ （联合高斯 ⇒ 条件高斯）

$$f_{x_1|x_2}(x_1|x_2) = \frac{f_{x_1x_2}(x_1,x_2)}{f_{x_2}(x_2)}$$ assume $\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$ $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$ $\Sigma_{12}^T = \Sigma_{21}$ Σ_{11}, Σ_{22} 对称.

$$= C\exp\left(-\frac{1}{2}(x_1^T-\mu_1^T, x_2^T-\mu_2^T)\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}^{-1}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}\right) \Big/ \exp\left(-\frac{1}{2}(x_2^T-\mu_2^T)\Sigma_{22}^{-1}(x_2-\mu_2)\right)$$

$$\begin{pmatrix}I & -\Sigma_{12}\Sigma_{22}\\ 0 & I\end{pmatrix}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}I & 0\\ -\Sigma_{22}^{-1}\Sigma_{21} & I\end{pmatrix} = \begin{pmatrix}\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0\\ 0 & \Sigma_{22}\end{pmatrix}$$

$$\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}^{-1} = \begin{pmatrix}I & 0\\ -\Sigma_{22}^{-1}\Sigma_{21} & I\end{pmatrix}\begin{pmatrix}(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0\\ 0 & \Sigma_{22}\end{pmatrix}\begin{pmatrix}I & -\Sigma_{12}\Sigma_{22}\\ 0 & I\end{pmatrix}$$

$$(x_1^T-\mu_1^T, x_2^T-\mu_2^T)\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}^{-1}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix} - (x_2^T-\mu_2^T)\Sigma_{22}^{-1}(x_2-\mu_2) = \left(x_1^T-\mu_1^T-(x_2^T-\mu_2^T)\Sigma_{22}^{-1}\Sigma_{21}\right)(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}$$
$$\left(x_1-\mu_1-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)\right)$$

投影

$$E(x_1|x_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) \sim \mu_1 + \frac{\sigma_{21}}{\sigma_{22}}(x_2-\mu_2)$$ $$\Sigma_{x_1|x_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Optimal 最优估计 ⟵ 对 Gaussian 等价 ⟶ Optimal Linear 最优线性估计

获取信息 ⇒ 方差减小

Linear Gaussian System

内在状态 (不能被直接观测) X Noise ε 可观测: Observation (Data) Y 从可观测量反推内在状态.

Model: $\underset{m\times 1}{Y} = \underset{m\times n}{A}\underset{n\times 1}{X} + \underset{m\times 1}{\varepsilon}$ $X\sim N(\mu, \Sigma_X)$ $\varepsilon \sim N(0, \Sigma_N)$ X, ε independent

$$\begin{pmatrix}X\\ Y\end{pmatrix} = \begin{pmatrix}I & 0\\ A & I\end{pmatrix}\begin{pmatrix}X\\ \varepsilon\end{pmatrix} \sim N \Rightarrow X|Y \sim N$$ $$E(X|Y) = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(Y-\mu_Y)$$

$$\Sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)^T] = E[(X-\mu)(AX+\varepsilon-A\mu)^T] = \Sigma_X A^T$$ $$\Sigma_{YY} = E[(Y-\mu_Y)(Y-\mu_Y)^T] = A\Sigma_X A^T + \Sigma_N$$

$$\Rightarrow E(X|Y) = \mu + \Sigma_X A^T(A\Sigma_X A^T+\Sigma_N)^{-1}(Y-A\mu)$$ $$\Sigma_{X|Y} = \Sigma_X - \Sigma_X A^T(A\Sigma_X A^T + \Sigma_N)^{-1}A\Sigma_X$$

e.g. $X, Y \overset{i.i.d}{\sim} N(0,1) \Rightarrow E(X+Y|X-Y) = ?$ $\begin{pmatrix} X+Y \\ X-Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$ $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(0, I) \Rightarrow \begin{pmatrix} X+Y \\ X-Y \end{pmatrix} \sim N(0, 2I)$

$\Rightarrow X+Y, X-Y$ independent $E(X+Y|X-Y) = E(X+Y) = 0$

7. Gaussian and Nonlinear

$X(t)$ Gaussian $Y(t) = g(X(t))$ g: Nonlinear, No Memory Generally $Y(t)$ Non-Gaussian $E(Y(t)) = ?$ $R_Y(t,s) = ?$

① Square $X(t)$ is GP. $E(X(t)) = 0$ w.w.s. $R_X(\tau)$. $Y(t) = X^2(t) \geq 0$ Non GP.

$E(Y(t)) = E(X^2(t)) = R_X(0) \geq 0$. $R_Y(t,s) = E(Y(t)Y(s)) = E(X^2(t)X^2(s))$

GP $X(t)$. $(X_1, X_2, X_3, X_4) \sim N$ $E(X_k) = 0$ $E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$

Characteristic Function $(X_1, \cdots, X_n)^T = X$ $\phi_X(\omega) = E(\exp(j\omega^T X)) = E(\exp(j\omega_1 X_1 + \cdots + j\omega_n X_n))$.

$\Rightarrow$ cal 矩 $E(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = \frac{1}{j^{\alpha_1+\cdots+\alpha_n}} \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial\omega_1^{\alpha_1}\cdots\partial\omega_n^{\alpha_n}} \phi_X(\omega) \Big|_{\omega=(0,\cdots,0)}$

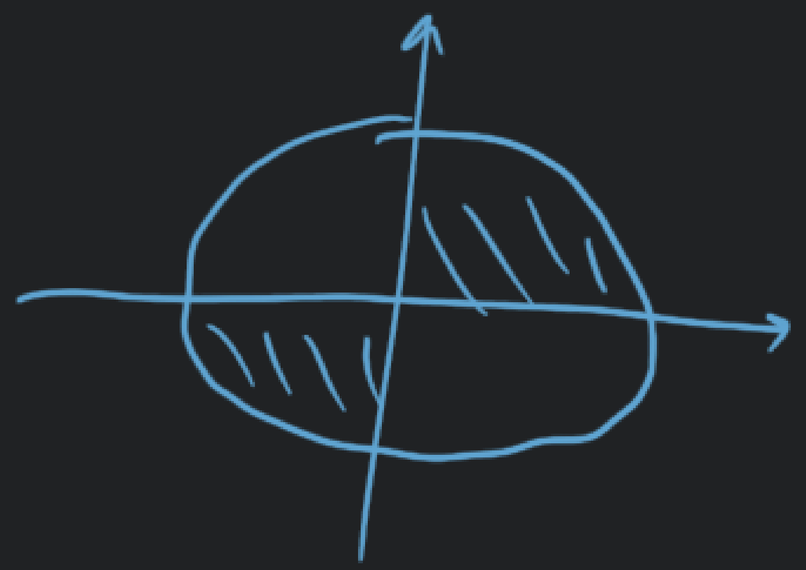
Cal $\frac{\partial^4}{\partial\omega_1\partial\omega_2\partial\omega_3\partial\omega_4} \exp\left(-\frac{1}{2}\sum_{i,j=1}^{4} \Sigma_{ij}\omega_i\omega_j\right)$

$\Rightarrow R_Y(t,s) = E(X^2(t))E(X^2(s)) + 2E(X(t)X(s))^2 = R_X^2(0) + 2R_X^2(t-s)$.

② Hard Limiter (Polar) $X(t)$ GP, $E(X(t)) = 0$, $Y(t) = \operatorname{sgn}(X(t)) = \begin{cases} 1 & X(t) > 0 \\ -1 & X(t) < 0 \end{cases}$.

$$E(Y(t)) = 1\cdot P(X(t)>0) - 1\cdot P(X(t)<0) = 0.$$

$$R_Y(t,s) = E(Y(t)Y(s)) = 1\cdot P(X(t)X(s)>0) + (-1)\cdot P(X(t)X(s)<0)$$ 记 $p = P(X(t)X(s)>0)$.



$X(t)$ $X(s)$ independent

$p = 1/2$

$X(t), X(s)$ non-independent

$p > 1/2$

$$f_{X(t)X(s)}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1}{\sigma_1}\right)^2 + \left(\frac{x_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1}{\sigma_1}\right)\left(\frac{x_2}{\sigma_2}\right)\right]\right)$$

$$p = \left(\int_0^\infty\int_0^\infty + \int_{-\infty}^0\int_{-\infty}^0\right) f_{X(t)X(s)}(x_1,x_2)\,dx_1\,dx_2.$$

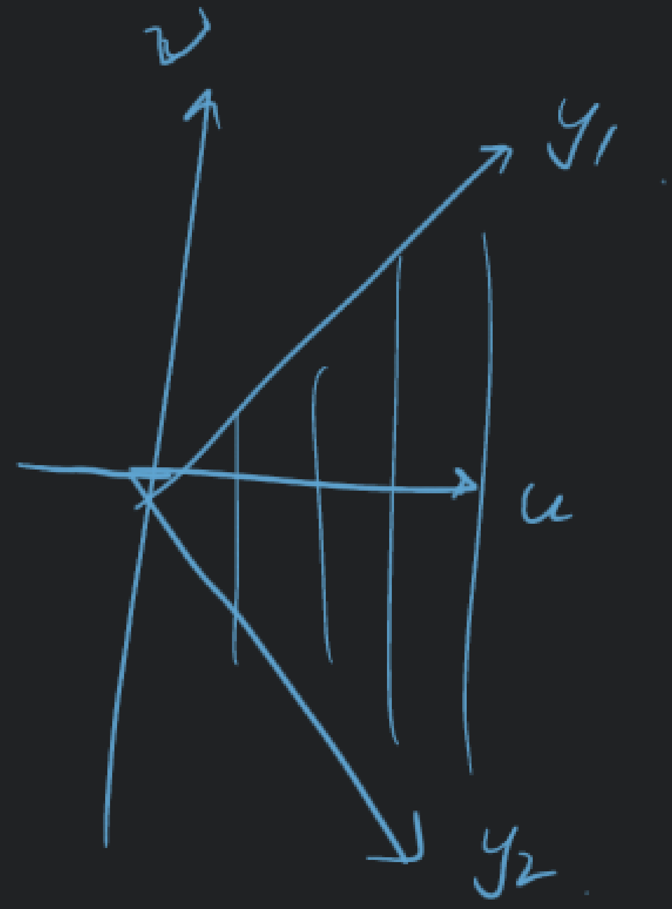
$$y_1 \triangleq \frac{x_1}{\sqrt{1-\rho^2}\,\sigma_1} \qquad y_2 = \frac{x_2}{\sqrt{1-\rho^2}\,\sigma_2}.$$

$$p = \int_0^\infty\int_0^\infty \frac{\sqrt{1-\rho^2}}{\pi}\exp\left(-\frac{1}{2}(y_1^2+y_2^2-2\rho y_1y_2)\right)dy_1dy_2$$

$y_1 \triangleq u+v \qquad y_2 \triangleq u-v.$

$$\Downarrow$$

$$\exp\left(-(1-\rho)u^2-(1+\rho)v^2\right) \qquad 2\,du\,dv$$



$$p = \iint_{\triangleleft} \frac{2\sqrt{1-\rho^2}}{\pi}\exp\left(-(1-\rho)u^2-(1+\rho)v^2\right)$$

$u' = \sqrt{1-\rho}\,u, \quad v' = \sqrt{1+\rho}\,v. \qquad u' = \rho\cos\theta \quad v' = \rho\sin\theta.$

$$\Rightarrow p = \frac{2}{\pi}\arctan\sqrt{\frac{1+\rho}{1-\rho}} = \frac{1}{\pi}\arccos(-\rho) = \frac{1}{2} + \frac{1}{\pi}\arcsin\rho.$$

$\rho>0 \Rightarrow p>1/2$

$\rho<0 \Rightarrow p<1/2.$

$$\Rightarrow R_Y(t,s) = \frac{2}{\pi}\arcsin\rho.$$

"Arcsin Law"

Price Theorem $\quad (X_1,X_2)\sim N(0,0,\sigma_1^2,\sigma_2^2,\rho) \quad g(X_1,X_2)$ Nonlinear，增长慢于高斯的衰减

$$\frac{\partial E(g(X_1,X_2))}{\partial\rho} = \sigma_1\sigma_2 E\left(\frac{\partial^2 g}{\partial X_1\partial X_2}(X_1,X_2)\right).$$

$$g(X_1,X_2) = \operatorname{sgn}(X_1)\operatorname{sgn}(X_2) \qquad \frac{\partial^2 g}{\partial X_1\partial X_2} = 4\delta(X_1)\delta(X_2). \qquad \frac{\partial E(g)}{\partial\rho} = 4\sigma_1\sigma_2\cdot\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} = \frac{2}{\pi\sqrt{1-\rho^2}}$$

$$\Rightarrow R_Y(t,s) = E(g)\Big|_{\rho} = E(g)\Big|_{\rho=0} + \frac{2}{\pi}\arcsin\rho = \frac{2}{\pi}\arcsin\rho = \frac{2}{\pi}\arcsin\left(\frac{R_X(t-s)}{R_X(0)}\right) \quad \text{for } X(t) \text{ w.w.s.}$$

$$E(X^2(t)X^2(s)) \qquad g(X_1,X_2) = X_1^2X_2^2 \qquad \frac{\partial^2 g}{\partial X_1\partial X_2} = 4X_1X_2 \qquad \frac{\partial}{\partial\rho}E(X^2(t)X^2(s)) = 4\sigma_1^2\sigma_2^2\rho \qquad E(X^2(t)X^2(s))\Big|_{\rho=0} = \sigma_1^2\sigma_2^2$$

$$\Rightarrow E(X^2(t)X^2(s)) = \sigma_1^2\sigma_2^2(1+2\rho^2) = 2R_X^2(t,s) + R_X^2(0)$$

Prove Price Theorem.

$$E(g(X_1,X_2)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(X_1,X_2)\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{X_1}{\sigma_1}\right)^2+\left(\frac{X_2}{\sigma_2}\right)^2-2\rho\left(\frac{X_1}{\sigma_1}\right)\left(\frac{X_2}{\sigma_2}\right)\right)\right]dx_1dx_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x_1,x_2)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi_{X_1X_2}(\omega_1,\omega_2)\exp(-j(\omega_1x_1+\omega_2x_2))\,d\omega_1d\omega_2\,dx_1dx_2$$

$$\phi_{X_1X_2}(\omega_1,\omega_2) = \exp\left(-\frac{1}{2}(\sigma_1^2\omega_1^2+\sigma_2^2\omega_2^2+2\rho\sigma_1\sigma_2\omega_1\omega_2)\right) \qquad (\text{因为 } \mu=(0,0)^T)$$

$$\frac{\partial E(g(X_1,X_2))}{\partial\rho} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx_1dx_2\, g(x_1,x_2)(-\sigma_1\sigma_2)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\omega_1d\omega_2\,\phi_{X_1X_2}(\omega_1,\omega_2)\exp(-j(\omega_1x_1+\omega_2x_2))\,\omega_1\omega_2$$

$$= \sigma_1\sigma_2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx_1dx_2\, g(x_1,x_2)\frac{\partial^2}{\partial X_1\partial X_2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi_{X_1X_2}(\omega_1,\omega_2)\exp(-j(\omega_1x_1+\omega_2x_2))\,d\omega_1d\omega_2$$

$$= \sigma_1\sigma_2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx_1dx_2\, g(x_1,x_2)\frac{\partial^2}{\partial x_1\partial x_2}f_{X_1,X_2}(X_1,X_2)$$

$$= \sigma_1\sigma_2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx_1dx_2\,\frac{\partial^2 g}{\partial X_1\partial X_2}(X_1,X_2)\, f_{X_1,X_2}(X_1,X_2) = \sigma_1\sigma_2 E\left(\frac{\partial^2 g}{\partial X_1\partial X_2}(X_1,X_2)\right) \qquad \#$$

③ $ReLu(x) = \begin{cases} x & x\geq 0 \\ 0 & x<0 \end{cases}$ $\quad X(t)$ G.P. $\quad E(X(t))=0 \quad Y(t) = ReLu(X(t))$

$$g(X_1,X_2) = ReLu(X_1)ReLu(X_2) \qquad \frac{\partial^2 g}{\partial X_1\partial X_2} = U(X_1)U(X_2) \qquad \left(U(x) = \begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}\right)$$

$E(U(X_1)U(X_2)) = P(X_1 \geq 0, X_2 \geq 0) = \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho$ $\quad \frac{\partial}{\partial\rho}h(\rho) = \sigma_1\sigma_2\left(\frac{1}{4} + \frac{1}{2\pi}\arcsin\rho\right)$ $\quad h(0) = \frac{1}{4}$

$\Rightarrow$ Solve $R_Y(t, s)$

总结：处理非线性过程：① 特征函数 ② 矩 ③ Price 定理

8. Application of Gaussian Process

① Discriminate Analysis (Classification)

N Categories $C_1, \cdots, C_N$ Data $X_1, \cdots, X_n$ $\exists i$ $X_k \in C_i \sim N(\mu_i, \Sigma_i)$ (为每个数据归类，每个类是一个 Gauss 分布)

Search i : $\min\limits_i (X_k - \mu_i)^T \Sigma_i^{-1} (X_k - \mu_i)$ (Malanokis Distance, 马氏距离)

Consider Prior knowledge? Bayesian $\max\limits_i P(C_i | X_k) = \max\limits_i \frac{P(X_k|C_i)P(C_i)}{P(X_k)} = \max\limits_i \frac{P(X_k|C_i)\boxed{P(C_i)}}{\sum_j P(X_k|C_j)\boxed{P(C_j)}}$ 先验知识

$\Leftrightarrow \max\limits_i \left[\log P(C_i) + \log P(X_k|C_i) - \log\left(\sum_j P(X_k|C_j)P(C_j)\right)\right]$ (MAP, 最大后验分类)

$\log P(X_k|C_i) = -\frac{1}{2}(X_k - \mu_i)^T \Sigma_i^{-1} (X_k - \mu_i) = X_k^T \beta_i + r_i - \frac{1}{2} X_k^T \Sigma_i^{-1} X_k$.

Assume $P(C_i) = \frac{1}{N}$ = Constant. (Don't have information about $P(C_i)$)

$\Leftrightarrow \max\limits_i \frac{\exp\left(-\frac{1}{2}(X_k - C_i)^T \Sigma_i^{-1} (X_k - C_i)\right)}{\sum_j \exp\left(-\frac{1}{2}(X_k - C_j)^T \Sigma_j^{-1} (X_k - C_j)\right)}$ (Nearest Centroid Classification, 最近中心分类)

$\Leftrightarrow \max\limits_i \frac{\exp(X_k^T \beta_i + r_i)}{\sum_j \exp(X_k^T \beta_j + r_j)}$ (Softmax, assume Σ_i all same)

② Gaussian Process Regression

$(X_1, \cdots, X_n) \longrightarrow X_{n+1}$

Time Series

1. Regard $(X_1, \cdots, X_n, X_{n+1})$ as Gaussian Process.
2. Estimation of Hyperparameters of this GP (Mean, Covariance).
3. $E(X_{n+1} | X_n, \cdots, X_1) = \mu_{n+1} + C_{(n+1),(1\cdots n)} C^{-1}_{(1:n)(1:n)} ((X_1, \cdots, X_n) - (\mu_1, \cdots, \mu_n))^T$.

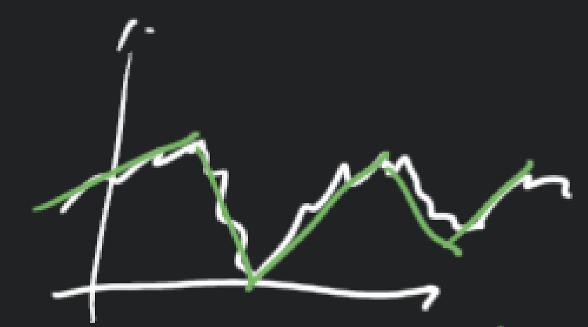
Assume $\mu_k = 0 \ \forall k$. $\quad C_{(1:n),(1:n)}(i,j) = E(X_i X_j) \xrightarrow{\text{kernal}} K(X_i, X_j)$ $\quad$ eg. $K(X_i, X_j) = \exp\left(-\frac{\|X_i - X_j\|^2}{\sigma_x^2}\right)$.

(Radial Basis, 镜像基)

③ Finance v.s. Gaussian Process (Brown Motion).

$B(t)$: GP. Independent Increment. $\quad B(t) - B(s) \sim N(0, \sigma^2(t-s)) \quad B(0) = 0$

Bachelier (1900) $\quad B(t) \longrightarrow$ Stock Price $\qquad$ Samuelson (193X) $\quad B(t) \longrightarrow \exp(\mu t + B(t)) \geq 0$



Geometric Brown Motion (几何布朗运动)

Ito (1944) $\quad f(t, B(t)) \Rightarrow df(t, B(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB(t) \quad \times$

$dB(t) \doteq B(t+dt) - B(t) \sim N(0, \sigma^2 dt) \qquad E(dB(t)) = 0 \quad E[(dB(t))^2] = \sigma^2 dt \quad \Rightarrow \boxed{dB(t) \sim \sigma \sqrt{dt}}$

$$\boxed{df(t, B(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB(t) + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} dt}$$

(Ito Formula) (Stochastic Calculus)

e.g. $\int_0^1 B(t) dB(t) \neq \int_0^1 d(\frac{1}{2} B^2(t)) \qquad d(\frac{1}{2} B^2(t)) = B(t) dB(t) + \frac{1}{2} dt. \Rightarrow \int_0^1 B(t) dB(t) = (\frac{1}{2} B^2(t) - \frac{1}{2} t)\Big|_0^1 = \frac{1}{2} B^2(1) - \frac{1}{2}$.

GBO $\quad S(t) = \exp(\mu t + \sigma^2 B(t)) \qquad dS(t) = S(t)((\mu + \frac{\sigma^4}{2}) dt + \sigma^2 dB(t))$.

(Black − Scholes − Merton) 197X, Option Pricing $V(t, S(t))$

Interest Rate

Portfolio (资产组合) $P(t) = V(t,S(t)) - \alpha S(t)$ 卖出股票，买入期权 $dP(t) = rP(t)dt$.

$$dP(t) = dV(t,S(t)) - \alpha\, dS(t) = \frac{\partial V}{\partial t}dt + \left(\frac{\partial V}{\partial S} - \alpha\right)dS(t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\underbrace{(dS(t))^2}_{=\sigma^4 S^2 dt} = rP(t)dt$$

消去随机性 $\alpha = \frac{\partial V}{\partial S}$.

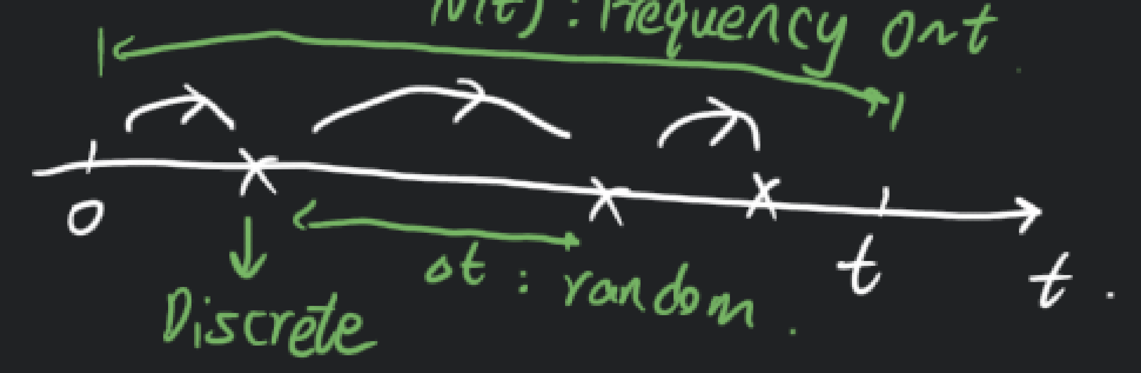
$$\Rightarrow dP(t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^4 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt = rP(t)dt$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^4 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$ (Black − Scholes Equation)

Boundary Condition 交割日 $V(T) = \mathrm{ReLu}(S(T) - K)$

Backward: $V(0) \xleftarrow{\text{Reference}} V(T)$

9. Possion Process



$N(t)$: Counting Process / Point Process $N(0) = 0$

Basic Condition: (1) Independent Increment $\forall t_1 < t_2 \le t_3 < t_4$ $X(t_4) - X(t_3)$, $X(t_2) - X(t_1)$ Independent.

(2) Stationary Increment $\forall t_1 < t_2$ $X(t_2) - X(t_1) \sim p(t_2 - t_1)$ ← Distribution

$P(N(t) = k) = ?$ $P(N(t_2) - N(t_1) = k) = ?$

① Choosing Tool: Moment Generating Functions (MGF) 矩母函数

X r.v. $P(X = x_k) = p_k$ $k = 1, 2, \cdots$ $\Rightarrow G_X(z) = E(z^X) = \sum_k z^k P(X = x_k) = \sum_k z^k p_k$ (z − Transform). $z \to \exp(j\omega) \to$ 特征函数

② $G_{N(t)}(z) = E(z^{N(t)})$ Differential Equation

$$\frac{1}{\Delta t}\left(G_{N(t+\Delta t)}(z) - G_{N(t)}(z)\right) = \frac{1}{\Delta t}\left[E\left(z^{N(t+\Delta t)} - z^{N(t)}\right)\right] = \frac{1}{\Delta t}E\left(z^{N(t)}\left(z^{N(t+\Delta t) - N(t)} - 1\right)\right)$$

$N(t) - N(0)$, $N(t+\Delta t) - N(t)$ independent $\Rightarrow$

$$= \frac{1}{\Delta t}G_{N(t)}(z)\, E\left(z^{N(t+\Delta t) - N(t)} - 1\right) = \frac{1}{\Delta t}G_{N(t)}(z)\, E\left(z^{N(\Delta t)} - 1\right)$$

$$E\left(z^{N(\Delta t)}-1\right) = \sum_{k=0}^{\infty} z^k P(N(\Delta t)=k) - 1 = P(N(\Delta t)=0) - 1 + zP(N(\Delta t)=1) + \sum_{k\geq 2} z^k P(N(\Delta t)=k)$$

$P(N(t)=0) = P(N(s)=0,\ N(t)-N(s)=0)\ (\forall s\in(0,t)) = P(N(s)=0)\,P(N(t-s)=0)$

Let $g(t) = P(N(t)=0) \Rightarrow g(t)=g(s)g(t-s)\ \forall s\in(0,t) \Leftrightarrow g(t+s)=g(t)g(s)\ \forall s,t>0$. Assume that $g(t)$ is continuous.

① $g(t) = \left(g(\tfrac{t}{2})\right)^2 \geq 0$ Assume $\exists x_0\ g(x_0)=0 \Rightarrow g(\frac{x_0}{2^n})=0 \xrightarrow{n\to\infty} g(0)=0 \Rightarrow g(t)=g(t)g(0)=0$ So assume $g(t)>0\ \forall t$

② $h(t)=\log g(t)$ $h(t+s)=h(t)+h(s)$. $\Rightarrow h(0)=0$ $\forall t\in\mathbb{N}^*\ h(t)=h(t-1)+h(1)=\dots=th(1)=\alpha t$ let $\alpha=h(1)$.

$\forall t\in\mathbb{Z}-\mathbb{N}$ $h(t)=0-h(-t)=-\alpha(-t)=\alpha t$. $\forall t\in\mathbb{Q}$ $t=\frac{n}{m}$ $n,m\in\mathbb{Z}$ $h(\frac{1}{m})=\frac{1}{m}h(1)=\frac{\alpha}{m}$. $h(\frac{n}{m})=\frac{\alpha n}{m}$

③ $t\in\mathbb{R}$ $\exists\{g_k\}\subseteq\mathbb{Q}$ $\lim_{n\to\infty} g_n=t$ $h(t)=h(\lim_{n\to\infty} g_n)=\lim_{n\to\infty} h(g_n)=\lim_{n\to\infty}(\alpha g_n)=\alpha t$. $\Rightarrow g(t)=\exp(-\alpha t)=P(N(t)=0)$. $(\alpha\to-\alpha)$

$P(N(t)=1) = P(N(s)=1, N(t-s)=0)+P(N(s)=0, N(t-s)=1) = P(N(s)=1)\,e^{-\alpha(t-s)} + P(N(t-s)=1)\,e^{\alpha}$ $\forall s\in(0,t)$

let $g(t)=P(N(t)=1)$ $\Rightarrow g(t+s)=g(t)\exp(-\alpha s)+g(s)\exp(-\alpha t)$ $\forall t,s>0$. $\Rightarrow \dfrac{g(t+s)}{\exp(-\alpha(t+s))} = \dfrac{g(t)}{\exp(-\alpha t)} + \dfrac{g(s)}{\exp(-\alpha s)}$.

$\Rightarrow g(t)=\beta t\exp(-\alpha t)$. Can prove $\beta=\alpha$.

考虑收敛域：$z\leq 1$ 内

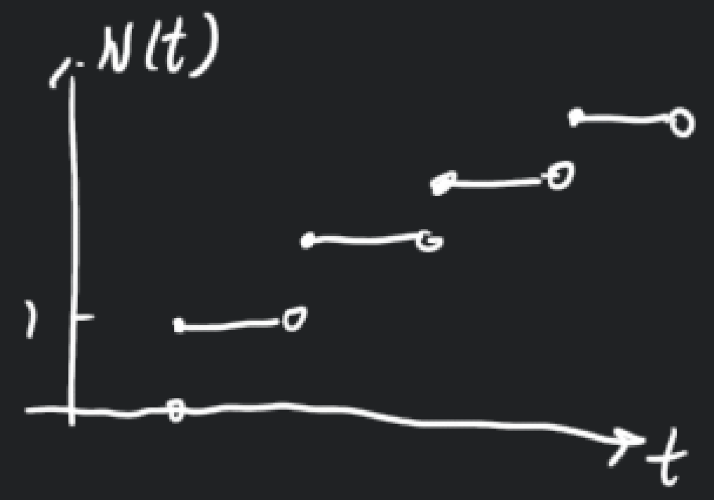
$$\left|\sum_{k\geq 2} z^k P(N(\Delta t)=k)\right| \leq \sum_{k\geq 2}|z|^k P(N(\Delta t)=k) \leq \sum_{k\geq 2} P(N(\Delta t)=k) = \underbrace{1-P(N(\Delta t)=0)}_{\sim\alpha\Delta t} - \underbrace{P(N(\Delta t)=1)}_{\sim\alpha\Delta t} = o(\Delta t)$$

$$\Rightarrow \frac{1}{\Delta t}E\left(z^{N(\Delta t)}-1\right) \xrightarrow{\Delta t\to 0} -\alpha+\alpha z=\alpha(z-1) \Longrightarrow \frac{\partial G_{N(t)}(z)}{\partial t} = G_{N(t)}(z)\,\alpha(z-1)$$ Boundary Condition $G_{N(0)}(z)=1$

$$\Rightarrow G_{N(t)}(z) = \exp[\alpha(z-1)t] = \sum_k z^k P(N(t)=k) = \exp(-\alpha t)\sum_k \frac{(\alpha t)^k}{k!}z^k.$$

$$\Rightarrow P(N(t)=k) = \frac{(\alpha t)^k}{k!}\exp(-\alpha t) \qquad P(N(t)-N(s)) = \frac{[\alpha(t-s)]^k}{k!}\exp(-\alpha(t-s)).$$ Possion Distribution

$E(N(t))=\alpha t$ $\alpha=E(N(t))/t$ (Intensity)



Jump Process

第1次跃迁时刻 T_1 的 c.d.f. (Probability Transformation) $F_{T_1}(t) = P(T_1 \le t) = 1 - P(N(t)=0) = 1-e^{-\lambda t}$ $\quad t \ge 0$.

$\Rightarrow$ p.d.f. $f_{T_1}(t) = \begin{cases} \lambda\exp(-\lambda t) & t\ge 0 \\ 0 & t<0 \end{cases}$ Exponential Distribution

事件发生的间隔 $T_n \overset{i.i.d.}{\sim} Exp(\lambda)$ $\quad E(T_1)=\frac{1}{\lambda}$ Non-Memory

e.g. $X(t) = (-1)^{N(t)}$ Random Telegraph $\quad R_X(t,s) = E(X(t)X(s)) = E((-1)^{N(t)+N(s)}) = E((-1)^{N(t)-N(s)+2N(s)}) = E((-1)^{N(t-s)})$

$R_X(t,s) = R_X(t-s) = \sum_{k=0}^{\infty} (-1)^k \frac{[\lambda(t-s)]^k}{k!}\exp(-\lambda(t-s)) = \exp(-2\lambda(t-s))$ $\quad (t>s)$.

e.g. 事件发生的时刻 S_n $\quad (S_n = \sum_{k=1}^{n} T_k)$ $\quad \phi_{S_n}(\omega) = E(\exp(j\omega S_n)) = [E(\exp(j\omega T_1))]^n = \phi_{T_1}^n(\omega)$.

$\phi_{T_1}(\omega) = \int_0^\infty \lambda\exp(-\lambda t)\exp(j\omega t)\,dt = \frac{\lambda}{\lambda-j\omega}$ $\quad \Rightarrow \phi_{S_n}(\omega) = \left(\frac{\lambda}{\lambda-j\omega}\right)^n \xrightarrow{\text{反变换}}$ 积分难处理

$F_{S_n}(t) = P(S_n \le t) = P(N(t)\ge n) = \sum_{k=n}^{\infty}\frac{(\lambda t)^k}{k!}\exp(-\lambda t)$ $\quad f_{S_n}(t) = \sum_{k=n}^{\infty}\left(\frac{(\lambda t)^{k-1}}{(k-1)!} - \frac{(\lambda t)^k}{k!}\right)\lambda e^{-\lambda t} = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}\exp(-\lambda t)$ $\quad (t\ge 0)$ Gamma Distribution

放弃平稳性的点过程 Assume $\lim_{\Delta t\to 0}\frac{P(N(t+\Delta t)-N(t)=0)-1}{\Delta t} = -\lambda(t)$ $\quad \Rightarrow \lim_{\Delta t\to 0}\frac{P(N(t+\Delta t)-N(t)=1)}{\Delta t} = \lambda(t)$

$\Rightarrow \lim_{\Delta t\to 0}\frac{1}{\Delta t}E(z^{N(t+\Delta t)-N(t)}-1) = \lambda(t)(z-1)$ $\quad \Rightarrow \frac{\partial}{\partial t}G_{N(t)}(z) = G_{N(t)}(z)\lambda(t)(z-1)$ $\quad G_{N(0)}(z) = 1$.

$\Rightarrow G_{N(t)}(z) = \exp\left[\int_0^t (z-1)\lambda(t)\,dt\right]$ $\quad P(N(t)=k) = \frac{(\int_0^t \lambda(t)dt)^k}{k!}\exp\left(-\int_0^t \lambda(s)ds\right)$ (Non-Homogeneous Possion) 非齐次泊松

$Y(t) = \sum_{k=1}^{N(t)} X_k$ $\quad X_k$ i.i.d. Independent of $N(t)$. $\quad N(t)$: Standard Possion

$G_{Y(t)}(z) = E(z^{Y(t)}) = E_N(E_X(z^{\sum_{k=1}^{N}X_k} \mid N(t)=n)) = E_{N(t)}(G_X(z)^{N(t)}) = G_{N(t)}(G_X(z))$ "Compound Possion" 复合泊松

$G_{N(t)}(z) = \exp(\lambda t(z-1)) \quad \Rightarrow \quad G_{Y(t)}(z) = \exp[\lambda t(G_X(z)-1)]$

e.g. 进校门总人数 $N(t)$ 性别分布 $\begin{pmatrix} M & F \\ p & 1-p \end{pmatrix}$ def $X_k \sim \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}$ 男生人数 $Y(t) = \sum_{k=1}^{N(t)} X_k$.

$G_X(z) = E(z^X) = z^1 \cdot p + z^0 \cdot (1-p) = pz + (1-p) \quad \Rightarrow G_{Y(t)}(z) = \exp[\lambda p t(z-1)]$ 即男生人数也服从 Possion 分布，强度 λp

Randomized "Thinned"

Possion 过程的和 $Y(t) = \overset{\lambda_1}{N_1(t)} + \overset{\lambda_2}{N_2(t)}$ independent $G_{Y(t)}(z) = G_{N_1(t)}(z) \cdot G_{N_2(t)}(z) = \exp[(\lambda_1+\lambda_2)t(z-1)]$ 也是 Possion，强度 $\lambda = \lambda_1 + \lambda_2$.

推论： $X_1 \sim Exp(\lambda_1)$ $X_2 \sim Exp(\lambda_2)$ independent $Y = \min(X_1, X_2) \sim Exp(\lambda_1 + \lambda_2)$

Possion 过程的差 $Y(t) = \overset{\lambda_1}{N_1(t)} - \overset{\lambda_2}{N_2(t)}$ independent $G_{Y(t)}(z) = G_{N_1(t)}(z) \cdot G_{N_2(t)}(-z) = \exp(\lambda_1 t(z-1)) \exp(\lambda_2 t(z^{-1}-1))$

$G_{Y(t)}(z) = \exp\left[(\lambda_1+\lambda_2)t\left(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}z + \frac{\lambda_2}{\lambda_1+\lambda_2}z^{-1}}_{G_X(z)} - 1\right)\right]$ $X \sim \begin{pmatrix} 1 & -1 \\ \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} \end{pmatrix}$ 是复合泊松

e.g. $N(t), \lambda$ S_n : Event Time Solve $E(S_4 | N(1)=2)$ $F_{S_4(t|N(1)=2)} = P(S_4 \le t | N(1)=2) = \frac{P(S_4 \le t, N(1)=2)}{P(N(1)=2)} = \frac{P(N(t) \ge 4, N(1)=2)}{P(N(1)=2)}$

$F_{S_4(t|N(1)=2)} = \frac{P(N(t)-N(1) \ge 2, N(1)=2)}{N(1)=2} = P(N(t-1) \ge 2) = 1 - \exp(-\lambda(t-1))(1+\lambda(t-1)) \quad (t>1)$

$E(S_4 | N(1)=2) = \int_1^{+\infty} dt \cdot t \cdot \frac{d(F-1)}{dt} = \int_1^{+\infty} (1-F)\,dt = 1 + \frac{2}{\lambda}$

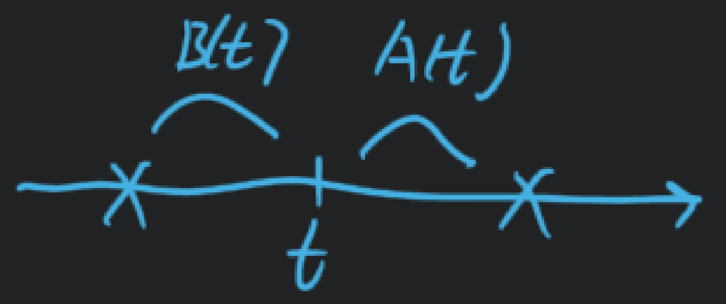
10. Filtering Possion

t之前一次事件与之后一次事件的时间间隔 $T(t) = S_{N(t)+1} - S_{N(t)}$

$F_{T(t)}(x) = P(T(t) \le x) = P(S_{N(t)+1} - S_{N(t)} \le x) = \sum_{k=0}^{+\infty} P(S_{n+1} - S_n | N(t)=n) P(N(t)=n) \neq \sum_{k=0}^{+\infty} (1-\exp(-\lambda x)) P(N(t)=n) = 1 - \exp(-\lambda x)$

问题在于 $\{S_k\}_{k=1}^{\infty}$ 与 $N(t)$ Not Independent!

$n=1$ $\quad F_{S_1|N(t)=1}(x) = P(S_1\le x \mid N(t)=1) = \dfrac{P(S_1\le x,\ N(t)=1)}{P(N(t)=1)} = \dfrac{P(N(x)=1,\ N(t-x)=0)}{P(N(t)=1)} = \dfrac{x}{t}$ $\quad f_{S_1|N(t)=1}(x) = \dfrac{1}{t} \to$ Uniform.



$F_{A(t),B(t)}(x,y) = P(A(t)\le x,\ B(t)\le y)$ $\quad P(A(t)>x,\ B(t)>y) = P(N(x+y)=0) = e^{-\lambda(x+y)}$

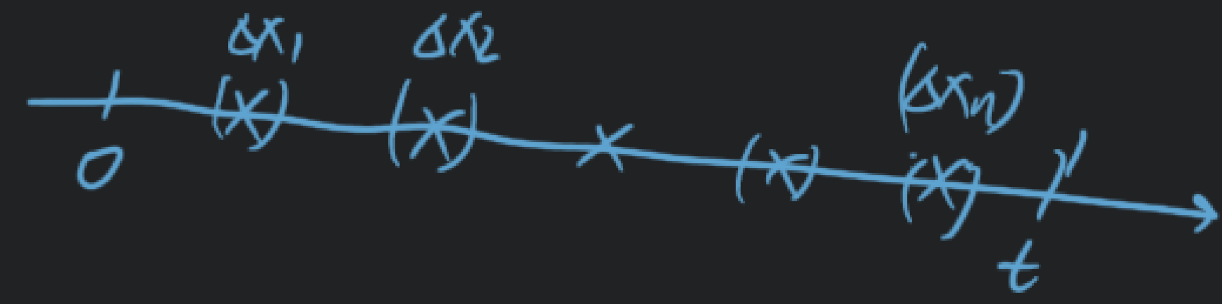
$P(A(t)>x) = P(A(t)>x,\ B(t)>0) = \exp(-\lambda x)$ $\quad P(B(t)>y) = e^{-\lambda y}$

$P(A(t)>x,\ B(t)\le y) = P(A(t)>x) - P(A(t)>x,\ B(t)>y) = e^{-\lambda x} - e^{-\lambda(x+y)}$

$P(A(t)\le x,\ B(t)\le y) = P(B(t)\le y) - P(A(t)>x,\ B(t)\le y) = (1-e^{-\lambda y}) - (e^{-\lambda x} - e^{-\lambda(x+y)}) = (1-e^{-\lambda x})(1-e^{-\lambda y})$

$F_{S_1,S_2,\cdots,S_n|N(t)=n}(x_1,x_2,\cdots,x_n) = P(S_1\le x_1,\ \cdots,\ S_n\le x_n \mid N(t)=n)$ 难以概率转换

"Microcell" Consider $\dfrac{P(x_1\le S_1\le x_1+\Delta x_1,\ \cdots,\ x_n\le S_n\le x_n+\Delta x_n \mid N(t)=n)}{\Delta x_1 \cdots \Delta x_n} \xrightarrow{\Delta\to 0} f_{S_1,\cdots,S_n|N(t)=n}(x_1,\cdots,x_n)$



$$= \lim_{\Delta\to 0} \frac{1}{\Delta_1\cdots\Delta_n} \frac{P(x_1\le S_1\le x_1+\Delta x_1,\ \cdots,\ N(t)=n)}{P(N(t)=n)} = \frac{1}{\Delta_1\cdots\Delta_n} \frac{\prod_{k=1}^{n} \lambda\Delta x_k \exp(-\lambda\Delta x_k)\, \exp\left(-\lambda\left(t-\sum_k \Delta x_k\right)\right)}{\frac{(\lambda t)^n}{n!}\exp(-\lambda t)}$$

$$= \frac{n!}{t^n}$$

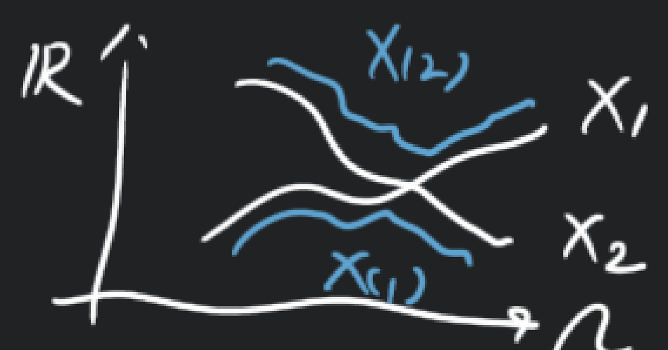
Consider $0\le x_1\le x_2\le\cdots\le x_n\le t$ $\Rightarrow f_{S_1,\cdots,S_n|N(t)=n}(x_1,\cdots,x_n) = \begin{cases} n!/t^n & 0\le x_1\le\cdots\le x_n\le t \\ 0 & \text{others} \end{cases}$

$\{S_k\}$与$N(t)$不独立

积分技巧： Symmetric Function $\quad g(x_1,\cdots,x_n) = g(x_{\sigma(1)},\cdots,x_{\sigma(n)})$ $\quad\forall\sigma$ Permutation Group

$$\int_0^t dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1\, g(x_1,\cdots,x_n) = \frac{1}{n!}\int_0^t dx_n \int_0^t dx_{n-1}\cdots\int_0^t dx_1\, g(x_1,\cdots,x_n).$$

Order Statistic $\quad X_1,\cdots,X_n$ i.i.d. $\xrightarrow{\text{排序}} X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$



$X_{(1)} = \min(X_1,X_2)$

$X_{(2)} = \max(X_1,X_2)$

$X_{(n)} = \max(X_1,\cdots,X_n)$ $\qquad X_{(1)} = \min(X_1,\cdots,X_n)$

$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, \dots, X_n \le x) = [F_{X_1}(x)]^n \Rightarrow f_{X_{(n)}}(x) = n[F_{X_1}(x)]^{n-1} f_{X_1}(x)$

$F_{X_{(1)}}(x) = 1 - P(X_{(1)} > x) = 1 - [1 - F_{X_1}(x)]^n \Rightarrow f_{X_{(1)}}(x) = n[1 - F_{X_1}(x)]^{n-1} f_{X_1}(x)$

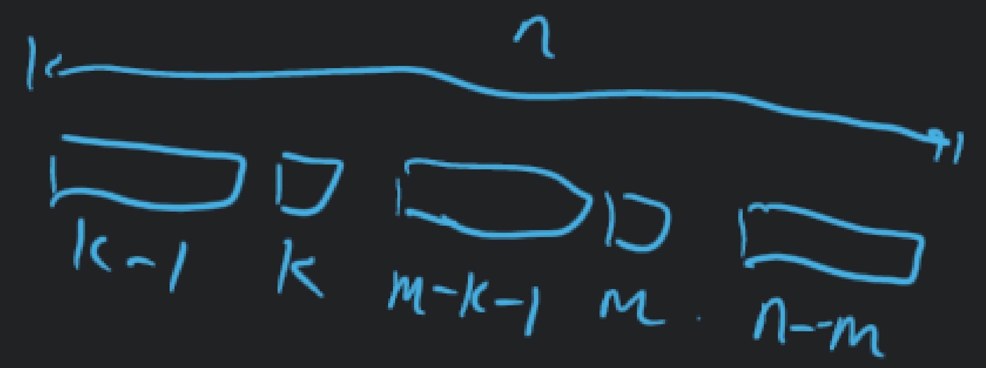
$f_{X_{(k)}}(x) = ?$

$$\binom{n}{k-1}\binom{n-k+1}{n-k} P(X_1, \dots, X_{k-1} \le x < X_k < x+\Delta x \le X_{k+1}, \dots, X_n) = \binom{n}{k-1}\binom{n-k+1}{n-k} (F_{X_1}(x))^{k-1} (1 - F_{X_1}(x))^{n-k} f_{X_1}(x)\Delta x.$$

$$\Rightarrow f_{X_{(k)}}(x) = \binom{n}{k-1}\binom{n-k+1}{n-k} (F_{X_1}(x))^{k-1} (1 - F_{X_1}(x))^{n-k} f_{X_1}(x)$$

$k < m$

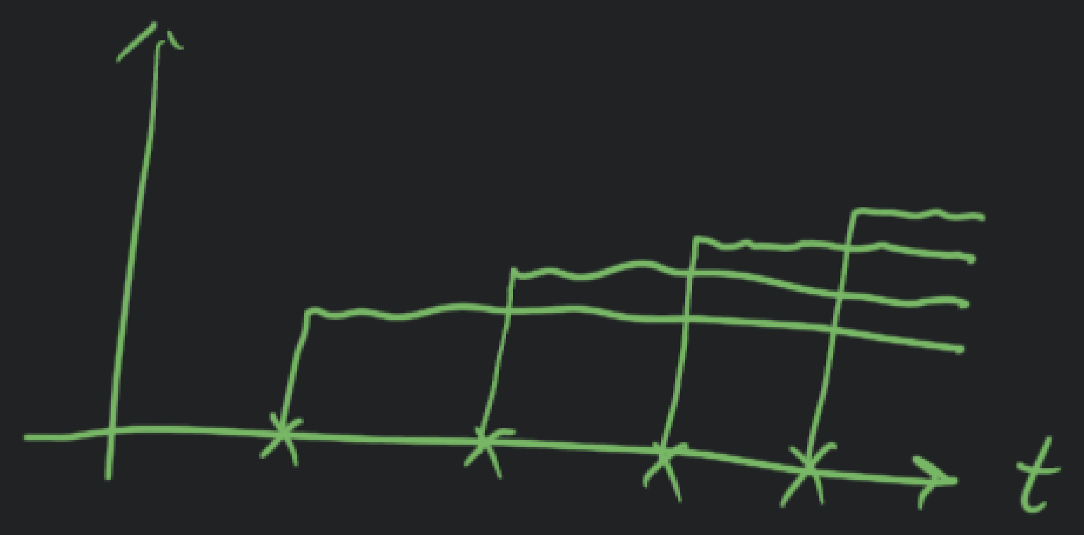
$$f_{X_{(k)}, X_{(m)}}(x_k, x_m) = \binom{n}{k-1}\binom{n-k+1}{m-k-1}\binom{n-m+2}{n-m}\binom{2}{1}\binom{1}{1} [F_X(x_k)]^{k-1} [F_X(x_m) - F_X(x_k)]^{m-k-1} [1 - F_X(x_m)]^{n-m} f_X(x_k) f_X(x_m)$$



$$f_{X_{(1)}, \dots, X_{(n)}}(x_1, \dots, x_n) = n! \prod_{k=1}^{n} f_X(x_k) \quad (x_1 \le \dots \le x_n)$$

$S_1, \dots, S_n \mid N(t) = n$ ~ Order Statistics of Uniforms. $\quad f_{S_k \mid N(t)=n} = \frac{1}{t} \quad \forall k \le n, \ >0$

□ 独立增量的本质：事件一经发生，影响不可改变



放弃独立增量假设： $Y(t) = \sum_{k=1}^{N(t)} X_k \longrightarrow Y(t) = \sum_{k=1}^{N(t)} X_k(t, \tau_k)$ Assume X, $N(t)$ Independent. $\{X_k\}$ i.i.d.

$$\Phi_{Y(t)}(\omega) = E\left[\exp\left(j\omega \sum_{k=1}^{N(t)} X_k(t, \tau_k)\right)\right] = E_{N,\tau}\left\{ E_X\left[\exp\left(j\omega \sum_{k=1}^{n} X_k(t, \tau_k)\right) \Big| N(t) = n, \tau_1, \dots, \tau_n\right]\right\}$$

$$= E\left(\prod_{k=1}^{N(t)} E\left[\exp(j\omega X_k(t, \tau_k))\right]\right) \qquad \text{let } B_X(t, \tau) = E_X(\exp(j\omega X_k(t, \tau)))$$

$$= E\left(\prod_{k=1}^{N(t)} B_k(t,\tau_k)\right) = E_N\left(E_\tau\left(\prod_{k=1}^{n} B_k(t,\tau_k) \Big| N(t)=n\right)\right) = E_N\left(\int_0^t d\tau_n \cdots \int_0^{\tau_2} d\tau_1 \ \frac{n!}{t^n} \prod_{k=1}^{n} B_k(t,\tau_k)\right)$$

$$= E_N\left(\int_0^t d\tau_n \cdots \int_0^t d\tau_1 \ \frac{1}{t^n}\prod_{k=1}^{n} B_k(t,\tau_k)\right) = E_N\left(\frac{1}{t^n}\prod_{k=1}^{n}\int_0^t d\tau_k\, B_k(t,\tau_k)\right) \xlongequal{i.i.d.} E_N\left(\left(\frac{1}{t}\int_0^t B(t,\tau)d\tau\right)^{N(t)}\right)$$

$$= \exp(\lambda t(z-1))\Big|_{z=\frac{1}{t}\int_0^t B(t,\tau)d\tau} = \exp\left[\lambda\left(\int_0^t [\phi_{X(t,\tau)}(\omega)d\tau - 1]d\tau\right)\right].$$

(Filtering Possion)
过滤泊松过程

$$E(Y(t)) = \frac{1}{j}\left(\frac{d}{d\omega}\phi_{Y(t)}(\omega)\right)\Big|_{\omega=0} = \lambda\int_0^t E[X(t,\tau)]d\tau$$

e.g. Bus Stop $[0,T]$ $t_1,\cdots,t_n$ 发车 到达车站乘客 $N(t),\lambda$. $\{t_k\}=?$ 使 Total Waiting Time Mean 最少.

Consider 间隔 s. $W=\sum_{k=1}^{N(s)}(s-\tau_k)$ $X(s,\tau_k)=s-\tau_k \Rightarrow E(W(s)) = \lambda\int_0^s (s-\tau)d\tau = \frac{\lambda}{2}s^2$.

$s_1,\cdots,s_n$ $\sum_{k=1}^{n} s_k = T$ $\min\limits_{\{s_k\}} \frac{\lambda}{2}\sum_{k=1}^{n} s_k^2 \Rightarrow s_1=\cdots=s_n=\frac{T}{n}$.

e.g. Park：来的人数 $N(t),\lambda$. 每个人逗留时间 $\{X_k\} \overset{i.i.d}{\sim} f(x)$ 求：公园里某时刻 t 的平均人数 $E(Y(t))$

$$Y(t)=\sum_{k=1}^{N(t)} sgn(X_k - t+\tau_k) \qquad E(sgn(X_k-t+\tau_k)) = P(X_k \geq t-\tau_k) = 1-F_X(t-\tau_k).$$

$$E(Y(t)) = \lambda\int_0^t \left(\int_{t-\tau}^{\infty} f(s)ds\right)d\tau$$

本题即排队问题

总结
- 标准 Possion $Y(t)=\sum_{k=1}^{N(t)} 1$
- 非齐次 Possion 放弃平稳性 $\lambda=\lambda(t)$
- 复合 Possion 每次事件发生 Z 跳变 $Y(t)=\sum_{k=1}^{N(t)} X_k$.
- 过滤 Possion 放弃独立增量 $Y(t)=\sum_{k=1}^{N(t)} X_k(t)$.

11. Markov Chain

$X_1, \cdots, X_n \Rightarrow$ Joint Distribution $P(X_1, \cdots, X_n) = P(X_n | X_1, \cdots, X_{n-1}) P(X_1, \cdots, X_{n-1})$

$$= \cdots = \left(\prod_{k=2}^{n} P(X_k | X_{k-1}, \cdots, X_1) \right) P(X_1)$$

Markov Assumption : $P(X_k | X_{k-1}, \cdots, X_1) = P(X_k | X_{k-1})$ 只与最近的时刻有关.

$$\Rightarrow P(X_1, \cdots, X_n) = \left(\prod_{k=2}^{n} P(X_k | X_{k-1}) \right) P(X_1)$$

两两独立 $\neq$ 相互独立

e.g. Coin Tossing $X_1, \cdots, X_n$ independent $A_i = \{X_i = X_{i+1}\}$ $i = 1, \cdots, n-1$ $A_n = \{X_1 = X_n\}$.

$$P(A_i \cap A_j) = \begin{cases} \frac{1}{4} & |i-j| = 1 \\ \frac{1}{4} & |i-j| > 1 \end{cases} = P(A_i) \cdot P(A_j)$$ Pairwise Independent 两两独立.

$$P(A_1 \cdots A_n) = \frac{1}{2^{n-1}} \neq \frac{1}{2^n} = \prod_{k=1}^{n} P(A_k)$$

$\{X_n\}$ Discrete Time, Discrete States Stochastic Process.

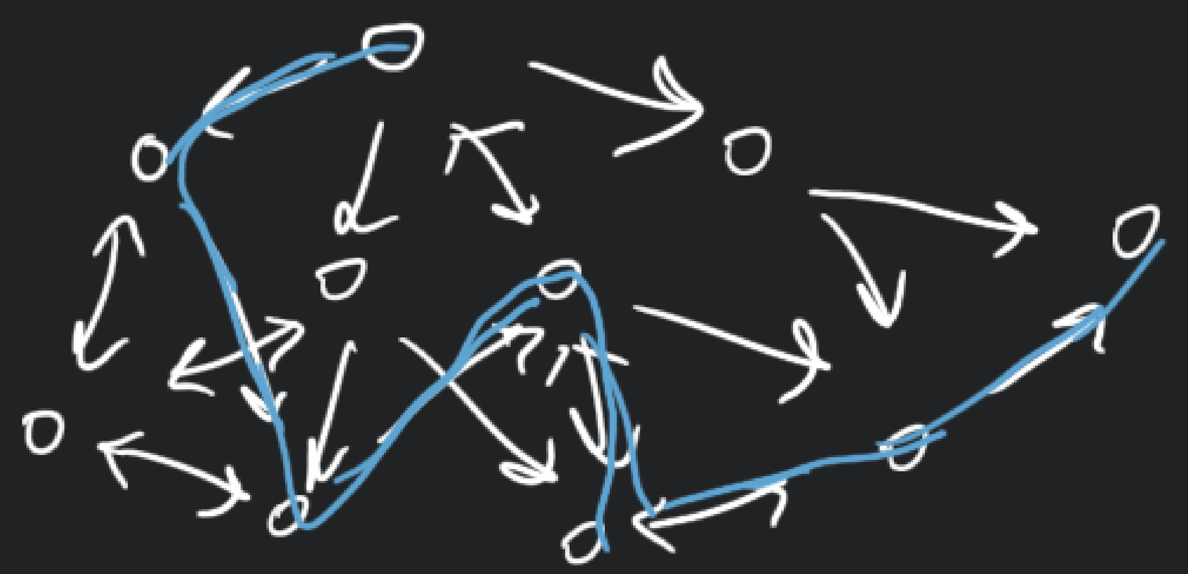
Markov Property $\Leftrightarrow P(X_n = x_n | X_{n-1} = x_{n-1}, \cdots, X_1 = x_1) = P(X_n = x_n | X_{n-1} = x_{n-1})$.

$P(\text{Future} | \text{Now}, \text{Past}) = P(\text{Future} | \text{Now}) \Leftrightarrow P(FP|N) = P(F|N) P(P|N)$ Future 和 Past 条件独立

def Transition Probabily $P_{ij}(m,n) \triangleq P(X_n=j|X_m=i)$ is Stationary $\Leftrightarrow P_{ij}(m,n)=P_{ij}(n-m)$

$P(F|PN)P(P/N)$

$P_{ij}(0)=\delta_{ij}$



Markov Chain: Directed Graph

Stationary: 转变概率只与起点，终点，步数有关，与转变时刻无关

Chapman-Kolmogorov $P_{ij}(m+n)=\sum_k P_{ik}(m)P_{kj}(n)$ 利用中间状态分类（空间分解）

Prove: $P_{ij}(m+n)=P(X_{m+n}=j|X_0=i)=\sum_k P(X_{m+n}=j, X_m=k|X_0=i)$

$=\sum_k P(X_{m+n}=j|X_m=k, X_0=i)P(X_m=k|X_0=i)$

$=\sum_k P(X_{m+n}=j|X_m=k)P(X_m=k|X_0=i)=\sum_k P_{ik}(m)P_{kj}(n)$

$\boxed{\text{Matrix-form C-K} \quad P(m+n)=P(m)P(n)}$ $\Rightarrow P(n)=[P(1)]^n$

---

① Reachable $i\to j \Leftrightarrow \exists n, P_{ij}(n)>0$ 可达是一个传递关系 $i\to j, j\to k \Rightarrow i\to k$

② Commutative $i\leftrightarrow j \Leftrightarrow i\to j, j\to i$ 相通也是传递关系

③ Closed Set. $C\subseteq S$ 状态集 is Closed $\Leftrightarrow \forall i\in C, j\notin C \Rightarrow i\not\to j$ "只能进，不能出"

Reduetion(约化)：只保留闭集，依然构成 Markov Chain

④ Irreducible (不可约) No Closed True Subset $\Leftrightarrow$ All States are Commutative $\forall i,j\in S,\ i\leftrightarrow j$

$P=P(1)=\left(p_{ij}\right)$ $\quad p_{ij}\geq 0 \quad \sum_j p_{ij}=1.$ $\qquad$ 可约: $\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix}$]闭集

12 Recurrent State (常返态)

First Passage Probability: $f_{ij}(n)=P(X_n=j, X_{n-1}\neq j, \dots, X_1\neq j \mid X_0=i)$ 首达概率.

$0\leq \sum_n f_{ij}(n)\leq 1$ 但对转移概率只约束 $0\leq \sum_n P_{ij}(n)\leq \infty$ (路径有重叠).

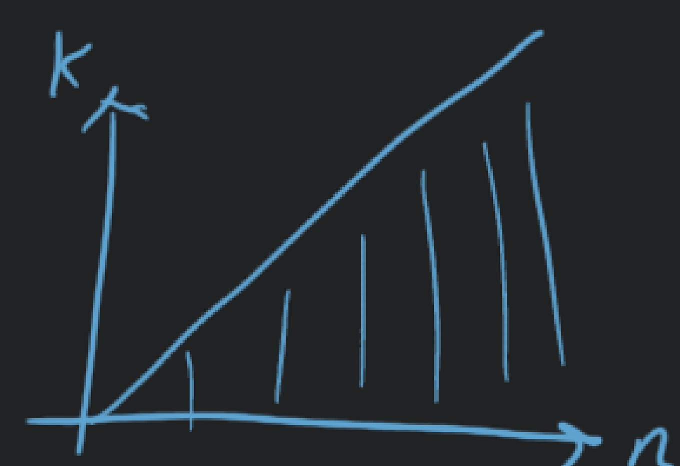
$i\in S$ is Recurrent (常返) $\iff \sum_{n=1}^{\infty} f_{ii}(n)=1$ $\qquad$ 非常返 Transient

$P_{ij}(n)=\sum_{k=1}^{n} f_{ij}(k)P_{jj}(n-k)$ $\quad$ 按首达时间对路径分类(时间分解) $\quad$ Convolution 卷积

$$P(X_n=j\mid X_0=i)=\sum_{k=1}^{n}P(X_n=j, \underset{\text{首达时间}}{T_{ij}}=k\mid X_0=i)=\sum_{k=1}^{n}P(X_n=j\mid T_{ij}=k, X_0=i)\,P(T_{ij}=k\mid X_0=i)$$

$$=\sum_{k=1}^{n}P(X_n=j\mid X_k=j, X_{k-1}\neq j, \dots, X_0=i)\,P(X_j=k, X_{j-1}\neq k, \dots, X_0=i\mid X_0=i)$$

$$=\sum_{k=1}^{n}P(X_n=j\mid X_k=j)\,f_{ij}(k)=\sum_{k=1}^{n}f_{ij}(k)P_{jj}(n-k)$$



z-Transform $\quad P_{ij}(z)=\sum_{n=0}^{\infty}P_{ij}(n)z^n=P_{ij}(0)+\sum_{n=1}^{\infty}P_{ij}(n)z^n=\delta_{ij}+\sum_{n=1}^{\infty}\sum_{k=1}^{n}f_{ij}(k)P_{jj}(n-k)z^n$

$$=\delta_{ij}+\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}f_{ij}(k)P_{jj}(n-k)z^n=\delta_{ij}+\sum_{k=1}^{\infty}f_{ij}(k)z^k\left(\sum_{n=0}^{\infty}P_{jj}(n)z^n\right)$$

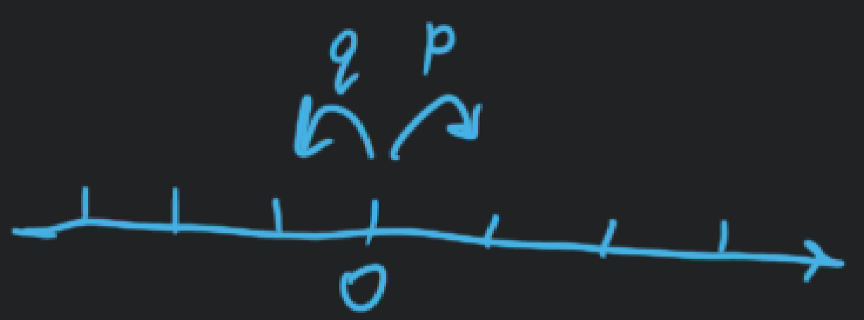
$F_{ij}(z)\triangleq\sum_{k=1}^{\infty}f_{ij}(k)z^k \Rightarrow P_{ij}(z)=\delta_{ij}+F_{ij}(z)P_{jj}(z).$

$i=j \quad P_{ii}(z)=1+F_{ii}(z)P_{ii}(z) \Rightarrow P_{ii}(z)=\dfrac{1}{1-F_{ii}(z)} \xrightarrow{z=1} \sum_{n=0}^{\infty}P_{ii}(n)=\dfrac{1}{1-\sum_{n=1}^{\infty}f_{ii}(n)}$ $\quad$ 发散?

令 $z \to 1^-$, Abel Lemma $\Rightarrow \sum_{n=0}^{\infty} P_{ii}(n)$ 发散

$\Rightarrow$ 常返的新定义　$i \in S$ is Recurrent $\Longleftrightarrow \sum_{n=0}^{\infty} P_{ii}(n)$ 发散

One Dimension Random Walk — 一维随机游走



$p+q=1$. 判断零点的常返性. $\sum_{n=0}^{\infty} P_{00}(n)$　　$P_{00}(n)=\begin{cases} 0 & n=2k+1 \\ \binom{2k}{k} p^k q^k & n=2k \end{cases}$

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \Rightarrow \sum_{k=0}^{\infty} \binom{2k}{k} p^k q^k \sim \sum_{k=0}^{\infty} (4pq)^k \frac{1}{\sqrt{k}}$　$4pq \le 1 \Rightarrow \begin{cases} p=q=1/2, & \text{零点常返} \\ p \neq q & \text{不常返} \end{cases}$

2d Random Walk

$$P_{00}(2n) = \sum_{k=0}^{n} \frac{(2n)!}{(k!)^2((n-k)!)^2} \left(\frac{1}{4}\right)^{2n} = \left(\frac{1}{4}\right)^{2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \left(\frac{1}{4}\right)^{2n} \left[\binom{2n}{n}\right]^2 \sim \frac{1}{n}$$

即：2维随机游走，零点常返.

3d Random Walk　$P_{00}(2n) \sim n^{-3/2}$ 收敛　　3维及以上零点不常返.

非常返　Transient　$\sum_{n=0}^{\infty} P_{ii}(n) < \infty \Rightarrow \lim_{n\to\infty} P_{ii}(n) = 0$.

设 j 非常返, $j \neq i$, $P_{ij}(z) = F_{ij}(z) P_{jj}(z) \xrightarrow{z \to 1} \sum_{n=0}^{\infty} P_{ij}(n) = \underbrace{\sum_{n=0}^{\infty} f_{ij}(n)}_{\le 1} \underbrace{\sum_{n=0}^{\infty} P_{jj}(n)}_{< \infty} \Rightarrow \sum_{n=0}^{\infty} P_{ij}(n) < \infty \Rightarrow \lim_{n\to\infty} P_{ij}(n) = 0 \quad \forall i, j$

$\forall i \in S$, 若 j 非常返, $\lim_{n\to\infty} P_{ij}(n) = 0$.　(极限情况下不需要考虑非常返态)

相通的状态常返性相同　$i \longleftrightarrow j \Rightarrow i$ 常返 $\Longleftrightarrow j$ 常返　Recurrent is a Class Property

$i \leftrightarrow j \Rightarrow \exists m,n>0 \quad P_{ij}(m)>0 \quad P_{ji}(n)>0$ $\quad \sum_{k=0}^{\infty} P_{ii}(m+n+k) \geq \sum_{k} P_{ij}(m) P_{jj}(k) P_{ji}(n) > C \sum_{k=0}^{\infty} P_{jj}(k) = \infty$

不可约链的每个节点常返性都相同.

Finite States 有限状态. $S=\{1,\cdots,N\}$. 不可能每个状态都非常返，即一定存在常返态.

$\forall n, \forall i, \quad 1=\sum_{j=1}^{N} P_{ij}(n) \Rightarrow 1=\lim_{n\to\infty}\sum_{j=1}^{N} P_{ij}(n) = \sum_{j=1}^{N}\lim_{n\to\infty} P_{ij}(n)$ 若每个状态都非常返 $\lim_{n\to\infty} P_{ij}(n)=0 \Rightarrow 1=0$，矛盾.

有限不可约链 Finite States + Irreducible $\Longrightarrow$ All States are Recurrent 每个状态都常返

记 T_i 为返回次数，构成随机事件 $T_i = \sum_{n=1}^{\infty} I_{[X_n=i]} \rightarrow \begin{cases} 1, & X_n=i \\ 0, & X_n\neq i \end{cases}$

$E(T_i) = E\left(\sum_{n=1}^{\infty} I_{[X_n=i]} \middle| X_0=i\right) = \sum_{n=1}^{\infty} P(X_n=i|X_0=i) = \sum_{n=1}^{\infty} P_{ii}(n)$ 常返 $\Rightarrow E(T_i)=\infty$ 返回次数的期望是无穷次.

$E(T_i) = \sum_{k=1}^{\infty} k\, P(T_i=k) = \sum_{k=1}^{\infty} k\, f_{ii}^{k}(1-f_{ii}) = \frac{f_{ii}}{1-f_{ii}}$ 常返 $\Rightarrow f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n) = 1$ "Zero – One" law

$g_{ii}(m) \triangleq P(T_i \geq m) = f_{ii}\, g_{ii}(m-1) = \cdots = (f_{ii})^m$ $\quad g_{ii} = \lim_{m\to\infty} g_{ii}(m) = P(T_i=\infty) = \begin{cases} 1, & f_{ii}=1 & 常返 \\ 0, & f_{ii}<1 & 非常返. \end{cases}$

i Recurrent $\quad i\to j \Rightarrow j\to i \Rightarrow j$ Recurrent. (常返态只访问常返态) (常返态总能回来)

$i\to j \Rightarrow \exists n, P_{ij}(n)>0$

$\forall j$，从 j 出发无穷多次到达 i 的概率 $g_{ji} = \sum_{k} P_{jk}(n)\, g \xrightarrow{j=i} g_{ii} = \sum_{k} P_{ik}(n) g_{ki} \quad \sum_{k} P_{ik}(n)=1 \quad g_{ii}=1$

$\Rightarrow \sum_{k} P_{ik}(n) = \sum_{k} P_{ik}(n) g_{ki}. \quad \sum_{k} \underbrace{(g_{ki}-1)}_{\leq 0} \underbrace{P_{ik}(n)}_{\geq 0} = 0 \Rightarrow \forall k, (g_{ki}-1)P_{ik}(n)=0 \quad P_{ij}(n)>0 \Rightarrow g_{ji}=1.$

(从 j 出发，无穷多次返回 i 的概率是 1)

13. Stationary Distribution

① $P_{ij}(n)$ 不一定有极限.

e.g. $\{0,1\}$ $P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $P_{00}(n)=P_{11}(n)=\{0,1,0,1,\cdots\}$ 没有极限.

$P_{01}(n)=P_{10}(n)=\{1,0,1,0,\cdots\}$

② Irreducible + i Recurrent $\Rightarrow$ $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}P_{ij}(k)=\frac{1}{\mu_j}$ 有极限 (Weak Ergodic Theorem)

| Cesaro Sum $a_n\to a \Rightarrow \frac{1}{n}\sum_{k=1}^{n}a_n\to a$

其中 $\mu_j=\sum_{n=1}^{\infty}nf_{jj}(n)=\begin{cases}<\infty & \text{Positive Recurrent}\\ =\infty & \text{Null Recurrent}\end{cases}$ 为平均首次返回时间.

③ Period: for state i, $d_i=\underset{\text{最大公因子}}{GCD}\{k\mid P_{ii}(k)>0\}$.

$d_i=1 \Rightarrow i$ is Non-Periodic. $\Rightarrow \exists N$, s.t. $n>N$, $P_{ii}(n)>0$

$i\leftrightarrow j \Rightarrow d_i=d_j$ Period is Class Property

$i\leftrightarrow j \Rightarrow \exists n,m$ $P_{ij}(m)>0$ $P_{ji}(n)>0$ $A_i=\{k\mid P_{ii}(k)>0\}$ $P_{ii}(m+n)\geq P_{ij}(m)P_{ji}(n)>0 \Rightarrow m+n\in A_i$

$\Rightarrow \underset{\text{整除}}{d_i\mid(m+n)}$ $\forall k\in A_j$, $P_{ii}(m+n+k)\geq P_{ij}(m)P_{jj}(k)P_{ji}(n)>0 \Rightarrow m+n+k\in A_i$ $d_i\mid(m+n+k)\Rightarrow d_i\mid k$

$\Rightarrow d_i$ is $\underset{\text{公因子}}{CD}(A_j) \Rightarrow d_i\mid d_j$ 又 $d_j\mid d_i \Rightarrow d_i=d_j$.

④ Irreducible + Non-Periodic $\Rightarrow$ $P_{ij}(n)$ 有极限.

$\begin{cases}P(n)=P(1)P(n-1) & \text{Backward}\\ P(n)=P(n-1)P(1) & \text{Forward}\end{cases}$

⑤ Irreducible + Non-Periodic + Recurrent $\quad P_{ij}(n) \xrightarrow{n\to\infty} \pi_j \quad \frac{1}{n}\sum_{k=1}^{n} P_{ij}(k) \xrightarrow{n\to\infty} \pi_j \quad \pi_j = \frac{1}{\sum_{n=1}^{\infty} n f_{jj}(n)}$.

$$\Rightarrow P(n) \longrightarrow \Pi = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_n \\ \pi_0 & \pi_1 & \cdots & \pi_n \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0 & \pi_1 & \cdots & \pi_n \end{pmatrix}.$$

$P(n) = P(1)P(n-1)$ 是恒等式,

$$P(n) = P(n-1)P(1) \Rightarrow (\pi_0, \cdots, \pi_n) = (\pi_0, \cdots, \pi_n)\begin{pmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \vdots \\ P_{n1} & \cdots & P_{nn} \end{pmatrix}$$

$\pi \triangleq (\pi_0, \cdots, \pi_n)$ is a row vector $\quad \pi = \pi P$ is a left hand equation (LHE)

方程总有非零解. $\quad \pi = \pi P \Leftrightarrow \pi^T = P^T \pi^T \Leftrightarrow (P^T - I)\pi^T = 0 \Leftrightarrow \det(P^T - I) = \det(P - I) = 0$.

$(P - I)\cdot \mathbf{1} = 0.\ (\sum_j P_{ij} = 1) \Rightarrow \det(P - I) = 0$

π is called Stationary Distribution. (不变分布) $\quad$ Assume $P(X_0 = i) = \pi_i \Rightarrow \forall n,\ P(X_n = i) = \pi_i$.

Irreducible + Recurrent. 若 $P_{ij}(n)$ 无极限, $\frac{1}{n}\sum_{k=1}^{n} P_{ij}(k) \xrightarrow{n\to\infty} \tilde{\pi}_j$ 仍满足 $\tilde{\pi} = \tilde{\pi} P$ (不变分布不一定是极限分布)

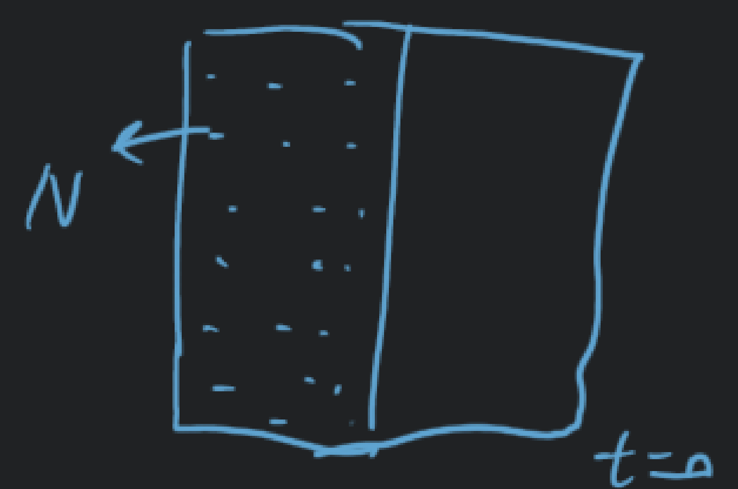
Detailed Balance $\quad \pi_i P_{ij} = \pi_j P_{ji} \quad \Rightarrow \pi_i = \underbrace{\sum_k \pi_k P_{ki}}_{= \pi_i P_{ik}} \Leftrightarrow \pi = \pi P$.

进出平衡

e.g. Irreducible, Recurrent, Non-Periodic.

$$(\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3) \quad \Rightarrow (\pi_1\ \pi_2\ \pi_3) = (1/3\ \ 1/3\ \ 1/3)$$

e.g. Ehrenfest Model $\quad$ State Space $S = \{0, \cdots, N\}$. 标记右边分子的数目.



Assume: ① Time Discretization Δt ② In Δt, Only one molecular change "state" and Equiprobable.

③
$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & \cdots & N \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ N \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1/N & 0 & \frac{N-1}{N} & \cdots & \cdots & 0 \\ 0 & 2/N & 0 & \frac{N-2}{N} & \cdots & 0 \\ & & 3/N & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \end{array}$$

$$⓪ \rightleftarrows ① \rightleftarrows ② \rightleftarrows ③ \rightleftarrows \cdots \rightleftarrows Ⓝ$$

④ $\pi = \pi P$
$$\begin{cases} \pi_0 = \frac{1}{N}\pi_1 \\ \pi_1 = \pi_0 + \frac{2}{N}\pi_2 \\ \pi_2 = \frac{N-1}{N}\pi_1 + \frac{3}{N}\pi_3 \\ \vdots \\ \pi_k = \frac{N-(k-1)}{N}\pi_{k-1} + \frac{k+1}{N}\pi_{k+1} \quad (1\le k\le N-1) \\ \pi_N = \frac{1}{N}\pi_{N-1} \end{cases} \Rightarrow \pi_k = \binom{N}{k}2^{-N}$$

⑤ Irreducible, Recurrent, but period $=2$. 所有状态正常返 ($\pi_k>0 \ \forall k$)

$\pi^0 = 2^{-N} \Rightarrow \sum_{n=1}^{\infty} n f_{00}(n) \sim 2^N >> 1$. (首次回到0状态的期望时间 $\sim 2^N >> 1$)

e.g. Random Walk on Undirected Graph 设N节点图，节点 i 的度（伸出的边数）为 n，则每条边的概率为 $1/n$. 求不变分布 π.

$p_{ij} = \frac{1}{d_i} \quad (i\to j)$

$\pi_i = \frac{d_i}{\sum_k d_k}$ Verify: $\sum_i \pi_i p_{ij} = \sum_{i,\{i\to j\}} \frac{d_i}{\sum_k d_k}\frac{1}{d_i} = \frac{d_j}{\sum_k d_k} \overset{!}{=} \pi_j$.

$T_i^A = \min\{n : X_n\in A \mid X_0 = i\}$. Absorbing Probability $p_i^A = P(T_i^A<\infty)$

Gambler Ruin Problem 计算赌博输光概率

$$\underset{\;0 \qquad n-1 \quad n \quad n+1}{\xrightarrow{\overset{1-p}{\curvearrowleft}\;\overset{p}{\curvearrowright}}}$$

$p_n = p\cdot p_{n+1} + (1-p)p_{n-1}$ 结论：即使 $p=1/2$，赌光概率仍是1.

$p_i^A = \sum_j p_{ij}p_j^A \Rightarrow \pi^A = (p_0^A,\dots,p_n^A)^T \quad \pi^A = P\cdot\pi^A$

$E(T_i^A) = \sum_j p_{ij}E(T_j^A) + 1$

14. Continuous Time Markov Chain

$\forall n\ \forall t_1, \cdots, t_n\quad X(t_1), \cdots, X(t_n) \in \{1, \cdots, n, \cdots\}$. $\quad P(X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, \cdots, X(t_1) = x_1) = P(X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1})$.

Markov Property $\Rightarrow$ Transition Probability $P_{ij}(s,t)$ $\qquad$ Stationary Transition $\quad P_{ij}(s,t) = P_{ij}(t-s)$ $\qquad \sum_j P_{ij}(t) = 1$

C-K $\quad P_{ij}(t+s) = \sum_k P_{ik}(s) P_{kj}(t)$ $\qquad P(t) = (P_{ij}(t))_{ij} \Rightarrow P(s+t) = P(t)P(s)$

Kolmogorov Forward-Backward Equations $\qquad$ No Minimal Time Unit.

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = P(t)\frac{P(\Delta t) - I}{\Delta t} = \frac{P(\Delta t) - I}{\Delta t} P(t) \qquad Q \triangleq \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t} \Rightarrow \begin{cases} \frac{d}{dt}P(t) = P(t)Q \\ \frac{d}{dt}P(t) = QP(t) \end{cases} \Rightarrow P(t) = \exp(Qt)$$

① $\forall i,\ Q_{ii} \le 0$ (对角线上元素 ≤ 0) $\qquad$ ② $\forall i \ne j\ \ Q_{ij} \ge 0$ (非对角线上元素 ≥ 0) $\qquad$ ③ $\sum_j Q_{ij} = 0$ (行和为0)

e.g. Possion

$$P_{ij}(t) = \begin{cases} \frac{(\lambda t)^{j-i}}{(j-i)!} \exp(-\lambda t) & j \ge i \\ 0 & j < i \end{cases} \qquad Q_{ii} = \lim_{\Delta t \to 0} \frac{P_{ii}(\Delta t) - 1}{\Delta t} = -\lambda \qquad Q_{i,i+1} = \lambda \qquad \text{else } Q_{ij} = 0$$

$$Q = \begin{pmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \\ & & & -\lambda \end{pmatrix}$$

Asymptotic Behavior $\quad \exists t_0,\ P_{ij}(t_0) > 0 \Rightarrow \forall t,\ P_{ij}(t) > 0$ $\qquad$ (理解：跳转不需要花时间，逗留时间可任意长)

$\lim_{t \to \infty} P(t) = \Pi$ 一定存在. $\qquad$ Irreducible : $\lim_{t \to \infty} P_{ij}(t) = \pi_j$.

$$\frac{d}{dt}P(t) = P(t)Q \xRightarrow{t \to \infty} \boxed{\underset{\text{Continuous}}{0 = \pi \cdot Q} \longleftrightarrow \underset{\text{Discrete}}{\pi = \pi \cdot P}}$$

PASTA : $X(t)$ CTMC (连续时间马氏链), $p^X(t)$ $N(t)$ Possion, λ, Independent of $x(t)$. $T_1, \cdots, T_n$. $Y_n = X(T_n)$ DTMC (离散时间) $\qquad \lim_{t \to \infty} p^Y = \lim_{t \to \infty} p^X(t)$.

$$p^Y = (P_{ij}(T_n - T_{n-1}))_{ij} = \int_0^\infty p^X(t) \exp(-\lambda t)\, \lambda\, dt = -p^X(t)\exp(-\lambda t)\Big|_0^\infty + \int_0^\infty \exp(-\lambda t) \frac{d}{dt} p^X(t)\, dt.$$

$= I + \int_0^\infty \exp(-\lambda t) Q P^x(t)dt = I + Q\int_0^\infty \exp(-\lambda t) P^x(t)dt \Rightarrow P^\gamma = I + QP^\gamma \cdot \frac{1}{\lambda}$ $\quad \pi \cdot P^\gamma = \pi + \pi Q \frac{1}{\lambda} P^\gamma$

$\Rightarrow \pi = \pi P^\gamma \Longleftrightarrow \pi Q = 0$

e.g.

Birth-Death Process ① Birth: Δt $\uparrow 1$ $p = \lambda\Delta t + o(\Delta t)$ $\uparrow \geq 2$ $o(\Delta t)$ $\uparrow 0$ $1-\lambda\Delta t + o(\Delta t)$

② Death Δt $\downarrow 1$ $p = \mu\Delta t + o(\Delta t)$ $\downarrow \geq 2$ $o(\Delta t)$ $\downarrow 0$ $1-\mu\Delta t + o(\Delta t)$

Birth, Death Independent

$P_{ii}(\Delta t) = (1-\lambda\Delta t + o(\Delta t))(1-\mu\Delta t + o(\Delta t)) + (\lambda\Delta t + o(\Delta t))(\mu\Delta t + o(\Delta t)) + (o(\Delta t))(o(\Delta t)) = 1-(\lambda+\mu)\Delta t + o(\Delta t)$

$P_{i,i+1}(\Delta t) = \lambda\Delta t + o(\Delta t)$ $\quad P_{i,i-1}(\Delta t) = \mu\Delta t + o(\Delta t)$ $\quad$ else $P_{ij} = o(\Delta t)$

$\Rightarrow Q = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t}$ $\quad Q_{ii} = -(\lambda+\mu)$ $\quad Q_{i,i+1} = \lambda$ $\quad Q_{i,i-1} = \mu$. $\quad$ Expansion: $\lambda, \mu \sim i$. λ_i, μ_i.

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \quad P(t) = \exp(Qt) = \sum_{k=0}^\infty \frac{Q^k t^k}{k!} \quad \pi Q = 0 \Rightarrow \begin{cases} \pi_0\lambda_0 = \pi_1\mu_1 \\ \pi_0\lambda_0 - \pi_1(\lambda_1+\mu_1) + \pi_2\mu_2 = 0 \\ \pi_1\lambda_1 - \pi_2(\lambda_2+\mu_2) + \pi_3\mu_3 = 0 \\ \vdots \end{cases}$$

$$\Rightarrow \pi_n = \left(\prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}}\right)\pi_0 \quad \sum_{n=0}^\infty \pi_n = 1 \Rightarrow \pi_0\left(\sum_{n=1}^\infty \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}} + 1\right) = 1. \Rightarrow \sum_{n=1}^\infty \sum_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}} < \infty.$$

Assume $\lambda_i = \lambda$, $\mu_i = \mu$. (排队问题 M/M/1) $\Rightarrow \sum_{n=1}^\infty \left(\frac{\lambda}{\mu}\right)^n < \infty \Rightarrow \mu > \lambda$ Death > Birth 排队系统进入稳态的关键

$\pi_0 = 1 - \frac{\lambda}{\mu}$ $\quad \pi_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$

Assume $\lambda_i = \lambda$ $\quad \mu_i = \begin{cases} i\mu & i \leq K \\ K\mu & i > K \end{cases}$ $\quad$ M/M/K 多柜台服务 $\quad \sum_{n=1}^\infty \sum_{l=0}^{n-1} \frac{\lambda_l}{\mu_{l+1}} = \sum_{n=1}^K \frac{\lambda^n}{n!\,\mu^n} + \sum_{n=K+1}^\infty \left(\frac{\lambda}{K\mu}\right)^n$ $\quad \lambda < K\mu$.

$$\pi_K = \frac{\frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K}{\sum_{n=0}^K \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \left(\frac{\lambda}{K\mu}\right)^n\left(1-\frac{\lambda}{K\mu}\right)}$$