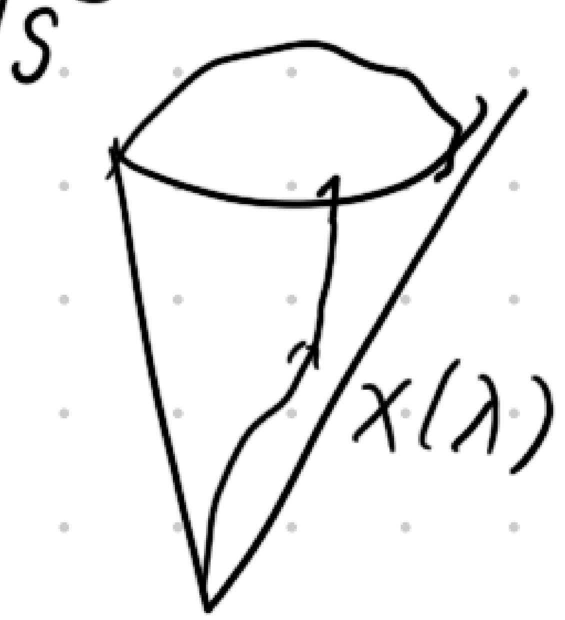
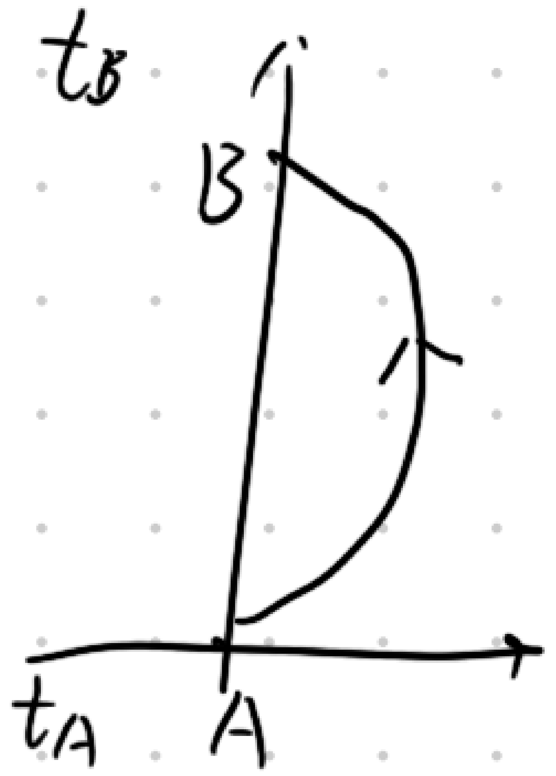
2. SR-Kinematics

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad x^\mu = (t, x, y, z) \qquad \eta_{\mu\nu} = \mathrm{diag}(-,+,+,+) \qquad d\tau^2 = -ds^2$



timelike curve: $\Delta\tau = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$



$\Delta\tau = t_{AB} = \int_A^B \sqrt{dt^2 - dr^2} < \int_A^B dt = t_B - t_A \Rightarrow$ 动钟变慢

4-vector $a^\mu = (a^0, a^1, a^2, a^3) \qquad \hat{e}_\mu = (\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3) \qquad \hat{a} = a^\mu \hat{e}_\mu$

$\hat{e}_\mu \cdot \hat{e}_\nu = \eta_{\mu\nu} \qquad \hat{a}\cdot\hat{b} = a^\mu b^\nu \eta_{\mu\nu} = a^\mu b_\mu = a_\mu b^\mu \quad (a_\mu = \eta_{\mu\nu} a^\nu)$

Lorentz 变换下: $x'^\mu = M^\mu{}_\nu x^\nu$

(1) 世界线, 选择参数 $\lambda = \tau$. 4-velocity $u^\mu = \frac{dx^\mu}{d\tau} \quad (u^\mu u_\mu = -1)$

4-momentum $p^\mu = m u^\mu = (E, \vec{p}) \qquad p^\mu p_\mu = -m^2 = -E^2 + \vec{p}^2$

$$M^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad a^\mu = \frac{d^2x^\mu}{d\tau^2} \qquad \text{4-force } f^\mu = m a^\mu = m\frac{d^2x^\mu}{d\tau^2}$$

free particle: $S = \int m\, d\tau = m \int_{\lambda_A}^{\lambda_B} d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$ E-L方程 $\frac{\partial L}{\partial x^\nu} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial(dx^\nu/d\lambda)}\right)$

$\lambda \to \tau \Rightarrow \frac{d^2x^\mu}{d\tau^2} = 0.$

(2) massless particle $\Delta\tau = 0$ $x^\mu(\lambda) = b^\mu \lambda$ $b^\mu = (1, \vec{b})$ $|\vec{b}|^2 = 1$ $\vec{b}$: "光线方向"

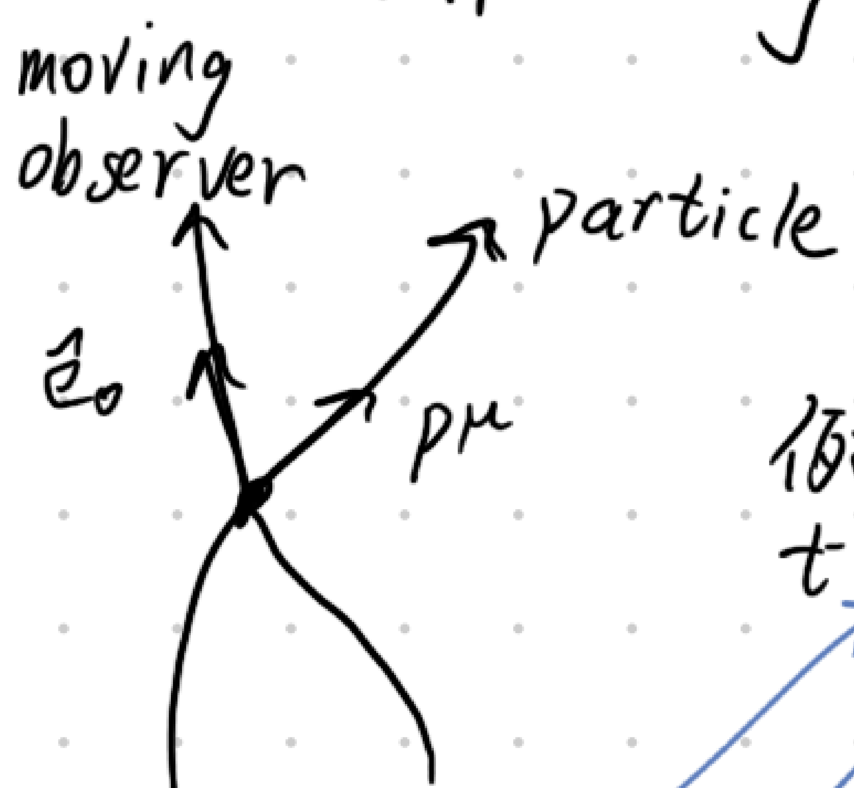
$u^\mu \triangleq \dfrac{dx^\mu}{d\lambda}$ $u^\mu = b^\mu$ $u^\mu u_\mu = 0$ $\dfrac{du^\mu}{d\lambda} = 0$ $(a^\mu = 0)$

$p^\mu \triangleq (E, \vec{p}) \propto u^\mu$ $p^\mu p_\mu = 0$ $E^2 = |\vec{p}|^2$ $p^\mu = \hbar k^\mu \xlongequal{\hbar = 1} k^\mu \Rightarrow$ dopple effect.

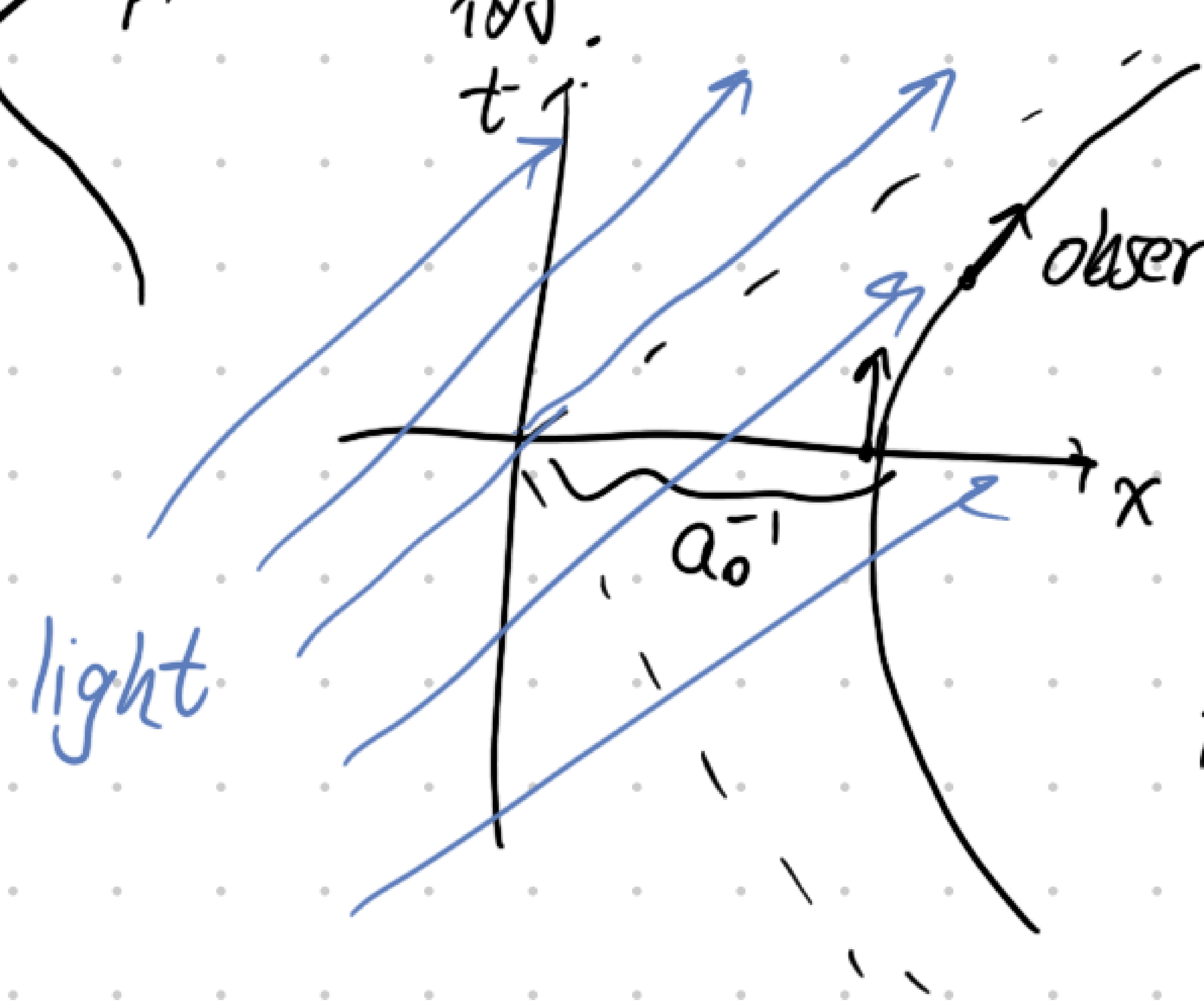
(3) moving (massive) observer 标架 $\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3$ $\hat{e}_0 = \hat{u}_{obs}$ $(\hat{u}_{obs} \cdot \hat{u}_{obs} = \hat{e}_0 \cdot \hat{e}_0 = -1)$

其他标架: $\hat{e}_i \cdot \hat{e}_0 = 0$ $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

计算 moving observer 观察到的物理量: 作点积求标架方向分量 e.g. $E = -\vec{p} \cdot \hat{e}_0$ $p^i = \vec{p} \cdot \hat{e}_i$.



例:



observer在自身瞬时静止系以加速度 a_0 运动

$$\begin{cases} t(\tau) = a_0^{-1} \sinh(a_0 \tau) \\ x(\tau) = a_0^{-1} \cosh(a_0 \tau) \end{cases}$$

$$\begin{cases} u^\mu = \dfrac{dx^\mu}{d\tau} = (\cosh(a_0\tau), \sinh(a_0\tau), 0, 0) \\ a^\mu = \dfrac{du^\mu}{d\tau} = (a_0 \sinh(a_0\tau), a_0 \cosh(a_0\tau), 0, 0) \end{cases} \quad \|\hat{a}\| = a_0$$

标架 $\hat{e}_0(\tau) = (\cosh(a_0\tau), \sinh(a_0\tau), 0, 0)$ $\hat{e}_2(\tau) = (0, 0, 1, 0)$

$\hat{e}_1(\tau) = (\sinh(a_0\tau), \cosh(a_0\tau), 0, 0)$ $\hat{e}_3(\tau) = (0, 0, 0, 1)$

dopple effect: $k^\mu = (\omega_0, \omega_0, 0, 0)$ observer 看来 $\omega = -\hat{k} \cdot \hat{e}_0 = \omega_0[\cosh(a_0\tau) - \sinh(a_0\tau)]$

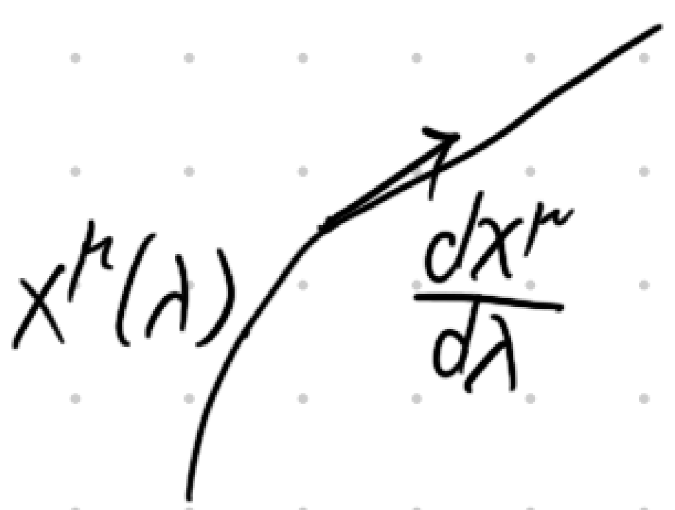
$\omega = \omega_0 e^{-a_0\tau}$ $\tau < 0$: $\omega > \omega_0$ blueshift $\tau > 0$: $\omega < \omega_0$ redshift $\tau = 0$: $\omega = \omega_0$

event horizon (事件视界): observer 无法观察到 $t>0$ 时刻从原点 $x=0$ 处发出的光.

3. tensor in SR $(a^\mu, \eta_{\mu\nu})$ 4-vector: $\vec{a} = a^\mu \vec{e}_\mu$ invariant under $x^\mu \to x^{\mu'}$

components

General definition of vectors



any function $f(x^\mu(\lambda)) \longrightarrow \frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} \equiv t^\mu \frac{\partial f}{\partial x^\mu}$

* Define the directional derivative $\vec{t} = t^\mu \frac{\partial}{\partial x^\mu}$ acting on f $\vec{t}: \mathcal{F} \to \mathbb{R}$.
* A general vector can be defined $\vec{a} = a^\mu \frac{\partial}{\partial x^\mu}$ basis: $\vec{e}_\mu = \frac{\partial}{\partial x^\mu}$
* changing of coord, basis $x^\mu \to x'^\mu$ $\vec{a} = a^\mu \frac{\partial}{\partial x^\mu} = a^\mu \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}$

transformation of a^μ: $a'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} a^\mu$ $\vec{e}_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\vec{e}'_\nu$ $\vec{e}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\vec{e}_\nu$

e.g. Lorentz. $x^{\mu'} = \Lambda^\mu{}_\nu x^\nu \Rightarrow a'^\mu = \Lambda^\mu{}_\nu a^\nu$ for $\vec{e}_\nu$ $\vec{e}'_\mu = \underbrace{\Lambda_\mu{}^\nu}_{=\frac{\partial x^\nu}{\partial x'^\mu}} \vec{e}_\nu$

($\Lambda_\mu{}^\nu$ is the inverse of $\Lambda^\mu{}_\nu$: $\Lambda_\mu{}^\rho \Lambda^\mu{}_\nu = \delta^\rho_\nu$)

* Tangent space T_p: all possible tangent vectors of curves at a point $p \in \mathbb{R}^{1,3}$
(切空间)
(linear space with basis $\vec{e}_\mu = \frac{\partial}{\partial x^\mu}$)

* Dual vector (1-form) $\hat{\omega}$: linear map $\hat{a} \longmapsto \hat{\omega}(\hat{a}) \in \mathbb{R}$ $\quad$ $\hat{\omega} \in Hom(T_p, \mathbb{R}) = T_p^*$ is another vector space. (cotangent space) (余切空间)

(对偶矢量) (1-形式)

$$\hat{\omega}(a\vec{v}_1 + b\vec{v}_2) = a\hat{\omega}(\vec{v}_1) + b\hat{\omega}(\vec{v}_2) \qquad (a\hat{\omega}_1 + b\hat{\omega}_2)(\vec{v}) = a\hat{\omega}_1(\vec{v}) + b\hat{\omega}_2(\vec{v})$$

Basis of dual vector : $\hat{\theta}^\mu$ $\qquad \hat{\theta}^\nu(\hat{e}_\mu) = \delta^\nu_\mu \qquad \hat{\omega} = \omega_\mu \hat{\theta}^\mu \qquad \hat{\omega}(\vec{v}) = \omega_\mu \hat{\theta}^\mu(v^\nu \hat{e}_\nu) = \omega_\mu v^\mu$.

e.g. gradient of a scalar $f(x^\mu)$ $\qquad \frac{df}{d\lambda} = t^\mu \frac{\partial f}{\partial x^\mu} \qquad df = dx^\mu \frac{\partial f}{\partial x^\mu} \qquad \underset{T_p^*}{df} : \underset{T_p}{\vec{t}} \rightarrow \underset{\mathbb{R}}{t^\mu \frac{\partial f}{\partial x^\mu}}$

basis $\hat{\theta}^\mu = dx^\mu \qquad dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$

T_p^* : all linear map $T_p \rightarrow \mathbb{R} \qquad \dim(T_p) = \dim(T_p^*) = \dim(\mathbb{R}^{1,3})$ $\leftarrow M$

T_p : all linear maps $T_p^* \rightarrow \mathbb{R}$.

* (k,l)-tensor $\quad$ a multi-linear map $\quad \vec{T} : \underbrace{T_p^* \otimes T_p^* \otimes \cdots \otimes T_p^*}_{k} \otimes \underbrace{T_p \otimes \cdots T_p}_{l} \longrightarrow \mathbb{R}$

Basis $\hat{e}_\mu, \hat{\theta}^\mu \qquad \vec{T} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \, \hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \cdots \otimes \hat{\theta}^{\nu_l}$

$\otimes$: tensor-product. def $\quad \underset{(k,l)}{\vec{T}}(\hat{\omega}^{(1)}, \ldots, \hat{\omega}^{(k)}, \vec{v}^{(1)} \ldots \vec{v}^{(l)}) \; \underset{(m,n)}{\vec{S}}(\hat{\omega}^{(k+1)}, \ldots, \hat{\omega}^{(k+m)}, \vec{v}^{(l+1)}, \ldots, \vec{v}^{(l+n)})$

$$= \underset{(k+m,\, l+n)}{(\vec{T} \otimes \vec{S})}(\hat{\omega}^{(1)} \ldots \hat{\omega}^{(k+m)} \vec{v}^{(1)} \ldots \vec{v}^{(l+n)})$$

$$(T\otimes S)^{\mu_1\cdots\mu_{k+m}}{}_{\nu_1\cdots\nu_{l+n}} = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\, S^{\mu_{k+1}\cdots\mu_{k+m}}{}_{\nu_{l+1}\cdots\nu_{l+n}}.$$

under Lorentz transformation $T'^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T^{\rho_1\cdots\rho_k}{}_{\sigma_1\cdots\sigma_l}\,\Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_k}{}_{\rho_k}\,\Lambda_{\nu_1}{}^{\sigma_1}\cdots\Lambda_{\nu_l}{}^{\sigma_l}$

* Invariant tensor: invariant under Λ transformation

(1) $\delta^\mu_\nu = \begin{cases} 0 & \mu\neq\nu \\ 1 & \mu=\nu \end{cases}$ (1,1) tensor.

$$\delta'^\mu_\nu = \Lambda^\mu{}_\rho\Lambda_\nu{}^\sigma\delta^\rho{}_\sigma = \Lambda^\mu{}_\rho\Lambda_\nu{}^\rho = \delta^\mu_\nu.$$

(2) metric tensor (度规张量) $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. $\eta'_{\mu\nu} = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\eta_{\rho\sigma} = \Lambda_\mu{}^\rho\eta_{\rho\sigma}(\Lambda_\sigma{}^\nu)^T = \eta_{\mu\nu}$ $(\Lambda\eta\Lambda^T=\eta)$.

Define inverse of $\eta_{\mu\nu}$: $\eta^{\mu\nu}$ $\quad \eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$ $\quad$ Inner product: $\eta_{\mu\nu}a^\mu b^\nu = \eta^{\mu\nu}a_\mu b_\nu$ is a scalar.

(3) Levi-Civita tensor (invariant for only $SO(1,3)$) $\quad \epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{even} \quad \text{偶置换} \\ -1 & \text{odd} \quad \text{奇置换} \\ 0 & \text{other cases} \end{cases}$

$$\epsilon'_{\mu\nu\rho\sigma} = \det(\Lambda)\,\epsilon_{\mu\nu\rho\sigma}$$

* Symmetric tensor if $T_{\mu_1\cdots\mu_k} = T_{\sigma(\mu_1)\cdots\sigma(\mu_k)}$ for any permutation (置换) $\sigma\in S_k$.

e.g. $\eta_{\mu\nu} = \eta_{\nu\mu}$

$\begin{cases} \text{General local coord trans} \quad x^\mu \to x'^\mu(x) \quad (\text{广义协变性, GR}) \\ \text{Global Lorentz trans:} \quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (\text{狭义协变性, SR}) \end{cases}$

* Symmetrication of idices

$$T_{(\mu_1,\cdots,\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{\sigma(\mu_1)\cdots\sigma(\mu_n)}$$

Anti: $T_{[\mu_1,\cdots,\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, T_{\sigma(\mu_1)\cdots\sigma(\mu_n)}$

(1) $X^{(\mu\nu)} Y_{\mu\nu} = X^{(\mu\nu)} Y_{(\mu\nu)}$

(2) $T_{\mu\nu\rho\sigma} = T_{[\mu\nu]\rho\sigma} + T_{(\mu\nu)\rho\sigma}$

4. EM and perfect fluids in tensors

(理想流体)

Maxswell: $\epsilon^{ijk}\partial_j B_k - \partial_0 E^i = 4\pi j^i \qquad \partial_i E^i = 4\pi\rho \qquad \epsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \qquad \partial_i B^i = 0$

* Current 4-vector $J^\mu = (\rho, j^1, j^2, j^3)$

Anti-Sym (2,0) $F^{\mu\nu}$ $\qquad F^{0i} = E^i \qquad F^{ij} = \epsilon^{ijk} B_k$

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= 4\pi J^\nu \\ \partial_{[\mu} F_{\nu\rho]} &= 0 \end{aligned}}$$

$F^{\mu\nu} = 2\partial^{[\mu} A^{\nu]}$

$$\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right) = 0 \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 4\pi A_\mu J^\mu$$ fields are A_μ.

e.g. perfect fluid 通量 $N^\mu = nU^\mu = (n, nV^1, nV^2, nV^3)$

Energy density $\rho = mn$ $T^{\mu\nu}$: the flux of p^μ across a surface of constant x^ν

静止系下: $T^{00} = \rho$

$T^{0i} = 0$ (理想流体，无热传导)

$T^{ii} = p_i$ (压强) T^{ij} $(i\neq j)$ 剪切应力 理想流体 $T^{ij} = 0$.

各向同性流体: $p_1 = p_2 = p_3 = p$

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

一般参考系: 不考虑压强 $T^{\mu\nu} = p^\mu N^\nu = \rho U^\mu U^\nu$

瞬时静止参考系下

加上压强 $\boxed{T^{\mu\nu} = (\rho+p)U^\mu U^\nu + p\eta^{\mu\nu}}$

e.g. (1) Dust. 尘埃 $p = 0$

(2) 光子气体 $p = \frac{1}{3}\rho$

(3) 真空能 $p = -\rho$

$$U_\nu \partial_\mu T^{\mu\nu} = U_\nu \partial_\mu(\rho+p)U^\mu U^\nu + U_\nu(\rho+p)(U^\nu\partial_\mu U^\mu + U^\mu \partial_\mu U^\nu) + U_\nu \partial^\nu p$$

$$= -U^\mu\partial_\mu(\rho+p) - (\rho+p)\partial_\mu U^\mu + U^\mu\partial_\mu p = -\partial_\mu(\rho U^\mu) - p\,\partial_\mu U^\mu = 0$$

$$\begin{cases} U_\nu U^\nu = -1 \\ \partial_\mu(U_\nu U^\nu) = 0 \\ U_\nu \partial_\mu U^\nu = 0 \end{cases}$$

(理想流体能量守恒方程)

非相对论极限 $U^\mu=(1,v^i)$ $|v^i|\ll 1$ $p\ll\rho$

$$U_\nu\partial_\mu T^{\mu\nu} = -\partial_t\rho - \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

5. Gravitation and geomety

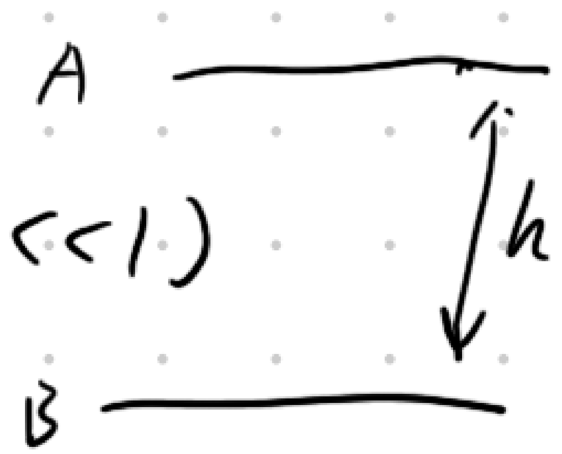
$\vec{F} = -\frac{GMm}{r^2}\hat{e}_r$ 引力势 $\Phi = -\frac{GM}{r}$ $\vec{g} = -\nabla\Phi = -\frac{GM}{r^2}\hat{e}_r$

$\Phi = -G\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$ $\int_S\vec{g}\cdot d\vec{S} = -\int_S\nabla\Phi\cdot d\vec{S} = -\int_V\nabla\cdot(\nabla\Phi)\,dV = -4\pi GM$

弱等效原理 惯性质量 = 引力质量

爱因斯坦等效原理 (EEP)：无法局域地区分加速度 $\vec{a}$ 和引力 $\vec{g}$.

引力红移：



$E_A = m(1-gh) = \omega_A < \omega_B$ $\frac{\omega_B}{\omega_A}\approx 1+gh$ redshift factor

$(gh\ll 1)$

$E_B = m = \omega_B$

加速度 $\vec{a} = -\vec{g}$ $(v\ll 1, gh\ll 1)$

$z_A = h+\frac{1}{2}gt^2$

$z_B = \frac{1}{2}gt^2$

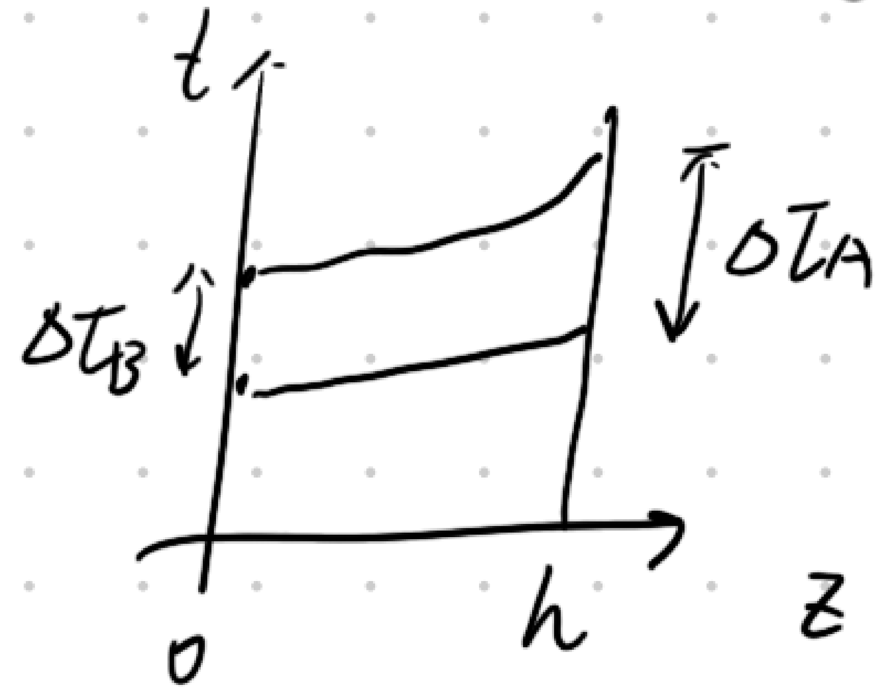
$t=0$ 和 $t=\Delta\tau_B$ $(\Delta\tau_B\ll h)$ 从B发出光信号.

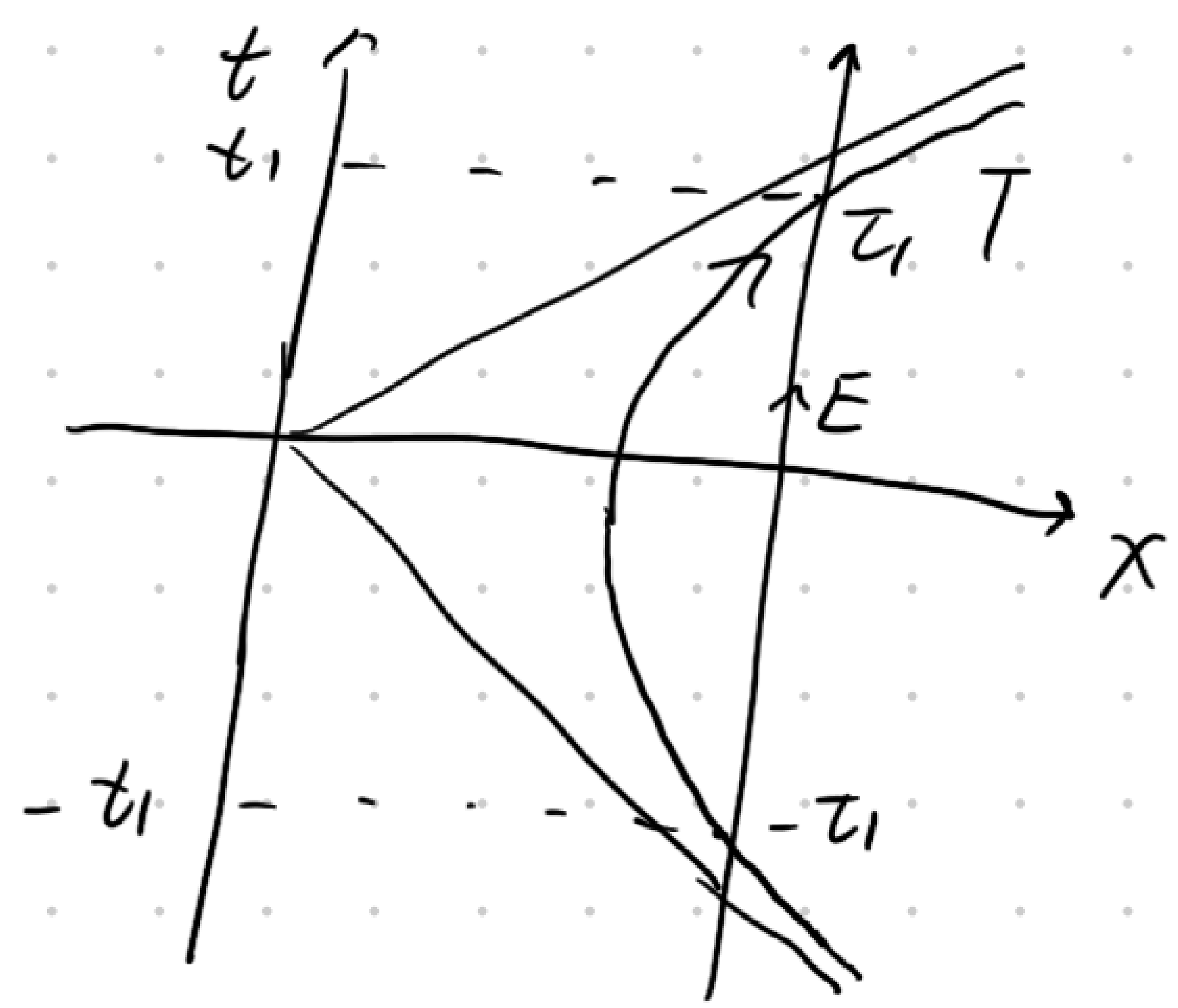
在A处 $t=t_1$ 与 $t=t_1+\Delta\tau_A$ 接收到

$z_A(t_1)-z_B(0)=t_1$ $z_A(t_1+\Delta\tau_A)-z_B(\Delta\tau_B) = t_1+\Delta\tau_A-\Delta\tau_B$

$$\Rightarrow \Delta\tau_A \approx \Delta\tau_B(1+gh) \qquad \omega \propto \frac{1}{\Delta\tau} \qquad \omega_B = \omega_A(1+gh)$$

引力场和加速度场中，时空都不是平直的.





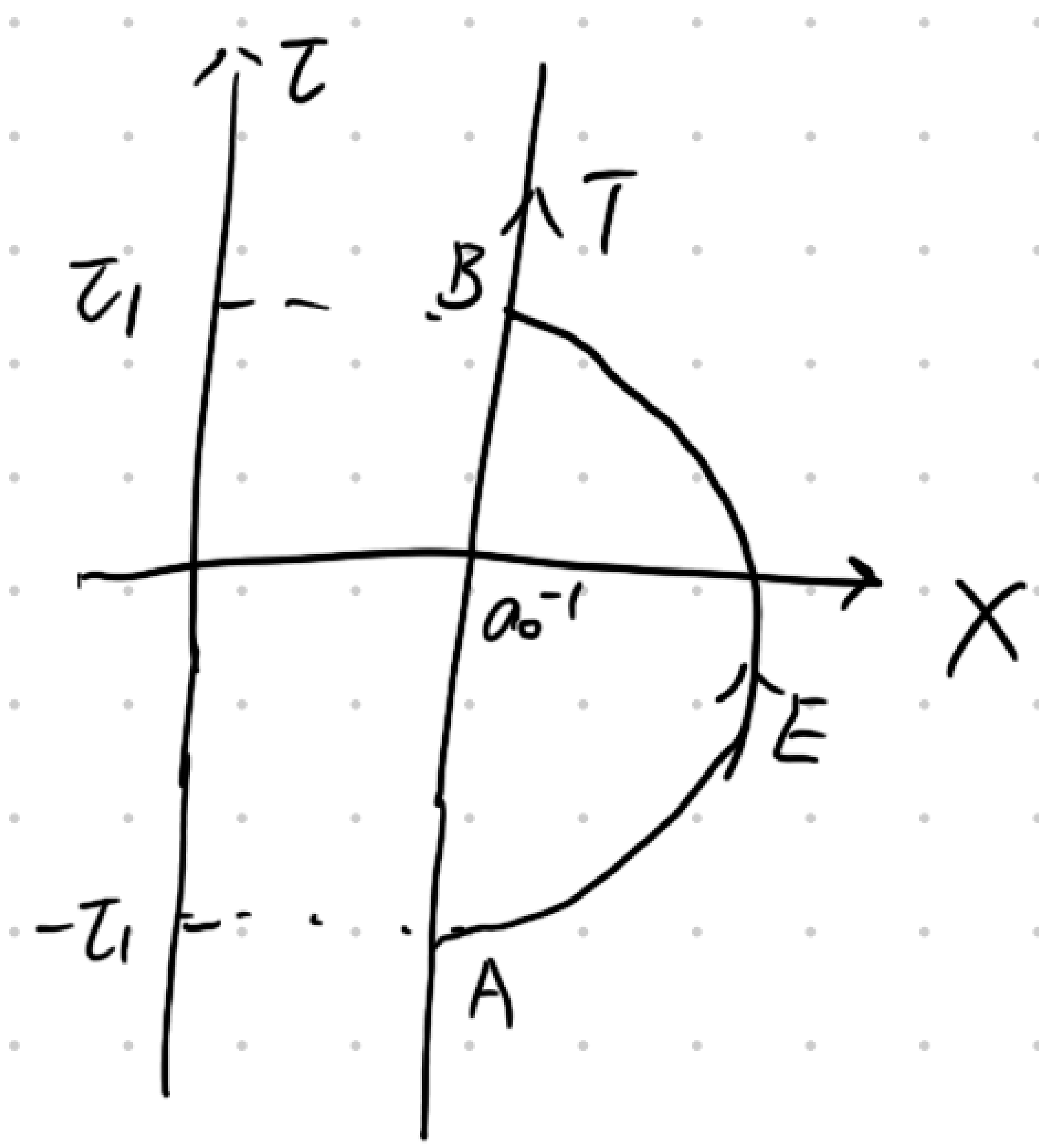
双生子详谬 T: $\begin{cases} t = a_0^{-1}\sinh(a_0\tau) \\ x = a_0^{-1}\cosh(a_0\tau) \end{cases}$

地球参考系: $-t_1 \to t_1 \qquad x_E = a_0^{-1}\cosh(a_0\tau_1)$.

$\tau_1 = a_0^{-1}\operatorname{arcsinh}(a_0 t_1)$

$2\tau_1 < 2t_1 = 2a_0^{-1}\sinh(a_0\tau_1)$.

加速参考系: $\tau = a_0^{-1}\operatorname{arctanh}\frac{t}{x} \qquad X = \sqrt{x^2-t^2} \qquad x = X\cosh a_0\tau \qquad t = X\sinh a_0\tau$，旅行者在 $X = a_0^{-1}$



地球: $X_E = \sqrt{x_E^2 - t^2} \qquad X_E^2 = x_E^2 - X_E^2\sinh^2 a_0\tau \Rightarrow X_E = \dfrac{x_E}{\cosh a_0\tau}$.

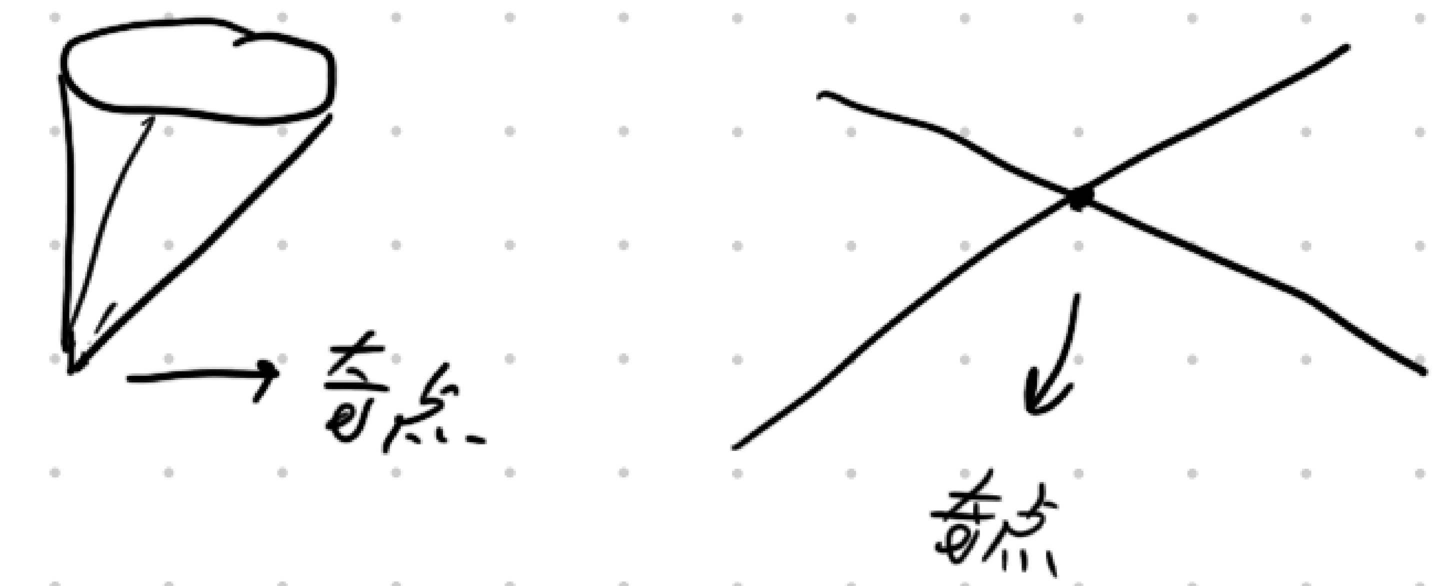
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad t = X\sinh a_0\tau \qquad x = X\cosh a_0\tau$.

$\Rightarrow ds^2 = -(a_0X)^2 d\tau^2 + dX^2 + dy^2 + dz^2$.

$\int_A^B dt = \int_{-\tau_1}^{\tau_1}\sqrt{(a_0X_E)^2 d\tau^2 - dX_E^2} = \int_{-\tau_1}^{\tau_1}\sqrt{(a_0X_E)^2 - \left(\frac{dX_E}{d\tau}\right)^2}\,d\tau$ 代入 $X_E = \dfrac{x_E}{\cosh a_0\tau}$

6. Manifolds 流形

(1) n-dim 流形 locally $\simeq \mathbb{R}^n$ not manifold:



拓扑空间是集合 X，其上指定开集 $\mathcal{T}=\{U_i\}$ 且：(1) $\phi, X \in \mathcal{T}$ (2) 任意并、有限交仍是开集.

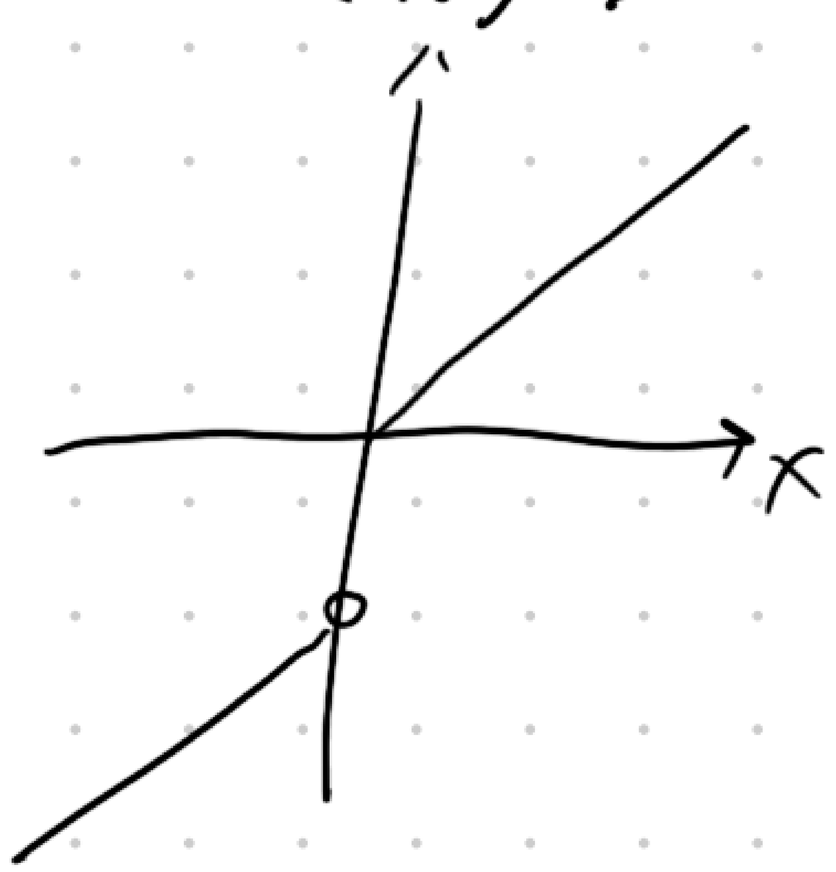
度量函数 $d: X \times X \to \mathbb{R}$
metric function

(1) $d(x,y) = d(y,x)$ (2) $d(x,y) \geq 0$，只在 $x=y$ 时取等.
(3) $d(x,y) + d(y,z) \geq d(x,z)$

度量空间：指定了一个度量函数的拓扑空间。可定义开集为 $U_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$ 和其并、交

continous map (连续映射)： $f: X \to Y$ 连续，任意一个开集的原像仍是开集.

e.g.



$$f(x) = \begin{cases} x, & x \geq 0 \\ x-1, & x < 0 \end{cases}$$

$f^{-1}((-0.1, 0.1)) = [0, 0.1)$ 不是开集.

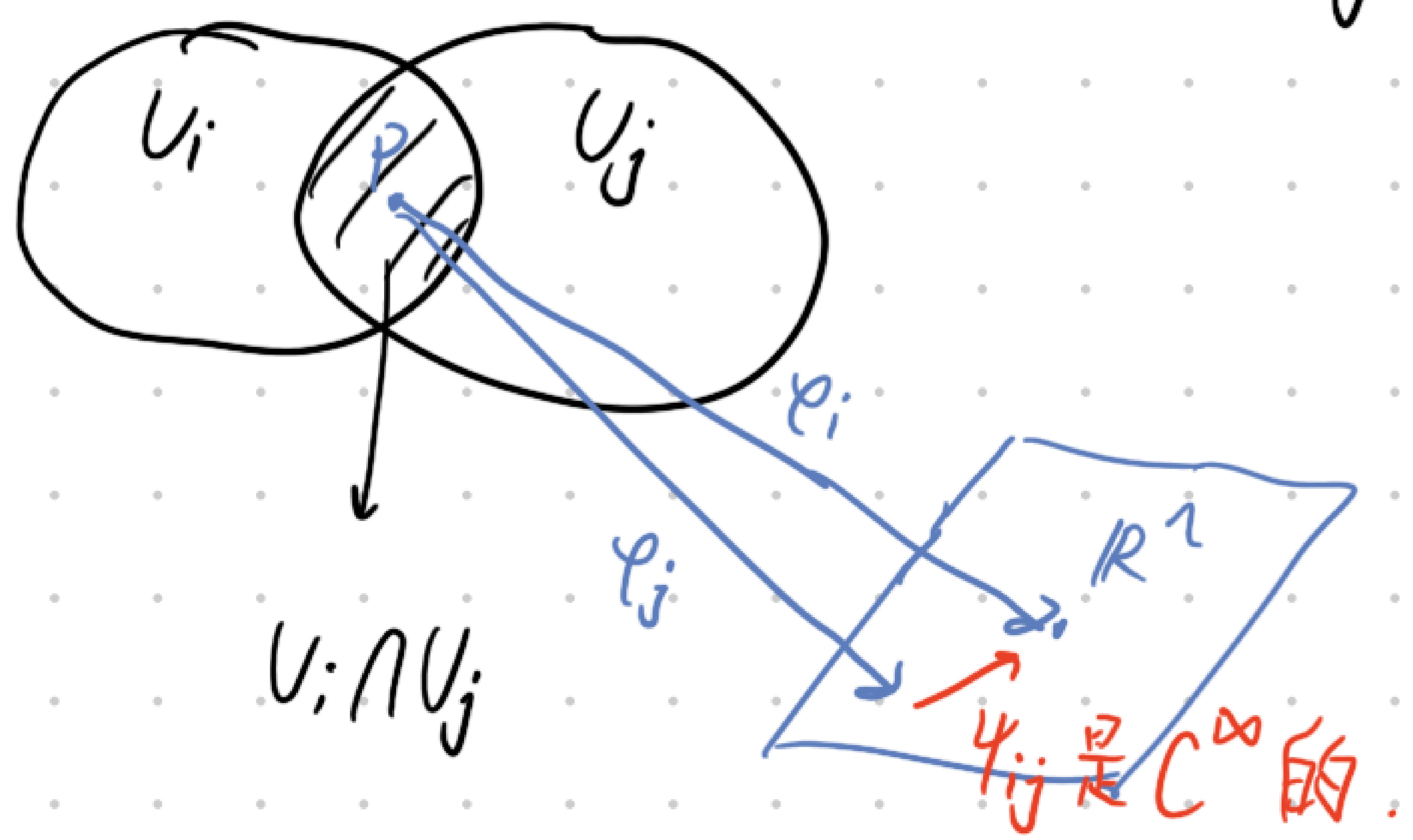
同胚：两个拓扑空间同胚，存在连续映射 $f: X \to Y$，且逆映射 $f^{-1}: Y \to X$ 也连续

(2)

微分流形 (differential manifold) M: (1) n-dim 拓扑空间 (2). 其上可描述 $\{(U_i, \varphi_i)\}$ (坐标卡, 地图条)

$\{U_i\}$ 可覆盖 M $\bigcup_i U_i = M$ φ_i: 是从 U_i 到 $U_i' \subset \mathbb{R}^n$ 的同胚映射.

(3) 光滑性. Given U_i and U_j st. $U_i \cap U_j \neq \emptyset$, 映射 $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$: $\varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ 是 C^∞ 的.

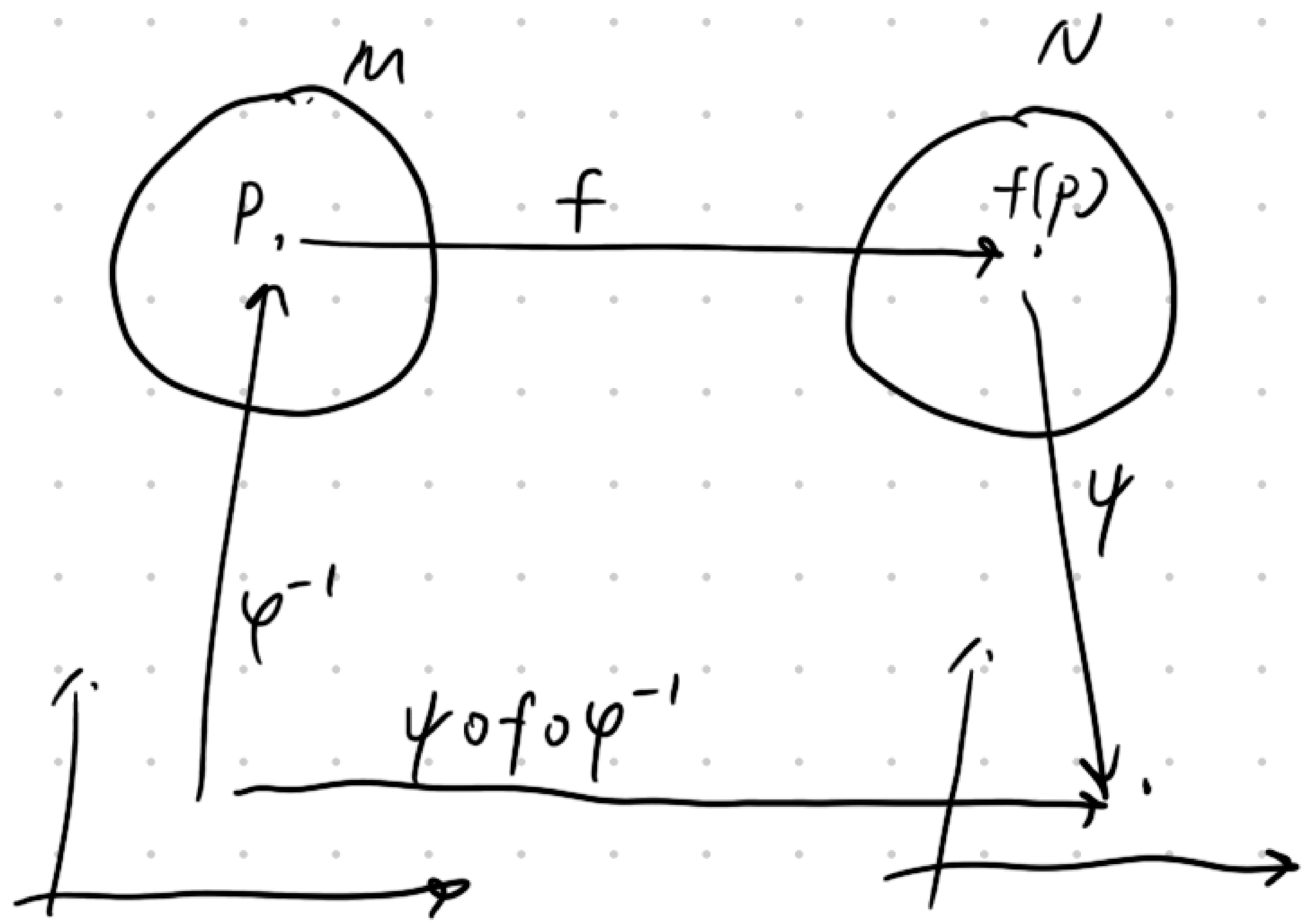


(3)

可微映射 (Differentiable map) $f: M \to N$ $\dim M = m$ $\dim N = n$ a point $p \in M$ is mapped to $f(p) \in N$.

M 上取 (U, φ) N 上取 (V, ψ) st. $p \in U$, $f(p) \in V$. 若 $\psi \circ f \circ \varphi^{-1}: U \to V$ 是 C^∞ 的,

称 f 在 p 点光滑.

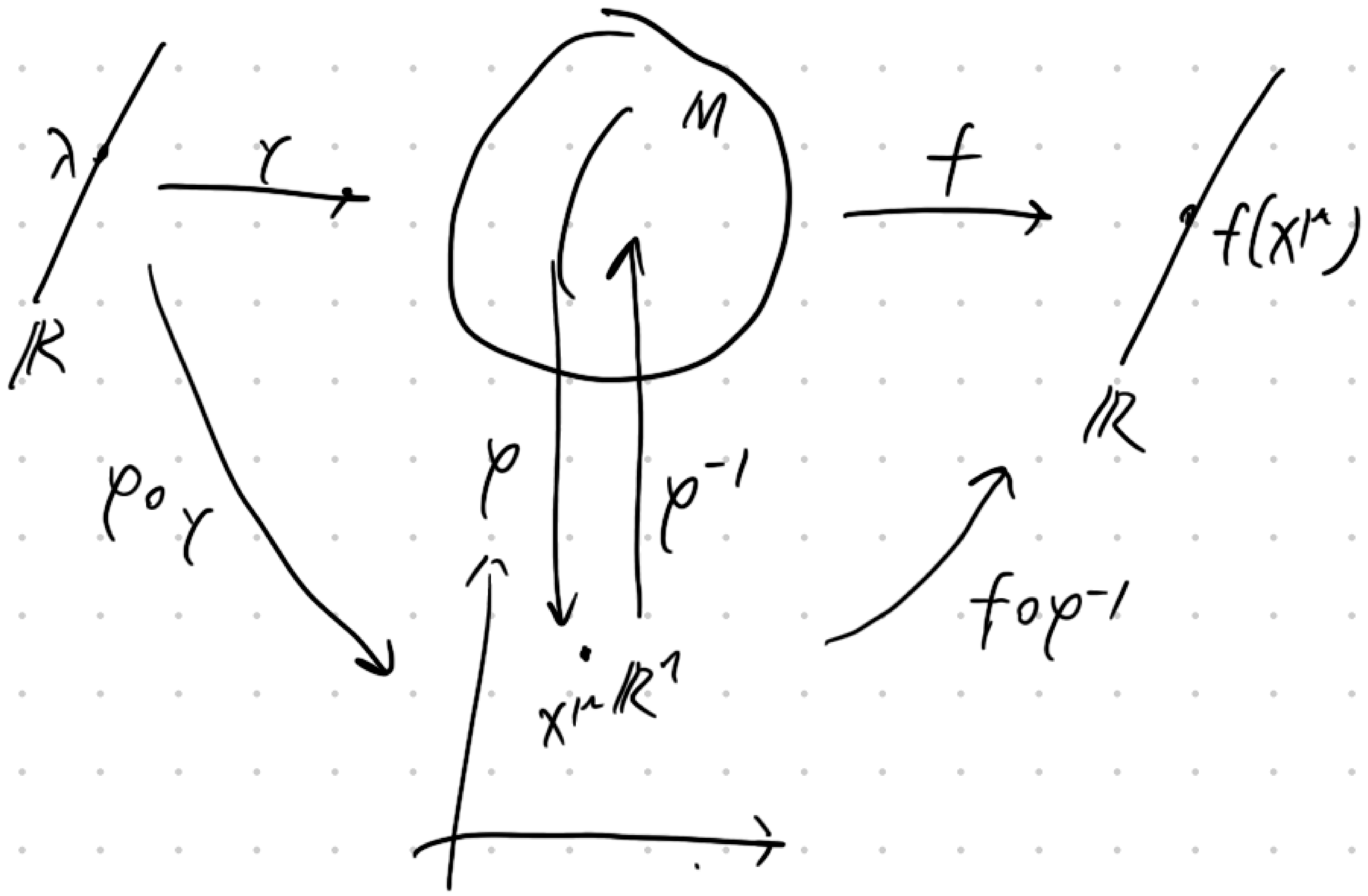


若 f 在 M 的每一点光滑, 称 f 是 $M \to N$ 的可微映射 (光滑映射).

若存在 $f: M \to N$, $f^{-1}: N \to M$ 都是可微的, 称 M, N 微分同胚.

(4) 流形上的张量 (Tensors on manifolds) M上的切空间 T_pM at p : a curve $\gamma:\mathbb{R}\to M$, 另有 $f: M\to\mathbb{R}$.

map $f\circ\gamma:\mathbb{R}\to\mathbb{R}$ $f(\lambda)=f(P(\lambda))$ $(P(\lambda)\in M)$. 方向导数 $\frac{d}{d\lambda}(f\circ\gamma)=\frac{d}{d\lambda}[(f\circ\varphi^{-1})\circ(\varphi\circ\gamma)]$



$(\varphi\circ\gamma)\ \mathbb{R}\to\mathbb{R}^n : \lambda\to x^\mu(\lambda)$

$(f\circ\varphi^{-1})\circ(\varphi\circ\gamma) : \lambda\to f(x^\mu(\lambda))$

$$\frac{d}{d\lambda}[(f\circ\varphi^{-1})\circ(\varphi\circ\gamma)]=\frac{dx^\mu}{d\lambda}\partial_\mu f$$

$\frac{d}{d\lambda}=\frac{dx^\mu}{d\lambda}\partial_\mu$ ∂_μ 是 T_pM 的一个基.

T_pM 有n个线性无关基矢 $\hat{e}_\mu=\partial_\mu$. 余切空间 $T_p^*M=\mathrm{Hom}(T_pM,\mathbb{R})$ 基: $\hat{\theta}^\mu=dx^\mu$ $\hat{\theta}^\mu(\hat{e}_\nu)=\delta^\mu{}_\nu$

$\frac{\partial}{\partial x^\mu}W^\nu=\partial_\mu W^\nu$ is not a tensor ($\partial_\mu\to\partial'_\mu$, $W^\nu\to W'^\nu$ $\partial'_\mu W'^\nu\neq\partial_\mu W^\nu$ 视为(1,1)张量按张量规则变化结果)

(5)

度规张量 (metric tensor) $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ 非退化: $\det|g_{\mu\nu}|=|g|\neq0$ (远离奇异点) → 来源于坐标选取 坐标奇点

对称: $g_{\mu\nu}=g_{\nu\mu}$ 有逆: $g^{\mu\nu}g_{\nu\sigma}=\delta^\mu{}_\sigma$ 升降指标、$g^{\mu\nu}V_\nu=V^\mu$ ⇒ 定义内积

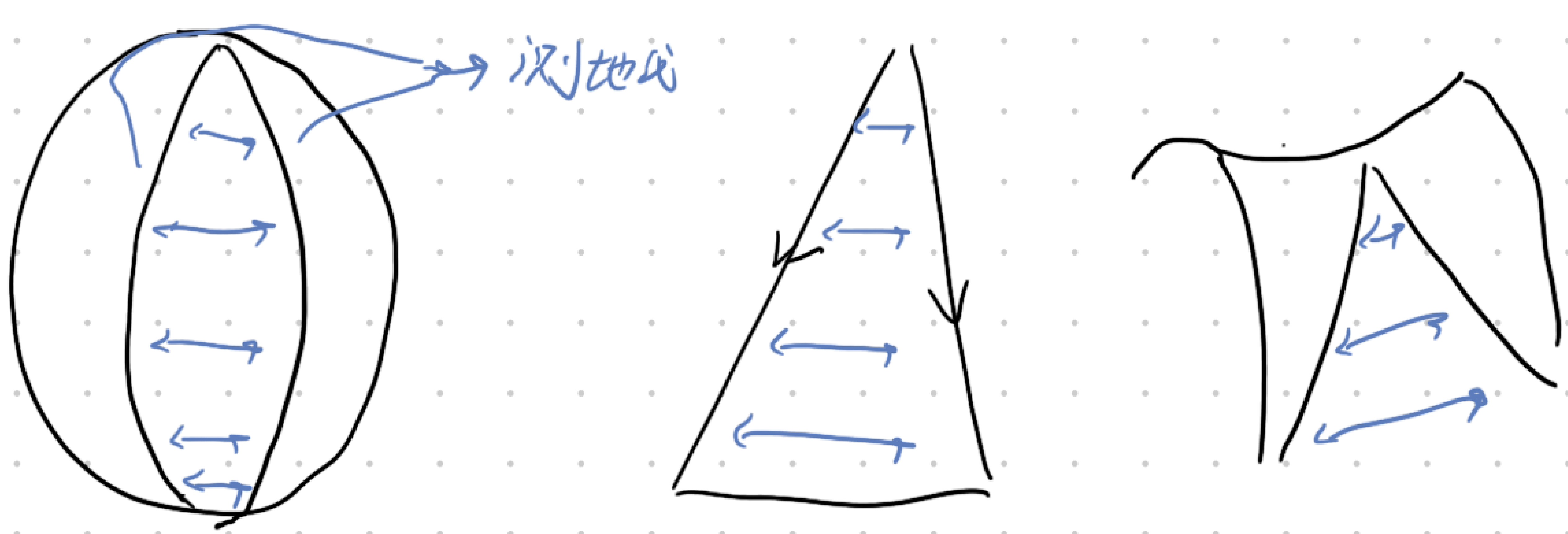
在某点 p 处 通过坐标变换, 可以局域地把 $g_{\mu\nu}$ 对角化 $g_{\mu\nu}=\mathrm{diag}(\underbrace{-1,\cdots,-1}_{s},\underbrace{+1,\cdots,+1}_{t})$ $s+t=n$.

$s=0$ 欧氏度规（黎曼流形） $s=1$ 洛伦兹度规.（伪黎曼流形）

S^2度规 $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ $ds^2 = dx^2 + dy^2 + \dfrac{(xdx+ydy)^2}{a^2-x^2-y^2}$ $ds^2 = \dfrac{a^2 d\rho^2}{a^2-\rho^2} + \rho^2 d\phi^2$.

应用：若 $g_{\mu\nu}$ 是对角的 $g_{\mu\nu} = \mathrm{diag}(g_{11}, g_{22})$ 面元：$dA = \sqrt{|g_{11}g_{22}|}\, dx^1 dx^2 = \sqrt{|g|}\, dx^1 dx^2$ $(g = \det(g))$

空间的曲率（正，零，负）. （外曲率）



正曲率 零曲率 负曲率

(6) n-dim 流形体元（$g_{\mu\nu}$ 对角） $d^nV = \sqrt{|g_{11}\cdots g_{nn}|}\, dx^1\cdots dx^n = \sqrt{|\det(g)|}\, dx^1\cdots dx^n$

坐标变换 $x^\mu \to x'^\mu$ $(d^nV)' = \sqrt{|g'|}\, dx'^1\cdots dx'^n = \sqrt{|g'|}\, J\, dx^1\cdots dx^n$ $J^\mu{}_\nu$的逆

$dx'^\mu = \left(\dfrac{\partial x'^\mu}{\partial x^\nu}\right) dx^\nu$ $g'_{\mu\nu} = J_\mu{}^\sigma J_\nu{}^\rho g_{\sigma\rho}$

$J^\mu{}_\nu$ 雅可比矩阵

$\sqrt{|g'|} = \sqrt{|g|}\,(\det J^\mu{}_\nu)^{-1} = \sqrt{|g|}\, J^{-1}$ $\Rightarrow d^nV$ 是不依赖于坐标选取的.

黎曼法坐标（RNC）　　EEP：任意时空可局域地看作惯性系（locally $g_{\mu\nu} \sim \eta_{\mu\nu}$）

(7)

在点 p 邻域，存在一个坐标变换，使 $g_{\mu\nu}|_p = \eta_{\mu\nu}$，$\partial_\sigma g_{\mu\nu}|_p = 0$　　($\partial_\sigma \partial_\rho g_{\mu\nu}|_p \neq 0$ in general)

"Proof"：$x^\mu \to x'^\mu$　　$g'_{\rho\sigma} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} g_{\mu\nu}$　　设 $x^\mu(p) = x'^\mu(p) = 0$

p 点邻域：$x^\mu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\Big|_p x'^\nu + \cdots$

$$g_{\mu\nu} = g_{\mu\nu}|_p + (\partial g)|_p \left(\frac{\partial x}{\partial x'}\right) x' + \cdots$$

$$g'_{\rho\sigma} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} g_{\mu\nu} + \cdots$$

(8)

微分形式（Differential Forms）　　p-form on n-dim manifold：$(0,p)$-tensor 全反对称

$$\sum_{\pi(\mu)} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

$$\boxed{A_p = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots dx^{\mu_p}}$$

M 上全体 p-forms 构成 $\Lambda^p(M)$　　$\dim \Lambda^p = \binom{n}{p} = \frac{n!}{p!(n-p)!}$

(1) wedge product　　p-form A 和 q-form B $\longrightarrow$ $(p+q)$-form $A \wedge B$

$$\boxed{(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}}$$

e.g. 1-form A & B：$(A \wedge B)_{\mu\nu} = 2 A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$

$$(B \wedge A)_{\mu_1 \cdots \mu_{p+q}} = (-)^{pq} (A \wedge B)_{\mu_1 \cdots \mu_{p+q}}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C) = \frac{(p+q+r)!}{p!\,q!\,r!} A_{[\cdots} B_{\cdots} C_{\cdots]}$$

(2) $d: \Lambda^p(M) \longrightarrow \Lambda^{p+1}(M)$ $\boxed{(dA)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}}$

for 0-form ϕ: $(d\phi)_\mu = \partial_\mu \phi$ 1-form A_μ: $(dA)_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

2-form $B_{\mu\nu}$: $(dB)_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$

for any p-form A: $d(dA) = 0$ 即 $d^2 = 0$ $(d_{p+1} d_p = 0)$

If $dA = 0 \Rightarrow A$ is closed. If $A = d\theta \Rightarrow A$ is exact. (For 1-form A, guage transformation $A_\mu \to A_\mu + \partial_\mu\lambda$, $A \to A + d\lambda$)

all exact forms are closed ($d^2\theta = 0$)

closed $\overset{?}{\nRightarrow}$ exact (Depends on 时空拓扑)

$$\longrightarrow \Lambda^{p-1}(M) \xrightarrow{d_{p-1}} \Lambda^p(M) \xrightarrow{d_p} \Lambda^{p+1}(M)$$

closed: $\mathrm{Ker}(d_p)$ exact p-form: $\mathrm{im}(d_{p-1})$ $H^p_{dR}(M, \mathbb{R}) = \dfrac{\mathrm{Ker}(d_p)}{\mathrm{im}(d_{p-1})}$

Liebniz rule: p-form A q-form B $d(A\wedge B) = dA\wedge B + (-)^p A\wedge dB$

dA is always a tensor ($\partial_\mu A_\nu$ is not)

Hodge star $*: \Lambda^p(M) \longrightarrow \Lambda^{n-p}(M)$

$$\boxed{(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}} A_{\nu_1\cdots\nu_p}}$$

$*(*A) = (-)^{s+p(n-p)} A$ s 是度规对角化后 (-1) 的个数.

Levi-Civita tensor in SR $\tilde{\epsilon}_{\mu_1 \cdots \mu_n} = \begin{cases} +1 & \text{even parm} \\ -1 & \text{odd} \\ 0 & \text{else} \end{cases}$

for any $n\times n$ matrix M $\quad \tilde{\epsilon}_{\mu_1' \cdots \mu_n'} |M| = \tilde{\epsilon}_{\mu_1 \cdots \mu_n} M^{\mu_1}{}_{\mu_1'} M^{\mu_2}{}_{\mu_2'} \cdots M^{\mu_n}{}_{\mu_n'}$

for $x^\mu \to x'^\mu$ 雅可比矩阵 $M^\mu{}_\nu = \dfrac{\partial x^\mu}{\partial x'^\nu}$

$$\tilde{\epsilon}_{\mu_1' \cdots \mu_n'} = |M|^{-1} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots\cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$ is not a tensor (is a tensor density 张量密度)

$\epsilon_{\mu_1 \cdots \mu_n} = \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \sqrt{|g|}$ is a tensor

$dx^0 \cdots dx^{n-1} = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$ is not a tensor.

but $\varepsilon = \epsilon_{12\cdots n}\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{|g|}\, dx^0 \cdots dx^{n-1}$ is a tensor

$$\varepsilon = \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \frac{\sqrt{|g|}}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \sqrt{|g|}\, d^n x$$ 在坐标变换下按张量变化

$$\int_M \mathcal{L}\, d^n x = \int_M \underbrace{\hat{\mathcal{L}}}_{\text{scalar}} \underbrace{\sqrt{|g|}\, d^n x}_{\text{scalar volumn form}} = \int_M \hat{\mathcal{L}} \underbrace{\varepsilon}_{n\text{-form}}$$

$\mathcal{L} = \hat{\mathcal{L}} \sqrt{|g|}$

More generally, for n-form $\omega^{(n)}$ on M $\int_M \omega^{(n)} \in \mathbb{R}$.

Generalized stokes theorem: for $(n-1)$ form $\omega^{(n-1)}$ a manifold M with boundary ∂M. $\dim M = n$, $\dim(\partial M) = n-1$

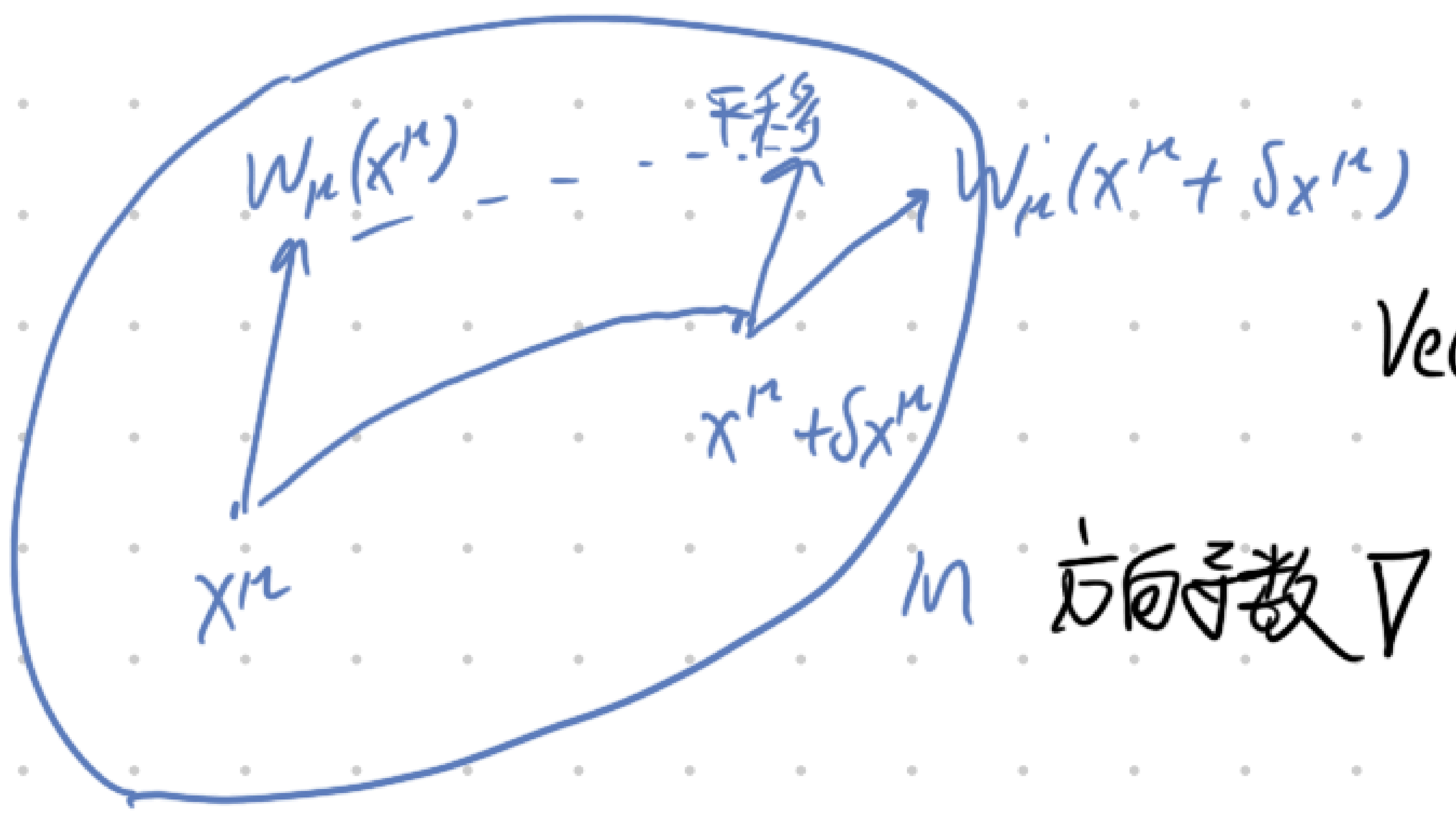
$$\int_M d\omega^{(n-1)} = \int_{\partial M} \omega^{(n-1)}$$

(9) Connection (联络)

$\partial_\mu W_\nu$ is not a tensor under $x^\mu \to x^{\mu'}$

$$\frac{\partial W'_\nu}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x^{\nu'}} W_\nu\right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial W_\nu}{\partial x^\mu} + \overbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^\nu}{\partial x^\mu \partial x^{\nu'}} W_\nu}^{\text{破坏 } \partial_\mu W_\nu \text{ 的tensor性}}$$

但 $(dW)_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$ is a tensor



Def modified derivative ∇_μ st $\nabla_\mu W_\nu$ is a tensor.

Vector field : $\hat{V}$: for any $p \in M \to \hat{V}(p) \in T_pM$

方向导数 $\nabla \{\hat{X}, \hat{Y}\} \longrightarrow \nabla_X \hat{Y} \longrightarrow$ output

$\underbrace{\hat{X}, \hat{Y}}_{\text{2 vector fields}}$ 1 vector field

satisfy : 线性 $\nabla_{\hat{X}}(\hat{Y}+\hat{Z}) = \nabla_{\hat{X}}\hat{Y} + \nabla_{\hat{X}}\hat{Z}$

$\nabla_{(\hat{X}+\hat{Y})}\hat{Z} = \nabla_{\hat{X}}\hat{Z} + \nabla_{\hat{Y}}\hat{Z}$

$\nabla_{f\hat{X}}\hat{Y} = f\nabla_{\hat{X}}\hat{Y}$

$\nabla_{\hat{X}}(f\hat{Y}) = \hat{X}[f]\hat{Y} + f\nabla_{\hat{X}}\hat{Y}$

$\hat{X}[f] \to X^\mu \partial_\mu f$

$\nabla_\mu \hat{e}_\nu \equiv \nabla_{\hat{e}_\mu} \hat{e}_\nu \equiv \hat{e}_\lambda \Gamma^\lambda_{\mu\nu} \longleftarrow \partial_\mu \hat{e}_\nu$ in the λ-component

Example. $\mathbb{R}^2$ $ds^2 = d\rho^2 + \rho^2 d\phi^2$ $\partial_\rho \hat{e}_\rho = 0$ $\partial_\phi \hat{e}_\rho = \frac{1}{\rho}\hat{e}_\phi$ $\partial_\rho \hat{e}_\phi = \frac{1}{\rho}\hat{e}_\phi$ $\partial_\phi \hat{e}_\phi = -\rho \hat{e}_\rho$

$\Rightarrow \Gamma^\rho_{\rho\rho} = \Gamma^\phi_{\rho\rho} = 0$ $\Gamma^\phi_{\rho\phi} = \frac{1}{\rho}$ $\Gamma^\rho_{\rho\phi} = 0$ $\Gamma^\phi_{\phi\rho} = \frac{1}{\rho}$ $\Gamma^\rho_{\phi\rho} = 0$ $\Gamma^\rho_{\phi\phi} = -\rho$ $\Gamma^\phi_{\phi\phi} = 0$

$$\nabla_{\hat{V}} \hat{W} = V^\mu \nabla_{\hat{e}_\mu}(W^\nu \hat{e}_\nu) = V^\mu\left(\frac{\partial W^\lambda}{\partial x^\mu}\hat{e}_\lambda + W^\nu \nabla_{\hat{e}_\mu}\hat{e}_\nu\right) = V^\mu\left(\frac{\partial W^\lambda}{\partial x^\mu} + W^\nu \Gamma^\lambda_{\mu\nu}\right)\hat{e}_\lambda \equiv V^\mu(\nabla_\mu W^\lambda)\hat{e}_\lambda$$

联络导数 $$\boxed{\nabla_\mu W^\lambda = (\nabla_\mu \hat{W})^\lambda = \partial_\mu W^\lambda + W^\nu \Gamma^\lambda_{\mu\nu}}$$

On tensors: scalar (0-form) $\nabla_\mu \phi = \partial_\mu \phi$ $\qquad \nabla_\mu (V^\nu W_\nu) = V^\nu \partial_\mu W_\nu + (\partial_\mu V^\nu) W_\nu = (\nabla_\mu V^\nu) W_\nu + V^\nu (\nabla_\mu W_\nu)$

1-form W_μ $\quad \nabla_\mu W_\nu = \partial_\mu W_\nu - W_\lambda \Gamma^\lambda_{\mu\nu}$

(k,l) tensor 个 $\quad \nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\lambda \cdots \nu_l} - \cdots$

e.g. $\nabla_\sigma \delta^\mu_\nu = \partial_\sigma \delta^\mu_\nu + \Gamma^\mu_{\sigma\lambda} \delta^\lambda_\nu - \Gamma^\lambda_{\sigma\nu} \delta^\mu_\lambda = 0$

$\boxed{\Gamma^\mu_{\nu\rho} \text{ is NOT a tensor}}$ $\qquad \Gamma^{\nu'}_{\mu'\rho'} = \Gamma^\nu_{\mu\rho} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\rho}{\partial x^{\rho'}} - \frac{\partial^2 x^{\nu'}}{\partial x^{\mu'} \partial x^{\rho'}}$

$T^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho} - \Gamma^\nu_{\rho\mu}$ is a tensor

Unique $\Gamma^\mu_{\nu\rho}$ with additional conditions (Christoffel connection).

(1) 无挠 $\quad T^\rho_{\mu\nu} = 0 \Rightarrow \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

(2) 度规相容性 $\quad \nabla_\rho g_{\mu\nu} = 0 \qquad \nabla_\mu V_\nu = \nabla_\mu (g_{\nu\rho} V^\rho) = \underbrace{(\nabla_\mu g_{\nu\rho})}_{=0} V^\rho + g_{\nu\rho} \overbrace{\nabla_\mu V^\rho}^{\text{tensor}} = \nabla_\mu V^\nu$

$$\begin{cases} \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \\ \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \\ \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \end{cases} \Rightarrow \boxed{\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}$$

散度 $\quad \nabla_\mu V^\mu \equiv \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu) \qquad \left(\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|} \right)$

For a general $n\times n$ matrix M, δM. $\delta \ln \mathrm{Det} M = \ln \mathrm{Det}(M+\delta M) - \ln \mathrm{Det} M = \ln \mathrm{Det}\, M^{-1}(M+\delta M)$

$$\delta \ln \mathrm{Det} M = \ln \mathrm{Det}(1+M^{-1}\delta M) = \ln(1+\mathrm{Tr}(M^{-1}\delta M)) + O(\delta M^2) = \mathrm{Tr}(M^{-1}\delta M) + O(\delta M^2)$$

$M_{\mu\nu} = g_{\mu\nu}$ $(M^{-1})_{\mu\nu} = g^{\mu\nu}$ $\Rightarrow \frac{1}{|g|}\partial_\nu |g| = \partial_\nu \ln|g| = g^{\mu\rho}\partial_\nu g_{\mu\rho} = 2\Gamma^\mu_{\mu\nu} = \frac{2}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}$

$$\nabla^2\phi = \nabla_\mu(\partial^\mu\phi) = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\,\partial^\mu\phi)$$

10. Parallel transport & geodesics (测地线)

$\hat{V}$ 沿曲线 $x^\mu(\lambda)$ 的方向导数 $\frac{D\hat{V}}{d\lambda} = \frac{dV^\mu}{d\lambda}\hat{e}_\mu + V^\mu\frac{D\hat{e}_\mu}{d\lambda} = \frac{dV^\mu}{d\lambda}\hat{e}_\mu + V^\mu\frac{dx^\nu}{d\lambda}\nabla_\nu\hat{e}_\mu = \left(\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}V^\sigma\frac{dx^\nu}{d\lambda}\right)\hat{e}_\mu \equiv \frac{DV^\mu}{d\lambda}\hat{e}_\mu$

$$\frac{DV^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}\left(\frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma}V^\sigma\right) = \frac{dx^\nu}{d\lambda}\nabla_\nu V^\mu$$

沿曲线平行移动: $\frac{D\hat{V}}{d\lambda} = 0 \iff \frac{dx^\mu}{d\lambda}\nabla_\mu V^\nu = 0$

对 (k,l) 张量 $\left(\frac{DT}{d\lambda}\right)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$ 平行移动 $=0$

$g_{\mu\nu}$ 的平行移动: $\frac{Dg_{\mu\nu}}{d\lambda} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma g_{\mu\nu} = 0$ 相容性

假设 $\hat{V}, \hat{W}$ 都沿 C 平行移动 $\Rightarrow \frac{D}{d\lambda}(V^\mu W_\mu) = \frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) = 0$ 即沿 C 内积不变.

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}V^\rho = 0$$

一阶 ODE，可解，确定一点 $V^\mu(0)$ 后有唯一解.

Geodesics 测地线 定义：切矢量平行移动 $\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda}=0 \Rightarrow \boxed{\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}=0}$ （测地线方程）

验证：作用量 $\tau=\int_{\lambda_A}^{\lambda_B} L\, d\lambda$ 取极值 （$L=\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \sqrt{-g_{\mu\nu}\dot x^\mu \dot x^\nu}$，类时曲线为例）

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot x^\mu}\right) = 0 \Leftrightarrow \frac{d}{d\lambda}\left(\frac{\partial L^2}{\partial \dot x^\mu}\right) - \frac{\partial L^2}{\partial x^\mu} = 2\frac{\partial L}{\partial \dot x^\mu}\frac{\partial L}{\partial \lambda}$$

$$\frac{d}{d\lambda}\left(2L\frac{\partial L}{\partial \dot x^\mu}\right) - 2L\frac{\partial L}{\partial x^\mu} = 2\frac{\partial L}{\partial \lambda}\frac{\partial L}{\partial \dot x^\mu}$$

$$\text{L.H.S.} = -2g_{\mu\rho}\ddot x^\rho - 2\partial_\sigma g_{\mu\rho}\dot x^\rho\dot x^\sigma + \partial_\mu g_{\rho\sigma}\dot x^\rho\dot x^\sigma = -2g_{\mu\rho}\ddot x^\rho - 2\dot x^\rho\dot x^\sigma g_{\mu\nu}\Gamma^\nu_{\rho\sigma}$$

$\text{R.H.S.} = 2\frac{\partial L}{\partial \dot x^\mu}\ddot\tau$ 选取参数 $\lambda=\tau$，R.H.S. $=0$

$\Rightarrow$ 测地线方程

Comment: (1) 测地线方程 $\xrightarrow{\text{反推}}$ $\Gamma^\mu_{\rho\sigma}$

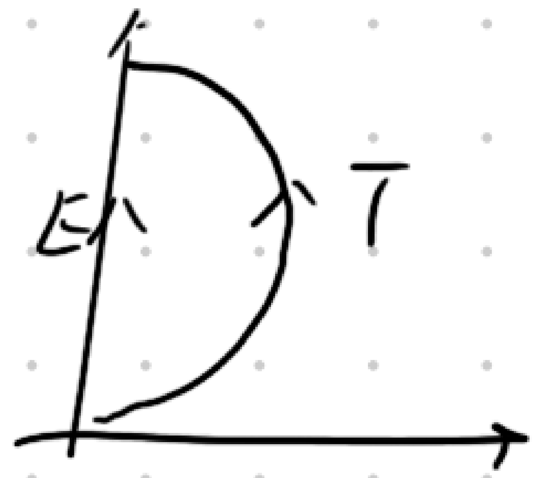
(2) GR中的 Newton 定律 不受力粒子走测地线运动.

$\hat a = \frac{D\hat u}{d\tau}$ $u^\mu = \frac{dx^\mu}{d\tau}$ $\hat a = 0 \Leftrightarrow \frac{D}{d\tau}\frac{dx^\mu}{d\tau}=0$ $\ddot x^\mu + \Gamma^\mu_{\rho\sigma}\dot x^\rho\dot x^\sigma = 0$

有外力：$\hat a = \hat f/m$ $\ddot x^\mu + \Gamma^\mu_{\rho\sigma}\dot x^\rho\dot x^\sigma = f^\mu/m$.

e.g. Lorentz force $qF^\mu{}_\nu\dot x^\nu$

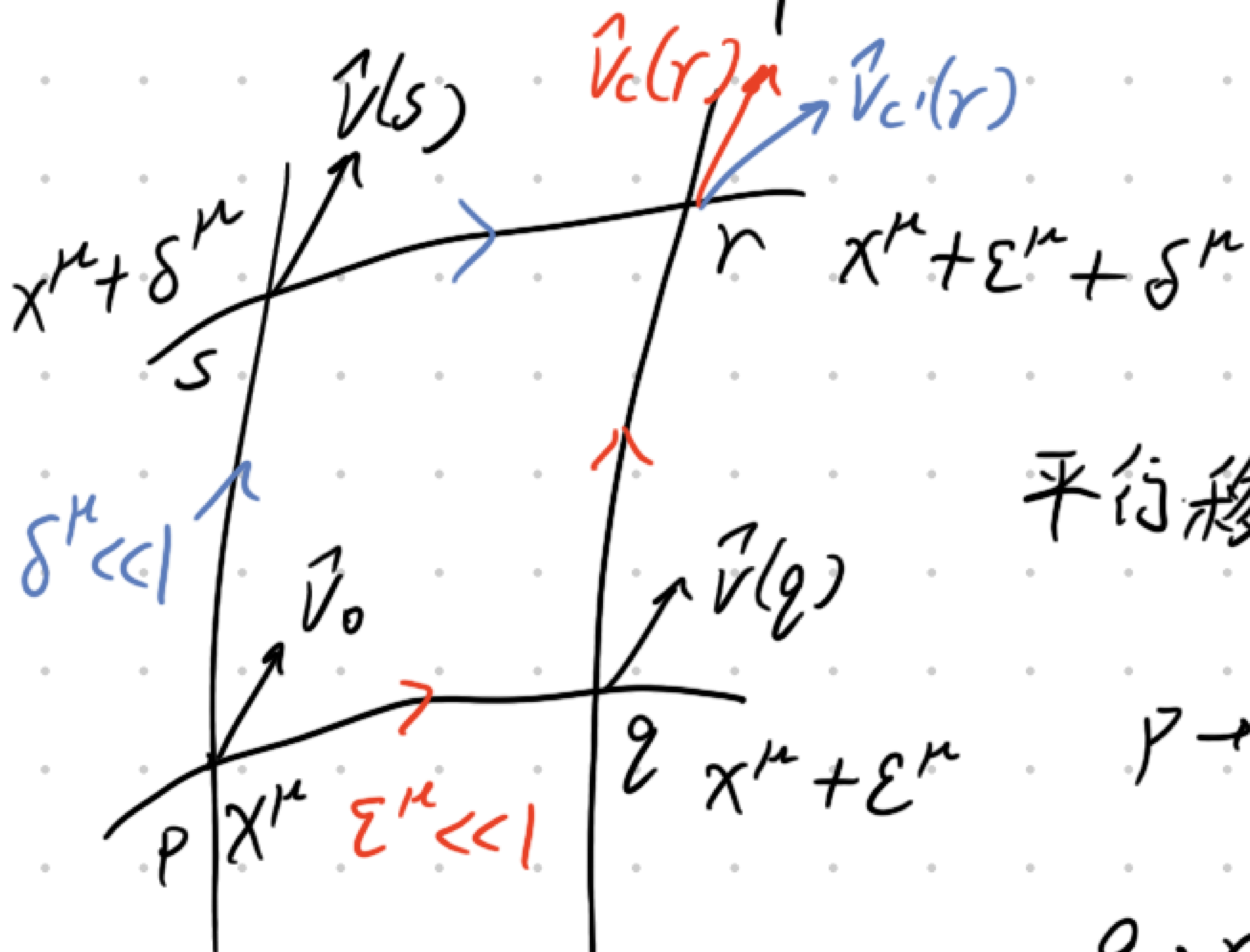
(3) 双生子佯谬 $\tau(T)<\tau(E)$ 测地线是T（E受支持力作用，不是测地线）



(4) 一点一切定一地 给定 $\gamma(0)=p$ 和 $\left.\frac{d\gamma(\tau)}{d\tau}\right|_p$ 定出一条测地线.

(5) 测地线完备性. （奇异点）

11. Curvature 曲率



$C: p\to q\to r$ $\quad \vec V_0 \to \vec V(q) \to \vec V_C(r)$

$C': p\to s\to r$ $\quad \vec V_0 \to \vec V(s) \to \vec V_{C'}(r)$ （假定路径在r处闭合） $(\varepsilon^\mu,\delta^\mu\ll1)$

平行移动 $\frac{dx^\nu}{d\lambda}\nabla_\nu V^\mu = (\partial_\nu V^\mu+\Gamma^\mu_{\nu\sigma}V^\sigma)\frac{dx^\nu}{d\lambda}=0 \Rightarrow \int\partial_\nu V^\mu dx^\nu = -\int\Gamma^\mu_{\nu\sigma}V^\sigma dx^\nu$

$p\to q$: $\frac{V^\mu(q)-V_0^\mu}{\varepsilon^\nu} = -\Gamma^\mu_{\nu\sigma}(p)V_0^\sigma \qquad V^\mu(q)=V_0^\mu-\Gamma^\mu_{\nu\sigma}(p)\varepsilon^\nu V_0^\sigma$

$q\to r$: $\frac{V_C^\mu(r)-V^\mu(q)}{\delta^\nu} = -\Gamma^\mu_{\nu\kappa}(q)V^\kappa(q) \qquad V_C^\mu(r)=V^\mu(q)-\Gamma^\mu_{\nu\kappa}(q)\delta^\nu V^\kappa(q)$

$$\Rightarrow V_C^\mu(r) = V_0^\mu - \Gamma^\mu_{\nu\kappa}(p)V_0^\kappa\varepsilon^\nu - (V_0^\kappa - \Gamma^\kappa_{\sigma\rho}(p)V_0^\rho\varepsilon^\sigma)(\Gamma^\mu_{\nu\kappa}(p)+\partial_\lambda\Gamma^\mu_{\nu\kappa}(p)\varepsilon^\lambda)\delta^\nu$$

$$= V_0^\mu - \Gamma^\mu_{\nu\kappa}(p)V_0^\kappa\varepsilon^\nu - \Gamma^\mu_{\nu\kappa}(p)V_0^\kappa\delta^\nu \underbrace{- V_0^\kappa\partial_\lambda\Gamma^\mu_{\nu\kappa}(p)\varepsilon^\lambda\delta^\nu + V_0^\rho\Gamma^\kappa_{\sigma\rho}(p)\Gamma^\mu_{\nu\kappa}(p)\varepsilon^\sigma\delta^\nu}_{-V_0^\kappa[\partial_\lambda\Gamma^\mu_{\nu\kappa}(p)-\Gamma^\rho_{\lambda\kappa}\Gamma^\mu_{\nu\rho}(p)]\varepsilon^\lambda\delta^\nu} + O(\varepsilon^3)$$

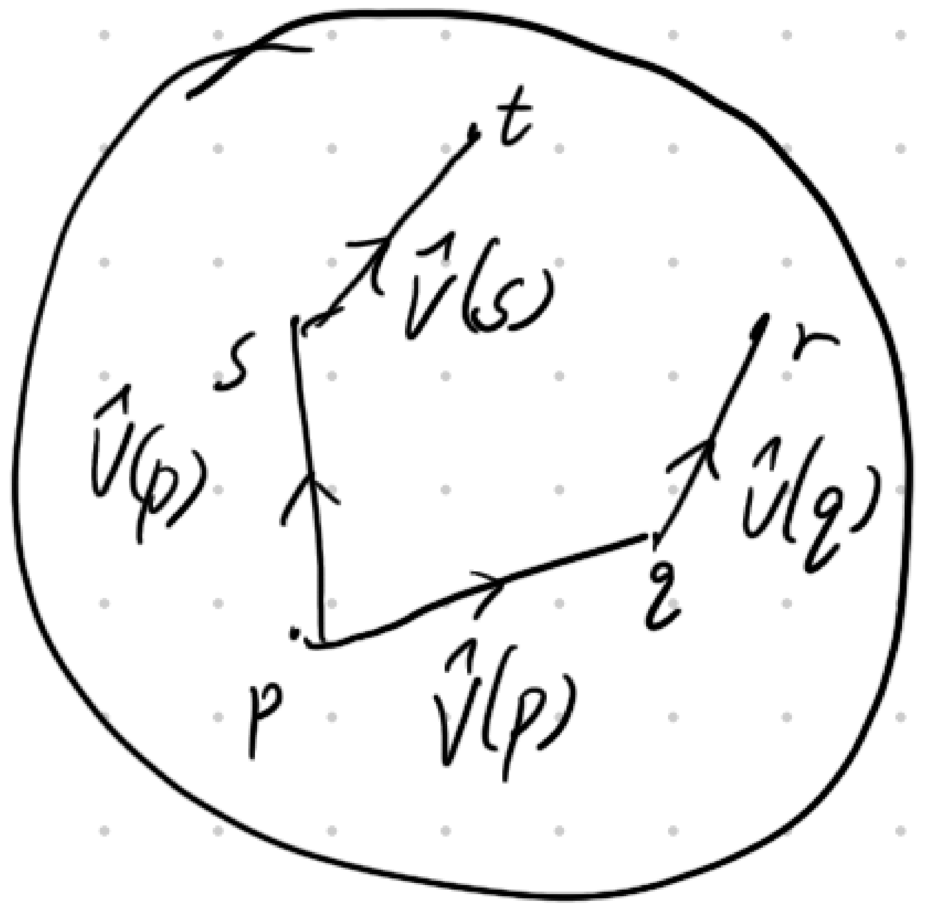
$\varepsilon \longleftrightarrow \delta \Rightarrow V_{c'}^{\mu}(r)$ $\quad V_{c'}^{\mu}(r) - V_c^{\mu}(r) \triangleq V_0^{\kappa} R^{\mu}{}_{\kappa\lambda\nu}(p)\,\varepsilon^{\lambda}\delta^{\nu}$

黎曼曲率张量 $R^{\mu}{}_{\kappa\lambda\nu} = \partial_\lambda \Gamma^{\mu}_{\nu\kappa} - \partial_\nu \Gamma^{\mu}_{\lambda\kappa} + \Gamma^{\rho}_{\nu\kappa}\Gamma^{\mu}_{\lambda\rho} - \Gamma^{\rho}_{\lambda\kappa}\Gamma^{\mu}_{\nu\rho}$

关于$\{\lambda,\nu\}$反对称、

$R^{\mu}{}_{\kappa\lambda\nu} = -R^{\mu}{}_{\kappa\nu\lambda}$

Lie 括号 $[\hat{U},\hat{V}]^{\mu} = U^{\nu}\partial_\nu V^{\mu} - V^{\nu}\partial_\nu U^{\mu}$



parallel transport of vector $\hat{w}$: $p \to q \to r \to t \to s \to p$

$\hat{w}(q) = (1 - \delta\nabla_{\hat{V}} + \frac{1}{2}\delta^2\nabla_{\hat{V}}\nabla_{\hat{V}})\,\hat{w}(p)$

$\hat{w}(r) = (1 - \varepsilon\nabla_{\hat{U}} + \frac{1}{2}\varepsilon^2\nabla_{\hat{U}}\nabla_{\hat{U}})\,\hat{w}(q)$

$\hat{w}(t) = (1 - \delta\varepsilon\,\nabla_{[\hat{U},\hat{V}]})\,\hat{w}(r)$

$\hat{w}(s) = (1 + \delta\nabla_{\hat{V}} + \frac{1}{2}\delta^2\nabla_{\hat{V}}\nabla_{\hat{V}})\,\hat{w}(t)$

$\hat{w}'(p) = (1 + \varepsilon\nabla_{\hat{U}} + \frac{1}{2}\varepsilon^2\nabla_{\hat{U}}\nabla_{\hat{U}})\,\hat{w}(s)$

$\delta W = \hat{w}'(p) - \hat{w}(p) = \left([\nabla_{\hat{U}},\nabla_{\hat{V}}] - \nabla_{[\hat{U},\hat{V}]}\right)\hat{w}(p)\cdot\delta\varepsilon$

$\hat{R}(\hat{X},\hat{Y},\hat{Z}) = \nabla_{\hat{X}}\nabla_{\hat{Y}}\hat{Z} - \nabla_{\hat{Y}}\nabla_{\hat{X}}\hat{Z} - \nabla_{[\hat{X},\hat{Y}]}\hat{Z}$

$[\nabla_\rho,\nabla_\sigma]X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = R^{\mu_1}{}_{\lambda\rho\sigma}X^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \cdots + R^{\lambda}{}_{\nu_1\rho\sigma}X^{\mu_1\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} + \cdots$

R对称性：

$R^{\mu}{}_{\nu\rho\sigma} = -R^{\mu}{}_{\nu\sigma\rho}$

$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$

$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$R_{\rho[\sigma\mu\nu]} = 0$

$R_{\mu\nu\rho\sigma}$ 独立分量：$\frac{1}{12}n^2(n^2-1)$

$n=4$，20个独立分量.

$n=3$，6个独立分量

$n=2$，1个独立分量.

$\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0$

$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma} + \partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho})$

Ricci tensor (里奇张量) $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\rho} R_{\mu\lambda\nu\rho} = g^{\lambda\rho} R_{\lambda\mu\rho\nu}$ $R_{\mu\nu} = R_{\nu\mu}$

Ricci scalar (里奇标量) $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$.

2维 $R_{\mu\nu\rho\sigma}$ 独立分量为1 $R_{\mu\nu\rho\sigma} = \frac{1}{2} R (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma})$

Einstein tensor (爱因斯坦张量) $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ satisfy: $\nabla^{\mu} G_{\mu\nu} = 0$

□ example of S^2 $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$ $\Gamma^{\mu}_{\nu\rho}$ 非零: $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$ $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$

$R^{\mu}{}_{\nu\rho\sigma}$ 只有一项非零: $R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$ $R_{\theta\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta = R_{\phi\theta\phi\theta}$

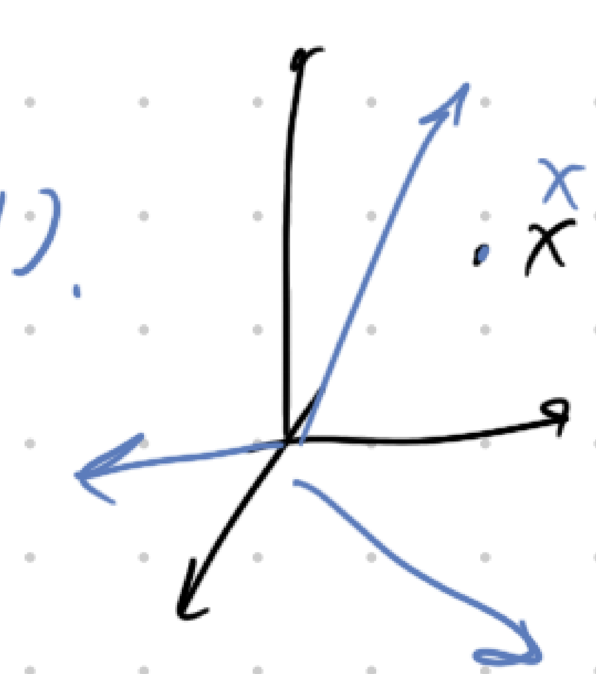
$R_{\theta\theta} = g^{\phi\phi} R_{\theta\phi\theta\phi} = 1$ $R_{\phi\phi} = g^{\theta\theta} R_{\phi\theta\phi\theta} = \sin^2\theta$ $R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}$ 是常量 (常曲率空间)

12. Killing vector and symmetry

直觉上: S^2 最大对称空间

$\boxed{\text{Isometry}}$ $\boxed{\text{保持度规形式不变的坐标变换}}$

$x^{\mu} \to x^{\mu'}$ $g'_{\alpha\beta}(x') = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu}(x)$ (same point on the manifold).



$a^2(d\theta^2 + \sin^2\theta d\phi^2)$

$\downarrow$

$a^2(d\theta'^2 + \sin^2\theta' d\phi'^2)$

Isometry: $g'_{\alpha\beta}(x') = g_{\alpha\beta}(x')$

无穷小生成元 $x'^{\alpha} = x^{\alpha} + \varepsilon \xi^{\alpha}(x) \quad \varepsilon \ll 1$ $\qquad g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}(x') = (\delta^{\alpha}_{\mu} + \varepsilon \partial_{\mu}\xi^{\alpha})(\delta^{\beta}_{\nu} + \varepsilon \partial_{\nu}\xi^{\beta})(g'_{\alpha\beta}(x) + \varepsilon \xi^{\gamma}\partial_{\gamma} g'_{\alpha\beta}(x))$

即 $g_{\mu\nu}(x) = g'_{\mu\nu}(x) + \varepsilon[\partial_{\mu}\xi^{\alpha} g'_{\alpha\nu} + \partial_{\nu}\xi^{\beta} g'_{\mu\beta} + \xi^{\gamma}\partial_{\gamma} g'_{\mu\nu}] + O(\varepsilon^2)$

$g_{\mu\nu}(x) = g'_{\mu\nu}(x) \Rightarrow \partial_{\mu}\xi_{\nu} - \xi^{\alpha}\partial_{\mu} g_{\alpha\nu} + \partial_{\nu}\xi_{\mu} - \xi^{\beta}\partial_{\nu} g_{\mu\beta} + \xi^{\gamma}\partial_{\gamma} g_{\mu\nu} = 0$

$\Rightarrow 2\partial_{(\mu}\xi_{\nu)} - 2\xi_{\gamma}\Gamma^{\gamma}_{\mu\nu} = 0 \Rightarrow \boxed{\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0 \quad \text{或} \quad \nabla_{(\mu}\xi_{\nu)} = 0}$

(Killing 场 ξ_{μ})

- Solve Killing eq e.g. if $\frac{\partial g_{\mu\nu}}{\partial x^1} = 0$ (时空度规不依赖于某一坐标) 存在常 Killing 场 $\begin{cases} \xi^1 = \text{constant } b \\ \xi^{\lambda}\ (\lambda \neq 1) = 0 \end{cases}$

e.g. $\mathbb{R}^2$ $\quad ds^2 = dx^2 + dy^2$ $\quad$ Killing vector $\hat{X} = \frac{\partial}{\partial x} = (1,0)$ $\quad \hat{Y} = \frac{\partial}{\partial y} = (0,1)$

$ds^2 = dr^2 + r^2 d\theta^2$ $\qquad \hat{R} = \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

$\mathbb{R}^2$ 只有3个相互独立 Killing 场 ($\hat{X}, \hat{Y}, \hat{R}$ 构成李代数) 两个 Killing vector 的对易也是 Killing vector

$[\hat{X}, \hat{Y}]^{\mu} = X^{\sigma}\nabla_{\sigma}Y^{\mu} - Y^{\sigma}\nabla_{\sigma}X^{\mu} = [X^{\sigma}\partial_{\sigma}, Y^{\rho}\partial_{\rho}]^{\mu}$

$[\hat{X}, \hat{Y}] = 0 \qquad [\hat{R}, \hat{X}] = (-y\partial_x + x\partial_y)\partial_x - \partial_x(-y\partial_x + x\partial_y) = -\partial_y = -\hat{Y}$

$[\hat{R}, \hat{Y}] = \hat{X}$ $\quad$ 记为 iso(2) (2平移 + 1旋转)

最大对称空间: Killing vector 张成的最大李代数维度 $\boxed{n\text{-dim 流形最多 } \frac{n(n+1)}{2} \text{ 个独立 Killing 场}}$

e.g. $\mathbb{R}^n$ 有 n个平移 killing vector ∂_i $(i=1,\cdots,n)$ $\frac{n(n-1)}{2}$旋转：$x^i\partial_j - x^j\partial_i$ $\Rightarrow$ 最大对称空间

常曲率空间（例：S^2）也是最大对称空间 $R = \pm\frac{n(n-1)}{L^2}$ $R_{\mu\nu\rho\sigma} = \pm\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$

2维流形均满足以上关于 R 的关系式.

正曲率：dS (de Sitter spacetime) 负曲率：AdS (Anti de Sitter spacetime)

* Killing vector & conserved quantities

$\xi_\nu u^\nu$ 沿测地线（u^ν是切矢 $u^\nu = \frac{dx^\nu}{d\tau}$） 方向导数 $u^\mu\nabla_\mu(\xi_\nu u^\nu) = u^\mu\xi_\nu\nabla_\mu u^\nu + \underbrace{u^\mu u^\nu\nabla_\mu\xi_\nu}_{=u^\mu u^\nu\nabla_{(\mu}\xi_{\nu)}=0}$

$= \xi_\nu(u^\mu\nabla_\mu u^\nu)$

$$u^\mu\nabla_\mu u^\nu = u^\mu\partial_\mu u^\nu + u^\mu\Gamma^\nu_{\mu\rho}u^\rho = \frac{dx^\mu}{d\tau}\frac{\partial x^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\rho}u^\mu u^\rho$$

$$= \frac{du^\nu}{d\tau} + \Gamma^\nu_{\mu\rho}u^\mu u^\rho = 0 \quad \text{(测地线方程)}$$

即 **每个 killing vector 对应一个沿着测地线守恒量.** e.g. $\mathbb{R}^n$中 $p^\nu = mu^\nu$ $\xi\cdot p$ 沿测地线守恒 $\Rightarrow$ 动量守恒

In general, no killing vector $\Longrightarrow$ no conservation of energy/momentum

13. Einstein's equation

SR → GR 广义协变性原理 $\partial_\mu \to \nabla_\mu$ $\eta_{\mu\nu} \to g_{\mu\nu}$

物理方程用张量形式写出，突出其协变性

作用量在局域坐标变换下不变

e.g. 牛一 平直时空： $\frac{d^2x^\mu}{d\lambda^2}=0$ 换为 $\left(\frac{dx^\nu}{d\lambda}\frac{\partial}{\partial x^\nu}\right)\frac{dx^\mu}{d\lambda}=0 \to \frac{dx^\nu}{d\lambda}\nabla_\nu\frac{dx^\mu}{d\lambda}=0$ （$u^\nu\nabla_\nu u^\mu=0$，测地线方程）

能-动量守恒 $\partial_\mu T^{\mu\nu}=0$ $\to \nabla_\mu T^{\mu\nu}=0$

微分形式 $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ 变换为 $\nabla_\mu A_\nu-\nabla_\nu A_\mu$ 不变

EM $\partial_\mu F^{\mu\nu}=-4\pi J^\nu \Rightarrow \nabla_\mu F^{\mu\nu}=-4\pi J^\nu$

但： $\partial_\mu\partial_\nu T^{\rho\sigma}\to\nabla_\mu\nabla_\nu T^{\rho\sigma}$ (μ,ν)不再对称（不能作直接替换，视具体物理而定）

引力 $\nabla^2\Phi=4\pi G\rho$ $\Phi\to?$

弱引力场：指 $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ $|h_{\mu\nu}|\ll1$ 非相对论：指 $\frac{dx^i}{d\tau}\ll\frac{dt}{d\tau}$ 静态： $\partial_0 g_{\mu\nu}=0$

非相对论极限下测地线方程： $\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{00}=0$ $\left(\frac{dt}{d\tau}\approx1\right)$ 弱场 静态下：

$$\Gamma^\mu_{00}=\frac12 g^{\mu\nu}(\partial_0 g_{\nu0}+\partial_0 g_{0\nu}-\partial_\nu g_{00})=-\frac12 g^{\mu\nu}\partial_\nu g_{00}=-\frac12\eta^{\mu\nu}\partial_\nu h_{00}$$

$$\Rightarrow \frac{d^2x^\mu}{dt^2} = \frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} \qquad \frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} = -\partial_i\Phi \qquad h_{00} = -2\Phi \qquad g_{00} = -(1+2\Phi)$$

$$ds^2 \approx -(1+2\Phi)dt^2 + d\vec{r}^{\,2}$$

由此猜测 $(\nabla^2 g)_{\mu\nu} \sim T_{\mu\nu}$ 　 $R_{\mu\nu} = \kappa T_{\mu\nu}$? ✗ 　 $\nabla^\mu T_{\mu\nu} = 0$ 但 in general $\nabla^\mu R_{\mu\nu} \neq 0$

由 $\nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0$ 即 $\nabla^\mu G_{\mu\nu} = 0$ 　 $(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu})$

$\Rightarrow$ Einstein 方程 　 $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu}$ 　 或写为 $G_{\mu\nu} = \kappa T_{\mu\nu}$ 　 $\kappa = ?$

确定 κ: 弱场，非相对论，静态时：

理想流体极限 　 $T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p g_{\mu\nu}$ 　 尘埃 $p=0$ 　 $T_{\mu\nu} = \rho U_\mu U_\nu$

静止参考系 　 $U^\mu = (U^0, 0, 0, 0)$ 　 由 $g_{\mu\nu}U^\mu U^\nu = -1$ 　 $g_{00} = -1 + h_{00}$ 　 $g_{00}(U^0)^2 = -1 \Rightarrow U^0 = 1 + \frac{1}{2}h_{00}$

$T_{\mu\nu}$ 只有一个非零量 $T_{00} = \rho$ 　 $T = g^{00}T_{00} = -\rho$

$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu}$ 与 $g^{\mu\nu}$ 缩并 　 $R - 2R = \kappa T \Rightarrow -R = \kappa T$

$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu})$ 　 $R_{00} = \kappa(\rho - \frac{1}{2}(-\rho)(-1)) = \frac{1}{2}\kappa\rho$

$$R^i{}_{0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{j0} + \underbrace{\Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0}}_{O(h^2)} = \partial_j \Gamma^i_{00} = -\frac{1}{2}\partial_j\partial^i h_{00}$$

$$R_{00} = R^i{}_{0i0} = -\frac{1}{2}\nabla^2 h_{00} = \nabla^2\Phi \quad \Rightarrow \quad \nabla^2\Phi = \frac{1}{2}\kappa\rho \quad 则\ \kappa = 8\pi G$$

Einstein equation: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ $\quad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)$

$T_{\mu\nu} = 0 \Rightarrow$ 真空 Einstein 方程 $R_{\mu\nu} = 0$ (not flat $R^\mu{}_{\nu\rho\sigma} \neq 0$)

14. Einstein Hilbert action

$$S = \int \underbrace{L\, d^n x}_{\text{tensor density}} = \int \underbrace{\mathcal{L}}_{\text{scalar}} \underbrace{\sqrt{|g|}\, d^n x}_{\text{scalar}}$$

$\mathcal{L}$: $R,\ R^2,\ R_{\mu\nu}R^{\mu\nu},\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \dots$

量纲: $[g_{\mu\nu}]: M^0$ $\quad [R], [R_{\mu\nu}][R_{\mu\nu\rho\sigma}]: L^{-2} = M^2$ $\quad (\hbar = 1 \sim mr)$

$[\mathcal{L}]: L^{-n} = M^n$ $\quad$ n-dim spacetime $\quad$ force $F \sim -\frac{GMm}{r^{n-2}} \sim ma$

$$[G] = \frac{[a][r^{n-2}]}{[m]} = [L]^{-1}[L]^{n-2}[L] = [L]^{n-2} = [M]^{2-n} \qquad \frac{R}{G} \sim [M]^n$$

$$\boxed{S_H = \frac{1}{16\pi G}\int \sqrt{|g|}\, R\, d^n x}$$

$$\delta S = \frac{1}{16\pi G}\underbrace{\int \sqrt{|g|}\, g^{\mu\nu}\delta R_{\mu\nu}\, d^n x}_{\delta S_1} + \frac{1}{16\pi G}\underbrace{\int g^{\mu\nu}R_{\mu\nu}\,\delta\sqrt{|g|}\, d^n x}_{\delta S_2}$$

$$+ \frac{1}{16\pi G}\underbrace{\int d^n x \sqrt{-g}\,\delta g^{\mu\nu} R_{\mu\nu}}_{\delta S_3}$$

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu)$$

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \delta\Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - (\lambda \leftrightarrow \nu)$$

$\{x^\mu\} \to \{x'^\mu\}$ $\quad\delta\Gamma^{\rho}_{\nu\mu}$ is a tensor.

$$\nabla_\lambda \delta\Gamma^{\rho}_{\nu\mu} = \partial_\lambda \delta\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}$$

$$\delta R^{\rho}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu})$$

$$\delta S_1 = \int d^nx\sqrt{|g|}\,g^{\mu\nu}[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})] = \int d^nx\sqrt{|g|}\,\nabla_\lambda \overbrace{[g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^{\sigma}_{\sigma\mu}]}^{V^\lambda}$$

$$= 0$$

δS_2 $\quad\ln(\det M) = \mathrm{Tr}(\ln M)$ $\quad\dfrac{\delta\det M}{\det M} = \mathrm{Tr}(M^{-1}\delta M)$ $\quad\delta g = g\,g^{\mu\nu}\delta g_{\mu\nu} = -g\,g_{\mu\nu}\delta g^{\mu\nu}$

因为 $\delta(\overbrace{g^{\mu\nu}g_{\mu\nu}}^{=n}) = 0$ $\quad\delta\sqrt{|g|} = \dfrac{1}{2\sqrt{|g|}}\delta(-g) = -\dfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}$

$$\delta S_2 = \int d^nx\, R\left(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\right)$$

$\Rightarrow \delta S = \int d^nx\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}$ $\quad\Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$ 即真空 Einstein 方程

$$g^{\mu\nu}(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) = \underbrace{g^{\mu\nu}R_{\mu\nu}}_{=R} - \tfrac{n}{2}R = 0 \Rightarrow R = 0 \quad (n>2) \qquad R_{\mu\nu} = 0$$

有物质场： $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ $\qquad S = \frac{1}{16\pi G}\int d^n x \sqrt{-g}\, R + S_M$

定义 $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$

标量场 ϕ： $S_\phi = \int d^n x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - V(\phi)\right]$

δS_ϕ (varing $g^{\mu\nu}$) $= \int d^n x \sqrt{-g}\, \delta g^{\mu\nu}\left[-\frac{1}{2}(\nabla_\mu\phi)(\nabla_\nu\phi)\right] + \int d^n x\, \delta\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)$

$= \int d^n x \sqrt{-g}\, \delta g^{\mu\nu}\left[-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + (-\frac{1}{2}g_{\mu\nu})(-\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - V(\phi))\right]$

$\Rightarrow T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - g_{\mu\nu}V(\phi)$

简化到 Minkonski spacetime： $g_{\mu\nu} \to \eta_{\mu\nu}$ $\quad \nabla_\mu \to \partial_\mu$

$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\phi\partial_\rho\phi - \eta_{\mu\nu}V(\phi) = \partial_\mu\phi\partial_\nu\phi + \eta_{\mu\nu}L$ $\qquad L = -\frac{1}{2}\partial^\rho\phi\partial_\rho\phi - V(\phi)$

即 $T_{\mu\nu} = -\left[\frac{\partial L}{\partial(\partial^\mu\phi)}\partial_\nu\phi - \eta_{\mu\nu}L\right]$

引入宇宙学常数 Λ $\quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$S_\Lambda = \frac{1}{16\pi G} \int d^n x \sqrt{-g} \, (-2\Lambda)$$

当成物质：$T_{\mu\nu,\Lambda} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$ as a perfect fluid $T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$

$\Rightarrow \rho = -p \quad p = -\frac{\Lambda}{8\pi G}$

观测值 $\Lambda = 10^{-122} > 0$ $\quad p < 0 \Rightarrow$ 负压强

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15. Properties of EH action & Einstein equations

d-dim $[G] \sim L^{d-2} = M^{2-d}$ $\quad$ Plank length $l_p = G^{\frac{1}{d-2}}$ $\quad t_p = G^{\frac{1}{d-2}}$ $\quad m_p = G^{-\frac{1}{d-2}}$

$d=4$, $\quad l_p = t_p = \sqrt{G} \quad m_p = \frac{1}{\sqrt{G}}$

(SI) $l_p = \sqrt{\frac{\hbar G}{c^3}} \sim 1.616 \times 10^{-35} m$ $\quad t_p = \frac{l_p}{c} = 5.341 \times 10^{-44} s$ $\quad m_p = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} kg$

$E_p = m_p c^2 = 1.227 \times 10^{19} GeV = 1.956 \times 10^4 J$

表征：量子引力开始起作用的能标.

After considering quantum connections.

$$S \sim \frac{1}{16\pi G}\int \sqrt{-g}\, d^3x \left(R + \underbrace{\frac{R^2}{m_p^2} + \frac{\nabla R\, \nabla R}{m_p^4} + \cdots}_{\text{高能 范围}}\right) \qquad [R] = M^2$$

类比：$\frac{1}{1+x} \approx 1-x$

物理意义：看作扰动场

线性引力近似 $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu} \ll 1|$ $\quad g_{\mu\nu} g^{\mu\nu} = \eta_{\mu\nu}\eta^{\mu\nu} \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\lambda}) = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\lambda}) = \frac{1}{2}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}).$$

$$R^\mu{}_{\nu\rho\sigma} \approx \partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} = \frac{1}{2}\partial_\rho(\partial_\sigma h_\nu{}^\mu + \partial_\nu h_\sigma{}^\mu - \partial^\mu h_{\sigma\nu}) - \frac{1}{2}\partial_\sigma(\partial_\rho h_\nu{}^\mu + \partial_\nu h_\rho{}^\mu - \partial^\mu h_{\rho\nu}).$$

$$= \frac{1}{2}(\partial_\rho\partial_\nu h_\sigma{}^\mu + \partial_\sigma\partial^\mu h_{\nu\rho} - \partial_\rho\partial^\mu h_{\sigma\nu} - \partial_\sigma\partial_\nu h^\mu{}_\rho)$$

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \frac{1}{2}(\partial_\rho\partial_\mu h_\nu{}^\rho + \partial_\nu\partial^\rho h_{\mu\rho} - \partial_\rho\partial^\rho h_{\nu\mu} - \partial_\nu\partial_\mu h^\rho{}_\rho) \qquad h \equiv h^\rho{}_\rho \text{ : 迹}$$

$$R = R^\mu{}_\mu = \frac{1}{2}(\partial_\rho\partial^\mu h_\mu{}^\rho + \partial_\mu\partial^\rho h^\mu{}_\rho - \partial_\rho\partial^\rho h_\mu{}^\mu - \partial_\mu\partial^\mu h) = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h$$

Einstein tensor $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

$$= \frac{1}{2}(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h)$$

拉氏量 $\mathcal{L} = \frac{1}{2}\left[\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\rho\sigma}\partial_\rho h^\mu{}_\sigma + \frac{1}{2}\eta^{\mu\nu}(\partial_\rho h^{\rho\sigma})(\partial_\nu h_{\mu\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right]$

规范对称性 $x'^\mu = x^\mu + \xi^\mu(x)$

$$\begin{cases} \dfrac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu \xi^\mu \\ \dfrac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu{}_\mu + \partial_\mu \xi^\nu \end{cases}$$

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} = (\delta^\rho{}_\mu - \partial_\mu \xi^\rho)(\delta^\sigma{}_\nu - \partial_\nu \xi^\sigma)(\eta_{\rho\sigma} + h_{\rho\sigma})$$
$$= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu .$$

$$\Rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\delta_\xi R_{\mu\nu\rho\sigma} = \delta\left[\frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} - \partial_\rho\partial_\mu h_{\nu\sigma}) - (\rho\leftrightarrow\sigma)\right]$$
$$= \frac{1}{2}(-\partial_\rho\partial_\nu\partial_\mu\xi_\sigma \underbrace{-\partial_\rho\partial_\nu\partial_\sigma\xi_\mu}_{=0} + \partial_\rho\partial_\mu\partial_\nu\xi_\sigma \underbrace{+\partial_\rho\partial_\mu\partial_\sigma\xi_\nu}_{=0}) - (\rho\leftrightarrow\sigma).$$
$$= 0$$

=0

由于规范不变性，可以选择合适规范. (Gauge fixing)

调和 (harmonic) 规范 $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$ $\quad \eta^{\mu\nu}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}) = 0 \Rightarrow 2\partial^\nu h_\nu{}^\rho - \partial^\rho h = 0$

规范自由度并未完全固定： $2\partial^\nu(h_\nu{}^\rho - \partial_\nu\xi^\rho - \partial^\rho\xi_\nu) - \partial^\rho(h - 2\partial_\nu\xi^\nu) = 0$ $\longrightarrow$ 可以相差满足 $\Box\xi^\mu = 0$ 的规范变换.

调和规范 $\partial^\nu h_\nu{}^\rho = \frac{1}{2}\partial^\rho h$ 下,

$$G_{\mu\nu} = \frac{1}{2}\left(\frac{1}{2}\partial_\nu\partial_\mu h + \frac{1}{2}\partial_\mu\partial_\nu h - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h + \eta_{\mu\nu}\Box h\right)$$

$$= -\frac{1}{2}\left(\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h\right)$$

Einstein eq. $G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T_{\mu\nu}$.

真空情形 $R_{\mu\nu} = 0 \Rightarrow \Box h_{\mu\nu} = 0$ 波动方程（真空中的引力波）

$T_{\mu\nu} \neq 0$ Define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ satisfy $\partial_\mu \bar{h}^{\mu}{}_{\nu} = 0$ (harmonic gauge)

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

线性引力 $S_{EH} \sim \frac{1}{G}\int d^4x \, (\partial h)^2 + (\partial h)^2 h + (\cdots)$.

def. $\tilde{h} = \frac{h}{\sqrt{G}} = h m_p$

$$S_{EH} \sim \int d^4x \, \underbrace{(\partial\tilde{h})^2}_{M^4} + \underbrace{(\partial\tilde{h})^2\tilde{h}}_{M^5}\frac{1}{m_p} + \cdots$$

高阶能量发散

需不断引入 $\frac{1}{m_p}$ 项，引力不可重整

是低能有效理论.

Solution to Einstein eq.

(1) $T_{\mu\nu} = 0$, $R_{\mu\nu} = 0$

$D < 4$, $R_{\mu\nu\rho\sigma} = 0$

$D \geq 4$, e.g. Schwarzschild solution.

(2) $T_{\mu\nu} \neq 0$ energy conditions (能量条件) $T_{\mu\nu}$上的条件，使得Einstein方程上的解有良好性质 (weak, null, dominant, strong)

e.g. 弱能量条件(WEC) for any timelike observer t^μ, $T_{\mu\nu}t^\mu t^\nu \geq 0$

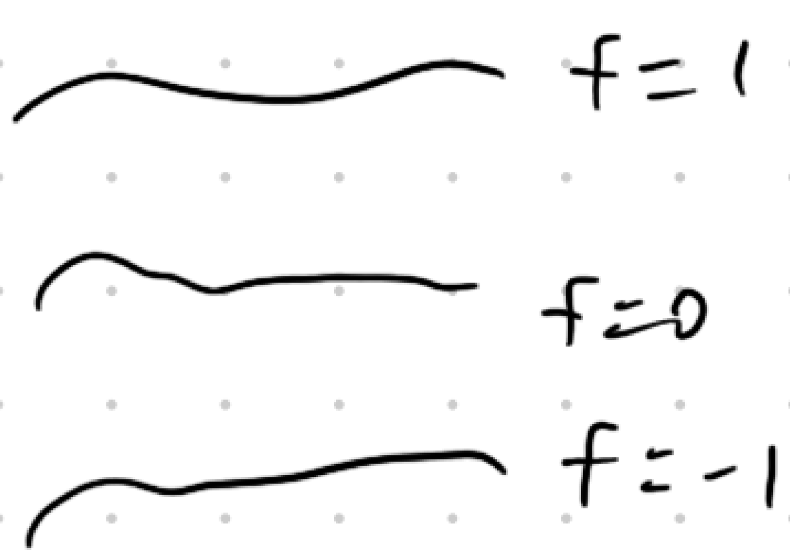
零能量条件(NEC) for any null vector l^μ, $T_{\mu\nu} l^\mu l^\nu \geq 0$

16. Schwarzschild solution (史瓦西解)

真空 Einstein 场方程 $R_{\mu\nu}=0$ 的严格解. 球对称 静态

- Stationary 稳态. 存在坐标系使 $\frac{\partial g_{\mu\nu}}{\partial x^0}=0$ ($g_{\mu\nu}$在某坐标系下不含时) $\Leftrightarrow$ 存在一个类时 Killing 场 $\xi^0=1$, $\xi^i=0$
- Static 静态. (1) 稳态 (2) $g_{0i}=0$ $\quad ds^2=g_{00}(dx^0)^2+g_{ij}\,dx^i dx^j$

$\Leftrightarrow$ 存在超曲面垂直的类时 Killing 场 (坐标无关的描述)



超曲面: $n-1$ 维空间, defined by $f(x^\mu)=C$ $\quad C\in\mathbb{R}$.

at a point $p\in\Sigma$, 法向是 $n_\mu \equiv \frac{\partial f}{\partial x^\mu}$

超曲面垂直: $X^\mu=\lambda(x)n^\mu = \lambda g^{\mu\nu}\frac{\partial f}{\partial x^\nu}$

$$X_\mu\partial_\nu X_\sigma = \lambda\frac{\partial f}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\left(\lambda\frac{\partial f}{\partial x^\sigma}\right) = \lambda \frac{\partial f}{\partial x^\mu}\frac{\partial\lambda}{\partial x^\nu}\frac{\partial f}{\partial x^\sigma} + \lambda^2\frac{\partial f}{\partial x^\mu}\frac{\partial^2 f}{\partial x^\nu\partial x^\sigma}$$

$X_{[\mu}\nabla_\nu X_{\sigma]} = X_{[\mu}\partial_\nu X_{\sigma]}=0 \Leftrightarrow X^\mu$ 是超曲面垂直的

在稳态时空, 类时 Killing 场 $X^\mu=\delta^\mu_0$ $\quad \partial_0 g_{\mu\nu}=0$.

$X_\mu = g_{\mu\nu} X^\nu = g_{\mu 0} \qquad |X|^2 = X_\mu X^\mu = g_{00}$

选取 $X_\mu = g_{00} \frac{\partial f}{\partial x^\mu} = g_{\mu 0}$ $\quad \mu = 0$ 时 $\Rightarrow \frac{\partial f}{\partial x^0} = 1 \Rightarrow f = x^0 + h(x^i)$ 且 $g_{i0} = g_{00} \partial_i h$

定义坐标变换 $x^0 \to x^{0'} = x^0 + h(x^i) \qquad x^i \to x^{i'} = x^i$.

$$g'_{00} = \frac{\partial x^\mu}{\partial x^{0'}} \frac{\partial x^\nu}{\partial x^{0'}} g_{\mu\nu} = g_{00}$$

$$g'_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \qquad \frac{\partial x^0}{\partial x^{0'}} = 1 \quad \frac{\partial x^0}{\partial x^{i'}} = -\partial_i h \quad \frac{\partial x^i}{\partial x^{i'}} = \delta^i_{i'} \quad \frac{\partial x^i}{\partial x^0} = 0$$

$$\partial_{0'} g'_{\mu'\nu'} = 0. \qquad g_{0'i'} = -\partial_i h \, g_{00} + g_{0i}$$

由于 $\partial_i h = \frac{g_{0i}}{g_{00}} \Rightarrow g_{0'i'} = 0$.

□ 史瓦西解 . 是真空中唯一的球对称解.

$R_{\mu\nu} = 0$ $\qquad$ 假定 $ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2$

$$R_{00} = [\partial_0^2 \beta + (\partial_0 \beta)^2 - 2\partial_0 \alpha \, \partial_0 \beta] + e^{2(\alpha-\beta)} [\partial_1^2 \alpha + (\partial_1 \alpha)^2 - \partial_1 \alpha \, \partial_1 \beta + \frac{2}{r} \partial_1 \alpha]$$

$$R_{11} = -[\partial_1^2 \alpha + (\partial_1 \alpha)^2 - \partial_1 \alpha \, \partial_1 \beta - \frac{2}{r} \partial_1 \beta] + e^{2(\beta-\alpha)} [\partial_0^2 \beta + (\partial_0 \beta)^2 - \partial_0 \alpha \, \partial_0 \beta]$$

$R_{01} = \frac{2}{r}\partial_0\beta \quad R_{22} = e^{-2\beta}[r(\partial_1\beta - \partial_1\alpha) - 1] + 1 \quad R_{33} = R_{22}\sin^2\theta.$ 上述 R 分量全为 0.

$R_{01} = 0 \Rightarrow \partial_0\beta = 0. \ \beta = \beta(r) \Rightarrow \partial_i(\alpha+\beta) = 0 \quad \alpha = -\beta(r) + g(t).$ 重新定义 t，可以使 $g(t) = 0$

静态解 $ds^2 = -e^{-2\beta(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2.$

$R_{22} = 0 \Rightarrow e^{-2\beta}(1 - 2r\partial_1\beta) = 1 \Rightarrow \frac{\partial}{\partial r}(re^{-2\beta}) = 1 \Rightarrow re^{-2\beta} = r + C \quad e^{-2\beta} = 1 + \frac{C}{r}.$

$$ds^2 = -(1+\frac{C}{r})dt^2 + (1+\frac{C}{r})^{-1}dr^2 + r^2 d\Omega^2.$$

考虑质量为 M 的星体附近时空，$C = ?$ 弱引力极限 $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h| \ll 1$

M

$g_{00} = -1 + h_{00} \quad h_{00} = -\frac{C}{r}.$

牛顿引力极限 $h_{00} = -2\frac{\Phi}{r}$ 牛顿引力势 $\Phi = -\frac{GM}{r}.$

$$\Rightarrow \boxed{ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2.}$$

定义 $R_s \equiv 2GM$ 史瓦西半径.

$r=0$ & $r=2GM$ 处奇异性？

$r=2GM$ 是坐标奇点（不是真奇点）（曲率标量不发散） $r=0$ 真正奇点、$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \to \infty \ (r\to 0)$

$r\to\infty,\quad ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2$ 渐近平坦

17. Geodesics in Schvazschild spacetime

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda}=0 \qquad \frac{d^2t}{d\lambda^2}+\frac{R_s}{r-R_s}\frac{dr}{d\lambda}\frac{dt}{d\lambda}=0 \quad \cdots$$

过于复杂 $\frac{d^2\theta}{d\lambda^2}+\frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}-\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2=0$

切矢量

对称性 (Killing vector K^μ) 沿着测地线 $\vec{u}\cdot\vec{K}$ 守恒 $\frac{d}{d\lambda}(u_\mu K^\mu)=0$

Static $\vec{\xi}=\partial_t$ $\quad \xi^\mu=(1,0,0,0) \quad \xi_\mu=g_{\mu\nu}\xi^\nu=\left(-\left(1-\frac{R_s}{r}\right),0,0,0\right) \quad u^\mu=\frac{dx^\mu}{d\lambda}$

$\Rightarrow E=-\xi_\mu u^\mu=\left(1-\frac{R_s}{r}\right)\frac{dt}{d\lambda}$ 定义：对质量为m的粒子，$r\to\infty$，$E=\frac{dt}{d\lambda}$ 观测能量 $E_0=-\vec{u}_{obs}\cdot m\vec{u}=mE$

对无质量粒子，E是 $r\to\infty$ 时光子能量.

SO(3) → 3个守恒角动量 $\quad \vec{\eta}=\partial_\phi \quad \eta^\mu=(0,0,0,1) \quad \eta_\mu=(0,0,0,r^2\sin^2\theta)$

$\Rightarrow L=\eta_\mu u^\mu=r^2\sin^2\theta\frac{d\phi}{d\lambda}$.

$$\frac{d\phi}{d\lambda}=\frac{L}{r^2\sin^2\theta} \Rightarrow r^2\frac{d^2\theta}{d\lambda^2}+2r\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}-\frac{L^2\cos\theta}{r^2\sin^3\theta}=0 \Rightarrow r^2\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right)=\frac{L^2\cos\theta}{\sin^3\theta}$$

$\Rightarrow \left(r^2\frac{d\theta}{d\lambda}\right)^2=-L^2\cot^2\theta+C$. $\quad \theta(\lambda_0)=\frac{\pi}{2}$，球对称性 $\frac{d\theta}{d\lambda}(\lambda_0)=0 \Rightarrow C=0$

且始终在 $\theta=\pi/2$ 平面上

此限制下 $E=\left(1-\frac{R_s}{r}\right)\frac{dt}{d\lambda}$，$L=r^2\frac{d\phi}{d\lambda}$ $\quad \vec{u}\cdot\vec{u}$ 沿着测地线守恒 (度规相容性)

$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\epsilon \qquad \epsilon = \begin{cases} 1 & \text{有质量粒子} \\ 0 & \text{无质量粒子} \end{cases}$

$-\left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon$

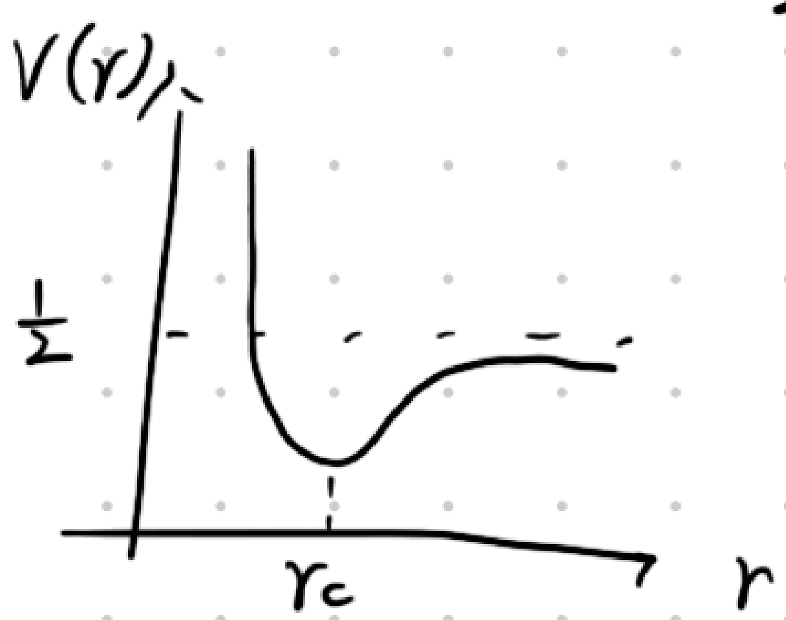
$\frac{dt}{d\lambda} = E\left(1-\frac{R_s}{r}\right)^{-1} \qquad \frac{d\phi}{d\lambda} = \frac{L}{r^2}$

$\Rightarrow -E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}+\epsilon\right) = 0 \Rightarrow \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \qquad V(r) = \frac{1}{2}\epsilon - \frac{GM}{r}\epsilon + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$

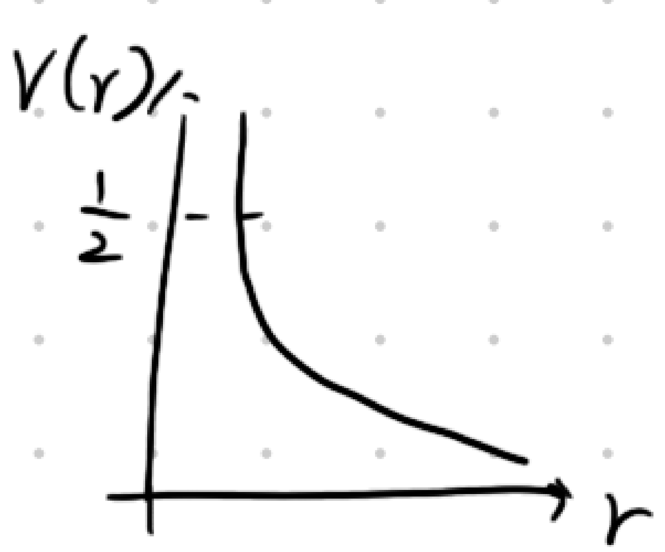
类比：牛顿力学下 $V(r)$，总能量 $\mathcal{E} = \frac{1}{2}E^2$，的粒子运动.

$\epsilon = 1 \qquad V(r) = \frac{1}{2} - \frac{GM}{r} + \underbrace{\frac{L^2}{2r^2}}_{\text{离心势}} - \underbrace{\frac{GML^2}{r^3}}_{\text{GR 项}}$

牛顿引力 $V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} \qquad r_c = \frac{L^2}{GM}$



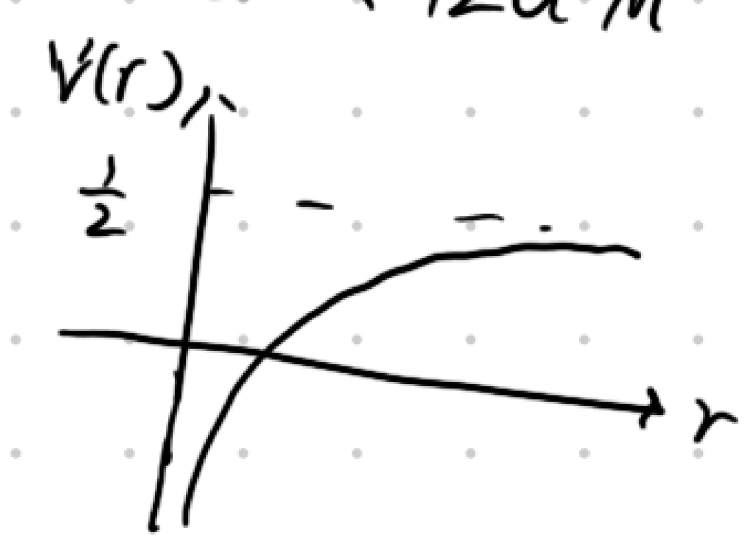
$\epsilon = 0$ 牛顿引力 $V(r) = \frac{L^2}{2r^2}$



GR中有质量粒子 $V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$

$\frac{\partial V(r)}{\partial r} = 0 \Rightarrow R_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2G^2M^2}}{R_s}$

(i) $L^2 < 12G^2M^2$ 无极值



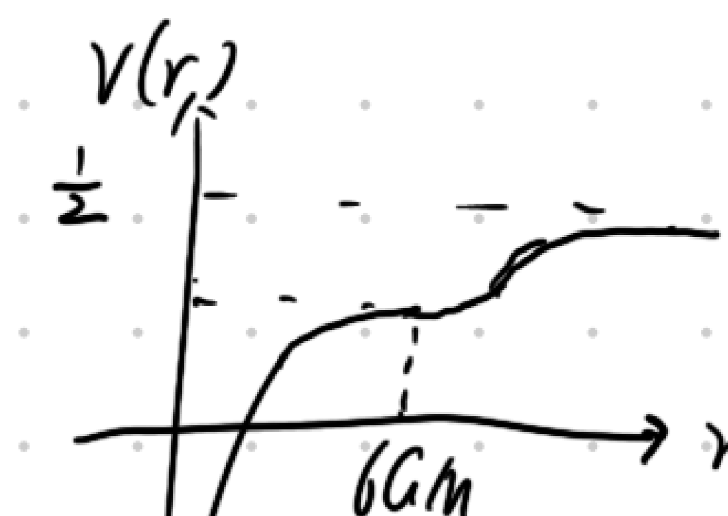
$\mathcal{E} > \frac{1}{2}$: escape to infinity

$\mathcal{E} < \frac{1}{2}$: falls to $r=0$

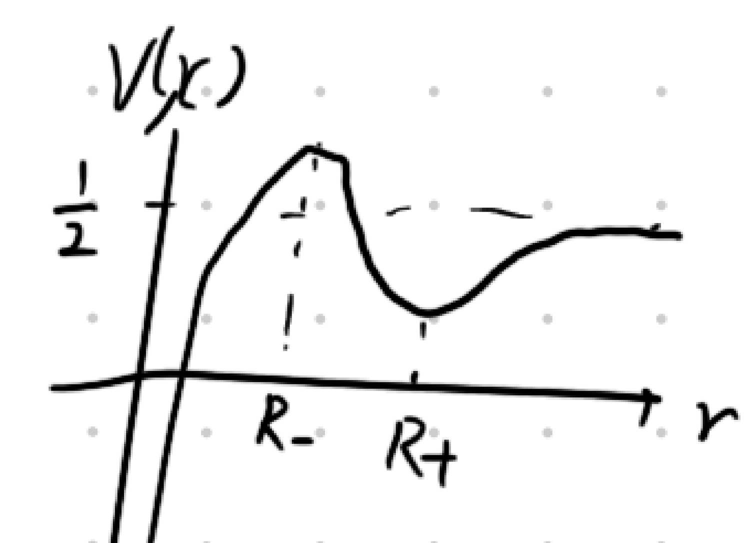
(ii) $L^2 = 12G^2M^2 \qquad R = 6GM$ 取极值 $\qquad V(6GM) = \frac{1}{2} - \frac{1}{18} < \frac{1}{2}$

最内层稳定圆周轨道 (ISCO)

$\frac{\partial^2 V}{\partial r^2} = 0 \qquad \frac{\partial^3 V}{\partial r^3} > 0$



(iii) $L^2 > 12G^2M^2$



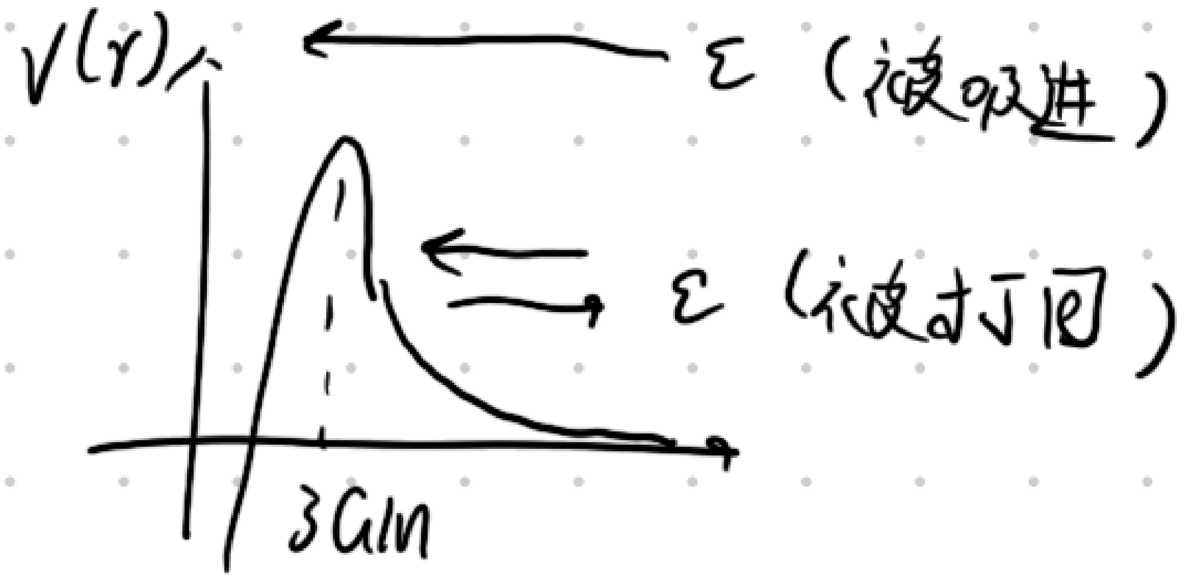
R_+ : 稳定圆轨道

R_- : 不稳定圆轨道

$R_+ > 6GM \qquad 3GM < R_- < 6GM$

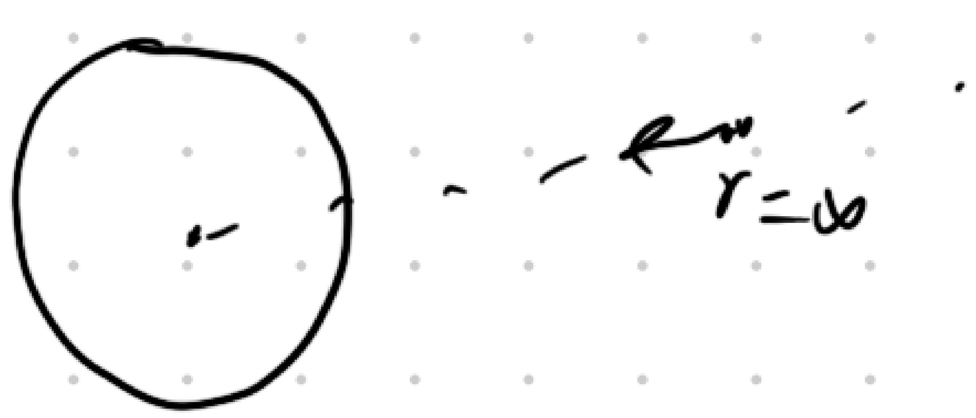
$E=0$ 无质量粒子 $V(r)=\frac{L^2}{2r^2}-\frac{GML^2}{r^3}$ 极值点 $R=3GM$ $V(3GM)=\frac{1}{54}\frac{L^2}{(GM)}$

光子半径



18. Geodesics and physical application

□ 有质量粒子 $m>0$，$L=0$ 的自由下落，从 $r=\infty$ 处开始，$E=\left.\frac{dt}{d\lambda}\right|_{r=\infty}=1$ （取 $\lambda=\tau$）



有效势 $V(r)=\frac{1}{2}-\frac{GM}{r}$ $\frac{1}{2}\dot{r}^2-\frac{GM}{r}=0 \Rightarrow \dot{r}=-\sqrt{\frac{R_s}{r}}=\frac{dr}{d\tau}$

$\frac{dt}{d\tau}=\frac{1}{1-R_s/r}$ $u^\mu=\left(\left(1-\frac{R_s}{r}\right)^{-1},\ -\sqrt{\frac{R_s}{r}},\ 0,0\right)$

$u^\mu u_\mu=g_{\mu\nu}u^\mu u^\nu=-\left(1-\frac{R_s}{r}\right)\left(1-\frac{R_s}{r}\right)^{-2}+\left(1-\frac{R_s}{r}\right)^{-1}\frac{R_s}{r}=-1$

$r=r_0\to r=r_1$ $(r_1<r_0)$ 固有时 $\sqrt{r}\,dr=-\sqrt{R_s}\,d\tau$ $\Delta\tau=\frac{2}{3\sqrt{R_s}}\left(r_0^{3/2}-r_1^{3/2}\right)$

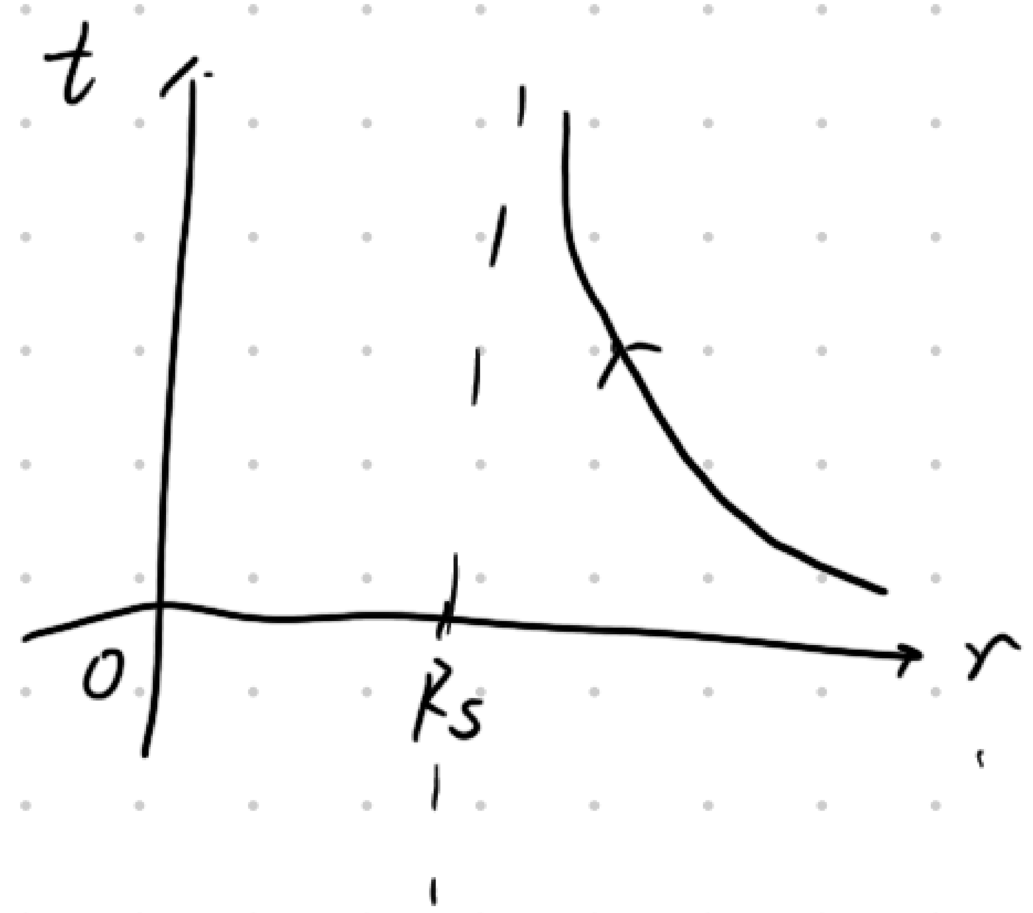
$\Delta\tau$ 从 $r=r_0$ 到 $r=0$，$\Delta\tau=\frac{2}{3\sqrt{R_s}}r_0^{3/2}$ 有限大.

坐标时 Δt from $r=r_0\to r_1$ by $r=\infty$ observer. $\frac{dt}{dr}=\frac{dt/d\tau}{dr/d\tau}=-\sqrt{\frac{r}{R_s}}\left(1-\frac{R_s}{r}\right)^{-1}$

$$\Delta t = \frac{2}{3\sqrt{R_s}}\left(r_0^{3/2} - r_1^{3/2}\right) + 2\sqrt{R_s}\left(r_0^{1/2} - r_1^{1/2}\right) + R_s \ln\left|\frac{\sqrt{r_1}+\sqrt{R_s}}{\sqrt{r_1}-\sqrt{R_s}} \cdot \frac{\sqrt{r_0}-\sqrt{R_s}}{\sqrt{r_0}+\sqrt{R_s}}\right| \qquad r_0, r_1 > R_s$$

若 $r_1 \to R_s$，$\Delta t \to \infty$

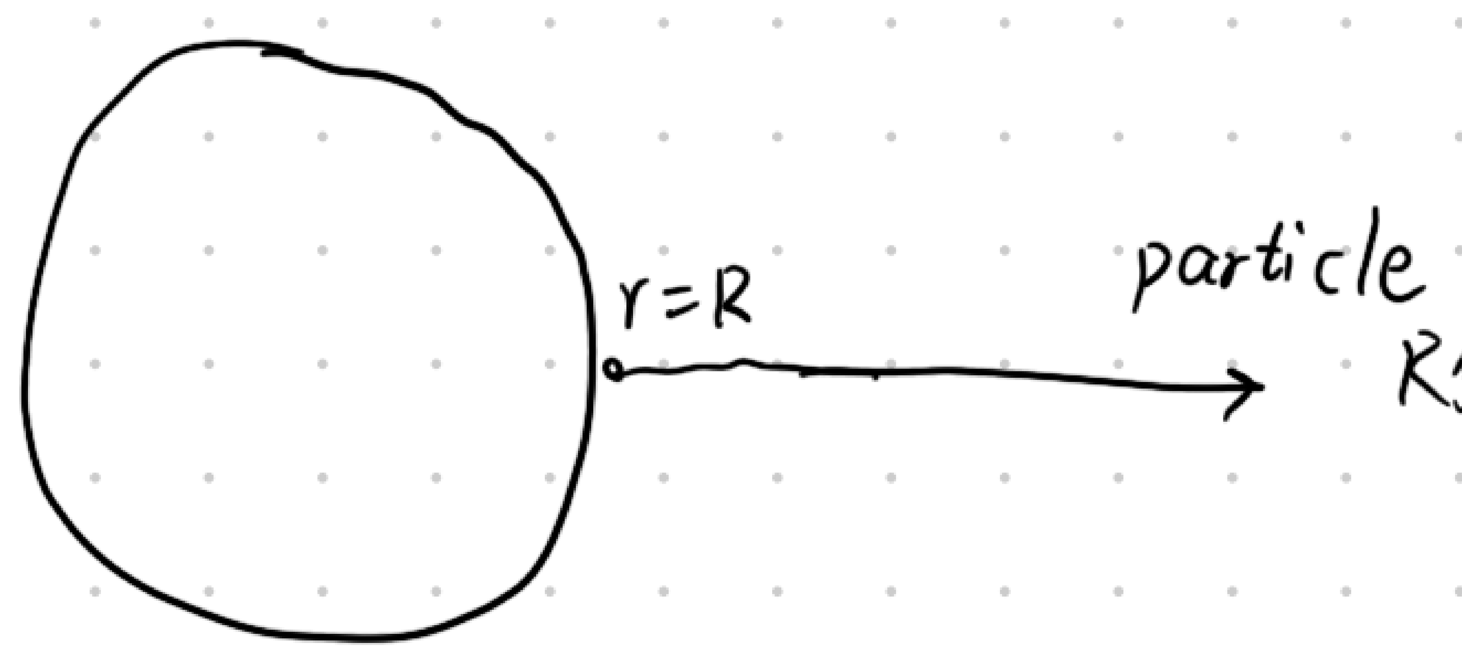
下落至R_s时，下落者可看到宇宙的无穷远时间之后



□ Escape velocity

$r=R$处 $u^\mu = \left(\left(1-\frac{R_s}{R}\right)^{-1}, \sqrt{\frac{R_s}{R}}, 0, 0\right)$ 为 $r=\infty$处观察者看到的.

逃离速度：在$r=R$处静止观察者看到的粒子速度



R处观察者4速度 $u^\mu_{obs}(R) = \left(\left(1-\frac{R_s}{R}\right)^{-\frac{1}{2}}, 0, 0, 0\right)$ （使 $u^\mu_{obs} u_{\mu,obs} = -1$）

观测到粒子能量 $\mathcal{E}_{obs} = -m\hat{u}\cdot\hat{u}_{obs} = m\left(1-\frac{R_s}{R}\right)^{-\frac{1}{2}} = \frac{m}{\sqrt{1-v_{escape}^2}} \Rightarrow v_{escape} = \sqrt{\frac{R_s}{R}} = \sqrt{\frac{2GM}{R}}$

与牛顿力学结果一致.

若 $R < 2GM = R_s$，$v_{escape} > 1 \Rightarrow$ 光也不能逃脱.

远处观察者看来，只能接受θ锥角以内发出的光

且距离R越近，看到的半径越小.

□ Gravitational redshift (time dilatime)

for static launcher at $r=R$ $(R>R_s)$ $\quad u^\mu_{obs} = ((1-\frac{R_s}{R})^{-\frac{1}{2}}, 0, 0, 0)$ $\quad \frac{dt}{d\tau} = (1-\frac{R_s}{R})^{-\frac{1}{2}}$

$\Delta\tau_{obs} = \Delta t\sqrt{1-\frac{R_s}{R}} < \Delta t$ 越接近黑洞，看远处时间流速越快

远处看接近黑洞处时间流速越慢

在$r=R$处的圆轨道 $(R>3GM)$ $\Rightarrow L^2 = \frac{GMR^2}{R-3GM}$ 由 $\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \Rightarrow E(R) = \frac{R-2GM}{\sqrt{R(R-3GM)}}$

$\frac{dt}{d\tau} = E(R)\left(1-\frac{R_s}{R}\right)^{-1} = \sqrt{\frac{R}{R-3GM}}$ $\quad \Delta\tau_{circ} = \Delta t\sqrt{1-\frac{3GM}{R}} < \Delta\tau_{obs} < \Delta t$

一般情况： 光子从A 发射，被B 接收

$\varepsilon(A) = -\vec{p}(A)\cdot\vec{u}(A) = -p_\mu(A)u^\mu(A)$

$\varepsilon(B) = -\vec{p}_\mu(B)u^\mu(B)$

红移因子 $\frac{\nu_B}{\nu_A} = \frac{\varepsilon(B)}{\varepsilon(A)} = \frac{p_\mu(B)u^\mu(B)}{p_\mu(A)u^\mu(A)}$ 其中$\vec{p}$是光子在A、B处的4动量，$\vec{u}(A)$，$\vec{u}(B)$是A、B的4速度

$p_\mu(A)$与$p_\mu(B)$联系： $\frac{dp_\mu}{d\lambda} - \Gamma^\nu_{\mu\rho}\, p_\nu p^\rho = 0$ （类测地线方程）

如果时空是稳态的，有 Killing vecter $\xi = \partial_t$ 沿测地线有守恒量 $\xi\cdot\vec{p} = p_0$

若 A & B 都静止不动 $u^\mu(A) = (u^0(A), 0, 0, 0)$ $\quad u^\mu(B) = (u^0(B), 0, 0, 0)$

$u^\mu u_\mu = -1$

$$\frac{V_B}{V_A} = \frac{P_0 u^0(B)}{P_0 u^0(A)} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}}$$

19. Experimental test of GR

近日点进动　　牛顿引力　$r = \dfrac{l}{1+e\cos(\phi-\phi_0)}$　　$l = \dfrac{L^2}{GM}$

GR：史瓦西度规，有质量粒子在星体周围走测地线.

守恒量：$\dfrac{d\phi}{d\lambda} = \dfrac{L}{r^2}$　　$\dfrac{dt}{d\lambda} = E\left(1-\dfrac{R_s}{r}\right)^{-1}$　　$\dfrac{1}{2}\left(\dfrac{dr}{d\lambda}\right)^2 + V(r) = \dfrac{1}{2}E^2$　　$V(r) = \dfrac{1}{2}\left(1-\dfrac{R_s}{r}\right)\left(\dfrac{L^2}{r^2}+1\right)$.

$$\dot{r}^2 + \left(1-\frac{R_s}{r}\right)\left(\frac{L^2}{r^2}+1\right) = E^2$$

$$\dot{r} = \frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} \Rightarrow \left(\frac{dr}{d\phi}\right)^2 + \left(1-\frac{R_s}{r}\right)r^2\left(1+\frac{r^2}{L^2}\right) = \frac{E^2 r^4}{L^2}$$

define $x = \dfrac{2L^2}{R_s r} = \dfrac{L^2}{GMr}$ $\Rightarrow \left(\dfrac{dx}{d\phi}\right)^2 + \left[\left(\dfrac{2L}{R_s}\right)^2 - \dfrac{4E^2L^2}{R_s^2}\right] - 2x + x^2 - \dfrac{R_s^2}{2L^2}x^3 = 0$.

acts on $\dfrac{d}{d\phi}$ $\Rightarrow$

$$\underbrace{\frac{d^2x}{d\phi^2} - 1 + x}_{\text{牛顿引力项}} = \underbrace{\frac{3R_s^2}{4L^2}x^2}_{\text{GR特有}}$$

微扰论　$x = x_0 + x_1$　α　$x_0 = 1 + e\cos\phi$　$\dfrac{d^2x_0}{d\phi^2} - 1 + x_0 = 0$　$|x_1| \ll |x_0|$.

$$\frac{d^2x_1}{d\phi^2} + x_1 = \frac{3R_s^2}{4L^2}x_0^2 = \alpha\left(1 + e^2\cos^2\phi + 2e\cos\phi\right)$$

$\Rightarrow x_1 = \alpha\left(1+\frac{1}{2}e^2 - \frac{1}{6}e^2\cos 2\phi + e\phi\sin\phi\right) \approx \alpha e\phi\sin\phi$

$x = x_0 + x_1 \approx 1 + e\cos\phi + \alpha e\phi\sin\phi \approx 1 + e\cos[(1-\alpha)\phi]$

轨道周期由 2π 变为 $2\pi/(1-\alpha) \approx 2\pi(1+\alpha)$ $\Delta\phi = 2\pi\alpha = \frac{3\pi R_s^2}{2L^2} = \frac{6\pi G^2M^2}{L^2}$

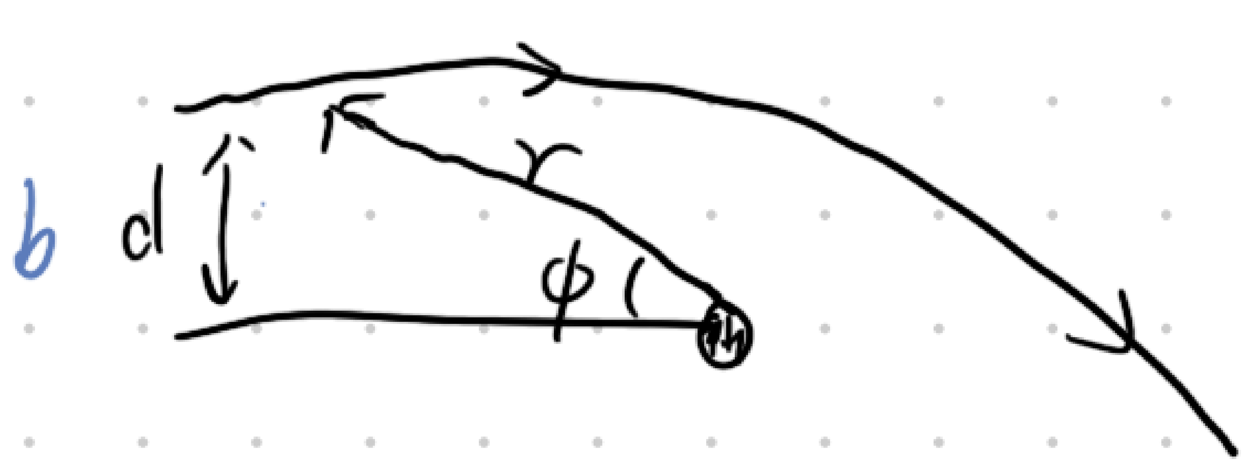
由近日点 $r_- = \frac{L^2}{GM(1+e)}$ $\Rightarrow \Delta\phi = \frac{6\pi GM}{r_-(1+e)}$ 水星 $\Delta\phi = 0.1038''$

20. Null geodesics

无质量粒子的零测地线 $E = \left(1-\frac{R_s}{r}\right)\frac{dt}{d\lambda}$ $L = r^2\frac{d\phi}{d\lambda}$ 但 not physical (依赖参数λ可随意选取)

$V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3} = \frac{L^2}{2r^3}(r-R_s)$ $\left(\frac{dr}{d\lambda}\right)^2 + 2V(r) = E^2$

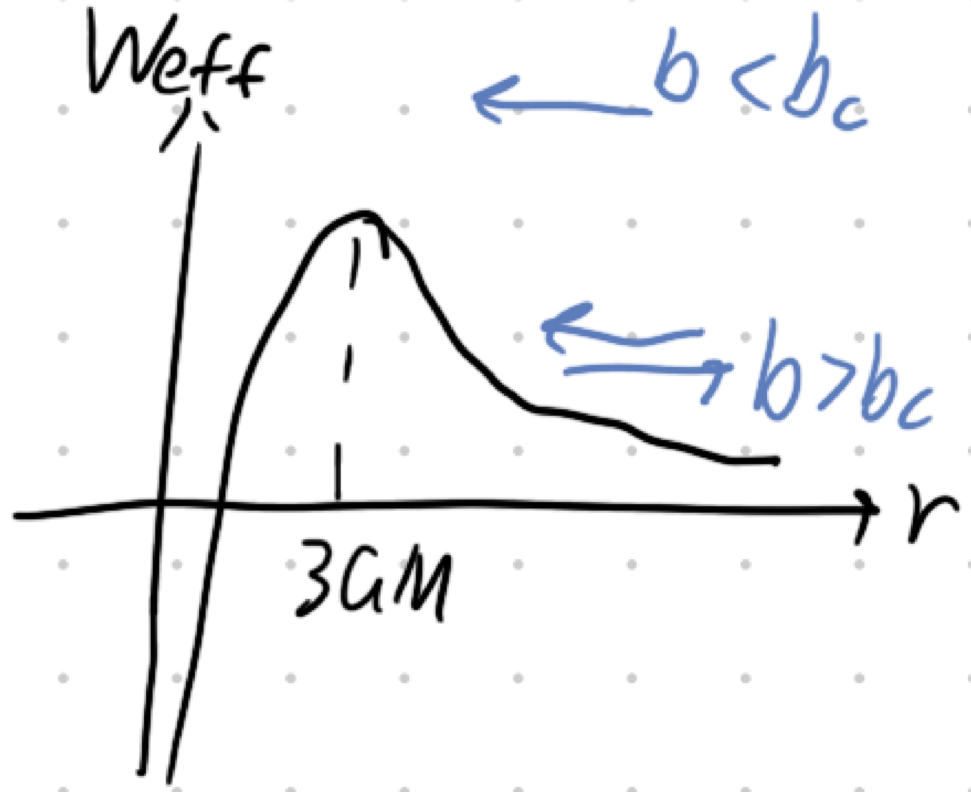
引入参数 $b = \frac{L}{E} = \frac{r^2}{r-R_s}\frac{d\phi}{dt}$ 是一个 physical 的守恒量.



建立如图极坐标系、 $r >> d$ 时 $\phi = \frac{d}{r}$ $\frac{d\phi}{dt} = -\frac{d}{r^2}\frac{dr}{dt}$ $\frac{dr}{dt} \to -1 \ (r\to\infty)$

$\Rightarrow \frac{d\phi}{dt} = \frac{d}{r^2}$ 又 $\frac{d\phi}{dt} = \frac{b}{r^2} \ (r\to\infty)$ $\Rightarrow d = b$

化简测地线方程: $\frac{1}{b^2} = \frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$ $W_{eff}(r) = \frac{1}{r^2}\left(1-\frac{R_s}{r}\right)$ 由于λ具有任意性,这里方程实际只依赖于b

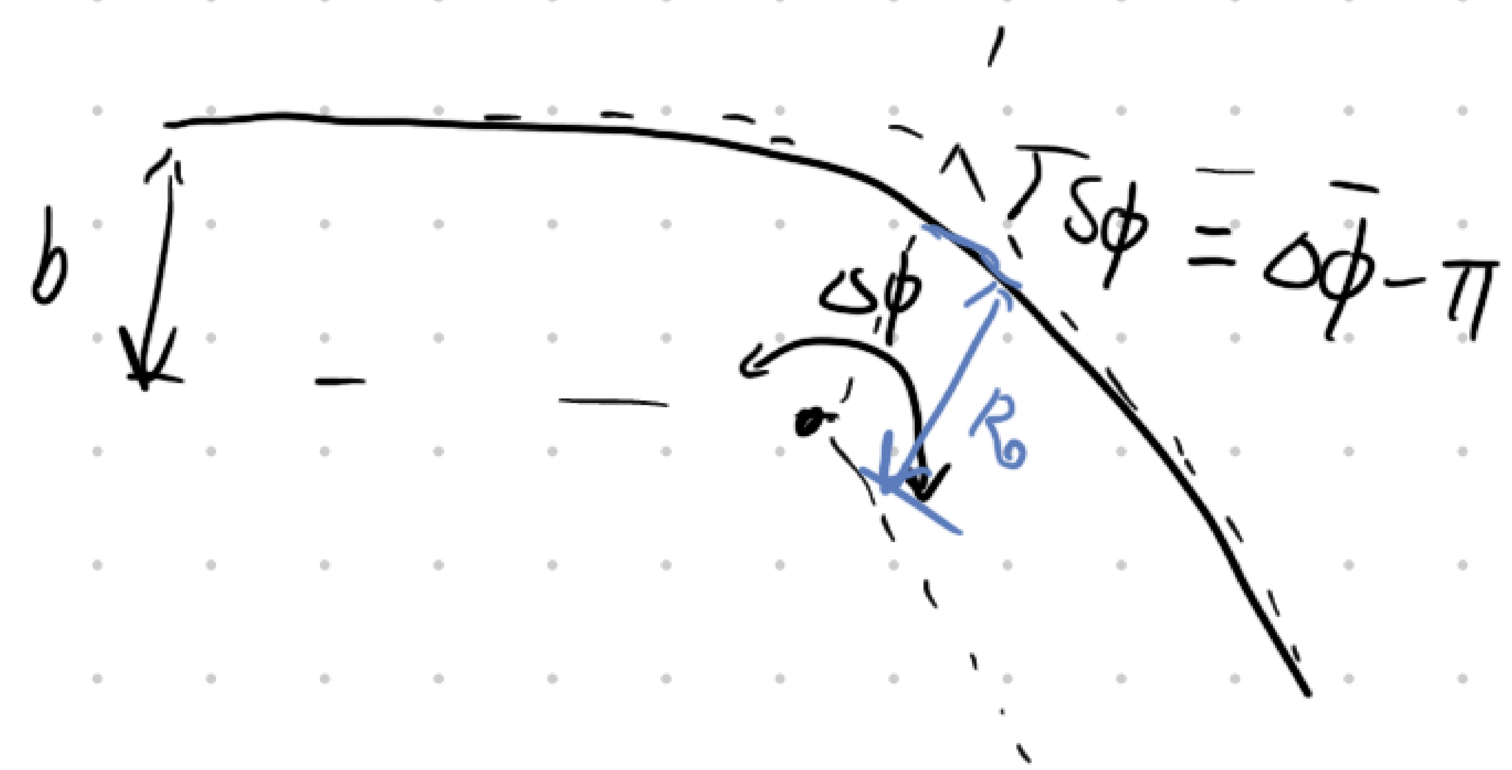


$W_{eff}(r)$ 只有1个极大值 $r=3GM$ $\quad W_{eff}(3GM)=\frac{1}{27G^2M^2}$

临界值 $b_c=3\sqrt{3}GM$ $\quad \frac{1}{b_c^2}=\frac{1}{27G^2M^2}$

$b>b_c$ 光子经过偏折回到∞处 $\quad b<b_c$ 光子被星体捕捉

Observed radius of BH (黑洞的视界半径) $\quad r_c=b_c=3\sqrt{3}GM=\frac{3\sqrt{3}}{2}R_s$.



求光子偏转角? $\quad \delta\phi=\Delta\phi-\pi$

$$\frac{dr}{d\lambda}=L\sqrt{\frac{1}{b^2}-W_{eff}(r)} \qquad L=r^2\frac{d\phi}{d\lambda}$$

$$\Rightarrow \frac{d\phi}{dr}=\frac{1}{r^2}\left[\frac{1}{b^2}-W_{eff}(r)\right]^{-1/2}$$

$$\Delta\phi=2\int_{R_0}^{\infty}\frac{dr}{\sqrt{r^4b^{-2}-r(r-R_s)}}$$

R_0 satisfy : $\frac{dr}{d\lambda}=0$, $W_{eff}(R_0)=\frac{1}{b^2}$ $\Rightarrow R_0^3-b^2(R_0-R_s)=0$

验证:若 $M=0$, $R_s=0$, $R_0=b$, $\Delta\phi=2\int_b^{\infty}\frac{dr}{\sqrt{r^4b^{-2}-r^2}}=\pi$ $\quad \delta\phi=0$

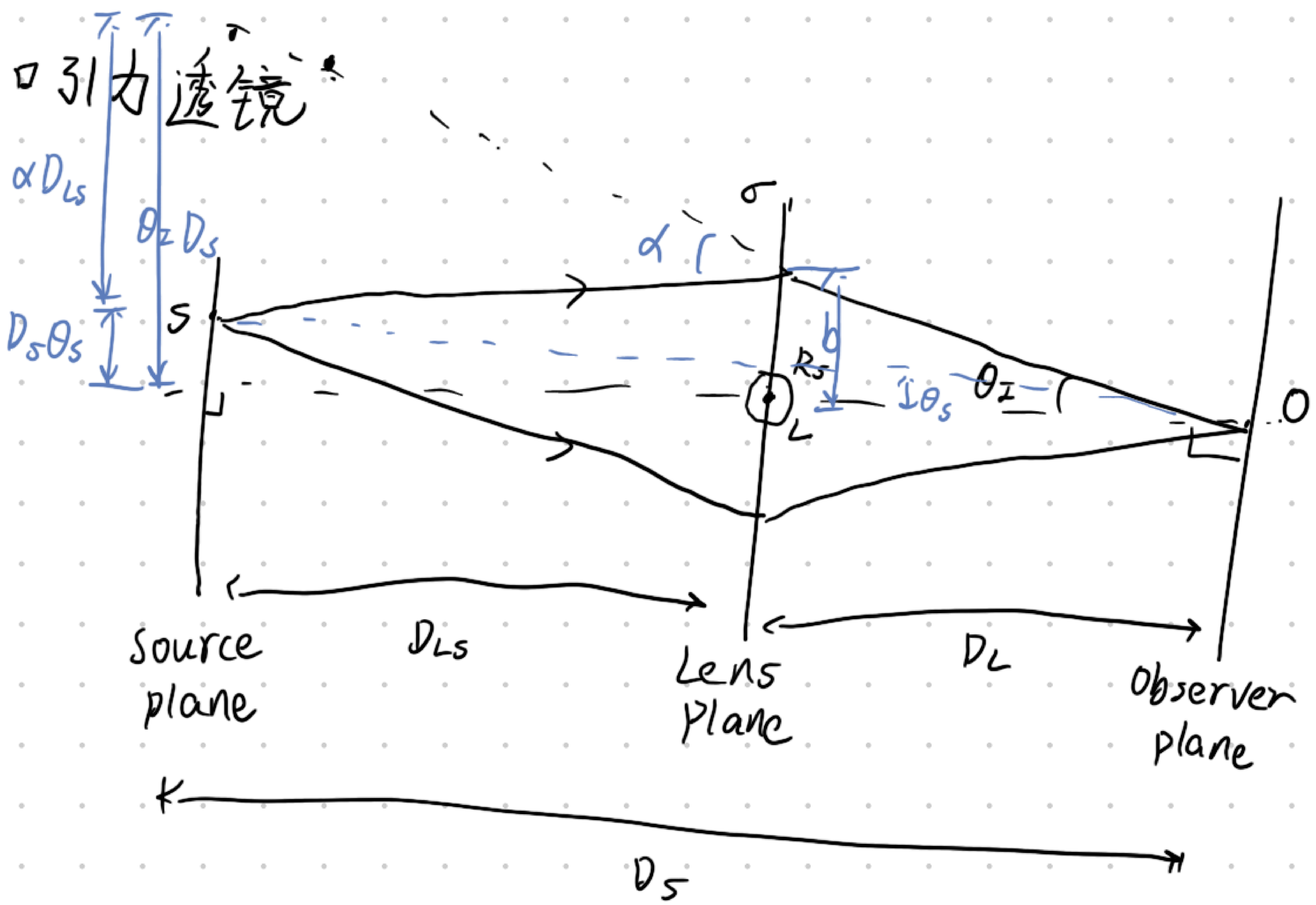
令 $r=\frac{1}{u}$, $\Delta\phi=2\int_0^{1/R_0}\frac{du}{\sqrt{b^{-2}-u^2+R_su^3}}$ $\quad b^{-2}=R_0^{-2}-R_sR_0^{-3}$ $\quad \Rightarrow \Delta\phi=2\int_0^{1/R_0}\frac{du}{[R_0^{-2}-R_sR_0^{-3}-u^2+R_su^3]^{1/2}}$

小M近似: $\delta\phi\ll 1$ $\quad \Delta\phi=\overbrace{\Delta\phi\big|_{M=0}}^{=\pi}+M\frac{\partial(\Delta\phi)}{\partial M}\Big|_{M=0}$

$$\delta\phi \approx R_s \cdot \int_0^{1/R_0} \frac{(R_0^{-3} - u^3)\, du}{[R_0^{-2} - R_s R_0^{-3} - u^2 + R_s u^2]^{3/2}} \Bigg|_{\substack{R_s = 0 \\ R_0 = b}} = R_s \int_0^{1/b} \frac{b^{-3} - u^3}{(b^{-2} - u^2)^{3/2}} du = \frac{2R_s}{b} = \frac{4GM}{b}$$

SI制： $\delta\phi = \dfrac{4GM}{bc^2}$ $\quad M = M_\odot,\ b = R_\odot,\ \delta\phi \approx 1.75''$

□ 引力透镜



$\alpha, \theta_I, \theta_s \ll 1 \quad \cos\alpha, \cos\theta_I, \cos\theta_s \approx 1 \quad \sin\alpha \approx \alpha$

$D_s = D_{Ls} + D_L \qquad \theta_I D_s = \alpha D_{Ls} + \theta_s D_s$

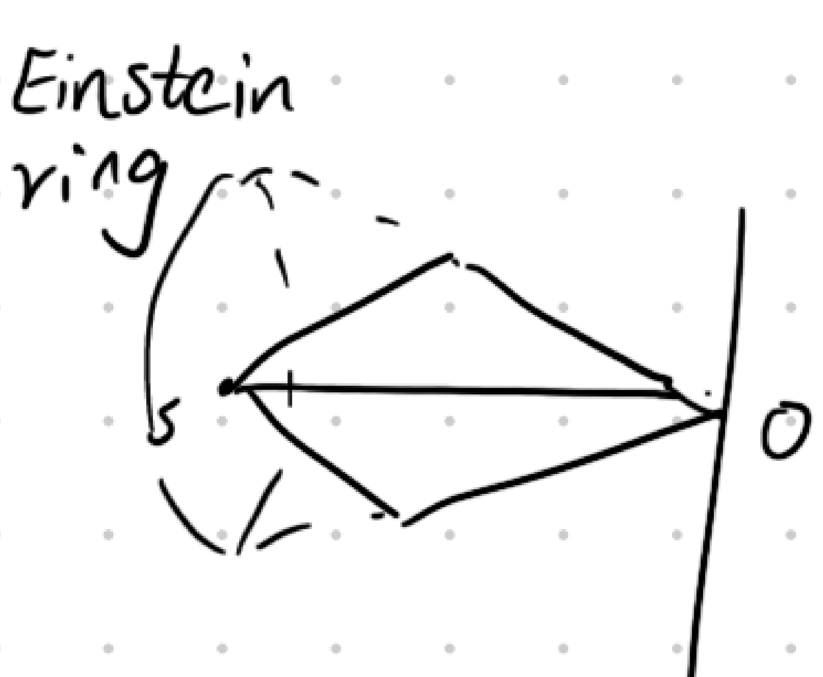
$\alpha = \dfrac{2R_s}{b} \qquad b = \theta_I D_L \qquad \alpha = \dfrac{2R_s}{\theta_I D_L}$

$\Rightarrow \theta_I = \theta_s + \dfrac{\alpha D_{Ls}}{D_s} = \theta_s + \dfrac{\theta_E^2}{\theta_I} \qquad \theta_E = \sqrt{\dfrac{2R_s D_{Ls}}{D_s D_L}}$ Einstein angle

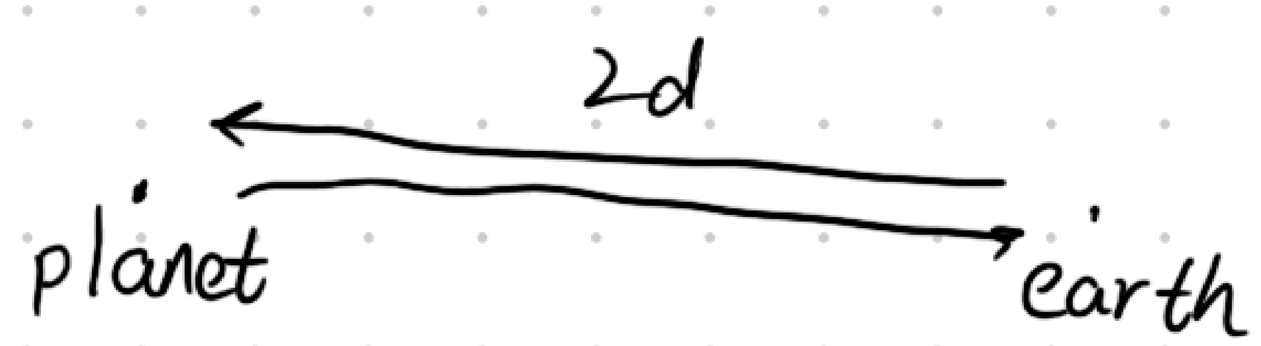
M作为星系，$M \sim 10^{11} M_\odot$，$R_s \sim 10^{11}$ km，$D_s, D_L, D_{Ls} \sim 10^{22}$ km

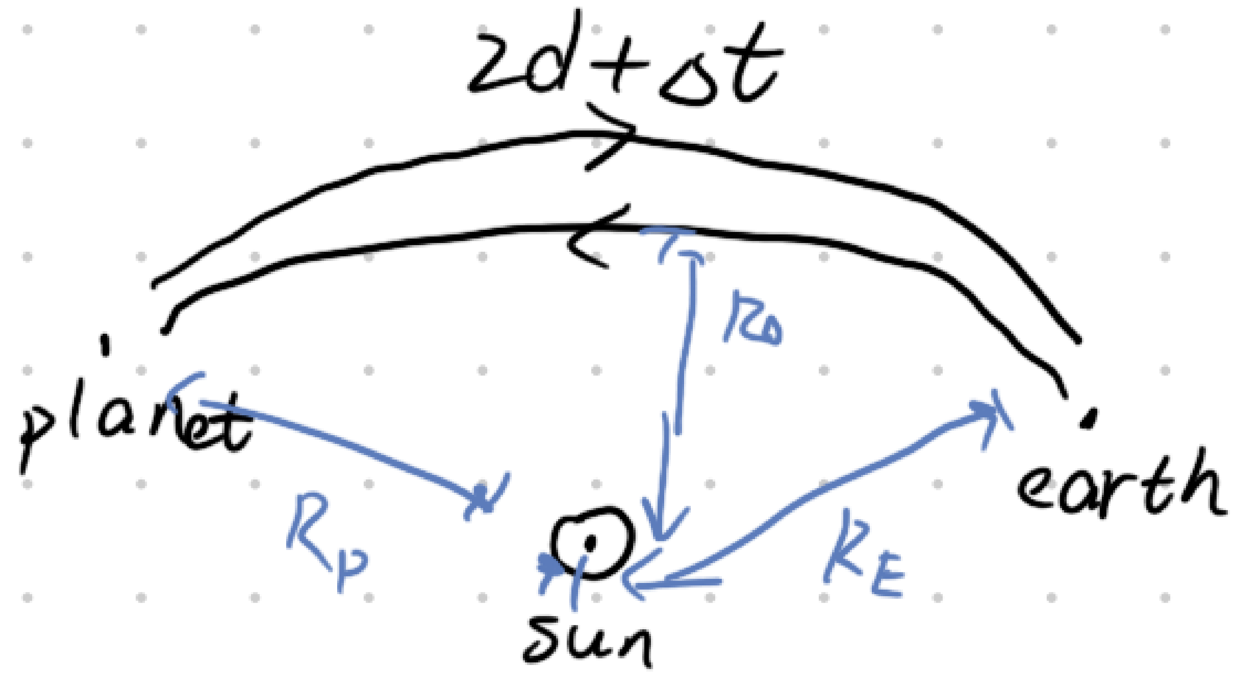
$\Rightarrow \theta_E = 1''$ 可被观测到

when $\theta_s = 0 \Rightarrow \theta_I = \theta_E$ observed S Einstein ring



Radar echo delay (雷达回波延迟)





$$\frac{dt}{d\lambda} = \frac{E}{1-R_s/r} \qquad \frac{dr}{d\lambda} = L\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{R_s}{r}\right)}$$

$$\frac{dt}{dr} = \left(1-\frac{R_s}{r}\right)^{-1}\left(1-\frac{b^2}{r^2}\left(1-\frac{R_s}{r}\right)\right)^{-\frac{1}{2}} \qquad b = \frac{L}{E}$$

$$\frac{t}{2} = \left(\int_{R_E}^{R_0} dr + \int_{R_0}^{R_p} dr\right)\frac{dt}{dr} = \left(-\int_{R_E}^{R_0} dr + \int_{R_0}^{R_p} dr\right)\left(1-\frac{R_s}{r}\right)^{-1}\left(1-\left(1-\frac{R_s}{r}\right)\frac{b^2}{r^2}\right)^{-\frac{1}{2}}$$

其中 $R_0^3 - b^2(R_0 - R_s) = 0$

$$\frac{t_0}{2} = (\cdots)\Big|_{\substack{M=0 \\ b=R_0}}$$

$$\Delta t = t - t_0 \approx 4GM\left[\ln\left(\frac{2R_E}{R_0}\right) + \ln\left(\frac{2R_p}{R_0}\right) + 1\right]$$

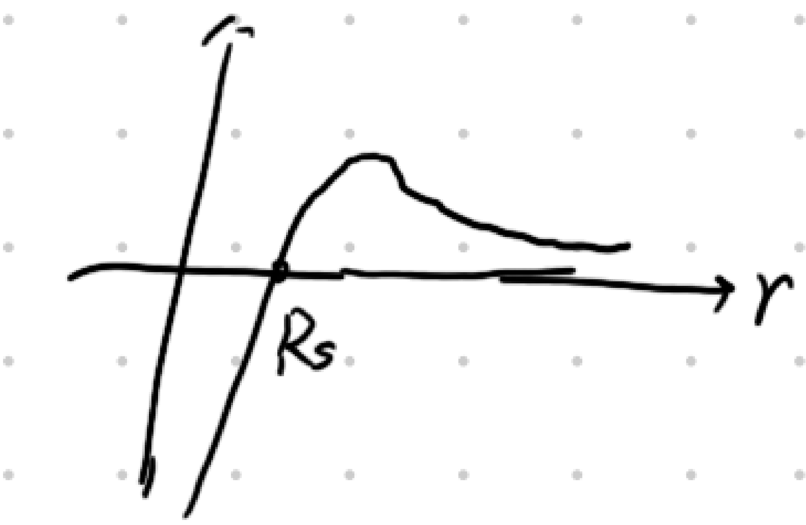
地球与金星之间：$\Delta t = 220\ \mu s$.

21. Schwareschild black hole (史瓦西黑洞)

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \quad (r>0) \qquad T_{\mu\nu}=0 \quad R_{\mu\nu}=0$$

史瓦西半径 $R_s = 2GM = \frac{2GM}{c^2}$

light can't escape when $r \le R_s$ $\qquad V(r) = \frac{L^2}{2r^3}(r-R_s)$



若星体半径 $R \le R_s$, appear to be black.

- Event horizon at $r = R_s$
  (1) signal cannot escape to $r=\infty$
  (2) Infinite gravitational redshift $\qquad \frac{\nu_E}{\nu_0} = \sqrt{\frac{g_{00}(r_0)}{g_{00}(r_E)}} = \sqrt{\frac{r}{r-R_s}} \qquad r\to R_s,\ \nu_0 \to 0$

- Killing vector ∂_t. outside of BH $r>R_s$ $\quad \partial_t$ is timelike ($g_{tt}<0$) (static spacetime)

  inside BH $r<R_s$, $g_{tt}>0$ $g_{rr}<0$ $\quad \partial_t$ is spacelike (不存在类时 Killing vector, not stationary)

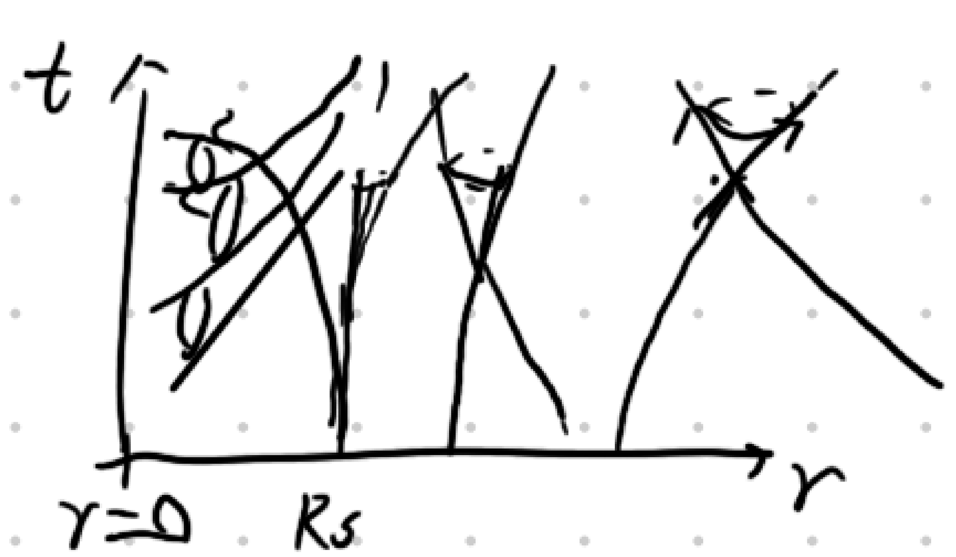
  ∂_r is timelike ($t \to$ 空间, $r \to$ 时间, r不可逆地减少)

- 对有质量观察者, in BH, $ds^2<0 \Rightarrow g_{rr}dr^2<0$ 必须落向 $r=0$. (走不了圆轨道)

- Null Geodesics & light cone (光锥)

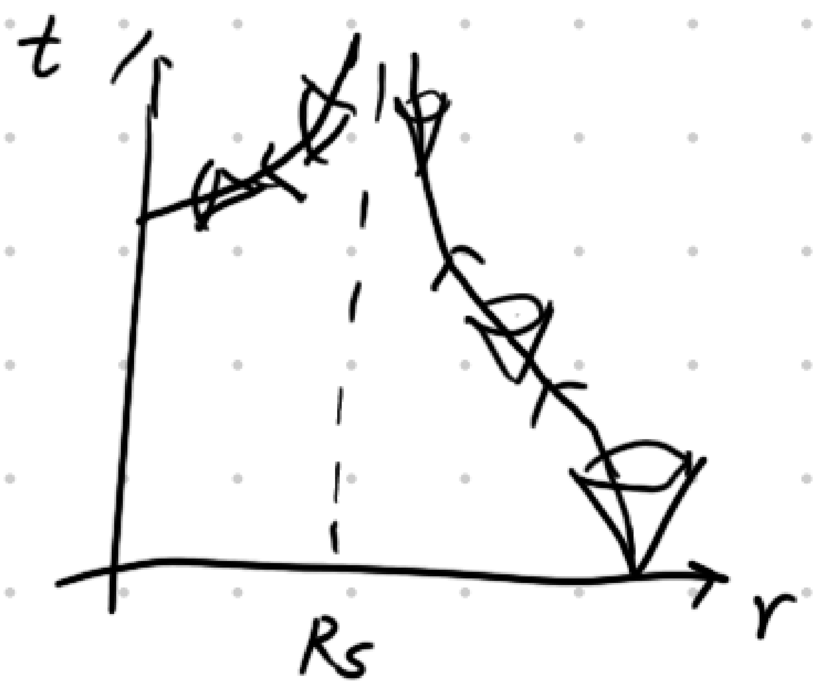
径向

$$ds^2 = 0 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 \Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{R_s}{r}\right)^{-1} = \begin{cases} \pm 1, & r\to\infty \\ \pm\infty, & r\to R_s \\ \pm 0, & r\to 0 \end{cases}$$



一旦进入视界, 粒子无法摆脱落入奇点的命运.

□ 有质量粒子 世界线



史瓦西坐标不够好 （AEF, REF 坐标…）

eg. Kruskal 坐标 $u' = \left(\frac{r}{R_s}-1\right)^{1/2} e^{(r+t)/2R_s}$ $v' = -\left(\frac{r}{R_s}-1\right)^{1/2} e^{(r-t)/2R_s}$ $(r \geq R_s)$

$ds^2 = -\frac{2R_s^3}{r} e^{-r/R_s} (du'dv' + dv'du') + r^2 d\Omega^2$ $(r \geq R_s)$

$v = \frac{1}{2}(u'+v') = \left(\frac{r}{R_s}-1\right)^{1/2} e^{r/2R_s} \sinh(t/2R_s)$ $u = \frac{1}{2}(u'-v') = \left(\frac{r}{R_s}-1\right)^{1/2} e^{r/2R_s} \cosh(t/2R_s)$ $(r \geq R_s)$

$\Rightarrow ds^2 = \frac{4R_s^3}{r} e^{-r/R_s} (-dv^2 + du^2) + r^2 d\Omega^2$

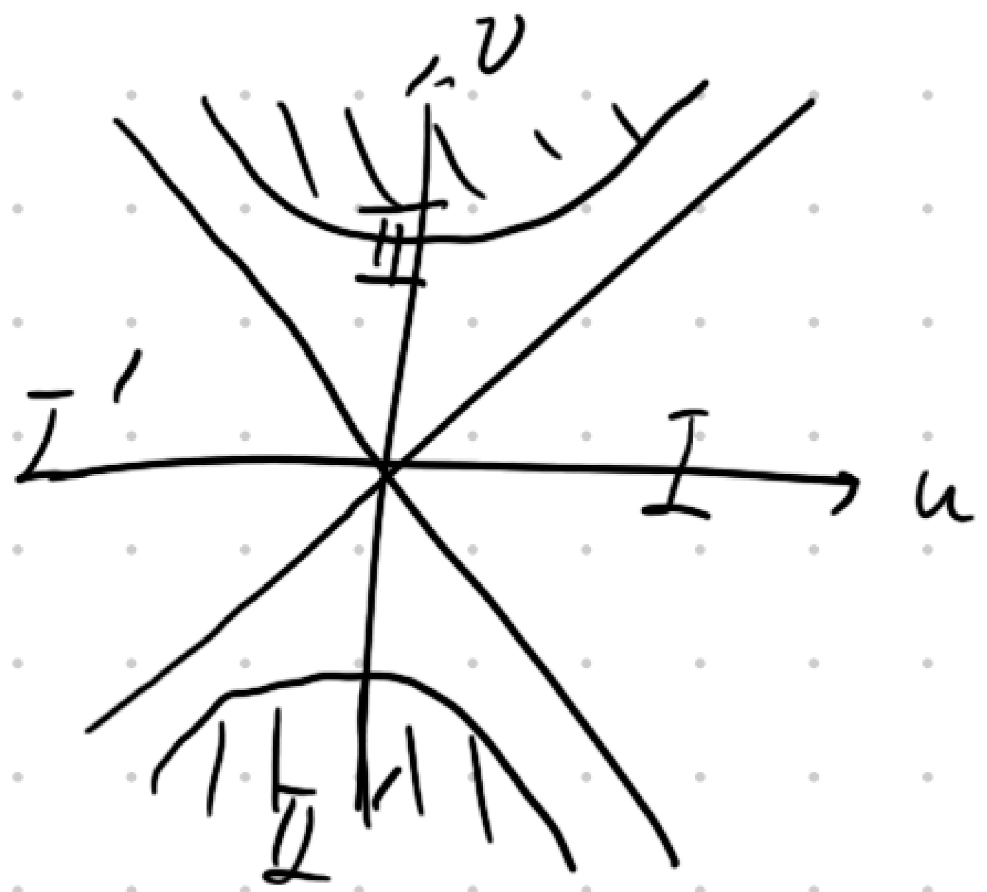
可经过共形变换 $(v, u, \Omega) \to (T, R, \Omega)$ $g_{\mu\nu} = g'_{\mu\nu} e^f$ $ds^2 \to -dT^2 + dR^2 + R^2 d\Omega^2$ 共形平坦

inside horizon $r < R_s$ $v = \sqrt{1-\frac{r}{R_s}}\, e^{r/2R_s} \cosh(t/2R_s)$ $u = \sqrt{1-\frac{r}{R_s}}\, e^{r/2R_s} \sinh(t/2R_s)$

常r $u^2 - v^2 = \text{Constant}$

$= -\left(1-\frac{r}{R_s}\right) e^{r/R_s}$

$\frac{v}{u} = \begin{cases} \tanh(t/R_s) & (r > R_s) \\ \tanh^{-1}(t/R_s) & (r < R_s) \end{cases}$



I. $u^2-v^2>0$ BH外部

II. $u^2-v^2<0$ BH内部 ($u^2-v^2\geq -1$ 给出边界，阴影表示延拓后的时空)

II'. 由II延拓 $t<0$, $r<R_s$ BH~~的时间~~反演 $\Rightarrow$ 白洞

I': 由I延拓 $t<0$, $r>R_s$

22. Schwarzschild black hole II

Kruskal metric $ds^2=\frac{4R_s^3}{r}e^{-r/R_s}(-dv^2+du^2)+r^2(d\theta^2+\sin^2\theta d\phi^2)$

▫ Wormhole connecting I & I'

space like hypersurface $v=0$ $(t=0)$ 上:

metric $ds^2=\frac{4R_s^3}{r}e^{-r/R_s}du^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$

$=(1-\frac{R_s}{r})^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$

in region I $(r\geq R_s)$

$v=\sqrt{\frac{r}{R_s}-1}\,e^{r/2R_s}\sinh(t/2R_s)$

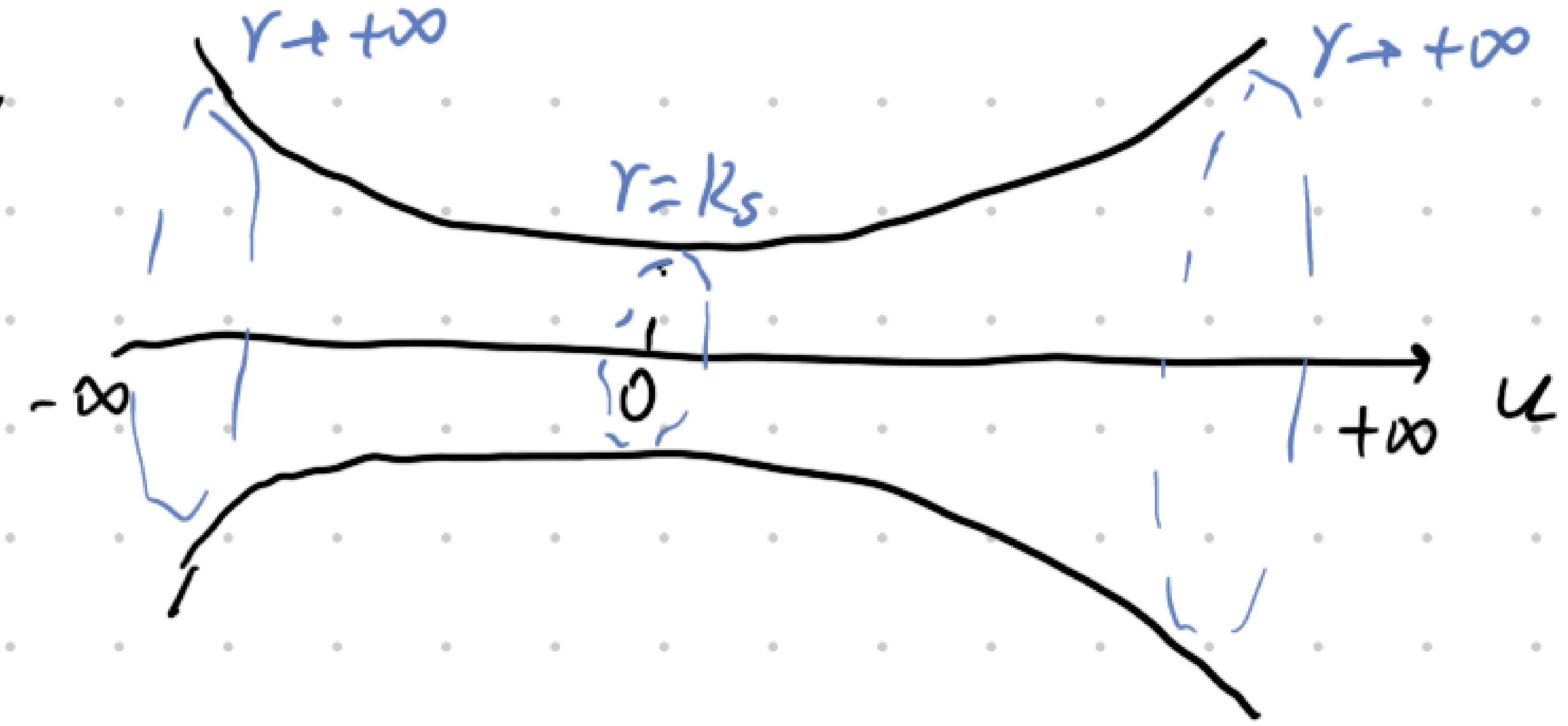
$u=\sqrt{\frac{r}{R_s}-1}\,e^{r/2R_s}\cosh(t/2R_s)$

fix $t=0$ $u\to\infty\Rightarrow r\to\infty$ far away from BH.

$u=0\Rightarrow r=R_s$ a S^2 with radius $r=R_s$ ($v=t=0$时r的最小值)

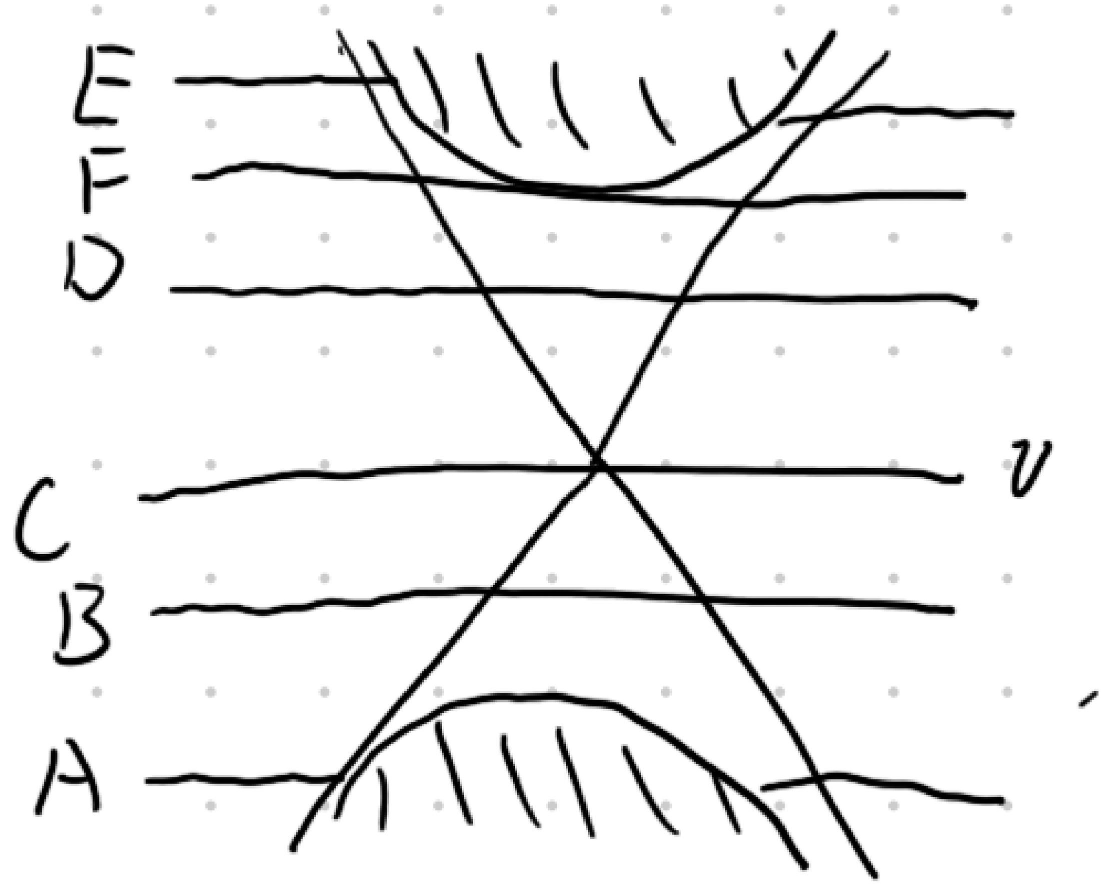
$u\to-\infty$ in region I', $u=\sqrt{\frac{r}{R_s}-1}\,e^{r/2R_s}\cosh(t/2R_s)\Rightarrow r\to+\infty$

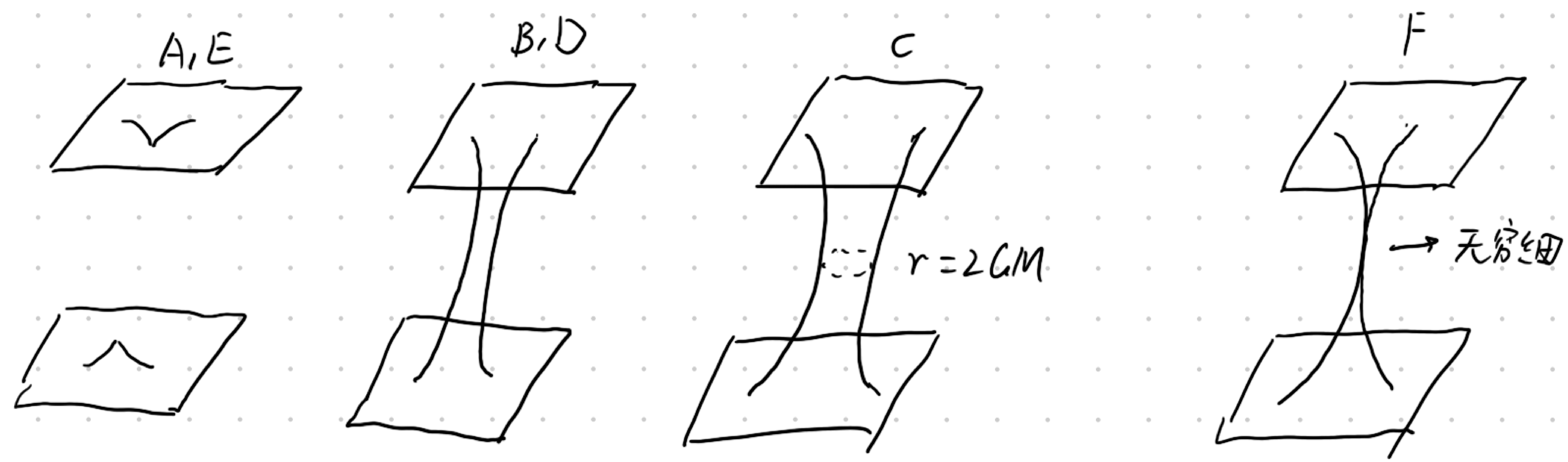
$t=0$,



wormhole (Einstein-Rosen bridge) 虫洞(爱因斯坦-罗森桥)

$t\neq 0$? fix v





Non-transversable (不可穿越性): 类空连接 / 虫洞关闭得太快，不足以让类时观测者穿过.

□ Surface gravity & Hawking temperature

史瓦西黑洞的视界面是 null hypersurface N (normal vector $\vec{\ell} \perp N$, $\vec{\ell}\cdot\vec{\ell}=0$).

Killing horizon: a null hypersurface, normal vector is a killing vector.

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$r=R_s$ 处 normal vector: ∂_t. 类光

Relation for event horizon & killing horizon

对一个稳态，渐近平坦时空，event horizon 是 killing horizon.

Surface gravity:

引理: on a null hypersurface N, the normal vector $l \perp N$ is a null geodesic,

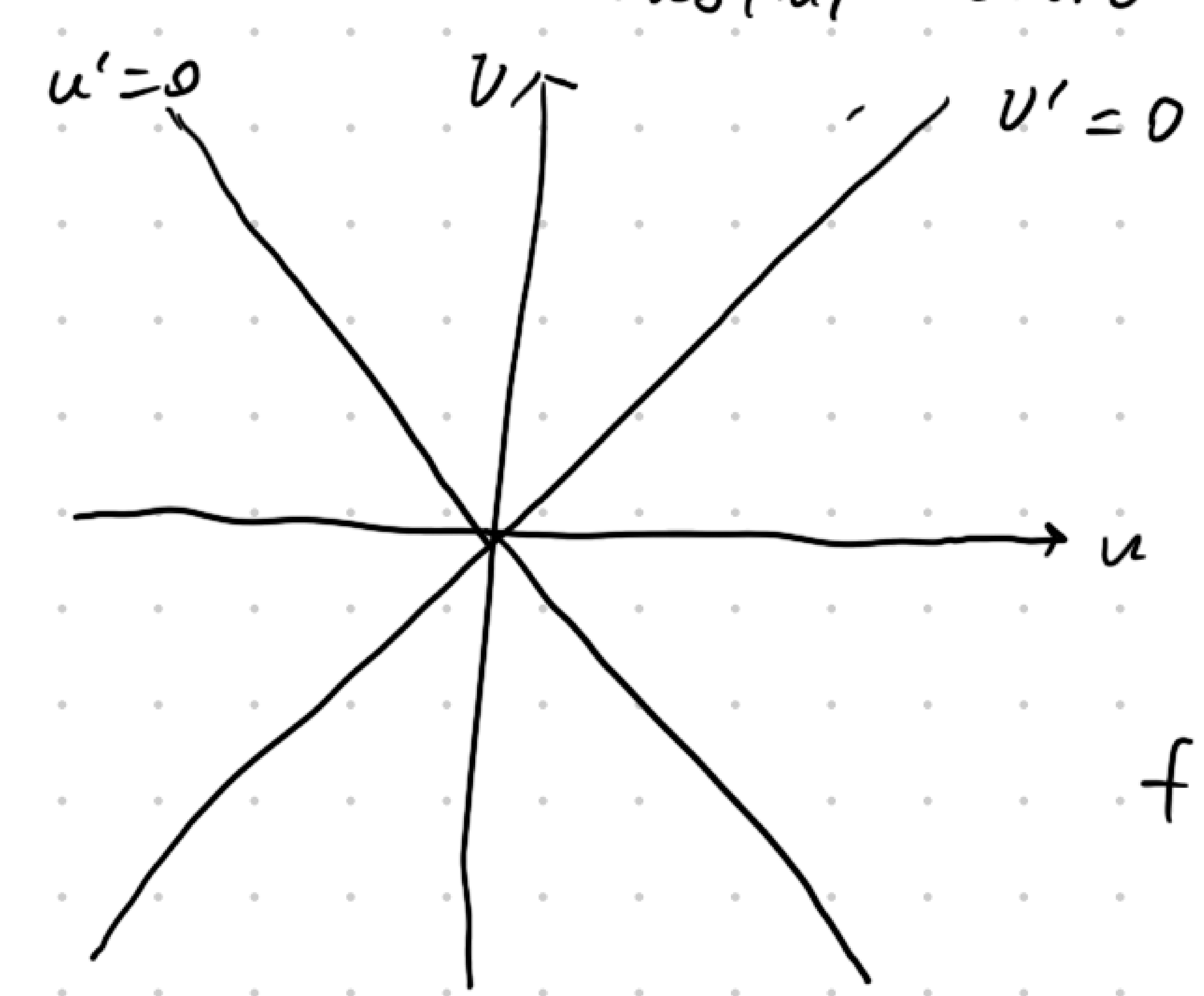
i.e. $l^\nu \nabla_\nu l^\mu = 0$

def N is a killing horizon, if killing vector $\xi \perp N$ i.e. $\xi = f l$ f is a function

$$\xi^\nu \nabla_\nu \xi^\mu = \xi^\nu \nabla_\nu (f l^\mu) = \xi^\nu (\underbrace{f \nabla_\nu l^\mu}_{=0} + l^\mu \partial_\nu f) = [(\xi^\nu \partial_\nu f)\cdot f^{-1}]\xi^\mu = \kappa \xi^\mu$$

$\kappa = (\xi^\nu \partial_\nu f) f^{-1} = \xi \cdot \partial \ln|f|$ is defined as the surface gravity.

Sch BH in Kruskal coord



$v = \frac{1}{2}(u'+v')$ $u = \frac{1}{2}(u'-v')$

$$\begin{cases} u' = (\frac{r}{R_s}-1)^{1/2} e^{(r+t)/2R_s} \\ v' = -(\frac{r}{R_s}-1)^{1/2} e^{(r-t)/2R_s} \end{cases}$$

$$\xi = \partial_t = \frac{\partial u'}{\partial t}\frac{\partial}{\partial u'} + \frac{\partial v'}{\partial t}\frac{\partial}{\partial v'} = \frac{u'}{2R_s}\frac{\partial}{\partial u'} - \frac{v'}{2R_s}\frac{\partial}{\partial v'}$$

on N $(v'=0)$ $\xi = \frac{u'}{2R_s}\frac{\partial}{\partial u'}$ normal vector of N, $l = \frac{\partial}{\partial u'}$

$f = \frac{u'}{2R_s}$ surface gravity on N: $\kappa = \xi \cdot \partial \ln|f| = \xi^{u'} \frac{\partial}{\partial u'} \ln|f| = \frac{1}{2R_s} = \frac{1}{4GM}$

surface gravity & Hawking temperature

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \quad (r \to R_s)$$

New coordinate $r = R_s + \frac{x^2}{4R_s}$ $(x \ll R_s)$ $\quad 1-\frac{R_s}{r} = \frac{x^2}{4R_s r} \approx \frac{x^2}{4R_s^2} = (\kappa x)^2$

$$dr = \frac{x\,dx}{2R_s} = \kappa x\,dx \qquad \left(1-\frac{R_s}{r}\right)^{-1} = \frac{1}{\kappa^2 x^2} = \frac{4R_s^2}{x^2}.$$

near horizon metric: $ds^2 = -(\kappa x)^2 dt^2 + dx^2 + \frac{1}{4\kappa^2} d\Omega^2.$

Rindler metric with constant acceleration $a = \kappa$ (for $r=\infty$ observer) (见第9页，加速参考系度规)

(for ∞ observer, near BH horizon → 匀加速 $a = \kappa$)

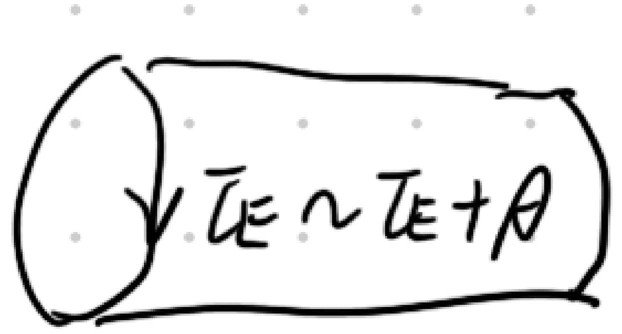
Wick notation into Euclidean signature $t = i\tau_E$ ← 虚时间

$$ds^2 = (\kappa x)^2 d\tau_E^2 + dx^2 + \frac{1}{4\kappa^2} d\Omega^2.$$

$dx^2 + x^2 d(\kappa\tau_E)^2$ "polar coord" $\sim dr^2 + r^2 d\theta^2$ only when $\theta \sim \theta + 2\pi$ smooth disk.

When $\kappa\tau_E \sim \kappa\tau_E + 2\pi$ spacetime has no singularity (奇点)

$\tau_E \sim \tau_E + \frac{2\pi}{\kappa}$ periodicity $\frac{2\pi}{\kappa} = \beta = \frac{1}{T}$ QFT with finite temperature $Z = \mathrm{tr}(e^{-\beta H}) = \sum_n \langle n | e^{-\beta H} | n \rangle$



跃迁振幅 $\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | e^{iH(t_f - t_i)} | x_i \rangle = \int Dx(t)\, e^{iS(x)}$ (路径积分)

$t = i\tau \Rightarrow \langle x_f | e^{-H(\tau_f - \tau_i)} | x_i \rangle = \int Dx(\tau)\, e^{-S_E(x)}$ $\quad S = \int_{t_i}^{t_f} dt\, L \quad S_E = \int_{\tau_i}^{\tau_f} d\tau\, L \quad S = iS_E$

周期条件 $x(0) = x(\beta) = X$ $\quad \langle X | e^{-\beta H} | X \rangle = \int_{x(0)=x(\beta)=X} Dx(\tau)\, e^{-S_E(x)}$

$Z = \int dX \langle X | e^{-\beta H} | X \rangle = \mathrm{Tr}\, e^{-\beta H}$.

(Unruh effect: constant acceleration $a = \kappa \simeq$ finite temperature $\beta = \frac{2\pi}{\kappa}$).

Hawking temperature $T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi G M} \propto \frac{1}{M}$ $\quad$ S.I. $T_H = \frac{\hbar c^3}{8\pi G M k_B}$ $\quad M \sim M_\odot,\ T_H \sim 6\times10^{-8} K$.

$J = \frac{|dU/dt|}{A} = \sigma T^4 \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^4} \quad \left|\frac{dU}{dt}\right| = \sigma T^4 \cdot 4\pi R_s^2 = \frac{\hbar c^6}{15360\pi G M^2} \propto M^{-2} = -\frac{dM}{dt} c^2$

$-\frac{dM}{dt} = \frac{a}{M^2} \Rightarrow t_{evap} = \frac{1}{a}\frac{M^3}{3} = \frac{5120\pi G^2 M^3}{\hbar c^4} \propto M^3 \quad M = M_\odot \quad t_{evap} \approx 10^{67} a$.

蒸发

理解蒸发 QFT 虚粒子 产生 $-E$ & E，黑洞吸收负能量粒子，正能量粒子逃逸

23. General Black Hole

□ Penrose diagram (彭罗斯图)

$x \to x'$ st. $\bar{g}_{\mu\nu} = \Lambda^2(x) g_{\mu\nu}$ $\Rightarrow$ null curves are invariant $\left(\bar{g}_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0\right)$

(only for 2 coordinates)

共形变换

e.g. flat Minkowski spacetime $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ $(-\infty < t < \infty,\ 0 \le r < +\infty)$

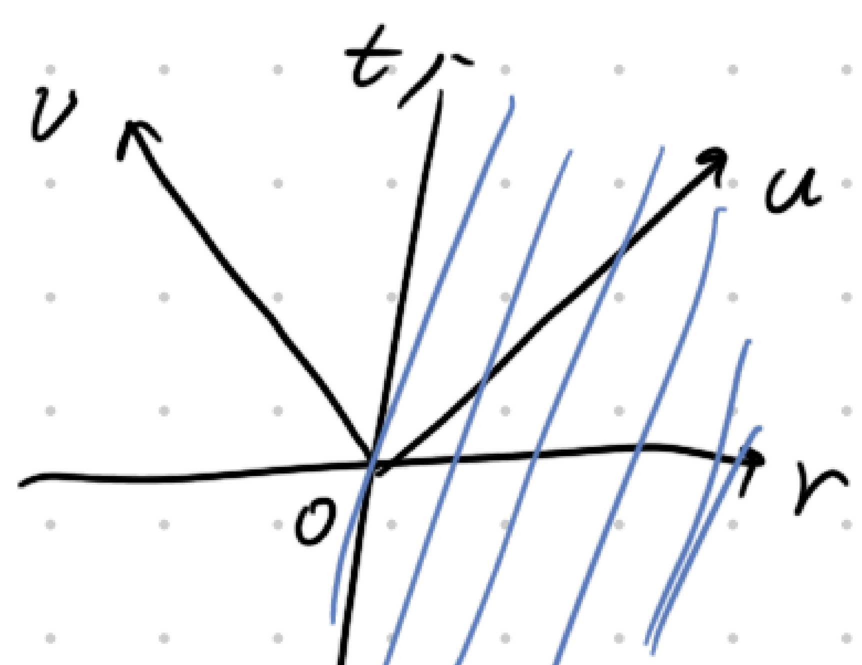
Null coord $u = \frac{1}{2}(t+r)$ $v = \frac{1}{2}(t-r)$ $-\infty < u, v \le +\infty$, $v \le u$

$ds^2 = -2(du\,dv + dv\,du) + (u-v)^2 d\Omega^2$.

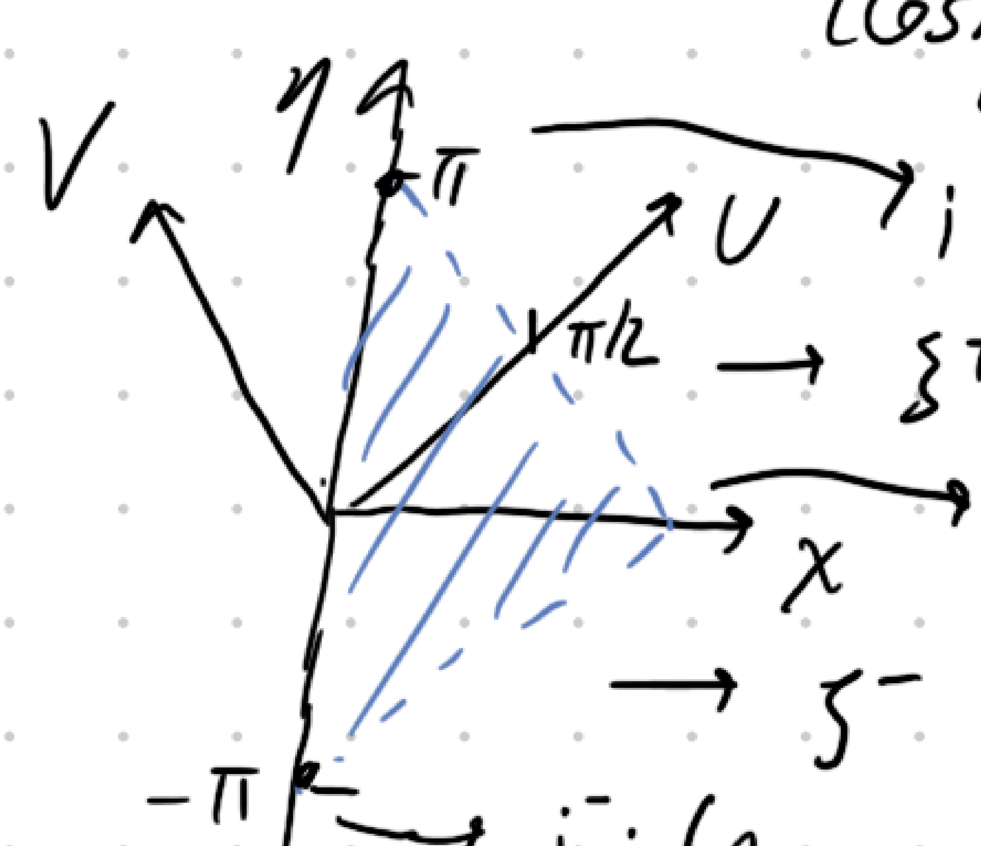
define $U = \arctan u$, $V = \arctan v$ $-\frac{\pi}{2} \le U, V \le \frac{\pi}{2}$, $V \le U$

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left[-2(dU\,dV + dV\,dU) + \sin^2(U-V)\,d\Omega^2\right]$$

redefine $\eta = U + V$, $\chi = U - V$. $ds^2 = \frac{4}{(\cos\eta + \cos\chi)^2}\left(-d\eta^2 + d\chi^2 + \sin^2\chi\, d\Omega^2\right)$



$\Longrightarrow$



i^+ ($\eta = \pi$, $\chi = 0$, $t = +\infty$, future timelike infinity)

$\mathscr{I}^+$ future null infinity $\eta = \pi - \chi$ scri-plus

i^0 ($\eta = 0$, $\chi = \pi$, $r = +\infty$ spacelike infinity)

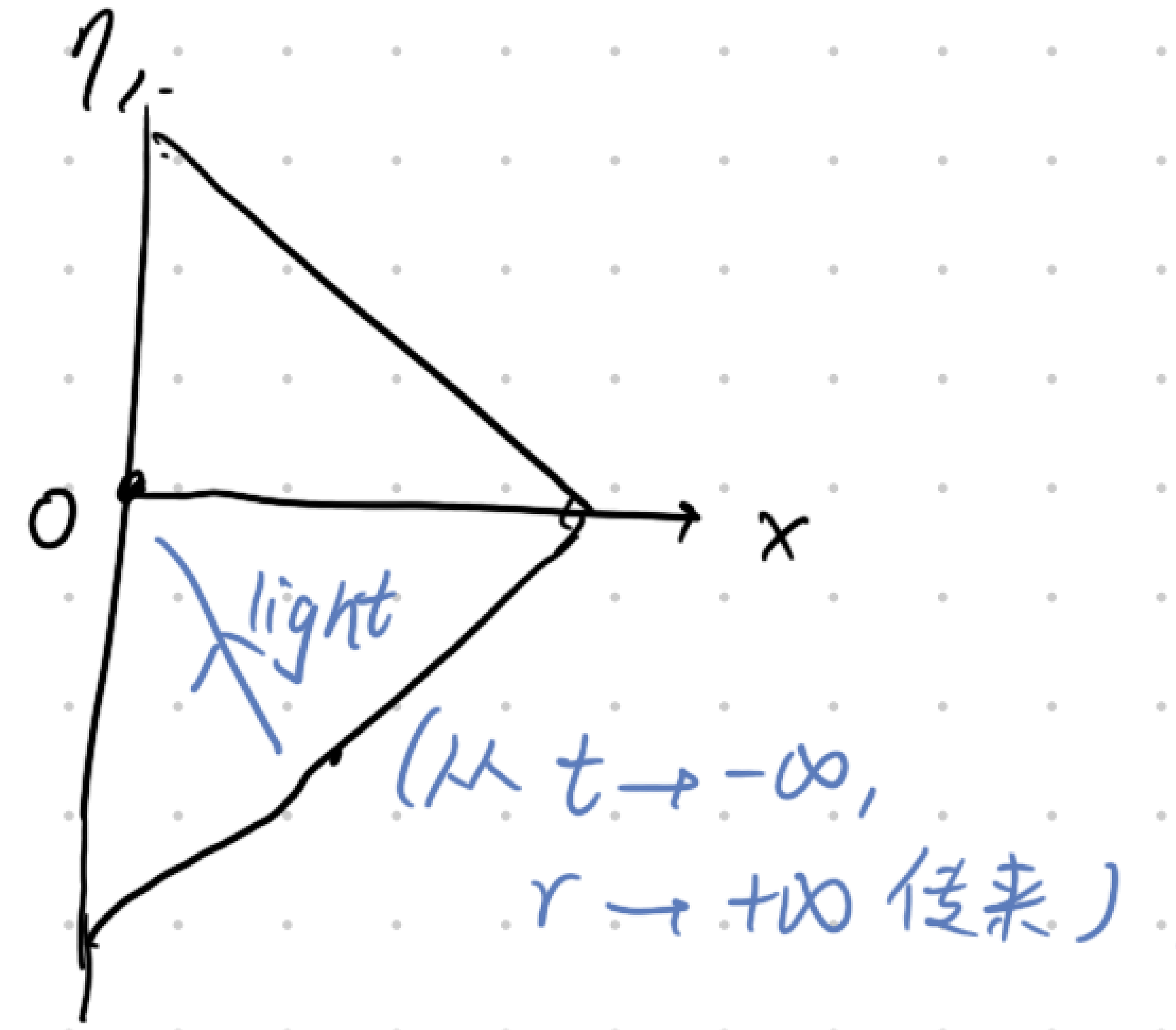
$\mathscr{I}^-$ past null infinity $\eta = -\pi + \chi$ scri-minus

i^-: ($\eta = -\pi$, $\chi = 0$, $t = -\infty$, past timelike infinity)

平直时空彭罗斯图

$\mathscr{J}^+$ $\eta=\pi-\chi$ e.g. $\eta=\chi=\frac{\pi}{2}$, $U=\frac{\pi}{2}, V=0$, $u\to+\infty, v=0$, $t,r\to+\infty$, $t=r$

$\mathscr{J}^-$ $\eta=-\pi+\chi$ e.g. $\eta=\chi=-\frac{\pi}{2}$, $r\to+\infty, t\to-\infty$ 且 $t+r=0$

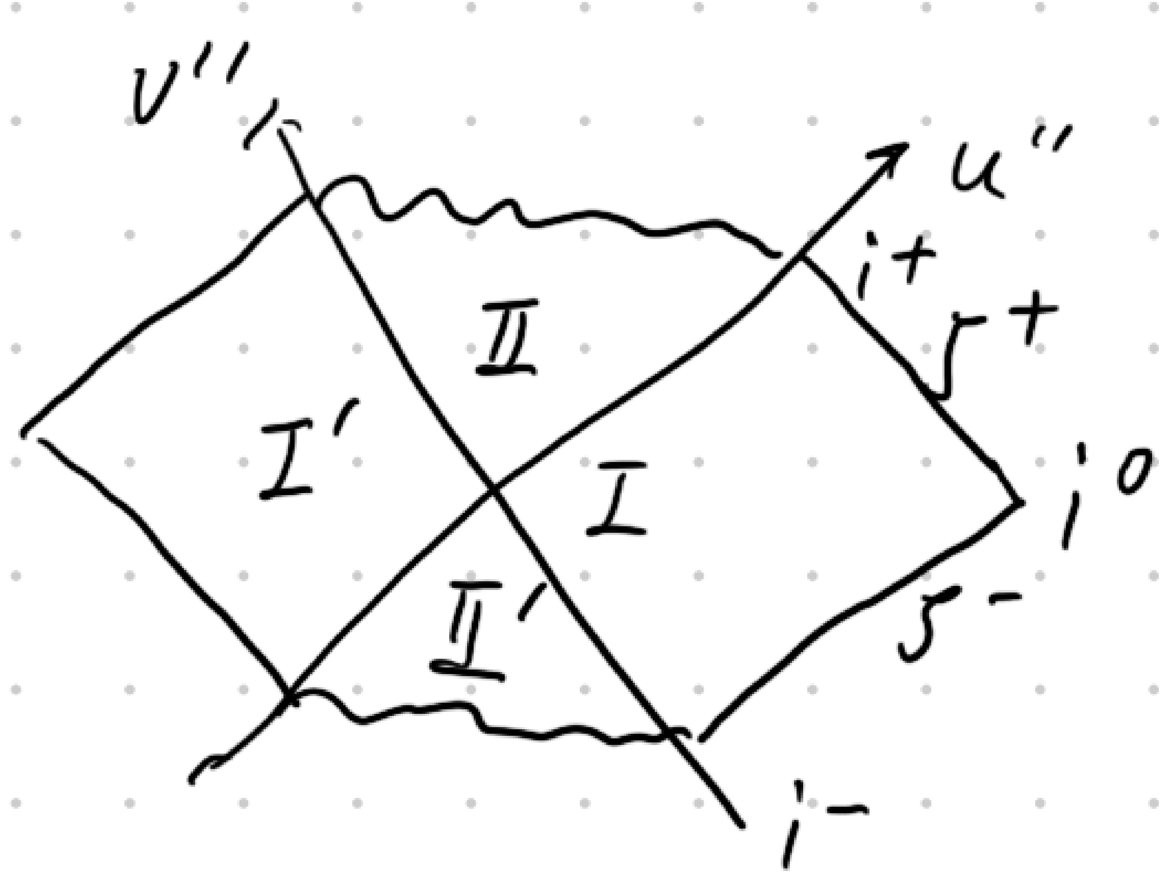


好处：把∞拉至有限距离，且不改变时空结构

□ Sch BH Kruskal coordinates $ds^2 = \frac{-2R_s^3}{r} e^{-r/R_s}(du'dv'+dv'du') + r^2 d\Omega^2$.

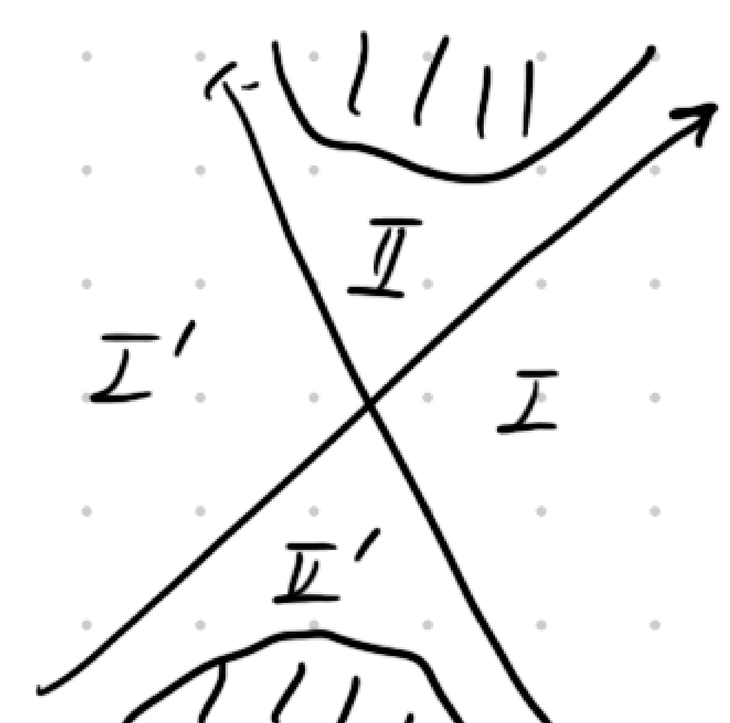
r is defined by $u'v'=(\frac{r}{R_s}-1)e^{r/R_s}$

$u''=\arctan u'$ $v''=\arctan v'$



波浪线：表示奇点 $r=0$

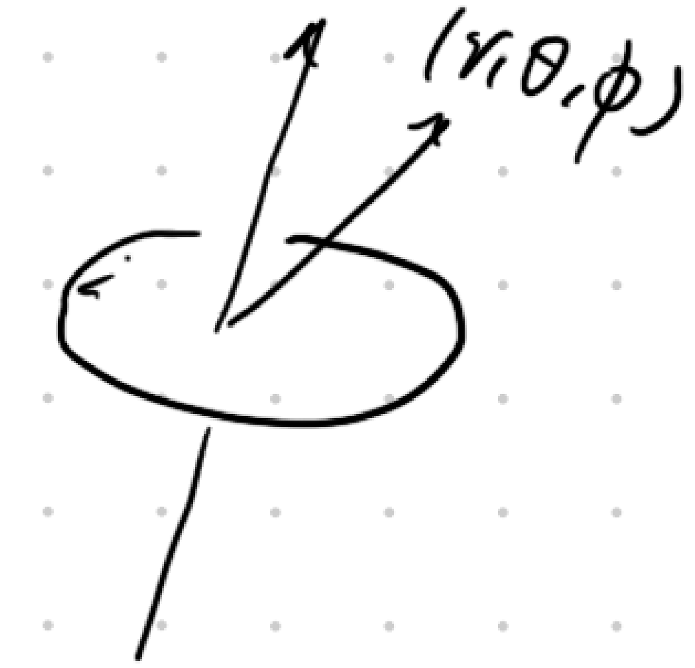
$\Longleftrightarrow$



24. Kerr black hole (Solution to vacumn Einstein's equation). 丧失时对称性，保留轴对称性.

metric of a rotating massive object (outside).

具有角动量.



黑洞吸收过程

$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta\, d\phi - dt)^2$$

$\Delta(r) = r^2 - 2GMr + a^2 \qquad \rho^2 = r^2 + a^2\cos^2\theta \qquad a = J/M$

J: BH 关于 ϕ-axis 角动量.

M: BH 质量.

Killing vector: $\xi = \partial_t$, $\eta = \partial_\phi$

defining $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$

$$\Rightarrow ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$\omega = \dfrac{2GMra}{\Sigma^2}$ angular velocity

outside of BH timelike Killing vector $\xi = \partial_t$: Stationary (稳态)

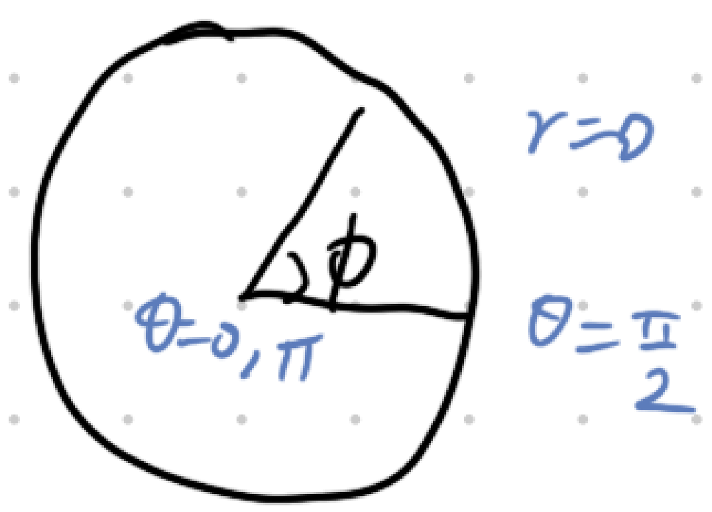
$g_{t\phi} \neq 0$ (无法通过坐标变换对角化) : not static (静态)

$r=0$: $\Delta = a^2 \quad \rho^2 = a^2\cos^2\theta \quad \Sigma^2 = a^4\cos^2\theta \quad ds^2 = -dt^2 + \cos^2\theta\, dr^2 + a^2\cos^2\theta\, d\theta^2 + a^2\sin^2\theta\, d\phi^2$

$\omega = 0$

$r=0$ is not a spatial point (不是奇点) is a disk with radius a.

outer bounding $\theta=\frac{\pi}{2}$.



奇点位置: $\rho=0 \Rightarrow r=0\ \&\ \theta=\frac{\pi}{2}$. ($R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\to\infty$) singularity is a ring

▷ Event horizon & Infinte "redshift" surface.

Event horizon $g^{rr}=0$ $\quad g_{rr}=\frac{\rho^2}{\Delta}=\infty \quad \Delta=r^2-2GMr+a^2=0$.

内部	horizon	外部
$g^{rr}, g_{rr}<0$		$g^{rr}, g_{rr}>0$

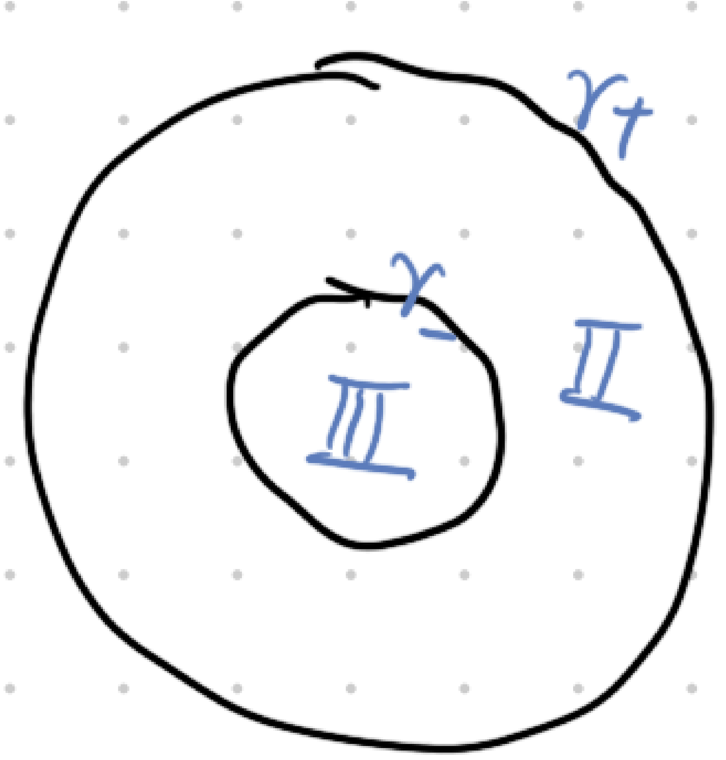
$$r_\pm = GM \pm \sqrt{(GM)^2-a^2}$$

1. $GM>a$ two horizons: inner r_-, outer r_+.

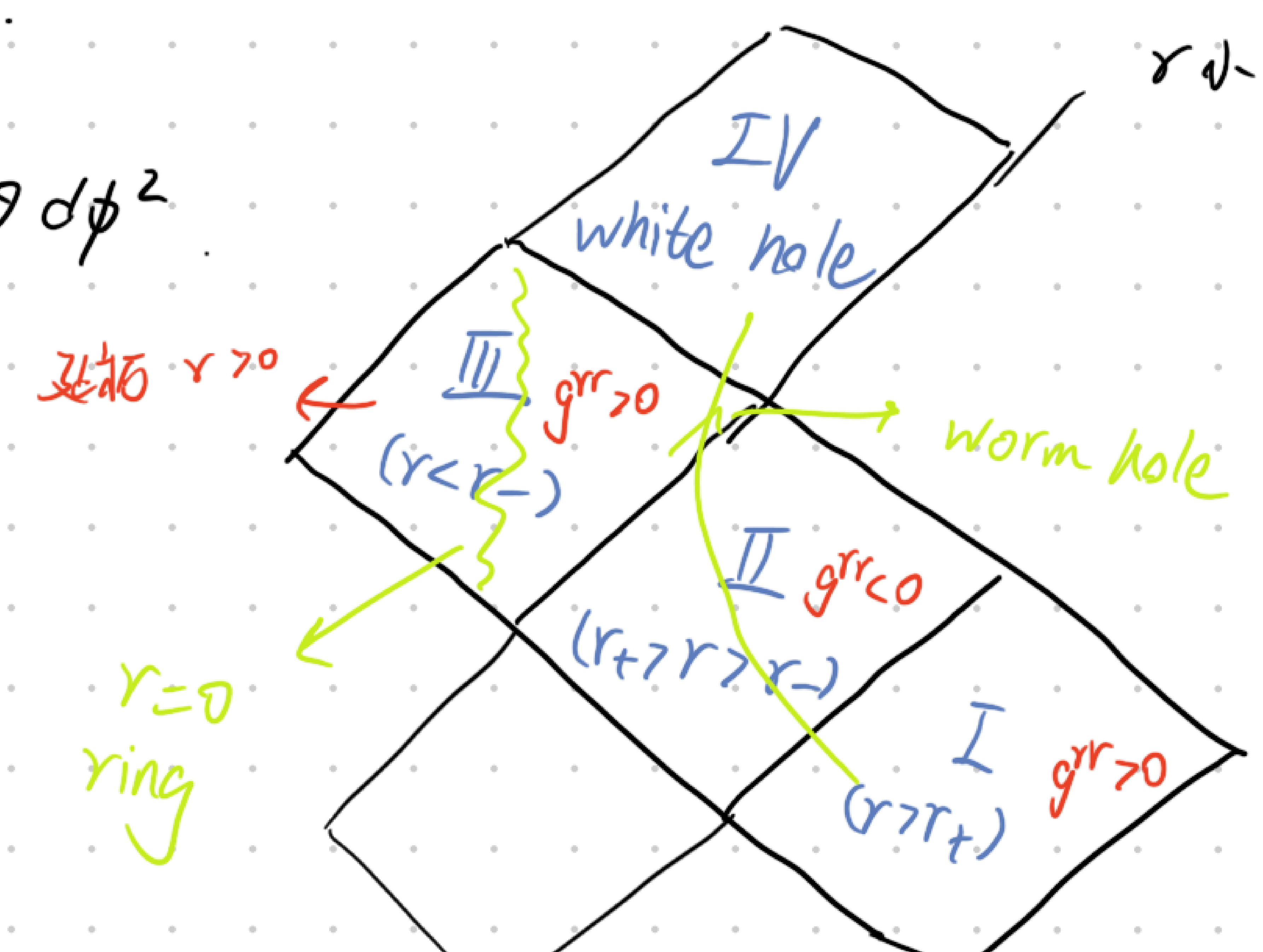
$r=r_\pm$, $ds^2=\rho_\pm^2 d\theta^2+\frac{(r_\pm^2+a^2)^2}{\rho_\pm^2}\sin^2\theta\, d\phi^2$.

内部: 类时观察者 $ds^2<0 \Rightarrow dr^2>0$

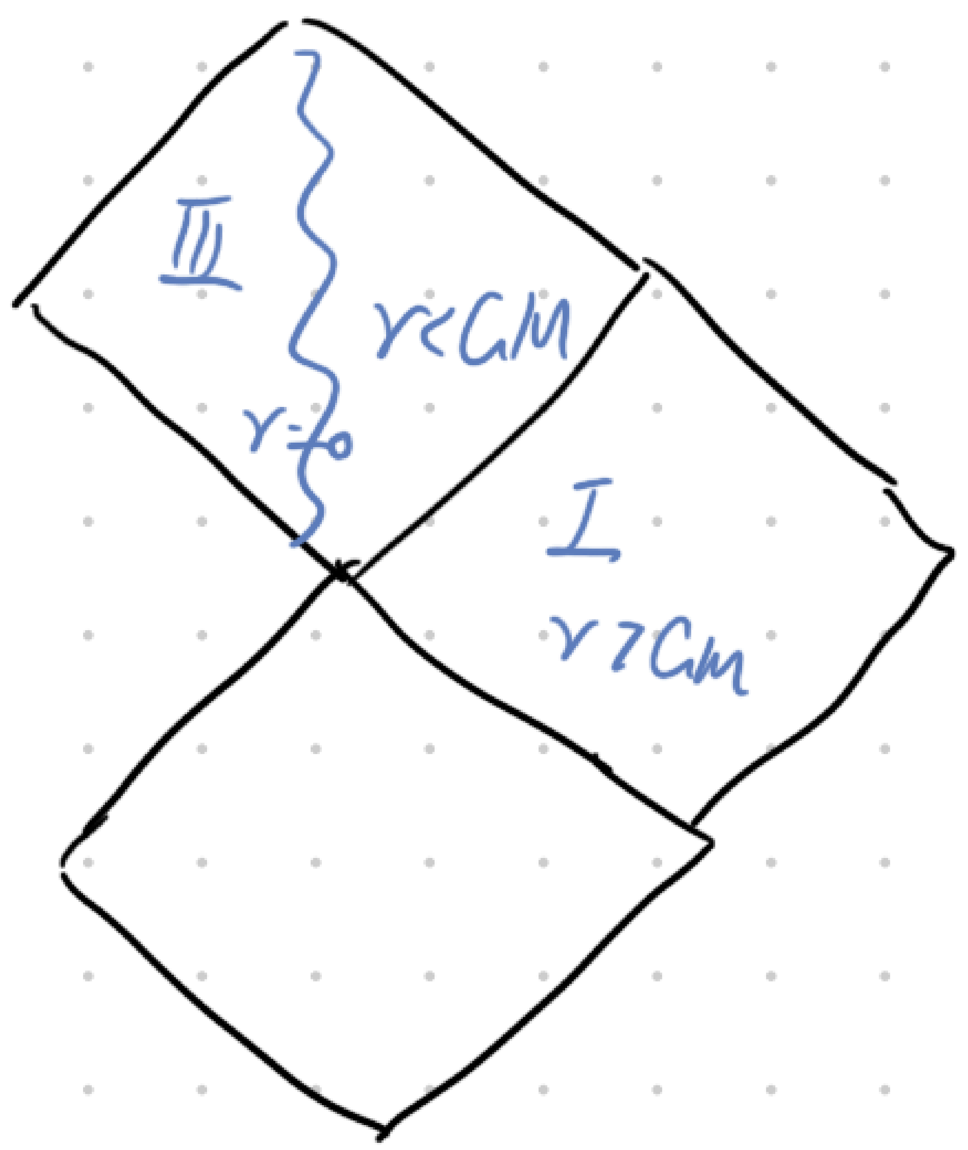
在视界内无法待在固定的 r 处，只能往 r 小的方向坠落.



Penrose diagram



2. $GM = a$ extremal Kerr BH $r_+ = r_- = a$.



3. $a > GM$ no event horizon

naked singularity at $r = \rho = 0$ (奇点未被视界包裹)

contracts Hawking's cosmic censorship hypothesis (宇宙监督准则).

- Geodesics Killing vector $\xi = \partial_t$, $\eta = \partial_\phi$. Consider simplification $\theta = \frac{\pi}{2}$ (赤道面上).

$u^\theta = 0$, $u^\mu = (u^t, u^r, 0, u^\phi)$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 - \frac{4aGM}{r}dt\,d\phi + \frac{r^2}{\Delta}dr^2 + \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)d\phi^2$$

$(\Delta = r^2 - 2GMr + a^2)$

Conserved quantities

energy density $E = -\xi \cdot u = -(g_{tt}u^t + g_{t\phi}u^\phi) = \left(1 - \frac{2GM}{r}\right)u^t + \frac{2aGM}{r}u^\phi$

angular momentum density $L = \eta \cdot u = g_{\phi t}u^t + g_{\phi\phi}u^\phi = -\frac{2aGM}{r}u^t + \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)u^\phi$

$$u^t = \frac{dt}{d\lambda} = \frac{1}{\Delta}\left(\left(r^2+a^2+\frac{2GMa^2}{r}\right)E - \frac{2GMaL}{r}\right) \qquad u^\phi = \frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left(\left(1-\frac{2GM}{r}\right)L + \frac{2GMa}{r}E\right)$$

$$u^\mu u_\mu = -\epsilon \qquad \begin{cases} \epsilon = 1 & \text{massive} \\ \epsilon = 0 & \text{massless} \end{cases}$$

$\epsilon = 1$ $\quad \frac{E^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff} \qquad V_{eff} = -\frac{GM}{r} + \frac{L^2-a^2(E^2-1)}{2r^2} - \frac{GM(L-aE)^2}{r^3}$

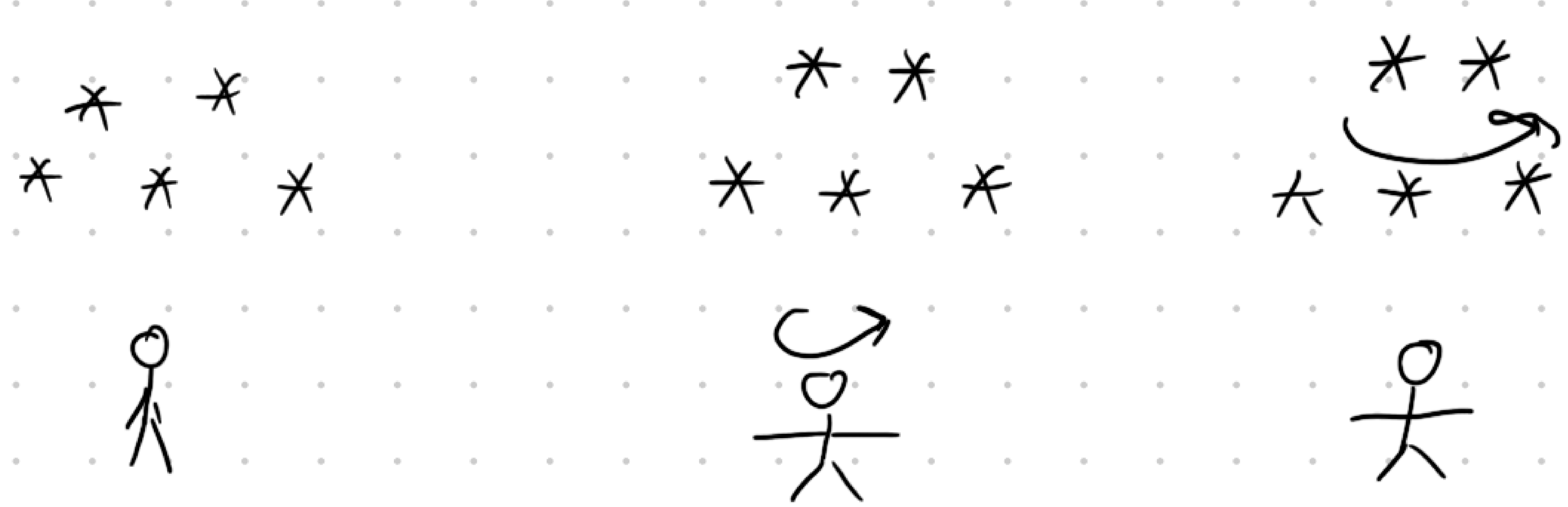
circular orbit $\frac{dr}{d\lambda}=0$ $\quad \frac{\partial V_{eff}}{\partial r} = 0$ $\qquad$ e.g. extremal $a=GM$ $\quad R = GM$ & $9GM$

closed timelike curve (CTC) (闭合类时曲线) (注意：时间上也闭合)

$\theta = \frac{\pi}{2}$, fix t & r $\quad ds^2 = \left(r^2+a^2+\frac{2GMa^2}{r}\right)d\phi^2 \quad (r \geq 0)$

若延拓至 $r<0$, $g_{\phi\phi}$ can be negative

- Mach's principle 马赫原理



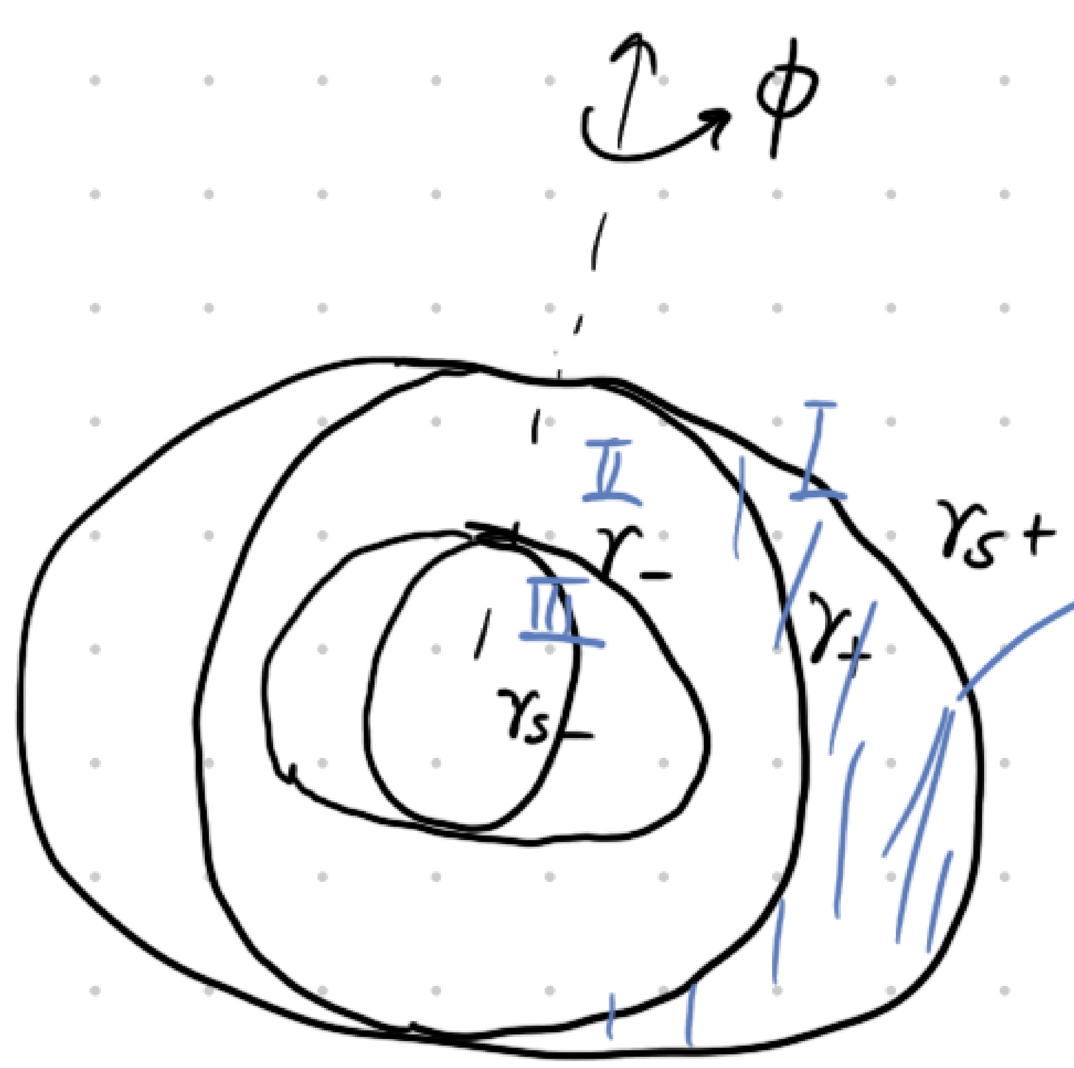
GR Mach's principle

Schwarzschild metric & Kerr metric

$\neq$

Kerr BH. outer horizon : $r_+ = GM + \sqrt{(GM)^2 - a^2}$ $\quad g^{rr} = 0$

Infinite red shift surface $g_{tt} = 0$ $\quad r_{s+} = GM + \sqrt{(GM)^2 - a^2\cos^2\theta} \geq r_+$



ergosphere (能层)

$g_{tt} > 0$, $g_{rr} > 0$

$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi$

timelike observer $ds^2 < 0$ $\quad g_{tt}, g_{rr}, g_{\theta\theta}, g_{\phi\phi} > 0 \Rightarrow g_{t\phi}dtd\phi < 0$

$g_{t\phi} < 0 \Rightarrow d\phi > 0$

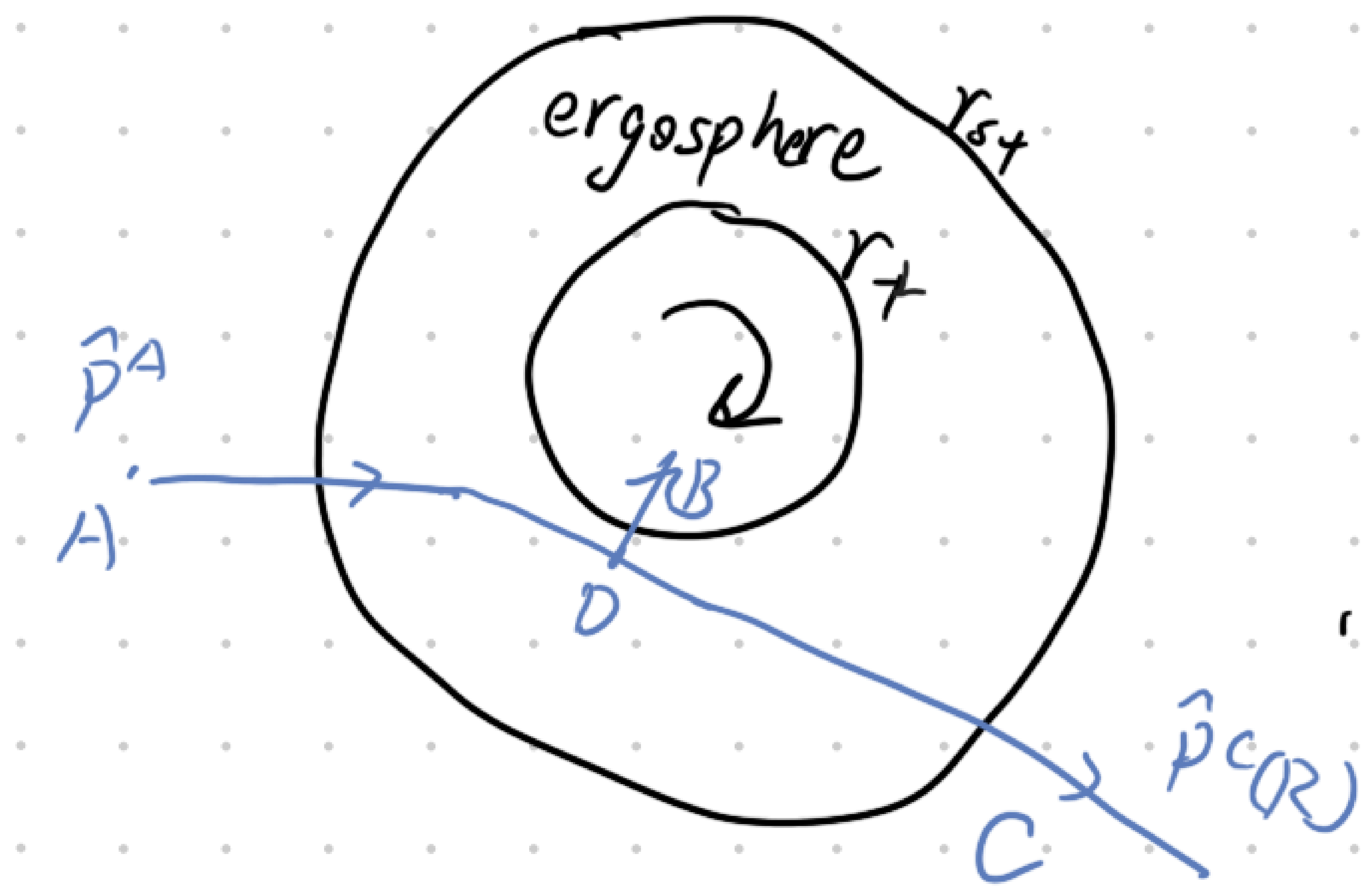
In ergosphere, observer must rotate with BH (same direction)

$g_{rr} > 0$, observer can still escape BH.

Permie process extract energy from BH

A particle is lauched to Kerr BH from $r=\infty$, $\vec{p}^{(A)}(\varepsilon)$.

for ∞ observer, energy of A $\quad E^A = -\vec{p}^A(\varepsilon)\cdot\vec{u}_{obs} = -p^A_t(\varepsilon)$



$u^\mu_{obs} = (1,0,0,0)$

at point D in ergosphere, $A \longrightarrow B + C$.

at D local conservation of 4-momentum $\quad \vec{p}^A(D) = \vec{p}^B(D) + \vec{p}^C(D)$.

C goes back to $r=\infty$, measured energy $E^C = -p^C_t(R)$

Killing vector $\partial_t \quad \Rightarrow \quad p^C_t(R) = p^C_t(D) \quad p^A_t(\varepsilon) = p^A_t(D) \quad\quad p^A_t(D) = p^B_t(D) + p^C_t(D)$

$\Rightarrow -E^A = p^B_t(D) - E^C \quad E^C = p^B_t(D) + E^A \quad p^B_t(D) = \vec{p}^B\cdot\vec{e}_t = p^{B,t}(D)\, g_{tt} + p^{B,\phi}(D)\, g_{\phi t}$

in ergosphere, $g_{tt} > 0$, $\quad p^B_t(D) = -E^B$ can be >0.

if $E^B < 0$, B can't escape from BH. $\rightsquigarrow E^C > E^A$

particle gains energy from BH

mass of BH is reduced

▫ In G.R. how to define conserved quantities e.g. mass M, angular momentum J, charge Q for a metric (BH)? (ADM mass)

Exists killing vector to define M & J ?

$d-1$ subspace V with boundary ∂V for each killing vector ξ define "Komar's integral".

$$Q_\xi(V) = \frac{c}{16\pi G}\oint_{\partial V} dS_{\mu\nu}\,\nabla^\mu \xi^\nu = \frac{c}{8\pi G}\int_V dS_\mu\, \nabla_\nu \nabla^\mu \xi^\nu$$

$(d-1)$ form $\propto \varepsilon_\mu^{\ \mu_1\cdots\mu_{d-1}}\, dS_{\mu_1\cdots\mu_{d-1}}$

Identity $\nabla_\nu \nabla_\mu \xi^\nu = R_{\mu\nu}\xi^\nu$ for Killing vector ξ^ν

$$\begin{cases} [\nabla_\nu, \nabla_\mu]\xi^\nu = R^\nu{}_{\rho\nu\mu}\xi^\rho = R_{\rho\mu}\xi^\rho = R_{\mu\nu}\xi^\nu \\ \nabla_{(\mu}\xi_{\nu)} = 0 \Rightarrow \nabla_\nu \xi^\nu = 0 \end{cases}$$

$$Q_\xi(V) = \frac{c}{8\pi G}\int_V dS_\mu\, R^\mu{}_\nu \xi^\nu$$ (using Einstein's eq. $R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}T)$)

$$= c\int_V dS_\mu \left(T^\mu{}_\nu \xi^\nu - \frac{1}{2}T\xi^\mu\right) = \int dS_\mu\, J^\mu(\xi)$$ satisfy : $\nabla_\mu J^\mu(\xi) = 0$

e.g. timelike Killing vector $\xi = \partial_t$ $\quad E(V) = \frac{1}{8\pi G}\oint_{\partial V} dS_{\mu\nu}\,\nabla^\mu \xi^\nu = M$

$\hat{\eta} = \partial\phi$, $J(V) = \frac{1}{16\pi G} \oint_{\partial V} dS_{\mu\nu} \nabla^\mu \eta^\nu$

M & J are associated with spacetime symmetry

Q electric charge 内部对称性 global gauge symmetry $\phi \to e^{i\gamma}\phi$

$Q(V) = \int_V j^{(d-1)} = \int_V d{*}F = \int_{\partial V} {*}F$ $\nabla_\mu F^{\mu\nu} = j^\nu \Rightarrow d({*}F) = j^{(d-1)}$

- No-hair theorem (无毛定理) all stationary BH solution in GR + EM can be completely characterized by the mass M, angular momentum J, electric charge Q.

$J = Q = 0$ Schwarzschild BH

$Q = 0, J \neq 0$ Kerr BH

$Q \neq 0, J = 0$ Reissner-Nordström (RN) BH

$Q, J \neq 0$ Kerr-Newman BH

BH thermodynamics (Stationary BH)

Hawking Black hole has 熵

M, J, Q

$S_{BH} = \frac{A}{4G} = \frac{A}{4}$ → surface area of event horizon

热力学量	BH
Energy U	mass M
Temperature T	surface gravity κ $\left(T_H = \frac{\kappa}{2\pi}\right)$
Entropy S	surface area A $\left(S_{BH} = \frac{A}{4}\right)$
μ	$J, Q, \ldots$

0th law T is constant in 热平衡 — κ is constant for stationary BH

1st law $dU = TdS + dW$ — $\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J$ $\quad \Omega_H = \frac{\delta M}{\delta J}$.

2nd law $\delta S \geq 0$ — $\delta A \geq 0$

3rd law $T=0$ can't be reached in 有限步 — $\kappa = 0$ extremal BH can't be reached 有限步

25. Gravitational wave

linear gravity limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $|h_{\mu\nu}| \ll 1$

$h_{\mu\nu}$ has gauge transformation $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Einstein's equation $\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \Box h = -16\pi G T_{\mu\nu}$ $(h = h^\mu{}_\mu \,,\ \Box = \partial_\mu \partial^\mu)$

harmonic gauge $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$ $\partial^\mu h = 2\partial^\nu h_\nu{}^\mu$

$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \implies \Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$ is a wave equation.

harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$

□ Vacumn gravitional wave solution $\Box \bar{h}_{\mu\nu} = 0$

plane wave solution $\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma}$ $C_{\mu\nu} = C_{\nu\mu}$

$\partial_\rho \bar{h}_{\mu\nu} = ik_\rho C_{\mu\nu} e^{ik_\sigma x^\sigma}$

$\Box \bar{h}_{\mu\nu} = 0 \implies -k_\rho k^\rho \bar{h}_{\mu\nu} = 0 \Rightarrow k_\rho k^\rho = 0.$

$\hat{k} = (\omega, \vec{k})$ $\hat{k}\cdot\hat{k} = 0 \Rightarrow \omega^2 = |\vec{k}|^2$ gravitational wave propagates at the speed of light.

$C_{\mu\nu}$ 有多少独立分量？ harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ $k^\mu C_{\mu\nu} = 0$

$C_{\mu\nu} = C_{\nu\mu} \longrightarrow$ 自由度 10.

进行规范变换 $h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ $\xi_\mu = B_\mu e^{ik_\sigma x^\sigma}$

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(h - 2\partial_\lambda \xi^\lambda)$$

$$= \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\lambda \xi^\lambda$$

$$\Rightarrow C'_{\mu\nu} = C_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu} k_\lambda B^\lambda$$

we can choose B_μ s.t. $C'^\mu{}_\mu = 0$ $C'_{0\nu} = 0$

$C'^\mu{}_\mu = 0 \Rightarrow k_\mu B^\mu = \frac{i}{2} C^\mu{}_\mu$ $(\eta_{\mu\nu}\eta^{\mu\nu} = 4)$

$C'_{00} = 0 \Rightarrow B_0 = -\frac{i}{2k_0}(C_{00} + \frac{1}{2}C^\mu{}_\mu)$

$C'_{0j} = 0 \Rightarrow B_j = \frac{1}{2k_0^2}(-2k_0 C_{0j} + k_j(C_{00} + \frac{1}{2}C^\mu{}_\mu))$

\# $k_\mu C^{\mu\nu} = 0$ $\downarrow$ -3 $(k_\mu C^{\mu 0}$ 不独立$)$

$C^\mu{}_\mu = 0$ $\downarrow$ -1

$C_{0\mu} = 0$ $\downarrow$ -4

独立限制自由度8个 $10 - 8 = 2$ 引力波独立的有2个.(同光波一样)

wave propagate along z-direction $k^\mu = (\omega, 0, 0, \omega)$ $k^\mu C_{\mu\nu} = 0$

$k^0 C_{0\mu} + k^3 C_{3\mu} = 0$ 又 $C^{\mu}{}_{\mu} = 0$ $\Rightarrow$ $C_{11} = -C_{22}$, $C_{12} = C_{21}$ 其余均为 0

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

transverse traceless gauge (横向无迹规范)

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h = h_{\mu\nu}$$

12 with source $T_{\mu\nu}$. $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$ (use Green's function)

$$\partial^{\mu}\partial_{\mu} G(x^{\sigma} - y^{\sigma}) = \delta^{(4)}(x^{\sigma} - y^{\sigma})$$

general solution $\bar{h}_{\mu\nu}(x^{\sigma}) = -16\pi G \int G(x^{\sigma} - y^{\sigma}) T_{\mu\nu}(y^{\sigma})\, d^4y$

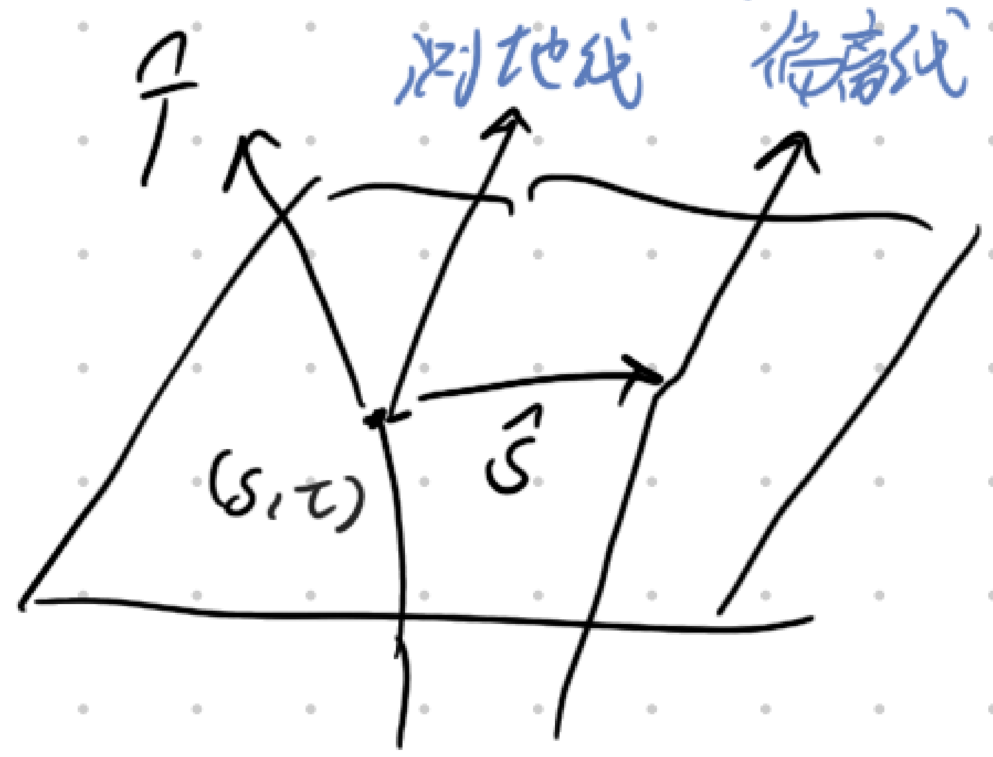
retarded Green's function $G(x^{\sigma} - y^{\sigma}) = -\dfrac{1}{4\pi|\vec{x}-\vec{y}|}\delta[|\vec{x}-\vec{y}| - (x^0 - y^0)]\,\theta(x^0 - y^0)$ $\theta(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G \int \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(t - |\vec{x}-\vec{y}|, \vec{y})\, d^3y$$

$$(t - |\vec{x}-\vec{y}|, \vec{y}) \xrightarrow{\text{light}} (t, \vec{x})$$

□ affect of gravitional wave on particles

geodesic deviation 测地线偏离



tangent vector along a geodesic $T^\mu = \dfrac{\partial x^\mu}{\partial \tau}$

deviation vector $S^\mu = \dfrac{\partial x^\mu}{\partial s}$.

$\hat{S}, \hat{T}$ 互相正交 $\quad [\hat{S}, \hat{T}] = 0 \quad S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$

geodesic deviation velocity $V^\mu = (\nabla_{\hat{T}} \hat{S})^\mu = T^\rho \nabla_\rho S^\mu$

geodesic deviation accaleration

$$a^\mu = (\nabla_{\hat{T}} \hat{V})^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu)$$

$$= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) = (T^\rho \nabla_\rho S^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$$= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu).$$

$$= S^\sigma \nabla_\sigma \underbrace{(T^\rho \nabla_\rho T^\mu)}_{=0} + R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$$= R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma = \frac{D^2 S^\mu}{d\tau^2} \quad \text{(geodesic deviation equation)}$$

gravitational wave (plane wave in vaccum) $u^\mu = (1,0,0,0) = T^\mu$

$R_{\mu\nu\rho\sigma} = \cdots = \frac{1}{2}\partial_0\partial_0 h_{\nu\sigma}$ (其余为0)

$$\frac{\partial^2 S^\mu}{\partial t^2} = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^\mu{}_\sigma \qquad h^\mu{}_\sigma = C^\mu{}_\sigma\, e^{ik_\lambda x^\lambda} \qquad C_{11},\ C_{12} = C_{21},\ C_{22} = -C_{11}.$$

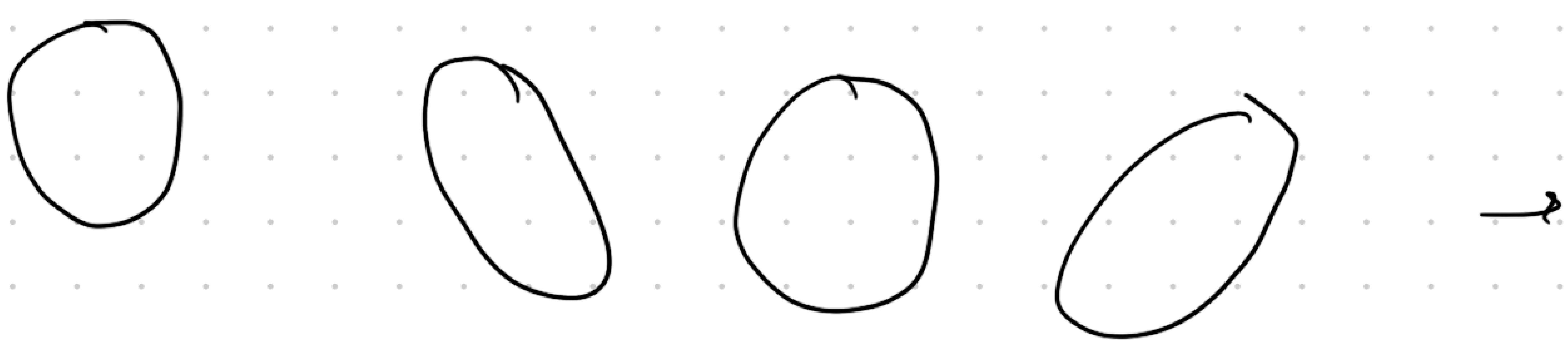
Two different modes

(1) $C_+ = C_{11} \neq 0 \quad C_{12} = 0$ (2) $C_\times = C_{12} \neq 0 \quad C_{11} = 0$.

(1) $S^1 = \left(1 + \frac{1}{2} C_+ e^{ik_\sigma x^\sigma}\right) S^1(0) \qquad S^2 = \left(1 - \frac{1}{2} C_+ e^{ik_\sigma x^\sigma}\right) S^2(0).$

(1)

(2)



26. FRW cosmology

- cosmological principle: the space is isotropic (各向同性) and homogeneous (均匀)

(最大对称空间假设 → Minkowski, dS, AdS 过强)

认为空间上是最大对称的，允许时间上的演化.

Comoving coordinate $ds^2 = -dt^2 + g_{ij}\,dx^i dx^j$ (膨胀的气球)

可验证: observer $x^\mu(\tau)$ $x^0 = t = \tau$ $x^i = \text{constant}$ goes along a geodesic.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \qquad u^\mu = \frac{dx^\mu}{d\tau} = (1,0,0,0) \qquad \Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \partial_\nu g_{00}) = 0$$

各向同性 & 均匀 $\Rightarrow$ $ds^2 = -dt^2 + a^2(t)\,h_{ij}\,dx^i dx^j$ $h_{ij} = h_{ij}(x^i)$ 与 t 无关

空间部分 $h_{ij}\,dx^i dx^j = C(r)\,dr^2 + r^2 d\Omega^2$ $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

验证: 3维最大对称空间满足 $R_{ijkl} = K(h_{ik}h_{jl} - h_{il}h_{jk})$ $R_{ij} = 2Kh_{ij}$ $R = 6K$

$$\Gamma^r_{rr} = \frac{C'}{2C} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{C} \qquad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{C} \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\phi\theta} = \cot\theta$$

$$R_{rr} = \frac{C'}{rC} \qquad R_{\theta\theta} = -\frac{1}{C} + 1 + \frac{rC'}{2C^2} \qquad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$$

$$\Rightarrow \quad \frac{C'}{rC} = 2KC \qquad 1 + \frac{rC'}{2C^2} - \frac{1}{C} = 2Kr^2 \qquad \Rightarrow C = \frac{1}{1-Kr^2}$$

$$\Rightarrow \quad ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

rescale r & $a(t)$

$$\begin{cases} K>0 & r \to \frac{r}{\sqrt{K}} \quad a(t) \to \frac{a(t)}{\sqrt{K}} & ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right) \\ K=0 & & ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \\ K<0 & r \to r/\sqrt{-K} \quad a(t) \to a(t)/\sqrt{-K} & ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1+r^2} + r^2 d\Omega^2\right) \end{cases}$$

$$K = \begin{cases} 1 & \text{负曲率} \quad \text{双曲、开放} \\ 0 & \text{零曲率} \quad \text{平直} \\ -1 & \text{正曲率} \quad \text{球面，闭合} \end{cases}$$

$ds^2 \longrightarrow$ Einstein equation $\quad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$

$$\Rightarrow R_{tt} = -\frac{3\ddot{a}}{a} \qquad R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1-Kr^2} \qquad R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K) \qquad R_{\phi\phi} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)\sin^2\theta$$

matter considered as perfect fluid $T_{\mu\nu} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu}$ 依时共动坐标： $u^\mu = (1,0,0,0)$

$u_\mu = (-1,0,0,0)$

$T_{tt} = \rho \quad T_{ti} = 0 \quad T_{ij} = p g_{ij} \quad T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p) \quad T = T^\mu{}_\mu = -\rho + 3p$

Einsten eq. tt component $-\frac{3\ddot{a}}{a} = 8\pi G(\rho + \frac{1}{2}(-\rho+3p)) = 4\pi G(\rho + 3p)$

$rr/\theta\theta/\phi\phi$: $\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - p)$

$$\Rightarrow \begin{cases} \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho+3p) \\ \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \end{cases}$$

$\rho(t) = \sum_i \rho_i(t) = \rho_m(t) + \rho_R(t) + \rho_\Lambda(t)$

matter (contain dark matter)

radiation

宇宙常数 (dark energy)

matter	$p = 0$
radiation	$p = \frac{1}{3}\rho$
c.c	$p = -\rho$

energy conservation for each component $\rho_i \propto a^{-3(1+w_i)}$ $\quad w_i = p_i/\rho_i$

matter $\rho_m(t) = \rho_m(0)\left(\frac{a_0}{a(t)}\right)^3$

radiation $\rho_R(t) = \rho_R(0)\left(\frac{a_0}{a(t)}\right)^4$

c.c. $\rho_\Lambda(t) = \rho_\Lambda(0)$

Hubble's constant (哈勃常数) $H(t) = \frac{\dot{a}(t)}{a(t)}$ (NOT a constant)

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \Rightarrow \begin{cases} H^2 = \frac{8\pi G}{3}\sum_i \rho_i(t) - \frac{K}{a^2} \\ \dot{H} = -4\pi G\left(\sum_i (1+w_i)\rho_i - \frac{K}{4\pi G a^2}\right) \end{cases}$$

def $\Omega_i(t) = \frac{8\pi G}{3H^2(t)}\rho_i(t)$

↓

归一化的能量密度

$$H^2 = H_0^2\left(\Omega_{m,0}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_{R,0}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_{\Lambda,0} + \Omega_K(t)\right)$$

Now $\Omega_{m,0} \approx 0.3$ $\Omega_{R,0} \approx 5\times 10^{-5}$ $\Omega_{\Lambda,0} \approx 0.7$

$$\Omega_K(t) = -\frac{K(t)}{H_0^2 a^2}$$

Geodesic in FRW universe $ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S^2(\chi)\, d\Omega^2\right]$

$$S(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1. \end{cases}$$

massless particle at $\chi = 0$ $\quad u_\phi = u_\theta = 0$ (satisfy geodesic eq.).

$r = S(\chi)$.

$u^\mu u_\mu = 0 \qquad \left(\dfrac{dt}{d\lambda}\right)^2 = a^2(t)\left(\dfrac{d\chi}{d\lambda}\right)^2 \qquad K^t = -a(t)\, K^\chi \qquad K_t = \dfrac{1}{a(t)} K_\chi \qquad K_\chi$ is a constant $\dfrac{du_\chi}{d\lambda} = 0$.

4动量分量

$\begin{cases} \text{at } t_E \text{, emits a photon with } \nu_E \\ \text{at } t_R \text{, received frequency } \nu_R. \end{cases}$

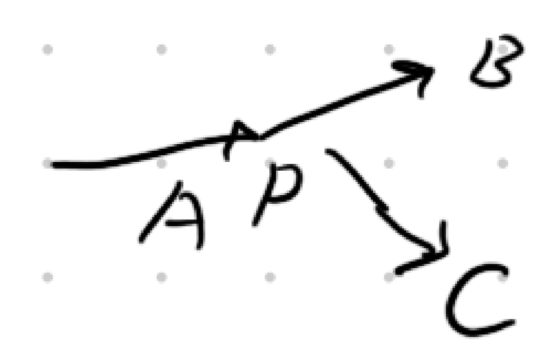
$$\frac{\nu_R}{\nu_E} = \frac{\hat{u}_R \cdot \hat{K}_R}{\hat{u}_E \cdot \hat{K}_E} = \frac{K_t^R}{K_t^E} = \frac{a(t_E)}{a(t_R)} \qquad t_R = t_E + \delta(t) \qquad \frac{a(t_R)}{a(t_E)} \simeq 1 + H(t_E)\,\delta t > 1 \Rightarrow \text{红移}$$

Review

GR $\begin{cases} \text{Geometry : curved differential manifold} \\ g_{\mu\nu} \longmapsto \Gamma^{\mu}_{\nu\rho} \longmapsto R_{\mu\nu\rho\sigma} \\ \text{Kinematics : geodesics / particle moving} \\ \text{Dynamics : Einstein's equation \& EH action} \\ \quad \text{solving vacumn Einstein's eq. / gravitional wave / cosmology.} \end{cases}$

12. SR (1) $g_{\mu\nu} \to \eta_{\mu\nu}$ SR $\quad \eta_{\mu\nu} \to g_{\mu\nu}$ GR

(2) In GR locally at a spacetime point, can define local inertia frame (局域惯性系).

(e.g. momentum conservation of $A \to B + C$ $\quad p^{\mu}_{A}(P) = p^{\mu}_{B}(P) + p^{\mu}_{C}(P)$ )

12 4-velocity (1) massive particle $\quad u^{\mu} = \dfrac{dx^{\mu}}{d\tau} \quad u^{\mu}u_{\mu} = -1$

for general metric comoving frame $\quad u^{\mu} = \left(\dfrac{1}{\sqrt{-g_{tt}}}, 0, 0, 0\right) \Rightarrow u^{\mu}u_{\mu} = -1$

4-momentum $\quad p^{\mu} = m u^{\mu} \quad p^{\mu}p_{\mu} = -m^2$

(2) massless particle $\quad u^{\mu}u_{\mu} = 0 \quad$ 4-wave vector $u^{\mu} = k^{\mu}$

$K^\mu = (\omega\ \vec{k})$ $K^\mu K_\mu = 0 \iff \omega^2 = |\vec{k}|^2$

lightlike/null curve $\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} = 0$ $u^\mu = \frac{dx^\mu}{d\lambda}$ choose λ s.t. $\frac{du^\mu}{d\lambda} = 0$.

- Measurement from an observer: define a local set of tetrads $\hat{e}_i$ $(i=0,1,2,3)$

 tetrad in general metric satisfy: $\hat{e}_i \cdot \hat{e}_j = \eta_{ij}$ $(-,+,+,+)$ $\Rightarrow g_{\mu\nu} e_i^\mu e_j^\nu = \eta_{ij}$

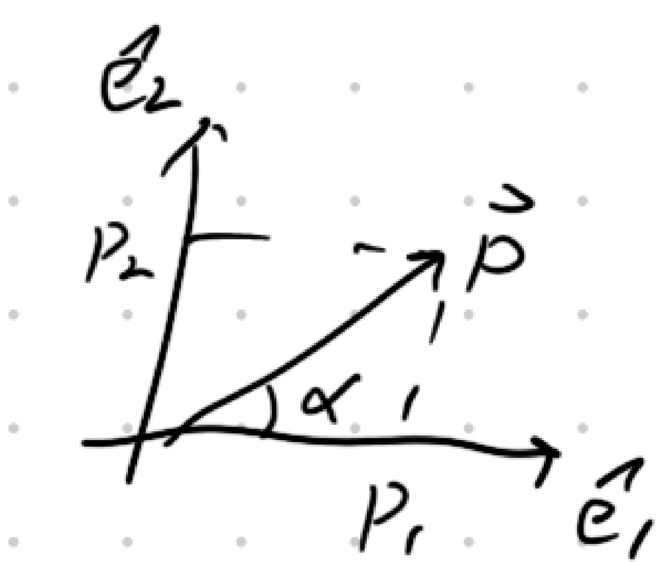
 $\hat{e}_0 = \hat{u}_{obs}$ along 4-velocity of observer measured energy $E = -\vec{p}\cdot\hat{e}_0$ $(-\vec{k}\cdot\hat{e}_0)$

 $\hat{e}_i$ $(i=1,2,3)$ measured momentum $p_i = \vec{p}\cdot\hat{e}_i$ $(\vec{k}\cdot\hat{e}_i)$

 广义协变性原理：观察者局域地看附近时空是闵氏时空.

 e.g. (1) redshift factor $\frac{\nu_E}{\nu_R} = \frac{-k_{E,\mu} u_E^\mu}{-k_{R,\mu} u_R^\mu}$

 (2) angle of momentum for an observer $\hat{u}_{obs}^R$, $\hat{e}_i$ $\tan\alpha = \frac{p_2}{p_1} = \frac{\vec{p}\cdot\hat{e}_2}{\vec{p}\cdot\hat{e}_1} = \frac{p_\mu e_2^\mu}{p_\mu e_1^\mu}$



- Tensors on manifold

 coordinate transformation: (1) transition between coordinate patch

 (2) diffeomorphism (微分同胚)

$x^\mu \to x^{\mu'}$ (p,q)-tensor transformation $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \left(\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\right)\cdots\left(\frac{\partial x^{\nu_1'}}{\partial x^{\nu_1}}\right)\cdots T^{\mu_1'\cdots\mu_p'}{}_{\nu_1'\cdots\nu_q'}$

Invariant tensor in SR : e.g $\eta_{\mu\nu}$ $(\eta'_{\mu\nu} = \eta_{\mu\nu})$ $\xrightarrow{\text{推广}}$ GR $g_{\mu\nu}$ (only under Killing vector)

GR metric vector $g_{\mu\nu}$

Length of line segment $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ volumn element in n-dim $dV = \sqrt{|g|}\, d^n x$.

Levi-civita tensor $\varepsilon_{\mu_1 \cdots \mu_n} = \tilde{\varepsilon}_{\mu_1 \cdots \mu_n} \sqrt{|g|}$ $\sqrt{|g|}\, d^n x = \frac{1}{n!} \varepsilon_{\mu_1 \cdots \mu_n} dx^1 \wedge \cdots \wedge dx^n$.

Differential form $A_p = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots dx^{\mu_p}$ $dA_p = 0$ $(d^2 = 0)$ $A_p \wedge B_q$ $*A_p$.

$g_{\mu\nu} \Rightarrow$ connection $\Gamma^\mu_{\nu\rho}$ $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ $\nabla_\rho g_{\mu\nu} = 0$

$\nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\lambda \sigma_2 \cdots \sigma_l} \cdots$

$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho})$

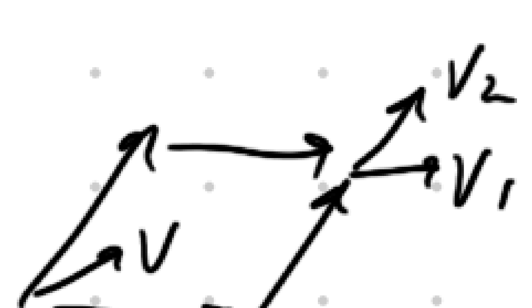
parallel transport of vector (field) $\frac{DV^\mu}{d\lambda} = 0 \Rightarrow \frac{dx^\nu}{d\lambda} \nabla_\nu V^\mu = 0$.

geodesic a curve that parallel transports its own tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$

$\frac{dx^\nu}{d\lambda} \nabla_\nu \frac{dx^\mu}{d\lambda} = 0$ $u^\nu \nabla_\nu u^\mu = 0$ $\Leftrightarrow \ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\rho \dot{x}^\sigma = 0$ $(\dot{u}^\mu + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0)$

变分定义: the length of geodesic is local extremum.

$\Gamma^\mu_{\nu\rho} \to R^\mu{}_{\nu\rho\sigma}$ $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma} \Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho} \Gamma^\mu_{\lambda\sigma}$



Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = g^{\lambda\rho} R_{\rho\mu\lambda\nu}$ Ricci scalar $R = R^\mu{}_\mu = R_{\mu\nu} g^{\mu\nu}$

$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$

maximal sym space $R_{\mu\nu\rho\sigma} \propto (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$

□ Symmety of M : Killing vector ξ^μ (使 $g'_{\alpha\beta}(x') = g_{\alpha\beta}(x')$ 的变换 $x^\nu \to x'^\mu$ 的生成元).

$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

if $\frac{\partial g_{\mu\nu}}{\partial x^1} = 0$ exist constant Killing vector $\xi^1 = 1$, other $\xi^\lambda = 0$ $(\lambda \neq 1)$

n-dim maximally symmetric space has $\frac{n(n+1)}{2}$ Killing vectors

dim of Lie algebra

conserved quantity $\xi^\mu u_\mu$ / $\xi^\mu p_\mu$ along geodesic

$$u^\mu \nabla_\mu(\xi^\nu u_\nu) = u^\mu u_\nu \nabla_\mu \xi^\nu + \underbrace{u^\mu \xi^\nu \nabla_\mu u_\nu}_{=0} = 0$$

redshift factor in stationary spacetime both E & R are static

$u_E^\mu = \left(\frac{1}{\sqrt{-g_{tt,E}}}, 0, 0, 0\right) \quad u_R^\mu = \left(\frac{1}{\sqrt{-g_{tt,R}}}, 0, 0, 0\right)$.

$$\frac{\nu_E}{\nu_R} = \frac{k_{E,\mu} u_E^\mu}{k_{R,\mu} u_R^\mu} = \frac{k_{t,E}}{k_{t,R}}\sqrt{\frac{g_{tt,R}}{g_{tt,E}}} = \sqrt{\frac{g_{tt,R}}{g_{tt,E}}}$$

□ Schwarzschild metric $ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$ $(R_s = 2GM)$

unique stationary, spherical symmetric solution to Einstein's eq.

$\xi = \partial_t$ only time like when $r > R_s$ (only stationary when $r > R_s$)

don't exist Global time like Killing vector

also has $\hat{\eta} = \partial_\phi$, $\eta^\mu = (0, 0, 0, 1)$

conserved quantity along geodesic $E = -\xi_\mu U^\mu = (1-\frac{R_s}{r})\frac{dt}{d\lambda}$ $\quad L = \eta_\mu U^\mu = r^2\sin\theta\frac{d\phi}{d\lambda}$.

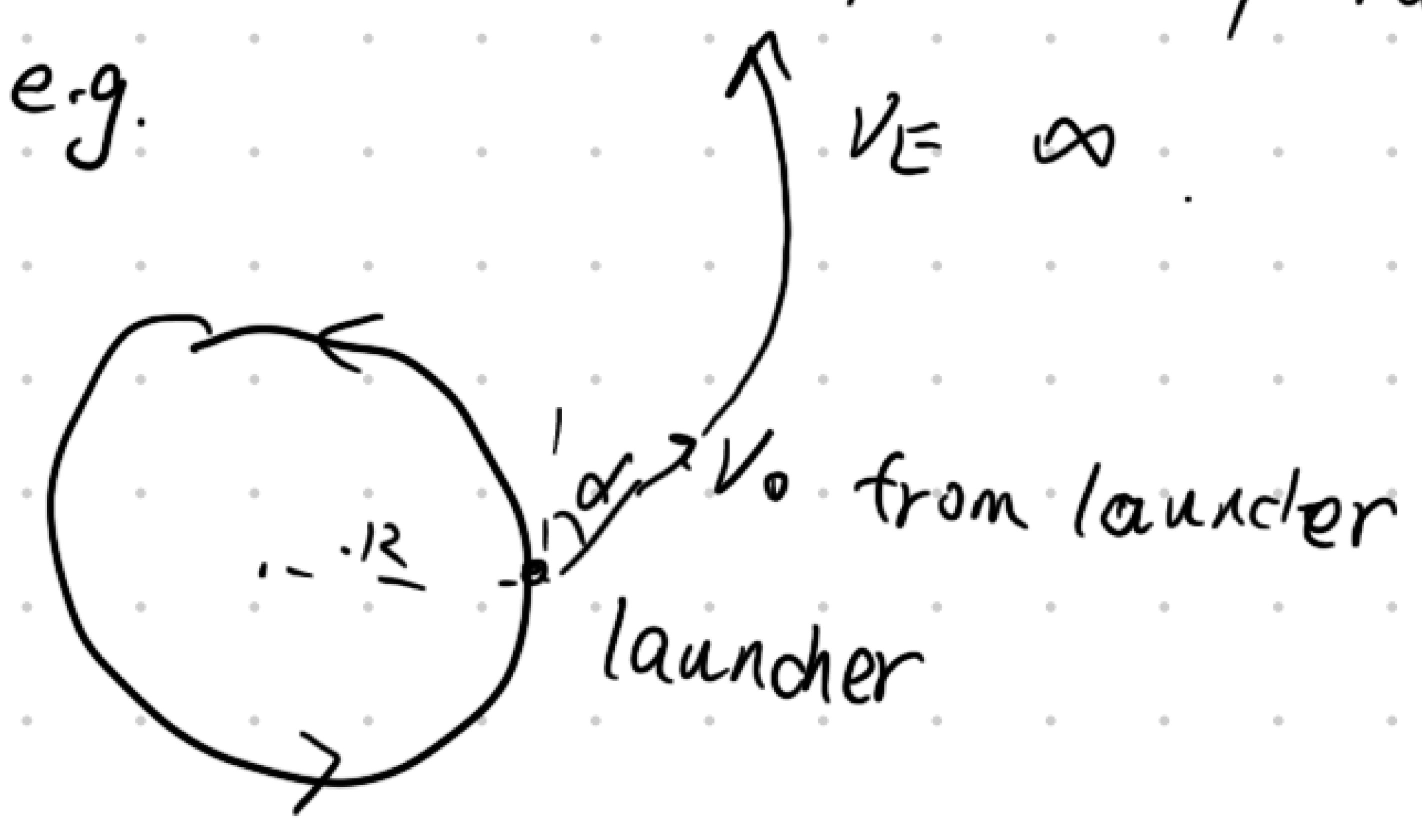
不失一般性 $\theta = \pi/2$. $\quad L = r^2\frac{d\phi}{d\lambda}$ $\quad U^\mu U_\mu = \begin{cases} -1 & \text{massive} \\ 0 & \text{massless} \end{cases}$

$\frac{1}{2}(\frac{dr}{d\lambda})^2 + V(r) = \frac{1}{2}E^2$ $\quad V(r) = \frac{1}{2}\epsilon - \frac{GM}{r}\epsilon + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$ $\quad \epsilon = \begin{cases} 1 \\ 0 \end{cases}$

circle orbit in geodesic $\frac{\partial V(r)}{\partial r} = 0$ $\quad \frac{dr}{d\lambda} = 0$ (fix $r = R$)

* massless impact parameter $b = \frac{L}{E} = \frac{d\phi}{dt}r^2$ (well-defined)

e.g.



launcher $U^\mu_{lau} = (\sqrt{\frac{R}{R-3GM}}, 0, 0, \frac{1}{R}\sqrt{\frac{GM}{R-3GM}})$

标架 $\hat{e}_0 = \hat{U}_{lau}$ $\quad \hat{e}_1 = (0, \sqrt{1-\frac{2GM}{R}}, 0, 0)$

$\hat{e}_3 = (e_3^t, 0, 0, e_3^\phi)$ satisfy: $\hat{e}_0\cdot\hat{e}_3 = 0$ $\quad \hat{e}_3\cdot\hat{e}_3 = 1$

$\Rightarrow e_3^t = \sqrt{\frac{GMR}{(R-3GM)(R-2GM)}}$ $\quad e_3^\phi = \frac{1}{R}\sqrt{\frac{R-2GM}{R-3GM}}$

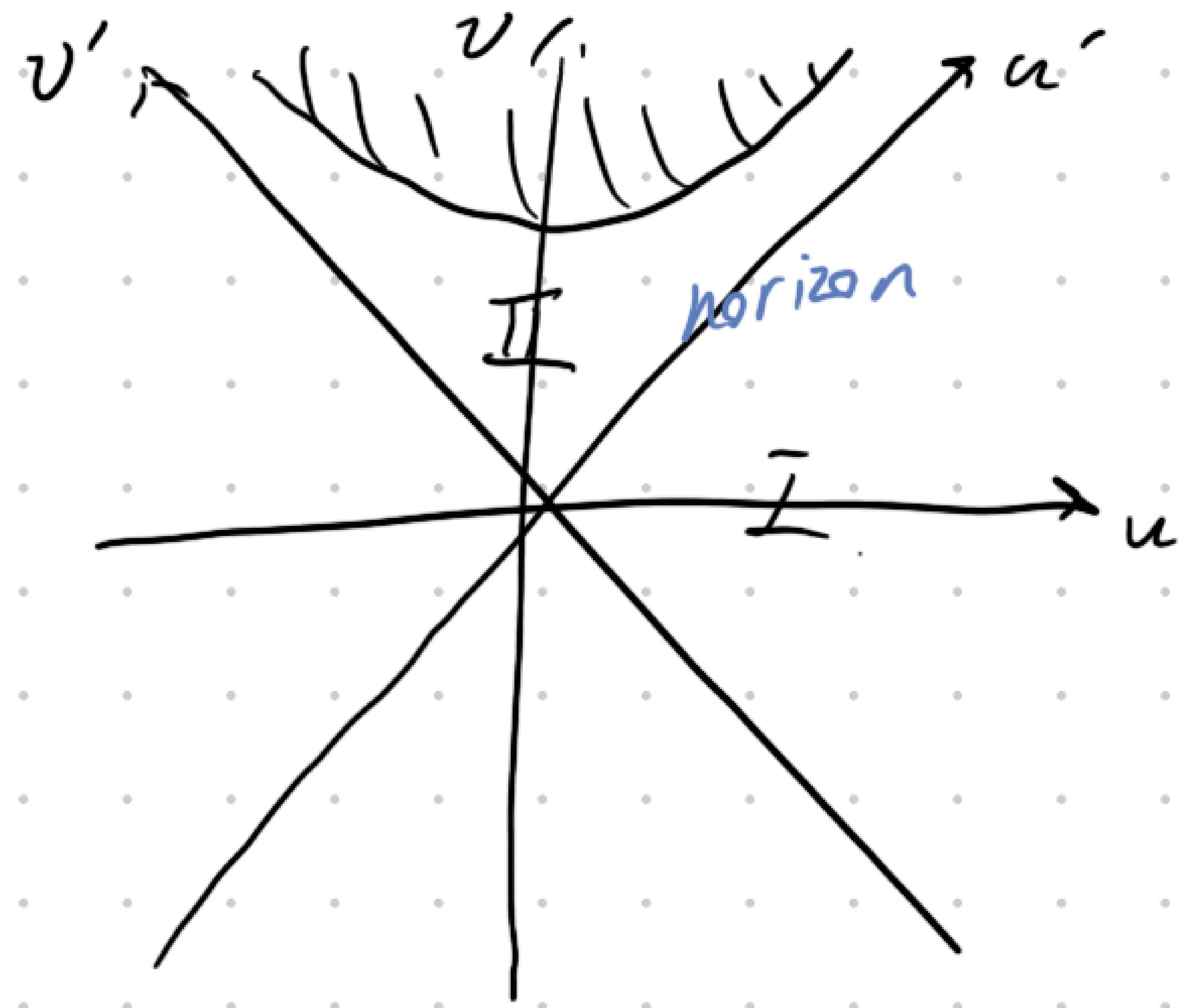
photon $K^\mu = (K^t, K^r, 0, K^\phi)$ frequency from launcher $2\pi\nu_0 = -\hat{e}_0\cdot\hat{K}$

angle $\tan\alpha = \frac{\hat{e}_1\cdot\hat{K}}{\hat{e}_3\cdot\hat{K}}$ $\quad \hat{K}\cdot\hat{K} = 0$

conserved: $\hat{\xi}\cdot\hat{K} = g_{tt}K^t$ along geodesic

$r\to\infty$ $\quad 2\pi\nu_E = -U_R^t K_{R,t} = -K_{R,t} = -g_{tt}K^t = (1-\frac{R_s}{R})K^t$

☑ BH Kruskal coordinate

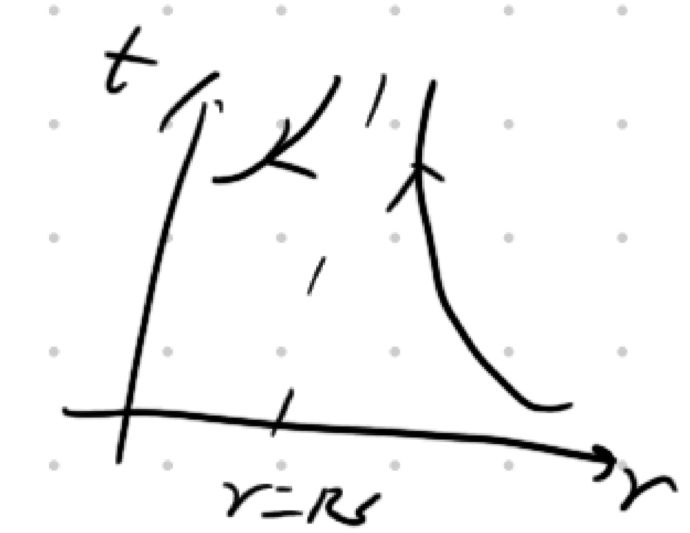


conformally flat for (v, u)

horizon $u = v$ $(v' = 0)$

light $u = \pm v$

horizon $v' = 0$



event horizon : infalling timelike/light cannot escape

$$ds^2 = \underset{>0}{g_{tt}} dt^2 + \underset{<0}{g_{rr}} dr^2 + r^2 d\Omega^2 < 0 \Rightarrow \left(\frac{dr}{d\tau}\right)^2 > 0$$

Killing horizon

null Killing vector $l \cdot l = 0$

Kruskal coord $\xi = \frac{\partial}{\partial t} = \frac{u'}{2R_s} \frac{\partial}{\partial u'} - \frac{v'}{2R_s} \frac{\partial}{\partial v'}$

at Killing horizon $v' = 0$ $\xi = \frac{u'}{2R_s} \frac{\partial}{\partial u'}$ normal vector $\vec{l} = \frac{\partial}{\partial u'}$ $l^\nu \nabla_\nu l^\mu = 0$ $\vec{l} \cdot \vec{l} = 0$

☑ Dynamics of GR

Einstein eq. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \Leftrightarrow R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu})$

$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_M$ define $T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$ $(\nabla^\mu T_{\mu\nu} = 0)$

e.g. $T_{\mu\nu} = \begin{cases} 0 & \text{vacumn} \\ (\rho+p) u^\mu u^\nu + p\eta^{\mu\nu} & \text{perfect fluid} \end{cases}$

□ Linearize GR $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $(|h_{\mu\nu}| \ll 1)$ $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

harmonic gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$ $(\partial_\mu A^\mu = 0)$

还有剩余自由度 $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho$ (consistent with $\partial_\mu \bar{h}^{\mu\nu} = 0$, $\partial_\mu \partial^\mu \xi_\nu = 0$)

最后：只有2个独立分量. $10 - 4 - 4 = 2$

□ Cosmology $ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$ $T_{\mu\nu} = (\rho+p) u_\mu u_\nu + p g_{\mu\nu}$ $(p = w_i \rho)$

$\rho \propto a^{-3(w_i+1)}$

$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \dot{\rho} + \frac{3\dot{a}}{a}(\rho+p) = 0$ $\dot{\rho} + \frac{3\dot{a}}{a}(w_i+1)\rho = 0$.

□ Future directions: classical numerical GR, cosmology, BH modified gravity $\Rightarrow$ gravitational wave

QG AdS / CFT

dS QG string theory ...