Analytical Investigation of the Bouncing Mass Problem: Exploring Dynamics and Energy Conservation

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Problem statement / Introduction

In classical mechanics, understanding how systems respond to different forces is fundamental. The focus here lies on a specific scenario: a mass falling while attached to a vertical, weightless spring. This problem, known as the "Bouncing Mass Problem," involves determining the path of a 1 kg mass dropped from a height of 1.2 meters, as it interacts with a spring with a rest length of 1 meter and a stiffness of 100 N/m.

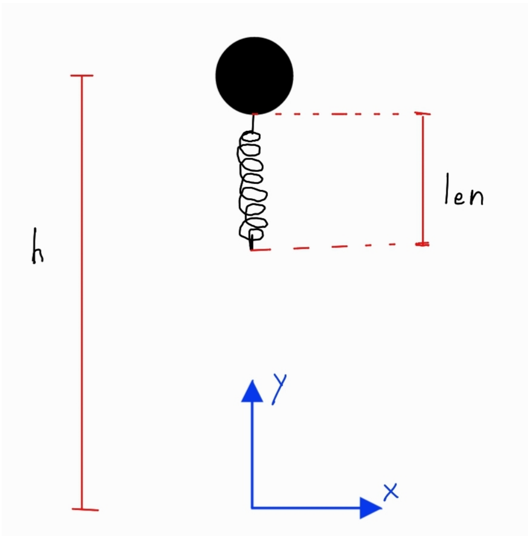

Figure 1: Illustration depicting the Bouncing Mass Problem - A "m" kg mass dropped from a height of "h" meters interacts with a vertical, weightless spring with a rest length of "len" meter and a stiffness of "k" N/m. The aim is to derive the equations of motion analytically, plot the trajectory using MATLAB, and analyze the energies involved to understand the dynamics of the system.

The objective is to find the equations of motion (EoM) describing the system's behavior. The steps involved include first deriving the EoM analytically, then plotting the analytical solutions using MATLAB, and finally analyzing the energies involved.

To solve this problem, MATLAB will be utilized to perform numerical computations and generate visualizations of the mass's trajectory. Through thorough analysis and discussion of the results, insights into the workings of this system will be sought.

Definitions

Table 1 gives an overview to the most important symbols, constants and variables

For all following calculations we assume that:

- Drag (resistance of air) is neglected
- The coordinate system is located at the bottom of 1 with the y-axis pointing

- Ball-movement is just considered in the y-direction (disturbances in horizontal direction e.g. due to wind are neglected)

Symbol	Property	Value	Unit
g	Gravitational acceleration	9.81	m/s^2
k	Spring stiffness	100	N/m
len	Spring resting length	1	m
h	Height of the drop	1.2	m
m	Mass of the ball	1	kg
y	Position of the ball	-	m
$\dot{y}$	Velocity of the ball	-	m/s
e_p	Gravitational potential energy	-	J
e_s	Elastic potential energy	-	J
e_k	Kinetic energy	-	J
t	Time	-	s

Table 1: Definitions of Symbols and Properties

Approach and Implementation

The problem can be divided in 2 different phases, the flight phase and the stance phase. in the first phase, the problem is simplified to a free falling mass, as the spring does not touch the ground and, therefore, has no relevance to the system. In the second phase, the spring touches the ground and start applying force to the mass. Each one of these phases have a different equation of motion. Furthermore, the flight phase was divide into the going up case (Case 1) and the going down case (Case 3). This division was used because the analytical solution is different for every initial condition. Therefore, to simulate the problem in Matlab it was necessary to derive each one of the equations for the different initial conditions. The cases and the equations will be described and shown in this section.

Case 1: Down Flight

In this case the ball is in a free fall dynamic motion, with a start at $h = 1.2m$.

Boundaries: $\dot{y} \leq 0$ and $y > 1m$

$$\ddot{y}(t) = -g \tag{1}$$

$$\dot{y}(t) = -gt + C_1 \tag{2}$$

$$y(t) = -\frac{gt^2}{2} + C_1 t + C_2 \tag{3}$$

Given the initial conditions:

$$\dot{y}(0) = 0 \rightarrow C_1 = 0 \tag{4}$$

$$y(0) = h \rightarrow C_2 = h \tag{5}$$

Therefore:

$$\dot{y}(t) = -gt \tag{6}$$

$$y(t) = -\frac{gt^2}{2} + h \tag{7}$$

Case 2: Stance

In this case, the spring touches the ground starts applying force to the system.

Boundaries: $y <= 1m$

$$m\ddot{y}(t)+ky = klen-mg \tag{8}$$

The EoM is a nonhomogeneous differential equation, therefore, to solve it analytically we have to consider a particular solution (y_p) and a homogeneous solution (y_h):

Starting by the particular solution, we can start assuming that the solution is constant.

$$y_p(t) = A$$

$$m\ddot{y}_p(t)+ky_p(t) = klen-mg \tag{9}$$
$$ky_p(t) = klen-mg \tag{10}$$
$$y_p(t) = len-\frac{mg}{k} \tag{11}$$

Now, for the homogeneous solution, it is assumed that:

$$y_h(t) = e^{\gamma t}$$

$$m\ddot{y}_h(t)+ky_h(t) = 0 \tag{12}$$
$$m\gamma^2 e + ke = 0 \tag{13}$$

Solving it for γ we can obtain the following result:

$$\gamma_1 = j\sqrt{\frac{k}{m}} \tag{14}$$
$$\gamma_2 = -j\sqrt{\frac{k}{m}} \tag{15}$$

The solution can be expressed in the following form:

$$y_h(t) = Ae^{\gamma_1 t} + Be^{\gamma_2 t}$$

It is defined than:

$$w = \sqrt{\frac{k}{m}}$$

And the expression of the homogeneous solution can be written as:

$$y_h(t) = C_1 cos(wt) + C_2 sin(wt); C_1 = A+B; C_2 = A-B \tag{16}$$

Finally, the full solution:

$$y(t) = y_h(t) + y_p(t) \tag{17}$$
$$y(t) = C_1 cos(wt) + C_2 sin(wt) + len - \frac{mg}{k} \tag{18}$$
$$\dot{y}(t) = -C_1 w sin(wt) + C_2 w cos(wt) \tag{19}$$

Given the following initial conditions for the third case:

$$y(0) = len \rightarrow C_1 = \frac{mg}{k} \tag{20}$$
$$\dot{y}(0) = -\sqrt{2(h-len)g} \rightarrow C_2 = -\frac{\sqrt{2(h-len)g}}{w} \tag{21}$$

Therefore:

$$y(t) = \frac{mg}{k}cos(wt) - \frac{\sqrt{2(h-len)g}}{w}sin(wt) + len - \frac{mg}{k} \tag{22}$$
$$\dot{y}(t) = -\frac{mg}{k}wsin(wt) - \sqrt{2(h-len)g}cos(wt) \tag{23}$$

Case 3: Up Flight

This case takes the same dynamics equations as the first one but with different initial conditions. The initial condition here is when the spring just lost contact with the ground.

Boundaries: $\dot{y} > 0$ and $y \geq 1$

$$\ddot{y}(t) = -g \tag{24}$$
$$\dot{y}(t) = -gt + C_1 \tag{25}$$
$$y(t) = -\frac{gt^2}{2} + C_1t + C_2 \tag{26}$$

Given the initial conditions:

$$\dot{y}(0) = \sqrt{2(h-len)g} \rightarrow C_1 = \sqrt{2(h-len)g} \tag{27}$$
$$y(0) = len \rightarrow C_2 = len \tag{28}$$

Therefore:

$$\dot{y}(t) = -gt + \sqrt{2(h-len)g} \tag{29}$$
$$y(t) = -\frac{gt^2}{2} + \sqrt{2(h-len)g}t + len \tag{30}$$

Matlab Implementation

As written before, each case is coupled to the other one in Matlab. This is done by resetting an auxiliary time variable each time the system changes phases, therefore the equations can be used following the given initial conditions, at time auxiliary equals zero. When the system leaves the current phase, the auxiliary time is than reset. The position and velocity is than stored in sequence. It was used a discretization of 10000 steps from 0s to 10s.

Energy of the system

With the systems position and velocity at each time step, it is possible to calculate it's energy. The energy of the system consist of gravitational potential energy (e_p), elastic potential energy (e_s) and kinetic energy (e_k).

$$e_p(t) = mgy(t) \tag{31}$$
$$e_s(t) = \frac{1}{2}k(len - y(t))^2 \tag{32}$$
$$e_k(t) = \frac{m}{\dot{y}^2(t)} \tag{33}$$
$$Total(t) = e_p(t) + e_s(t) + e_k(t) \tag{34}$$

Results

The results were plotted in Matlab and are shown in this section.

The figure 2 shows the position and the velocity of the mass for each time step of the simulation. According to the results, the system reached the maximum point equals to the start point at 1.2 m; the minimum point at 0.6808m; maximum velocity of 2.2105 m/s and minimum velocity of -2.2105 m/s.

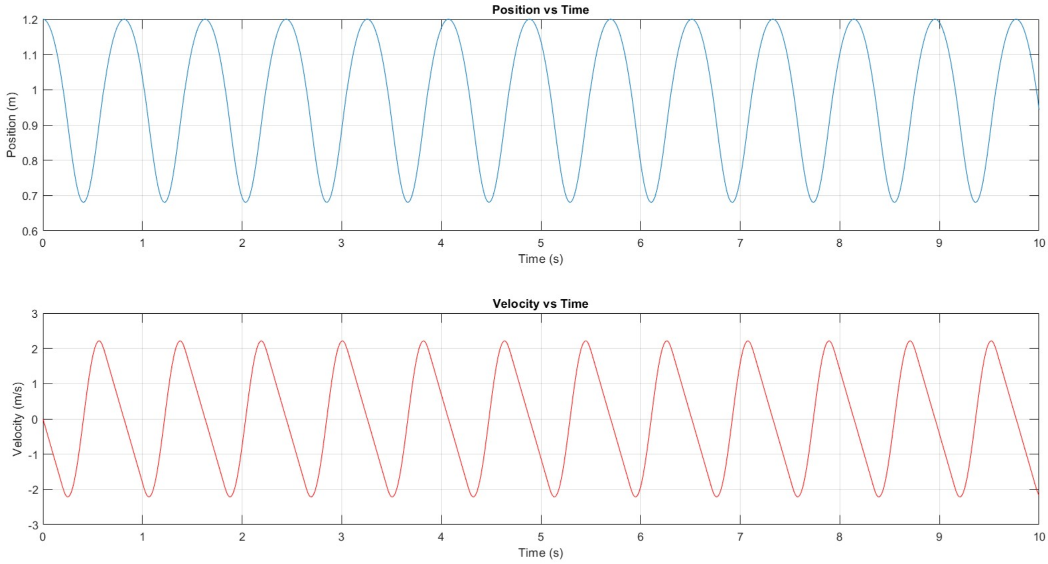

Figure 2: Graph of the position (m) and the velocity (m/s) of the mass in each time step

The energies are shown in figure 3. It can be seen that the total energy stays the same at every time step, reaching the value of 11.772 J.

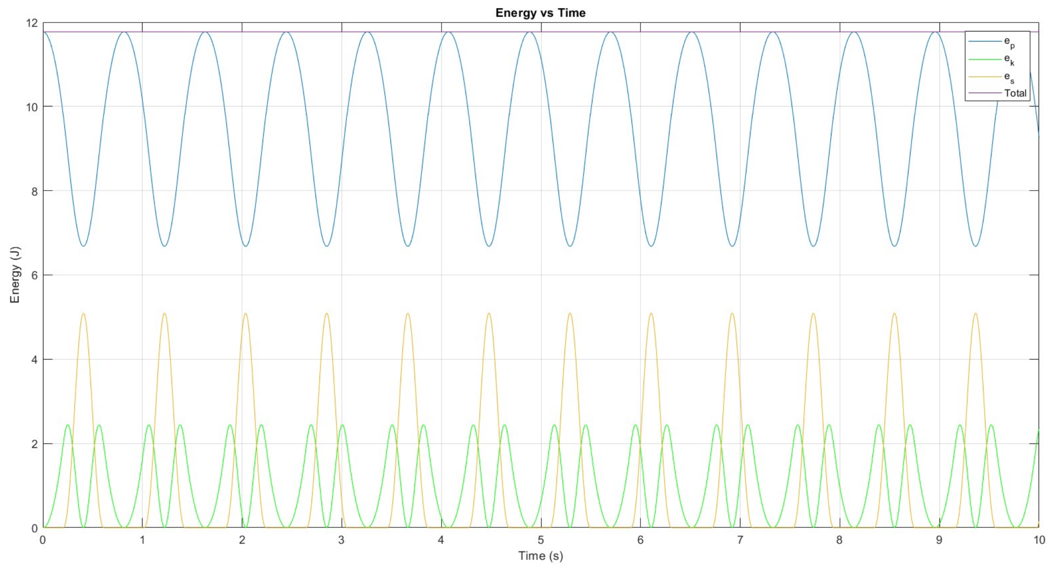

Figure 3: Graph of the energies of the system and the total energy in each time step. The total energy in purple; Kinetic energy in green; Elastic potential energy in yellow; Gravitational potential energy in blue.

Discussion

The results obtained from both the analytical solution and simulation appear to be satisfactory and align well with expectations derived from physical principles. Notably, the total energy of the system remained constant throughout, as anticipated for a system with only conservative forces.

Regarding the method chosen to tackle the problem in MATLAB, overall, it proved effective. However, an initial challenge arose concerning the coupling of different phases within the simulation. This obstacle was successfully addressed by increasing the number of time steps from 1000 to 10000. This adjustment allowed for a more precise representation of the system's dynamics.

While solving analytical solutions by hand and plotting them in MATLAB may not be the most efficient method for simulating this particular problem, it nonetheless served as a valuable exercise. Despite MATLAB's capability to numerically solve the equations of motion, engaging in manual derivations and plotting offered insights into the underlying mathematical principles governing the system. Moreover, it provided an opportunity to reinforce problem-solving skills and deepen understanding through hands-on practice.

In summary, the combination of analytical and numerical approaches employed in this study yielded comprehensive insights into the behavior of the system. The successful validation of results against physical expectations and the resolution of computational challenges highlight the robustness of our methodology. Moving forward, further exploration could focus on refining the numerical simulation techniques to optimize efficiency while maintaining accuracy. Additionally, future studies may benefit from investigating more complex systems or exploring alternative numerical methods to enhance computational efficiency.

AI usage declaration

tools used

ChatGPT

used prompts

I have to write an essay based on the following problem: (problem copied from moodle) The first part of my essay is problem statement and introduction, please write that for me.

Now I need you to write a discussion for that essay, I want you to note the following points in it: (3 points that I wanted to talk about)

Write a title for my essay.

Prompts asking for latex content and tips

comments

It was really useful for writing with latex, searching for some expressions. Also useful for writing the discussion given in the prompt the detailed information of what I wanted to write.