CHEN 461 HW9

March 31, 2023

[ ]:
```
# imports
import control

import numpy as np
import matplotlib.pyplot as plt
import matplotlib_inline

from sympy.abc import t, s
from sympy import symbols, simplify
from sympy.series import limit
from sympy import matrices
from sympy.integrals import integrate

%matplotlib inline
matplotlib_inline.backend_inline.set_matplotlib_formats('png', 'pdf')
```

1 Problem 10.8

1.1 Part A

Real PD:

$G(s) = k_c \left(\frac{1+\tau_D s}{1+\alpha\tau_D s} \right)$

$E(s) = \frac{1}{s}$

$U(s) = k_c \left(\frac{1+\tau_D s}{s(1+\alpha\tau_D s)} \right)$

$U(s) = k_c \left(\frac{1}{s} + \frac{\tau_D(1-\alpha)}{(1+\alpha\tau_D s)} \right)$

$u(t) = k_c \left(\mathcal{H}(t) + \frac{\tau_D(1-\alpha)}{\alpha\tau_D} \right) \exp\left(-\alpha^{-1} \frac{t}{\tau_D} \right)$

Real PD response:

$$\boxed{\frac{u(t)}{k_c} = \mathcal{H}(t) + \frac{(1-\alpha)}{\alpha} \exp\left(-\alpha^{-1} \frac{t}{\tau_D} \right)}$$

Ideal PD:

$G(s) = k_c (1 + \tau_D s)$

$U(s) = k_c \left(\frac{1}{s} + \tau_D\right)$

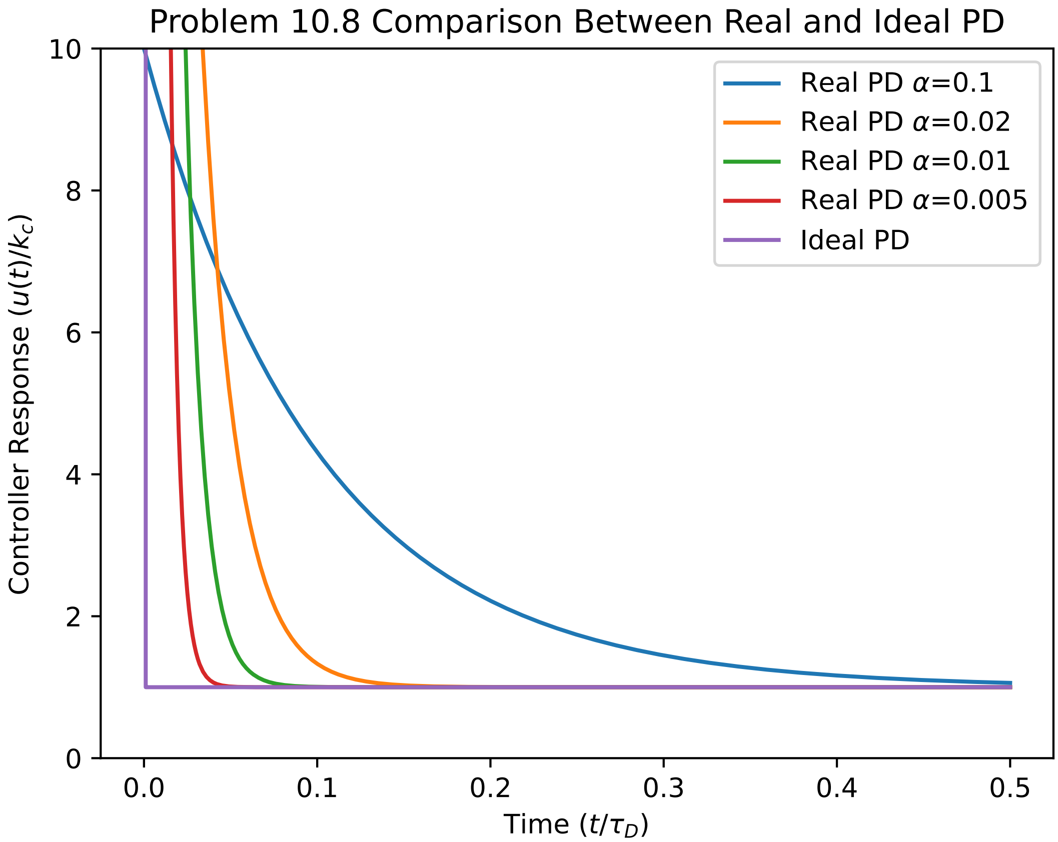
Ideal PD response:

$$\boxed{\frac{u(t)}{k_c} = \mathcal{H}(t) + \tau_D \delta(t)}$$

1.1.1 Simulation

[ ]:
```python
# response simulation
alpha_values = [0.1, 0.02, 0.01, 0.005]

def real_pd(t, alpha):
    return 1 + (1 - alpha) / alpha * np.exp(-t / alpha)

t_values = np.linspace(0, .5, 500)

for a in alpha_values:
    plt.plot(t_values, real_pd(t_values, a), label=fr"Real PD $\alpha$={a}")

ideal_pd_response = np.ones(t_values.shape[0])
ideal_pd_response[0] = 1e300

plt.plot(t_values, ideal_pd_response, label="Ideal PD")

plt.ylim([0, 10])
plt.xlabel(r"Time ($t/\tau_D$)")
plt.ylabel(r"Controller Response ($u(t)/k_c$)")
plt.title("Problem 10.8 Comparison Between Real and Ideal PD")
plt.legend()
```

[ ]:
```
<matplotlib.legend.Legend at 0x21e38395310>
```



The closer that α is to zero, the closer closer that the Real PD is to the Ideal PD. The smaller the α value, the faster that the Real PD reaches the set point.

1.2 Part B

AR and phase as a function of frequency:

Real PD:

$$\frac{\mathrm{AR}}{k_c} = \frac{\sqrt{\left(\tau_D \omega\right)^2 + 1}}{\sqrt{\alpha^2 \left(\tau_D \omega\right)^2 + 1}}$$

$$\phi = \tan^{-1}\left(\tau_D \omega\right) - \tan^{-1}\left(\alpha \tau_D \omega\right)$$

Ideal PD:

$$\frac{\mathrm{AR}}{k_c} = \sqrt{\left(\tau_D \omega\right)^2 + 1}$$

$$\phi = \tan^{-1}(\tau_D \omega)$$

Transfer functions:

Real PD:

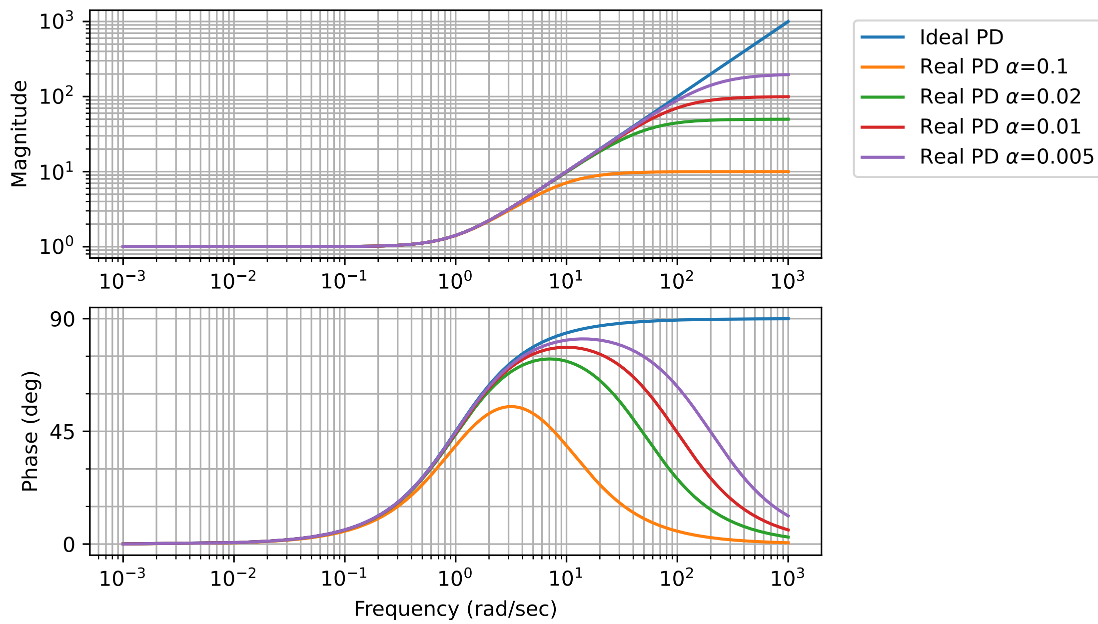
$$G(s) = k_c \left(\frac{1 + \tau_D s}{1 + \alpha \tau_D s} \right)$$

Ideal PD:

$$G(s) = k_c (1 + \tau_D s)$$

1.2.1 Bode Diagrams

[ ]:
```python
# bode plots
ideal_pd = control.tf([1, 1], [1])

bode_kwargs = {
    'wrap_phase': True,
    'omega': np.linspace(1e-3, 1e3, int(1e5)),
}

control.bode_plot(ideal_pd, **bode_kwargs, label="Ideal PD")

for a in alpha_values:
    real_pd = control.tf([1, 1], [a, 1])
    control.bode_plot(real_pd, **bode_kwargs, label=fr"Real PD $\alpha$={a}")

plt.legend(loc="upper right", bbox_to_anchor=(1.4, 2.2))
plt.show()
```



In the magnitude plot, the real PD flattens past a $\tau_D\omega$ value of 10, while the ideal PD keeps increasing. As α approaches 0, the real PD begins to approach the same curve as the ideal PD in the magnitue plot.

In the phase plot, the real PD phase peaks around $\tau_D\omega = 10$ and then returns to 0°. In contrast, the ideal PD starts at 0° and steps up to 90°. As α approaches 0, the real PD begins to approach the same curve as the ideal PD in the phase plot.

In both the magnitude and phase plots, unless $\alpha = 0$ the real PD only approximates the ideal PD over a certain frequency range.

2 Problem 10.10

2.1 Part A

$G(s) = k_c\left(1 + \frac{1}{\tau_I s} + \frac{\tau_D s}{\tau_F s+1}\right)$

$U(s) = k_c\left(E(s) + \frac{1}{\tau_I s}E(s) + \frac{\tau_D s}{\tau_F s+1}E(s)\right)$

$\frac{de_I}{dt} = e$

$sE_I = E$

First order filter in front of derivative.

$\tau_F\frac{de_F}{dt} + e_F = e$

$E_F = \frac{E}{\tau_F s+1}$

$\mathcal{L}^{-1}\left\{U(s)\right\} = \mathcal{L}^{-1}\left\{k_c\left(E(s) + \frac{E_I}{\tau_I} + \tau_D s E_F\right)\right\}$

$u(t) = k_c \left(e + \frac{e_I}{\tau_I} + \tau_D \frac{de_F}{dt}\right)$

2.1.1 State space models:

$$\frac{de_I}{dt} = e$$

$$\frac{de_F}{dt} = \frac{e - e_F}{\tau_F}$$

$$u = k_c \left(e + \frac{e_I}{\tau_I} + \frac{\tau_D}{\tau_F}\left(e - e_F\right)\right)$$

2.2 Part B

$$\frac{d}{dt}\begin{bmatrix} e_I \\ e_F \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{\tau_F} \end{bmatrix}\begin{bmatrix} e_I \\ e_F \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{\tau_F} \end{bmatrix} e$$

$$u = \begin{bmatrix} \frac{k_c}{\tau_I} & -k_c\frac{\tau_D}{\tau_F} \end{bmatrix}\begin{bmatrix} e_I \\ e_F \end{bmatrix} + k_c\left(1 + \frac{\tau_D}{\tau_F}\right) e$$

2.2.1 Define the state-space system in matrix form

[ ]:
```
# state space matrices
tau_F, tau_I, k_c, tau_D, T_s = symbols('tau_F, tau_I, k_c, tau_D, T_s')

A = matrices.Matrix([
    [0, 0],
    [0, -1/tau_F]
])
B = matrices.Matrix([
    [1],
    [1/tau_F]
])
C = matrices.Matrix([
    [k_c/tau_I, -k_c*tau_D/tau_F]
])
D = k_c * (1 + tau_D/tau_F)
```

2.2.2 Compute A_d symbolically

$$A_d = e^{AT_s}$$

[ ]:
```
# compute A_d
A_d = (A * T_s).exp()
A_d
```

[ ]:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{T_s}{\tau_F}} \end{bmatrix}$$

2.2.3 Compute B_d symbolically

$$B_d = \int_0^{T_s} e^{At'} B dt'$$

[ ]:
```
# compute B_d
B_d = integrate((A*t).exp() * B, (t, 0, T_s))
B_d
```

[ ]: $\begin{bmatrix} T_s \\ 1 - e^{-\frac{T_s}{\tau_F}} \end{bmatrix}$

Discretization:

$$\boxed{A_d = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{T_s}{\tau_F}} \end{bmatrix}}$$

$$\boxed{B_d = \begin{bmatrix} T_s \\ 1 - e^{-\frac{T_s}{\tau_F}} \end{bmatrix}}$$

3 Problem 11.2

3.1 Part A

$Y = AG + WG'$

$B = Y - AM$

$E = Y_{sp} - B$

$E = Y_{sp} + AM - Y$

$A = QE$

$A = QY_{sp} + AMQ - QY$

$A(1 - MQ) = QY_{sp} - QY$

$Y(1 - MQ) = GQY_{sp} - GQY + W(1 - MQ)G'$

$Y(1 - MQ + GQ) = GQY_{sp} + W(1 - MQ)G'$

$$\boxed{Y = \frac{GQ}{1 - MQ + GQ} Y_{sp} + \frac{G'(1 - MQ)}{1 - MQ + GQ} W}$$

3.2 Part B

$Y = AG + WG'$

$A = C_E E - C_Y Y$

$E = Y_{sp} - Y$

$A = C_E Y_{sp} - C_E Y - C_Y Y$

$Y = G C_E Y_{sp} - G C_E Y - G C_Y Y + W G'$

$Y(1 + G C_E + G C_Y) = G C_E Y_{sp} + W G'$

$$\boxed{Y = \frac{C_E G}{1 + C_E G + C_Y G} Y_{sp} + \frac{G'}{1 + C_E G + C_Y G} W}$$

3.3 Part C

$Y = AG + G'W$

$A = EG_c + G_{ff}W$

$Y = EG_cG + G_{ff}GW + G'W$

$E = Y_{sp} - Y$

$Y = Y_{sp}G_cG - YG_cG + G_{ff}GW + G'W$

$Y(1 + G_cG) = Y_{sp}G_cG + G_{ff}GW + G'W$

$$\boxed{Y = \frac{GG_c}{1 + GG_c} Y_{sp} + \frac{GG_{ff} + G'}{1 + GG_c} W}$$

4 Problem 11.8

4.1 Part A

Feedback loop transfer function:

$$G(s) = \frac{G_c G_p}{1 + G_c G_p}$$

Process transfer function:

$$G_p(s) = \frac{k_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Controller transfer function:

$$G_c(s) = k_c \left(\frac{1 + \tau_D s}{1 + \alpha\tau_D s} \right)$$

4.1.1 Define feedback transfer function symbolically

[ ]:
```
# define feedback transfer function
tau, zeta, tau_D, alpha = symbols('tau, zeta, tau_D, alpha')
k_c, k_p = symbols('k_c, k_p')

G_p = k_p / (tau**2 * s**2 + 2 * zeta * tau * s + 1)
G_c = k_c * (1 + tau_D * s) / (1 + alpha * tau_D * s)

G_feedback = simplify(G_c * G_p / (1 + G_c * G_p))

(G_feedback)
```

[ ]:

$$\frac{k_c k_p \left(s\tau_D + 1\right)}{k_c k_p \left(s\tau_D + 1\right) + \left(\alpha s\tau_D + 1\right)\left(s^2\tau^2 + 2s\tau\zeta + 1\right)}$$

Feedback transfer function:

$$\boxed{G(s) = \frac{k_c k_p \left(s\tau_D + 1\right)}{k_c k_p \left(s\tau_D + 1\right) + \left(\alpha s\tau_D + 1\right)\left(s^2\tau^2 + 2s\tau\zeta + 1\right)}}$$

4.2 Part B

Find offset by Final Value Theorem:

$$\lim_{s\to 0^+} sY(s) = \lim_{t\to\infty} y(t)$$

$G(s) = \frac{G_c G_p}{1+G_c G_p}$

$Y(s) = G(s)Y_{sp}(s)$

In deviation form:

$y_{sp}(t) = \mathcal{H}(t)$

$Y_{sp}(s) = \frac{1}{s}$

$Y(s) = \frac{G(s)}{s}$

$sY(s) = s\frac{G(s)}{s} = G(s)$

Offset is $y_{sp}(t) - y(t)$ at ∞

$y(t) = \lim_{s\to 0^+} G(s)$

Offset:

$$1 - \lim_{s\to 0^+} G(s)$$

[ ]:
```
# evaluate offset
simplify(1 - limit(G_feedback, s, 0))
```

[ ]: $\dfrac{1}{k_c k_p + 1}$

Offset:

$$\boxed{\text{offset} = \frac{1}{k_c k_p + 1}}$$