# CHEN 461 HW11

April 26, 2023

[ ]:
```python
from sympy import symbols, exp, lambdify, simplify, expand, solve, latex
from sympy.integrals import inverse_laplace_transform
import sympy

from numpy import linspace
from matplotlib.pyplot import plot, grid, xlabel, ylabel, legend, title, xlim
from control import tf, margin, step_response

import matplotlib_inline
%matplotlib inline
matplotlib_inline.backend_inline.set_matplotlib_formats('png', 'pdf')
```

## 1 Problem 14.7

### 1.1 Part A

poles: $\boxed{0}$

zeros:

$1 - \gamma \exp\left(-\theta s\right) = 0$

$\frac{1}{\gamma} = \frac{1}{\exp(\theta s)}$

$s = \boxed{\frac{\ln \gamma}{\theta}}$

The reboiler is not BIBO stable. The poles are not all real and negative.

### 1.2 Part B

$U(s) = \frac{1}{s}$

$Y(s) = \frac{1 - \gamma e^{-s\theta}}{s^2}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1 - \gamma e^{-s\theta}}{s^2}\right\}$

$$\boxed{y(t) = -\gamma\left(t - \theta\right)\mathcal{H}\left(t - \theta\right) + t\mathcal{H}\left(t\right)}$$

[ ]:
```python
s, t, theta, gamma = symbols("s, t, theta, gamma", real=True)
```

[ ]:
```
Y = (1 - gamma * exp(-theta * s)) / s**2
y = inverse_laplace_transform(Y, s, t)
y
```

[ ]:

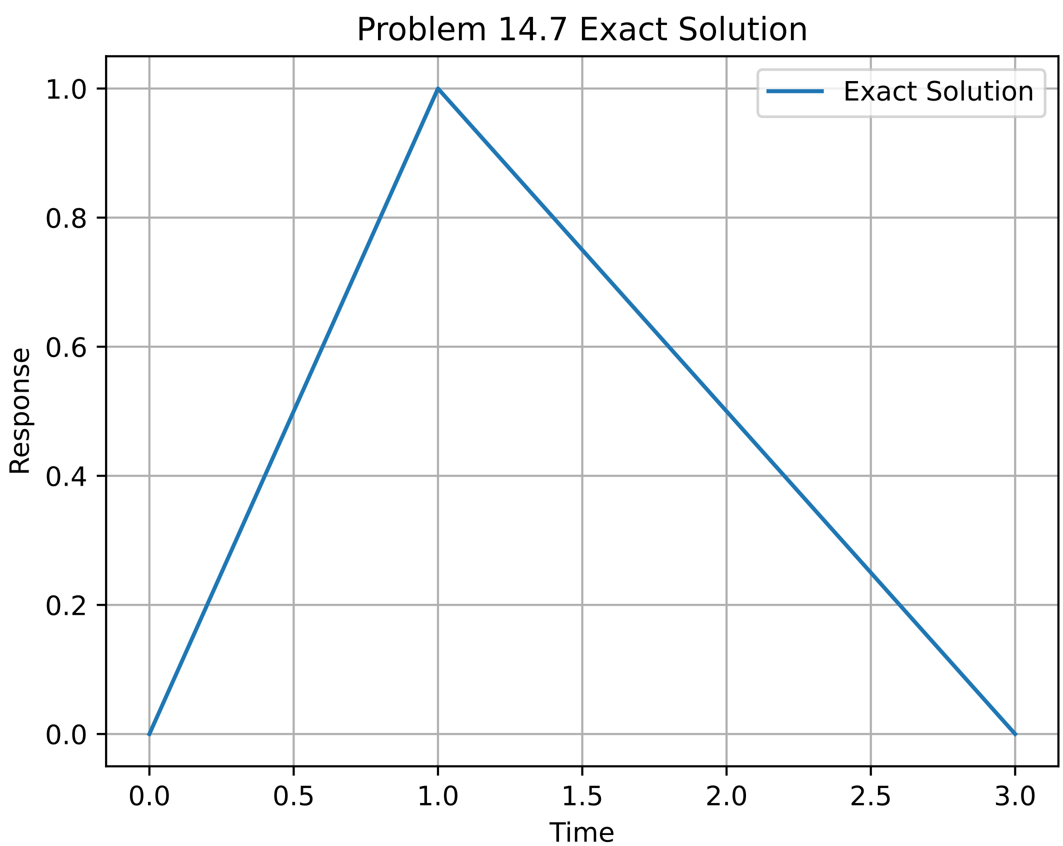
$$-\gamma\left(t-\theta\right)\theta\left(t-\theta\right)+t\theta\left(t\right)$$

[ ]:
```
y_lambda = lambdify(t, y.subs({gamma: 1.5, theta: 1}), "numpy")

t_range = linspace(0, 3, 100)

plot(t_range, y_lambda(t_range), label="Exact Solution")
grid(which="both")
xlabel("Time")
ylabel("Response")
title("Problem 14.7 Exact Solution")
legend()
```

[ ]: <matplotlib.legend.Legend at 0x1aff5df9810>




The response goes to 0 as $t$ goes to $\infty$.

### 1.3 Part C

1st order Pade

$\frac{1-\frac{s\theta}{2}}{1+\frac{s\theta}{2}}$

$$Y(s) = \frac{-\frac{\gamma\left(-\frac{s\theta}{2}+1\right)}{\frac{s\theta}{2}+1}+1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-\frac{\gamma\left(-\frac{s\theta}{2}+1\right)}{\frac{s\theta}{2}+1}+1}{s^2}\right\}$$

$$\boxed{y(t) = \left(-\gamma\theta + \left(\gamma\theta - t\left(\gamma - 1\right)\right)e^{\frac{2t}{\theta}}\right)e^{-\frac{2t}{\theta}}}$$

2nd order Pade

$\frac{\frac{s^2\theta^2}{12}-\frac{s\theta}{2}+1}{\frac{s^2\theta^2}{12}+\frac{s\theta}{2}+1}$

$$Y(s) = \frac{-\frac{\gamma\left(\frac{s^2\theta^2}{12}-\frac{s\theta}{2}+1\right)}{\frac{s^2\theta^2}{12}+\frac{s\theta}{2}+1}+1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-\frac{\gamma\left(\frac{s^2\theta^2}{12}-\frac{s\theta}{2}+1\right)}{\frac{s^2\theta^2}{12}+\frac{s\theta}{2}+1}+1}{s^2}\right\}$$

$$\boxed{y(t) = \left(-2\gamma\theta\sin\left(\frac{\sqrt{3}t}{\theta}+\frac{\pi}{6}\right) + \left(\gamma\theta - t\left(\gamma - 1\right)\right)e^{\frac{3t}{\theta}}\right)e^{-\frac{3t}{\theta}}}$$

[ ]:
```
pade_1 = (1 - theta * s / 2) / (1 + theta * s / 2)
pade_2 = (1 - theta * s / 2 + theta**2 * s**2 / 12) / (1 + theta * s / 2 +
  ↪theta**2 * s**2 / 12)

Y_p1 = (1 - gamma * pade_1) / s**2
y_p1 = inverse_laplace_transform(Y_p1, s, t)
y_p1
```

[ ]: $\left(-\gamma\theta + \left(\gamma\theta - t\left(\gamma - 1\right)\right)e^{\frac{2t}{\theta}}\right)e^{-\frac{2t}{\theta}}\theta\left(t\right)$

[ ]:
```
Y_p2 = (1 - gamma * pade_2) / s**2
y_p2 = inverse_laplace_transform(Y_p2, s, t)
y_p2
```

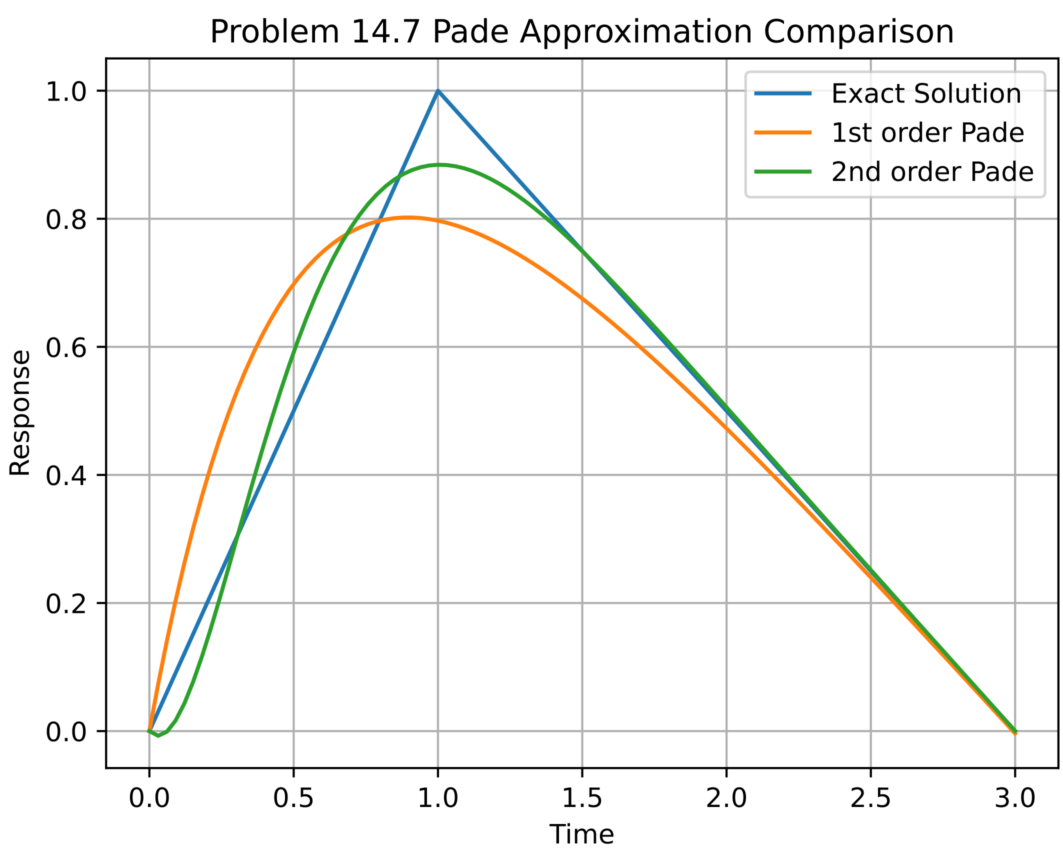
[ ]: $\left(-2\gamma\theta\sin\left(\frac{\sqrt{3}t}{\theta}+\frac{\pi}{6}\right) + \left(\gamma\theta - t\left(\gamma - 1\right)\right)e^{\frac{3t}{\theta}}\right)e^{-\frac{3t}{\theta}}\theta\left(t\right)$

[ ]:
```
y_p1_lambda = lambdify(t, y_p1.subs({gamma: 1.5, theta: 1}), "numpy")
y_p2_lambda = lambdify(t, y_p2.subs({gamma: 1.5, theta: 1}), "numpy")
```

```
t_range = linspace(0, 3, 100)

plot(t_range, y_lambda(t_range), label="Exact Solution")
plot(t_range, y_p1_lambda(t_range), label="1st order Pade")
plot(t_range, y_p2_lambda(t_range), label="2nd order Pade")
grid(which="both")
xlabel("Time")
ylabel("Response")
title("Problem 14.7 Pade Approximation Comparison")
legend()
```

```
[ ]: <matplotlib.legend.Legend at 0x1aff8997b90>
```




## 2 Problem 14.8

$G_c = k_c$

$G = \frac{G_c G_p}{1+G_c G_p}$

Use $\gamma = 1.5$ and $\theta = 1$

1st order Pade

$G_p = \frac{- \frac{\gamma \left(- \frac{s \theta}{2} + 1\right)}{\frac{s \theta}{2} + 1} + 1}{s}$

$G = \frac{k_{c} \left(1.0 - 2.5 s\right)}{k_{c} \left(1.0 - 2.5 s\right) - s \left(s + 2\right)}$

common denominator $= - 2.5 k_{c} s + 1.0 k_{c} - s^{2} - 2 s$

$a_0 = -1$

$a_1 = - 2.5 k_{c} - 2$

$a_2 = k_{c}$

$k_{c} < -0.8$

$\boxed{k_{c} > 0}$

2nd order Pade

$G_p = \frac{- \frac{\gamma \left(\frac{s^{2} \theta^{2}}{12} - \frac{s \theta}{2} + 1\right)}{\frac{s^{2} \theta^{2}}{12} + \frac{s \theta}{2} + 1} + 1}{s}$

$G = \frac{k_{c} \left(0.5 s^{2} - 15.0 s + 6.0\right)}{k_{c} \left(0.5 s^{2} - 15.0 s + 6.0\right) - s \left(s^{2} + 6 s + 12\right)}$

common denominator $= 0.5 k_{c} s^{2} - 15.0 k_{c} s + 6.0 k_{c} - s^{3} - 6 s^{2} - 12 s$

$a_0 = -1$

$a_1 = 0.5 k_{c} - 6$

$a_2 = - 15.0 k_{c} - 12$

$a_3 = 6.0 k_{c}$

$B_1 = - 7.5 k_{c}^{2} + 90.0 k_{c} + 72.0$

$C_1 = a_3$

$k_{c} < 12$

$-0.75 < k_{c} < 12.75$

$k_{c} > 0$

$\boxed{0 < k_{c} < 12.75}$

Finding the exact stability range involves finding the crossover frequency and then finding the AR at the crossover frequency.

[ ]:
```
k_c = symbols("k_c")

G_p1 = ((1 - gamma * pade_1) / s).subs({gamma: 1.5, theta: 1})

G = simplify(k_c * G_p1 / (1 + k_c * G_p1))
G
```

[ ]:

$$\frac{k_c\left(1.0 - 2.5s\right)}{k_c\left(1.0 - 2.5s\right) - s\left(s + 2\right)}$$

[ ]:
```
den = expand((k_c*(1.0 - 2.5*s) - s*(s + 2)))
den
```

[ ]: $-2.5k_cs + 1.0k_c - s^2 - 2s$

[ ]:
```
a_0 = den.coeff(s**2)
a_1 = den.coeff(s)
a_2 = expand(den - a_0 * s**2 - a_1 * s)
```

[ ]:
```
solve(a_1 > 0, k_c)
```

[ ]: $-\infty < k_c \wedge k_c < -0.8$

[ ]:
```
solve(a_2 > 0, k_c)
```

[ ]: $0 < k_c \wedge k_c < \infty$

[ ]:
```
k_c = symbols("k_c")

G_p2 = ((1 - gamma * pade_2) / s).subs({gamma: 1.5, theta: 1})

G = simplify(k_c * G_p2 / (1 + k_c * G_p2))
G
```

[ ]:
$$\frac{k_c\left(0.5s^2 - 15.0s + 6.0\right)}{k_c\left(0.5s^2 - 15.0s + 6.0\right) - s\left(s^2 + 6s + 12\right)}$$

[ ]:
```
den = expand((k_c*(0.5*s**2 - 15.0*s + 6.0) - s*(s**2 + 6*s + 12)))
den
```

[ ]: $0.5k_cs^2 - 15.0k_cs + 6.0k_c - s^3 - 6s^2 - 12s$

[ ]:
```
a_0 = den.coeff(s**3)
a_1 = den.coeff(s**2)
a_2 = den.coeff(s)
a_3 = expand(den - a_0 * s**3 - a_1 * s**2 - a_2 * s)
```

[ ]:
```
B_1 = simplify(a_1 * a_2 - a_0 * a_3)
C_1 = a_3
```

[ ]:
```
solve(a_1 > 0, k_c)
```

[ ]: $12.0 < k_c \wedge k_c < \infty$

[ ]:
```
solve(B_1 > 0, k_c)
```

[ ]: $-0.752777206453654 < k_c \wedge k_c < 12.7527772064537$

[ ]:
```
solve(C_1 > 0, k_c)
```

[ ]: $0 < k_c \wedge k_c < \infty$

# 3 17.1

## 3.1 Part A

$G = \frac{2}{s(s+1)^2}$

$L = \frac{2k_c}{s(s+1)^2}$

$|\mathrm{L}(i\omega)| = \frac{2k_c}{\omega(\omega^2+1)}$

$\arg L(i\omega) = -90 - 2\tan^{-1}\omega$

Find $\omega_p$

$-90 - 2\tan^{-1}\omega_p = -180$

$\omega_p = 1$

$\mathrm{GM} = \frac{1}{\frac{2k_c}{\omega_p(\omega_p^2+1)}}$

$2 = \frac{1}{\frac{2k_c}{1(1^2+1)}}$

$\boxed{k_c = 0.5}$

[ ]:
```
# vefify gain margin is 2
k = 0.5
L = tf(2 * k, [1, 2, 1, 0])
margin(L)
```

[ ]: (2.0, 21.386389751875043, 1.0, 0.6823278038280193)

## 3.2 Part B

$\mathrm{PM} = -90 - 2\tan^{-1}\omega_g + 180$

$30 = -90 - 2\tan^{-1}\omega_g + 180$

$\omega_g = 0.5774$

$1 = \frac{2k_c}{\omega_g(\omega_g^2+1)}$

$1 = \frac{2k_c}{0.5774\cdot(0.5774^2+1)}$

$\boxed{k_c = 0.3842}$

[ ]:
```
# verify phase margin is 30
k = 0.38490017946
L = tf(2 * k, [1, 2, 1, 0])
margin(L)
```

[ ]: (2.598076211351632, 29.99999999997854, 1.0, 0.5773502691898753)

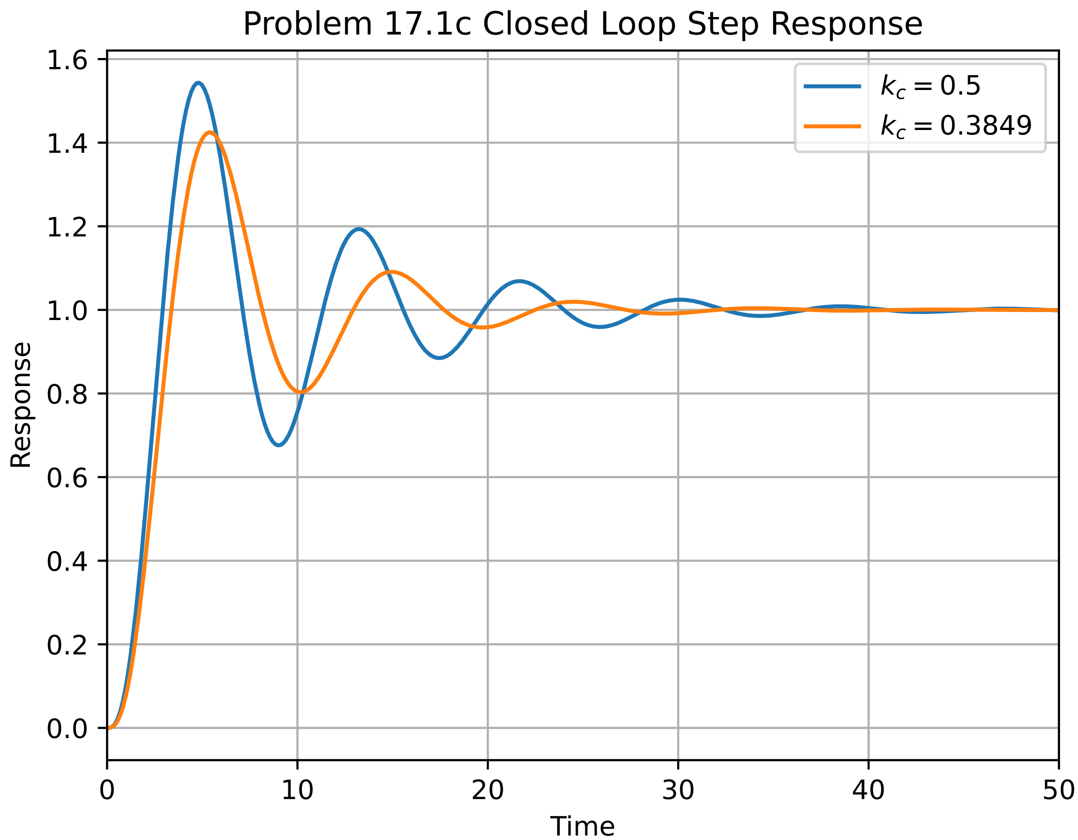
### 3.3 Part C

[ ]:
```
k = 0.5
L = tf(2 * k, [1, 2, 1, 0])
G = L / (1 + L)
t, y = step_response(G)
plot(t, y, label=r"$k_c=0.5$")

k = 0.38490017946
L = tf(2 * k, [1, 2, 1, 0])
G = L / (1 + L)
t, y = step_response(G)
plot(t, y, label=r"$k_c=0.3849$")

grid(which="both")
xlabel("Time")
ylabel("Response")
title("Problem 17.1c Closed Loop Step Response")
legend()
xlim([0, 50])
```

[ ]: (0.0, 50.0)




### 3.4 Part D

$|\mathrm{L}(i\omega_p)| < 1$

$\frac{2k_c}{1\cdot(1^2+1)} < 1$

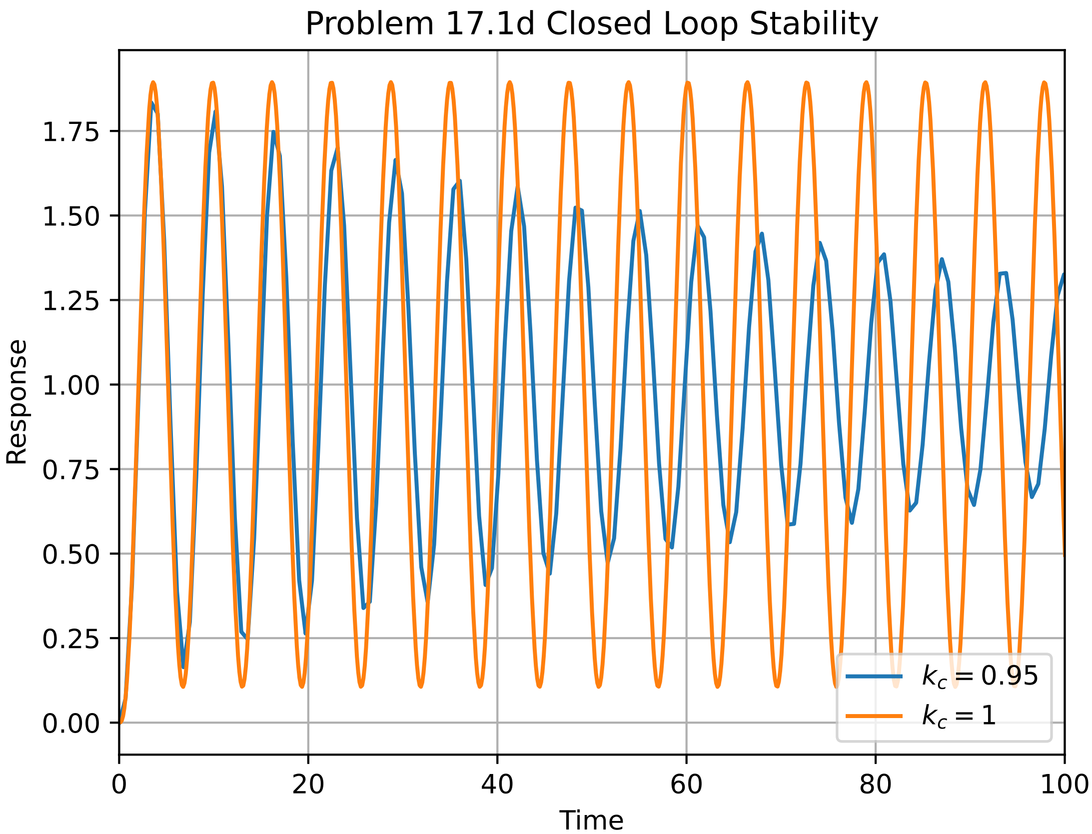
$\boxed{k_c < 1}$

[ ]:
```
# verify stability
k = 0.95
L = tf(2 * k, [1, 2, 1, 0])
G = L / (1 + L)
t, y = step_response(G)
plot(t, y, label=r"$k_c=0.95$")

k = 1
L = tf(2 * k, [1, 2, 1, 0])
G = L / (1 + L)
t, y = step_response(G)
plot(t, y, label=r"$k_c=1$")
```

```
grid(which="both")
xlabel("Time")
ylabel("Response")
title("Problem 17.1d Closed Loop Stability")
legend(loc="lower right")
xlim([0, 100])
```

[ ]: (0.0, 100.0)




For $k_c < 1$ the response dies down, while for $k_c = 1$ the response oscillates forever.

# 4 17.9

## 4.1 Part A

$\omega_p = 1$

$|G_p(i\omega_p)| = 0.5$

$k_c|G_p(i\omega_p)| < 1$

$\boxed{k_c < 2}$

Offset:

offset $= \frac{1}{1+k_p k_c}$

$k_p \approx 5$

offset $= \boxed{\dfrac{1}{1+5k_c}}$

### 4.2 Part B

$|\mathrm{L}(i\omega_p)| \approx 0.5$

$\boxed{\mathrm{GM} = 0.5}$

$\arg L(i\omega_g) = -150$

$\boxed{\mathrm{PM} = 30}$

### 4.3 Part C

$G_c = k_c\,(1 + 25s)$

$|G_c(i\omega)| = k_c\sqrt{25^2\omega^2 + 1}$

$\arg G_c(i\omega) = -\tan^{-1}(25\omega)$

$\arg G_c(i\omega_p) = -180$

$\omega_p = 0$

$k_c\sqrt{25^2 \cdot 0^2 + 1} < 1$

$\boxed{k_c < 1}$