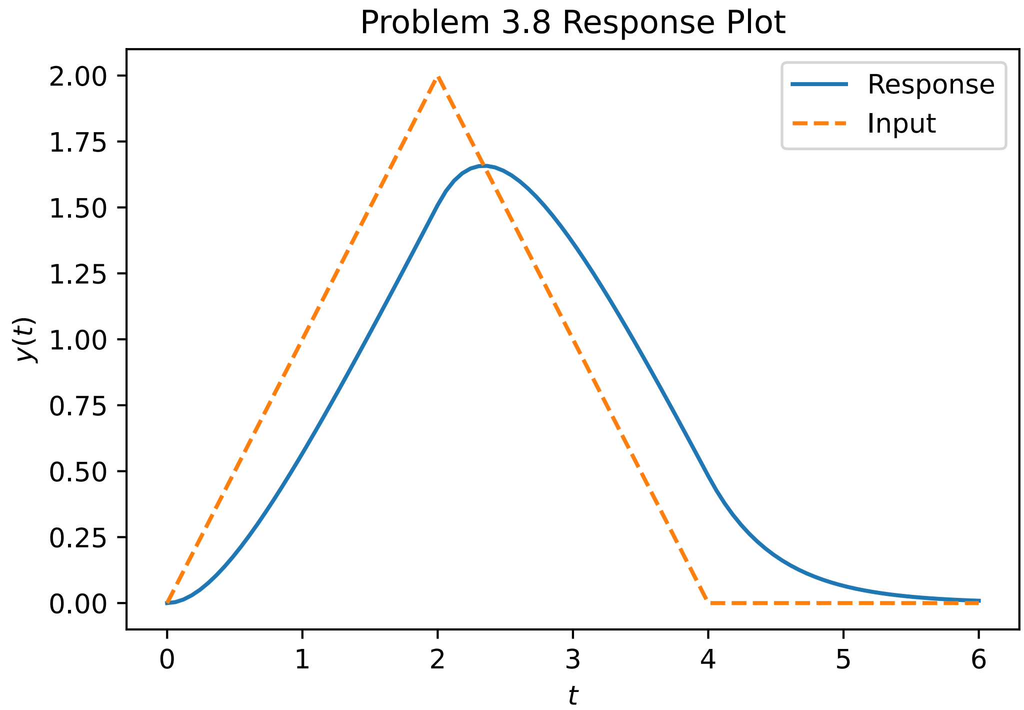
1. Problem 3.8

$$\tau\frac{dy}{dt} + y(t) = ku(t)$$
$$u(t) = t\mathcal{H}(t) - t\mathcal{H}(t-2) + (-t+4)\mathcal{H}(t-2) - (-t+4)\mathcal{H}(t-4)$$
$$u(t) = t\mathcal{H}(t) - 2(t-2)\mathcal{H}(t-2) + (t-4)\mathcal{H}(t-4)$$
$$0.5\cdot\frac{dy}{dt} + y(t) = t\mathcal{H}(t) - 2(t-2)\mathcal{H}(t-2) + (t-4)\mathcal{H}(t-4)$$
$$\mathcal{L}\left\{0.5\cdot\frac{dy}{dt} + y(t)\right\} = \mathcal{L}\left\{t\mathcal{H}(t) - 2(t-2)\mathcal{H}(t-2) + (t-4)\mathcal{H}(t-4)\right\}$$
$$0.5sY(s) + Y(s) = \frac{1}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2}$$
$$Y(s)\left(\frac{s}{2}+1\right) = \frac{1}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2}$$
$$\frac{1}{\left(\frac{s}{2}+1\right)s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{\frac{s}{2}+1}$$
$$1 = A\left(\frac{s^2}{2}+s\right) + B\left(\frac{s}{2}+1\right) + Cs^2$$
$$\frac{A}{2} + C = 0$$
$$A + \frac{B}{2} = 0$$
$$B = 1$$
$$A = -\frac{1}{2}$$
$$C = -\frac{1}{4}$$
$$\begin{aligned} Y(s) = &\frac{1}{2}\left(\frac{2}{s^2} - \frac{1}{s+2} - \frac{1}{s}\right) \\ &-\frac{1}{2}\left(\frac{2}{s^2} - \frac{1}{s+2} - \frac{1}{s}\right)\left(2e^{-2s}\right) \\ &+\frac{1}{2}\left(\frac{2}{s^2} - \frac{1}{s+2} - \frac{1}{s}\right)\left(e^{-4s}\right)\end{aligned}$$

Take the inverse Laplace Transform of the above equation.

$$\boxed{y(t) = \frac{1}{2}\left[\left(2t-9+e^{8-2t}\right)\mathcal{H}(t-4) - 2\left(2t-5+e^{4-2t}\right)\mathcal{H}(t-2) + \left(2t-1+e^{-2t}\right)\mathcal{H}(t)\right]}$$

Solution plot:



Code to produce the output plot:

```
import numpy as np
import matplotlib.pyplot as plt
from sympy.integrals import inverse_laplace_transform, laplace_transform
from sympy.abc import t, s
from sympy.functions import exp, Heaviside
from sympy import lambdify
# input a sympy expression
u_expr = t*Heaviside(t) - 2*(t-2)*Heaviside(t-2) + (t-4)*Heaviside(t-4)
# convert sympy expression u(t) into lambda func
u = lambdify(t, u_expr, 'numpy')
# lambdify won't work on Heaviside() with sympy version <1.11
# inverse laplace transform Y(s)
y_expr = inverse_laplace_transform( (1 - 2*exp(-2*s) + exp(-4*s)) / s**2 / (s/2 + 1), s, t)
# convert sympy expression y(t) into lambda func
y = lambdify(t, y_expr, 'numpy')
t_range = np.linspace(0, 6, 100)
# plotting
plt.plot(t_range, y(t_range))
plt.plot(t_range, u(t_range), '--')
plt.xlabel(r'$t$')
plt.ylabel(r'$y(t)$')
plt.title(r'Problem 3.8 Response Plot')
plt.legend(['Response', 'Input'])
```

2. Problem 3.9

Response:

$$y(t) = \frac{kM}{\epsilon}\left(1 - e^{\frac{-t}{\tau}}\right)$$

Assume the sensor is perfectly calibrated:

$$k = 1$$
$$\epsilon = 1\text{min}$$

At t =1 min:

$$T \approx 39°\text{C}$$
$$T_0 = 20$$
$$\frac{M}{\epsilon} = 50 - 20 = 30$$
$$T = 19$$
$$19 = 30 \cdot \left(1 - e^{\frac{-1}{\tau}}\right)$$
$$\boxed{\tau = 0.997\text{min}}$$

3. Problem 3.10

Input:

$$
\begin{aligned}
T_L(t) &= 20 + \sin(2t) \\
\omega &= 2 \\
M &= 1 \\
\text{output amplitude} &= \frac{kM}{\sqrt{1+(\omega\tau)^2}} \\
\text{output amplitude} &= 0.1 \text{ from plot}
\end{aligned}
$$

Assume the sensor is perfectly calibrated:

$$
\begin{aligned}
k &= 1 \\
0.1 &= \frac{1}{\sqrt{1+(2\tau)^2}}
\end{aligned}
$$

Solve for τ:

$$\boxed{\tau = 4.975}$$

4. Problem 3.12

$$\tau \frac{dT}{dt} + T(t) = ku(t)$$
$$\tau = 10\text{s} = \frac{10\text{s}}{60\text{s/min}} = \frac{1}{6}\text{min}$$

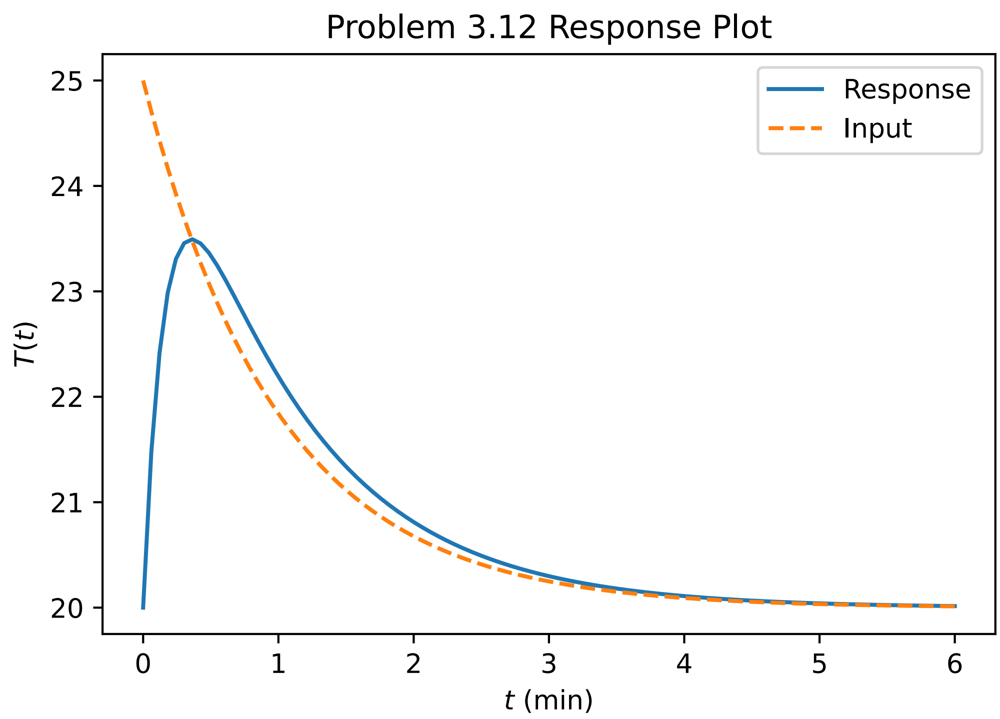
Assume the sensor is perfectly calibrated:

$$k = 1$$
$$u(t) = 20 + 5e^{-t}$$

Make $T_0 = 20°\text{C}$:

$$T(0) = 0, \text{ 0 initial condition}$$
$$u(t) = 5e^{-t}$$
$$\frac{1}{6}\frac{dT}{dt} + T(t) = 5e^{-t}$$
$$\mathcal{L}\left\{\frac{1}{6}\frac{dT}{dt} + T(t)\right\} = \mathcal{L}\left\{5e^{-t}\right\}$$
$$\frac{sT(s)}{6} + T(s) = \frac{5}{s+1}$$
$$T(s) = \frac{5}{(s+1)(\frac{s}{6}+1)}$$
$$\frac{5}{(s+1)(\frac{s}{6}+1)} = \frac{A}{s+1} + \frac{B}{\frac{s}{6}+1}$$
$$5 = A\left(\frac{s}{6}+1\right) + B(s+1)$$
$$\frac{A}{6} + B = 0$$
$$A + B = 5$$
$$A = 6$$
$$B = -1$$
$$T(s) = \frac{6}{s+1} + \frac{-1}{\frac{s}{6}+1}$$
$$\mathcal{L}^{-1}\left\{T(s)\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s+1} - \frac{6}{s+6}\right\}$$
$$T(t) = 6\left(e^{-t} - e^{-6t}\right)$$

Solution plot:




```python
import numpy as np
import matplotlib.pyplot as plt
from sympy.integrals import inverse_laplace_transform, laplace_transform
from sympy.abc import t, s
from sympy.functions import exp, Heaviside
from sympy import lambdify
# input function
u = lambda t: 20 + 5 * np.exp(-t)
# inverse laplace transform Y(s)
y_expr = inverse_laplace_transform(5 / (s + 1) / (s/6 + 1), s, t)
# convert sympy expression y(t) into lambda func
y = lambdify(t, y_expr, 'numpy')
# lambdify won't work on Heaviside() with sympy version <1.11
t_range_min = np.linspace(0, 6, 100)
# plotting
plt.plot(t_range_min, y(t_range_min) + 20) # +20 to add initial condition back
plt.plot(t_range_min, u(t_range_min), '--')
plt.xlabel(r'$t$ (min)')
plt.ylabel(r'$T(t)$')
plt.title(r'Problem 3.12 Response Plot')
plt.legend(['Response', 'Input'])
```