CHEN 461 HW9

March 30, 2023

```
[ ]: import control

     import numpy as np
     import matplotlib.pyplot as plt

     from sympy.abc import t, s
     from sympy import symbols, simplify, expand
     from sympy.series import limit
     from sympy import matrices
     from sympy.integrals import integrate
```

1 Problem 10.8

1.1 Part a

Real PD:

$G(s) = k_c \left(\frac{1+\tau_D s}{1+\alpha\tau_D s}\right)$

$E(s) = \frac{1}{s}$

$U(s) = k_c \left(\frac{1+\tau_D s}{s(1+\alpha\tau_D s)}\right)$

$U(s) = k_c \left(\frac{1}{s} + \frac{\tau_D(1-\alpha)}{(1+\alpha\tau_D s)}\right)$

$u(t) = k_c \left(1 + \frac{\tau_D(1-\alpha)}{\alpha\tau_D}\right) \exp\left(-\alpha^{-1}\frac{t}{\tau_D}\right)$

Real PD response:

$$\frac{u(t)}{k_c} = 1 + \frac{(1-\alpha)}{\alpha} \exp\left(-\alpha^{-1}\frac{t}{\tau_D}\right)$$

Ideal PD:

$G(s) = k_c \left(1 + \tau_D s\right)$

$U(s) = k_c \left(\frac{1}{s} + \tau_D\right)$

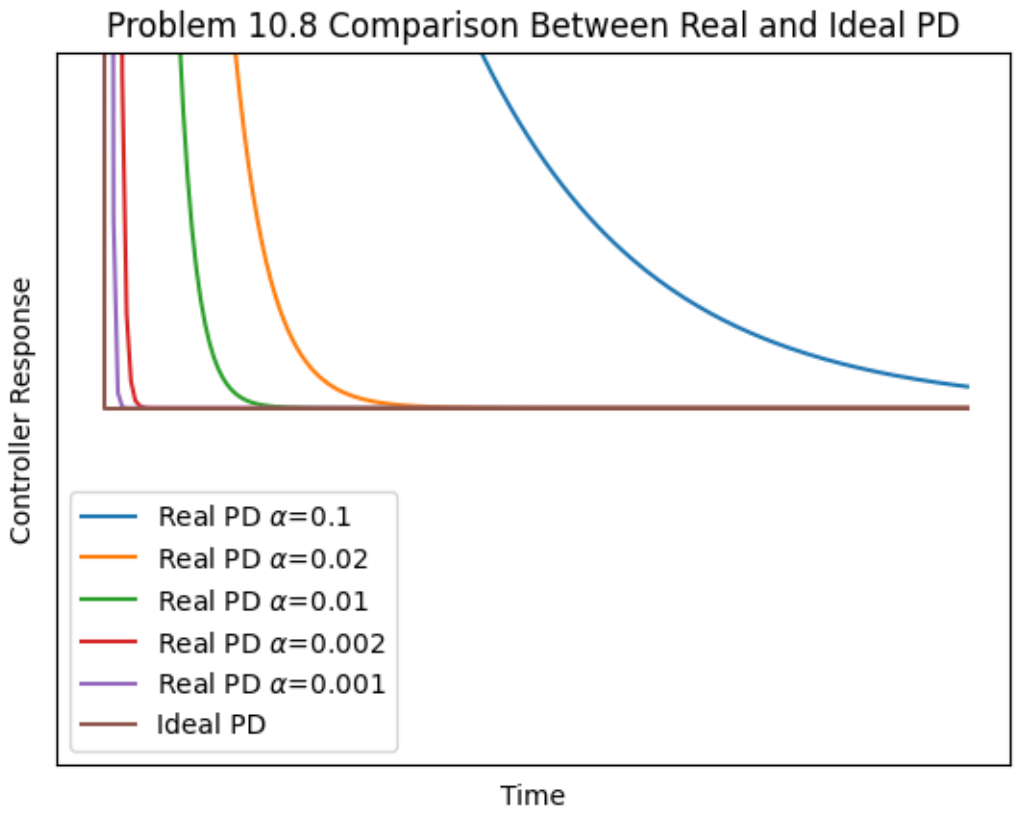
Ideal PD response:

$$\frac{u(t)}{k_c} = 1 + \delta\left(t\right)$$

1.1.1 Simulation

[ ]:
```
alpha = [0.1, 2e-2, 1e-2, 2e-3, 1e-3]

def real_pd(t, alpha):
    return 1 + (1 - alpha) / alpha * np.exp(-t / alpha)

t_values = np.linspace(0, .5, 200)

for a in alpha:
    plt.plot(t_values, real_pd(t_values, a), label=r"Real PD $\alpha$="+f"{a}")

ideal_pd_response = np.ones(t_values.shape[0])
ideal_pd_response[0] = 1e300

plt.plot(t_values, ideal_pd_response, label="Ideal PD")

plt.ylim([0, 2])
plt.xlabel("Time")
plt.ylabel("Controller Response")
plt.title("Problem 10.8 Comparison Between Real and Ideal PD")
plt.xticks([], [])
plt.yticks([], [])
plt.legend()
```

[ ]:
```
<matplotlib.legend.Legend at 0x2207b9bb1d0>
```



The closer that α is to zero, the closer closer that the Real PD is to the Ideal PD. The smaller the α value, the faster that the Real PD reaches the set point.

1.2 Part B

Real PD:

$$G(s) = k_c \left(\frac{1 + \tau_D s}{1 + \alpha \tau_D s} \right)$$

Ideal PD:

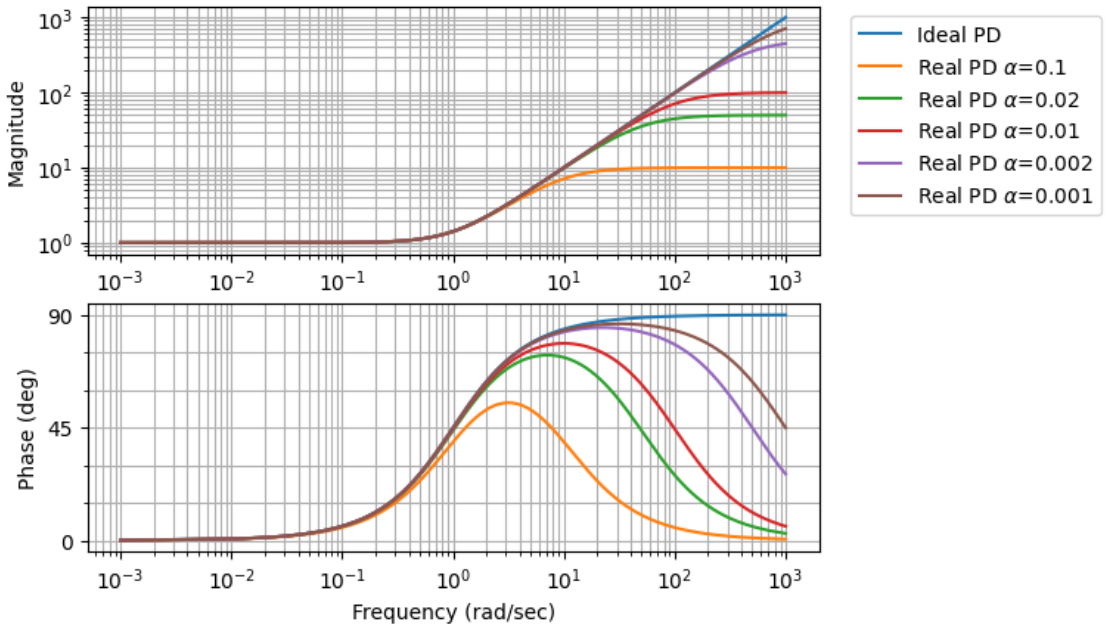
$$G(s) = k_c \left(1 + \tau_D s \right)$$

1.2.1 Bode Diagrams

[ ]:
```
# alpha = [0.1, 0.02, 0.01, 0.005, 0.005/2, 1e-6]
ideal_pd = control.tf([1, 1], [1])

w = np.linspace(1e-3, 1e3, int(1e5))
```

```
mag, phase, omega = control.bode_plot(ideal_pd, omega=w, wrap_phase=True,␣
 ↪label="Ideal PD")

for a in alpha:
    real_pd = control.tf([1, 1], [a, 1])
    mag, phase, omega = control.bode_plot(real_pd, omega=w, wrap_phase=True,␣
 ↪label=r"Real PD $\alpha$="+f"{a}")

plt.legend(loc="upper right", bbox_to_anchor=(1.4, 2.2))
```

[ ]: <matplotlib.legend.Legend at 0x2207ba7e910>



In the magnitude plot, the real PD flattens past a $\tau_D\omega$ value of 10, while the ideal PD keeps increasing. As α approaches 0, the real PD begins to approach the same curve as the ideal PD in the magnitue plot.

In the phase plot, the real PD phase peaks around $\tau_D\omega = 10$ and then returns to 0°. In contrast, the ideal PD starts at 0° and steps up to 90°. As α approaches 0, the real PD begins to approach the same curve as the ideal PD in the phase plot.

2 Problem 10.10

2.1 Part A

2.1.1 State space models:

$$\frac{de_I}{dt} = e$$

$$\frac{de_F}{dt} = \frac{e - e_F}{\tau_F}$$

$$u = k_c \left(e + \frac{e_I}{\tau_I} + \frac{\tau_D}{\tau_F} (e - e_F) \right)$$

2.2 Part B

$$\frac{d}{dt} \begin{bmatrix} e_I \\ e_F \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{\tau_F} \end{bmatrix} \begin{bmatrix} e_I \\ e_F \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{\tau_F} \end{bmatrix} e$$

$$u = \begin{bmatrix} \frac{k_c}{\tau_I} & -k_c \frac{\tau_D}{\tau_F} \end{bmatrix} \begin{bmatrix} e_I \\ e_F \end{bmatrix} + k_c \left(1 + \frac{\tau_D}{\tau_F} \right) e$$

2.2.1 Define the state-space system in matrix form

[ ]:
```
tau_F, tau_I, k_c, tau_D, T_s = symbols('tau_F, tau_I, k_c, tau_D, T_s')

A = matrices.Matrix([
    [0, 0],
    [0, -1/tau_F]
])
B = matrices.Matrix([
    [1],
    [1/tau_F]
])
C = matrices.Matrix([
    [k_c/tau_I, -k_c*tau_D/tau_F]
])
D = k_c * (1 + tau_D/tau_F)
```

2.2.2 Compute A_d symbolically

$$A_d = e^{AT_s}$$

[ ]:
```
A_d = (A * T_s).exp()
A_d
```

[ ]:
$$\begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{T_s}{\tau_F}} \end{bmatrix}$$

2.2.3 Compute B_d symbolically

$$B_d = \int_0^{T_s} e^{At'} B dt'$$

[ ]:
```
B_d = integrate((A*t).exp() * B, (t, 0, T_s))
B_d
```

[ ]:
$$\begin{bmatrix} T_s \\ 1 - e^{-\frac{T_s}{\tau_F}} \end{bmatrix}$$

3 Problem 11.2

3.1 Part A

$Y(s)\frac{1-M(s)Q(s)+Q(s)G(s)}{1-M(s)Q(s)} = \frac{Q(s)G(s)}{1-M(s)Q(s)}Y_{sp}(s) + W(s)G'(s)$

$$Y(s) = \frac{Q(s)G(s)}{1 - M(s)Q(s) + Q(s)G(s)}Y_{sp}(s) + \frac{(1 - M(s)Q(s))\,G'(s)}{1 - M(s)Q(s) + Q(s)G(s)}W(s)$$

3.2 Part B

$$Y(s) = \frac{C_E(s)G(s)}{(1 + C_E(s)G(s) - C_Y(s)G(s))}Y_{sp}(s) + \frac{W(s)}{(1 + C_E(s)G(s) - C_Y(s)G(s))}G'(s)$$

3.3 Part C

$$Y(s) = \frac{G'(s) + G(s)G_{ff}(s)}{1 + G_c(s)G(s)}W(s) + \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}Y_{sp}(s)$$

4 Problem 11.8

4.1 Part A

Feedback loop transfer function:

$$G(s) = \frac{G_cG_p}{1 + G_cG_p}$$

Process transfer function:

$$G_p(s) = \frac{k_p}{\tau^2s^2 + 2\zeta\tau s + 1}$$

Controller transfer function:

$$G_c(s) = k_c\left(\frac{1 + \tau_Ds}{1 + \alpha\tau_Ds}\right)$$

4.1.1 Define feedback transfer function symbolically

[ ]:
```
tau, zeta, tau_D, alpha = symbols('tau, zeta, tau_D, alpha')
k_c, k_p = symbols('k_c, k_p')
```

```
G_p = k_p / (tau**2 * s**2 + 2 * zeta * tau * s + 1)
G_c = k_c * (1 + tau_D * s) / (1 + alpha * tau_D * s)

G_feedback = simplify(G_c * G_p / (1 + G_c * G_p))

G_feedback
```

[ ]:

$$\frac{k_c k_p \left(s\tau_D + 1\right)}{k_c k_p \left(s\tau_D + 1\right) + \left(\alpha s\tau_D + 1\right)\left(s^2\tau^2 + 2s\tau\zeta + 1\right)}$$

4.2 Part B

Find offset by Final Value Theorem:

$$\lim_{s\to 0^+} sY(s) = \lim_{t\to\infty} y(t)$$

$G(s) = \frac{G_c G_p}{1+G_c G_p}$

$Y(s) = G(s)Y_{sp}(s)$

In deviation form:

$y_{sp}(t) = 1$

$Y_{sp}(s) = \frac{1}{s}$

$Y(s) = \frac{G(s)}{s}$

$sY(s) = s\frac{G(s)}{s} = G(s)$

Offset is $y_{sp}(t) - y(t)$ at ∞

$y(t) = \lim_{s\to 0^+} G(s)$

Offset:

$$1 - \lim_{s\to 0^+} G(s)$$

[ ]:
```
simplify(1 - limit(G_feedback, s, 0))
```

[ ]:

$$\frac{1}{k_c k_p + 1}$$