1. Problem 4.8

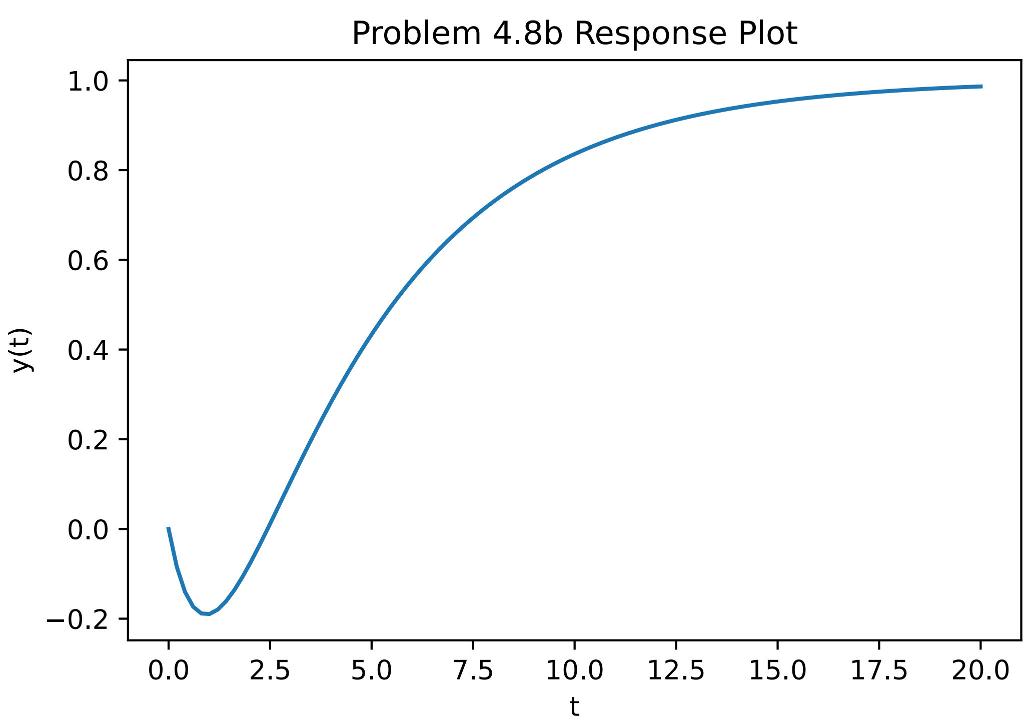
   (a)

$$
\begin{aligned}
Y_1(s) &= \frac{k_1}{\tau_1 s + 1} U(s) \\
Y_2(s) &= \frac{-k_2}{\tau_2 s + 1} U(s) \\
Y(s) &= \left(\frac{k_1}{\tau_1 s + 1} - \frac{k_2}{\tau_2 s + 1} \right) U(s) \\
U(s) &= \frac{M}{s} \\
\mathcal{L}^{-1}\left\{Y(s)\right\} &= \mathcal{L}^{-1}\left\{ \left(\frac{k_1}{\tau_1 s + 1} - \frac{k_2}{\tau_2 s + 1} \right) \frac{M}{s} \right\} \\
&\boxed{y(t) = k_1 M \left(1 - e^{-t/\tau_1}\right) - k_2 M \left(1 - e^{-t/\tau_2}\right)} \\
\frac{dy}{dt} &= \frac{k_1 M}{\tau_1} e^{-t/\tau_1} - \frac{k_2 M}{\tau_2} e^{-t/\tau_2} \\
&\boxed{\frac{dy}{dt}(0) = \frac{k_1 M}{\tau_1} - \frac{k_2 M}{\tau_2}}
\end{aligned}
$$

   As $t \to \infty$

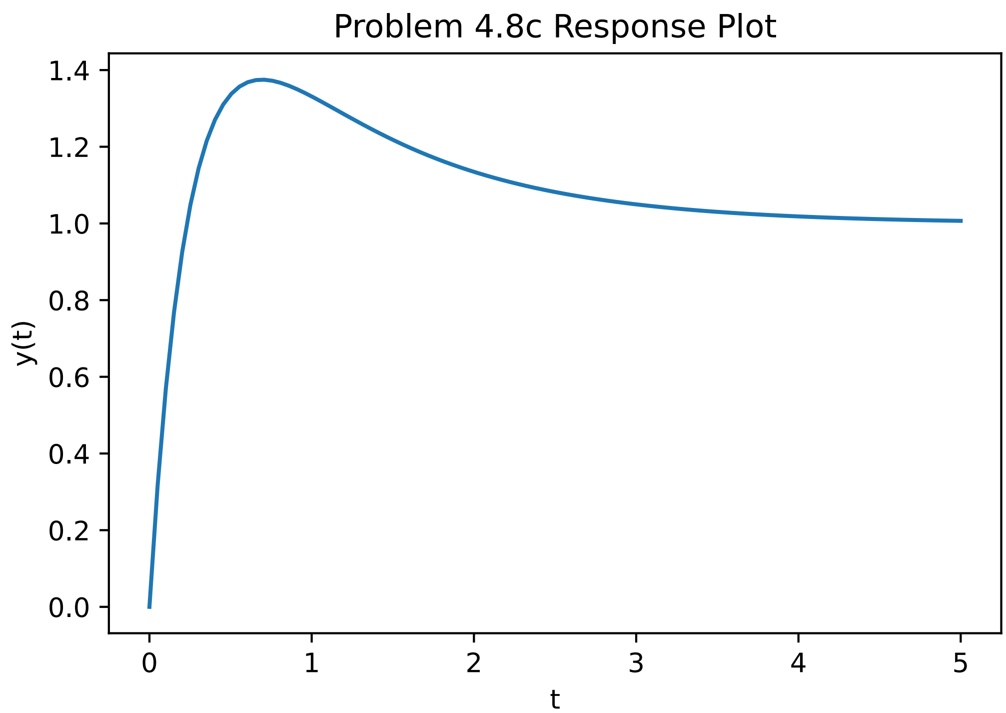
$$\boxed{y(t) = \frac{k_1 M}{\tau_1} - \frac{k_2 M}{\tau_2}}$$

   (b) Plot:



   Initially, the second block's output is larger than the first block's output. As a result, the second block's output pulls the combined output negative. The first output eventually outpaces the second output, and the output reaches a new higher steady state.

(c) Plot:



A similar situation to that of part b occurs in part c. Except, the first output is initially larger than the second output. The combined output reaches a peak before the second output matches the first output and pulls the combined output to a new steady state, lower than the peak but higher than the initial.

Plotting code:

```
import numpy as np
import matplotlib.pyplot as plt

k_1 = 2
tau_1 = 4
k_2 = 1
tau_2 = 1
M = 1

func = lambda t: k_1 * M * (1 - np.exp(-t / tau_1)) - k_2 * M * (1 - np.exp(-t / tau_2))

t_ran = np.linspace(0, 20, 100)
plt.plot(t_ran, func(t_ran))
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.8b Response Plot")

k_1 = 2
tau_1 = 1/4
k_2 = 1
tau_2 = 1
```

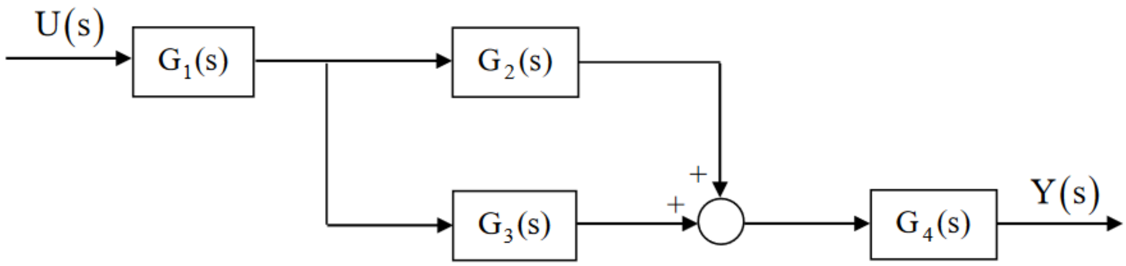
```
M = 1

func = lambda t: k_1 * M * (1 - np.exp(-t / tau_1)) - k_2 * M * (1 - np.exp(-t / tau_2))

t_ran = np.linspace(0, 5, 100)
plt.plot(t_ran, func(t_ran))
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.8c Response Plot")
```

2. Problem 2

   Calculate the overall transfer function of the system represented by the following block diagram:



$$
\begin{aligned}
Y_1(s) &= G_1(s)U(s) \\
Y_2(s) &= (G_2(s) + G_3(s))U_2(s) \\
Y_2(s) &= (G_2(s) + G_3(s))\,G_1(s)U(s) \\
Y(s) &= G_4(s)U_3(s) \\
Y(s) &= G_4(s)\,(G_2(s) + G_3(s))\,G_1(s)U(s) \\
&\boxed{Y(s) = (G_2(s) + G_3(s))\,G_1(s)G_4(s)U(s)}
\end{aligned}
$$

3. Problem 5.8

$$
\begin{aligned}
kM &\approx 38 \\
A \approx 47 - 38 &= 9 \\
\frac{A}{kM} &= \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \\
\frac{9}{38} &= \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \\
&\boxed{\zeta = 0.4168} \\
&T \approx 20 \\
&T = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \\
&20 = \frac{2\pi\tau}{\sqrt{1-0.4168^2}} \\
&\boxed{\tau = 2.893} \\
M = 0.5 \cdot 76 &= 38 \\
&\boxed{k = 1}
\end{aligned}
$$

4. Problem 4.2

(a) State space:

$$
\begin{aligned}
V\frac{dC_R}{dt} &= FC_{in,R} - FC_R - Vk_1C_R \\
V\frac{dC_P}{dt} &= Vk_1C_R - FC_P - Vk_2C_P
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathcal{L}\left\{V\frac{dC_R}{dt}\right\} &= \mathcal{L}\left\{FC_{in,R} - FC_R - Vk_1C_R\right\} \\
VC_R(s)s &= FC_{in,R}(s) - FC_R(s) - Vk_1C_R(s) \\
C_R(s) &= \frac{F}{Vs + F + Vk_1}C_{in,R}(s) \\
&\boxed{G_1(s) = \frac{F}{Vs + F + Vk_1}} \\
\mathcal{L}\left\{V\frac{dC_P}{dt}\right\} &= \mathcal{L}\left\{Vk_1C_R - FC_P - Vk_2C_P\right\} \\
VC_P(s)s &= Vk_1C_R(s) - FC_P(s) - Vk_2C_P(s) \\
C_P(s)(Vs + F + Vk_2) &= Vk_1C_R(s) \\
C_P(s)(Vs + F + Vk_2) &= \frac{FVk_1}{Vs + F + Vk_1}C_{in,R}(s) \\
C_P(s) &= \frac{FVk_1}{(Vs + F + Vk_1)(Vs + F + Vk_2)}C_{in,R}(s) \\
&\boxed{G_2(s) = \frac{FVk_1}{(Vs + F + Vk_1)(Vs + F + Vk_2)}}
\end{aligned}
$$

(c) Partial fraction decomposition:

$$\begin{aligned}
\frac{1}{(Vs+F+Vk_1)(Vs+F+Vk_2)} &= \frac{A}{Vs+F+Vk_1} + \frac{B}{Vs+F+Vk_2} \\
1 &= A(Vs+F+Vk_2) + B(Vs+F+Vk_1) \\
1 &= s(AV+BV) + (AF+AVk_2+BF+BVk_1) \\
AV+BV &= 0 \\
B &= -A \\
AF+AVk_2+BF+BVk_1 &= 1 \\
AF+AVk_2-AF-AVk_1 &= 1 \\
AV(k_2-k_1) &= 1 \\
A &= \frac{1}{V(k_2-k_1)} \\
B &= -\frac{1}{V(k_2-k_1)} \\
G(s) &= \frac{FVk_1}{V(k_2-k_1)}\left(\frac{1}{Vs+F+Vk_1} - \frac{1}{Vs+F+Vk_2}\right) \\
G(s) &= \frac{Fk_1}{V(k_2-k_1)}\left(\frac{1}{s+\frac{F}{V}+k_1} - \frac{1}{s+\frac{F}{V}+k_2}\right)
\end{aligned}$$

Impulse input:

$$\begin{aligned}
C_{in,R} &= \frac{n_R}{V}\delta(t) \\
C_{in,R}(s) &= \frac{n_R}{V} \\
\mathcal{L}^{-1}\left\{C_P(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{Fk_1}{V(k_2-k_1)}\left(\frac{1}{s+\frac{F}{V}+k_1} - \frac{1}{s+\frac{F}{V}+k_2}\right)\frac{n_R}{V}\right\} \\
&\boxed{C_P(t) = \frac{Fk_1 n_R}{V^2(k_2-k_1)}\left(e^{-\left(\frac{F}{V}+k_1\right)t} - e^{-\left(\frac{F}{V}+k_2\right)t}\right)} \\
C_R(s) &= \frac{F}{Vs+F+Vk_1}\frac{n_R}{V} \\
\mathcal{L}^{-1}\left\{C_R(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{Fn_R}{V^2}\frac{1}{s+\frac{F}{V}+k_1}\right\} \\
&\boxed{C_R(t) = \frac{Fn_R}{V^2}e^{-\left(\frac{F}{V}+k_1\right)t}}
\end{aligned}$$

(d) Reversible

State Space:

$$\begin{aligned}
V\frac{dC_R}{dt} &= FC_{in,R} - FC_R - Vk_1C_R + Vk_{-1}C_P \\
V\frac{dC_P}{dt} &= Vk_1C_R - FC_P - Vk_2C_P - Vk_{-1}C_P
\end{aligned}$$

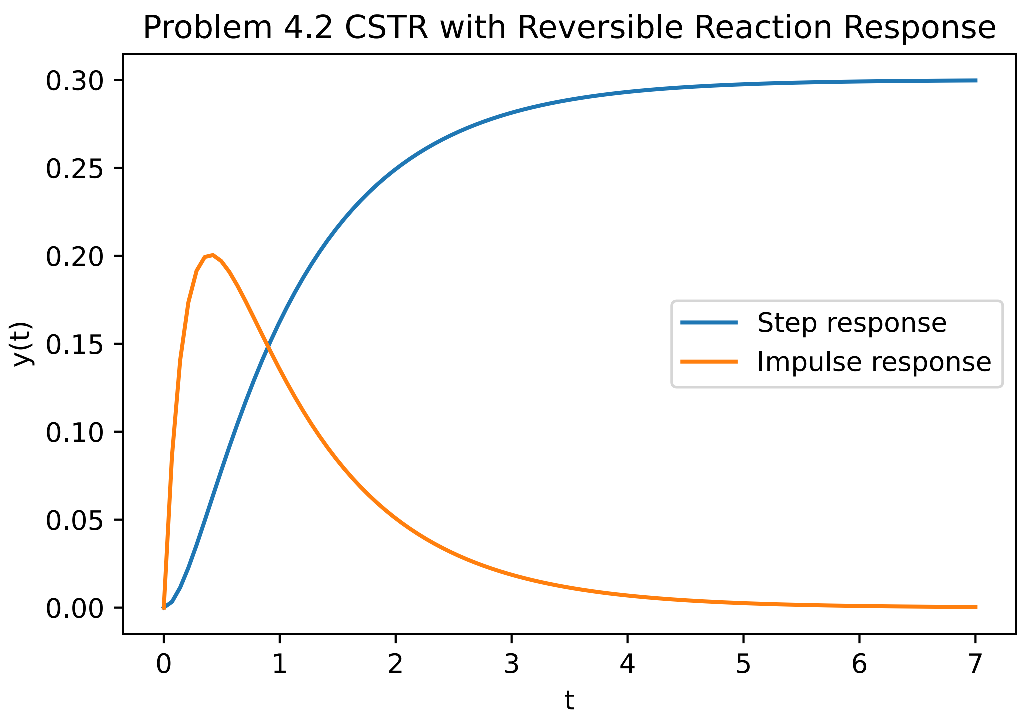
Transfer Function:

$$
\begin{aligned}
\mathcal{L}\left\{V\frac{dC_R}{dt}\right\} &= \mathcal{L}\left\{FC_{in,R} - FC_R - Vk_1C_R + Vk_{-1}C_P\right\} \\
VsC_R(s) &= FC_{in,R}(s) - C_R(s)(F + Vk_1) + Vk_{-1}C_P(s) \\
C_R(s)(Vs + F + Vk_1) &= FC_{in,R}(s) + Vk_{-1}C_P(s) \\
C_R(s) &= \frac{F}{Vs + F + Vk_1}C_{in,R}(s) + \frac{Vk_{-1}}{Vs + F + Vk_1}C_P(s) \\
\mathcal{L}\left\{V\frac{dC_P}{dt}\right\} &= \mathcal{L}\left\{Vk_1C_R - FC_P - Vk_2C_P - Vk_{-1}C_P\right\} \\
VsC_P(s) &= Vk_1C_R(s) - FC_P(s) - Vk_2C_P(s) - Vk_{-1}C_P \\
C_P(s)(Vs + F + Vk_2 + Vk_{-1}) &= Vk_1C_R(s) \\
C_P(s)(Vs + F + Vk_2 + Vk_{-1}) &= \frac{FVk_1}{Vs + F + Vk_1}C_{in,R}(s) + \frac{V^2k_{-1}k_1}{Vs + F + Vk_1}C_P(s) \\
\frac{FVk_1}{Vs + F + Vk_1}C_{in,R}(s) &= C_P(s)\left((Vs + F + Vk_2 + Vk_{-1}) - \frac{V^2k_{-1}k_1}{Vs + F + Vk_1}\right) \\
(FVk_1)C_{in,R}(s) &= C_P(s)\big((Vs + F + Vk_2 + Vk_{-1})(Vs + F + Vk_1) - V^2k_{-1}k_1\big) \\
C_P(s) &= \frac{FVk_1}{(Vs + F + Vk_2 + Vk_{-1})(Vs + F + Vk_1) - V^2k_{-1}k_1}C_{in,R}(s) \\
&\boxed{G(s) = \frac{FVk_1}{(Vs + F + Vk_2 + Vk_{-1})(Vs + F + Vk_1) - V^2k_{-1}k_1}}
\end{aligned}
$$

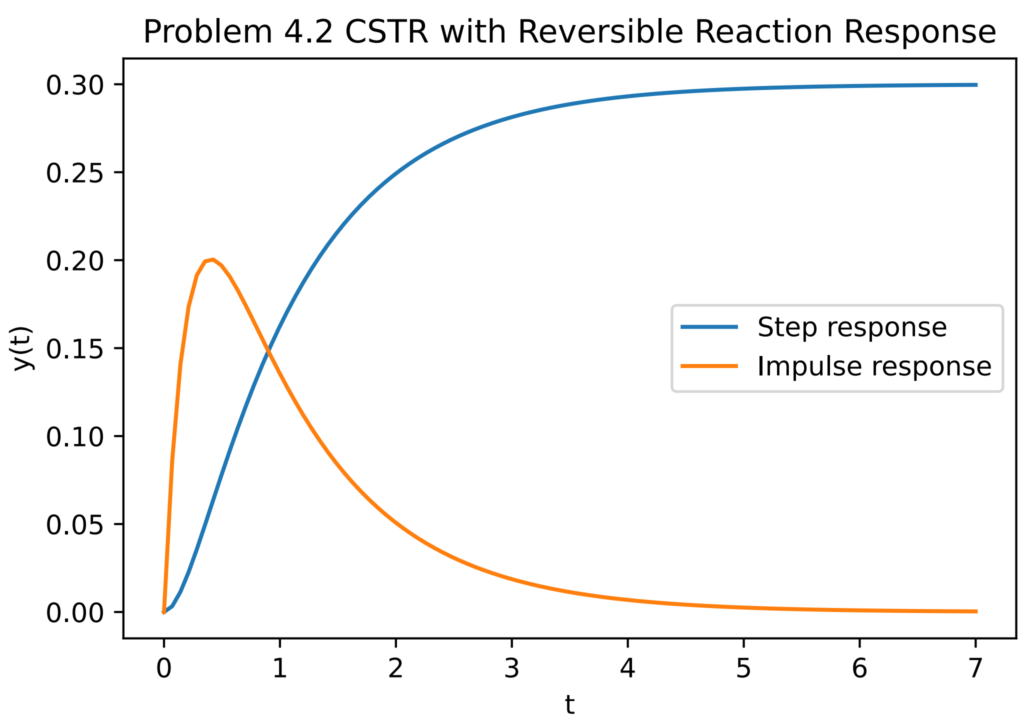
(Matlab)

$$
\begin{aligned}
V\frac{dC_R}{dt} &= FC_{in,R} - FC_R - Vk_1C_R + Vk_{-1}C_P \\
V\frac{dC_P}{dt} &= Vk_1C_R - FC_P - Vk_2C_P - Vk_{-1}C_P \\
\frac{dC_R}{dt} &= \frac{F}{V}C_{in,R} - \frac{F}{V}C_R - k_1C_R + k_{-1}C_P \\
\frac{dC_P}{dt} &= k_1C_R - \frac{F}{V}C_P - k_2C_P - k_{-1}C_P \\
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1u \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_2u \\
y &= c_1x_1 + c_2x_2 + du \\
\frac{dC_R}{dt} &= -\left(\frac{F}{V} + k_1\right)C_R + k_{-1}C_P + \frac{F}{V}C_{in,R} \\
\frac{dC_P}{dt} &= -\left(\frac{F}{V} + k_2 + k_{-1}\right)C_P + k_1C_R \\
C_P &= C_P \\
\frac{d}{dt}\begin{bmatrix} C_R \\ C_P \end{bmatrix} &= \begin{bmatrix} -\left(\frac{F}{V} + k_1\right) & k_{-1} \\ k_1 & -\left(\frac{F}{V} + k_2 + k_{-1}\right) \end{bmatrix}\begin{bmatrix} C_R \\ C_P \end{bmatrix} + \begin{bmatrix} \frac{F}{V} \\ 0 \end{bmatrix}C_{in,R} \\
C_P &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} C_R \\ C_P \end{bmatrix} + 0 \cdot du \\
&\frac{F}{V} = 0.5 \\
&k_1 = 3 \\
&k_{-1} = 1.25 \\
&k_2 = 0.75
\end{aligned}
$$

Response plot:



Response plot from manual transfer function:



Transfer function output:
TransferFunctionContinuous(array([1.5]), array([1., 6., 5.]), dt: None)

$$G(s) = \frac{1.5}{s^2 + 6s + 5}$$

Code for solving the system

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.signal as signal

# make scipy.signal look like matlab control systems toolbox
ss = signal.lti
step = signal.step
impulse = signal.impulse
tf = signal.TransferFunction

# constants
F_V = 0.5
k_1 = 3.
k_b = 1.25
k_2 = 0.75

# matrices
A = np.array([
    [-(F_V + k_1), k_b],
    [k_1, -(F_V + k_2 + k_b)]
])
B = np.array([
    [F_V],
    [0]
])
C = np.array([0, 1])
D = 0

# define state space system
sys = ss(A, B, C, D)

# compute step/impulse response
t_step, y_step = step(sys)

t_impulse, y_impulse = impulse(sys)

# plot
plt.plot(t_step, y_step, label="Step response")
plt.plot(t_impulse, y_impulse, label="Impulse response")
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.2 CSTR with Reversible Reaction Response")
plt.legend()
plt.show()

# compute transfer function
transfer_func = tf(sys)
```

```python
print(transfer_func)

# manually create system from transfer function
sys_manual = ss(transfer_func.num, transfer_func.den)

# comput step/impulse response
t_step, y_step = step(sys_manual)

t_impulse, y_impulse = impulse(sys_manual)

plt.plot(t_step, y_step, label="Step response")
plt.plot(t_impulse, y_impulse, label="Impulse response")
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.2 CSTR with Reversible Reaction Response")
plt.legend()
plt.show()
```