Learning with Guarantees

Parametrising Neural Feedback Policies with Stability & Robustness Certificates

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Doctor of Philosophy – Defence

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THE UNIVERSITY OF SYDNEY
Australian Centre for Robotics

Outline

1. Motivation
2. Background: robust neural networks
3. Lipschitz-bounded policy networks
4. React to surprises with Youla-REN (Theory)
5. React to surprises with Youla-REN (Application)
6. Robust recurrent deep networks
7. Conclusions

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Deep Reinforcement Learning (RL)

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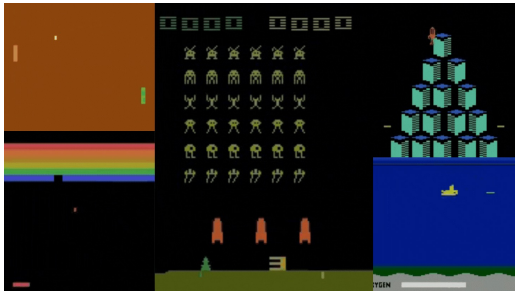

[1] Mnih et al., Human-level control through deep reinforcement learning. *Nature* (2015).

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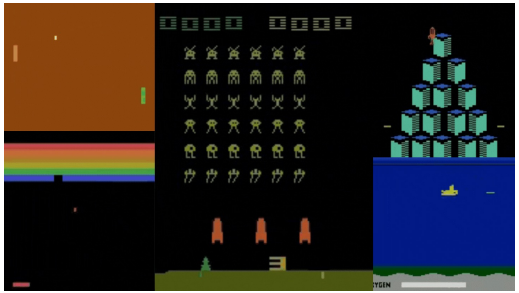

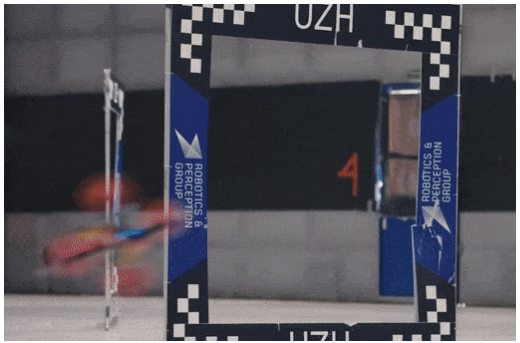

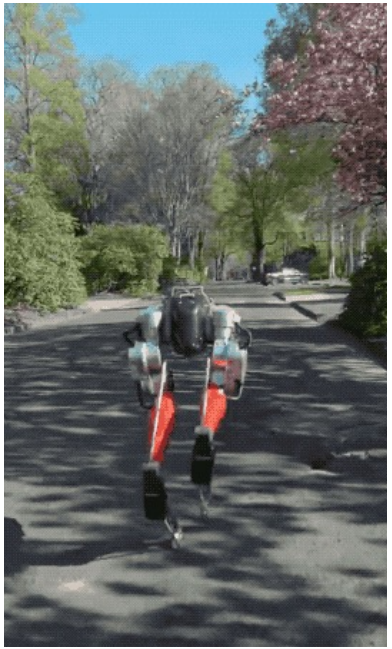

[1] Mnih et al., Human-level control through deep reinforcement learning. *Nature* (2015).
[2] Foehn et al., Agilicious: Open-Source and Open-Hardware Agile Quadrotor for Vision-Based Flight. *Science Robotics* (2022).
[3] Miki et al., Learning robust perceptive locomotion for quadrupedal robots in the wild. *Science Robotics* (2022).
[4] Siekmann et al., Sim-to-real learning of all common bipedal gaits via periodic reward composition. *ICRA* (2021).

No Closed-Loop Guarantees

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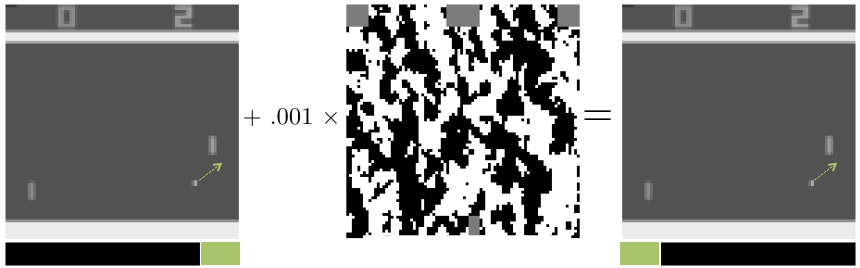
- Deep RL is not robust to small perturbations



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No Closed-Loop Guarantees

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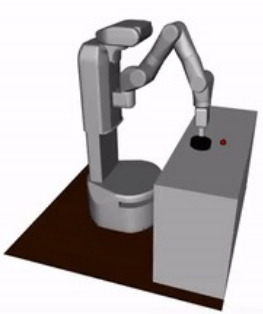
- Can improve robustness with empirical methods



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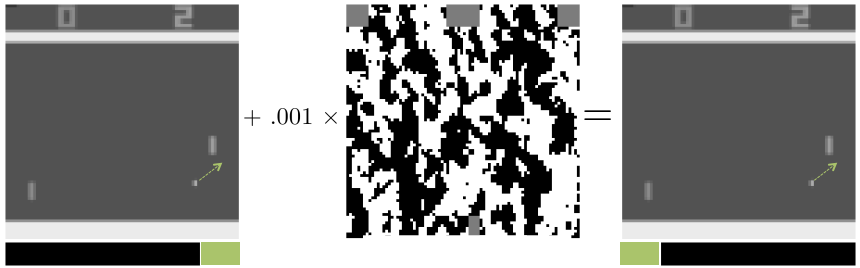

action taken: **down**
original input

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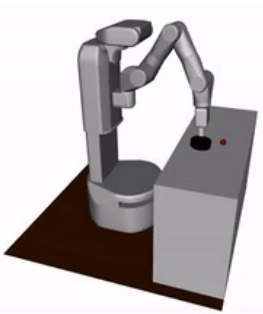
No Closed-Loop Guarantees

- Deep RL is not robust to small perturbations
- Can improve robustness with empirical methods
- …but without *guarantees*, still closed-loop unstable!



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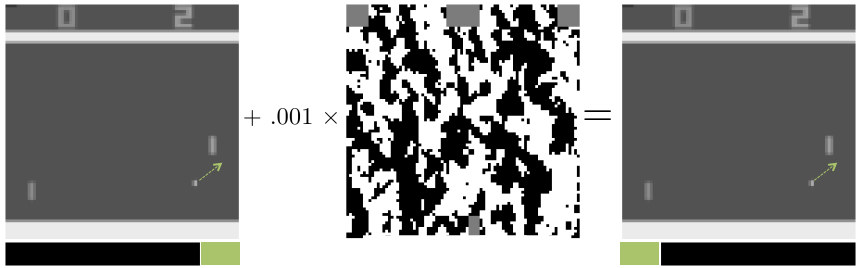

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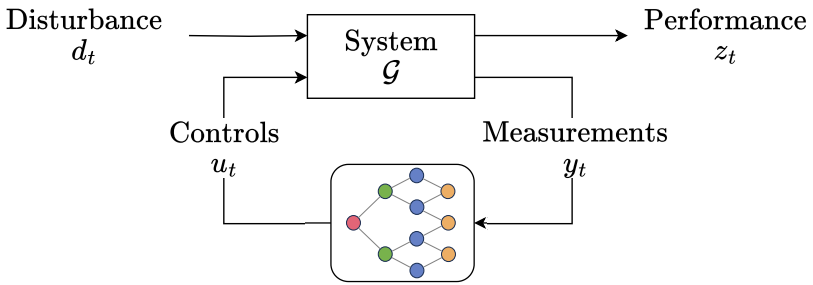

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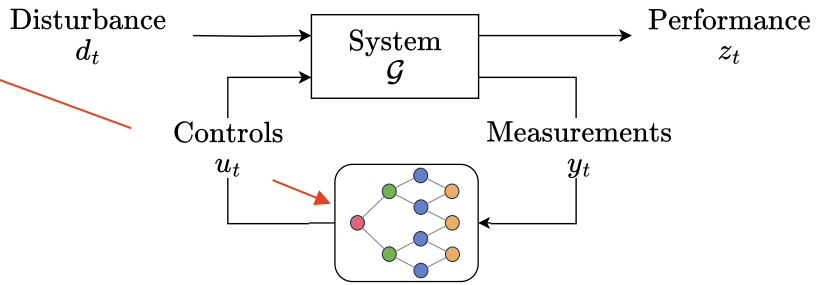
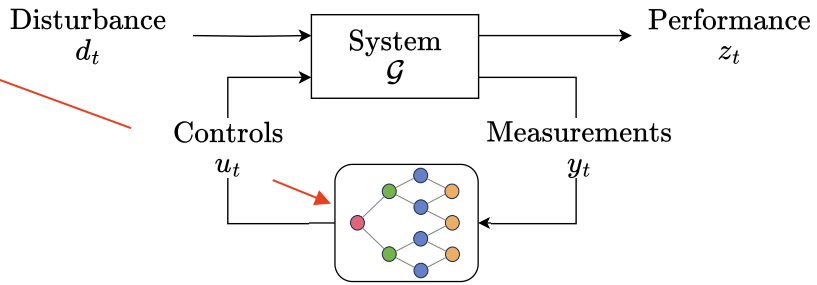


Black-Box Methods

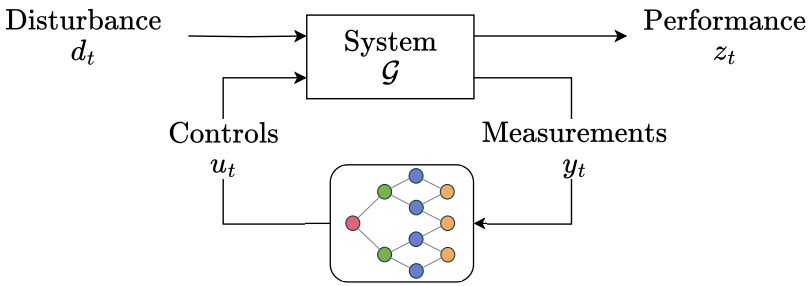
- Deep RL relies on black-box Neural Networks (NNs)
- Problem:
  - No prior system knowledge in the policy architecture
  - No prior system knowledge in the policy parametrisation
- …so closed-loop stability is *not explicitly considered!*



Question

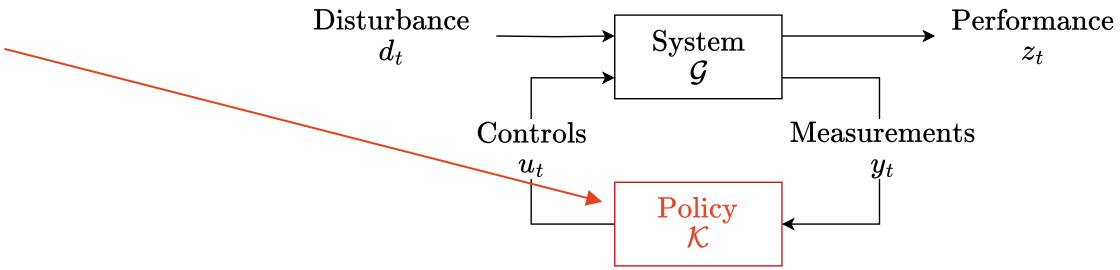
Can we learn robustly-stabilising neural feedback policies with closed-loop guarantees for nonlinear systems?

Stable-by-Design Neural Feedback Policies



Stable-by-Design Neural Feedback Policies

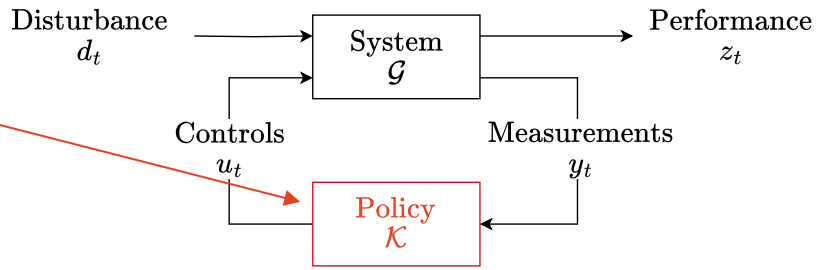
Construct a new *policy class*:



Stable-by-Design Neural Feedback Policies

Construct a new *policy class*:

1. Closed-loop robust stability by construction
2. Differentiable, scalable parametrisation
3. Expressive parametrisation



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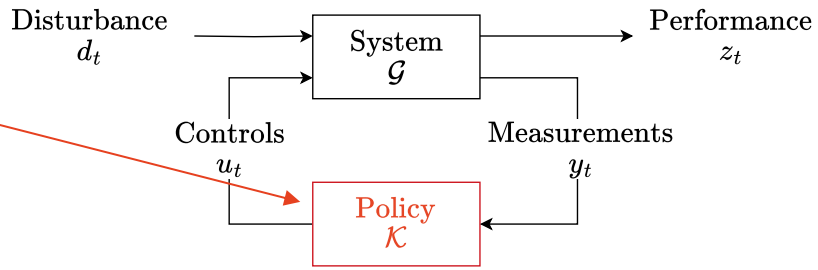
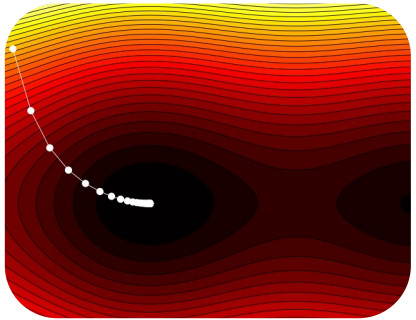


Directly learn controllers with robust stability guarantees using standard ML/deep RL tools



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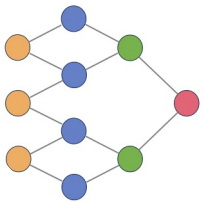

Neural networks

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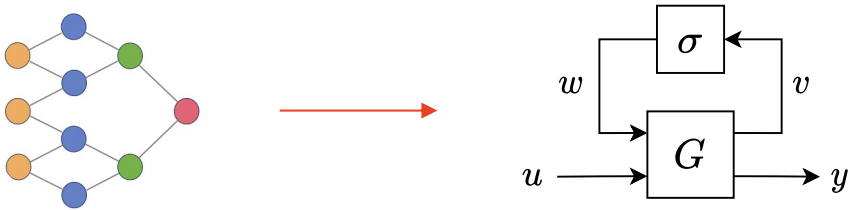

Neural networks + Robust control theory

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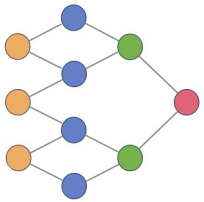

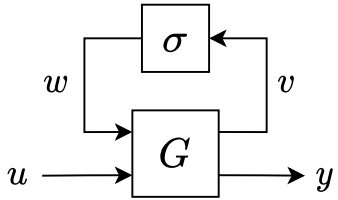

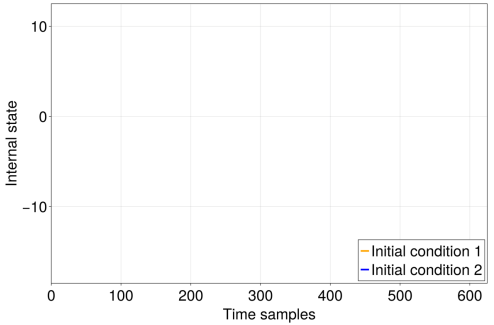

Neural networks + Robust control theory = Provable guarantees of smooth, stable response

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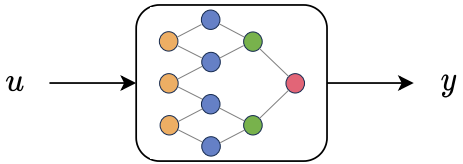
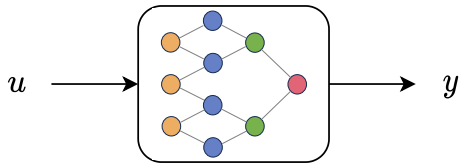
Input-Output Robustness: Lipschitz Continuity



Input-Output Robustness: Lipschitz Continuity

- Lipschitz systems respond smoothly to changes in their inputs:



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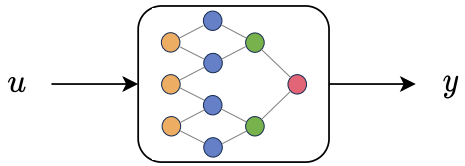
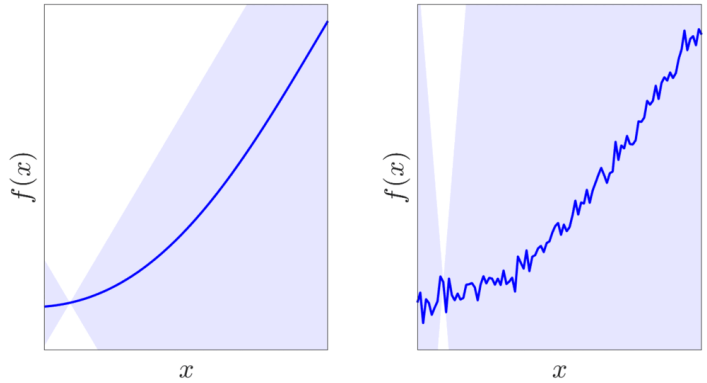


- Small Lipschitz constant ➔ smooth response
- Large Lipschitz constant ➔ sensitive to perturbations



[9] Pauli et al., Neural Network Training Under Semidefinite Constraints, *CDC* (2022).

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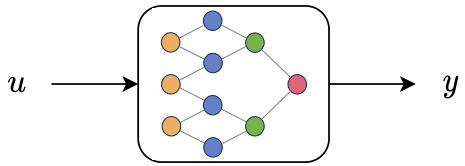
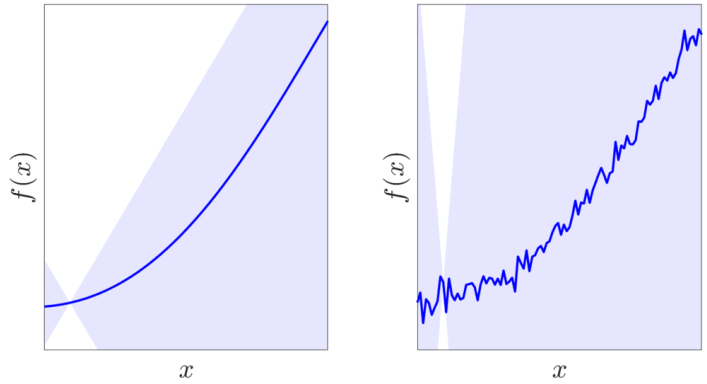


- Small Lipschitz constant ➔ smooth response
- Large Lipschitz constant ➔ sensitive to perturbations
- Generalise to other I/O properties with IQCs[10]



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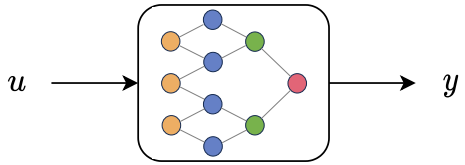
[10] Megretski & Rantzer, System Analysis via Integral Quadratic Constraints, *TAC* (1997).

Internal Stability: Contraction



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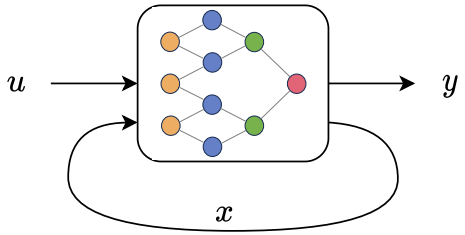

Internal Stability: Contraction

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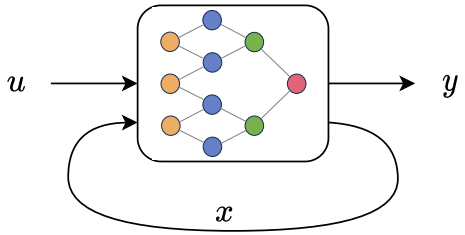


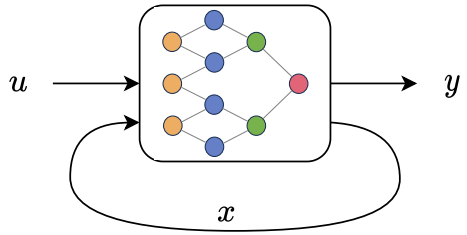
Internal Stability: Contraction

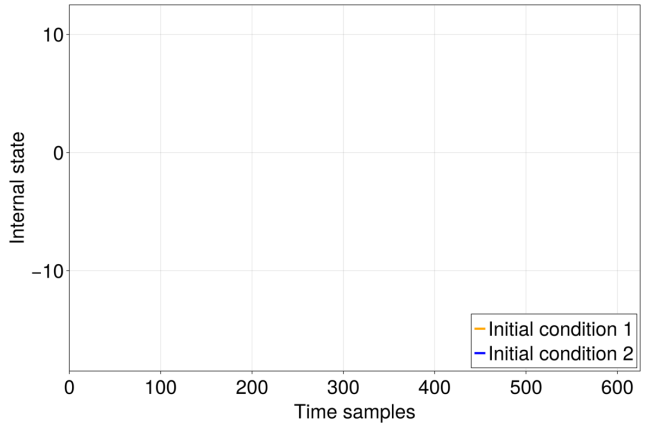
- Contracting systems forget their initial conditions exponentially

- For a discrete-time contracting system ($\Delta u = 0$)

$$|\Delta x_t| \leq \beta \alpha^t |\Delta x_0|, \quad \alpha \in [0, 1), \ \beta \geq 0$$

- I.e., incremental exponential stability





[11] Lohmiller & Slotine, On Contraction Analysis for Nonlinear Systems. *Automatica* (1998).

Stable & Robust Neural Networks

- Our suite of robust neural networks @ the ACFR[8]:

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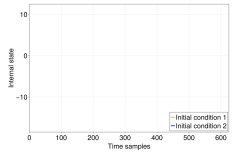

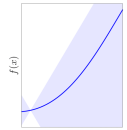

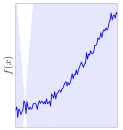

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  - Static (feedforward) and dynamic (recurrent) models
  - Built-in guarantees of incremental stability (contraction) & robustness (Lipschitz, IQC)

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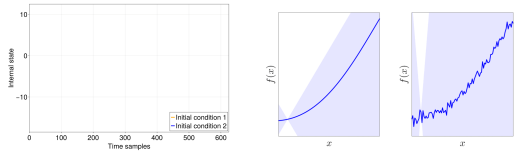

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  - Static (feedforward) and dynamic (recurrent) models
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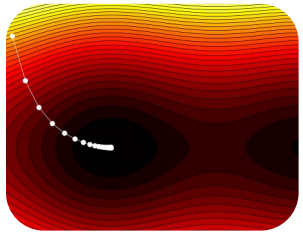
  - **Directly parametrised** to allow unconstrained optimisation
  - Universal approximators of all contracting & Lipschitz systems



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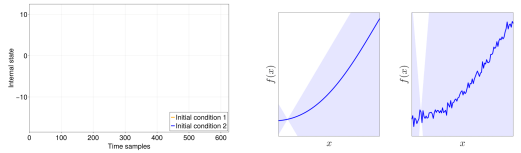
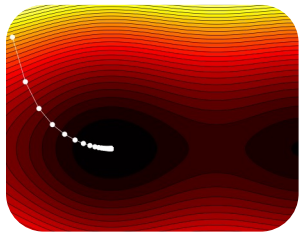
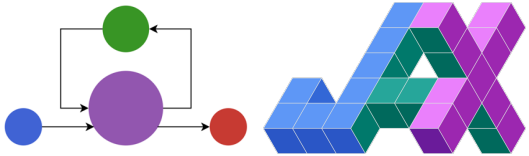
Stable & Robust Neural Networks



- Our suite of robust neural networks @ the ACFR[8]:
  - Static (feedforward) and dynamic (recurrent) models
  - Built-in guarantees of incremental stability (contraction) & robustness (Lipschitz, IQC)
  - **Directly parametrised** to allow unconstrained optimisation
  - Universal approximators of all contracting & Lipschitz systems



- Open-source and ready for use in Julia and Python (JAX)[8,12]



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[12] Barbara et al., RobustNeuralNetworks.jl: a Package for Machine Learning and Data-Driven Control with Certified Robustness, *JuliaCon Proceedings* (2025).

Examples of Robust Neural Networks

Two main robust NN architectures used in this talk:

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   - Static, feedforward DNNs (MLP, CNN, etc…)
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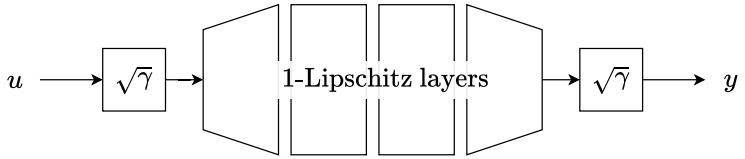

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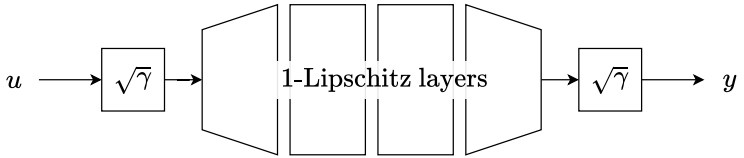
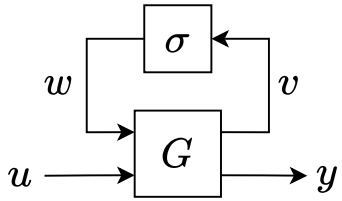


2. Recurrent Equilibrium Network (REN)[14]
   - Recurrent / dynamic models
   - Certified contraction and Lipschitz bounds
   - Universal approximator for contracting & Lipschitz systems



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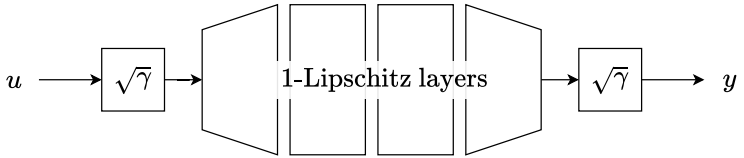

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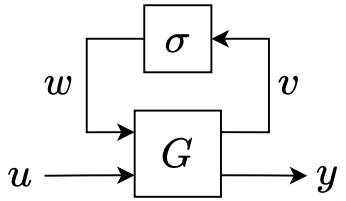

For more, see Manchester et al., *Ann. Rev. Control, Robotics, Autonomous Systems* (2025).

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Lipschitz-Bounded Policy Networks

Construct a new *policy class*:

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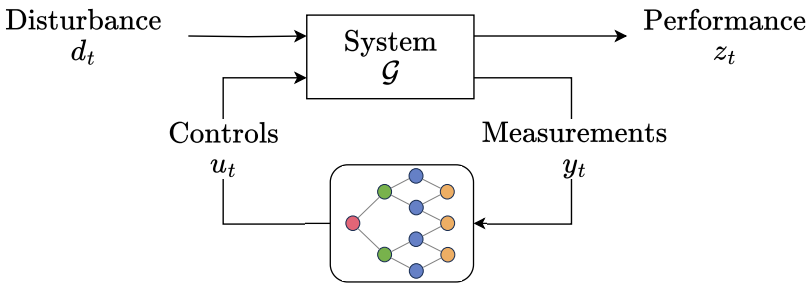
Directly learn controllers with robust stability guarantees using standard ML/deep RL tools



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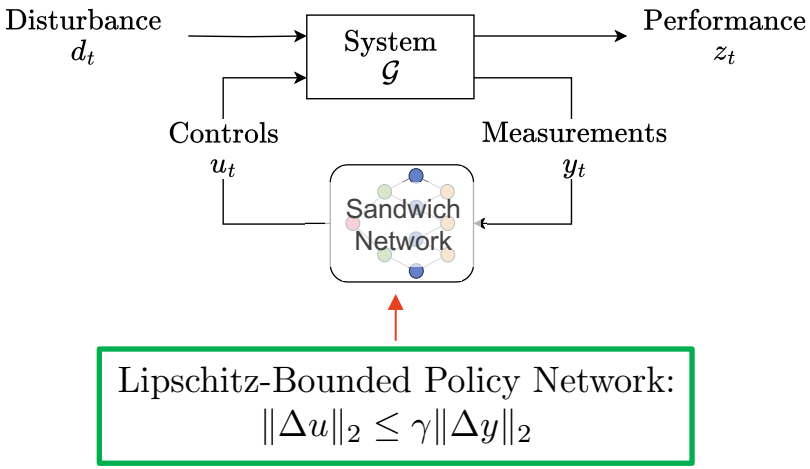

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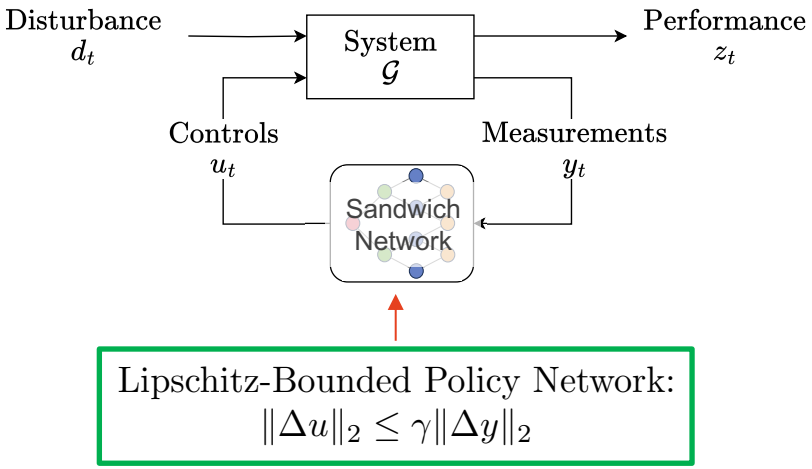

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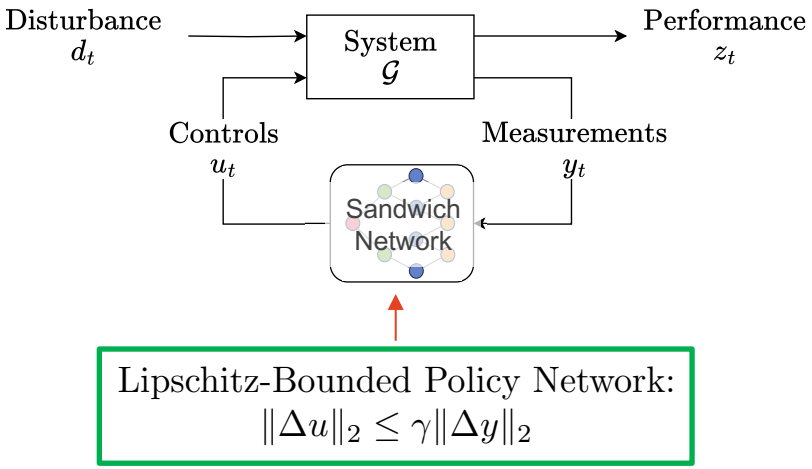

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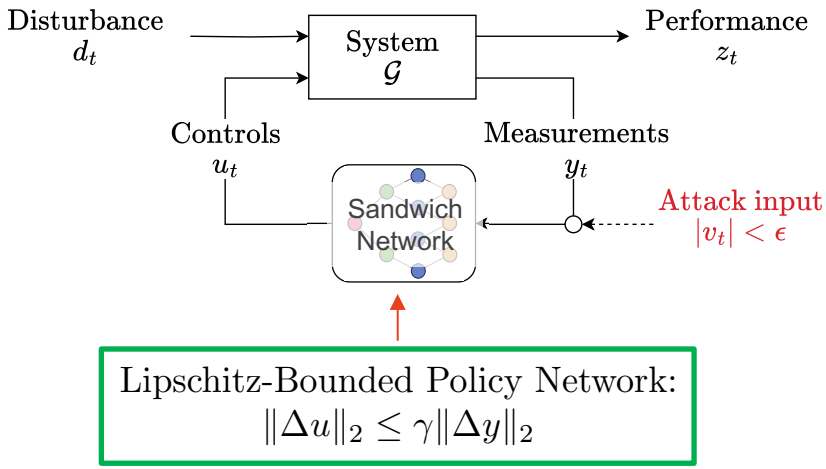

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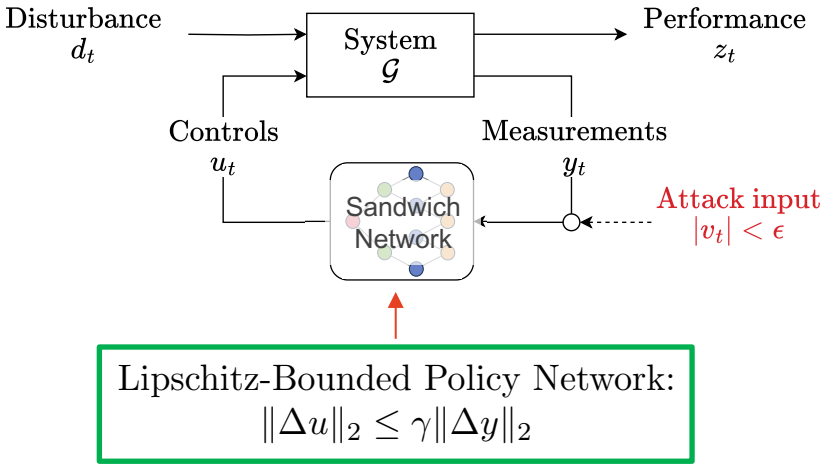

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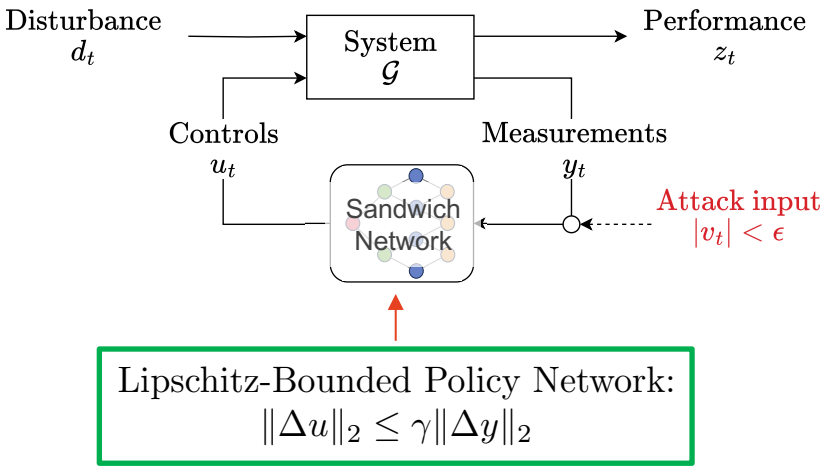

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Playing Atari Pong (Noise-Free)

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Trained RL Agent

Unconstrained CNN

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Tuneable Robustness to Adversarial Attacks

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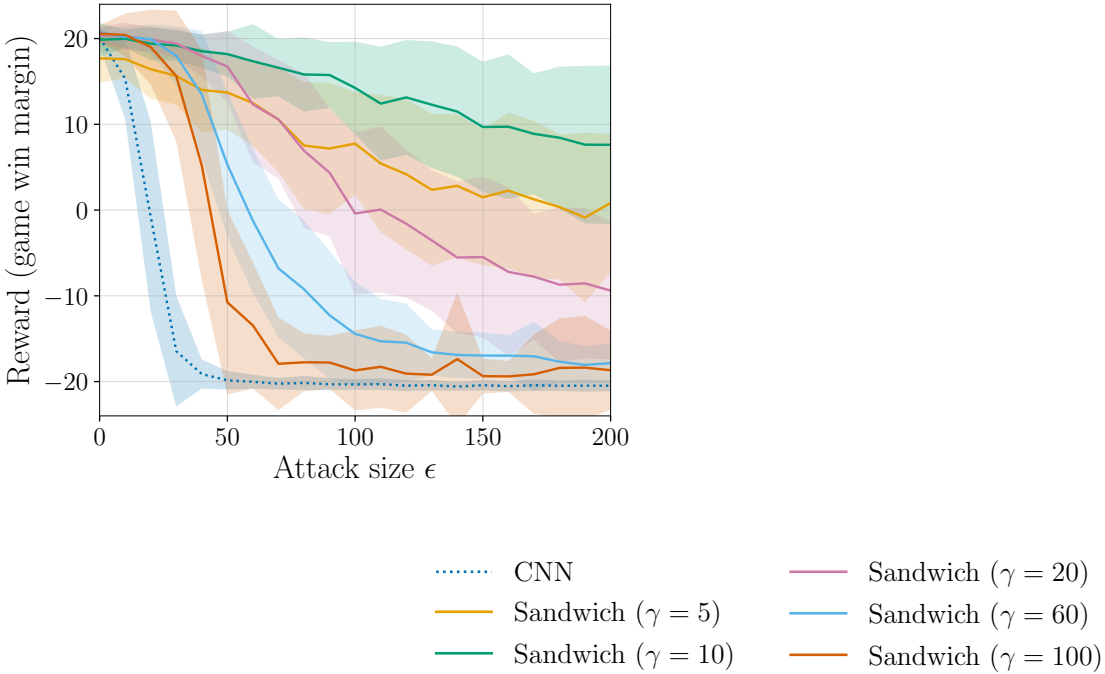

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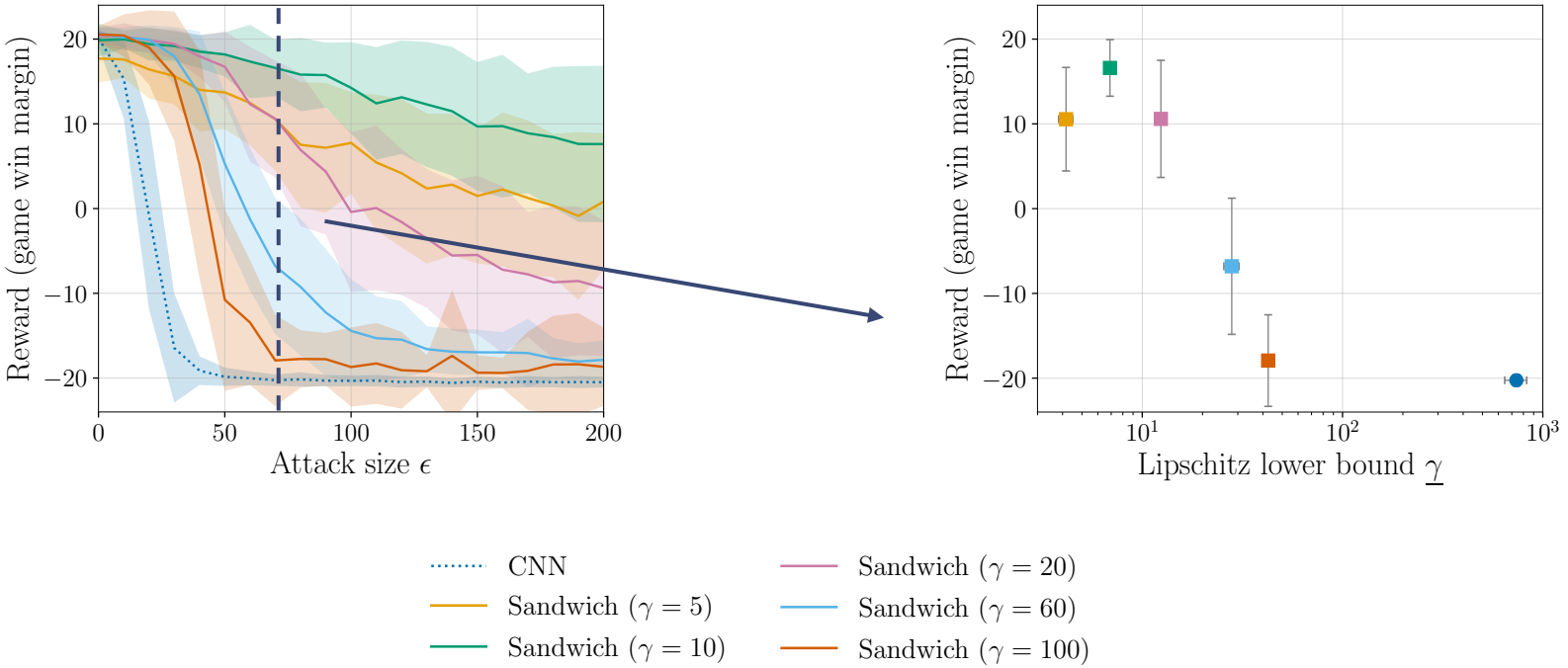

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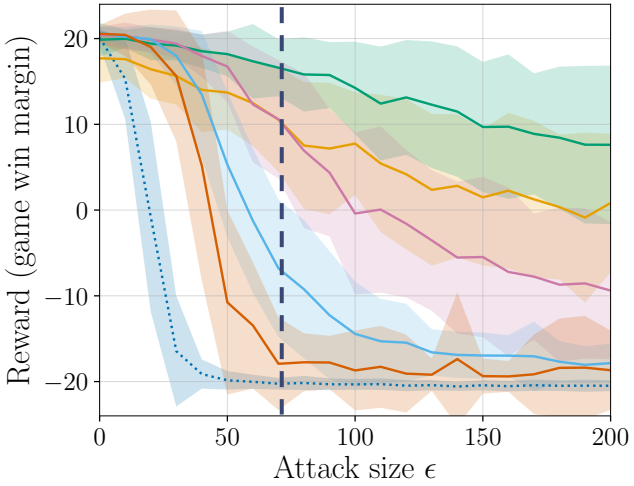

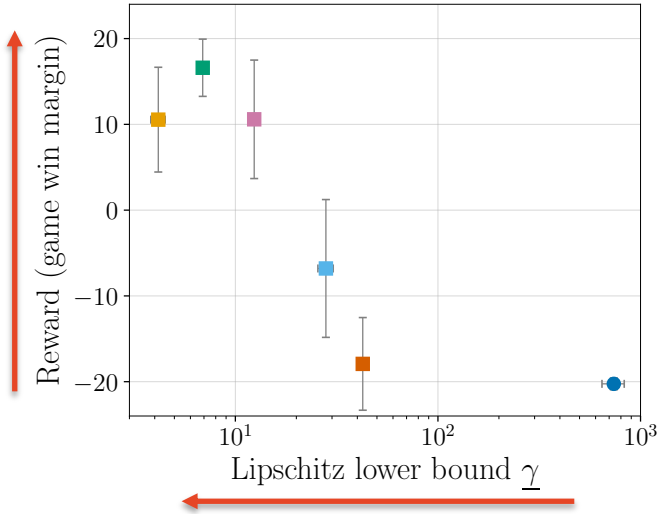

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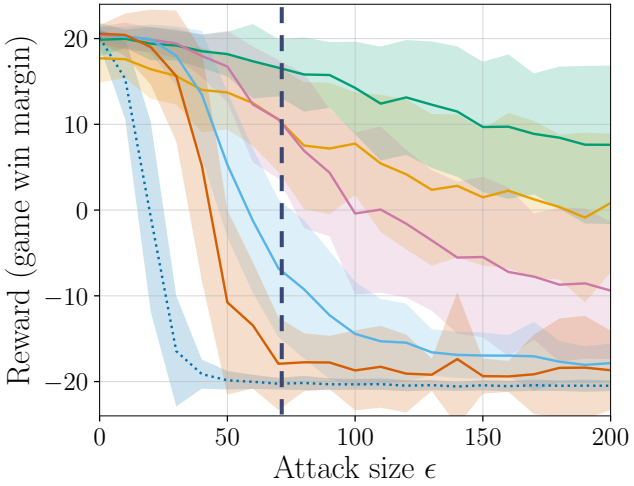

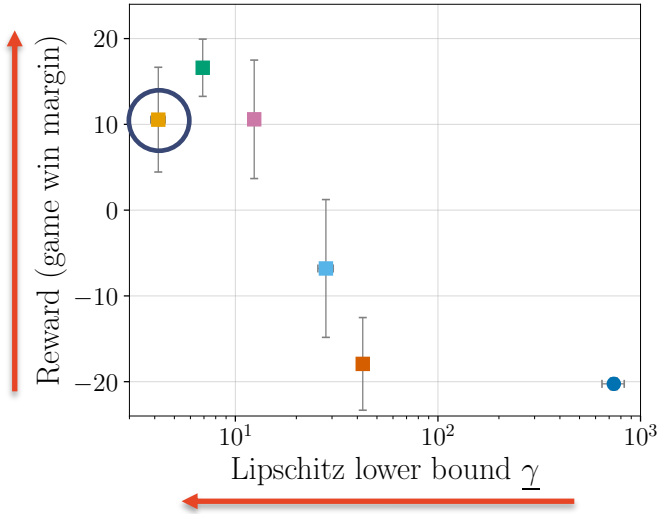

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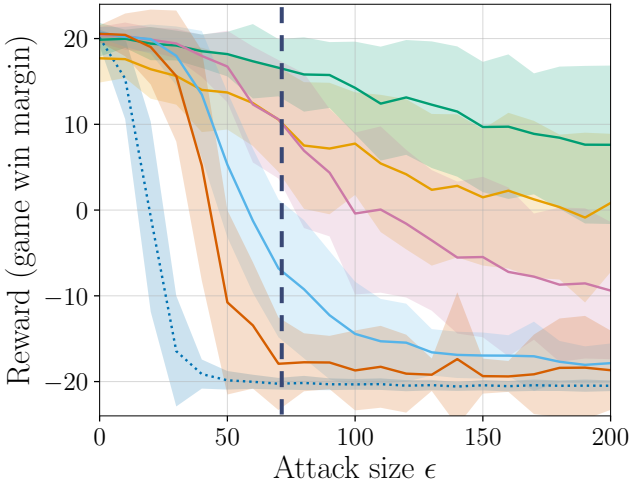

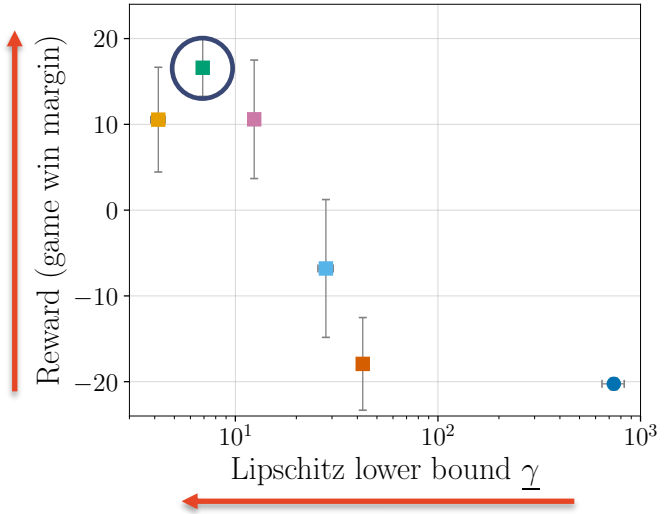

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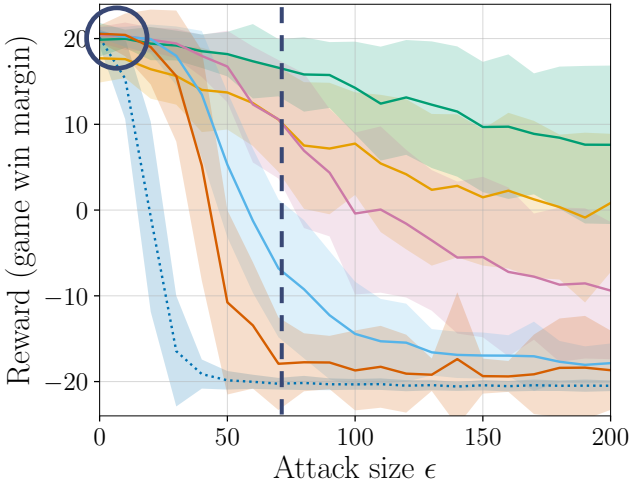

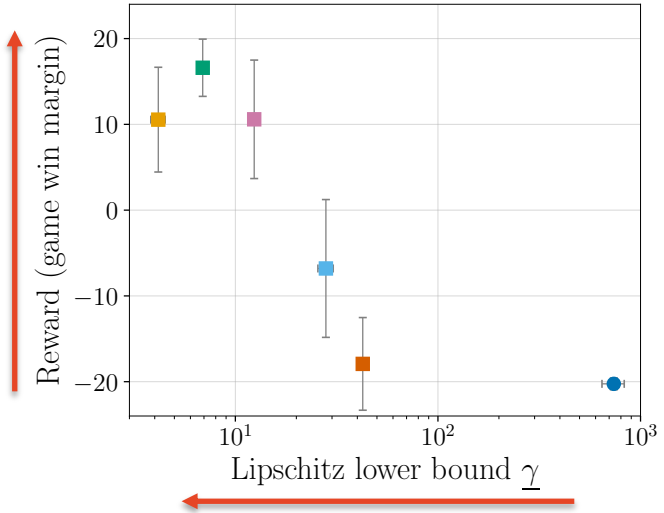

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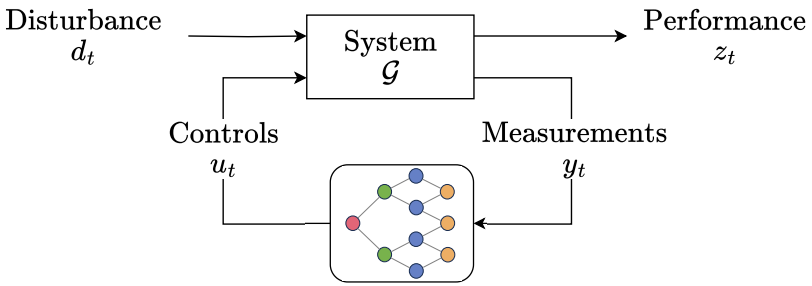

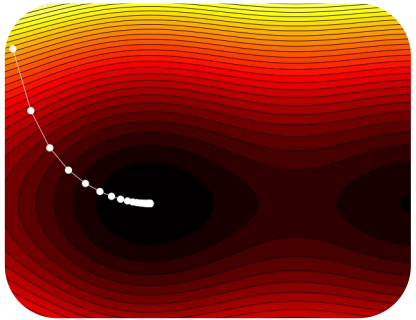

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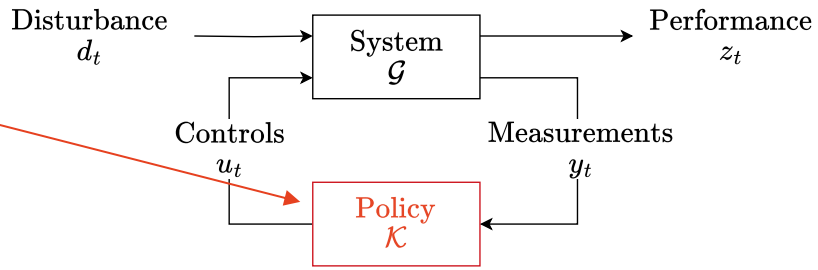

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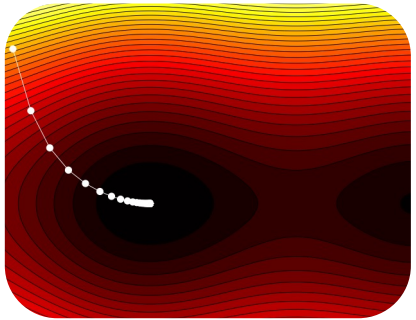

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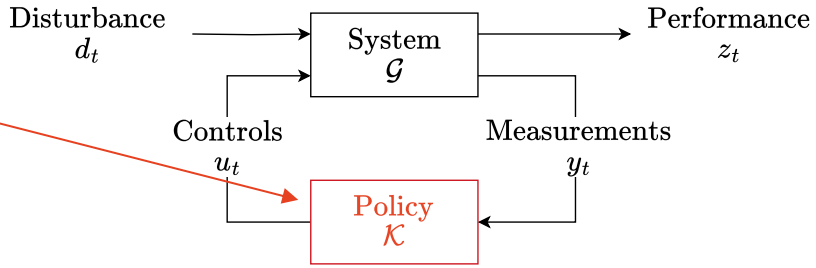

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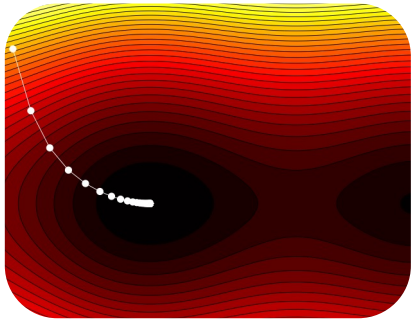

Policy Classes for Deep RL

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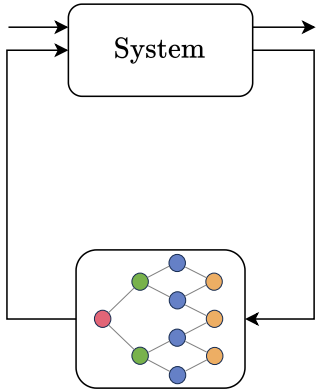

Vanilla RL

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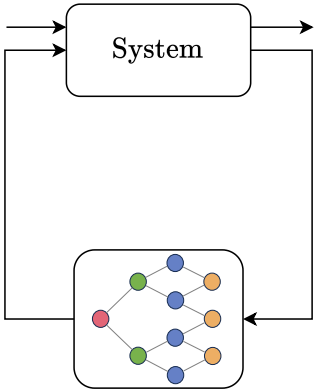

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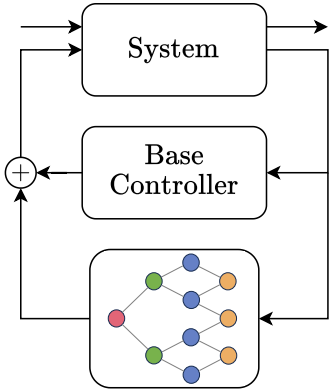

Residual RL[16]

[16] Johannik et al., Residual Reinforcement Learning for Robot Control, *ICRA* (2019).

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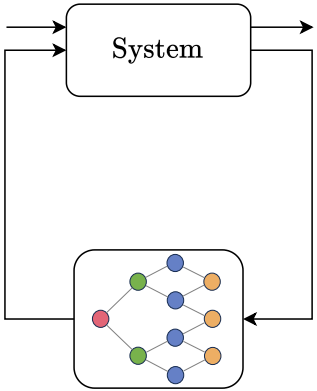

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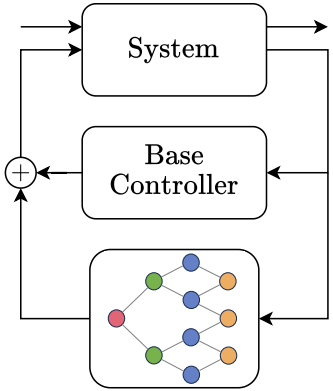

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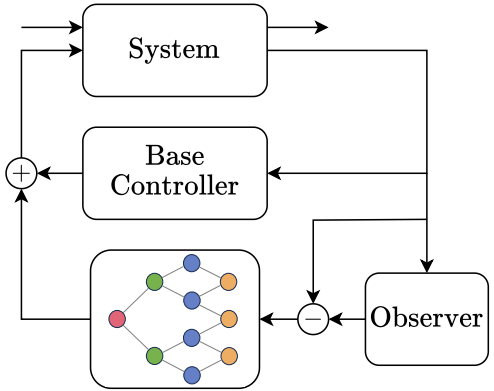

Youla-REN (ours)[17]

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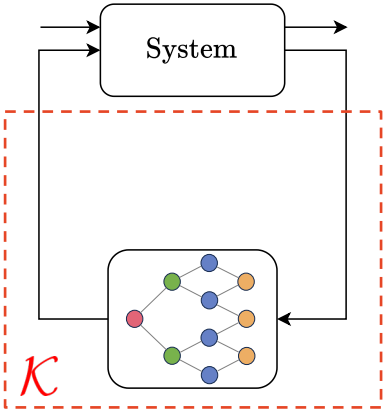
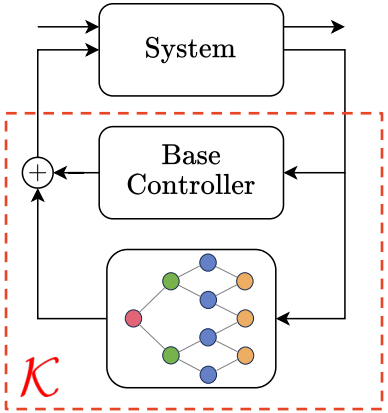
Policy Classes for Deep RL



Vanilla RL



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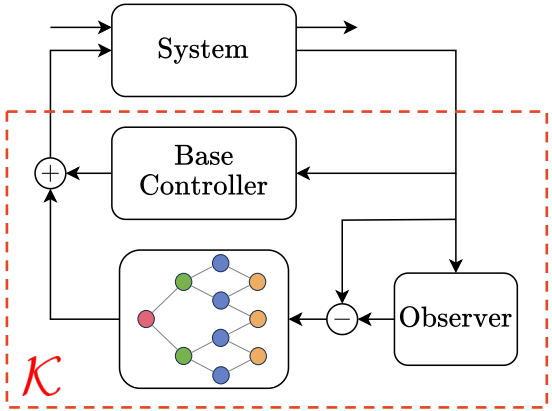

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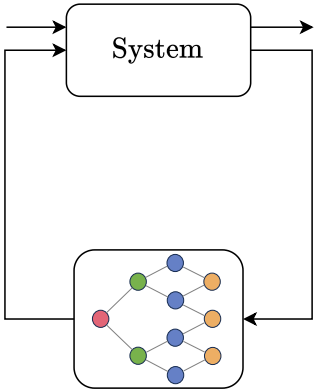

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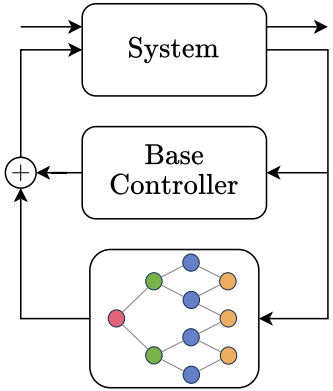

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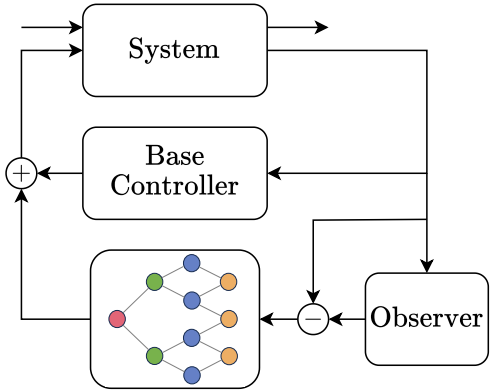

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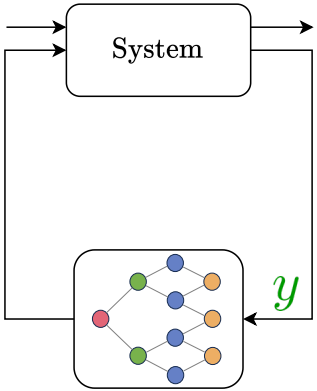

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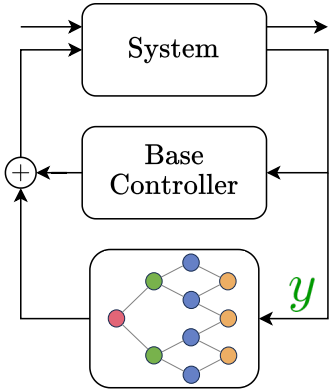

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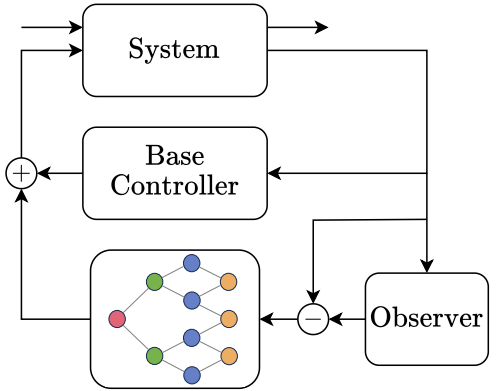

Youla-REN (ours)[17]

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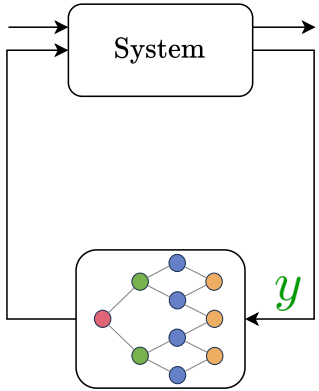

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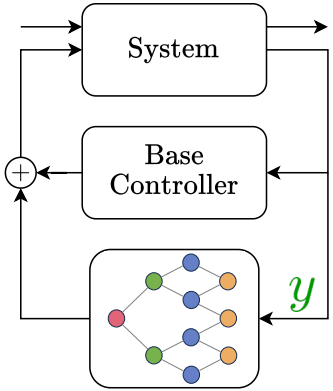

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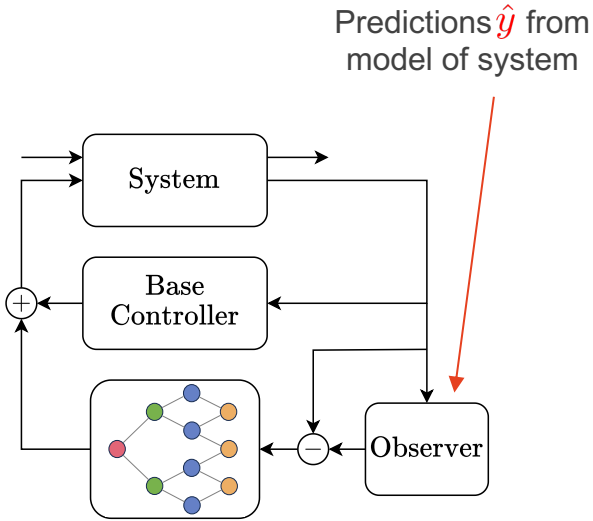

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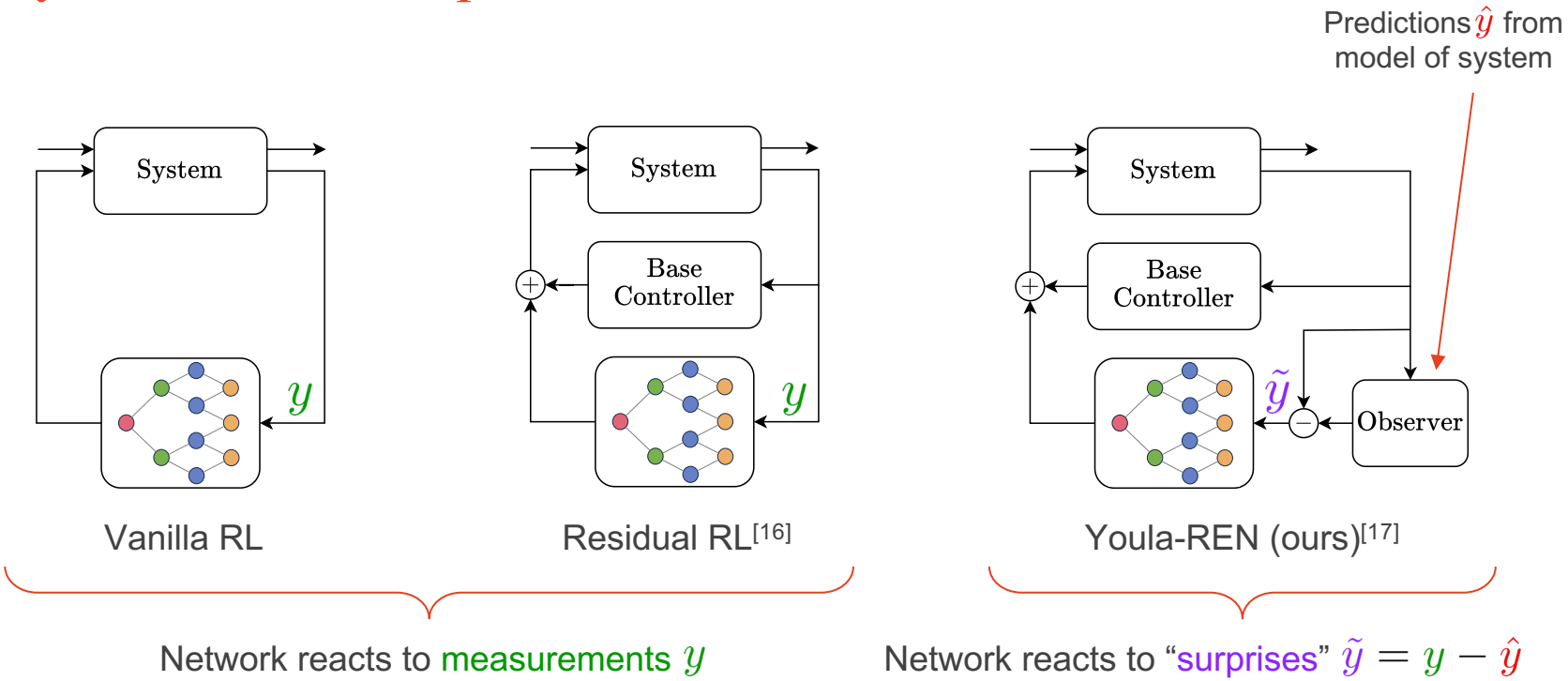

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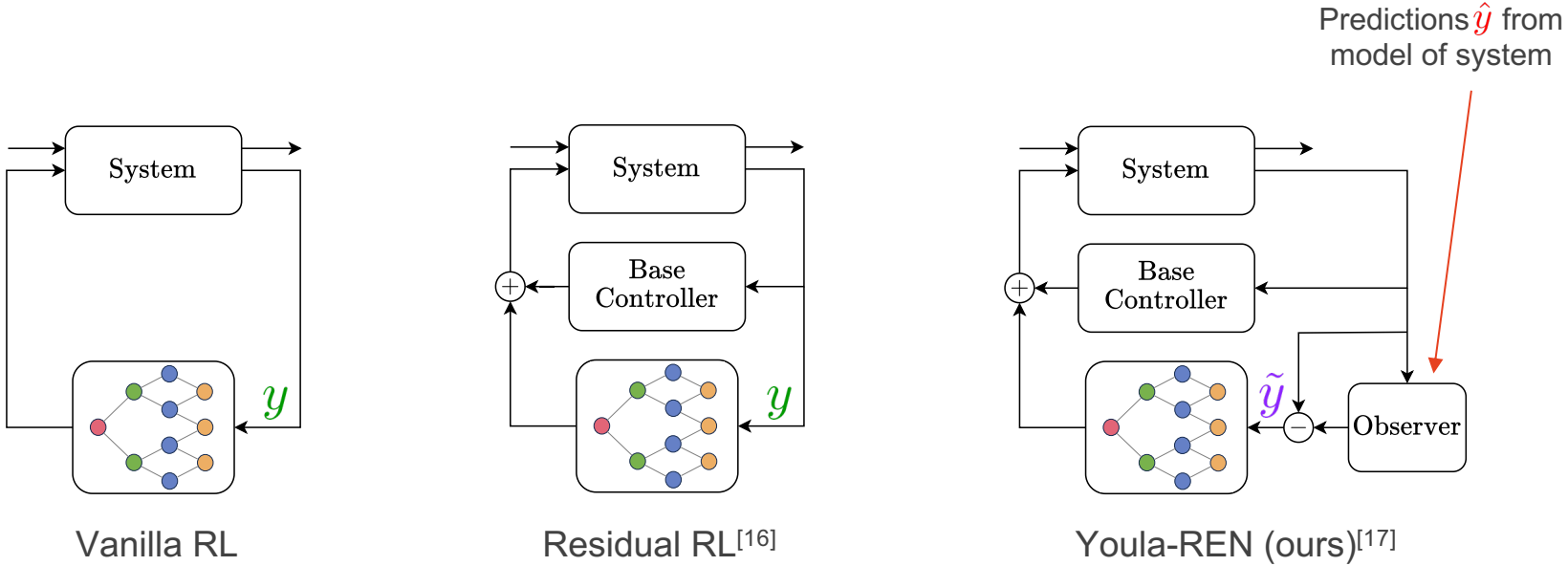

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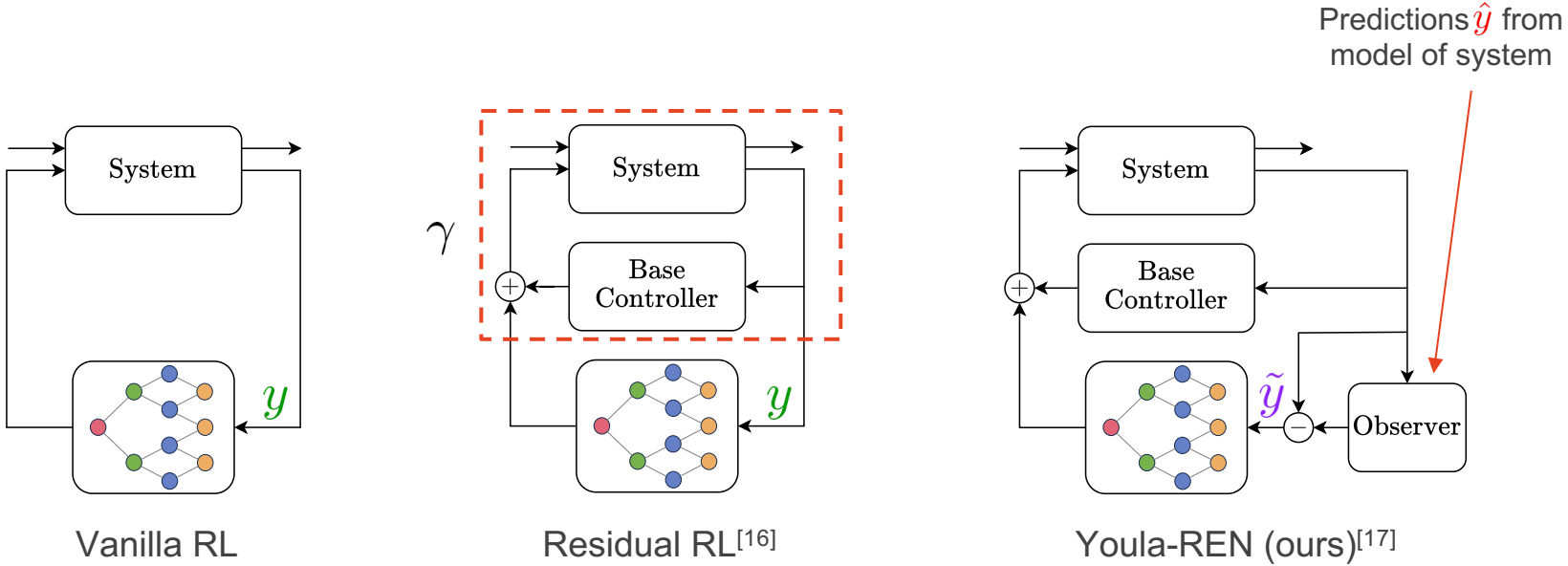

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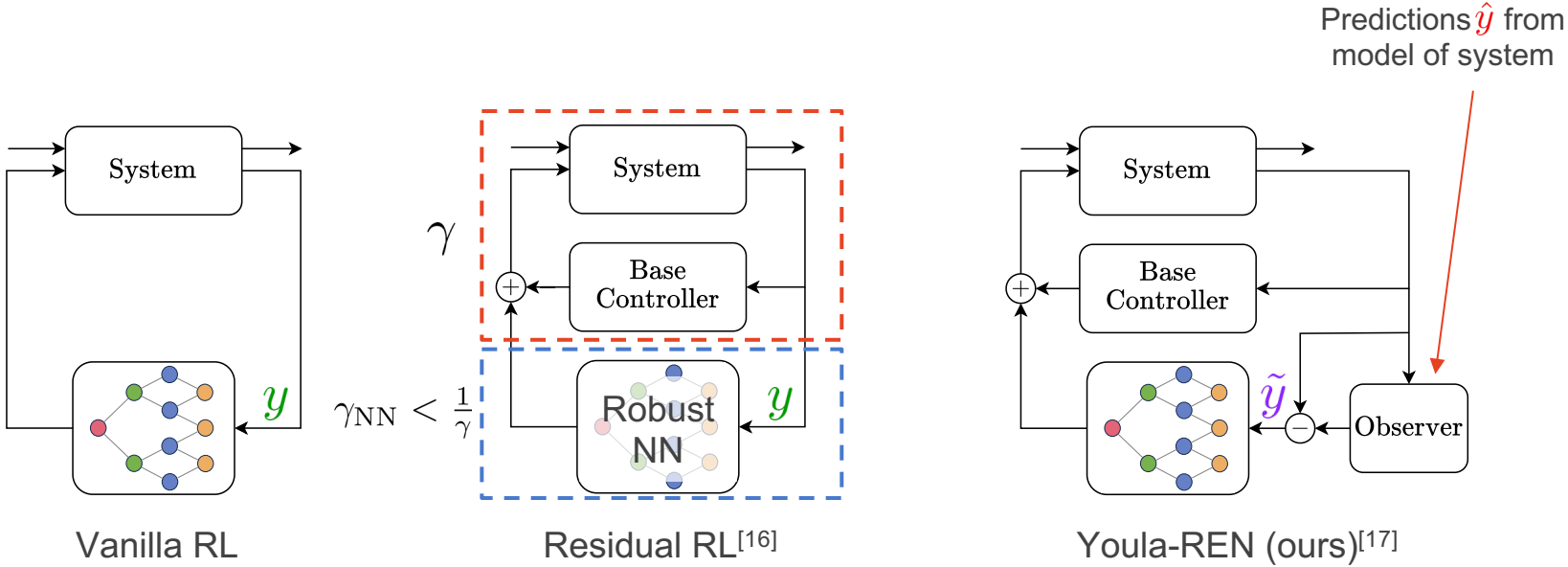

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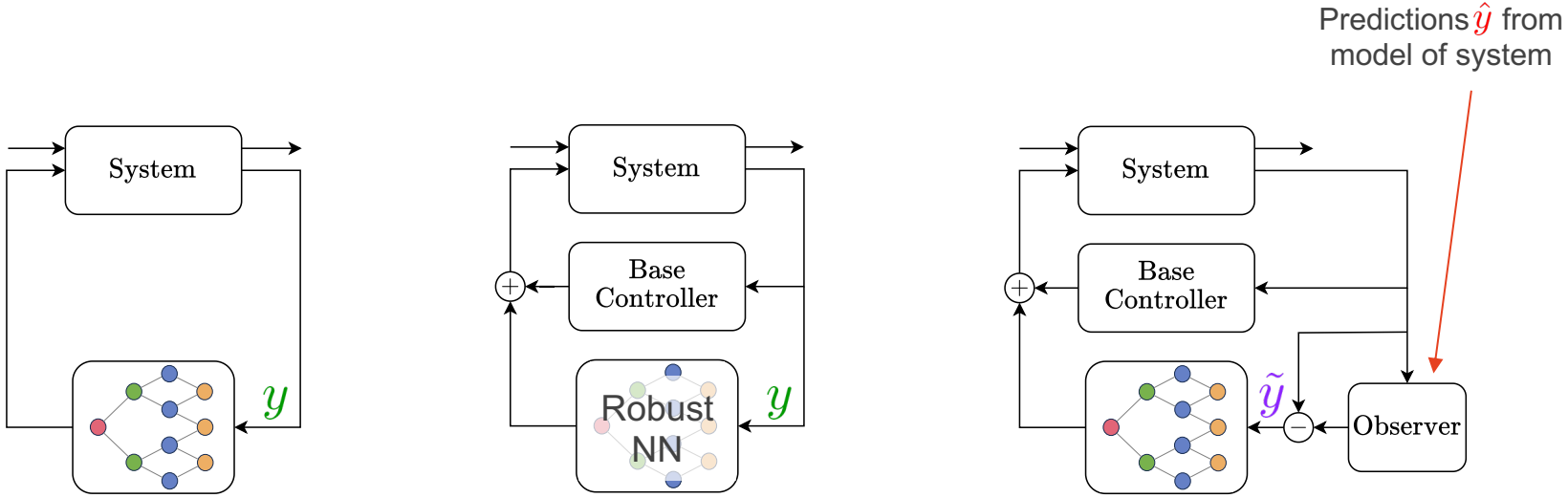

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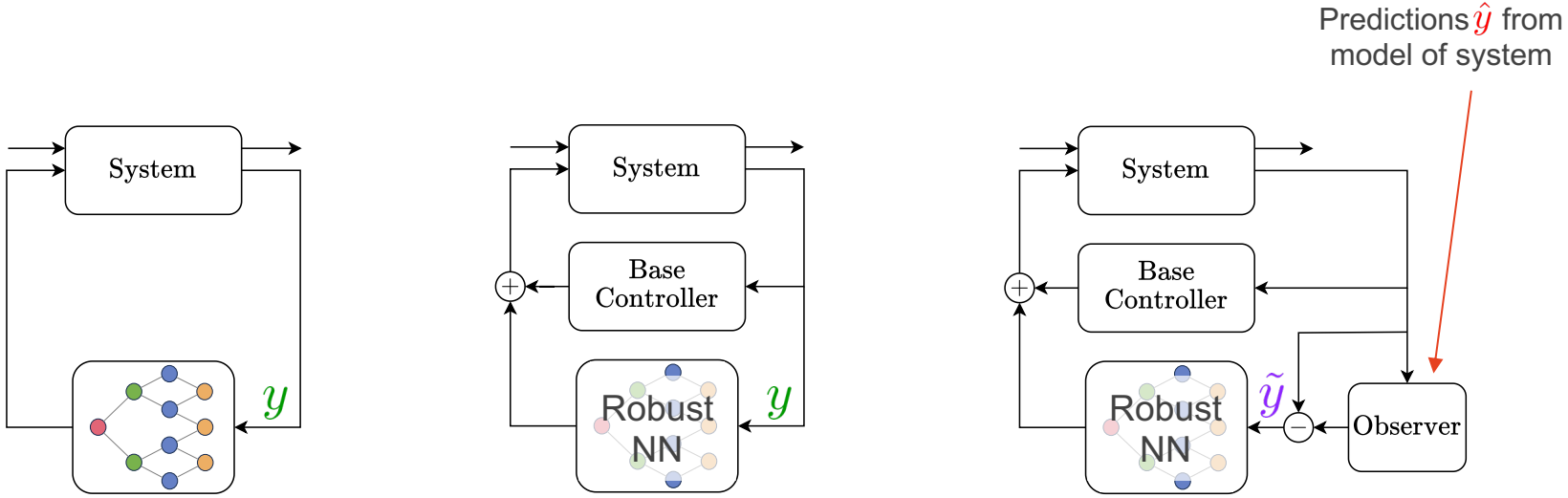
Policy Classes for Deep RL



- React to measurements y: closed-loop stability via small-gain theorem
- React to surprises $\tilde{y}$: closed-loop stability without gain restrictions (under assumptions)

[16] Johannik et al., Residual Reinforcement Learning for Robot Control, *ICRA* (2019).
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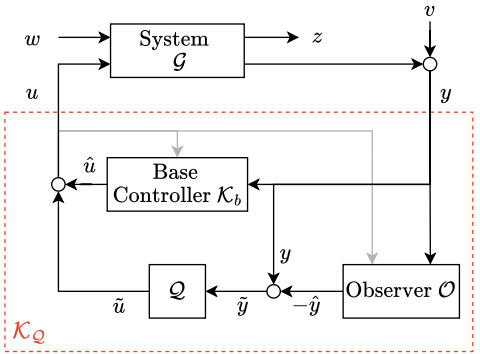
Youla Parametrisation Guarantees Closed-Loop Stability

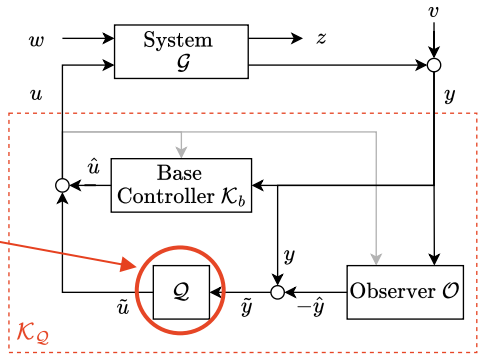


Youla Parametrisation Guarantees Closed-Loop Stability

"Youla Parameter" Q

Nonlinear (possibly dynamic) system.

Eg: neural network, or more general system



Youla Parametrisation Guarantees Closed-Loop Stability

Theorem. *Consider the LTI system*

$$\mathcal{G}: \begin{cases} x_{t+1} = Ax_t + Bu_t + w_t \\ y_t = Cx_t + v_t \end{cases}$$

together with an observer and base controller

$$\mathcal{O}, \mathcal{K}_b: \begin{cases} \hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y - C\hat{x}_t) \\ \hat{u}_t = -K\hat{x}_t \end{cases}$$

designed such that $(A - BK)$ and $(A - LC)$ are stable. Writing $d = [w; v]$ and $z = [x; u]$, then the following statement is true for the closed-loop system under the Youla controller $u = \hat{u} + \mathcal{Q}(y - C\hat{x})$:

A controller $u = \mathcal{K}(y)$ with a valid state-space representation can achieve a contracting and Lipschitz closed loop $d \mapsto z$ ***if and only if*** *it can be written in the form of the Youla controller* ***with a contracting and Lipschitz*** *$\mathcal{Q}$.*

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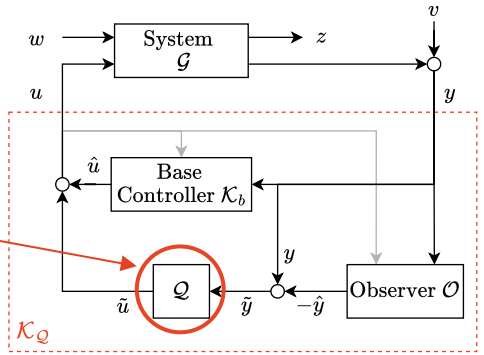

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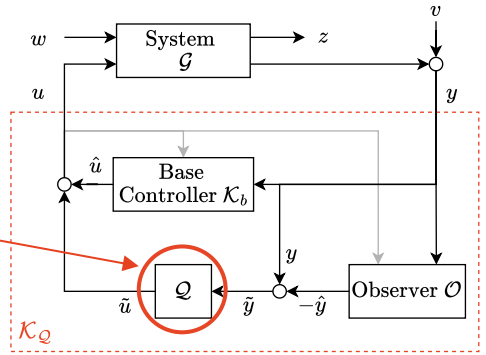


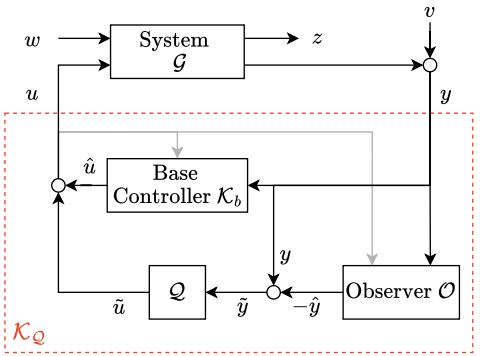
What it means:

- Stabilising controller $\iff$ Youla controller with a stable (contracting & Lipschitz) Q
- No gain restrictions on Q
- For linear[18] and nonlinear[19] controllers

I.e., the policy class $\mathcal{K}_\mathcal{Q}$ covers **all and only** stabilising controllers for LTI systems!

[18] Doyle, Matrix Interpolation Theory and Optimal Control, *PhD Thesis, Berkley* (1984).
[19] Wang, Barbara, Revay, & Manchester, Learning Over All Stabilizing Nonlinear Controllers for a Partially-Observed Linear System, *L-CSS* (2022).

Contracting & Lipschitz Closed Loops for Nonlinear Systems



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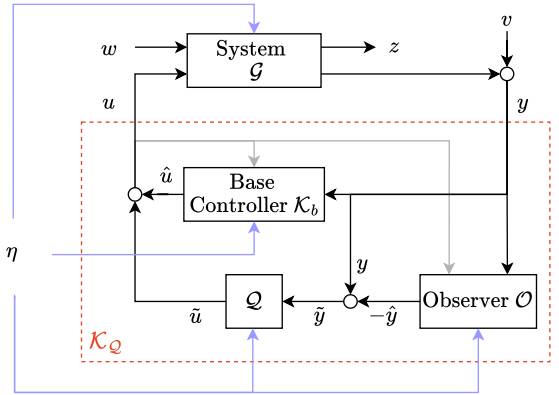

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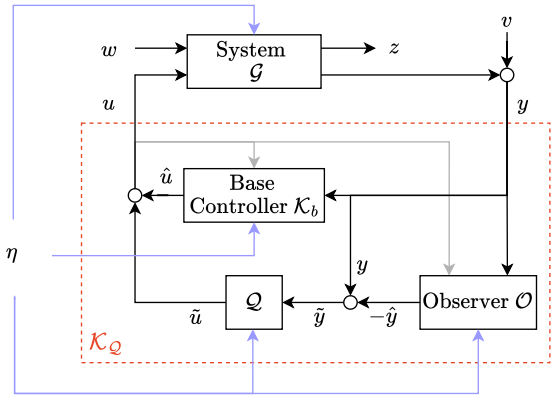

$$\mathcal{G}: \begin{cases} x_{t+1} = f(x_t, \eta_t, u_t) + w_t \\ y_t = h(x_t) + v_t \end{cases}$$

$$\mathcal{K}_b: \begin{cases} s_{t+1} = f_o(s_t, \eta_t, u_t, y_t) \\ \hat{u}_t = k(s_t, \eta_t, y_t) \end{cases}$$

$$\mathcal{O}: \begin{cases} \hat{x}_{t+1} = f_o(\hat{x}_t, \eta_t, u_t, y_t) \\ \hat{y}_t = h(\hat{x}_t) \end{cases}$$

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- *$\mathcal{K}_b$ achieves a contracting & Lipschitz closed loop $(\eta, d, \tilde{u}) \mapsto z$ with $\mathcal{G}$.*
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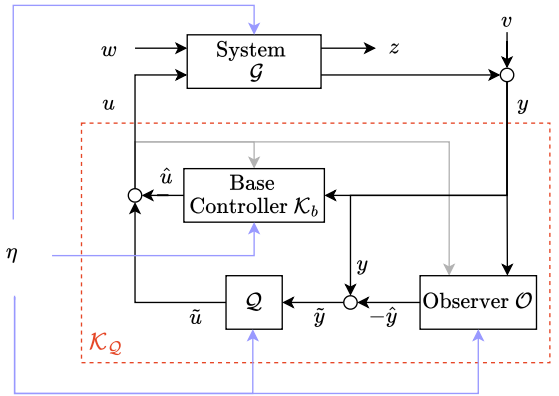
- *Observer correctness: $f(x_t, \eta_t, u_t) = f_o(s_t, \eta_t, u_t, h(x_t))$.*
- *All systems have piecewise-diff'ble dynamics and Lipschitz output maps.*

Then, the following statements are true for the closed-loop system under the Youla controller $u = \hat{u} + \mathcal{Q}(y - \hat{y})$ ***with a contracting and Lipschitz*** *$\mathcal{Q}$:*



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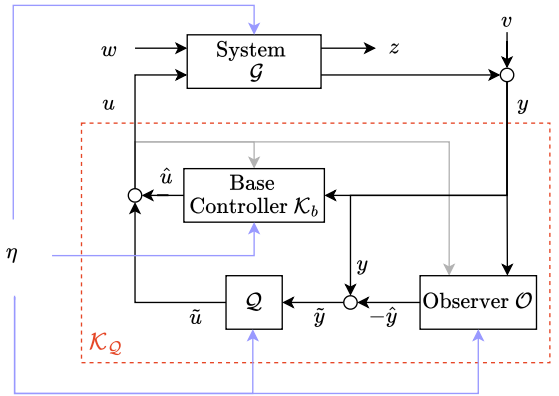

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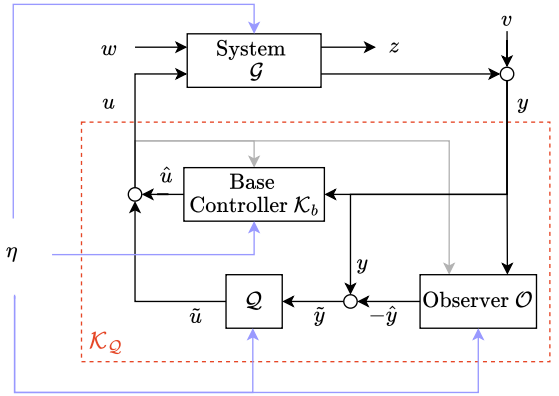
Then, the following statements are true for the closed-loop system under the Youla controller $u = \hat{u} + \mathcal{Q}(y - \hat{y})$ ***with a contracting and Lipschitz*** *$\mathcal{Q}$:*

1. *With zero disturbances (i.e., $d \equiv 0$, $x_0 = \hat{x}_0$), the closed-loop system is contracting and $\eta \mapsto z$ is Lipschitz.*



$$\mathcal{G}: \begin{cases} x_{t+1} = f(x_t, \eta_t, u_t) + w_t \\ y_t = h(x_t) + v_t \end{cases}$$

$$\mathcal{K}_b: \begin{cases} s_{t+1} = f_o(s_t, \eta_t, u_t, y_t) \\ \hat{u}_t = k(s_t, \eta_t, y_t) \end{cases}$$

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[17] Barbara et al., React to Surprises: Stable-by-Design Neural Feedback Control and the Youla-REN, *arXiv* (2025).

Contracting & Lipschitz Closed Loops for Nonlinear Systems

Theorem. *Consider the partially-observed, nonlinear system $\mathcal{G}$ together with the base controller and observer $\mathcal{K}_b$ and $\mathcal{O}$ (respectively) which satisfy the following assumptions, where $d = [w; v]$ and $z = [x; u]$:*

- *$\mathcal{K}_b$ achieves a contracting & Lipschitz closed loop $(\eta, d, \tilde{u}) \mapsto z$ with $\mathcal{G}$.*
- *$\mathcal{O}$ is contracting and the map $(\eta, u, y) \mapsto \hat{x}$ is Lipschitz.*
- *Observer correctness: $f(x_t, \eta_t, u_t) = f_o(s_t, \eta_t, u_t, h(x_t))$.*
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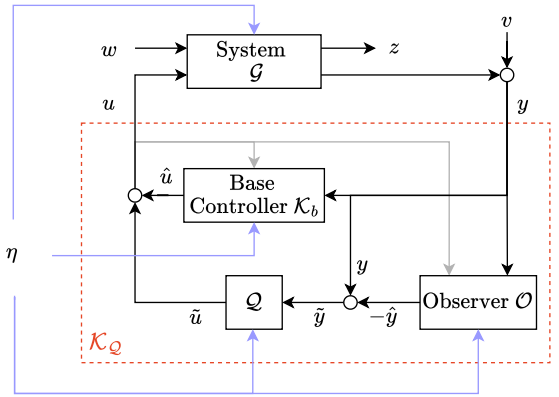
Then, the following statements are true for the closed-loop system under the Youla controller $u = \hat{u} + \mathcal{Q}(y - \hat{y})$ ***with a contracting and Lipschitz*** *$\mathcal{Q}$:*

1. *With zero disturbances (i.e., $d \equiv 0$, $x_0 = \hat{x}_0$), the closed-loop system is contracting and $\eta \mapsto z$ is Lipschitz.*
2. *Let * denote any zero-disturbance closed-loop trajectory. Then with non-zero disturbances (i.e., $d \not\equiv 0$, $x_0 \neq \hat{x}_0$), the closed-loop system satisfies*

$$\|z - z^\star\|_T \leq \gamma(\|\eta - \eta^\star\|_T + \|d\|_T) + \kappa(\bar{x}_0, \bar{x}_0^\star)$$

for all $T \in \mathbb{N}$, $\bar{x}_0, \eta$, where $\gamma \in \mathbb{R}^+$, $\kappa(\bar{x}_0, \bar{x}_0^\star) \geq 0$, $\kappa(\bar{x}_0^\star, \bar{x}_0^\star) = 0$. We call this property **d-tube contracting and Lipschitz**.



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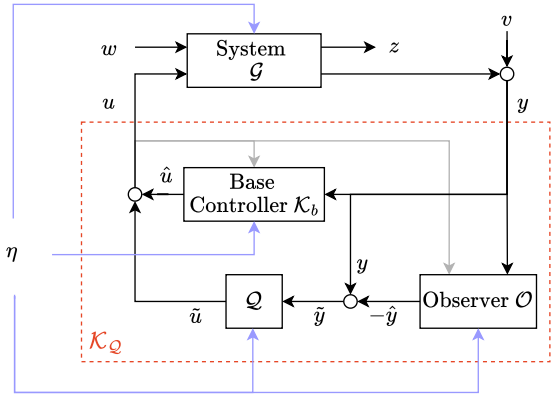

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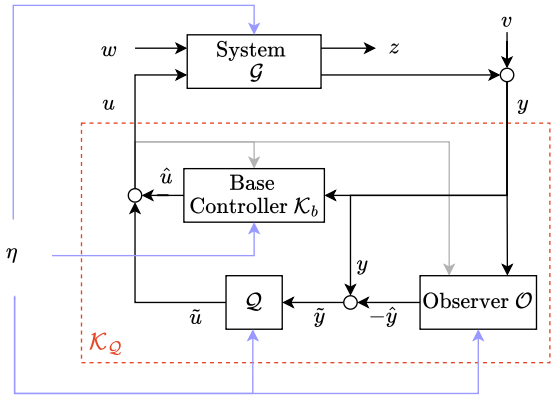

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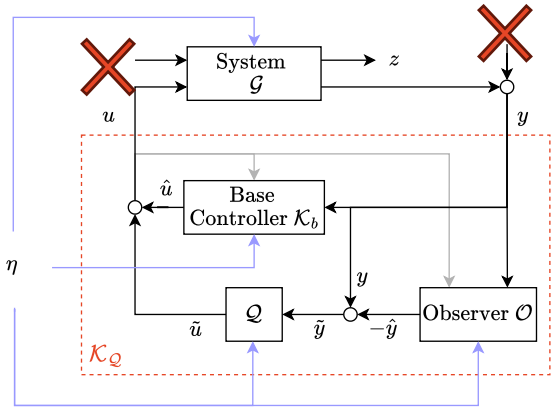

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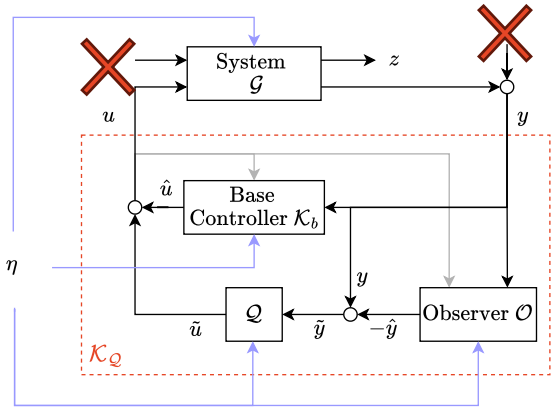

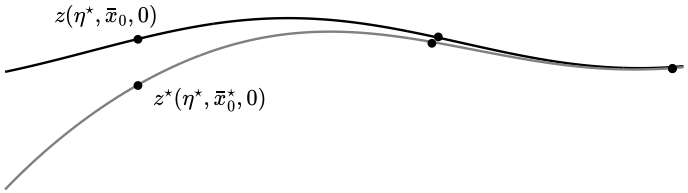

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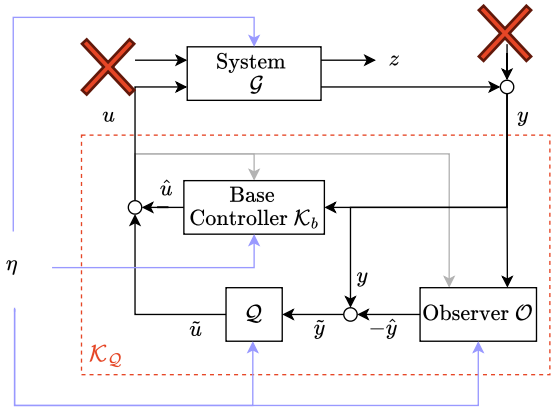

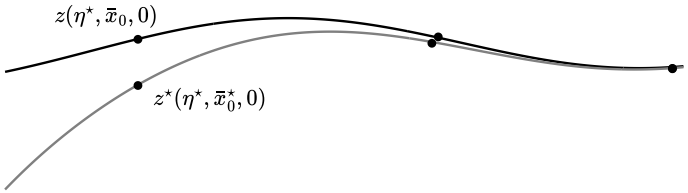

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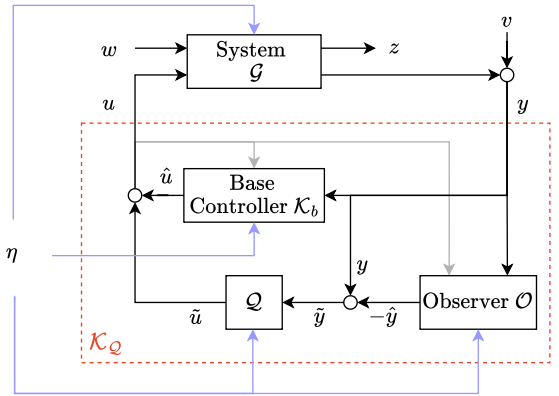

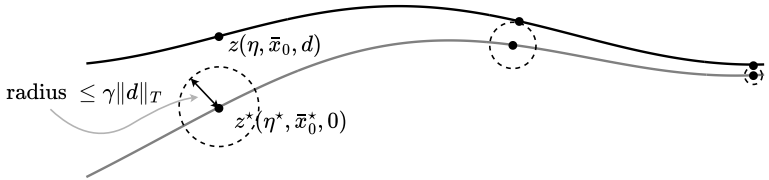

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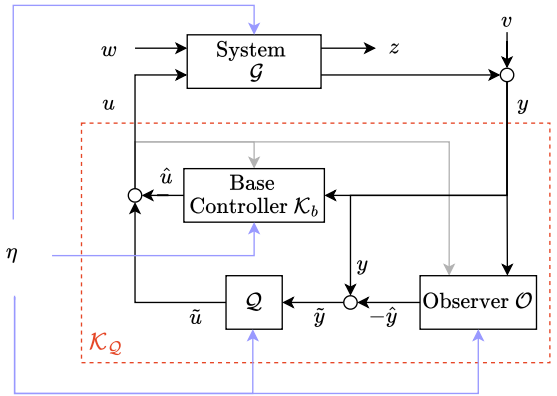

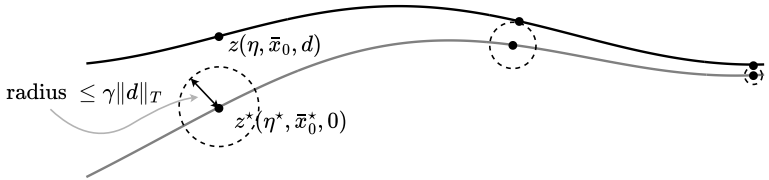

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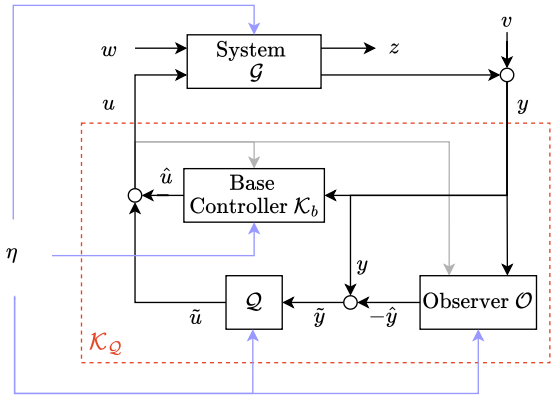
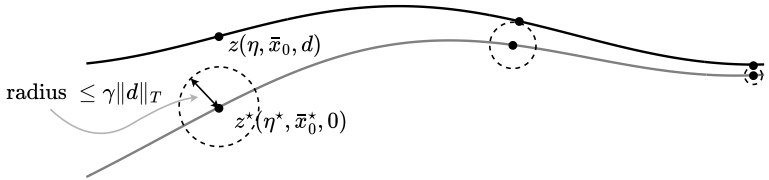
- Converse results: see thesis or paper[17] ☺





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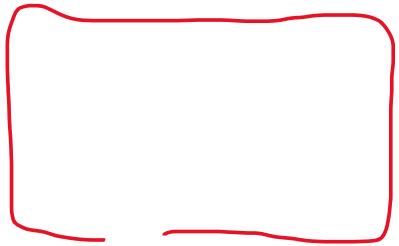

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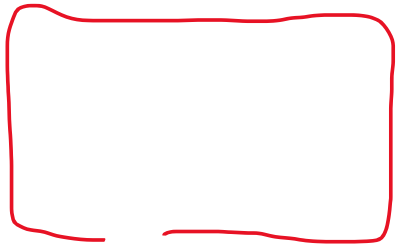

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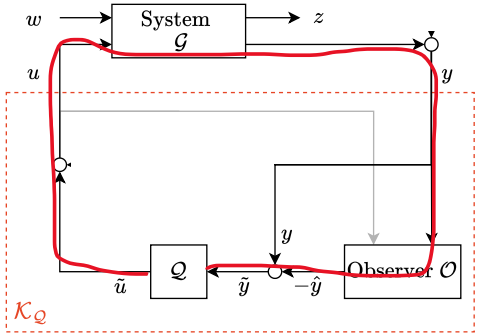

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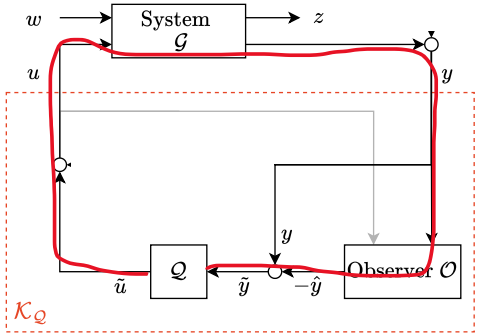

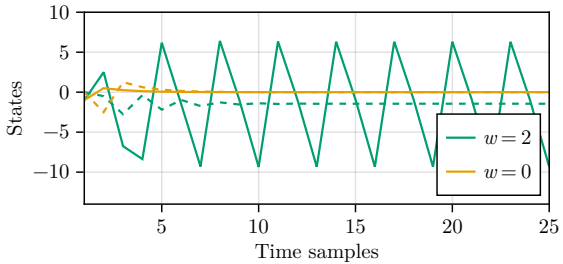

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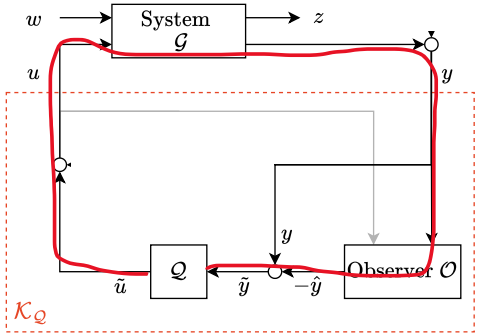

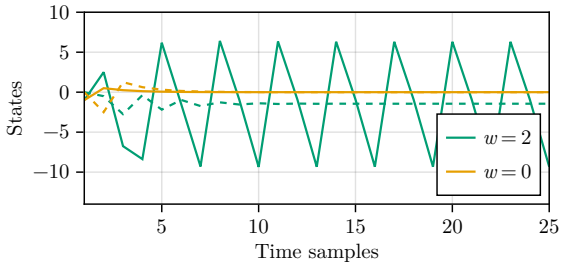

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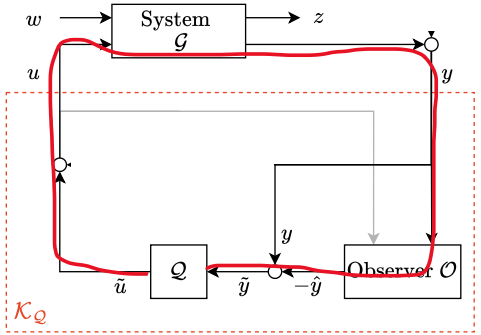

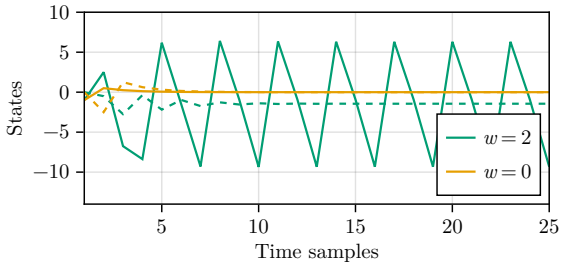

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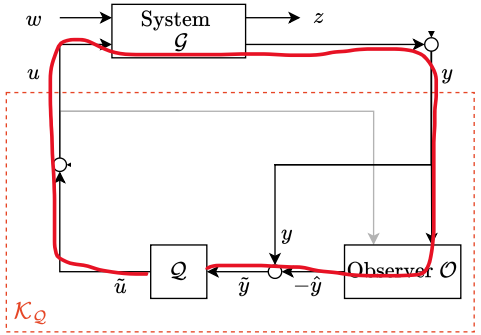

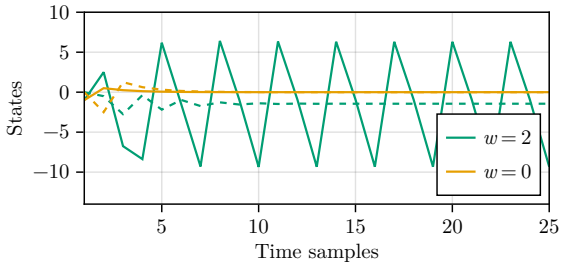

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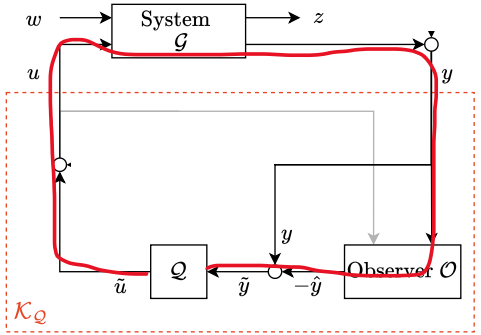

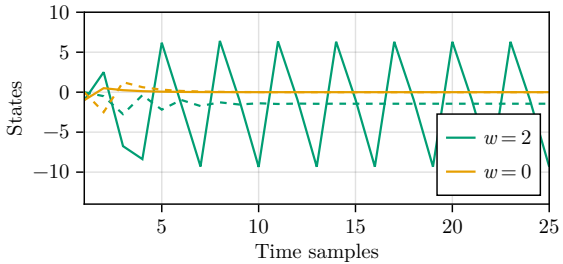

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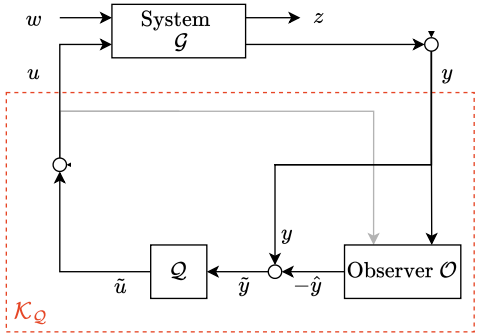

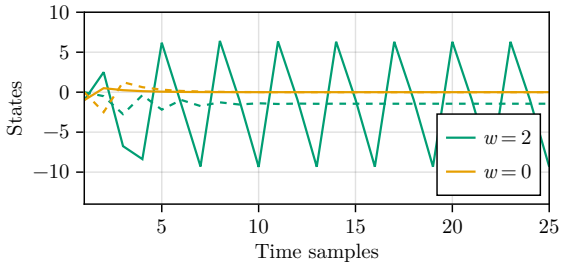

- Linear case: $\hat{x}$ terms cancel out

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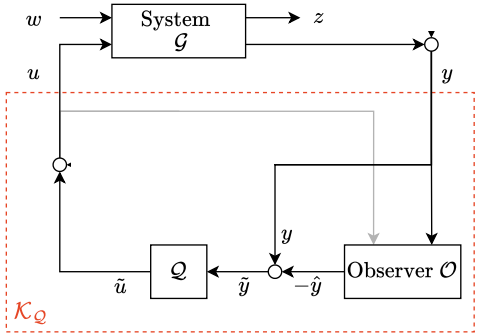

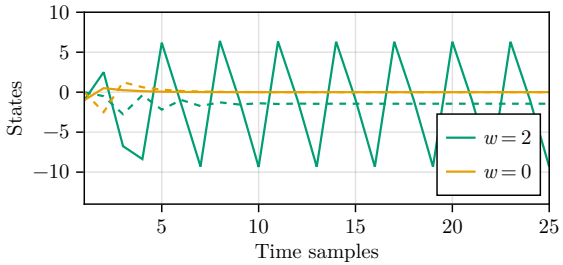

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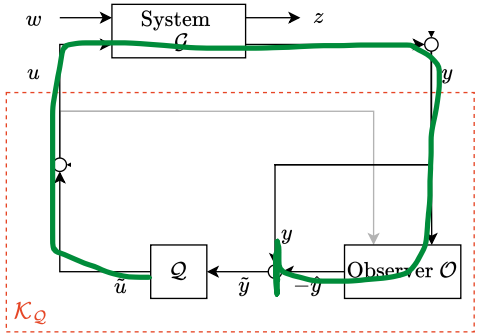

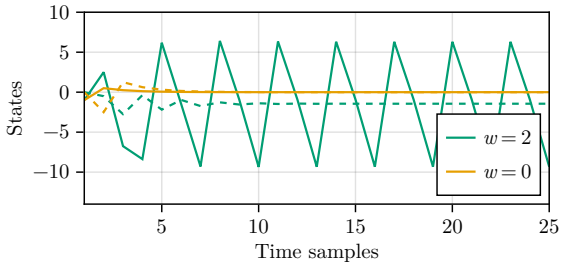

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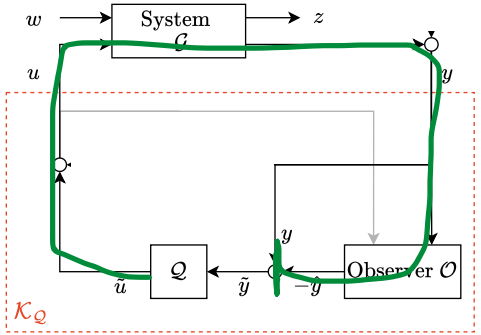

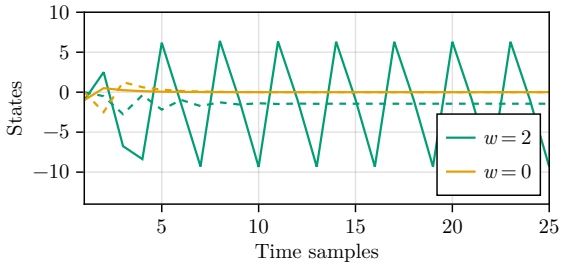

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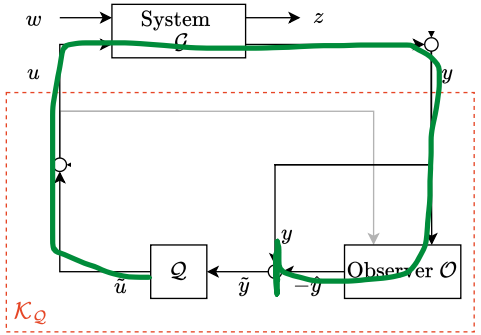

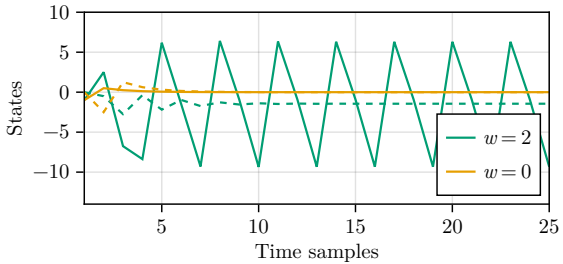

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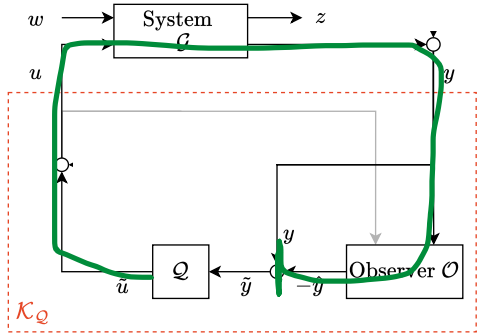

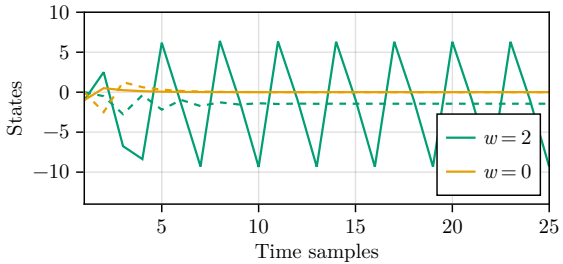

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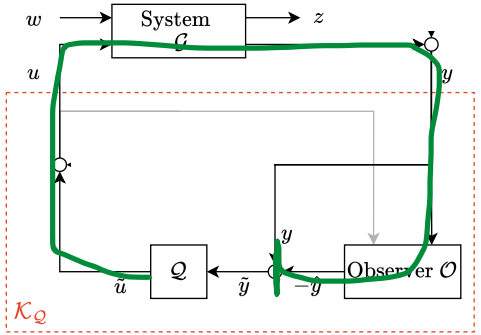

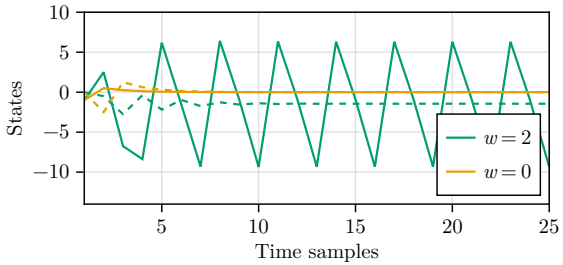

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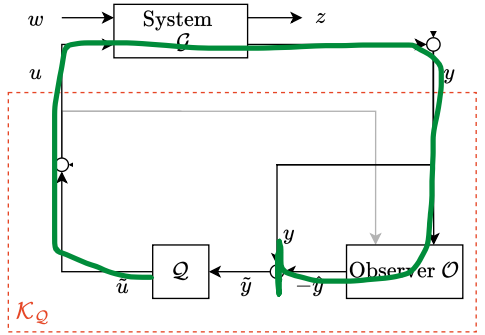

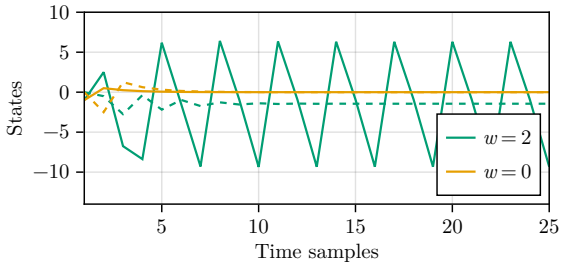

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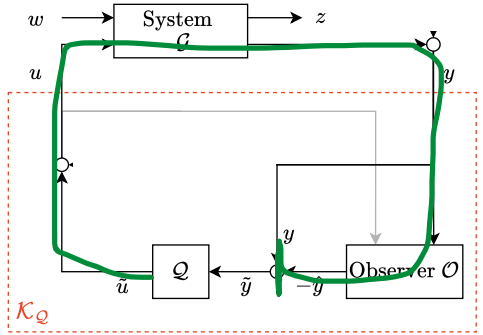

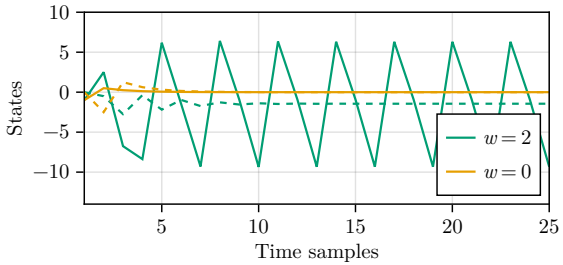

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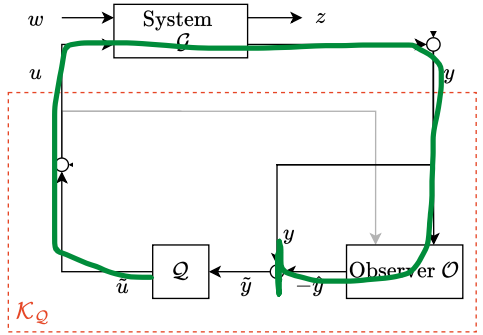

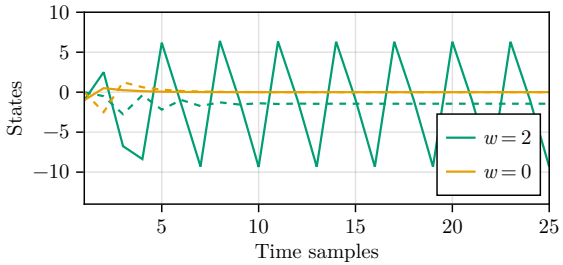

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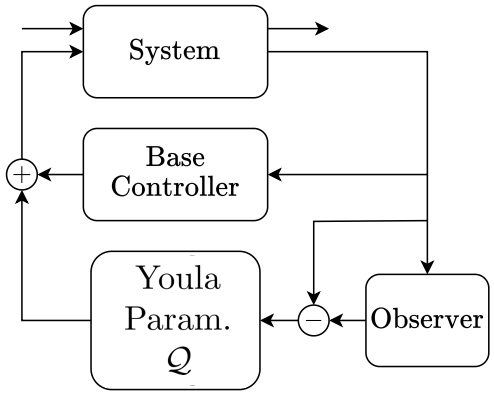
Outline

1. Motivation
2. Background: robust neural networks
3. Lipschitz-bounded policy networks
4. React to surprises with Youla-REN (Theory)
5. React to surprises with Youla-REN (Application)
6. Robust recurrent deep networks
7. Conclusions

The Youla-REN



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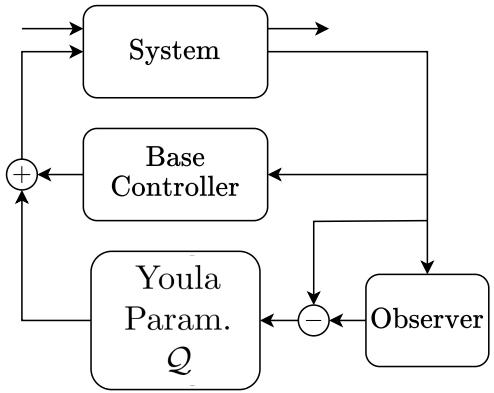

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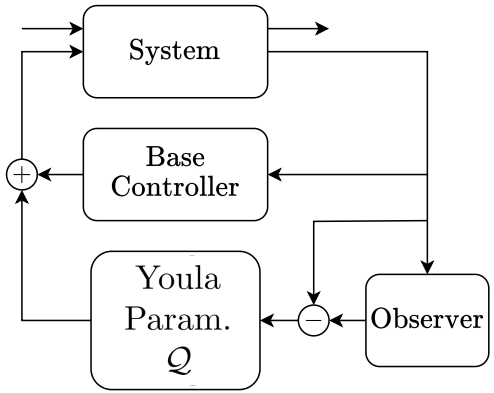

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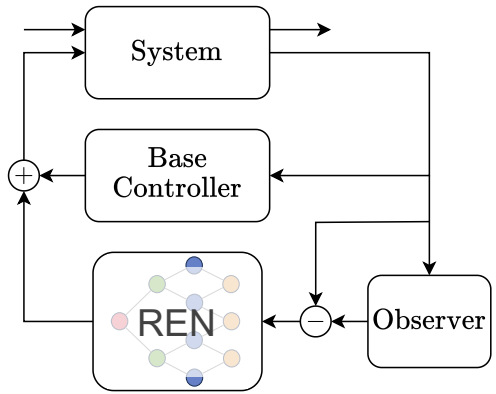

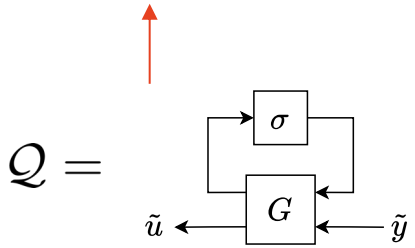

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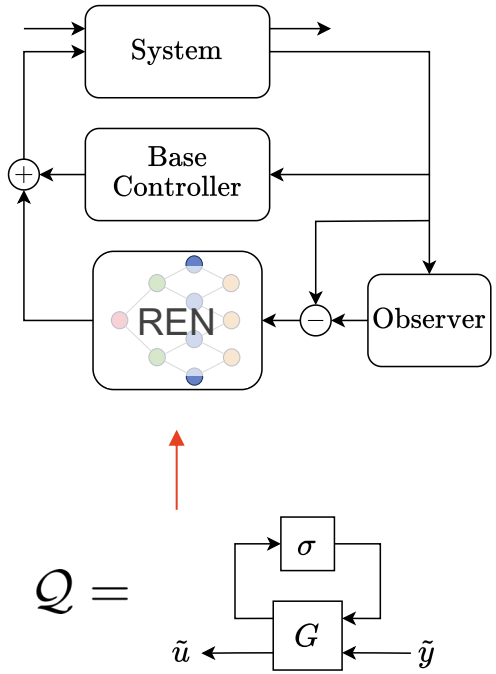

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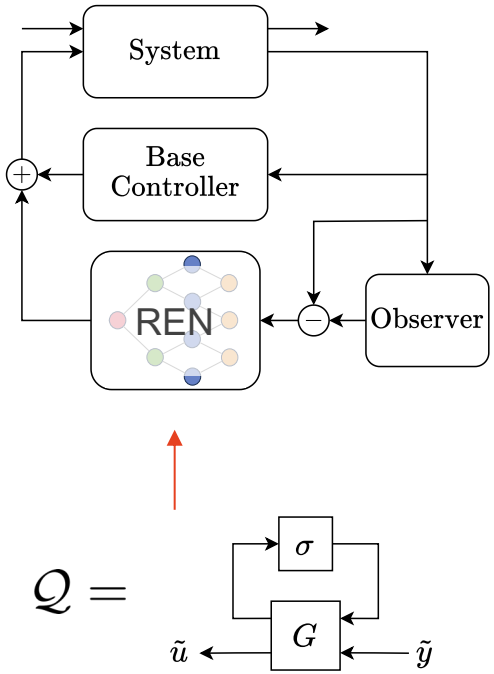

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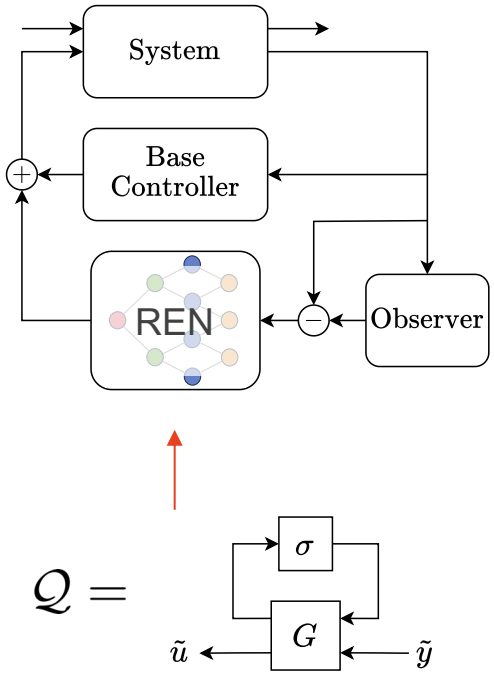
- Practical implementation:
  - Design base controller and observer first
  - If model uncertainty, restrict gain of REN



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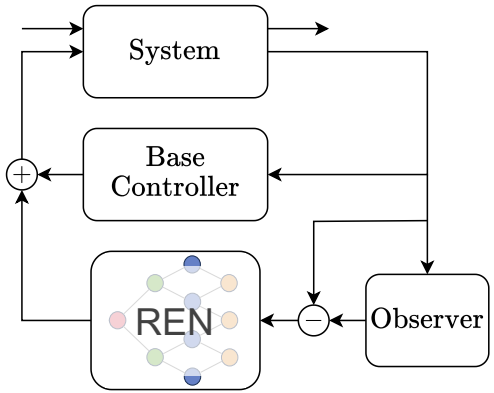

Learning with Stability: Example 1

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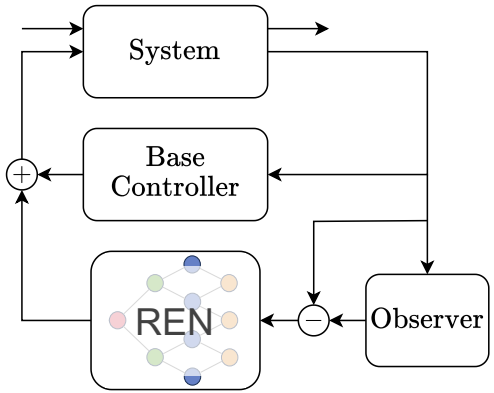

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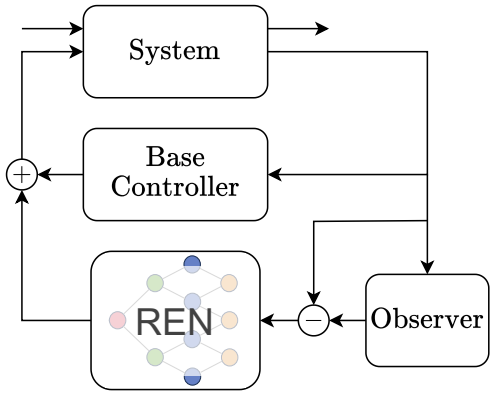

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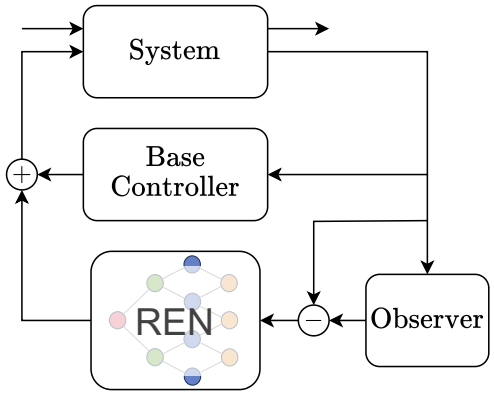
$$\min \textstyle\sum_t \left(|u_t| + 500 \max(|u_t| - u_{\max}, 0)\right)$$

- Optimal policy is u = 0 (closed-loop unstable)



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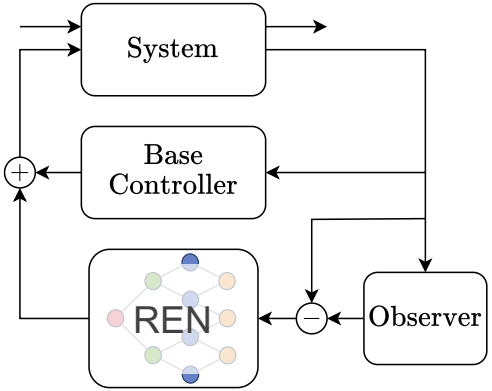
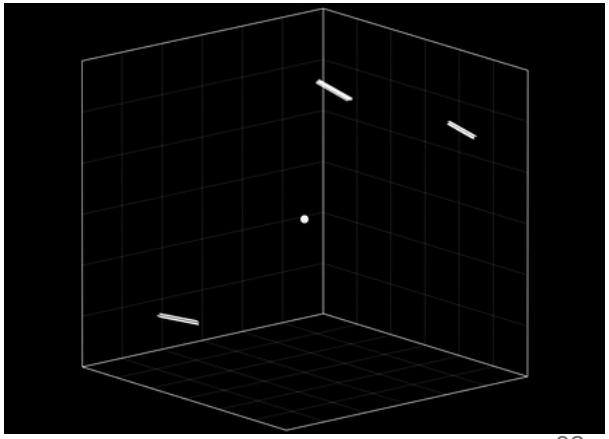
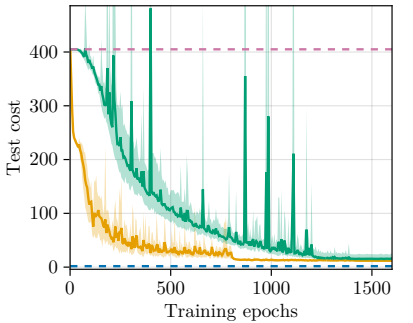
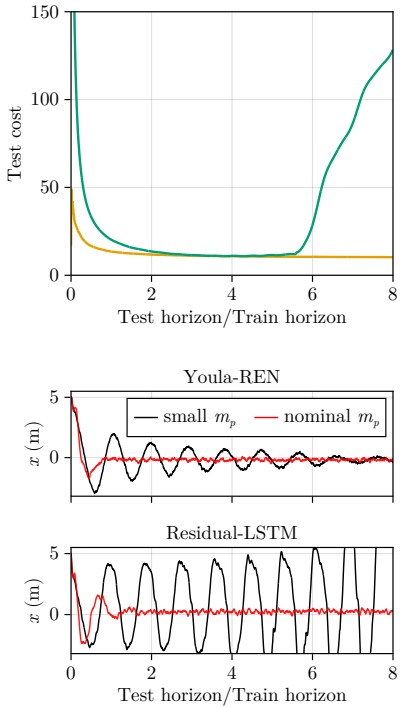




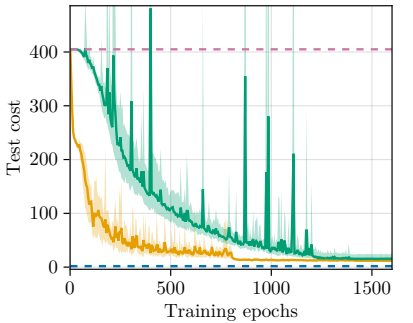
Learning with Stability: Example 2

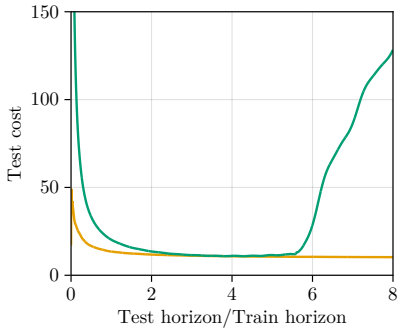


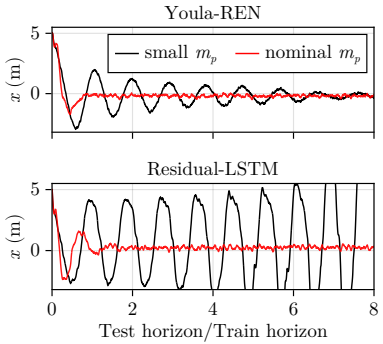


Learning with Stability: Example 2

- Linearised cart-pole, uncertain pole mass

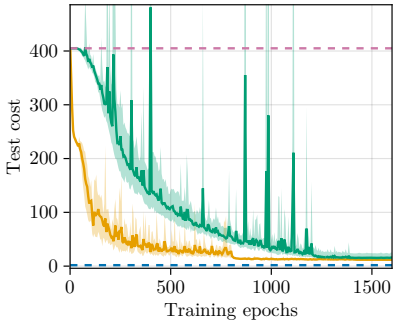


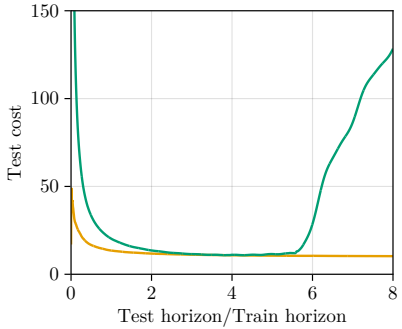


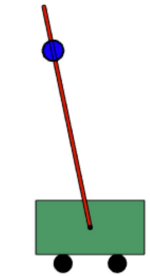


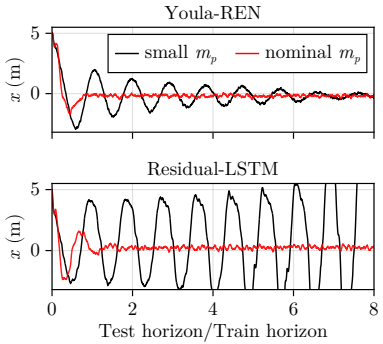
Learning with Stability: Example 2

- Linearised cart-pole, uncertain pole mass
- Youla-REN is closed-loop stable during training



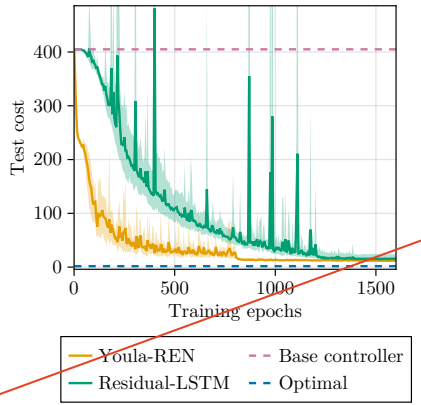


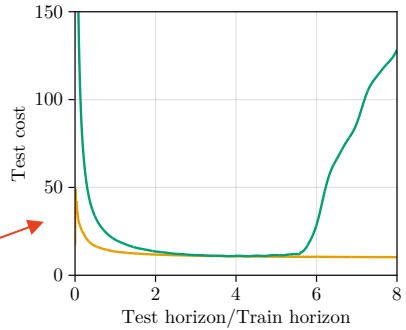


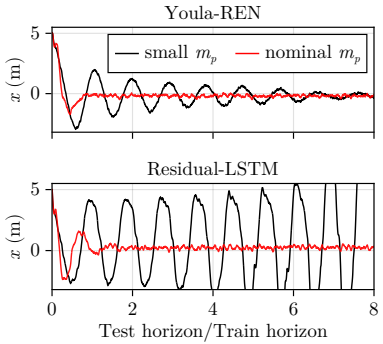


Learning with Stability: Example 2

- Linearised cart-pole, uncertain pole mass
- Youla-REN is closed-loop stable during training
- Youla-REN is closed-loop stable for long test horizons







Learning with Stability: Example 2

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- Youla-REN is closed-loop stable during training
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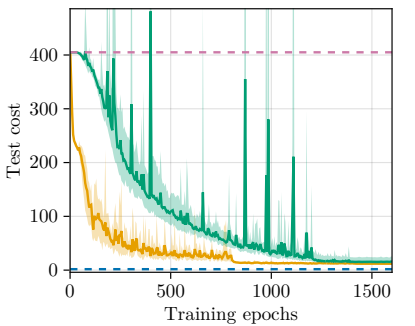
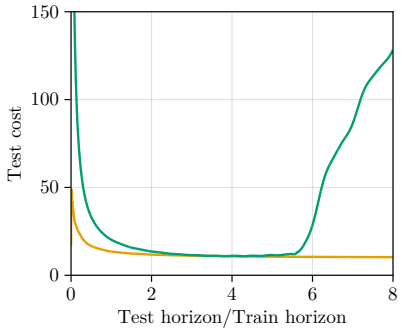
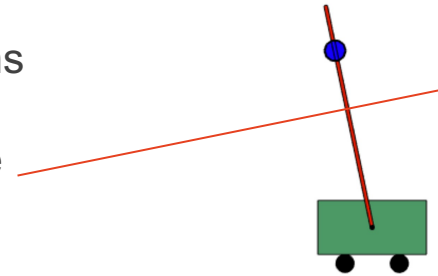
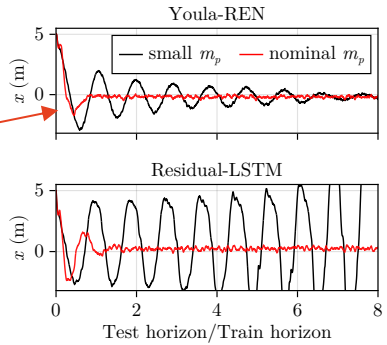
- Youla-REN "adapts" to the uncertain parameter









Outline

1. Motivation
2. Background: robust neural networks
3. Lipschitz-bounded policy networks
4. React to surprises with Youla-REN (Theory)
5. React to surprises with Youla-REN (Application)
6. Robust recurrent deep networks
7. Conclusions

Why Robust Recurrent Deep Network (R2DN)?

[21] Barbara et al., R2DN: Scalable Parametrization of Contracting and Lipschitz Recurrent Deep Networks, *arXiv* (2025).

Why Robust Recurrent Deep Network (R2DN)?

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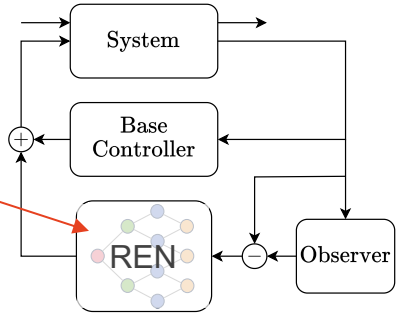

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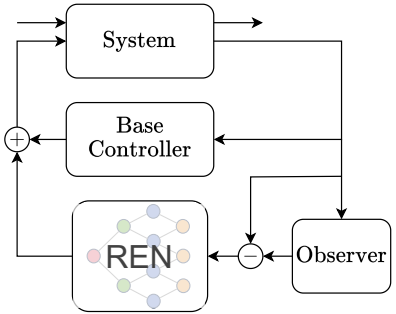
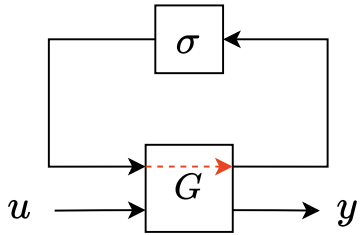


REN:



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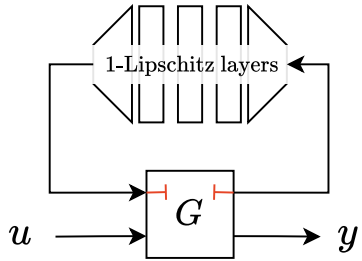

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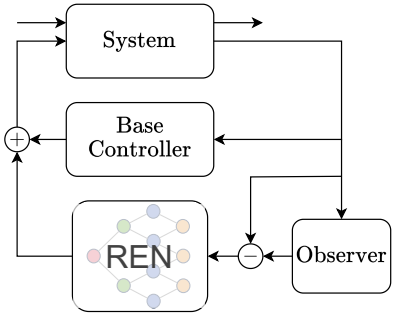

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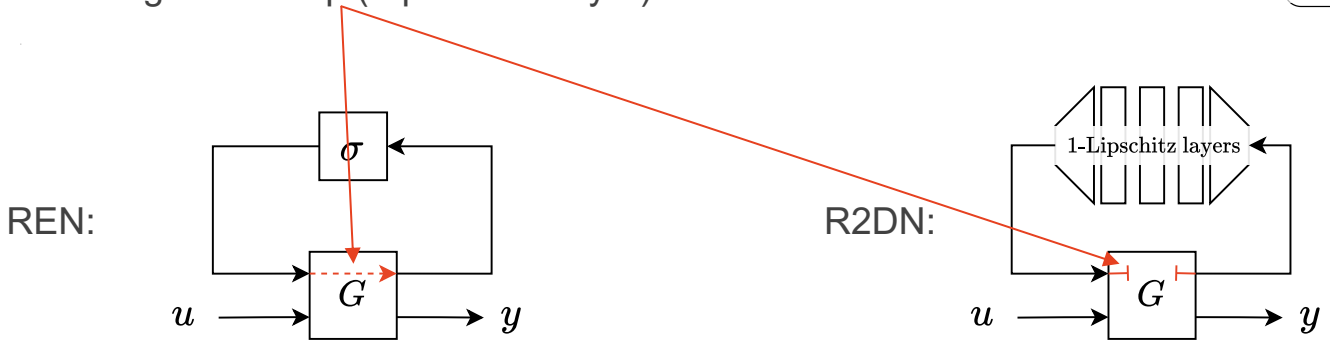
  - No algebraic loop (equilibrium layer)



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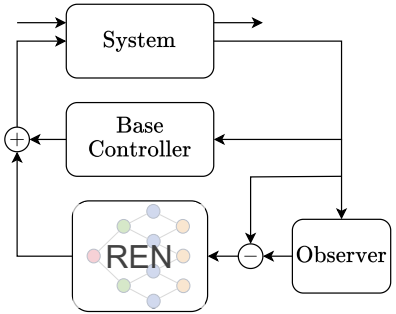

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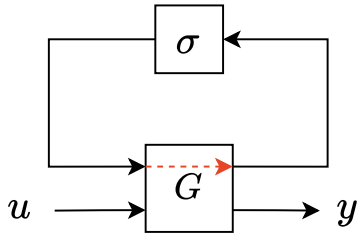

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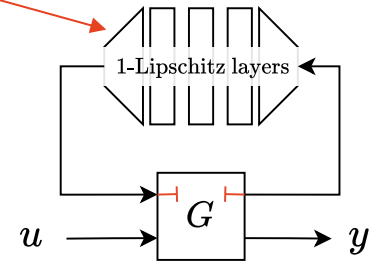

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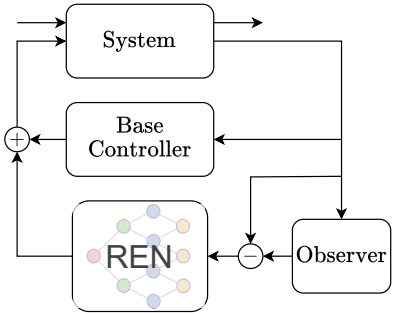

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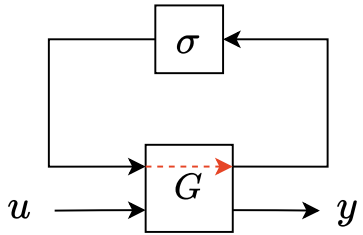

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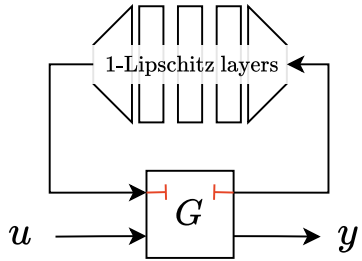

- Main contribution[21]: direct parametrisation of contracting & Lipschitz R2DNs

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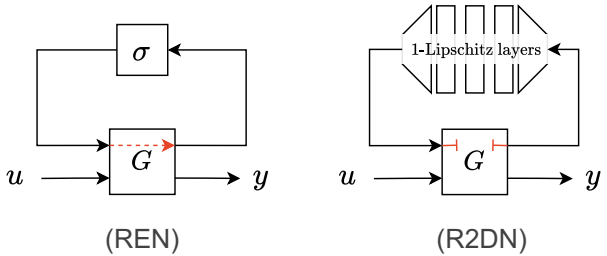
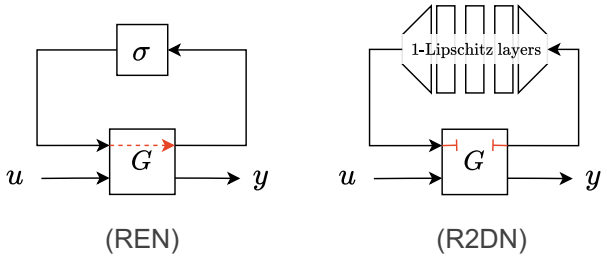
Better Scalability, Similar Performance



(REN) (R2DN)

Better Scalability, Similar Performance

- Main benefits:
  - No algebraic loop → efficient GPU computation
  - No algebraic loop → scalable
  - Any 1-Lipschitz network → flexible



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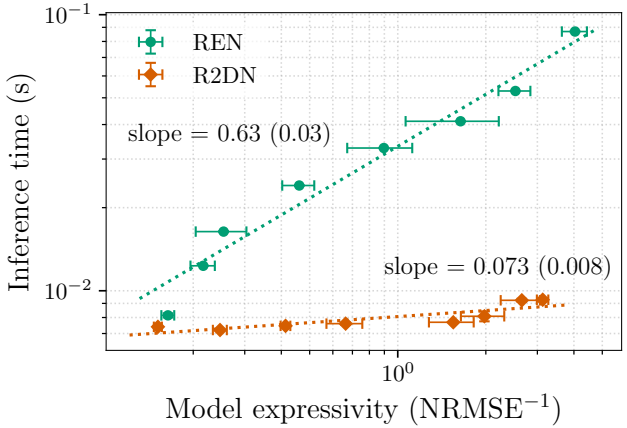

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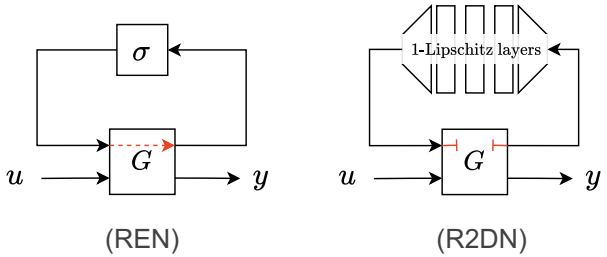
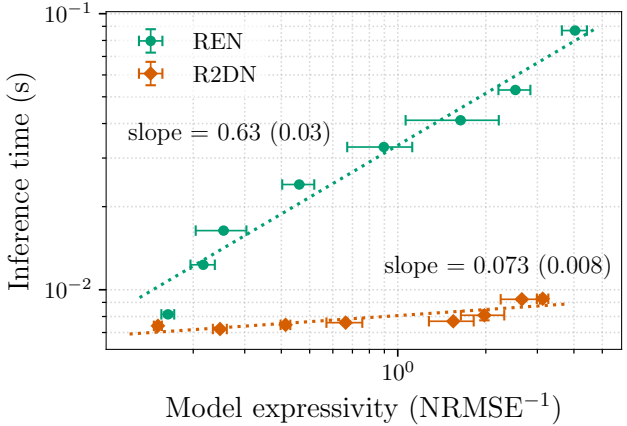
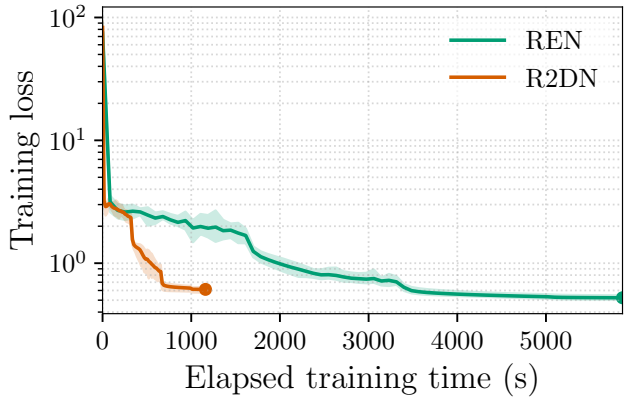


- Possible limitation (in theory): more conservative. In practice…





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Conclusions

- Robust NNs → learning-based control with stability & robustness guarantees

- The right choice of policy parametrisation and architecture enables:

    1. Closed-loop stability certificates
    2. Smooth, unconstrained parametrisations
    3. Non-restrictive parametrisations (under assumptions)

- “Plug-and-play” with standard deep RL and machine-learning pipelines

Future Work

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  - Develop more scalable parametrisations
  - Bigger library of behavioural properties & network architectures

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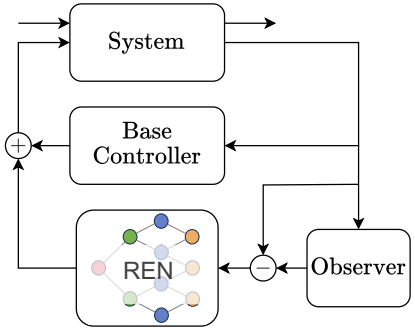
  - Designing good observers?



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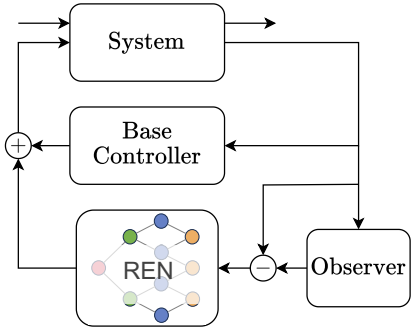

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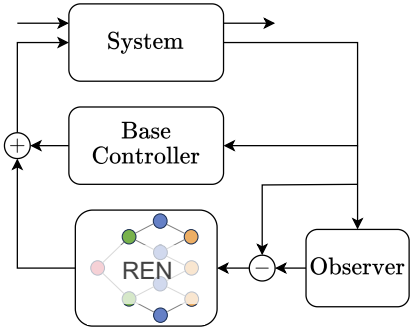
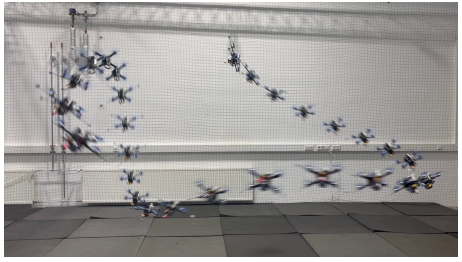
- Applications to robotics[22]





[22] Pries & Ryll, Learning Robust Agile Flight Control with Stability Guarantees, *arXiv* (2025).