1. Problem 6.13

$$
\begin{aligned}
V\frac{dC_R}{dt} &= F(C_{R0}-C_R) - Vk_1C_R + Vk_2C_P - Vk_3C_R \\
V\frac{dC_P}{dt} &= -FC_P + Vk_1C_R - Vk_2C_P \\
\frac{dC_R}{dt} &= -\left(\frac{F}{V}+k_1+k_3\right)C_R + k_2C_P + \frac{F}{V}C_{R0} \\
\frac{dC_P}{dt} &= k_1C_R - \left(\frac{F}{V}+k_2\right)C_P \\
y &= C_P \\
A &= \begin{bmatrix} -\left(\frac{F}{V}+k_1+k_3\right) & k_2 \\ k_1 & -\left(\frac{F}{V}+k_2\right) \end{bmatrix} \\
B &= \begin{bmatrix} \frac{F}{V} \\ 0 \end{bmatrix} \\
C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D &= 0 \\
G(s) &= c\,(sI-A)^{-1}\,b + d \\
sI-A &= \begin{bmatrix} s+\left(\frac{F}{V}+k_1+k_3\right) & -k_2 \\ -k_1 & s+\left(\frac{F}{V}+k_2\right) \end{bmatrix} \\
\mathrm{Adj}(sI-A) &= \begin{bmatrix} s+\left(\frac{F}{V}+k_2\right) & k_2 \\ k_1 & s+\left(\frac{F}{V}+k_1+k_3\right) \end{bmatrix} \\
\det(sI-A) &= \left(s+\left(\frac{F}{V}+k_1+k_3\right)\right)\left(s+\left(\frac{F}{V}+k_2\right)\right) - k_1k_2 \\
\det(sI-A) &= \left(s+\frac{F}{V}+k_1+k_3\right)\left(s+\frac{F}{V}+k_2\right) - k_1k_2 \\
\theta &= \det(sI-A) \\
(sI-A)^{-1} &= \theta^{-1}\begin{bmatrix} s+\left(\frac{F}{V}+k_2\right) & k_2 \\ k_1 & s+\left(\frac{F}{V}+k_1+k_3\right) \end{bmatrix} \\
G(s) &= c\theta^{-1}\begin{bmatrix} s+\left(\frac{F}{V}+k_2\right) & k_2 \\ k_1 & s+\left(\frac{F}{V}+k_1+k_3\right) \end{bmatrix}\begin{bmatrix} \frac{F}{V} \\ 0 \end{bmatrix} \\
G(s) &= c\theta^{-1}\begin{bmatrix} \frac{F}{V}\left(s+\left(\frac{F}{V}+k_2\right)\right) \\ \frac{F}{V}k_1 \end{bmatrix} \\
G(s) &= \theta^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{F}{V}\left(s+\left(\frac{F}{V}+k_2\right)\right) \\ \frac{F}{V}k_1 \end{bmatrix} \\
G(s) &= \frac{\frac{F}{V}k_1}{\theta} \\
G(s) &= \frac{\frac{F}{V}k_1}{\left(s+\frac{F}{V}+k_1+k_3\right)\left(s+\frac{F}{V}+k_2\right) - k_1k_2}
\end{aligned}
$$

2. Problem 6.7

State space model:

$$A_1 \frac{dh_1}{dt} = F_{in} - \frac{h_1 - h_2}{R_1}$$
$$A_2 \frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$

Assume $R_1 = R_2 = 1$

$$\frac{dh_1}{dt} = -\frac{h_1}{A_1} + \frac{h_2}{A_1} + \frac{F_{in}}{A_1}$$
$$\frac{dh_2}{dt} = \frac{h_1}{A_2} - \frac{2h_2}{A_2}$$
$$A = \begin{bmatrix} -\frac{1}{A_1} & \frac{1}{A_1} \\ \frac{1}{A_2} & -2\frac{h_2}{A_2} \end{bmatrix}$$
$$B = \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix}$$
$$D = 0$$
$$y = h_2$$
$$\text{Or,}$$
$$y = h_1$$

Finding h_2

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Finding h_1

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

3. Problem 6.8

(a)

$$\begin{aligned}
\tau_1 \frac{dx_1}{dt} + x_1 &= ku \\
\tau_2 \frac{dx_2}{dt} + x_2 &= kx_1 \\
\frac{dx_1}{dt} &= \frac{-1}{\tau_1} x_1 + \frac{k}{\tau_1} u \\
\frac{dx_2}{dt} &= \frac{k}{\tau_2} x_1 - \frac{1}{\tau_2} x_2 \\
A &= \begin{bmatrix} \frac{-1}{\tau_1} & 0 \\ \frac{k}{\tau_2} & \frac{-1}{\tau_2} \end{bmatrix} \\
B &= \begin{bmatrix} \frac{k}{\tau_1} \\ 0 \end{bmatrix} \\
C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D &= 0 \\
sI - A &= \begin{bmatrix} s + \frac{1}{\tau_1} & 0 \\ -\frac{k}{\tau_2} & s + \frac{1}{\tau_2} \end{bmatrix} \\
\mathrm{Adj}(sI - A) &= \begin{bmatrix} s + \frac{1}{\tau_2} & 0 \\ \frac{k}{\tau_2} & s + \frac{1}{\tau_1} \end{bmatrix} \\
\det(sI - A) &= \left(s + \frac{1}{\tau_2}\right)\left(s + \frac{1}{\tau_1}\right) \\
(sI - A)^{-1} &= \begin{bmatrix} \frac{s+\frac{1}{\tau_2}}{\left(s+\frac{1}{\tau_2}\right)\left(s+\frac{1}{\tau_1}\right)} & 0 \\ \frac{\frac{k}{\tau_2}}{\left(s+\frac{1}{\tau_2}\right)\left(s+\frac{1}{\tau_1}\right)} & \frac{s+\frac{1}{\tau_1}}{\left(s+\frac{1}{\tau_2}\right)\left(s+\frac{1}{\tau_1}\right)} \end{bmatrix} \\
(sI - A)^{-1} &= \begin{bmatrix} \frac{\tau_1}{\tau_1 s+1} & 0 \\ \frac{k\tau_1}{(\tau_2 s+1)(\tau_1 s+1)} & \frac{\tau_2}{\tau_2 s+1} \end{bmatrix}
\end{aligned}$$

4. Problem 6.8 MATLAB

   The purpose of this problem is to use MATLAB as a matrix calculator.

   You will use the matrix operations and matrix functions in MATLAB, including the expm function for the matrix exponential.

   Consider the CSTR example of p. 172, with state equation given by Eq. (6.1.5), where F = 10 l/min, V = 100 l, $k_1 = 0.1$ min^{-1}, $k_2 = 0.05$ min^{-1}, $k_3 = 0.02$ min^{-1}

   a) How are the input and state vector related at steady state? If the inlet concentration of the reactant is steady and equal to $C_{R0,s} = 2$ mol/l, calculate the corresponding steady state values of C_R, C_I and C_P.

   b) Use formulas from Table 6.2 (p. 188) and Table 6.3 (p.190) to calculate and plot

   • the response of C_R, C_I and C_P to a step change in C_{R0} of size M = 2 mol/l

   • the response of C_R, C_I and C_P to a ramp change in C_{R0} of slope M = 0.5 mol/min

   • the response of C_P to a sinusoidal change in C_{R0} of amplitude M = 1 mol/l and frequency $\omega =$ 0.2 min^{-1} all in deviation form.

   c) Calculate the zero-order hold discretization of the system (Eqns. (6.7.5) and (6.7.6) on p. 193) for sampling period $T_s = 0.2$ min.

<u>Hint for c)</u>: $\begin{bmatrix} A_d & b_d \end{bmatrix} = \begin{bmatrix} e^{AT_s} & \int_0^{T_s} e^{A\tau} b\, d\tau \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} T_s}$, as an immediate consequence of the identity $e^{\begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} t} = \begin{bmatrix} e^{At} & \int_0^{t} e^{A\tau} b\, d\tau \\ 0 & 1 \end{bmatrix}$. Can you prove this identity?