1. Problem 4.8

(a)

$$\begin{aligned} Y_1(s) &= \frac{k_1}{\tau_1 s + 1} U(s) \\ Y_2(s) &= \frac{-k_2}{\tau_2 s + 1} U(s) \\ Y(s) &= \left(\frac{k_1}{\tau_1 s + 1} - \frac{k_2}{\tau_2 s + 1} \right) U(s) \\ U(s) &= \frac{M}{s} \end{aligned}$$

Linear combination

$$\boxed{y(t) = k_1 M \left(1 - e^{-t/\tau_1}\right) - k_2 M \left(1 - e^{-t/\tau_2}\right)}$$

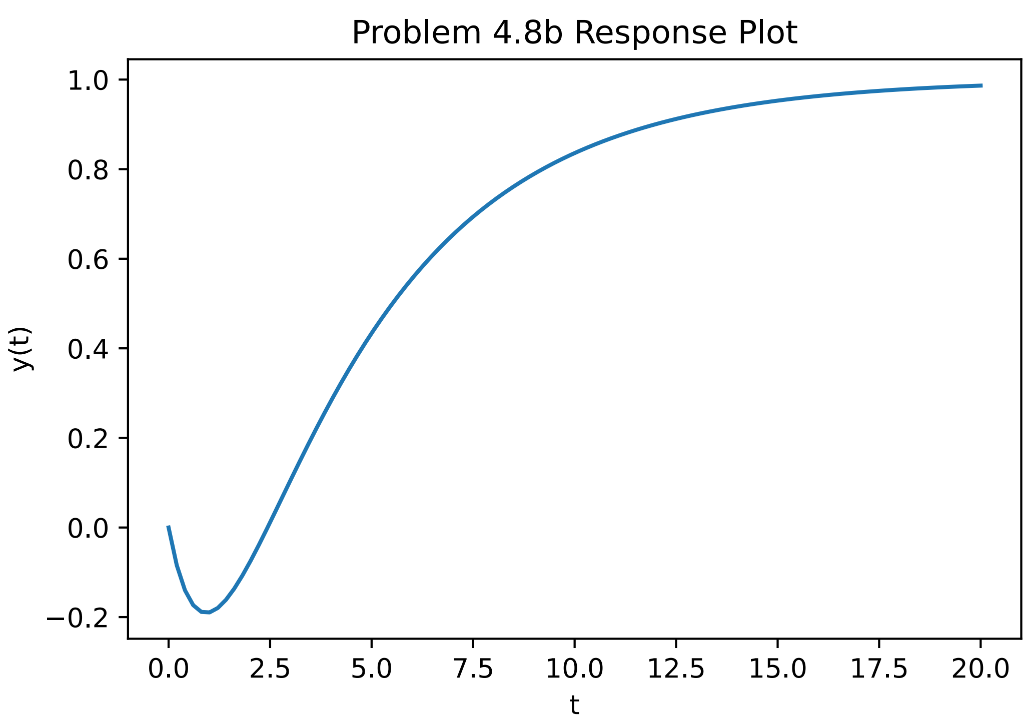
$$\frac{dy}{dt} = \frac{k_1 M}{\tau_1} e^{-t/\tau_1} - \frac{k_2 M}{\tau_2} e^{-t/\tau_2}$$

$$\boxed{\frac{dy}{dt}(0) = \frac{k_1 M}{\tau_1} - \frac{k_2 M}{\tau_2}}$$

As $t \to \infty$

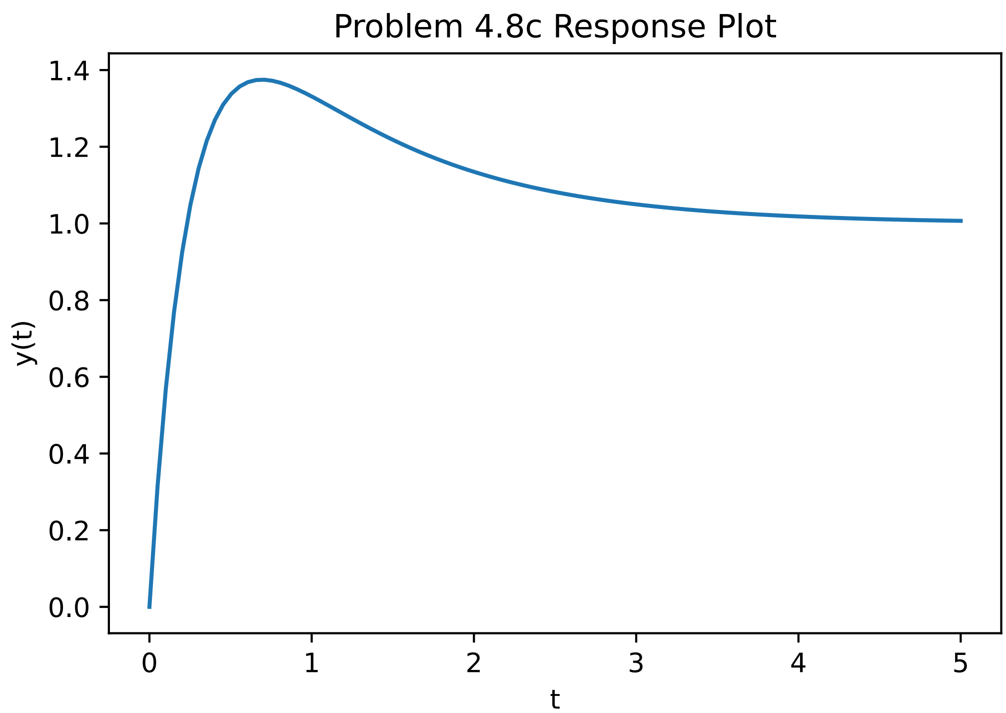
$$\boxed{y(t) = \frac{k_1 M}{\tau_1} - \frac{k_2 M}{\tau_2}}$$

(b) Plot:



Initially, the second block's output is larger than the first block's output. As a result, the second block's output pulls the combined output negative. The first output eventually outpaces the second output, and the output reaches a new higher steady state.

(c) Plot:



A similar situation to that of part b occurs in part c. Except, the first output is initially larger than the second output. The combined output reaches a peak before the second output matches the first output and pulls the combined output to a new steady state, lower than the peak but higher than the initial.

Plotting code:

```
import numpy as np
import matplotlib.pyplot as plt

k_1 = 2
tau_1 = 4
k_2 = 1
tau_2 = 1
M = 1

func = lambda t: k_1 * M * (1 - np.exp(-t / tau_1)) - k_2 * M * (1 - np.exp(-t / tau_2))

t_ran = np.linspace(0, 20, 100)
plt.plot(t_ran, func(t_ran))
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.8b Response Plot")

k_1 = 2
tau_1 = 1/4
k_2 = 1
tau_2 = 1
```

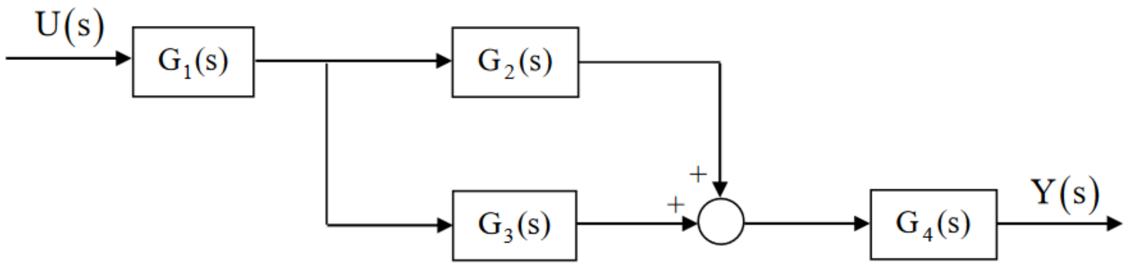
```python
M = 1

func = lambda t: k_1 * M * (1 - np.exp(-t / tau_1)) - k_2 * M * (1 - np.exp(-t / tau_2))

t_ran = np.linspace(0, 5, 100)
plt.plot(t_ran, func(t_ran))
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.8c Response Plot")
```

2. Problem 2

Calculate the overall transfer function of the system represented by the following block diagram:



$$
\begin{aligned}
Y_1(s) &= G_1(s)U(s)\\
Y_2(s) &= (G_2(s)+G_3(s))U_2(s)\\
Y_2(s) &= (G_2(s)+G_3(s))\,G_1(s)U(s)\\
Y(s) &= G_4(s)U_3(s)\\
Y(s) &= G_4(s)\,(G_2(s)+G_3(s))\,G_1(s)U(s)\\
&\boxed{Y(s) = (G_2(s)+G_3(s))\,G_1(s)G_4(s)U(s)}
\end{aligned}
$$

3. Problem 5.8

$$
\begin{aligned}
kM &\approx 38 \\
A \approx 47 - 38 &= 9 \\
\frac{A}{kM} &= \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \\
\frac{9}{38} &= \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \\
&\boxed{\zeta = 0.4168} \\
&T \approx 20 \\
&T = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \\
&20 = \frac{2\pi\tau}{\sqrt{1-0.4168^2}} \\
&\boxed{\tau = 2.893} \\
M = 0.5 \cdot 76 &= 38 \\
&\boxed{k = 1}
\end{aligned}
$$

4. Problem 4.2 Additional Question:

   For the reaction scheme of question d), suppose that F/V = 0.5, k1 = 3, k–1 = 1.25, k2 = 0.75.

   → Put your model in matrix form, i.e. in the form of Eqns. (5.6.3) and (5.6.4), using deviation variables

   → Use the Control Systems Toolbox of MATLAB to:

   • calculate and plot the response to a unit step change in Cin,R and to a unit impulse change in Cin,R

   • calculate the transfer function

   • calculate and plot the unit step response and unit impulse response based on the transfer function

   (a)

$$
\begin{aligned}
V\frac{dC_R}{dt} &= FC_{in,R} - FC_R - Vk_1C_R \\
V\frac{dC_P}{dt} &= Vk_1C_R - FC_P - Vk_2C_P
\end{aligned}
$$

   (b)

$$
\begin{aligned}
VC_R(s)s &= FC_{in,R}(s) - FC_R(s) - Vk_1C_R(s) \\
C_R(s) &= \frac{F}{Vs+F+Vk_1}C_{in,R}(s) \\
VC_P(s)s &= Vk_1C_R(s) - FC_P(s) - Vk_2C_P(s) \\
C_P(s)(Vs+F+Vk_2) &= Vk_1C_R(s) \\
C_P(s)(Vs+F+Vk_2) &= \frac{FVk_1}{Vs+F+Vk_1}C_{in,R}(s) \\
C_P(s) &= \frac{FVk_1}{(Vs+F+Vk_1)(Vs+F+Vk_2)}C_{in,R}(s) \\
G(s) &= \frac{FVk_1}{(Vs+F+Vk_1)(Vs+F+Vk_2)} \\
G(s) &= \frac{FVk_1}{V^2s^2 + 2FVs + V^2sk_2 + F^2 + FVk_2 + V^2k_1s + FVk_1 + V^2k_1k_2} \\
G(s) &= \frac{FVk_1}{V^2s^2 + s(2FV + V^2k_2 + V^2k_1) + F^2 + FVk_2 + FVk_1 + V^2k_1k_2} \\
G(s) &= \frac{\frac{V}{F}k_1}{\left(\frac{V}{F}\right)^2 s^2 + s\left(2\frac{V}{F} + \left(\frac{V}{F}\right)^2(k_2+k_1)\right) + \left(\frac{V}{F}\right)^2 + \frac{F}{V}(k_2+k_1) + \left(\frac{V}{F}\right)^2 k_1k_2} \\
G(s) &= \frac{\frac{V}{F}k_1}{\left(\frac{V}{F}\right)^2 s^2 + s\left(2\frac{V}{F} + \left(\frac{V}{F}\right)^2(k_2+k_1)\right) + \left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)}
\end{aligned}
$$

(c)

$$
\begin{aligned}
G(s) &= \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1} \\
k &= \frac{\frac{V}{F}k_1}{\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)} \\
\tau &= \sqrt{\frac{\left(\frac{V}{F}\right)^2}{\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)}} \\
2\tau\zeta &= \frac{\left(2\frac{V}{F}+\left(\frac{V}{F}\right)^2(k_2+k_1)\right)}{\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)} \\
2\tau\zeta &= \frac{\left(2\frac{V}{F}+\left(\frac{V}{F}\right)^2(k_2+k_1)\right)}{\sqrt{\left(\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)\right)^2}} \\
2\zeta\frac{V}{F} &= \frac{\left(2\frac{V}{F}+\left(\frac{V}{F}\right)^2(k_2+k_1)\right)}{\sqrt{\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)}} \\
\zeta &= \frac{1+\frac{V}{2F}(k_2+k_1)}{\sqrt{\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)}} \\
\tau &= \frac{\frac{V}{F}}{\sqrt{\left(\frac{V}{F}+k_1\right)\left(\frac{V}{F}+k_2\right)}}
\end{aligned}
$$

(d) Reversible

State Space:

$$
\begin{aligned}
V\frac{dC_R}{dt} &= FC_{in,R} - FC_R - Vk_1C_R + Vk_{-1}C_P \\
V\frac{dC_P}{dt} &= Vk_1C_R - FC_P - Vk_2C_P - Vk_{-1}C_P
\end{aligned}
$$

Transfer Function:

$$
\begin{aligned}
VsC_R(s) &= FC_{in,R}(s) - C_R(s)(F+Vk_1) + Vk_{-1}C_P(s) \\
C_R(s)(Vs+F+Vk_1) &= FC_{in,R}(s) + Vk_{-1}C_P(s) \\
C_R(s) &= \frac{F}{Vs+F+Vk_1}C_{in,R}(s) + \frac{Vk_{-1}}{Vs+F+Vk_1}C_P(s) \\
VsC_P(s) &= Vk_1C_R(s) - FC_P(s) - Vk_2C_P(s) - Vk_{-1}C_P \\
C_P(s)(Vs+F+Vk_2+Vk_{-1}) &= Vk_1C_R(s) \\
C_P(s)(Vs+F+Vk_2+Vk_{-1}) &= \frac{FVk_1}{Vs+F+Vk_1}C_{in,R}(s) + \frac{V^2k_{-1}k_1}{Vs+F+Vk_1}C_P(s) \\
\frac{FVk_1}{Vs+F+Vk_1}C_{in,R}(s) &= C_P(s)\left((Vs+F+Vk_2+Vk_{-1}) - \frac{V^2k_{-1}k_1}{Vs+F+Vk_1}\right) \\
(FVk_1)C_{in,R}(s) &= C_P(s)\left((Vs+F+Vk_2+Vk_{-1})(Vs+F+Vk_1) - V^2k_{-1}k_1\right)
\end{aligned}
$$

$$C_P(s) = C_{in,R}(s)\frac{FVk_1}{V^2s^2 + s(2FV + V^2(k_1 + k_2 + k_{-1})) + (F^2 + FV(k_1 + k_2 + k_{-1}) + V^2k_1k_2)}$$

$$G(s) = \frac{FVk_1}{V^2s^2 + s(2FV + V^2(k_1 + k_2 + k_{-1})) + (F^2 + FV(k_1 + k_2 + k_{-1}) + V^2k_1k_2)}$$

(Matlab)

$$V\frac{dC_R}{dt} = FC_{in,R} - FC_R - Vk_1C_R + Vk_{-1}C_P$$

$$V\frac{dC_P}{dt} = Vk_1C_R - FC_P - Vk_2C_P - Vk_{-1}C_P$$

$$\frac{dC_R}{dt} = \frac{F}{V}C_{in,R} - \frac{F}{V}C_R - k_1C_R + k_{-1}C_P$$

$$\frac{dC_P}{dt} = k_1C_R - \frac{F}{V}C_P - k_2C_P - k_{-1}C_P$$

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1u$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2u$$

$$y = c_1x_1 + c_2x_2 + du$$

$$\frac{dC_R}{dt} = -\left(\frac{F}{V} + k_1\right)C_R + k_{-1}C_P + \frac{F}{V}C_{in,R}$$

$$\frac{dC_P}{dt} = -\left(\frac{F}{V} + k_2 + k_{-1}\right)C_P + k_1C_R$$

$$C_P = C_P$$

$$\frac{d}{dt}\begin{bmatrix} C_R \\ C_P \end{bmatrix} = \begin{bmatrix} -\left(\frac{F}{V} + k_1\right) & k_{-1} \\ k_1 & -\left(\frac{F}{V} + k_2 + k_{-1}\right) \end{bmatrix}\begin{bmatrix} C_R \\ C_P \end{bmatrix} + \begin{bmatrix} \frac{F}{V} \\ 0 \end{bmatrix} C_{in,R}$$

$$C_P = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} C_R \\ C_P \end{bmatrix} + 0 \cdot du$$

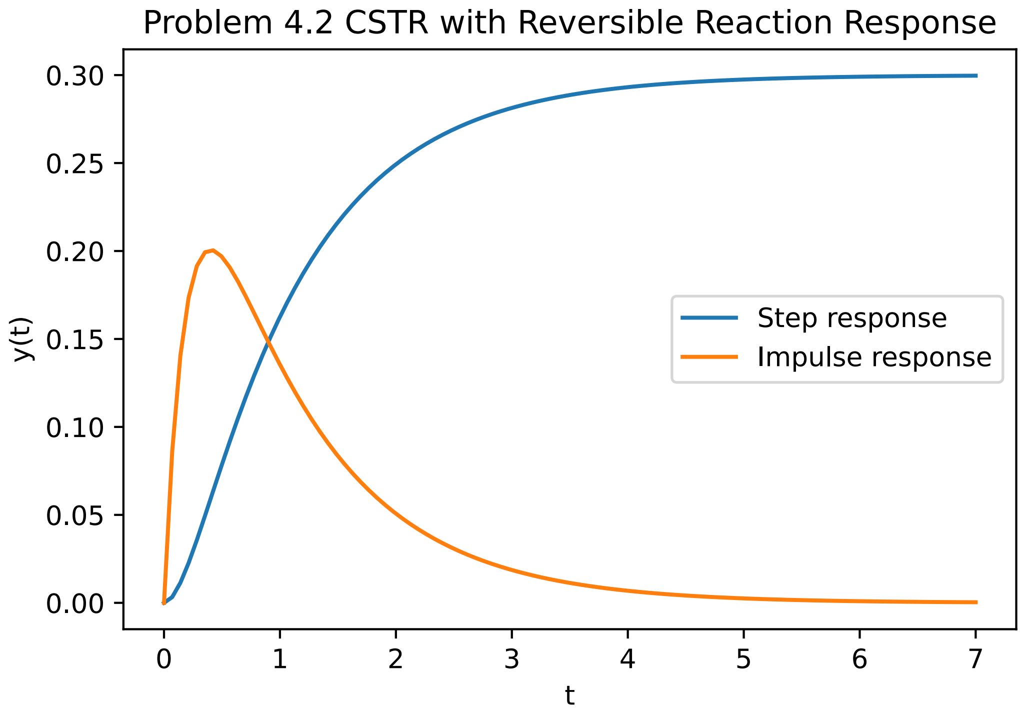
$$\frac{F}{V} = 0.5$$

$$k_1 = 3$$

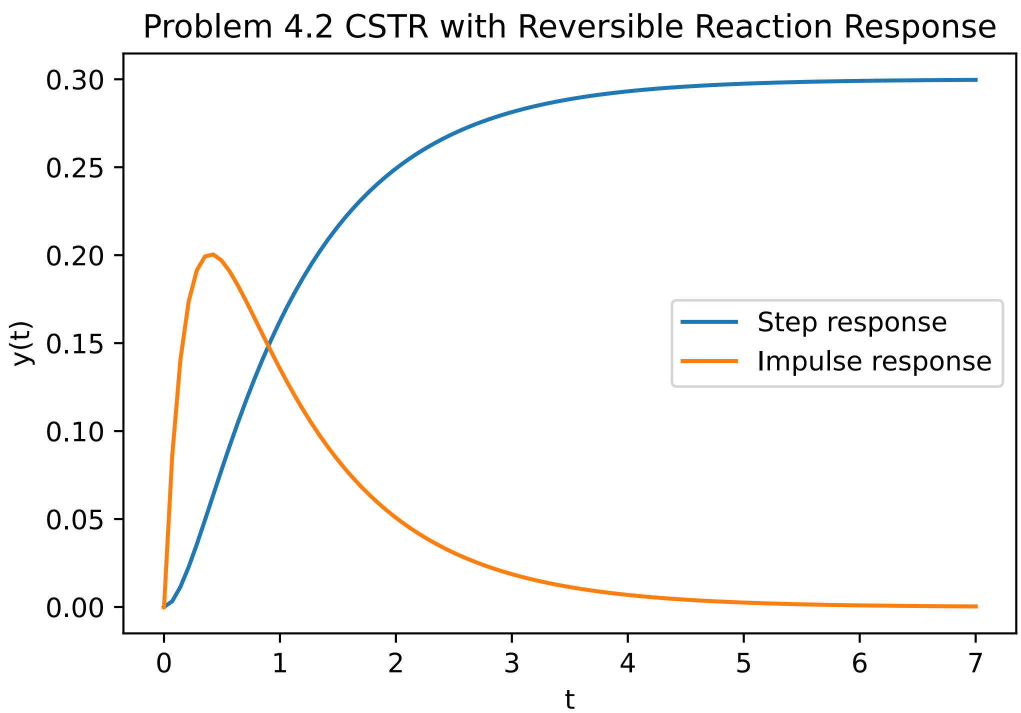
$$k_{-1} = 1.25$$

$$k_2 = 0.75$$

Response plot:



Response plot from manual transfer function:



Transfer function output:
TransferFunctionContinuous(array([1.5]), array([1., 6., 5.]), dt: None)

$$G(s) = \frac{1.5}{s^2 + 6s + 5}$$

Code for solving the system

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.signal as signal

# constants
F_V = 0.5
k_1 = 3
k_b = 1.25
k_2 = 0.75

# matrices
A = np.array([
    [-(F_V + k_1), k_b],
    [k_1, -(F_V + k_2 + k_b)]
])
B = np.array([
    [F_V],
    [0]
])
C = np.array([0, 1])
D = 0

# define state space system
sys = signal.StateSpace(A, B, C, D)

# compute step/impulse response
t_step, y_step = sys.step()

t_impulse, y_impulse = sys.impulse()

# plot
plt.plot(t_step, y_step, label="Step response")
plt.plot(t_impulse, y_impulse, label="Impulse response")
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.2 CSTR with Reversible Reaction Response")
plt.legend()

# compute transfer function
transfer_func = sys.to_tf()

print(transfer_func)

# manually create system from transfer function
sys_manual = signal.lti(transfer_func.num, transfer_func.den)

# comput step/impulse response
t_step, y_step = sys_manual.step()
```

```python
t_impulse, y_impulse = sys_manual.impulse()

plt.plot(t_step, y_step, label="Step response")
plt.plot(t_impulse, y_impulse, label="Impulse response")
plt.xlabel(r"t")
plt.ylabel(r"y(t)")
plt.title("Problem 4.2 CSTR with Reversible Reaction Response")
plt.legend()
```