Maxwell方程组

$$\begin{cases} \nabla\cdot\vec{E} = 4\pi\rho \\ \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \end{cases}$$

SI $\Leftrightarrow$ Gauss

$$\vec{E}' = \frac{1}{\sqrt{4\pi\varepsilon_0}}\vec{E} \qquad \vec{B}' = \sqrt{\frac{\mu_0}{4\pi}}\vec{B} \qquad \rho' = \sqrt{4\pi\varepsilon_0}\,\rho \qquad \vec{J}' = \sqrt{4\pi\varepsilon_0}\,\vec{J}.$$

$\vec{E}'$ ↓ SI制　　$\vec{E}$ ↓ Gauss

介质

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad H = \vec{B} - 4\pi\vec{M}.$$

$$\vec{D}' = \sqrt{\frac{\varepsilon_0}{4\pi}}\vec{D} \qquad \vec{P}' = \sqrt{4\pi\varepsilon_0}\,\vec{P} \qquad \vec{H}' = \frac{1}{\sqrt{4\pi\mu_0}}\vec{H} \qquad \vec{M}' = \sqrt{\frac{4\pi}{\mu_0}}\vec{M}.$$

$$\begin{cases} \nabla\cdot\vec{D} = 4\pi\rho_f \\ \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}_f \end{cases}$$

$$\nabla\cdot\vec{P} = -\rho_b \qquad \vec{J}_b = \frac{\partial\vec{P}}{\partial t}.$$

$$\nabla\times\vec{M} = \frac{1}{c}(\vec{J}_t - \vec{J}_f) - \frac{1}{c}\frac{\partial\vec{P}}{\partial t} = \frac{1}{c}(\vec{J}_t - \vec{J}_f - \vec{J}_b)$$
$$= \frac{1}{c}\vec{J}_m.$$

均匀介质

$\vec{D} = \varepsilon \vec{E}$ $\vec{B} = \mu \vec{H}$ $\vec{P} = \chi_e \vec{E}$ $\vec{M} = \chi_m \vec{H}$ $\chi_e = \frac{\varepsilon - 1}{4\pi}$ $\chi_m = \frac{\mu - 1}{4\pi}$

对于真实介质，响应不是瞬时的，但仍假设响应是局域的。

$\vec{P}(t) = \int_{-\infty}^{\infty} dt' \chi_e(t-t') \vec{E}(t')$ $\vec{M}(t) = \int_{-\infty}^{\infty} dt' \chi_m(t-t') \vec{H}(t')$ $\chi(\tau) = 0 \ (\tau < 0)$ (因果)

频域：$\vec{P}(\omega) = \chi_e(\omega) \vec{E}(\omega)$ $\vec{M}(\omega) = \chi_m(\omega) \vec{E}(\omega)$

色散 相速度 $v_p = \frac{c}{n} = \frac{\omega}{k}$ $k = \frac{\omega}{v_p} = \frac{\omega}{c} n(\omega)$

群速度 $v_g = \frac{d\omega}{dk} = \frac{c}{n(\omega) + \omega(dn/d\omega)}$

正常 $\frac{dn}{d\omega} > 0$ $v_g < v_p$ 反常 $\frac{dn}{d\omega} < 0$ $v_g > v_p$ $\frac{dn}{d\omega} \to +\infty$ 时 $v_g \to 0$

若 $\frac{dn}{d\omega}\omega + n < 1$，$v_g > c$

Kramers - Kronig 关系

$\vec{D}(\omega) = \varepsilon(\omega) \vec{E}(\omega)$ $\varepsilon(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(\tau) e^{+i\omega\tau} d\tau$ 因果关系 $\tau < 0$ 时 $\varepsilon(\tau) = 0$

$\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(\omega) e^{-i\omega\tau} d\omega \Rightarrow \varepsilon(\omega)$ 在上半复平面解析无奇点。

于是由复变中的 Possion 公式，$\varepsilon(\omega)$ 的实部与虚部并不独立。 ω 在上半复平面时，

$$\varepsilon(\omega)-1=\frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{\varepsilon(\omega')-1}{\omega'-\omega}d\omega' \quad (\text{Cauchy 公式}) \qquad 0=\frac{1}{\pi i}P\int_{-\infty}^{\infty}\frac{\varepsilon(\omega')-1}{\omega'-\omega^*}d\omega'$$

注意这里不能直接取 ω 在实轴上，因为 ω 在积分围道上，采取数理方法中的技巧，

$$\frac{1}{2}[\varepsilon(\omega)-1]=\frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{\varepsilon(\omega')-1}{\omega'-\omega}d\omega'.$$

进而 $$\mathrm{Re}\{\varepsilon(\omega)\}=1+\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\{\varepsilon(\omega')\}}{\omega'-\omega}d\omega$$

$$\mathrm{Im}\{\varepsilon(\omega)\}=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}\{\varepsilon(\omega)\}-1}{\omega'-\omega}d\omega.$$

注意要取 $\varepsilon(\omega)-1$ 因为 $\lim\limits_{\omega\to\infty}\varepsilon(\omega)\neq 0$ 无法应用大圆弧引理.

边界条件

$\nabla \to \vec{n}$ 为1指向2的单位法矢

矢量 $\vec{c} \to \vec{c}_2-\vec{c}_1$

体密度 $\to$ 面密度

守恒律

电荷：$\frac{\partial \rho_t}{\partial t} + \nabla\cdot \vec{J}_t = 0$

能量：$u = \frac{1}{8\pi}\left(|\vec{E}|^2 + |\vec{B}|^2\right)$ $\quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$ $\quad \frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = 0$

动量：$T_{ij} = -\frac{1}{4\pi}\left[(\vec{E}\vec{E})_{ij} + (\vec{B}\vec{B})_{ij} - \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)\delta_{ij}\right]$

$\overleftrightarrow{T} = -\frac{1}{4\pi}(\vec{E}\vec{E} + \vec{B}\vec{B}) + u\overleftrightarrow{I}$ 对称

$\vec{P}^{(field)} = \int d^3x\ \vec{g}(\vec{x})$ $\quad \vec{g}(\vec{x}) = \frac{1}{4\pi c}(\vec{E}\times\vec{B}) = \frac{1}{c^2}\vec{S}$

$\frac{d\vec{P}^{(src)}}{dt} + \frac{\partial \vec{P}^{(field)}}{\partial t} = -\int d^3x\ \partial_j T_{ij}\ \vec{e}^i$

对称性

线性

宇称 $\quad \vec{r}\to -\vec{r}$ $\quad \nabla \to -\nabla$ $\quad \vec{E}\to -\vec{E}$ $\quad \vec{B}\to\vec{B}$

反演 $\quad \partial_t \to -\partial_t$ $\quad \vec{E}\to\vec{E}$ $\quad \vec{B}\to -\vec{B}$ $\quad \vec{A}\to -\vec{A}$ $\quad \vec{J}\to -\vec{J}$

协变

规范 $\vec{B}=\nabla\times\vec{A}$ $\vec{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$$\begin{cases}\phi\to\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t}\\ \vec{A}\to\vec{A}+\nabla\Lambda\end{cases}$$

$A_\mu=(-\phi,\vec{A})$ Lorenz 规范：$\partial_\mu A^\mu=\partial^\mu A_\mu=\frac{1}{c}\frac{\partial\phi}{\partial t}+\nabla\cdot\vec{A}=0$.

此规范下 $\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-4\pi\rho_t$ $\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t}=-\frac{4\pi}{c}\vec{j}_t$.

库仑规范：$\nabla\cdot\vec{A}=0$. 此规范下 $\nabla\cdot\vec{E}=-\nabla^2\phi=4\pi\rho_t$

唯一性定理

Dirichlet 边界条件 ∂V 上 ϕ 已知. Neumann 边界条件 ∂V 上 $\frac{\partial\phi}{\partial n}$ 已知.

Green 定理

$$\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)\,d^3x=\int_{\partial V}(\phi\nabla\psi-\psi\nabla\phi)\cdot d\vec{s}.$$

取 $\phi(\vec{r}')$ 为静电势 $\nabla'^2\phi=-4\pi\rho_t$. ψ 为 V 中的 Green 函数 $G(\vec{r},\vec{r}')$ $\nabla'^2G(\vec{r},\vec{r}')=-4\pi\delta(\vec{r}-\vec{r}')$.

$G(\vec{r},\vec{r}')=\frac{1}{|\vec{r}-\vec{r}'|}+F(\vec{r},\vec{r}')$ $\nabla'^2F(\vec{r},\vec{r}')=0$ 对 G 施加 V 上的边界条件可唯一确定 G.

$\vec{r}$：场点 $\vec{r}'$：源点.

取 $\vec{r}'$ 为积分变量，∇' 只对 $\vec{r}'$ 作用，于是

$$\phi(\vec{r}) = \int_V \rho(\vec{r}')G(\vec{r},\vec{r}')d^3\vec{r}' + \frac{1}{4\pi}\int_{\partial V}\left[G(\vec{r},\vec{r}')\nabla'\phi(\vec{r}') - \phi(\vec{r}')\nabla'G(\vec{r},\vec{r}')\right]\cdot d\vec{s}'$$

若 ϕ 在 ∂V 上满足 Dirichlet 边界条件，取 $G_D(\vec{r},\vec{r}')\big|_{\partial V} = 0$

$$\Rightarrow \phi(\vec{r}) = \int_V \rho(\vec{r}')G_D(\vec{r},\vec{r}')d^3\vec{r}' - \frac{1}{4\pi}\int_{\partial V}\phi(\vec{r}')\nabla'G(\vec{r},\vec{r}')\cdot d\vec{s}'$$

若 ϕ 在 ∂V 上满足 Neumann 边界条件，由 $\int_V \nabla'^2 G_N(\vec{r},\vec{r}')d^3r' = \int_{\partial V}\nabla' G_N(\vec{r},\vec{r}')d^3r' = -4\pi$

可以取 $\nabla' G_N(\vec{r},\vec{r}')_{\partial V} = -\frac{4\pi}{A}$

于是 $\phi(\vec{r}) = \int_V \rho(\vec{r}')G_N(\vec{r},\vec{r}')d^3r' + \frac{1}{4\pi}\int_{\partial V} G_N(\vec{r},\vec{r}')\nabla'\phi(\vec{r}')\cdot d\vec{s}' + \langle\phi\rangle\big|_{\partial V}$

在 Dirichlet 边界条件下，$G_D(\vec{r},\vec{r}') = G_D(\vec{r}',\vec{r})$，但 Neumann 边界条件下 $G_N(\vec{r},\vec{r}')$ 不一定可交换

但总可选取适当的附加条件使 $G_N(\vec{r},\vec{r}')$ 可交换

G_D 可由接地导体电像求出，G_N 与一般的导体电像不同

本征函数

Laplace方程在球坐标中的轴对称解：Legendre多项式. $P_l(x)$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l \qquad P_0(x)=1 \qquad P_l(1)=1 \qquad \int_{-1}^{1} P_k(x) P_l(x)\,dx = \frac{2}{2l+1}\delta_{kl}.$$

$$P_l(-x) = (-1)^l P_l(x). \qquad \phi(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{C_l}{r^{l+1}} + D_l r^l\right) P_l(\cos\theta).$$

通解：定义 $Y_l^m(\theta,\varphi)$ $\quad \nabla^2_{\theta,\varphi} Y_l^m(\theta,\varphi) = l(l+1) Y_l^m(\theta,\varphi)$ $\quad Y_l^{-m}(\theta,\varphi) = (-1)^m Y_l^{m*}(\theta,\varphi)$

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) Y_l^m(\theta,\varphi).$$

对于 $\phi = \dfrac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$ （源点 (r',θ',φ') 场点 (r,θ,φ)）

$r>r'$时，$\phi = \sum_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}} P_l(\cos\gamma)$ $\qquad$ $r<r'$时 $\phi = \sum_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}} P_l(\cos\gamma).$

$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'$ $\qquad$ $\phi(r,\theta,\varphi)$又可写成

$$\begin{cases} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \dfrac{1}{r^{l+1}} B_{lm}(r',\theta',\varphi')\, Y_l^m(\theta,\varphi) & r>r' \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^l A_{lm}(r',\theta',\varphi')\, Y_l^m(\theta,\varphi) & r<r'. \end{cases}$$

有如下加法公式：$\boxed{P_l(\cos\gamma) = \sum_{m=-l}^{l}\dfrac{4\pi}{2l+1}[Y_l^m]^*(\theta',\varphi')\, Y_l^m(\theta,\varphi)}$

因此 $\phi = \begin{cases} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} Y_l^{m*}(\theta',\varphi') Y_l^m(\theta,\varphi) \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r^l}{r'^{l+1}} \frac{4\pi}{2l+1} Y_l^{m*}(\theta',\varphi') Y_l^m(\theta,\varphi) \end{cases}$

球形边界的 Dirichlet Green 函数 $G_D(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{a/r'}{|\vec{r}-(a^2/r'^2)\vec{r}'|} \quad (r, r' > a)$

$r > r'$ 时，$G_D(\vec{r},\vec{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\left[\frac{r'^l}{r^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r'^{l+1}}\right] Y_l^{m*}(\theta',\varphi') Y_l^m(\theta,\varphi)$

$r < r'$ 时 $G_D(\vec{r},\vec{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\left[\frac{r^l}{r'^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r'^{l+1}}\right] Y_l^{m*}(\theta',\varphi') Y_l^m(\theta,\varphi)$

多极展开

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} - x'^i\partial_i\frac{1}{r} + \frac{1}{2!}x'^i x'^j \partial_i\partial_j\frac{1}{r} - \frac{1}{3!}x'^i x'^j x'^k \partial_i\partial_j\partial_k\frac{1}{r} + \cdots$$

$\partial_i r = \frac{x_i}{r}$ $\quad \nabla^2\frac{1}{r} = \partial_i\partial_i\frac{1}{r} = 0$ 因此对 $\partial_i\partial_j\frac{1}{r}$ $\partial_i\partial_j\partial_k\frac{1}{r}$ $\partial_i\partial_j\partial_k\partial_l\frac{1}{r}$ …… 缩并其中任意两指标结果为 0.

因此可以把 $x'^i x'^j$ $x'^i x'^j x'^k$ 等约化为无迹张量而不产生任何影响：

$$x'^i x'^j - \frac{1}{3}r'^2\delta_{ij} \qquad x'^i x'^j x'^k - \frac{1}{5}(x'^i\delta^{jk} + x'^j\delta^{ik} + x'^k\delta^{ij})r'^2$$

得到 $\phi(r) = \frac{Q}{r} - p^i\partial_i\frac{1}{r} + \frac{1}{3}\frac{1}{2!}Q^{ij}\partial_i\partial_j\frac{1}{r} - \frac{1}{5}\frac{1}{3!}Q^{ijk}\partial_i\partial_j\partial_k\frac{1}{r} + \cdots$

无迹的好处是，我们有 $\phi(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{n}}{r^2} + \frac{1}{2}\frac{Q^{ij} n_i n_j}{r^3} + \frac{1}{2}\frac{Q^{ijk} n_i n_j n_k}{r^4} + \cdots$

对偶极子，$E_i = -\partial_i \frac{p^j x_j}{r^3} = \frac{3p^j n_j n_i - p^j \delta_{ij}}{r^3}$

2^l 极矩有 $2l+1$ 个独立分量

使用球谐函数展开：$\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{B_{lm}}{r^{l+1}} Y_l^m(\theta,\varphi)$

或者可以直接展开 $\phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{q_{lm}}{2l+1}\frac{1}{r^{l+1}} Y_l^m(\theta,\varphi)$

其中 $q_{lm} = \int \rho(\vec{r}') r'^l \, Y_l^{m*}(\theta',\varphi') \, d^3r'$

$q_{lm}^* = (-1)^m q_{l,-m}$ 即 $\{q_{lm}\}$ 对给定的 l 有 $2l+1$ 个独立实数

外场中能量的多极展开

$$V_{ext} = \int \rho(\vec{r}) \Phi(\vec{r}) \, d^3r = Q\,\Phi(0) + p^i E_i(0) - \frac{1}{6} Q^{ij} \partial_i E_j(0)$$

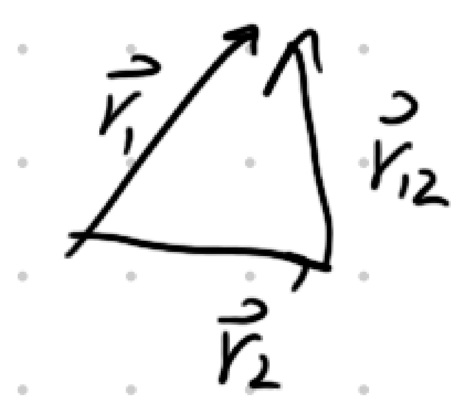
静磁场

选取库仑规范 $\nabla\cdot\vec{A} = 0$ 于是 $\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{j}$ $\Rightarrow \vec{A} = \frac{1}{c}\int \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$

$$\Rightarrow \vec{B}(\vec{r}) = \nabla\times\vec{A}(\vec{r}) = \frac{1}{c}\int \nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{j}(\vec{r}')\,d^3r' = \frac{1}{c}\int \vec{j}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d^3r'$$

$$d\vec{B}(\vec{r}) = \frac{I}{c}\frac{d\vec{l}\times\vec{r}}{|\vec{r}|^3} \qquad B(\vec{r}) = \frac{q}{c}\frac{\vec{v}\times\vec{r}}{|\vec{r}|^3}$$

安培定律： $d\vec{F} = \frac{I_1}{c}d\vec{l}_1\times\vec{B} = \frac{I_1I_2}{c^2}\frac{d\vec{l}_1\times(d\vec{l}_2\times\vec{r}_{12})}{r_{12}^3}$ $\quad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$



$$\vec{F}_{12} = \frac{I_1I_2}{c^2}\oint\oint\frac{d\vec{l}_1\times(d\vec{l}_2\times\vec{r}_{12})}{r_{12}^3} = -\frac{I_1I_2}{c^2}\oint\oint\frac{d\vec{l}_1\cdot d\vec{l}_2}{r_{12}^3}\vec{r}_{12}$$

$$\vec{F} = \frac{1}{c}\int\vec{j}(\vec{r})\times\vec{B}(\vec{r})\,d^3r \qquad \tau = \int\vec{r}\times d\vec{F} = \frac{1}{c}\int\vec{r}\times[\vec{j}(\vec{r})\times\vec{B}(\vec{r})]\,d^3r$$

$$\vec{F} = \frac{1}{c}q\vec{v}\times\vec{B}(\vec{r}_q)$$

磁介质下 $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}_f + 4\pi\nabla\times\vec{M} = -\nabla^2\vec{A}$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{1}{c}\int\frac{\vec{j}_f(\vec{r}') + c\nabla\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3r' \qquad \vec{j}_M = c\nabla\times\vec{M}(\vec{r}')$$

若问题中不存在自由电流 $\nabla\times\vec{H} = 0$ 可以记 $\vec{H} = -\nabla\varphi_m$ 若 $\vec{B} = \mu\vec{H} \Rightarrow \nabla^2\Phi_m = 0$

磁场与局域电流

$$m = \frac{IS}{c}$$

结果：对稳恒电流0阶项为0，一阶项为 $\vec{A}(\vec{r}) = \frac{\vec{m}\times\vec{r}}{r^3}$ $\vec{m} = \frac{1}{2c}\int \vec{r}' \times \vec{j}(\vec{r}')\, d^3r'$ $\vec{M} = \frac{1}{2c}\vec{r}\times\vec{j}$.

$$\vec{B} = \nabla\times\vec{A} = \frac{3(\vec{m}\cdot\vec{n})\vec{n} - \vec{m}}{r^3}.$$

考虑原点后 $\vec{B} = \frac{3(\vec{m}\cdot\vec{n})\vec{n} - \vec{m}}{r^3} + \frac{8\pi}{3}\vec{m}\,\delta(\vec{r})$

外磁场对电流分布作用的力：$\vec{F} = \nabla(\vec{m}\cdot\vec{B}) = -\nabla U$ $U = -\vec{m}\cdot\vec{B}$.

单位制转换

Gauss	SI
c	$\frac{1}{\sqrt{\varepsilon_0\mu_0}}$
$\vec{E},\ \varphi$	$\sqrt{4\pi\varepsilon_0}\,(\vec{E},\ \varphi)$
$\vec{D}$	$\sqrt{\frac{4\pi}{\varepsilon_0}}\ \vec{D}$
$q,\rho,I,\vec{J},\vec{P},\vec{p}$	$\frac{1}{\sqrt{4\pi\varepsilon_0}}(q,\rho,I,\vec{J},\vec{P},\vec{p})$
$\vec{B},\ \Phi_m,\ \vec{A}$	$\sqrt{\frac{4\pi}{\mu_0}}\,(\vec{B},\ \Phi_m,\ \vec{A})$
$\vec{H}$	$\sqrt{4\pi\mu_0}\ \vec{H}$
$m,\vec{M}$	$\sqrt{\frac{\mu_0}{4\pi}}\ (\vec{m},\vec{M})$
$\epsilon,\ \mu$	$\frac{\epsilon}{\varepsilon_0},\ \frac{\mu}{\mu_0}$
$\chi_e,\ \chi_m$	$\frac{1}{4\pi}(\chi_e,\ \chi_m)$
$\sigma,\ S,\ C$	$\frac{1}{4\pi\varepsilon_0}(\sigma,\ S,\ C)$
$\rho,\ R,\ L$	$4\pi\varepsilon_0\ (\rho,R,L)$.

$\nabla^2 \vec{A} = (\nabla^2 A^\mu) \vec{e}_\mu$ 在曲线坐标系下不适用，此时需利用

$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ 计算 $\nabla^2 \vec{A}$.

下半部分

洛伦兹变换 $\vec{r}' = \vec{r} + \frac{(\gamma-1)}{v^2}(\vec{r}\cdot\vec{v})\vec{v} - \gamma\vec{v}t$ $t' = \gamma(t - \vec{v}\cdot\vec{r})$

Lorentz 变换 $\Lambda^\mu{}_\nu$ 保持度规 $\eta_{\mu\nu}$ 不变： $\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta^{\mu\nu} = \eta^{\rho\sigma}$ 或写为 $\Lambda^\mu{}_\nu \Lambda_\mu{}^\sigma = \delta^\sigma_\nu$

$\Lambda^\mu{}_\nu$ 有6个独立分量，3个表示 rotation，3个表示 boost，构成 $O(1,3)$ 群 $\det|\Lambda|\ \pm 1$

闵氏度规 $\eta^{\mu\nu}\eta_{\mu\sigma} = \delta^\nu_\sigma$

4-vector 定义为满足Lorentz变换：$A'^\mu = \Lambda^\mu{}_\nu A^\nu$ 的 $\{A^\mu\}$ 逆变   反变换：$A^\nu = \Lambda_\mu{}^\nu A'^\mu$

$A'_\mu = \Lambda_\mu{}^\nu A_\nu$ 协变

$A_\mu = \Lambda^\nu{}_\mu A'_\nu$

基矢 $\partial_\mu = (\partial_t, \nabla)$ 是协变的 $\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$

体元不变 $dt'dx'dy'dz' = ||\Lambda^\mu{}_\nu||\,dt\,dx\,dy\,dz = dt\,dx\,dy\,dz$

张量与张量缩并 略 $\Box = \partial^\mu\partial_\mu = \partial_\mu\partial^\mu = -\frac{\partial^2}{\partial t^2} + \nabla^2$

$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ $d\tau^2 = -ds^2 > 0$ $U^\mu = \frac{dx^\mu}{d\tau}$ 是 逆变 4-vector

类时世界线 $v<1$，$d\tau = \frac{1}{\gamma}dt$. $U^\mu = (\gamma, \gamma\vec{u})$ $U^\mu U_\mu = \frac{ds}{d\tau} = -1$. $\gamma = \sqrt{1-u^2}$

$p^\mu = mU^\mu = (m\gamma, m\gamma\vec{u}) = (E, \vec{p})$ $p^\mu p_\mu = -m^2$ $f^\mu = \frac{dp^\mu}{d\tau}$ 瞬时系下 $f^\mu = (0, \vec{F})$

$A^\mu = (\phi, \vec{A})$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $J^\mu = (\rho, \vec{J})$ 电荷守恒：$\partial_\mu J^\mu = 0$

$F_{\mu\nu} = -F_{\nu\mu}$

$A^\mu \to A^\mu + \partial^\mu \lambda$ 称为规范变换，不改变 $F_{\mu\nu}$.　　Lorentz 规范：$\partial_\mu A^\mu = 0$.

场方程 $\begin{cases} \nabla\cdot\vec{E} = 4\pi\rho \\ \nabla\times\vec{B} - \dfrac{\partial\vec{E}}{\partial t} = 4\pi\vec{J} \end{cases} \Rightarrow \boxed{\partial_\mu F^{\mu\nu} = -4\pi J^\nu}$　　Lorentz 规范下：$\Box A^\nu = -4\pi J^\nu$.

$$\begin{cases} \dfrac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 4\pi\rho \\ \dfrac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = 4\pi\vec{J} \end{cases}$$

比安基方程 $\begin{cases} \nabla\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0 \\ \nabla\cdot\vec{B} = 0 \end{cases} \Rightarrow \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$　　$\partial_{[\rho} F_{\mu\nu]} = 0$

$$\Updownarrow$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = \frac{\partial\vec{A}}{\partial t} + (\nabla\phi)_i = -E_i.$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B_k.$$

$$\begin{cases} \vec{E}' = \gamma(\vec{E} + \vec{v}\times\vec{B}) - \dfrac{\gamma-1}{v^2}(\vec{v}\cdot\vec{E})\vec{v} \\ \vec{B}' = \gamma(\vec{B} - \vec{v}\times\vec{E}) - \dfrac{\gamma-1}{v^2}(\vec{v}\cdot\vec{B})\vec{v}. \end{cases}$$

$$\begin{cases} \vec{E}_{\parallel}' = \vec{E}_{\parallel} & \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{v}\times\vec{B}) \\ \vec{B}_{\parallel}' = \vec{B}_{\parallel} & \vec{B}_{\perp}' = \gamma(\vec{B}_{\perp} - \vec{v}\times\vec{E}). \end{cases}$$

Lorentz force $f^\mu = eF^{\mu\nu}U_\nu$ $\Rightarrow \frac{d\vec{p}}{dt} = e(\vec{E}+\vec{u}\times\vec{B})$

free particle lorentz-invariant action $S = -m\int_{\tau_1}^{\tau_2} d\tau = -m\Delta\tau$ $\delta S = 0$

$S = -m\int_{t_1}^{t_2}(1-\dot{x}^i\dot{x}^i)^{1/2}dt \Rightarrow L = -m(1-\dot{x}^i\dot{x}^i)^{1/2}$ $\delta S = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0$

$F_{\mu\nu}$ 下 $S = \int_{\tau_1}^{\tau_2}(-m\,d\tau + eA_\mu dx^\mu)$ Lorentz 协变 $L = -m(1-\dot{x}^j\dot{x}^j)^{1/2} - e\phi + eA_i\dot{x}^i$

$\delta S = 0 \Rightarrow \frac{d\vec{p}}{dt} = e(\vec{E}+\vec{v}\times\vec{B})$ $f^\mu = eF^{\mu\nu}U_\nu$

S 是规范不变的 $A_\mu \to A_\mu + \partial_\mu\lambda$ $S \to S + \int_{\tau_1}^{\tau_2} e\partial_\mu\lambda\, dx^\mu = S + \underbrace{e[\lambda(\tau_2)-\lambda(\tau_1)]}_{\text{常数项}}$

$\pi_i = \frac{\partial L}{\partial \dot{x}_i} = m(1-\dot{x}^j\dot{x}^j)^{-1/2}\dot{x}_i + eA_i = p_i + eA_i$

$H = \pi_i\dot{x}_i - L = m\gamma + e\phi = \sqrt{(\pi_i - eA_i)^2 + m^2} + e\phi$ $v \ll 1 \Rightarrow H \approx \frac{1}{2m}(\pi_i - eA_i)^2 + e\phi$

EM场的2个洛伦兹不变量 $I_1 = F_{\mu\nu}F^{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)$ $I_2 = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4\vec{E}\cdot\vec{B}$

其中 $\epsilon^{0123} = -1$ $(\epsilon_{0123} = 1)$ $\epsilon^{\mu\nu\rho\sigma}$ 是赝张量，在洛伦兹变换下

$\epsilon'^{\mu\nu\rho\sigma} = (\det\Lambda)\,\epsilon^{\mu\nu\rho\sigma}$ 只要 Λ 没有宇称变换，$\epsilon^{\mu\nu\rho\sigma}$ 就与一般张量无区别

无源情形

$L=\int \mathcal{L}\, d^3x \Rightarrow S=\int \mathcal{L}\, dx^4$ S洛伦兹不变 $\Rightarrow \mathcal{L}$也是 取 $\mathcal{L}=-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$ (2nd-order formalism).

这里 $F_{\mu\nu}$ 只是 $\partial_\mu A_\nu-\partial_\nu A_\mu$ 的记号，设4维边界面上 $\delta A_\nu=0$ 令 $\delta S=0 \Rightarrow \partial_\mu F^{\mu\nu}=0$ (场方程).

比安基方程已包括在 $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ 中.

1st-order formalism: $S_{f.o.}=\frac{1}{4\pi}\int\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-F^{\mu\nu}\partial_\mu A_\nu\right)d^4x$ 这里 F与A视为独立的，

对 $F^{\mu\nu}$ 变分，并要求 $\delta F^{\mu\nu}$ 具有全反对称性，$\delta S_{f.o}=0 \Rightarrow F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.

对 A_ν 变分 $\Rightarrow \partial_\mu F^{\mu\nu}=0$.

有源情形 取 $\mathcal{L}=-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}+J^\mu A_\mu$

点电荷情形 $J^\mu=q\,\delta(\vec{x}-\vec{x}_0)\frac{dx_0^\mu}{dt}$. $\int J^\mu A_\mu d^4x$ 得到 $q\int_{path}A_\mu dx^\mu$ 即带电粒子于电磁场的作用量.

电荷守恒 $\Rightarrow S$ 规范不变 (或差个常数) (电荷守恒 $\Leftrightarrow$ 规范对称性!). $Q=\int_{t=const}J^\mu d\Sigma_\mu$

标量场 在 $\partial_\mu A^\mu$ 规范下 $\Box\phi=0$ (修改: $\Box\phi-m^2\phi=0$，构造短程相互作用) 平凡: 加上质量项 涨落变为短程

量子: 把 ϕ 视为波函数 即为 K-G方程

对应 $\mathcal{L}=-\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi)-\frac{1}{2}m^2\phi^2$

假设 $\mathcal{L}=\mathcal{L}(\phi,\partial_\nu\phi)$ 不显式依赖 x_ν $\Rightarrow$ E-L方程: $\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\right)=0$.

$\partial_\rho \mathcal{L} = \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \partial_\rho \phi \right]$ 构造 $T_\rho{}^\nu = -\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \partial_\rho \phi + \delta_\rho^\nu \mathcal{L} \Rightarrow \partial_\nu T_\rho{}^\nu = 0$. $T^{\mu\nu}$ 即能动张量.

对 free massive 标量场，$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial^\sigma \phi)(\partial_\sigma \phi) - \frac{1}{2}m^2 \eta_{\mu\nu}\phi^2$. 这里 $T_{\mu\nu}$ 是对称的.

$\partial_\nu T_\rho{}^\nu = 0 \Rightarrow P^\mu = \int_{t=\text{const}} T^{\mu 0} d\Sigma_0 = \int_{t=\text{const}} T^{\mu\nu} d\Sigma_\nu$ 是守恒 4-vector.

$$\left(\frac{dP^\mu}{dt} = \int \partial_0 T^{\mu 0} d^3x = -\int \partial_i T^{\mu i} d^3x = -\int_S T^{\mu i} dS_i = 0 \right)$$

$T^{00} = -\frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \partial^0 \phi - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \partial_0 \phi - \mathcal{L}$ 即能量密度. $P^0 = \int T^{00} d^3x$ 为总能量, P^μ 为4动量.

$T_\rho{}^\nu \longrightarrow T_\rho{}^\nu + \partial_\sigma \psi_\rho{}^{\nu\sigma}$ ($\psi_\rho{}^{\nu\sigma}$ 是任意关于ν,σ反对称的张量，$\psi_\rho{}^{\nu\sigma} = -\psi_\rho{}^{\sigma\nu}$，且在空间无穷远处 $\to 0$)

$\partial_\nu \partial_\sigma \psi_\rho{}^{\nu\sigma} = 0 \Rightarrow \partial_\nu [T_\rho{}^\nu + \partial_\sigma \psi_\rho{}^{\nu\sigma}] = 0$，即若只要求 $\partial_\nu T_\rho{}^\nu = 0$，$T_\rho{}^\nu$ 不唯一.

但 P^μ 不变 $\left(\int \partial_\sigma \psi^{\mu 0\sigma} d\Sigma_0 = \int \partial_i \psi^{\mu 0 i} d^3x = 0 \right)$

$T^{\mu\nu}$ 可通过要求4角动量 $M^{\mu\nu}$ 守恒来"唯一"确定 $M^{\mu\nu} = \int_t (x^\mu dP^\nu - x^\nu dP^\mu) = \int_t (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) d\Sigma_\rho$

$\frac{dM^{\mu\nu}}{dt} \Longleftrightarrow \partial_\rho (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) = 0 \Longleftrightarrow T^{\mu\nu} = T^{\nu\mu}$. 调整 $\partial_\sigma \psi^{\mu\nu\sigma}$ 总可以做到

$T^{\mu\nu} + \partial_\sigma \psi^{\mu\nu\sigma} = T^{\nu\mu} + \partial_\sigma \psi^{\nu\mu\sigma} \Rightarrow \partial_\sigma(\psi^{\mu\nu\sigma} - \psi^{\nu\mu\sigma}) = T^{\nu\mu} - T^{\mu\nu}$ $\partial_\sigma\psi^{\mu\nu\sigma}$ 共16个独立分量

$\partial_\sigma\psi^{\mu\nu\sigma}$ 仍有6个未定的（共形？规范不变？） 6个方程加上 $\partial_\nu(\partial_\sigma\psi^{\mu\nu\sigma})=0$ 4个方程

· 之后总假定 $T^{\mu\nu}$ 对称（物理要求，不能有∞大角加速度，能流密度＝动量密度）

$\partial_\nu T^{\mu\nu} = 0$ $\mu=0 \Rightarrow \frac{\partial}{\partial t}T^{00} + \partial_j T^{0j} = 0$ $\frac{\partial}{\partial t}\int_V T^{00} d^3x = -\int_S T^{0j} dS_j \rightarrow T^{0j}$ 为能流，由 $T^{0j} = T^{j0}$

$\Rightarrow$ 能流密度 ＝ 动量密度 $\mu=i \Rightarrow \frac{\partial}{\partial t}T^{i0} + \partial_j T^{ij} = 0$ $\frac{\partial}{\partial t}\int_V T^{i0} d^3x = -\int_S T^{ij} dS_j$

T^{ij}：3级 动量流密度.

上述对标量场讨论推广到 vector field A_σ. $T_\rho{}^\nu = -\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\sigma}\partial_\rho A_\sigma + \delta_\rho^\nu \mathcal{L}$. $\mathcal{L} = \mathcal{L}(A_\sigma, \partial_\nu A_\sigma)$

代入无源电磁场 $\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$ 由 $\delta\mathcal{L} = -\frac{1}{8\pi}F^{\mu\nu}\delta F_{\mu\nu} = -\frac{1}{8\pi}F^{\mu\nu}[\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu]$

$$= -\frac{1}{4\pi}F^{\mu\nu}\partial_{[\mu}\delta A_{\nu]} = -\frac{1}{4\pi}F^{\mu\nu}\partial_\mu\delta A_\nu.$$

$\Rightarrow \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} = -\frac{1}{4\pi}F^{\mu\nu}$. $\Rightarrow T_\rho{}^\nu = \frac{1}{4\pi}F^{\nu\sigma}\partial_\rho A_\sigma - \frac{1}{16\pi}\delta_\rho^\nu F_{\sigma\lambda}F^{\sigma\lambda}$.

$\Rightarrow T^{\mu\nu} = \frac{1}{4\pi}F^{\nu\sigma}\partial^\mu A_\sigma - \frac{1}{16\pi}\eta^{\mu\nu}F_{\sigma\lambda}F^{\sigma\lambda}$. choose $\psi^{\mu\nu\sigma} = -\frac{1}{4\pi}A^\mu F^{\nu\sigma}$.

$$\Rightarrow T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\nu\sigma}F^\mu{}_\sigma - \frac{1}{4}\eta^{\mu\nu}F_{\sigma\lambda}F^{\sigma\lambda}\right)$$

$T^{\mu\nu}$ 无迹： $\eta_{\mu\nu}T^{\mu\nu}=0$ （共形不变性）　　$T^{\mu\nu}$ 规范不变.　　物理上要求：$\begin{cases}\text{共形}\\ \text{规范不变}\\ \text{对称}\end{cases}$ 才是真实的 $T^{\mu\nu}$.

$$T^{\mu\nu}=\begin{pmatrix} W & S_j \\ S_i & \sigma_{ij}\end{pmatrix} \qquad W=\frac{1}{8\pi}(\vec{E}^2+\vec{B}^2) \qquad \vec{S}=\frac{1}{4\pi}\vec{E}\times\vec{B}$$

$$\sigma_{ij}=-\frac{1}{4\pi}(E_iE_j+B_iB_j)+W\delta_{ij}.$$

带电粒子与场的耦合

考虑质量为 m 的粒子，速度为 $\vec{v}$，质量密度 $\varepsilon = m\,\delta(\vec{r}-\vec{r}_0)$.　　$T^{00}=\gamma\varepsilon=\varepsilon\frac{dt}{d\tau}$.

3-动量密度 $T^{0i}=\varepsilon\frac{dt}{d\tau}\frac{dx^i}{dt}=\frac{\varepsilon}{\gamma}\frac{dt}{d\tau}\frac{dx^i}{d\tau}$.　　$\frac{\varepsilon}{\gamma}$ 洛伦兹不变　$\frac{\varepsilon}{\gamma}=\frac{m\,\delta(\vec{r}-\vec{r}_0)\,d\tau}{dt}\propto\frac{1}{dt\,dx\,dy\,dz}$

根据洛伦兹协变，$T^{\mu\nu}=\frac{\varepsilon}{\gamma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\varepsilon\frac{dx^\mu}{d\tau}\frac{dx^\nu}{dt}$.

粒子在场中运动，$T^{\mu\nu}_{tot}=T^{\mu\nu}_{e.m}+T^{\mu\nu}_{part.}$ 和守恒　但分别不守恒

对有源情况 $\partial_\nu T^{\mu\nu}_{e.m}=-F^\mu{}_\nu J^\nu$　不守恒.

$$\partial_\nu T^{\mu\nu}_{part.}=\underbrace{\partial_\nu\left(\varepsilon\frac{dx^\nu}{dt}\right)}_{=0}\frac{dx^\mu}{d\tau}+\varepsilon\frac{dx^\nu}{dt}\partial_\nu\left(\frac{dx^\mu}{d\tau}\right)=\varepsilon\frac{dx^\nu}{dt}\partial_\nu U^\mu=\varepsilon\frac{dU^\mu}{dt}$$

由 $m\frac{dU^\mu}{d\tau}=qF^\mu{}_\nu U^\nu$　　$\varepsilon\frac{dU^\mu}{dt}=\rho F^\mu{}_\nu\frac{dx^\nu}{dt}=F^\mu{}_\nu J^\nu$.　$\Rightarrow\ \partial_\nu T^{\mu\nu}_{tot.}=0$.

匀速运动电荷的场

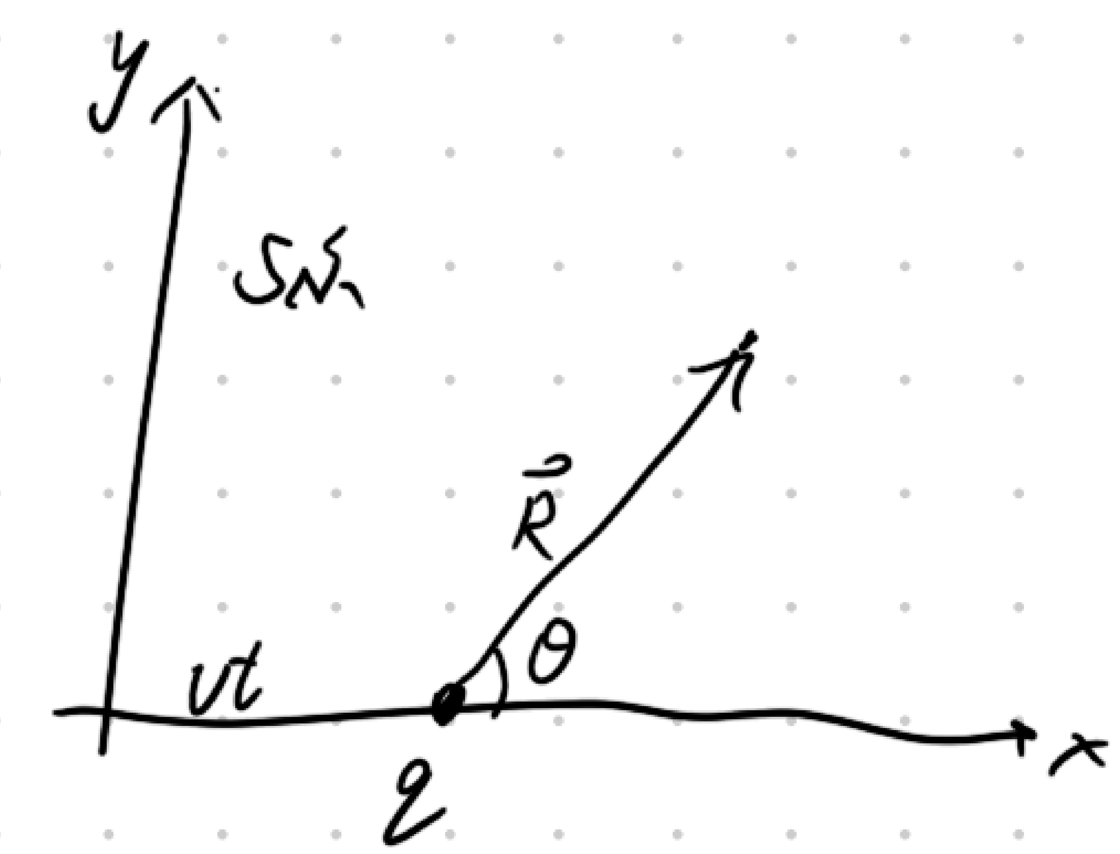
S'系: $\phi' = \frac{e}{r'}$ $\quad \vec{A}' = 0$ $\quad$ S系: $\phi = \gamma(\phi' + \vec{v}\cdot\vec{A}')$ $\quad \vec{A} = \vec{A}' + \frac{\gamma-1}{v^2}(\vec{v}\cdot\vec{A}')\vec{v} + \gamma\vec{v}\phi'$

$\Rightarrow \phi = \gamma\phi' = \frac{e\gamma}{r'}$ $\quad \vec{A} = \gamma\vec{v}\phi' = \frac{\gamma e}{r'}\vec{v}$

设 $\vec{v} = v\hat{x}$ $\quad r'^2 = \gamma^2(x-vt)^2 + y^2 + z^2$ $\quad \Rightarrow \phi = \frac{e}{R_*}$ $\quad \vec{A} = \frac{e\vec{v}}{R^*}$

$R^{*2} = (x-vt)^2 + (1-v^2)(y^2+z^2)$ $\quad$ $v\to 0$时 $R^* = R$

S'系: $\vec{E}' = \frac{e\vec{r}'}{r'^3}$ $\quad \vec{B}' = 0$ $\quad \Rightarrow \begin{cases} \vec{E} = \frac{e\gamma\vec{r}'}{r'^3} - \frac{\gamma-1}{v^2}\frac{e\vec{v}\cdot\vec{r}'}{r'^3}\vec{v} \\ \vec{B} = \gamma\vec{v}\times\vec{E}' = \frac{e\gamma\vec{v}\times\vec{r}'}{r'^3} \end{cases}$



$\vec{v} = v\hat{x}$时 $\vec{R}$(源点→场点) $= (x-vt, y, z)$

$\Rightarrow \vec{E} = \frac{e\gamma\vec{R}}{r'^3} = \frac{e(1-v^2)\vec{R}}{R^{*3}}$ $\quad$ 记 θ 为 $\vec{R}$ 与 x 轴夹角

$R_*^2 = R^2 - v^2(y^2+z^2) = R^2(1-v^2\sin^2\theta)$ $\quad \Rightarrow \vec{E} = \frac{e\vec{R}}{R^3}\boxed{\frac{1-v^2}{(1-v^2\sin^2\theta)^{3/2}}}$ 相对论因子

$E_\parallel = \frac{e(1-v^2)}{R^2}$ ↓ $\quad E_\perp = \frac{e(1-v^2)^{-1/2}}{R^2}$ ↗, $\quad \vec{v}\times\vec{E} = \gamma\vec{v}\times\vec{E}'$

$v\sim 1$时, $E_\parallel \to 0$ $\quad E_\perp \to \infty$ $\quad$ 记 $\psi = \frac{\pi}{2} - \theta$ $\quad |\vec{E}| = \frac{e}{R^2}\frac{1-v^2}{[1-v^2(1-\frac{1}{2}\psi^2)]^{3/2}} \approx \frac{e}{R^2}\frac{1-v^2}{(1-v^2+\frac{1}{2}\psi^2)^{3/2}}$ $\quad (\psi \ll 1)$

峰的角宽度 $\Delta\psi \sim \sqrt{1-v^2}$.

$$\vec{B} = \gamma \vec{v} \times \vec{E}' = \vec{v} \times \vec{E} = \frac{e(1-v^2)\vec{v}\times\vec{R}}{R_*^3}$$

库仑势下电荷运动的相对论结果

$\phi = \frac{Q}{r}$ $\vec{A} = 0$ 采用极坐标 $L = -m \underbrace{(1-\dot{r}^2 - r^2\dot{\varphi}^2)^{1/2}}_{\gamma^{-1}} - \frac{k}{r}$ $k = eQ$.

$l = \frac{\partial L}{\partial \dot{\varphi}} = \gamma m r^2 \dot{\varphi}$ 守恒 $\Rightarrow m r^2 \frac{d\varphi}{d\tau} = l$. $\mathcal{E} = \sqrt{m^2 + m^2\gamma^2 \vec{v}^2} + \frac{k}{r}$ 守恒.

$(\mathcal{E} - \frac{k}{r})^2 = m^2\left[\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 + 1\right]$ 取 $r = \frac{1}{u}$ $\Rightarrow (\mathcal{E} - ku)^2 = l^2(u'^2 + u^2) + m^2$. $u' = \frac{du}{d\tau}$.

进一步求解略去. 州门是进动角: $\delta\varphi = 2\pi\left[\left(1 - \frac{k^2}{l^2}\right)^{-\frac{1}{2}} - 1\right] \approx \frac{\pi k^2}{l^2}$.

电磁波真空无源情形 $\Box\vec{E} = \Box\vec{B} = 0$ $\vec{B} = \vec{n}\times\vec{E}$ $\vec{n}\cdot\vec{B} = \vec{n}\cdot\vec{E} = \vec{B}\cdot\vec{E} = 0$ $E = B$

$W = \frac{1}{4\pi}E^2$ $\vec{S} = \vec{n}W$ 定义 $n^\mu = (1, \vec{n})$ $\vec{n}\cdot\vec{n} = 1 \Rightarrow n^\mu n_\mu = 0$ n^μ 称为 null vector

$\vec{n}\cdot\vec{r} - t = n_\mu x^\mu$ $T^{ij} = \frac{1}{4\pi}n_i n_j E^2 = n_i n_j W \Rightarrow T^{\mu\nu} = n^\mu n^\nu W$

$P^\mu = \int_{t=\text{const}} T^{\mu\nu} d\Sigma_\nu = \int T^{\mu 0} d^3x = n^\mu \int W d^3x = n^\mu \mathcal{E}$ P^μ 也是 null vector

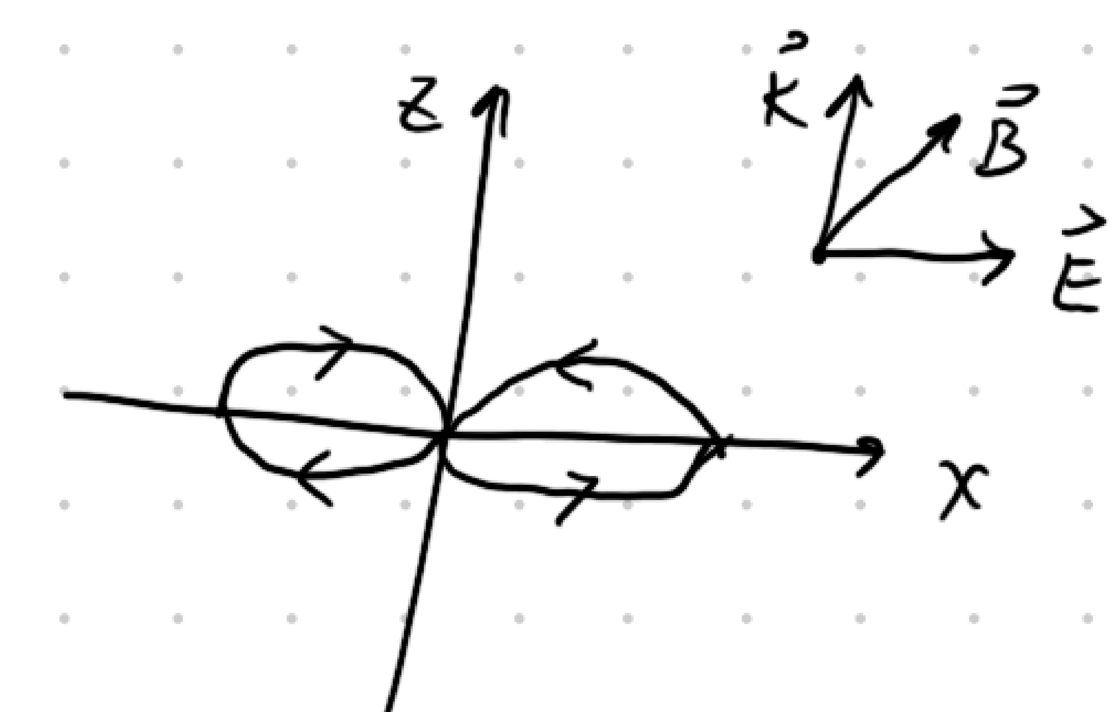
单色情形 $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ $k = \omega$ $\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}$ $k^\mu = (\omega, \vec{k}) = \omega n^\mu$

k^μ 是 null vector $\vec{k}\cdot r - \omega t = k_\mu x^\mu$ 写为 $k\cdot x$ $\langle W\rangle = \frac{1}{8\pi}|E_0|^2 = \frac{1}{8\pi}|B_0|^2 = \frac{1}{8\pi}(\vec{E}\cdot\vec{E}^*)$

$\langle\vec{S}\rangle = \frac{1}{8\pi}\vec{E}\times\vec{B}^* = \frac{1}{8\pi}\vec{n}\,\vec{E}\cdot\vec{E}^* = \vec{n}\langle W\rangle$

线偏振电磁波中点电荷运动 $\vec{E} = E_0\cos\omega(z-t)\,\hat{x}$ $\vec{B} = E_0\cos\omega(z-t)\,\hat{y}$ $\begin{cases} m\ddot{x} = eE_0\cos\omega t \\ m\ddot{z} = e\dot{x}E_0\cos\omega t \end{cases}$

$x = \alpha\cos\omega t$, $z = \beta\sin 2\omega t$ $\alpha = -\frac{eE_0}{m\omega^2}$ $\beta = -\frac{e^2E_0^2}{8m^2\omega^3}$



圆偏振与椭圆偏振 设 $\vec{E} = (a_1 e^{i\delta_1}, a_2 e^{i\delta_2}, 0)e^{i\omega(z-t)}$

Stokes 参量 (s_0, s_1, s_2, s_3) $s_0^2 = s_1^2 + s_2^2 + s_3^2$

$s_0 = E_xE_x^* + E_yE_y^* = a_1^2 + a_2^2$ $s_1 = E_xE_x^* - E_yE_y^* = a_1^2 - a_2^2$

$s_2 = 2\,\mathrm{Re}(E_x^*E_y) = 2a_1a_2\cos(\delta_1 - \delta_2)$ $s_3 = 2\,\mathrm{Im}(E_x^*E_y) = 2a_1a_2\sin(\delta_2 - \delta_1)$

平面波的一般叠加　　辐射规范：令 $\phi=0$　　单色平面波 $\vec{A} = a\vec{e}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$　　$\vec{e}$：极化矢量，

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = i a\omega\, \vec{e}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{B} = i a\, \vec{k}\times\vec{e}\ e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

一般：$\vec{A} = \sum_{\lambda=1}^{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\left[\vec{e}_\lambda(\vec{k})\, a_\lambda(\vec{k})\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + c.c.\right]$　　其中 $\omega=|\vec{k}|$，$\vec{k}\cdot\vec{e}_\lambda(\vec{k})=0$

$\vec{e}_\lambda\cdot\vec{e}_{\lambda'}(\vec{k})=\delta_{\lambda\lambda'}$，

也可以用圆偏振基 $\vec{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\vec{e}_1 \pm i\,\vec{e}_2)$ 张开.　　$\vec{\epsilon}_+\cdot\vec{\epsilon}_+ = \vec{\epsilon}_-\cdot\vec{\epsilon}_- = 0$　　$\vec{\epsilon}_+\cdot\vec{\epsilon}_- = 1$

角动量张量 $M^{ij} = \int (x^i S^j - x^j S^i)\, d^3x$　　$L_i = \frac{1}{2}\epsilon_{ijk}M^{jk} = \int \epsilon_{ijk}\, x^j S^k\, d^3x$.

$$\Rightarrow \vec{L} = \frac{1}{4\pi}\int \vec{r}\times(\vec{E}\times\vec{B})\, d^3x = \frac{1}{4\pi}\int \Big(\underbrace{\vec{E}\times\vec{A}}_{\text{“自旋”}} - \underbrace{A_i(\vec{r}\times\nabla)E_i}_{\text{“轨道角动量”}}\Big)\, d^3x$$

自旋不依赖原点选取

$$\vec{L}_{spin} = \frac{1}{4\pi}\int \vec{E}\times\vec{A}\, d^3x$$

关心平均自旋　$\langle \vec{L}_{spin}\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{L}_{spin}\, dt$.　　$\langle e^{i\nu t}\rangle = \delta_{\nu 0}$.

$$\Rightarrow \langle \vec{L}_{spin}\rangle = \frac{1}{2\pi}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\vec{k}\left(|a_+(\vec{k})|^2 - |a_-(\vec{k})|^2\right).$$

$$\langle \mathcal{E}\rangle = \frac{1}{2\pi}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\omega^2\left(|a_+(\vec{k})|^2 + |a_-(\vec{k})|^2\right).$$

即　$\langle \vec{L}_{spin}\rangle_{k,\lambda} = \frac{1}{2\pi}\vec{k}\,|a_\lambda(\vec{k})|^2(\mathrm{sgn}\,\lambda)$　　$\langle \mathcal{E}\rangle_{\vec{k},\lambda} = \frac{1}{2\pi}\omega^2|a_\lambda(\vec{k})|^2$.

自旋沿 $\vec{k}$ 方向分量　$\sigma = \frac{1}{\omega}\langle \mathcal{E}\rangle_{\vec{k},\lambda}\,\mathrm{sgn}(\lambda)$.　　$\mathcal{E} = |\sigma|\omega$.　　量子 → 光子自旋

辐射规范与 Lorentz 规范不冲突. Lorentz 规范使 A^μ 有3个独立分量, 只要 λ 满足 $\Box\lambda=0$, A^μ 仍满足 Lorentz 规范.

设 $A^\mu = a^\mu e^{ik\cdot x}$ $(\Box A^\mu=0)$ $\partial_\mu A^\mu = 0 \Rightarrow k_\mu a^\mu = 0$ 选择 $\lambda = i h e^{ik\cdot x}$ $(\Box\lambda=0)$.

$\Rightarrow a^\mu \to a^\mu - h k^\mu$. 取 $h = -\frac{a_0}{\omega}$. 即可让 $a_0 = 0$ (辐射规范) 辐射 + Lorentz 规范完全确定 A^μ.

因此沿 $\vec{k}$ 方向传播的电磁波有2个独立分量.

波导

截面在 xOy 平面, 轴为 z 轴. 定义 $\vec{m} = (0,0,1)$ $\vec{\nabla}_\perp = (\partial_x, \partial_y, 0)$

$\vec{E} = \vec{E}_\perp + \vec{m}E_z$ $\vec{B} = \vec{m}B_z$ $\Rightarrow$ $\nabla_\perp\cdot\vec{E}_\perp = -ikE_z$ $\nabla_\perp\cdot\vec{B}_\perp = -ikB_z$.

$ik\vec{E}_\perp + i\omega\vec{m}\times\vec{B}_\perp = \nabla_\perp E_z$.

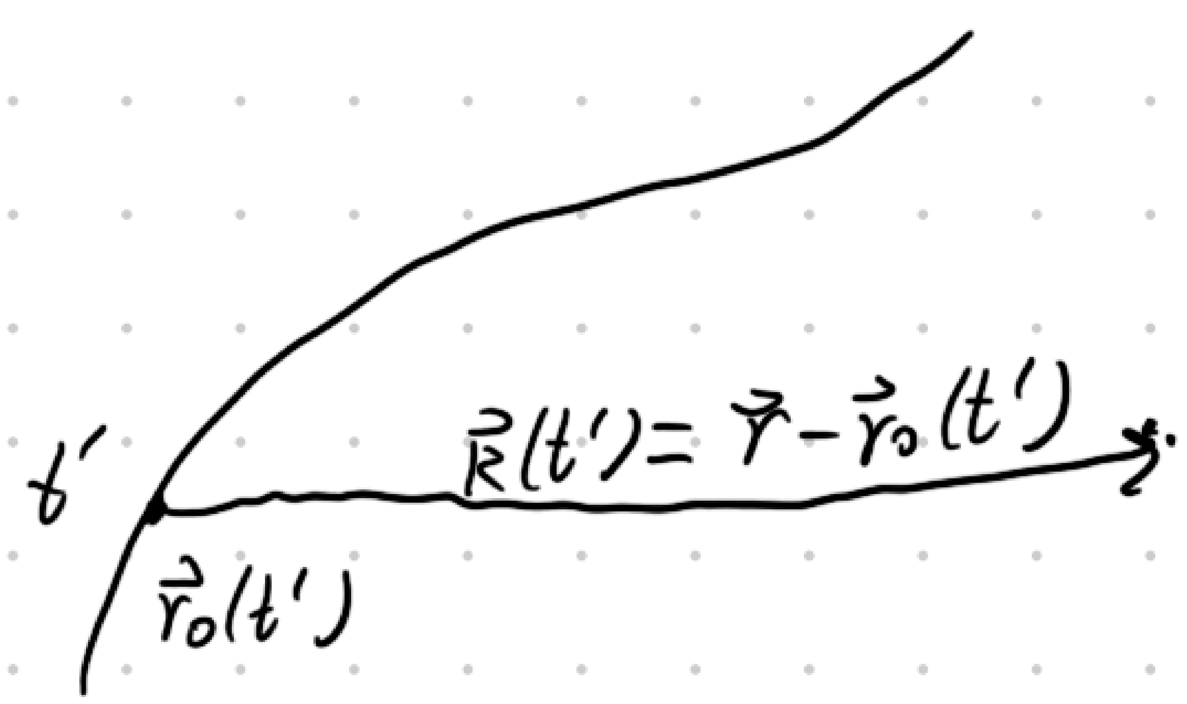
推迟势 $\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = -4\pi\rho$ Green 函数 $\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = -4\pi e(t)\,\delta^3(\vec{R}) \Rightarrow \phi = \frac{e(t-R)}{R}$

一般解
$$\boxed{\begin{aligned}\phi(\vec{r},t) &= \int \frac{\rho(\vec{r}\,',t-R)}{R}\,d^3r' \\ \vec{A}(\vec{r},t) &= \int \frac{\vec{J}(\vec{r}\,',t-R)}{R}\,d^3r'\end{aligned}}$$
其中 $R = |\vec{r}-\vec{r}\,'|$

Lienard-Wiechert 势

运动电荷(可以有加速)的场   推迟效应：t时刻$\vec{r}$处的场由之前t'时刻决定，

$t - t' = |\vec{r} - \vec{r}_0(t')|$.   粒子速度 $v<1$ 时只有一个解 t'.



在t'粒子的瞬时静止系下，场点$\vec{r}$势 $\phi = \frac{e}{R(t')}$ $\vec{A} = 0$

设在另一惯性系下 粒子 4速度为 U^μ 定义 $R^\mu = (t-t', \vec{r}-\vec{r}_0(t')) = (t-t', \vec{R}(t'))$

在t'粒子瞬时静止系，$U^\mu = (1, \vec{0})$ $A^\mu = \frac{e\,U^\mu}{(-U^\nu R_\nu)}$ 此形式在其他惯性系也对.

一般 $U^\mu = \gamma(1, \vec{v})$，$\phi(\vec{r},t) = \frac{e\gamma}{\gamma(t-t') - \gamma\vec{v}\cdot\vec{R}} = \frac{e}{R - \vec{v}\cdot\vec{R}}$

$\vec{A}(\vec{r},t) = \frac{e\gamma\vec{v}}{\gamma(t-t') - \gamma\vec{v}\cdot\vec{R}} = \frac{e\vec{v}}{R - \vec{v}\cdot\vec{R}}$

注意 右边量均为t'时刻的，即 $\vec{R} = \vec{R}(t')$ $\vec{v} = \frac{d\vec{r}_0(t')}{dt'}$ 且 $\vec{R}(t') = \vec{r} - \vec{r}_0(t')$ $|\vec{R}(t')| = t - t'$.

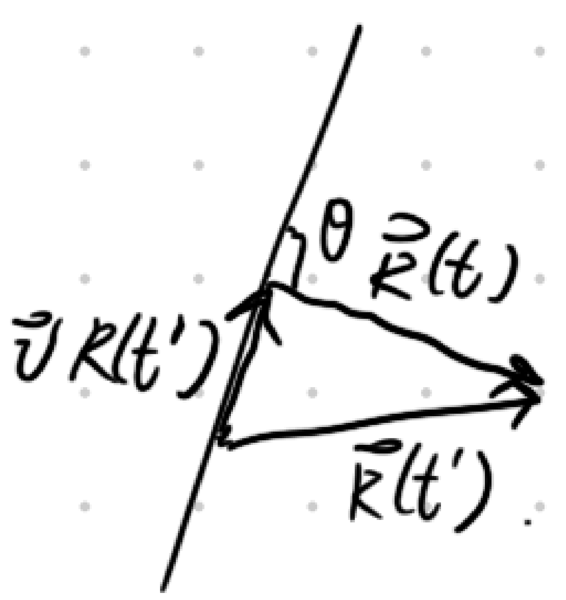
$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$，$\vec{B} = \nabla\times\vec{A}$.   利用 $\frac{\partial R(\vec{r},t)}{\partial t} = \frac{\partial R(\vec{r},t')}{\partial t'}\frac{\partial t'(\vec{r},t)}{\partial t}$，$R(\vec{r},t') = |\vec{r}-\vec{r}_0(t')|$ $\frac{\partial R(\vec{r},t')}{\partial t'} = -\frac{\vec{v}\cdot\vec{R}}{R}$

由 $t-t'=R(\vec{r},t')$ $\Rightarrow$ $1-\frac{\partial t'}{\partial t}=\frac{\partial R(\vec{r},t')}{\partial t}=-\frac{\vec{v}\cdot\vec{R}}{R}\frac{\partial t'}{\partial t}$ $\Rightarrow$ $\frac{\partial t'}{\partial t}=\left(1-\frac{\vec{v}\cdot\vec{R}}{R}\right)^{-1}$

$\Rightarrow \frac{\partial R}{\partial t}=-\frac{\vec{v}\cdot\vec{R}}{R-\vec{v}\cdot\vec{R}}$ $\quad \frac{\partial t'}{\partial x_i}=-\frac{R_i}{R-\vec{v}\cdot\vec{R}}$ $\quad \frac{\partial R}{\partial x_i}=\frac{R_i}{R-\vec{v}\cdot\vec{R}}$

$$\Rightarrow \vec{E}=\underbrace{\frac{e(1-v^2)(\vec{R}-\vec{v}R)}{(R-\vec{v}\cdot\vec{R})^3}}_{一阶运动贡献\sim\frac{1}{R^2}}+\underbrace{\frac{e\vec{R}\times[(\vec{R}-\vec{v}R)\times\vec{a}]}{(R-\vec{v}\cdot\vec{R})^3}}_{二阶运动贡献\sim\frac{1}{R}\Rightarrow 辐射}\qquad \vec{B}=\frac{\vec{R}\times\vec{E}}{R}$$

方程右边均为 t' 时的量.



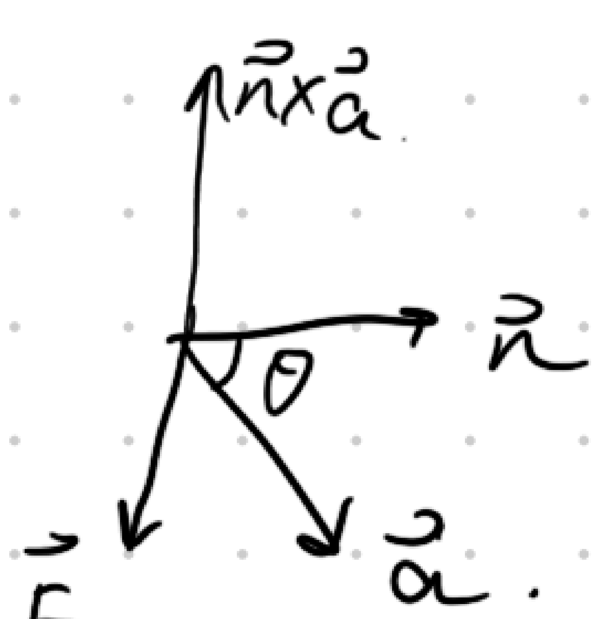
粒子匀速时 $\vec{R}(t')-\vec{v}R(t')=\vec{R}(t)=\vec{r}-\vec{r}_0(t)$ $\quad R(t')-\vec{v}\cdot\vec{R}(t')=R(t')(1-v^2)-\vec{v}\cdot\vec{R}(t)$.

设 $\langle\vec{R}(t),\vec{v}\rangle=\theta$. $\quad R^2(t')=v^2R^2(t')+2vR(t')R(t)\cos\theta+R^2(t)$ $\Rightarrow$ 解出 $R(t')$

$\Rightarrow R(t')-\vec{v}\cdot\vec{R}(t')=R(t)\sqrt{1-v^2\sin^2\theta}$.

$\Rightarrow \vec{E}(t)=\frac{e\vec{R}(t)}{R^3(t)}\frac{1-v^2}{(1-v^2\sin^2\theta)^{3/2}}$ 与之前结果相同.

$v\ll 1$ 时 二阶项 $\vec{E}=\frac{e\,\vec{n}\times(\vec{n}\times\vec{a})}{R}$ $\quad(\vec{n}=\vec{R}/R)$ $\quad\vec{E}$ 在 $\vec{n}$ 与 $\vec{a}$ 张成的平面内，且垂直于 $\vec{n}$.



$\vec{B}=\vec{n}\times\vec{E}$ $\quad \vec{S}=\frac{1}{4\pi}\vec{E}\times\vec{B}=\frac{1}{4\pi}\left(E^2\vec{n}-(\vec{n}\cdot\vec{E})\vec{E}\right)=\frac{1}{4\pi}E^2\vec{n}$.

记 $\theta=\langle\vec{n},\vec{a}\rangle$. $\quad \vec{E}=\frac{e}{R}(a\vec{n}\cos\theta-\vec{a})$ $\quad\Rightarrow \vec{S}=\frac{e^2a^2\sin^2\theta}{4\pi R^2}\vec{n}$.

$$\frac{dP}{d\Omega} = \frac{e^2a^2}{4\pi}\sin^2\theta \qquad P = \frac{2}{3}e^2a^2$$ (非相对论加速电荷的拉莫尔公式)

注意 P是洛伦兹标量(能量/时间) $P = \frac{2e^2}{3m^2}\left(\frac{d\vec{p}}{dt}\right)^2 = \frac{2e^2}{3m^2}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$ (这里 $\gamma = 1$)

即P的标量形式，对任何参考系适用，现在 $v \neq 0$, $\frac{dp^\mu}{d\tau} = m\gamma\left(\gamma^3\vec{v}\cdot\vec{a},\ \gamma^3(\vec{v}\cdot\vec{a})\vec{v} + \gamma\vec{a}\right)$

$$\Rightarrow P = \frac{2}{3}e^2\gamma^6\left[a^2 - (\vec{v}\times\vec{a})^2\right]$$ (相对论形式拉莫尔公式)

只有相对论形式 P与 $\vec{v}$ 有关

低速情形 相当于取 $v \to 0$

注意这里P是后文 $P(t')$ 即粒子在 dt' 中辐射的能量，因为它才是标量

$\vec{a} \parallel \vec{v}$ 时 利用 $\frac{dp}{dt} = m\gamma^3 a \Rightarrow P = \frac{2e^2}{3m^2}\left(\frac{dp}{dt}\right)^2 = \frac{2e^2}{3m^2}\left(\frac{d\varepsilon}{dx}\right)^2$

energy-loss factor $= \frac{P}{(d\varepsilon/dt)} = \frac{2e^2}{3m^2v}\frac{d\varepsilon}{dx} \sim \frac{2e^2}{3m^2}\frac{d\varepsilon}{dx}$

$\vec{a} \perp \vec{v}$ 时 即匀速圆周运动 $\left|\frac{d\vec{p}}{d\tau}\right| = \gamma\omega|\vec{p}|$ $\quad \frac{dp^0}{d\tau} = 0 \Rightarrow P = \frac{2}{3}e^2\gamma^4\omega^2v^2$

$$P = \frac{2e^2\gamma^4v^4}{3R^2}$$ 每周期 $\Delta\varepsilon = \frac{2\pi RP}{v} = \frac{4\pi e^2\gamma^4v^3}{3R}$

辐射角分布 $\vec{S} = \frac{1}{4\pi}E^2\vec{n} = \frac{e^2}{4\pi R^2}\left|\frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^3}\right|^2\vec{n}$

$$\frac{dP(t)}{d\Omega} = [\vec{n}\cdot\vec{S}R^2]_{ret}.$$ → 标记在 t' 时刻的量， $t-t'=R(t')$ 为场点在 dt, $d\Omega$ 内接收能量.

$$\frac{d\varepsilon}{d\Omega} = \int_{T_1}^{T_2}[R^2\vec{n}\cdot\vec{S}]_{ret}\,dt = \int_{T_1'}^{T_2'}[R^2\vec{n}\cdot\vec{S}]_{ret}\frac{dt}{dt'}\,dt'$$

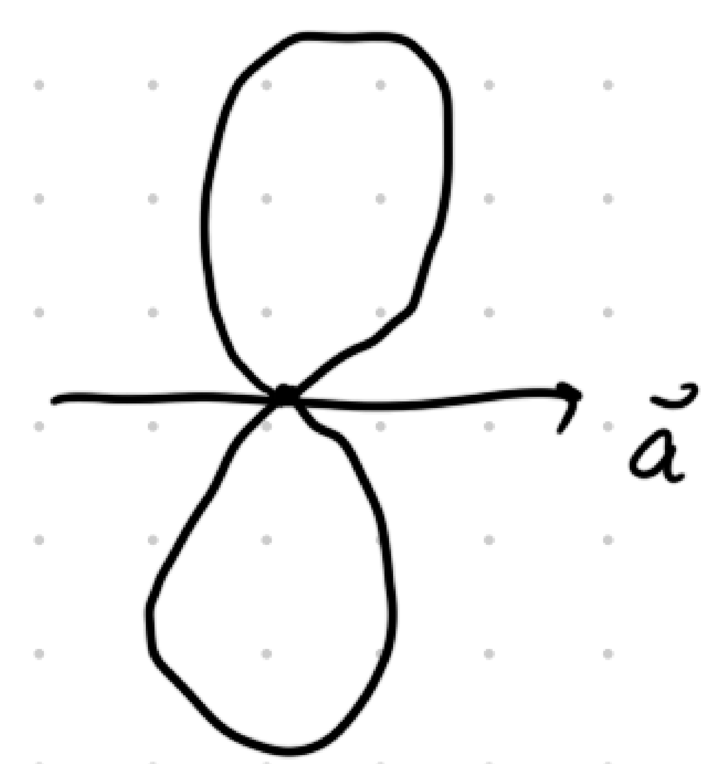
$$\frac{dP(t')}{d\Omega} = [R^2\vec{n}\cdot\vec{S}]_{ret}\frac{dt}{dt'} = (1-\vec{n}\cdot\vec{v})[R^2\vec{n}\cdot\vec{S}]_{ret}$$ 为粒子在 dt', $d\Omega$ 内辐射功率.

$$= \frac{e^2}{4\pi}\frac{|\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]|^2}{(1-\vec{n}\cdot\vec{v})^5}$$

1. $\vec{a}\parallel\vec{v}$. $\theta=\langle\vec{a},\vec{n}\rangle$.

$$\frac{dP(t')}{d\Omega} = \frac{e^2a^2}{4\pi}\frac{\sin^2\theta}{(1-v\cos\theta)^5}.$$

$v\to 0$



高能极限 $v=\sqrt{1-\gamma^{-2}}$ $\theta\ll 1$

$$\frac{dP(t')}{d\Omega} \simeq \frac{8e^2a^2\theta^2}{\pi(\gamma^{-2}+\theta^2)^5} \simeq \frac{8e^2a^2\gamma^8}{\pi}\frac{(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5}$$

$v\to 1$

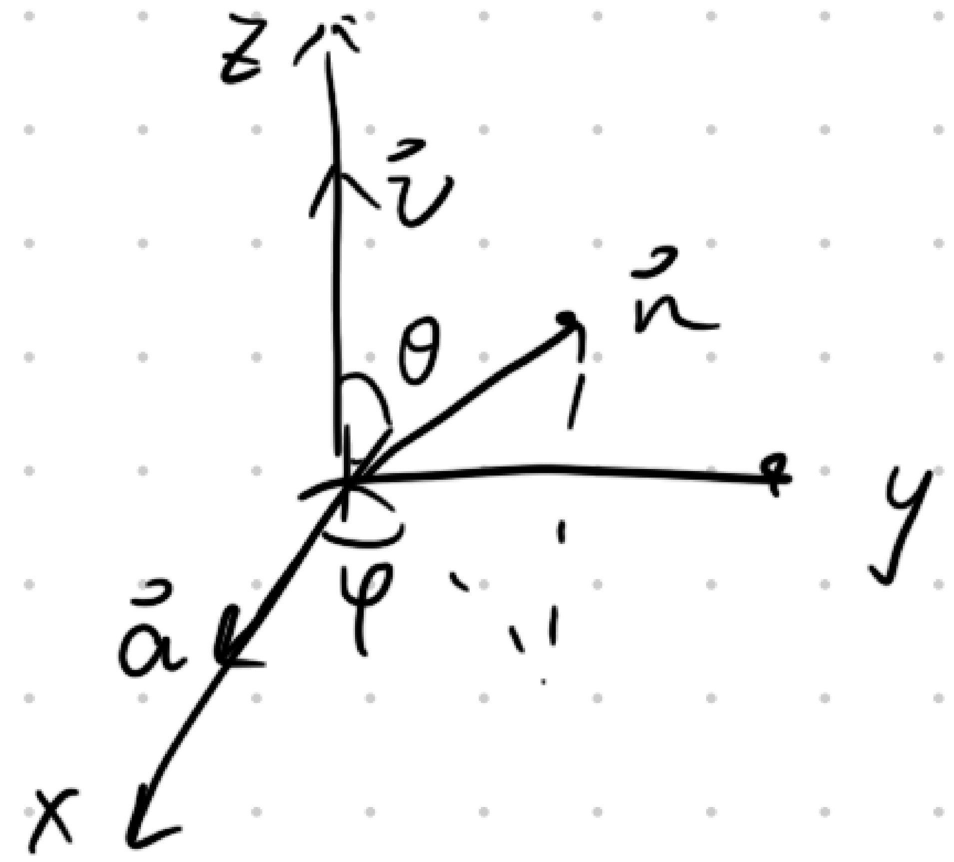
$\theta_{max}\sim\frac{1}{2\gamma}$ $(\gamma\gg 1)$

$\Delta\theta\sim\frac{1}{\gamma}$.

$$P(t') = \int\frac{dP(t')}{d\Omega}d\Omega = \frac{2}{3}e^2\gamma^6a^2.$$ 与之前结果相同

2. $\vec{a}\perp\vec{v}$.

$$\frac{dP(t')}{d\Omega} = \frac{e^2a^2}{4\pi(1-v\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-v\cos\theta)^2}\right]$$



$$P(t') = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \, \frac{dP(t')}{d\Omega} = \frac{2}{3} e^2 \gamma^4 a^2$$ 与之前结果相同.

a 一定时 $\vec a \parallel \vec v \ \propto \gamma^6$ $\quad \vec a \perp \vec v \ \propto \gamma^4$.

$\left|\frac{d\vec p}{dt'}\right|$ 一定时 $\vec a \parallel \vec v \ \propto \gamma^2$ $\quad \vec a \perp \vec v \ \propto \gamma^0$

频谱分析 这里关心观察者得到的辐射频谱, 使用 $\frac{dP(t)}{d\Omega}$. $\quad \vec a(t) \triangleq \frac{1}{\sqrt{4\pi}}[R\vec E]_{ret}$.

$$\frac{dP(t)}{d\Omega} = |\vec a(t)|^2 \qquad \frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec a(t)|^2 dt. \qquad \vec g(\omega) = \mathcal{F}\{\vec a(t)\}$$ (假设粒子只在有限时间段加速, 从而 $dW/d\Omega$ 收敛)

$$\Rightarrow \frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec g(\omega)|^2 d\omega = \int_0^{\infty} d\omega \, \frac{d^2 I(\omega, \vec n)}{d\omega \, d\Omega}$$

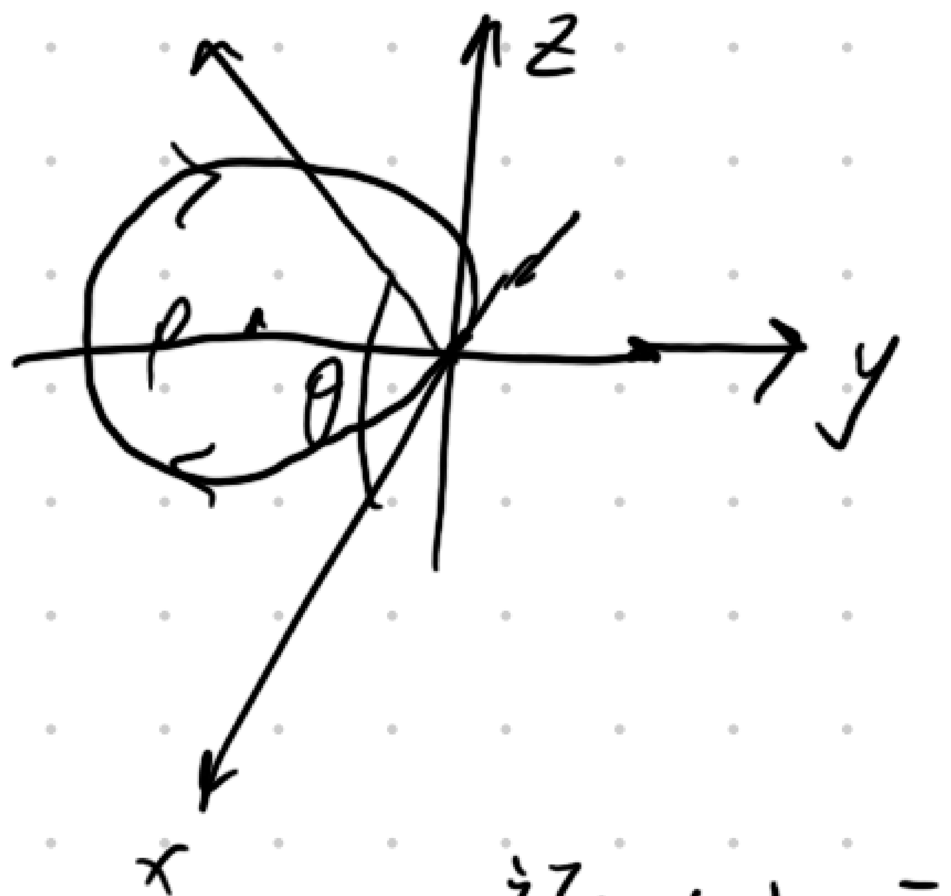
考虑 $\vec a(t)$ 是实的 $\Rightarrow \vec g(-\omega) = \vec g^*(\omega) \qquad \frac{d^2 I(\omega,\vec n)}{d\omega\, d\Omega} = 2|\vec g(\omega)|^2$.

$$\vec g(\omega) = \frac{e}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{\vec n \times [(\vec n - \vec v)\times \vec a]}{(1-\vec n\cdot\vec v)^3}\right]_{ret} dt = \frac{e}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega(t' + R(t'))} \frac{\vec n \times [(\vec n - \vec v)\times \vec a]}{(1-\vec n\cdot\vec v)^2} dt'$$

远场近似: $R(t') \approx r - \vec n \cdot \vec r_0(t')$ $\qquad$ 利用 $\frac{\vec n \times [(\vec n - \vec v)\times \vec a]}{(1-\vec n\cdot\vec v)^2} = \frac{d}{dt'}\left[\frac{\vec n \times (\vec n \times \vec v)}{1 - \vec n\cdot\vec v}\right]$

$$\Rightarrow \frac{d^2 I(\omega, \vec n)}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi}\left|\int_{-\infty}^{\infty} \vec n \times (\vec n \times \vec v)\, e^{i\omega(t' - \vec n\cdot \vec r_0(t'))} dt'\right|^2$$

$\vec a \perp \vec v$ 情形



略去 t' 上标， $\vec r_0 = (\rho\sin\frac{vt}{\rho}, \rho\cos\frac{vt}{\rho} - \rho, 0)$ 粒子在 xOy 平面运动，

$\vec n = (\cos\theta, 0, \sin\theta)$ $v \sim 1$ $(\gamma \gg 1)$ 极端相对论情形 $\Rightarrow \theta \ll 1$ 并设 $t \ll \frac{\rho}{v}$

即关心粒子在原点附近

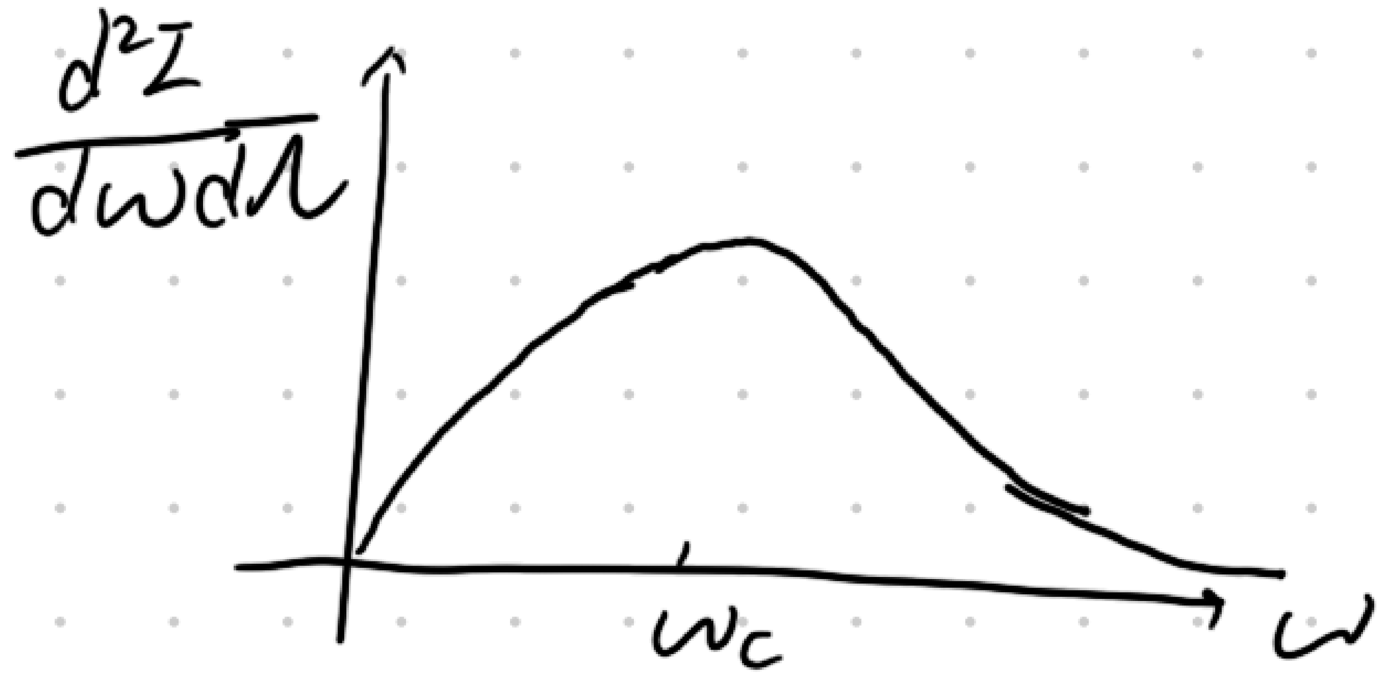
$$\frac{d^2 I}{d\omega d\Omega} \approx \frac{e^2\omega^2\rho^2}{3\pi^2}(\gamma^{-2}+\theta^2)^2\left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\xi)\right]$$

$\xi = \frac{1}{3}\omega\rho(\gamma^{-2}+\theta^2)^{3/2}$

$K_\nu(\xi)$ 为变形 Bessel 函数 $K_\nu(x) \to \begin{cases} \frac{1}{2}\Gamma(\nu)\left(\frac{2}{x}\right)^\nu & x\to 0 \\ \sqrt{\frac{\pi}{2x}}\, e^{-x} & x \to \infty \end{cases}$

记 $\omega_c = \frac{3\gamma^3}{\rho} = 3\omega_0\left(\frac{\varepsilon}{m}\right)^3$ $\left(\xi(\omega_c, \theta=0)=1, \ \omega_0 = v/\rho \approx 1/\rho\right)$

对 $\theta = 0$， $\omega \ll \omega_c$ 时 $\frac{d^2 I}{d\omega d\Omega} \propto (\omega\rho)^{2/3}$ $\omega \gg \omega_c$ 时 $\frac{d^2 I}{d\omega d\Omega} \propto \frac{\omega}{\omega_c} e^{-2\omega/\omega_c}$



峰值在 $\omega = \omega_c$ 附近.

周期运动的频谱 $e^{-i\omega\vec n\cdot\vec r_0(t)}$ 依赖时间因子 $H(t) = \sum_{n=-\infty}^{\infty} b_n e^{-in\omega_0 t} \to h(\omega) = \sqrt{2\pi}\sum_n b_n \delta(\omega - n\omega_0)$

$\vec a(t) = \sum_n \vec a_n e^{-in\omega_0 t}$. $\vec a_n = \frac{1}{T}\int_0^T e^{in\omega_0 t}\vec a(t)dt$ $\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{T}\int_0^T |\vec a(t)|^2 dt = \sum_{n=-\infty}^{\infty}|\vec a_n|^2 = \sum_{n\geq 1}\frac{dP_n}{d\Omega}$

$\frac{dP_n}{d\Omega} = |\vec a_n|^2 + |\vec a_{-n}|^2 = 2|\vec a_n|^2$ ($n=0$ 对应静场) 远场近似下：

$$\frac{dP_n}{d\Omega} = \frac{e^2 n^2\omega_0^4}{4\pi^2}\left|\int_0^T \vec n\times(\vec n\times\vec v)e^{in\omega_0(t-\vec n\cdot\vec r_0(t))}dt\right|^2 \simeq \frac{e^2 n^2\omega_0^4}{4\pi^2}\left|\int_0^T \vec v\times\vec n\, e^{in\omega_0(t-\vec n\cdot\vec r_0(t))}dt\right|^2$$

Cerenkov 辐射　介质中 $(\epsilon>1,\ \mu\approx1)$　恢复 c：$t\to\frac{ct}{\sqrt{\epsilon}}$　$\omega\to\frac{\omega}{c}\sqrt{\epsilon}$　$\rho\to\frac{\rho}{\sqrt{\epsilon}}$　$\vec{E}\to\sqrt{\epsilon}\vec{E}$　$\vec{B}\to\frac{\vec{B}}{\sqrt{\epsilon}}$　$c\to\frac{c}{\sqrt{\epsilon}}$

$\vec{v}\to\frac{\vec{v}}{c}\sqrt{\epsilon}$

$$\frac{d^2I(\omega,\vec{n})}{d\omega\, d\Omega}=\frac{e^2\omega^2\sqrt{\epsilon}}{4\pi^2c^3}\left|\int_{-\infty}^{\infty}\vec{n}\times(\vec{n}\times\vec{v})\,e^{i\omega(t'-\sqrt{\epsilon}\,\vec{n}\cdot\vec{r}_0(t')/c)}\,dt'\right|^2$$

对匀速运动电荷 $\vec{r}_0(t')=\vec{v}t'$

$$\frac{d^2I(\omega,\vec{n})}{d\omega\, d\Omega}=\frac{e^2\sqrt{\epsilon}}{c^3}v^2\sin^2\theta\,|\delta(1-\sqrt{\epsilon}(v/c)\cos\theta)|^2$$

$\cos\theta=\langle\vec{n},\vec{v}\rangle$

$\Rightarrow$ 所有辐射集中在角度 $\cos\theta_c=\frac{c}{v\sqrt{\epsilon}}$ (Cerenkov Angle)

更现实：粒子在 $t'=-T$ 进入介质薄板，$t'=T$ 离开.

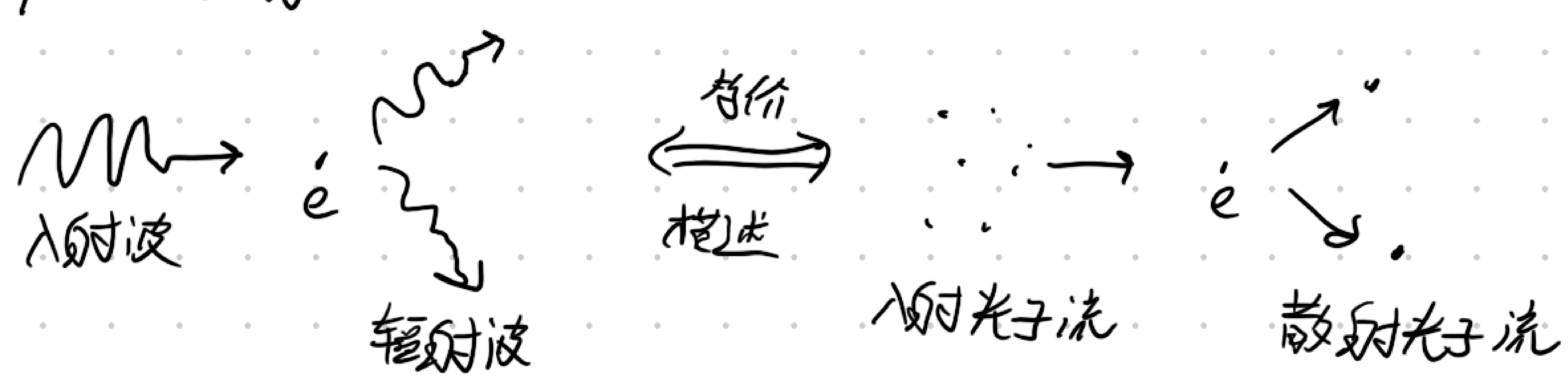
$$\Rightarrow\frac{d^2I(\omega,\vec{n})}{d\omega\, d\Omega}=\frac{e^2\omega^2\sqrt{\epsilon}\,v^2T^2\sin^2\theta}{\pi^2c^3}\left(\frac{\sin[\omega T(1-\sqrt{\epsilon}(v/c)\cos\theta)]}{\omega T(1-\sqrt{\epsilon}(v/c)\cos\theta)}\right)^2$$

$$\frac{dI}{d\omega}=\int\frac{d^2I}{d\omega\, d\Omega}d\Omega\approx\frac{2e^2v\omega T\sin^2\theta_c}{c^2}$$

(Franck-Tamm 关系)

单位长度辐射 $\frac{d^2I}{d\omega\, dl}=\frac{e^2\omega}{c^2}\sin^2\theta_c=\frac{e^2\omega}{c^2}\left(1-\frac{c^2}{v^2\epsilon}\right)$

Thompson 散射



非相对论情形 $\frac{dP}{d\Omega}=\frac{e^2a^2}{4\pi}\sin^2\Theta$　$\Theta=\langle\vec{a},\vec{n}\rangle$

设入射：$\vec{E}=E_0\vec{\epsilon}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$　$\vec{k}\parallel\hat{z}$.　$\vec{\epsilon}=(\cos\psi,\sin\psi,0)$

$\vec{n}$ 用球坐标 (θ,φ) 描述.

$m\vec{a}=e\vec{E}$　$\Rightarrow\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{e^4}{8\pi m^2}|E_0|^2[1-\sin^2\theta\cos^2(\varphi-\psi)]$.

若入射光为自然光，无固定偏振方向，$\langle\langle\frac{dP}{d\Omega}\rangle\rangle_t = \frac{e^4}{16\pi m^2}|E_0|^2(1+\cos^2\theta)$

散射截面 $\frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{\text{能流密度} \to \frac{|E_0|^2}{8\pi}} = \frac{e^4}{2m^2}(1+\cos^2\theta)$ 　总截面 $\sigma = \frac{8\pi e^4}{3m^2}$.

辐射系统

局域振荡源产生的场． 将源换到频域，考虑单频：$\rho(\vec r,t) = \rho(\vec r)e^{-i\omega t}$ 　$\vec J(\vec r,t) = \vec J(\vec r)e^{-i\omega t}$

推迟势为 $\phi(\vec r,t) = e^{-i\omega t}\underbrace{\int \frac{\rho(\vec r')}{|\vec r-\vec r'|} e^{ik|\vec r-\vec r'|} d^3r'}_{\phi(\vec r)}$ 　$\vec A(\vec r,t) = e^{-i\omega t}\underbrace{\int \frac{\vec J(\vec r')}{|\vec r-\vec r'|} e^{ik|\vec r-\vec r'|} d^3r'}_{\vec A(\vec r)}$

设振荡源是局域的，考虑远离源点的区域，$\vec E \sim \vec E(\vec r)e^{-i\omega t}$ 　$\vec B \sim \vec B(\vec r)e^{-i\omega t}$.

$\nabla\times\vec B = \frac{\partial \vec E}{\partial t} + 4\pi\vec J \Rightarrow \vec E = \frac{i}{k}\nabla\times\vec B$. 　$\vec B = \nabla\times\vec A$.

$|\vec r| \gg |\vec r'|$: $\frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|} = \underbrace{\frac{1}{r}e^{ikr}}_{\text{电偶极子项}} + \left(\frac{1}{r^2} - \frac{ik}{r}\right)(\vec n\cdot\vec r')e^{ikr} + \cdots$

第一阶：$\vec A(\vec r) = \frac{1}{r}e^{ikr}\int \vec J(\vec r')d^3r'$ 　$0 = \int \partial_i'(x_j' J_i(\vec r'))d^3r' = \int J_j(\vec r')d^3r' + \int x_j' \underbrace{\nabla'\cdot\vec J(\vec r')}_{=ik\rho(\vec r')} d^3r'$

电偶极子项：$= -\frac{ik}{r}e^{ikr}\vec p$. 　(不出现单极项：电荷守恒，不会振荡)

$\vec B = \nabla\times\vec A(\vec r) = k^2(\vec n\times\vec p)\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)$

$$\vec{E} = \frac{i}{k}\nabla\times\vec{B} = -k^2\vec{n}\times(\vec{n}\times\vec{p})\frac{e^{ikr}}{r} + [3(\vec{n}\cdot\vec{p})\vec{n}-\vec{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}$$

近场(静态区): $kr \ll 1$ 　$r \ll \lambda$　　中间场(感应区): $kr \sim 1$ 　$r \sim \lambda$　　远场(辐射区): $kr \gg 1$ 　$r \gg \lambda$

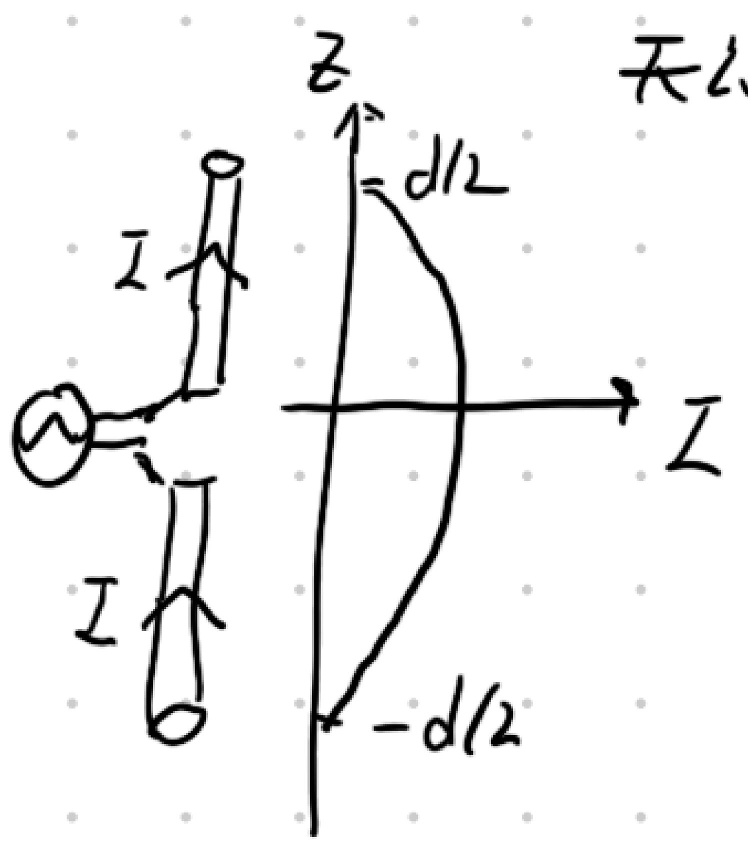
静态区 $\vec{B}\sim 0$　$\vec{E}\sim\vec{E}_{static}(\vec{r})$ 即 静态电偶极子场

辐射区: $\vec{B}_{rad} = k^2(\vec{n}\times\vec{p})\frac{e^{ikr}}{r}$　　$\vec{E}_{rad} = -k^2\vec{n}\times(\vec{n}\times\vec{p})\frac{e^{ikr}}{r}$　　$\vec{B}_{rad}, \vec{E}_{rad}, \vec{n}$ 两两垂直, $|\vec{B}_{rad}| = |\vec{E}_{rad}|$

(在辐射区 ∇ 只作用于 e^{ikr}, 可作代换 $\nabla\to ik\vec{n}$ 快速计算)

$\langle\vec{S}\rangle = \frac{1}{8\pi}\vec{E}\times\vec{B}^*$　　$\frac{dP}{d\Omega} = \langle\vec{S}\rangle\cdot\vec{n}r^2 = \frac{1}{8\pi}|\vec{B}|^2 r^2 d\Omega = \frac{k^4}{8\pi}|\vec{n}\times\vec{p}|^2 = \frac{k^4}{8\pi}p^2\sin^2\theta$　　$\theta = \langle\vec{p},\vec{n}\rangle$

$P = \frac{1}{3}k^4|\vec{p}|^2$



天线辐射　$kd \ll 1$ ($d \ll \lambda$)　设 $I(z,t) = I(z)e^{-i\omega t} = I_0\left(1-\frac{2|z|}{d}\right)e^{-i\omega t}$

$$\frac{\partial I(z,t)}{\partial z} + \frac{\partial\lambda(z,t)}{\partial t} = 0 \Rightarrow \lambda(z) = -\frac{i}{\omega}\frac{\partial I(z)}{\partial z} = \pm\frac{2iI_0}{\omega d}$$ ($z>0$ 取正, $z<0$ 取负)

$p = \int_{d/2}^{-d/2} z\lambda(z)dz = \frac{iI_0 d}{2\omega}$　　$\frac{dP}{d\Omega} = \frac{I_0^2(kd)^2}{32\pi}\sin^2\theta$　　$P = \frac{1}{12}I_0^2(kd)^2$

第二阶: $\vec{A}(\vec{r}) = e^{ikr}\left(\frac{1}{r^2}-\frac{ik}{r}\right)\int(\vec{n}\cdot\vec{r}')\vec{J}(\vec{r}')d^3r'$

$$\begin{aligned} n_j x_j' J_i &= \frac{1}{2}(J_i x_j' - J_j x_i')n_j + \frac{1}{2}(J_i x_j' + J_j x_i')n_j \\ &= \frac{1}{2}\epsilon_{ijk}\epsilon_{lmk}J_l x_m' n_j + \frac{1}{2}(J_i x_j' + J_j x_i')n_j \\ &= -\epsilon_{ijk}n_j M_k + \frac{1}{2}(J_i x_j' + J_j x_i')n_j \end{aligned}$$

$$0=\int \partial_k'(x_i'x_j'n_jJ_k)\,d^3r' = \int (x_i'J_j + x_j'J_i)n_j\,d^3r' + i\omega\int x_i'x_j' n_j\rho\,d^3r'$$

$$\vec{m} = \int \vec{M} d^3r' = \frac{1}{2}\int \vec{r}'\times\vec{J}(\vec{r}')d^3r'$$

$$\Rightarrow \vec{A}(\vec{r}) = e^{ikr}\left(\frac{ik}{r}-\frac{1}{r^2}\right)\left[\vec{n}\times\vec{m} + \frac{ik}{2}\int \vec{r}'(\vec{n}\cdot\vec{r}')\rho(\vec{r}')d^3r'\right.$$

磁偶极子项 $\vec{A}(\vec{r}) = e^{ikr}\left(\frac{ik}{r}-\frac{1}{r^2}\right)\vec{n}\times\vec{m}$ 将电偶极子项中 $\vec{p}\to\vec{m}$, $\vec{E}\to\vec{B}$ $\vec{B} = -\frac{i}{k}\nabla\times\vec{E}$

即 $\vec{p}\to\vec{m}$ $\vec{E}\to\vec{B}$ $\vec{B}\to-\vec{E}$

电四极子项 $\vec{A}(\vec{r}) = -\frac{1}{2}k^2\frac{e^{ikr}}{r}\int \vec{r}'(\vec{n}\cdot\vec{r}')\rho(\vec{r}')d^3r'$ 辐射区 $\nabla\to ik\vec{n}$

$\vec{B} = ik\vec{n}\times\vec{A}$ $\vec{E} = \frac{i}{k}\nabla\times\vec{B} = -\vec{n}\times\vec{B} = -ik\vec{n}\times(\vec{n}\times\vec{A})$ 定义 $Q(\vec{n})_i = Q_{ij}n_j$

$$\left[\frac{1}{3}\vec{n}\times\vec{Q}(\vec{n})\right]_i = \frac{1}{3}\epsilon_{ijk}n_jQ_{kl}n_l = \frac{1}{3}\epsilon_{ijk}n_jn_l\int(3x_kx_l - r^2\delta_{kl})\rho(\vec{r})d^3r = \int(\vec{n}\times\vec{r})_i(\vec{n}\cdot\vec{r})\rho(\vec{r})d^3r$$

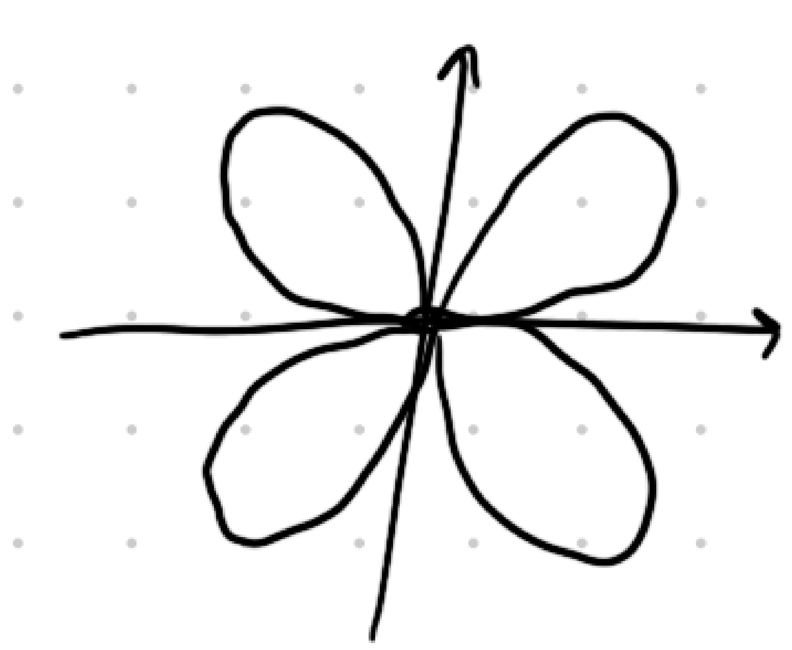
$$\Rightarrow \vec{B} = -\frac{ik^3}{6r}e^{ikr}\,\vec{n}\times\vec{Q}(\vec{n}) \qquad \frac{dP}{d\Omega} = \frac{1}{8\pi}(\vec{E}\times\vec{B}^*)\cdot\vec{n}r^2 = \frac{k^6}{288\pi}|\vec{n}\times\vec{Q}(\vec{n})|^2 = \frac{k^6}{288\pi}(Q_{ki}Q_{kj}^*n_in_j - Q_{ij}Q_{kl}^*n_in_jn_kn_l)$$

$$P = \frac{k^6}{288\pi}\left[Q_{ki}Q_{kj}^*\int n_in_j\,d\Omega - Q_{ij}Q_{kl}^*\int n_in_jn_kn_l\,d\Omega\right] = \frac{k^6}{360}Q_{ij}Q_{ij}^*$$

总可将 Q_{ij} 对角化：$Q_{ij} = \begin{pmatrix} Q_1 & & \\ & Q_2 & \\ & & Q_3\end{pmatrix}$ 且 $Q_1+Q_2+Q_3=0$

$$\frac{dP}{d\Omega} = \frac{k^6}{288\pi}\left(Q_1^2n_1^2 + Q_2^2n_2^2 + Q_3^2n_3^2 - (Q_1n_1^2 + Q_2n_2^2 + Q_3n_3^2)^2\right)$$

特别地，当 $Q_1 = Q_2 = -\frac{1}{2}Q_3$ 时 有 $\frac{dP}{d\Omega} = \frac{k^6Q_3^2}{512\pi}\sin^2 2\theta$



线性天线　设 $I(\vec{r},t) = I_0 \sin k(d/2-|z|)\, e^{-i\omega t}\, \vec{z}$.　$|z| \le \frac{1}{2}d$　但不假设 $d \ll \lambda$.

$$\vec{A}(\vec{r}) = \int \frac{\vec{J}(\vec{r}', t-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} d^3r' \approx \vec{z}\, \frac{I_0 e^{ikr}}{r} \int_{-d/2}^{d/2} \sin k(d/2-|z|)\, e^{-ikz\cos\theta} dz = \vec{z}\, \frac{2I_0 e^{ikr}}{r} \frac{\cos(\frac{1}{2}kd\cos\theta) - \cos(\frac{1}{2}kd)}{\sin^2\theta}$$

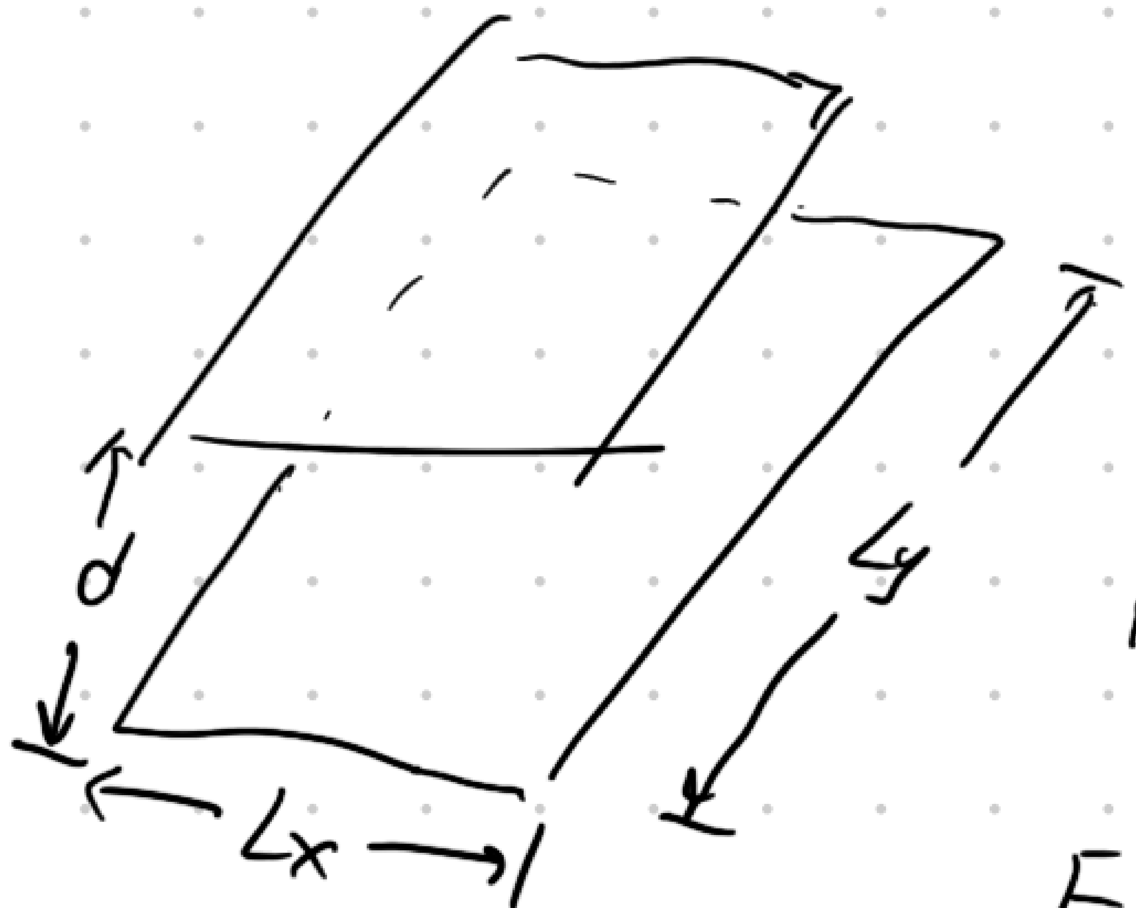
辐射区 $$\frac{dP}{d\Omega} = \frac{r^2}{8\pi}|\vec{B}|^2 = \frac{r^2}{8\pi} k^2 |\vec{A}|^2 \sin^2\theta = \frac{I_0^2}{2\pi}\left[\frac{\cos(\frac{1}{2}kd\cos\theta) - \cos(\frac{1}{2}kd)}{\sin\theta}\right]^2$$　$\theta = \langle \hat{n}, \vec{z} \rangle$.

$kd \ll 1$ ($d \ll \lambda$): $$\frac{dP}{d\Omega} \simeq \frac{I_0^2 (kd)^2 \sin^2\theta}{128\pi}$$

$kd = \pi$ ($d = \frac{\lambda}{2}$): $$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi} \frac{\cos^2(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}$$

$kd = 2\pi$ ($d = \lambda$): $$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi} \frac{\cos^4(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

Casimir force　"Vaccum" 系统处于基态，没有办法从中提取能量.



电磁真空：E-M 场处于基态（$\neq \vec{E},\vec{B}=0$，零点能）.

$$\begin{cases} \varepsilon_0(\omega,k) = \frac{1}{2}\hbar\omega \\ \omega = kc \end{cases}$$

$d\to\infty$, $E_{0,free}$ inf　　$E_0(d)$ inf.

$\Delta E = E_0(d) - E_{0,free}$ finite.

$k_x = \frac{m\pi}{L_x}$　$k_y = \frac{n\pi}{L_z}$　$k_z = \frac{l\pi}{d}$　　$\varepsilon_{mnl} = \frac{1}{2}\hbar c\sqrt{\left(\frac{m\pi}{L_x}\right)^2+\left(\frac{n\pi}{L_z}\right)^2+\left(\frac{l\pi}{d}\right)^2}$　$(m,n,l\geq 0)$

$$E_0(d) = \sum_l{}' \sum_{mn} \varepsilon_{mnl}\cdot \underset{\substack{\uparrow \\ \text{spin}}}{2}$$

"$\sum'$" 若 $l=0$，没有 2 因子.

$$\sum_{k_x,k_y} \to \frac{L_xL_y}{(2\pi)^2}\iint dk_x\, dk_y$$

$$E_0(d) = \frac{L_xL_y}{4\pi^2}\hbar c\sum_l{}'\int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y\sqrt{k_x^2+k_y^2+\left(\frac{l\pi}{d}\right)^2}$$

$$= \frac{L_xL_y}{2\pi}\hbar c\sum_l{}'\int_0^{+\infty}dk\cdot k\sqrt{k^2+\left(\frac{l\pi}{d}\right)^2} = \frac{L_xL_y}{4\pi}\hbar c\sum_l{}'\int_0^{+\infty}d\xi\sqrt{\xi+\left(\frac{l\pi}{d}\right)^2}$$

$$E_0(d) = \frac{\hbar cL_xL_y}{4\pi}\sum_l{}'\int d\xi\,\frac{\pi}{d}\sqrt{\underbrace{\frac{\xi d^2}{\pi^2}}_{\eta}+l^2} = \frac{\hbar cL_xL_y\pi^2}{4d^3}\sum_l{}'\int_0^{+\infty}d\eta\sqrt{\eta+l^2}$$

$E_{0,free}$: $\sum_l{}' \to \int dl$　　$E_{0,free} = \frac{\hbar cL_xL_y\pi^2}{4d^3}\int d\eta\int dl\sqrt{\eta+l^2}$

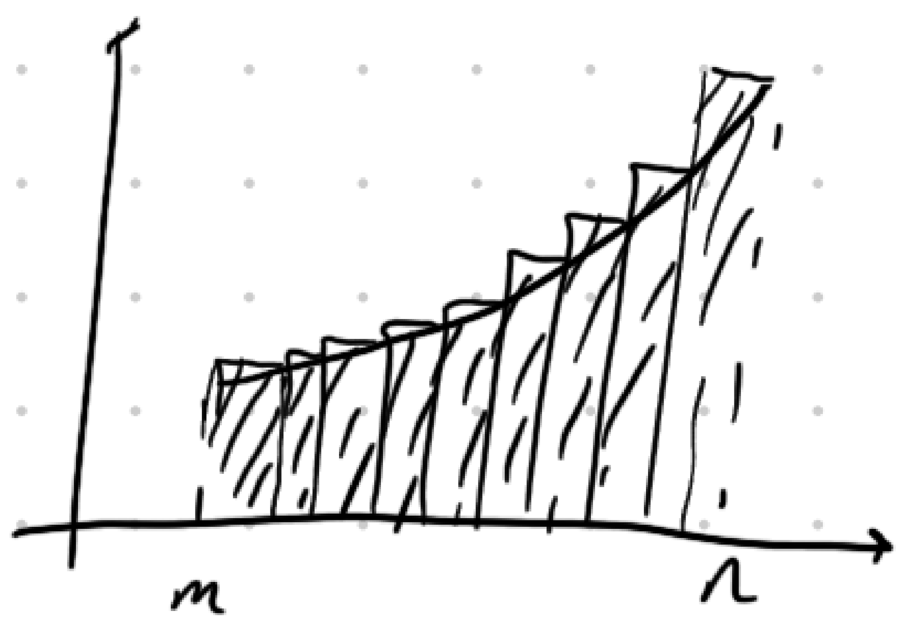
$$\Delta E = \frac{\hbar cL_xL_y\pi^2}{4d^3}\left[\sum_l{}'\int d\eta\sqrt{\eta+l^2} - \int d\eta\int dl\sqrt{\eta+l^2}\right]$$

def $F(l) = \int_0^\infty d\eta \sqrt{\eta + l^2}$ $\qquad \Delta E = \frac{\hbar c L_x L_y \pi^2}{4 d^3} \left[\frac{1}{2} F(0) + \sum_{l=1}^{\infty} F(l) - \int_0^{+\infty} dl\, F(l) \right]$

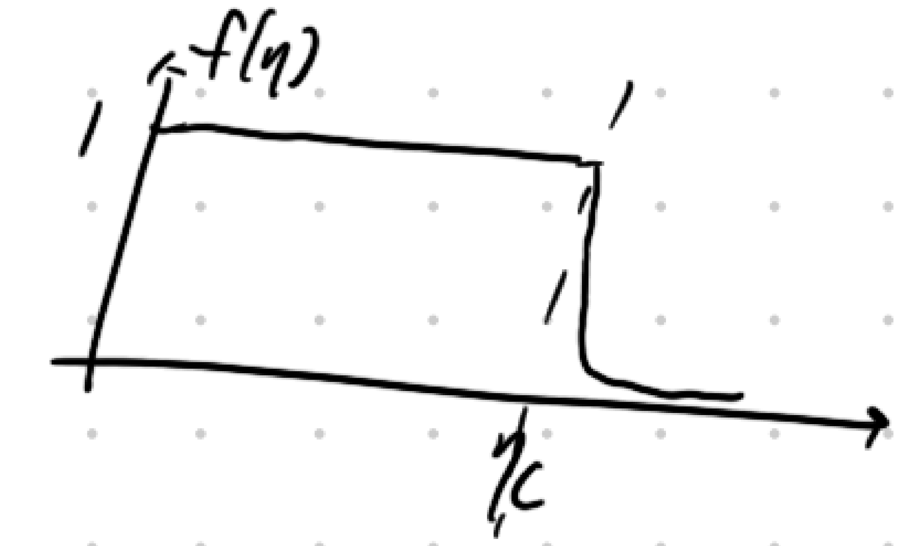
Euler-Bernoulli 公式 $\qquad I = \int_m^n f(x) dx \qquad S = \sum_{l=m+1}^{n} f(l) \qquad S - I = \sum_{k=1}^{N} \frac{B_k}{k!} \left[f^{(k-1)}(n) - f^{(k-1)}(m) \right] + \text{Error}^{(N)}$

B_k : Bernoulli 数. $\qquad \frac{y}{e^y - 1} = \sum_\nu B_\nu \frac{y^\nu}{\nu!} \qquad B_1 = \frac{1}{2} \qquad \Rightarrow \sum_{l=1}^{\infty} F(l) - \int_0^\infty dl\, F(l) = -\frac{1}{2} F(0) - \frac{1}{2} B_2 F'(0)$

$$- \frac{1}{3!} B_3 F^{(2)}(0) - \frac{1}{4!} B_4 F^{(3)}(0) + \cdots + \mathcal{F}(\infty)$$



$F(l) = \int_0^\infty d\eta \sqrt{\eta + l^2} \rightarrow \int_0^\infty d\eta \sqrt{\eta + l^2}\, f(\eta) \qquad f(\eta)$: 截断函数



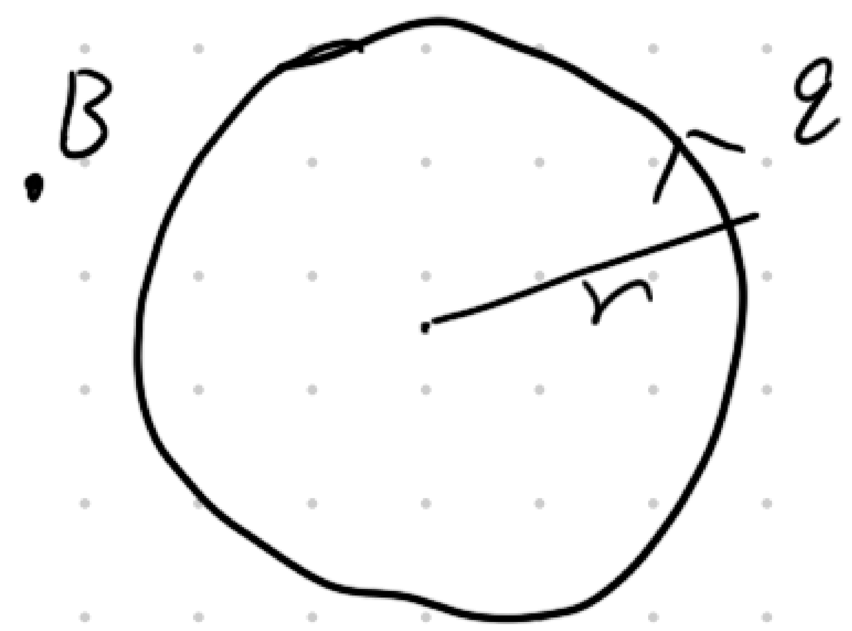
$\eta_c \sim \omega_c^2 \qquad$ For $\omega \to \infty$, $n(\omega) \to 1$ (紫外透明，边界失效)

$\mathcal{F}(\infty) \to 0$.

$F(l) = \int_{l^2}^{\infty} d\alpha \sqrt{\alpha}\, f(\alpha) \qquad F'(l) = -\sqrt{l^2}\,(2l) f(0) = -2l^2 \qquad l \to 0,\ F'(l) = F''(l) = 0 \qquad F^{(3)}(0) = -4$

$$\Rightarrow \Delta E = \frac{\hbar c L_x L_y \pi^2}{4 d^3} \left[-\frac{1}{4!} B_4 (-4) \right] \qquad B_4 = -\frac{1}{30} \qquad \Rightarrow \Delta E = -\frac{\hbar c L_x L_y \pi^2}{720\, d^3}$$

Casimir force $\qquad f = -\frac{\partial E}{\partial d} = \frac{\hbar c L_x L_y \pi^2}{240\, d^4} \sim \frac{1}{d^4}$



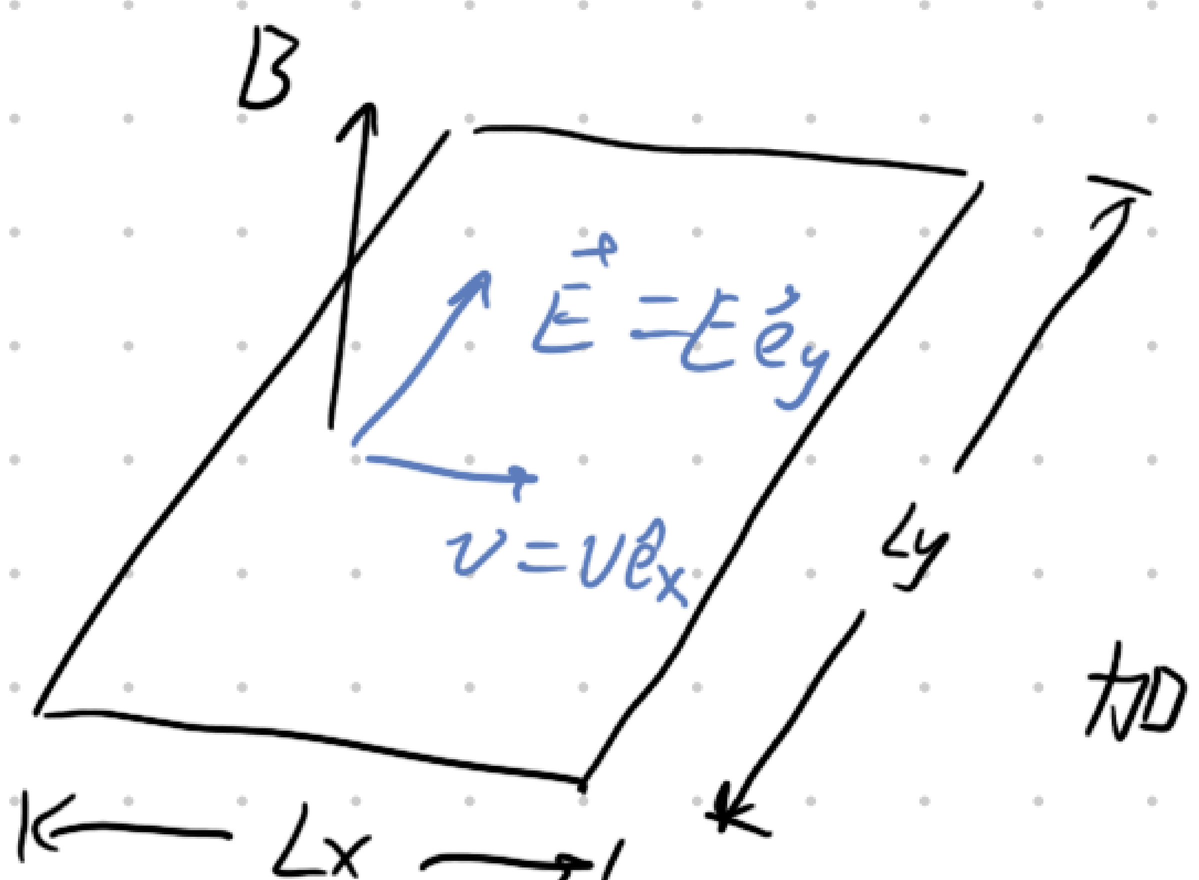
$I = \oint_C \vec{\pi}\cdot d\vec{r} = \oint (m\gamma\vec{u} + q\vec{A})\cdot d\vec{l} = m\gamma u(2\pi r) - q\Phi = qBr(2\pi r) - q\Phi = q\Phi$

正则动量

$I_n = nh \Rightarrow q\Phi_n = nh$ $\quad r_n = \sqrt{\frac{nh}{\pi qB}}$ $\quad p_n = qBr_n = \sqrt{\frac{nqBh}{\pi}} = \sqrt{2nqB\hbar}$

$\varepsilon_n = \sqrt{m^2+p_n^2} = \sqrt{m^2+2nqB\hbar}$ 非相对论极限: $\varepsilon_n \to \frac{p_n^2}{2m} = \frac{nqB\hbar}{m}$

实际上 $\varepsilon_n \to (n+\frac{1}{2})\hbar\omega$ $\quad I_n \to I_n = (n+\frac{1}{2})h$ (WKB 近似) $\quad \omega = qB/m$

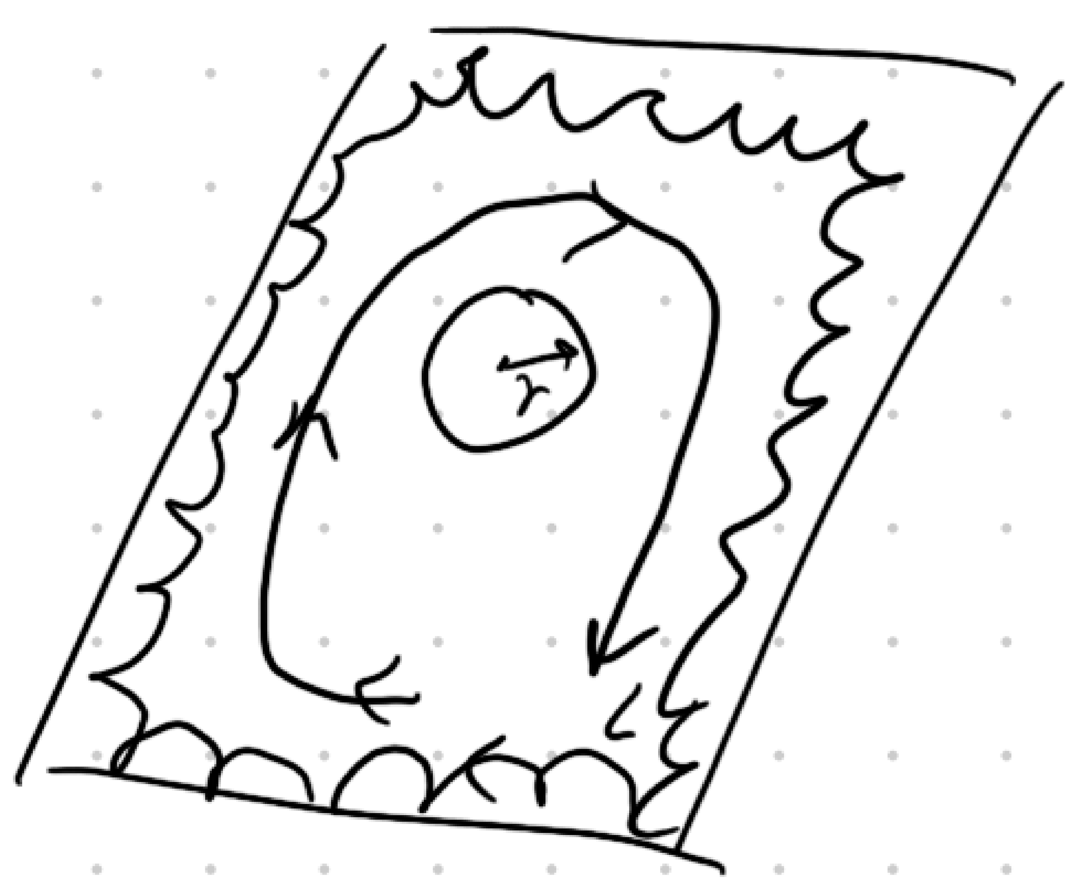


$\varepsilon_n = (n+\frac{1}{2})\hbar\omega$ 每能级简并 $N_0 = \frac{\Phi_{total}}{\Phi_0} = \frac{BL_xL_y}{h/e} = \frac{eBL_xL_y}{h}$

电子数 $N = n_0 N_0 = n_0\frac{eBL_xL_y}{h}$

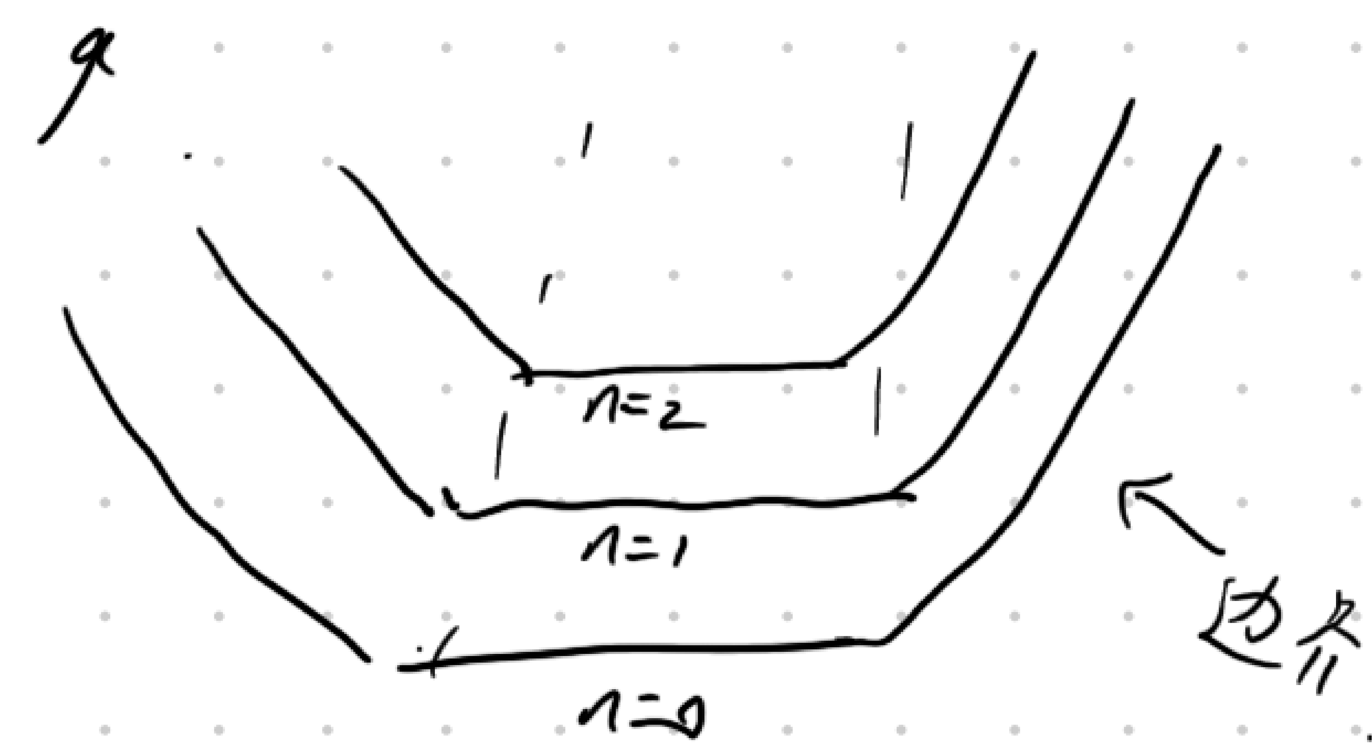
加电场 $\vec{E} = E\vec{e}_y$ 电子漂移 $\vec{v} = \frac{E}{B}\vec{e}_x$. $\quad I_x = \sigma_{xy}E_y$ $\quad \sigma_{xy} = -n_0\frac{e^2}{h}$.

若 $q = e > 0$, $\sigma_{xy} = n_0\frac{e^2}{h}$



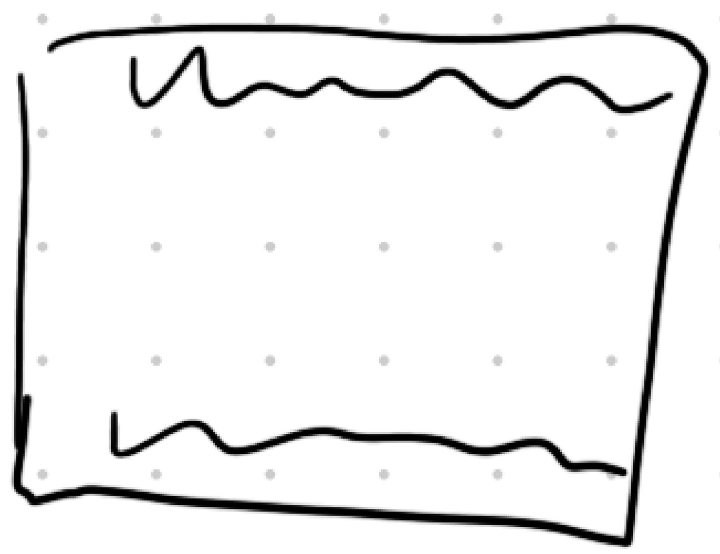
$\pi L_{edge} = nh$ $\quad L_{edge} < L_{in} \Rightarrow \pi_{edge} > \pi_{in}$ $\quad \varepsilon_{edge}\uparrow$

边界形成手性流



Topological field theory

QHE



$$J_x = \sigma_{xy} E_y \qquad \sigma_{xy} = \frac{ne^2}{h} \qquad S_{topo} = \frac{1}{2}\sigma_{xy}\int d^2x\, dt\ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \qquad (\mu,\nu,\rho) = (0,1,2)$$

δS_{topo} 对 A_μ 作变分

$$\delta S_{topo} = \frac{1}{2}\sigma_{xy}\int d^2x\, dt\ \epsilon^{\mu\nu\rho}\left[(\delta A_\mu)\partial_\nu A_\rho + A_\mu \partial_\nu(\delta A_\rho)\right]$$

$$= \frac{1}{2}\sigma_{xy}\int d^2x\, dt\ \epsilon^{\mu\nu\rho}(\partial_\nu A_\rho\ \delta A_\mu - \partial_\nu A_\mu\ \delta A_\rho)$$

$$= \frac{1}{2}\sigma_{xy}\int d^2x\, dt\ \left[\epsilon^{\rho\nu\mu} - \epsilon^{\mu\nu\rho}\right](\partial_\nu A_\mu)\ \delta A_\rho .$$

$$= -\sigma_{xy}\int d^2x\, dt\ \epsilon^{\mu\nu\rho}\partial_\nu A_\mu\, \delta A_\rho .$$

$$\Rightarrow J^\rho = \frac{\delta \mathcal{L}_{topo}}{\delta A_\rho} = -\sigma_{xy}\epsilon^{\mu\nu\rho}\partial_\nu A_\mu = \sigma_{xy}\epsilon^{\mu\nu\rho}\partial_\mu A_\nu .$$

$$\rho = 1,\ J^1 = \sigma_{xy}(\epsilon^{021}\partial_0 A_2 + \epsilon^{201}\partial_2 A_0) = \sigma_{xy}(-\partial_t A_y - \partial_y \phi) = \sigma_{xy}E_y .$$

$$J^0 = \rho = \sigma_{xy}(\epsilon^{120}\partial_1 A_2 + \epsilon^{210}\partial_2 A_1) = \sigma_{xy} B_z .$$