hw10

April 7, 2023

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
import matplotlib_inline

from scipy.integrate import solve_ivp

%matplotlib inline
matplotlib_inline.backend_inline.set_matplotlib_formats('png', 'pdf')
```

1 Problem 12.12

$T_1 = \frac{T_0}{\tau s+1} + \frac{kG_c}{\tau s+1}E$

$T_3 = \frac{T_1}{(\tau s+1)^2}$

$E = T_{sp} - T_3$

$T_3 = \frac{\frac{T_0}{\tau s+1} + \frac{kG_c}{\tau s+1}E}{(\tau s+1)^2}$

$T_3 = \frac{T_0}{(\tau s+1)^3} + \frac{kG_c}{(\tau s+1)^3}E$

$T_3 = \frac{T_0}{(\tau s+1)^3} + \frac{kG_c}{(\tau s+1)^3}T_{sp} - \frac{kG_c}{(\tau s+1)^3}T_3$

$T_3\left(1 + \frac{kG_c}{(\tau s+1)^3}\right) = \frac{T_0}{(\tau s+1)^3} + \frac{kG_c}{(\tau s+1)^3}T_{sp}$

$T_3\left((\tau s+1)^3 + kG_c\right) = T_0 + kG_cT_{sp}$

$T_3 = \frac{1}{(\tau s+1)^3+kG_c}T_0 + \frac{kG_c}{(\tau s+1)^3+kG_c}T_{sp}$

1.1 Part A

1.1.1 P controller

$G_c = k_c$

$T_3 = \frac{1}{(\tau s+1)^3+kk_c}T_0 + \frac{kG_c}{(\tau s+1)^3+kk_c}T_{sp}$

Denominator $= \tau^3s^3 + 3\tau^2s^2 + 3\tau s + 1 + kk_c$

Routh array;

$a_0 = \tau^3 \ a_2 = 3\tau$

$a_1 = 3\tau^2 \ a_3 = 1 + kk_c$

$B_1 = \frac{9\tau^3 - \tau^3(1+kk_c)}{3\tau^2}$

$C_1 = 1 + kk_c$

$\frac{9\tau^3 - \tau^3(1+kk_c)}{3\tau^2} > 0$

$9 > 1 + kk_c$

$k_c < \frac{8}{k}$

$1 + kk_c > 0$

$k_c > -\frac{1}{k}$

$$\boxed{-\frac{1}{k} < k_c < \frac{8}{k}}$$

1.1.2 PD controller

$G_c = k_c\left(1 + \tau_D s\right)$

Denominator $= \tau^3 s^3 + 3\tau^2 s^2 + 3\tau s + 1 + kk_c\left(1 + \tau_D s\right)$

Routh array;

$a_0 = \tau^3 \ a_2 = 3\tau + k_c\tau_D$

$a_1 = 3\tau^2 \ a_3 = 1 + kk_c$

$B_1 = \frac{3\tau^2(3\tau + k_c\tau_D) - \tau^3(1+kk_c)}{3\tau^2}$

$\frac{3\tau^2(3\tau + k_c\tau_D) - \tau^3(1+kk_c)}{3\tau^2} > 0$

$3\left(3\tau + k_c\tau_D\right) - \tau\left(1 + kk_c\right) > 0$

$8\tau + k_c\left(3\tau_D - \tau k\right) > 0$

$k_c\left(\tau - \tau k\right) > -8\tau$

$k_c\left(1 - k\right) > -8$

$k_c > \frac{8}{k-1}$

$$\boxed{k_c > -\frac{1}{k}}$$

Adding the derivative action to the proportional only controller has a stabilizing effect, and thus the PD controller can operate with a broader range of k_c values than the P only controller.

1.2 Part B

T_0 step change

$T_0(s) = \frac{M}{s}$

$T_3 = \frac{1}{(\tau s+1)^3 + kk_c} \cdot \frac{M}{s}$

Final value theorem:

$$\lim_{s\to 0^+} sY(s) = \lim_{t\to\infty} y(t)$$

$sT_3 = \frac{M}{(\tau s+1)^3 + kk_c}$

$\lim_{s\to 0^+} sT_3 = \lim_{s\to 0^+} \frac{M}{(\tau s+1)^3 + kk_c} = \frac{M}{1+kk_c}$

Final $T_0 = 0$

Offset $= 0 - T_3 = 0 - \frac{M}{1+kk_c}$

Largest $k_c = \frac{8}{k}$

Offset $= -\frac{M}{1+k\frac{8}{k}}$

Smallest possible offset $= \boxed{-\frac{M}{9}}$

2 Problem 12.13

2.1 Part A

$C_P = \frac{G_c G_P}{1+G_c G_P} C_{P_{sp}} + \frac{G'}{1+G_c G_P} C_{in,R}$

$G' = \frac{2}{(s+1)(s+2)}$

$G_P = \frac{4-s}{(s+1)(s+2)}$

$C_P = \frac{G_c \frac{4-s}{(s+1)(s+2)}}{1+G_c \frac{4-s}{(s+1)(s+2)}} C_{P_{sp}} + \frac{\frac{2}{(s+1)(s+2)}}{1+G_c \frac{4-s}{(s+1)(s+2)}} C_{in,R}$

$C_P = \frac{G_c(4-s)}{s^2+3s+2+G_c(4-s)} C_{P_{sp}} + \frac{2}{s^2+3s+2+G_c(4-s)} C_{in,R}$

2.1.1 P controller

$G_c = k_c$

Denominator $= s^2 + 3s + 2 + k_c(4-s) = s^2 + (4k_c+2)s + 4k_c + 2$

Routh array:

$a_0 = 1$ $a_2 = 4k_c + 2$

$a_1 = 3 - k_c$ $a_3 = 0$

$B_1 = 4k_c + 2$

$k_c > -\frac{1}{2}$

$3 - k_c > 0$

$k_c < 3$

$\boxed{-\frac{1}{2} < k_c < 3}$

2.1.2 PI controller

$G_c = k_c\left(1 + \frac{1}{\tau_I s}\right) = k_c\left(1 + \frac{4}{s}\right)$

Denominator $= s^2 + 3s + 2 + k_c\left(1 + \frac{4}{s}\right)(4 - s) = s^3 + (3 - k_c) + 2s + 16k_c$

Routh array:

$a_0 = 1$ $a_2 = 2$

$a_1 = 3 - k_c$ $a_3 = 16k_c$

$B_1 = \frac{6 - 2k_c - 16k_c}{3 - k_c}$

$\frac{6 - 2k_c - 16k_c}{3 - k_c} > 0$

$6 - 2k_c - 16k_c > 0$

$k_c < \frac{1}{3}$

$C_1 = 16k_c$

$16k_c > 0$

$k_c > 0$

$\boxed{0 < k_c < \frac{1}{3}}$

Adding the integral action to the proportional only controller has a destabilizing effect on the system, and thus the PI controller requires a narrower range of k_c to operate with stability.

2.2 Part B

$C_P = \frac{k_c(4-s)}{s^2+3s+2+k_c(4-s)} C_{P_{sp}} + \frac{2}{s^2+3s+2+k_c(4-s)} C_{in,R}$

$C_{P_{sp}} = \frac{M}{s}$

$C_P = \frac{k_c(4-s)}{s^2+3s+2+k_c(4-s)} \cdot \frac{M}{s}$

Final value theorem:

$$\lim_{s\to 0^+} sY(s) = \lim_{t\to\infty} y(t)$$

$sC_P = \frac{Mk_c(4-s)}{s^2+3s+2+k_c(4-s)}$

$\lim_{s\to 0^+} sC_P = \frac{2Mk_c}{2k_c+1}$

Offset $= c_{P_{sp}}(t) - c_P(t) = M - \lim_{t\to\infty} c_P(t)$

Offset $= M - \frac{2Mk_c}{2k_c+1} = M\left(\frac{2k_c+1}{2k_c+1} - \frac{2k_c}{2k_c+1}\right)$

Offset $= \frac{M}{2k_c+1}$

Maximum $k_c = 3$

Minimum offset $= \boxed{\dfrac{M}{7}}$

3 Problem 13.4

3.0.1 State space models

$A_1 \frac{dh_1}{dt} = -\frac{h_1}{R_1} + k_v u + F_w$

$A_2 \frac{dh_2}{dt} = \frac{h_1}{R_1} - \frac{h_2}{R_2}$

$\tau_m \frac{dh_{2,m}}{dt} + h_{2,m} = h_2$

3.0.2 Real PID state space

$\frac{de_I}{dt} = e$

$\frac{de_f}{dt} = -\frac{1}{\alpha\tau_D} e_f + \frac{1}{\alpha\tau_D} e$

$u = \frac{k_c}{\tau_I} e_I + k_c \left(1 - \frac{1}{\alpha}\right) e_f + \frac{k_c}{\alpha} e$

$e = h_{2,sp} - h_{2,m}$

$\frac{de_I}{dt} = h_{2,sp} - h_{2,m}$

$\frac{de_f}{dt} = -\frac{1}{\alpha\tau_D} e_f + \frac{1}{\alpha\tau_D} \left(h_{2,sp} - h_{2,m}\right)$

$u = \frac{k_c}{\tau_I} e_I + k_c \left(1 - \frac{1}{\alpha}\right) e_f + \frac{k_c}{\alpha} \left(h_{2,sp} - h_{2,m}\right)$

$A_1 \frac{dh_1}{dt} = -\frac{h_1}{R_1} + k_v \left(\frac{k_c}{\tau_I} e_I + k_c \left(1 - \frac{1}{\alpha}\right) e_f + \frac{k_c}{\alpha} \left(h_{2,sp} - h_{2,m}\right)\right) + F_w$

3.1 State space model that describes the system

$$\frac{dh_1}{dt} = -\frac{1}{A_1 R_1} h_1 - \frac{k_c k_v}{A_1 \alpha} h_{2,m} + \frac{k_c k_v}{A_1 \tau_I} e_I + \left(\frac{k_c k_v}{A_1} - \frac{k_c k_v}{A_1 \alpha}\right) e_f + \frac{1}{A_1} F_w + \frac{k_c k_v}{A_1 \alpha} h_{2,sp}$$

$$\frac{dh_2}{dt} = \frac{1}{A_2 R_1} h_1 - \frac{1}{A_2 R_2} h_2$$

$$\frac{dh_{2,m}}{dt} = \frac{1}{\tau_m} h_2 - \frac{1}{\tau_m} h_{2,m}$$

$$\frac{de_I}{dt} = -h_{2,m} + h_{2,sp}$$

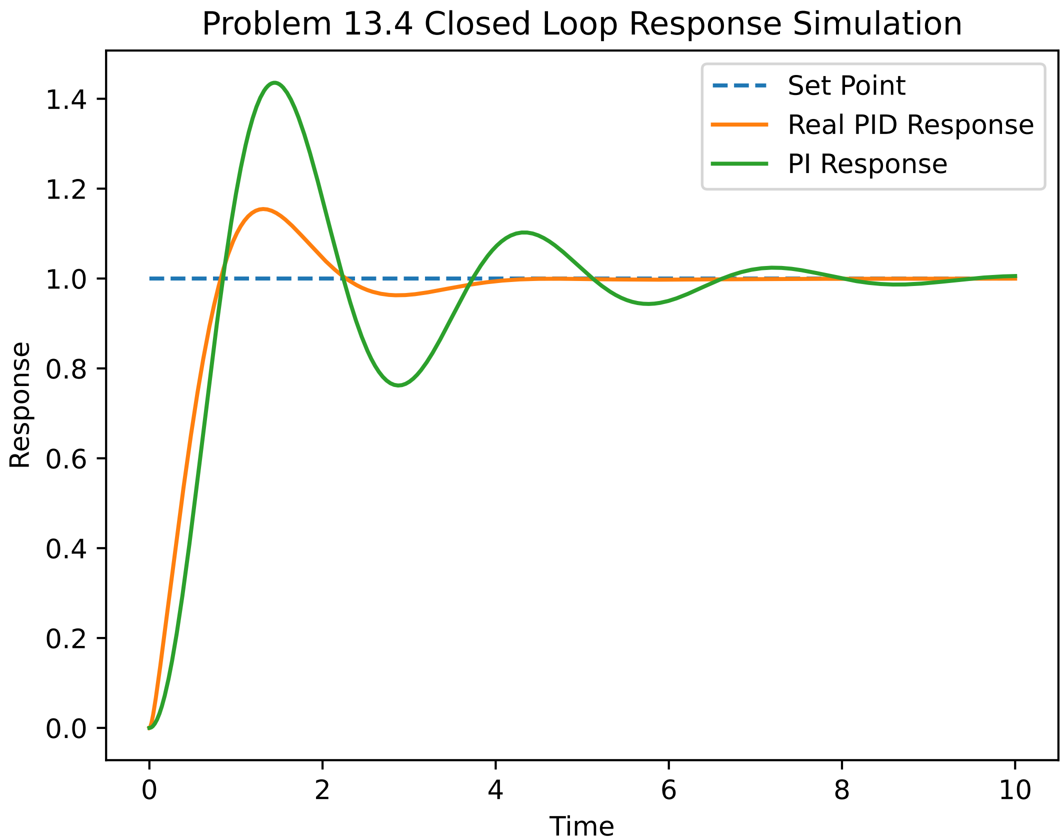
$$\frac{de_f}{dt} = -\frac{1}{\alpha\tau_D} h_{2,m} - \frac{1}{\alpha\tau_D} e_f + \frac{1}{\alpha\tau_D} h_{2,sp}$$

3.2 System Simulation

[ ]:
```python
# 13.4 simulation
ode_kwargs = {
    'method': 'Radau',
    'rtol': 1e-8,
    'atol': 1e-8,
}

t_range = [0, 10]

initial_cond = [0, 0, 0, 0, 0]

def real_pid_ode(t, y):
    f = y*0

    h_1 = y[0]
    h_2 = y[1]
    h_2m = y[2]
    e_I = y[3]
    e_f = y[4]

    h_2sp = 1
    F_w = 0

    k_c = 5
    tau_I = 2
    tau_D = 0.25
    alpha = 0.1
    tau_m = 0.1
    A_1, A_2, R_1, R_2, k_v = 1, 1, 1, 1, 1

    e = h_2sp - h_2m

    u = k_c * (e_I / tau_I + (1 - 1 / alpha) * e_f + e / alpha)

    f[0] = (-h_1 / R_1 + k_v * u + F_w) / A_1
    f[1] = (h_1 / R_1 - h_2 / R_2) / A_2
    f[2] = (h_2 - h_2m) / tau_m
    f[3] = e
    f[4] = (e - e_f) / (alpha * tau_D)

    return f

def pi_ode(t, y):
    f = y*0
```

```
    h_1 = y[0]
    h_2 = y[1]
    h_2m = y[2]
    e_I = y[3]

    h_2sp = 1
    F_w = 0

    k_c = 5
    tau_I = 2
    tau_m = 0.1
    A_1, A_2, R_1, R_2, k_v = 1, 1, 1, 1, 1

    e = h_2sp - h_2m

    u = k_c * (e_I / tau_I + e)

    f[0] = (-h_1 / R_1 + k_v * u + F_w) / A_1
    f[1] = (h_1 / R_1 - h_2 / R_2) / A_2
    f[2] = (h_2 - h_2m) / tau_m
    f[3] = e

    return f

real_pid_sol = solve_ivp(real_pid_ode, t_range, initial_cond, **ode_kwargs)

pi_sol = solve_ivp(pi_ode, t_range, initial_cond[:4], **ode_kwargs)

plt.plot(real_pid_sol.t, np.ones(real_pid_sol.t.shape[0]), '--', label='Set␣
 ↪Point')
plt.plot(real_pid_sol.t, real_pid_sol.y[1], label='Real PID Response')
plt.plot(pi_sol.t, pi_sol.y[1], label="PI Response")
plt.xlabel('Time')
plt.ylabel('Response')
plt.title('Problem 13.4 Closed Loop Response Simulation')
plt.legend()
```

[ ]: <matplotlib.legend.Legend at 0x1b8ffb041d0>


Problem 13.4 Closed Loop Response Simulation
Set Point
Real PID Response
PI Response
Response
Time
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0
2
4
6
8
10