CHEN 461 HW10

April 10, 2023

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib_inline

from scipy.integrate import solve_ivp

%matplotlib inline
matplotlib_inline.backend_inline.set_matplotlib_formats('png', 'pdf')
```

1 Problem 12.12

$T_1 = \frac{T_0}{\tau s+1} + \frac{kG_c}{\tau s+1}E$

$T_3 = \frac{T_1}{(\tau s+1)^2}$

$E = T_{sp} - T_3$

$T_3 = \frac{\frac{T_0}{\tau s+1} + \frac{kG_c}{\tau s+1}E}{(\tau s+1)^2}$

$T_3 = \frac{T_0}{(\tau s+1)^3} + \frac{kG_c}{(\tau s+1)^3}E$

$T_3 = \frac{T_0}{(\tau s+1)^3} + \frac{kG_c}{(\tau s+1)^3}T_{sp} - \frac{kG_c}{(\tau s+1)^3}T_3$

$T_3\left(1 + \frac{kG_c}{(\tau s+1)^3}\right) = \frac{T_0}{(\tau s+1)^3} + \frac{kG_c}{(\tau s+1)^3}T_{sp}$

$T_3\left((\tau s+1)^3 + kG_c\right) = T_0 + kG_cT_{sp}$

$T_3 = \frac{1}{(\tau s+1)^3+kG_c}T_0 + \frac{kG_c}{(\tau s+1)^3+kG_c}T_{sp}$

1.1 Part A

1.1.1 P controller

$G_c = k_c$

$T_3 = \frac{1}{(\tau s+1)^3+kk_c}T_0 + \frac{kG_c}{(\tau s+1)^3+kk_c}T_{sp}$

Denominator $= \tau^3s^3 + 3\tau^2s^2 + 3\tau s + 1 + kk_c$

Routh array;

$a_0 = \tau^3 \ a_2 = 3\tau$

$a_1 = 3\tau^2 \ a_3 = 1 + kk_c$

$B_1 = \frac{9\tau^3 - \tau^3(1+kk_c)}{3\tau^2}$

$C_1 = 1 + kk_c$

$\frac{9\tau^3 - \tau^3(1+kk_c)}{3\tau^2} > 0$

$9 > 1 + kk_c$

$k_c < \frac{8}{k}$

$1 + kk_c > 0$

$k_c > -\frac{1}{k}$

$$\boxed{-\frac{1}{k} < k_c < \frac{8}{k}}$$

1.1.2 PD controller

$G_c = k_c (1 + \tau_D s)$

Denominator $= \tau^3 s^3 + 3\tau^2 s^2 + 3\tau s + 1 + kk_c (1 + \tau_D s)$

Routh array;

$a_0 = \tau^3 \ a_2 = 3\tau + k_c \tau_D$

$a_1 = 3\tau^2 \ a_3 = 1 + kk_c$

$B_1 = \frac{3\tau^2(3\tau + k_c\tau_D) - \tau^3(1+kk_c)}{3\tau^2}$

$\frac{3\tau^2(3\tau + k_c\tau_D) - \tau^3(1+kk_c)}{3\tau^2} > 0$

$3(3\tau + k_c \tau_D) - \tau(1 + kk_c) > 0$

$8\tau + k_c (3\tau_D - \tau k) > 0$

$k_c (\tau - \tau k) > -8\tau$

$k_c (1 - k) > -8$

$k_c > \frac{8}{k-1}$

$$\boxed{k_c > -\frac{1}{k}}$$

Adding the derivative action to the proportional only controller has a stabilizing effect, and thus the PD controller can operate with a broader range of k_c values compared to the P only controller.

1.2 Part B

T_0 step change

$T_0(s) = \frac{M}{s}$

$T_3 = \frac{1}{(\tau s+1)^3 + kk_c} \cdot \frac{M}{s}$

Final value theorem:

$$\lim_{s\to 0^+} sY(s) = \lim_{t\to\infty} y(t)$$

$sT_3 = \frac{M}{(\tau s+1)^3 + kk_c}$

$\lim_{s\to 0^+} sT_3 = \lim_{s\to 0^+} \frac{M}{(\tau s+1)^3 + kk_c} = \frac{M}{1+kk_c}$

Final $T_0 = 0$

Offset $= 0 - T_3 = 0 - \frac{M}{1+kk_c}$

Largest $k_c = \frac{8}{k}$

Offset $= -\frac{M}{1+k\frac{8}{k}}$

Smallest possible offset $= \boxed{-\frac{M}{9}}$

2 Problem 12.13

2.1 Part A

$C_P = \frac{G_c G_P}{1+G_c G_P} C_{P_{sp}} + \frac{G'}{1+G_c G_P} C_{in,R}$

$G' = \frac{2}{(s+1)(s+2)}$

$G_P = \frac{4-s}{(s+1)(s+2)}$

$C_P = \frac{G_c \frac{4-s}{(s+1)(s+2)}}{1+G_c \frac{4-s}{(s+1)(s+2)}} C_{P_{sp}} + \frac{\frac{2}{(s+1)(s+2)}}{1+G_c \frac{4-s}{(s+1)(s+2)}} C_{in,R}$

$C_P = \frac{G_c(4-s)}{s^2+3s+2+G_c(4-s)} C_{P_{sp}} + \frac{2}{s^2+3s+2+G_c(4-s)} C_{in,R}$

2.1.1 P controller

$G_c = k_c$

Denominator $= s^2 + 3s + 2 + k_c(4-s) = s^2 + (4k_c+2)s + 4k_c + 2$

Routh array:

$a_0 = 1$ $a_2 = 4k_c + 2$

$a_1 = 3 - k_c$ $a_3 = 0$

$B_1 = 4k_c + 2$

$k_c > -\frac{1}{2}$

$3 - k_c > 0$

$k_c < 3$

$\boxed{-\frac{1}{2} < k_c < 3}$

2.1.2 PI controller

$G_c = k_c\left(1 + \frac{1}{\tau_I s}\right) = k_c\left(1 + \frac{4}{s}\right)$

Denominator $= s^2 + 3s + 2 + k_c\left(1 + \frac{4}{s}\right)(4 - s) = s^3 + (3 - k_c) + 2s + 16k_c$

Routh array:

$a_0 = 1$ $a_2 = 2$

$a_1 = 3 - k_c$ $a_3 = 16k_c$

$B_1 = \frac{6-2k_c-16k_c}{3-k_c}$

$\frac{6-2k_c-16k_c}{3-k_c} > 0$

$6 - 2k_c - 16k_c > 0$

$k_c < \frac{1}{3}$

$C_1 = 16k_c$

$16k_c > 0$

$k_c > 0$

$\boxed{0 < k_c < \frac{1}{3}}$

Adding the integral action to the proportional only controller has a destabilizing effect on the system, and thus the PI controller requires a narrower range of k_c values to operate with stability.

2.2 Part B

$C_P = \frac{k_c(4-s)}{s^2+3s+2+k_c(4-s)} C_{P_{sp}} + \frac{2}{s^2+3s+2+k_c(4-s)} C_{in,R}$

$C_{P_{sp}} = \frac{M}{s}$

$C_P = \frac{k_c(4-s)}{s^2+3s+2+k_c(4-s)} \cdot \frac{M}{s}$

Final value theorem:

$$\lim_{s\to 0^+} sY(s) = \lim_{t\to\infty} y(t)$$

$sC_P = \frac{Mk_c(4-s)}{s^2+3s+2+k_c(4-s)}$

$\lim_{s\to 0^+} sC_P = \frac{2Mk_c}{2k_c+1}$

Offset $= c_{P_{sp}}(t) - c_P(t) = M - \lim_{t\to\infty} c_P(t)$

Offset $= M - \frac{2Mk_c}{2k_c+1} = M\left(\frac{2k_c+1}{2k_c+1} - \frac{2k_c}{2k_c+1}\right)$

Offset $= \frac{M}{2k_c+1}$

Maximum $k_c = 3$

Minimum offset $= \boxed{\dfrac{M}{7}}$

3 Problem 13.4

3.0.1 State space models

$A_1 \frac{dh_1}{dt} = -\frac{h_1}{R_1} + k_v u + F_w$

$A_2 \frac{dh_2}{dt} = \frac{h_1}{R_1} - \frac{h_2}{R_2}$

$\tau_m \frac{dh_{2,m}}{dt} + h_{2,m} = h_2$

3.0.2 Real PID state space

$\frac{de_I}{dt} = e$

$\frac{de_f}{dt} = -\frac{1}{\alpha\tau_D} e_f + \frac{1}{\alpha\tau_D} e$

$u = \frac{k_c}{\tau_I} e_I + k_c \left(1 - \frac{1}{\alpha}\right) e_f + \frac{k_c}{\alpha} e$

$e = h_{2,sp} - h_{2,m}$

$\frac{de_I}{dt} = h_{2,sp} - h_{2,m}$

$\frac{de_f}{dt} = -\frac{1}{\alpha\tau_D} e_f + \frac{1}{\alpha\tau_D} \left(h_{2,sp} - h_{2,m}\right)$

$u = \frac{k_c}{\tau_I} e_I + k_c \left(1 - \frac{1}{\alpha}\right) e_f + \frac{k_c}{\alpha} \left(h_{2,sp} - h_{2,m}\right)$

$A_1 \frac{dh_1}{dt} = -\frac{h_1}{R_1} + k_v \left(\frac{k_c}{\tau_I} e_I + k_c \left(1 - \frac{1}{\alpha}\right) e_f + \frac{k_c}{\alpha} \left(h_{2,sp} - h_{2,m}\right)\right) + F_w$

3.1 State space model that describes the system

$$\frac{dh_1}{dt} = -\frac{1}{A_1 R_1} h_1 - \frac{k_c k_v}{A_1 \alpha} h_{2,m} + \frac{k_c k_v}{A_1 \tau_I} e_I + \left(\frac{k_c k_v}{A_1} - \frac{k_c k_v}{A_1 \alpha}\right) e_f + \frac{1}{A_1} F_w + \frac{k_c k_v}{A_1 \alpha} h_{2,sp}$$

$$\frac{dh_2}{dt} = \frac{1}{A_2 R_1} h_1 - \frac{1}{A_2 R_2} h_2$$

$$\frac{dh_{2,m}}{dt} = \frac{1}{\tau_m} h_2 - \frac{1}{\tau_m} h_{2,m}$$

$$\frac{de_I}{dt} = -h_{2,m} + h_{2,sp}$$

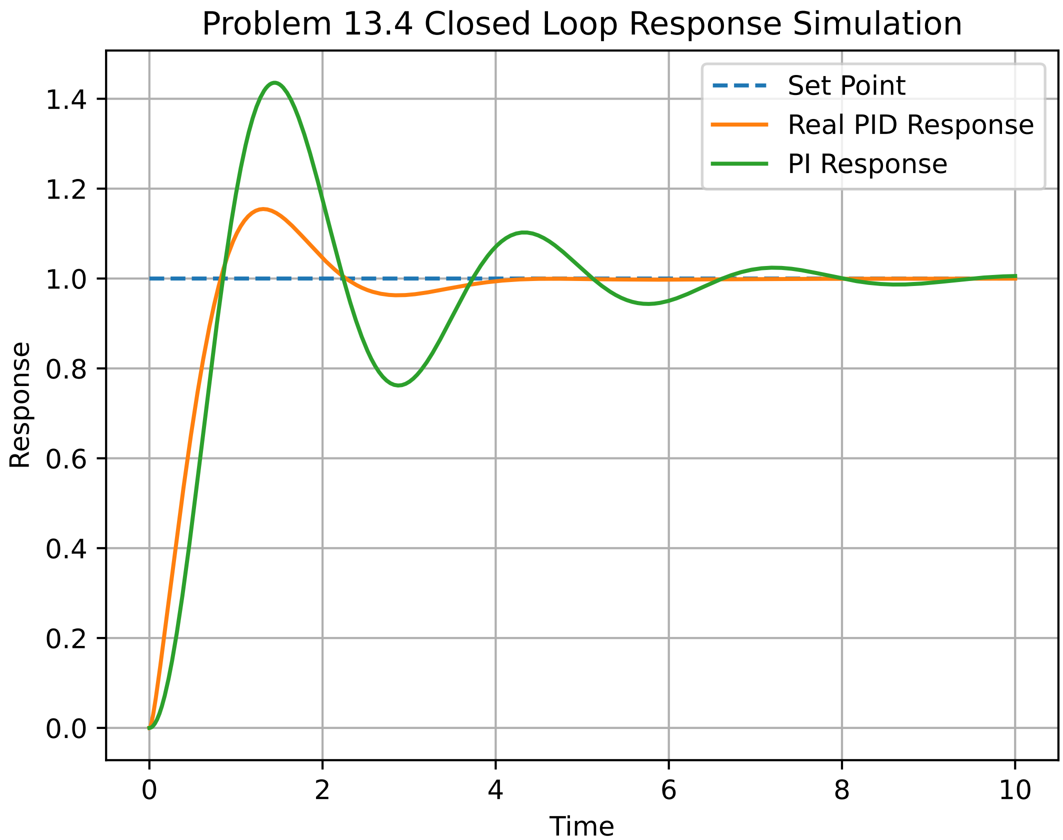
$$\frac{de_f}{dt} = -\frac{1}{\alpha\tau_D} h_{2,m} - \frac{1}{\alpha\tau_D} e_f + \frac{1}{\alpha\tau_D} h_{2,sp}$$

3.2 System Simulation

[ ]:
```python
# 13.4 simulation
ode_kwargs = {
    'method': 'Radau',
    'rtol': 1e-8,
    'atol': 1e-8,
}

t_range = [0, 10]

initial_cond = [0, 0, 0, 0, 0]

def real_pid_ode(t, y):
    f = y*0

    h_1 = y[0]
    h_2 = y[1]
    h_2m = y[2]
    e_I = y[3]
    e_f = y[4]

    h_2sp = 1
    F_w = 0

    k_c = 5
    tau_I = 2
    tau_D = 0.25
    alpha = 0.1
    tau_m = 0.1
    A_1, A_2, R_1, R_2, k_v = 1, 1, 1, 1, 1

    e = h_2sp - h_2m

    u = k_c * (e_I / tau_I + (1 - 1 / alpha) * e_f + e / alpha)

    f[0] = (-h_1 / R_1 + k_v * u + F_w) / A_1
    f[1] = (h_1 / R_1 - h_2 / R_2) / A_2
    f[2] = (h_2 - h_2m) / tau_m
    f[3] = e
    f[4] = (e - e_f) / (alpha * tau_D)

    return f

def pi_ode(t, y):
    f = y*0
```

```
    h_1 = y[0]
    h_2 = y[1]
    h_2m = y[2]
    e_I = y[3]

    h_2sp = 1
    F_w = 0

    k_c = 5
    tau_I = 2
    tau_m = 0.1
    A_1, A_2, R_1, R_2, k_v = 1, 1, 1, 1, 1

    e = h_2sp - h_2m

    u = k_c * (e_I / tau_I + e)

    f[0] = (-h_1 / R_1 + k_v * u + F_w) / A_1
    f[1] = (h_1 / R_1 - h_2 / R_2) / A_2
    f[2] = (h_2 - h_2m) / tau_m
    f[3] = e

    return f

real_pid_sol = solve_ivp(real_pid_ode, t_range, initial_cond, **ode_kwargs)

pi_sol = solve_ivp(pi_ode, t_range, initial_cond[:4], **ode_kwargs)

plt.plot(real_pid_sol.t, np.ones(real_pid_sol.t.shape[0]), '--', label='Set␣
 ↪Point')
plt.plot(real_pid_sol.t, real_pid_sol.y[1], label='Real PID Response')
plt.plot(pi_sol.t, pi_sol.y[1], label="PI Response")
plt.xlabel('Time')
plt.ylabel('Response')
plt.title('Problem 13.4 Closed Loop Response Simulation')
plt.grid()
plt.legend()
```

[ ]:
```
<matplotlib.legend.Legend at 0x28e7e864810>
```



3.2.1 Comparison

Both the PI and PID controller are able to get the system to the set point. The PID controller is able to get the system to the set point faster than PI controller. The PI controller has more variation from the set point than the PID controller.