1. Problem 3.2

   (a)

$$V\frac{dC_R}{dt} = FC_{in,R} - FC_R - Vk_R C_R^n$$
$$C_R^n \approx C_{R,s}^n + nC_{R,s}^{n-1}(C_R - C_{R,s})$$

Deviation form:

$$\overline{C}_R^n = nC_{R,s}^{n-1}\overline{C}_R$$
$$V\frac{d\overline{C}_R}{dt} = F\overline{C}_{in,R} - F\overline{C}_R - Vk_R\overline{C}_R^n$$
$$V\frac{d\overline{C}_R}{dt} = F\overline{C}_{in,R} - F\overline{C}_R - Vk_R nC_{R,s}^{n-1}\overline{C}_R$$
$$V\frac{d\overline{C}_R}{dt} + \overline{C}_R\left(F + Vk_R nC_{R,s}^{n-1}\right) = F\overline{C}_{in,R}$$
$$\boxed{\frac{V}{F + Vk_R nC_{R,s}^{n-1}}\frac{d\overline{C}_R}{dt} + \overline{C}_R = \frac{F}{F + Vk_R nC_{R,s}^{n-1}}\overline{C}_{in,R}}$$

   (b)

$$\tau = \frac{V}{F + Vk_R nC_{R,s}^{n-1}}$$
$$k = \frac{F}{F + Vk_R nC_{R,s}^{n-1}}$$

Transfer function:

$$\boxed{G(s) = \frac{F}{Vs + F + Vk_R nC_{R,s}^{n-1}}}$$

   (c)

Step input:

$$\overline{C}_{in,R}(t) = M\mathcal{H}(t)$$

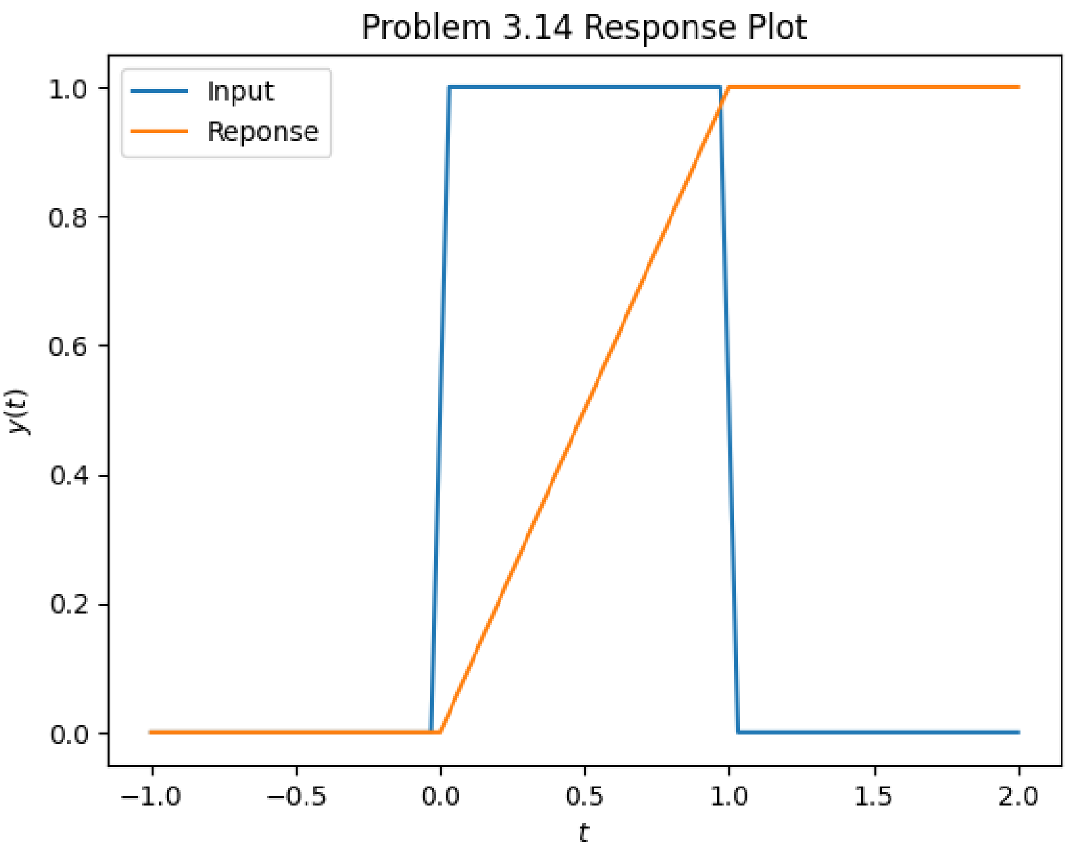
Step response:

$$y(t) = kM(1 - e^{-t/\tau})$$
$$\boxed{\overline{C}_R(t) = \frac{FM}{F + Vk_R nC_{R,s}^{n-1}}\left(1 - e^{-t/\left(\frac{V}{F + Vk_R nC_{R,s}^{n-1}}\right)}\right)}$$

2. Problem 3.14

$$
\begin{aligned}
A\frac{d\overline{h}}{dt} &= \overline{F}_{in} \\
\overline{F}_{in} &= \frac{M}{\epsilon}(\mathcal{H}(t) - \mathcal{H}(t-\epsilon)) \\
\mathcal{L}\left\{A\frac{d\overline{h}}{dt}\right\} &= \mathcal{L}\left\{\frac{M}{\epsilon}(\mathcal{H}(t) - \mathcal{H}(t-\epsilon))\right\} \\
As\overline{H}(s) &= \frac{M}{\epsilon}\left(\frac{1}{s} - \frac{e^{-\epsilon s}}{s}\right) \\
\mathcal{L}^{-1}\left\{\overline{H}(s)\right\} &= \mathcal{L}\left\{\frac{M}{A\epsilon}\left(\frac{1}{s^2} - \frac{e^{-\epsilon s}}{s^2}\right)\right\} \\
&\boxed{\overline{h}(t) = \frac{M}{A\epsilon}(t - (t-\epsilon)\mathcal{H}(t-\epsilon))}
\end{aligned}
$$

Here is a plot of what $\overline{h}(t)$ would look like:



The response ramps up and reaches a new steady state. $\overline{h}(t)$ will not return to 0 as $t \to \infty$.

Here is the code that generates the plot:

```
from sympy import lambdify, Heaviside
from sympy.abc import t

import numpy as np
import matplotlib.pyplot as plt

# convert sympy expressions to lambda func
u_expr = Heaviside(t) - Heaviside(t - 1)
u = lambdify(t, u_expr, 'numpy')

y_expr = t * Heaviside(t) - (t - 1) * Heaviside(t - 1)
y = lambdify(t, y_expr, 'numpy')

# plotting
t_ran = np.linspace(-1, 2, 100)

plt.plot(t_ran, u(t_ran))
plt.plot(t_ran, y(t_ran))
plt.xlabel(r"$t$")
plt.ylabel(r"$y(t)$")
plt.legend(['Input', 'Reponse'])
plt.title('Problem 3.14 Response Plot')
```

3. Problem 3.17

(a)

$$
\begin{aligned}
\mathcal{L}\left\{RC\frac{dV_C}{dt}+V_C\right\} &= \mathcal{L}\left\{V_{in}\right\} \\
RCsV_C(s)+V_C(s) &= V_{in}(s) \\
V_{out} &= V_{in}-V_C \\
V_C &= V_{in}-V_{out} \\
RCsV_{in}(s)-RCsV_{out}(s)+V_{in}(s)-V_{out}(s) &= V_{in}(s) \\
V_{out}(s)(RCs+1) &= RCsV_{in}(s) \\
V_{out}(s) &= \frac{RCs}{RCs+1}V_{in}(s) \\
\Aboxed{G(s) &= \frac{RCs}{RCs+1}}
\end{aligned}
$$

(b)

Response to step input:

$$
\begin{aligned}
V_C(t) &= kM\left(1-e^{-t/\tau}\right) \\
\tau &= RC \\
k &= 1 \\
V_C(t) &= M\left(1-e^{-t/(RC)}\right) \\
V_{in}(t) &= M\mathcal{H}(t) \\
V_{out}(t) &= V_{in}(t)-M\left(1-e^{-t/(RC)}\right) \\
\Aboxed{V_{out}(t) &= M\mathcal{H}(t)-M\left(1-e^{-t/(RC)}\right)}
\end{aligned}
$$

(c)

$$
V_{in}(t) = M\sin(\omega t)
$$

For large t:

$$
\begin{aligned}
V_C(t) &= \frac{kM}{\sqrt{1+(\omega\tau)^2}}\sin\left(\omega t-\tan^{-1}(\omega\tau)\right) \\
\tau &= RC \\
k &= 1 \\
V_C(t) &= \frac{M}{\sqrt{1+(\omega RC)^2}}\sin\left(\omega t-\tan^{-1}(\omega RC)\right) \\
V_{out}(t) &= V_{in}(t)-\frac{M}{\sqrt{1+(\omega RC)^2}}\sin\left(\omega t-\tan^{-1}(\omega RC)\right) \\
\Aboxed{V_{out}(t) &= M\sin(\omega t)-\frac{M}{\sqrt{1+(\omega RC)^2}}\sin\left(\omega t-\tan^{-1}(\omega RC)\right)}
\end{aligned}
$$

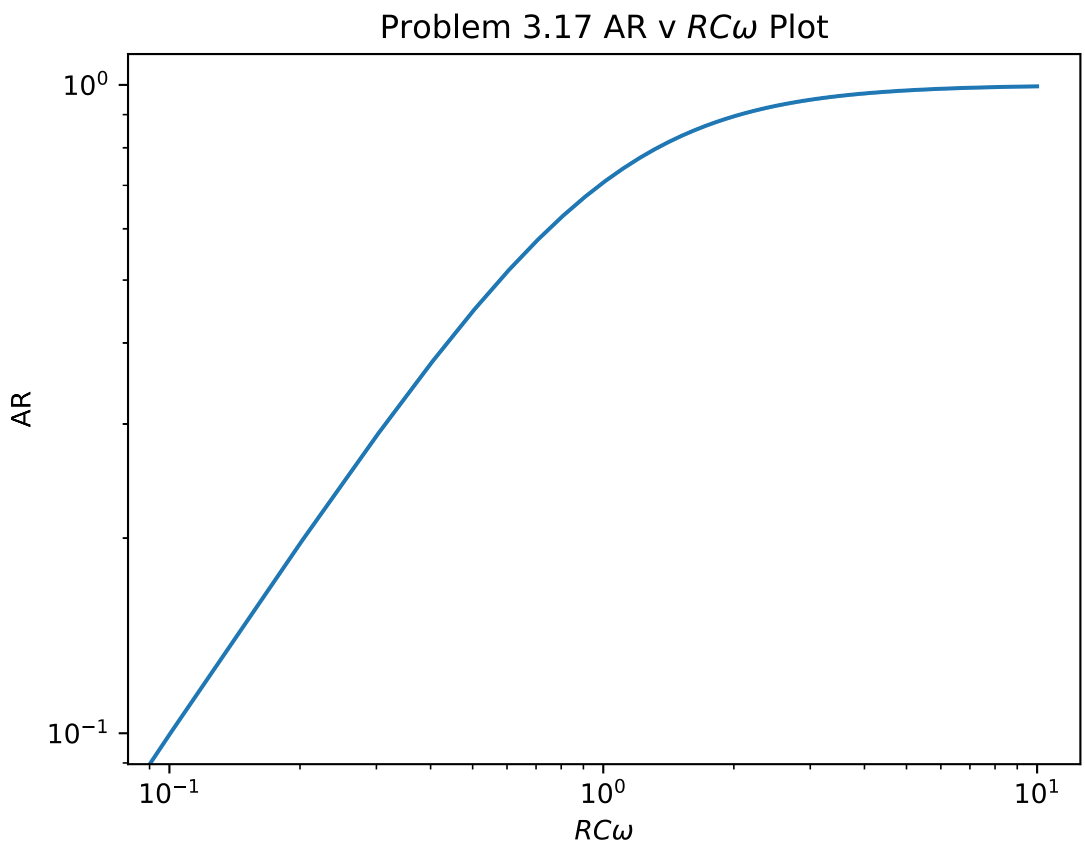
Account for phase shift:

$$\text{Amplitude} = \frac{M}{\sqrt{1+(\omega RC)^2}} \cdot \omega RC$$

$$\text{AR} = \frac{\frac{M}{\sqrt{1+(\omega RC)^2}} \cdot \omega RC}{M}$$

$$\boxed{\text{AR} = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}$$

Plot of AR v $RC\omega$:



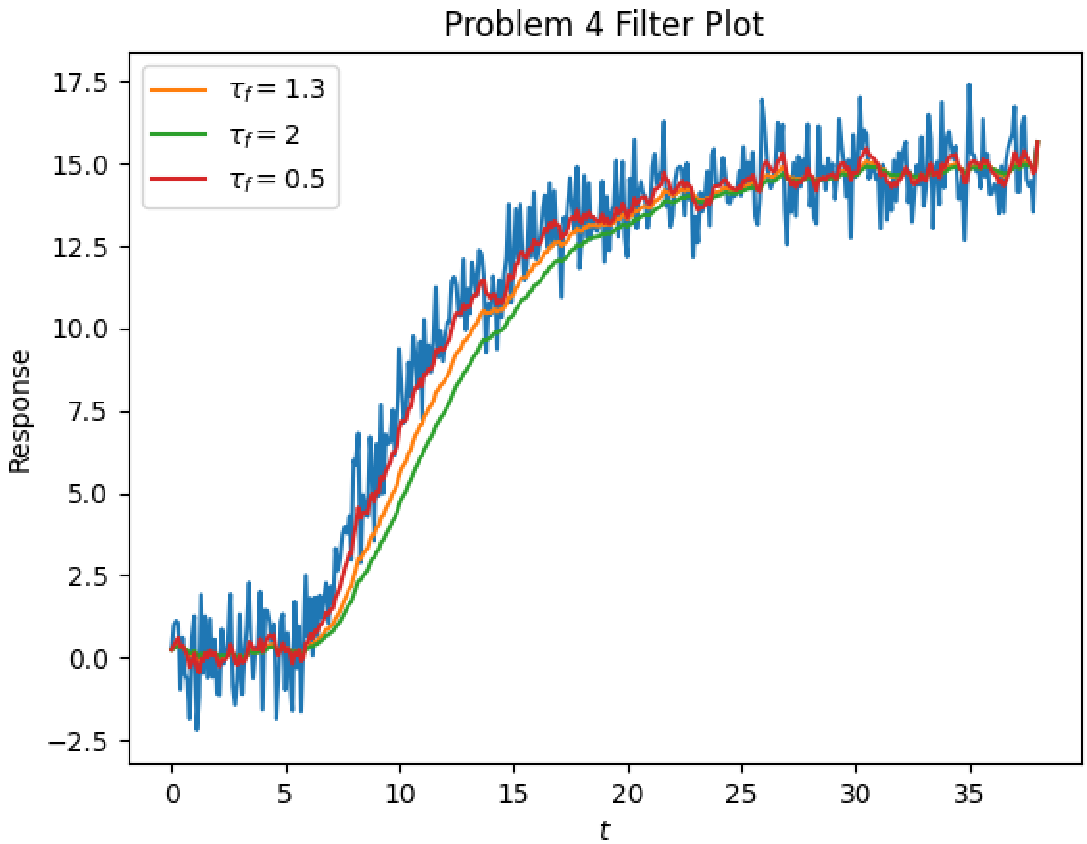
The curve is flat at high frequency, "passing" those frequencies, and ramping the lower frequency up. The plot shows why the filter is called high pass.

4. (a)

Implement algorithm:

$$\sigma_f[j] = (1-f)\sigma_f[j-1] + f\sigma[j]$$
$$f = 1 - e^{\frac{-T_s}{\tau_f}}$$

Here is a plot of the data with different τ_f values:



At $\tau_f = 0.5$ the data is not filtered enough; there is still significant variation. At $\tau_f = 1.3$, the data is much smoother. At $\tau_f = 2$, the data is even smoother, but the filtered data diverges significantly. $\tau_f = 1.3$ is a good value if accuracy is desired. $\tau_f = 2$ would be better if more smoothing of the data is desired.

The code below implements the algorithm and produces the plot.

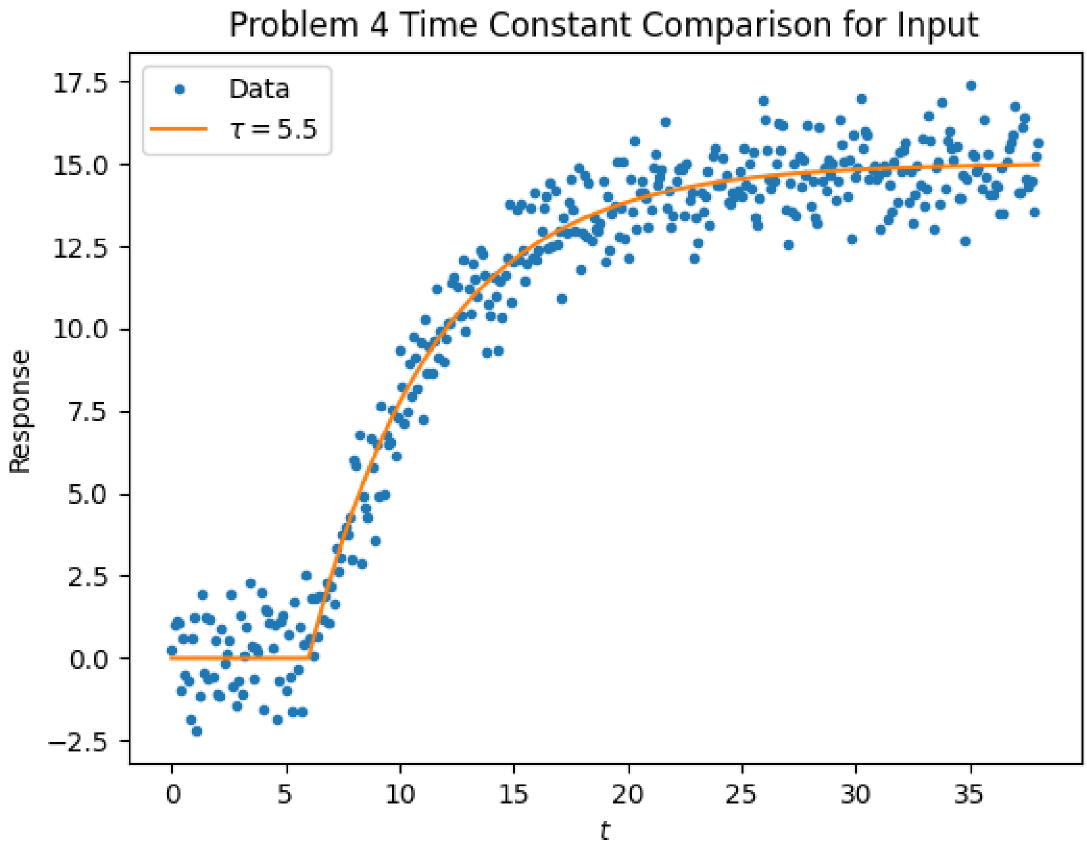
```
import numpy as np
from scipy.io import loadmat
import matplotlib.pyplot as plt
import math

# load data
data_file = loadmat('datafile.mat')

# low pass filter function
def lpf(t, y, tau_f):
    filt_y = y.copy()

    T_s = t[1] - t[0]
    f = 1 - math.exp(-T_s[0]/tau_f)

    # filter algorithm
    for i in range(1, len(y)-1):
        filt_y[i] = (1 - f) * filt_y[i-1,0] + f * y[i,0]

    return filt_y

# filter with different tau_f values
filt_1 = lpf(data_file['t'], data_file['y_data'], 1.3)
filt_2 = lpf(data_file['t'], data_file['y_data'], 2)
filt_3 = lpf(data_file['t'], data_file['y_data'], 0.5)

# plotting
plt.plot(data_file['t'], data_file['y_data'])
plt.plot(data_file['t'], filt_1, label=r"$\tau_f=1.3$")
plt.plot(data_file['t'], filt_2, label=r"$\tau_f=2$")
plt.plot(data_file['t'], filt_3, label=r"$\tau_f=0.5$")
plt.xlabel(r'$t$')
plt.ylabel('Response')
plt.legend()
plt.title('Problem 4 Filter Plot')
```

(b) For a step input, one τ is the time at which the response reaches approximately 63.2% of its maximum value. Looking at the data, the steady-state maximum is approximately 15. 63.2% of 15 is 9.48. The response is 9.48 at approximately 11.5 min. The input started at 6 min, and so $\tau = 11.5 - 6 = 5.5$ min.

The equation below is the response for a step input assuming a perfectly calibrated sensor.

$$y(t) = M \cdot \left(1 - e^{-t/\tau}\right)$$

Test this equation against the data with $M = 15$ and $\tau = 5.5$.



The response output with $\tau = 5.5$ fits the data reasonably well.

$\boxed{\tau = 5.5}$ is a suitable value.

The code below creates the output for the plot.

```python
import numpy as np
from scipy.io import loadmat
import matplotlib.pyplot as plt
import math

# load data
data_file = loadmat('datafile.mat')

from sympy import Heaviside, exp, lambdify
from sympy.abc import t

# plot data
plt.plot(data_file['t'], data_file['y_data'], '.')

# create a response formula with tau=5.5
resp_expr = (15 * (1 - exp(-(t - 6) / 5.5))) * Heaviside(t - 6)
# convert expression to lamdba func
resp = lambdify(t, resp_expr, 'numpy')

# plotting
plt.plot(data_file['t'], resp(data_file['t']))
plt.title('Problem 4 Time Constant Comparison for Input')
plt.legend(['Data', r"$\tau=5.5$"])
plt.xlabel(r"$t$")
plt.ylabel("Response")
```