THE CHARACTERIZATION OF INFINITE EULERIAN GRAPHS, A SHORT AND COMPUTABLE PROOF

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Abstract. In this paper we present a short proof of a theorem by Erdős, Grünwald and Weiszfeld on the characterization of infinite graphs which admit infinite Eulerian trails. In addition, we extend this result with a characterization of which finite trails can be extended to infinite Eulerian trails.

Our proof is computable and yields an effective version of this theorem. This exhibits stark contrast with other classical results in the theory of infinite graphs which are not effective.

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1. Introduction

Eulerian trails -which arose from the Königsberg bridge problem- are those trails which visit every edge of a graph exactly once. In contrast to Hamiltonian paths and other graph theoretic objects, Eulerian trails are very tractable from an algorithmic point of view. In this paper we shall see that this tractability still holds for infinite trails on infinite graphs.

Let us recall the following classical result, which provides a criterion on the existence of closed and non closed Eulerian trails for finite graphs.

Theorem (Euler's theorem)**.** *Let Γ be a finite graph. Then Γ admits a closed Eulerian trail if and only if it is connected and all vertices have even degree.*

Moreover, Γ admits an Eulerian trail from u to $v \neq u$ if and only if it is connected and u, v are the only vertices with odd degree.

König asked for a generalization of this result to infinite graphs and infinite Eulerian trails, and a relatively simple characterization was announced in 1936 by Erdős, Grünwald and Weiszfeld. This theorem completely characterizes which graphs admit one-sided infinite Eulerian trails, whose vertex set is indexed by $\mathbb{N}$, and two-sided infinite Eulerian trails, whose vertex set is indexed by $\mathbb{Z}$ (an example is shown in Figure 1 on page 4). The graphs under consideration are undirected,

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multiple edges with the same endpoints are allowed (multigraphs), and vertices may have infinite degree.

This theorem anticipated the important concept of *ends*, defined later in [4, 5]. In this paper we will use the following definition. The **number of ends** of a graph Γ is the supremum of the number of infinite connected components of $\Gamma - E$, where E ranges over all finite sets of edges of Γ. There are other definitions of ends for infinite graphs. They are all equivalent as long as we restrict ourselves to locally finite graphs, a discussion on this topic can be found in [2]. The definition considered here captures the right notion for our discussion on infinite trails beyond locally finite graphs. For example, it makes clear that a graph Γ which admits a one-sided infinite Eulerian trail T must have one end. Let us sketch an argument here.

Let E be a finite set of edges in Γ. We need to check that $\Gamma - E$ has only one infinite connected component, for which we take e be the first edge visited by T after visiting all edges in E. Now the set of edges visited by T after e induce an infinite connected component in $\Gamma - E$, as T shows how to join any pair of vertices there. This must be the only infinite connected component in $\Gamma - E$ because there are only finitely many edges visited by T before e. This finishes the argument.

Let us review other evident restrictions on a graph which admits a one-sided infinite Eulerian trail. For example, it is clear that the graph must be connected, and its edge set must be countably infinite. In addition the vertex where the trail starts must have either odd or infinite degree, and the remaining vertices must have even or infinite degree. This list of conditions, which we call $\mathscr{E}_1$, in fact constitutes a characterization.

Theorem 1.1 ([3])**.** *A graph Γ admits a one-sided infinite Eulerian trail if and only if it satisfies the following set of conditions, called $\mathscr{E}_1$.*

- *$E(\Gamma)$ is countable, infinite, and Γ is connected.*
- *There exists at most one vertex with odd degree, and there exists at least one vertex which has odd or infinite degree.*
- *Γ has one end.*

Let us now review the case of two-sided infinite Eulerian trails. It is clear that a graph which admits a two-sided infinite Eulerian trail must be connected, its edge set must be countably infinite, and every vertex must have even or infinite degree. Moreover it must have one or two ends, the argument being similar to the one provided above. A less evident observation is the following: if we remove a finite set of edges E which induces an even[1] subgraph, then there remains only one infinite connected component. This is an easy parity argument which we defer for now. This list of conditions, which we call $\mathscr{E}_2$, in fact constitutes a characterization.

Theorem 1.2 ([3])**.** *A graph Γ admits a two-sided infinite Eulerian trail if and only if it satisfies the following set of conditions, called $\mathscr{E}_2$.*

- *$E(\Gamma)$ is countable, infinite, and Γ is connected.*
- *The degree of each vertex is infinite or even.*
- *Γ has one or two ends. Moreover, If E is a set of edges which induces an even subgraph, then $\Gamma - E$ has one infinite connected component.*

The main contribution of this paper is a short proof of Theorem 1.1 and Theorem 1.2. We extend these results by providing a characterization of which trails can be extended to one-sided or two-sided infinite Eulerian trails (Corollary 3.7 and Corollary 3.14). As consequence, infinite Eulerian trails can be defined locally, by successively extending finite trails, and we have a certain level of control in this process.

[1] A finite graph all of whose vertices have even degree.

Our proof is computable, we exhibit algorithms to compute infinite Eulerian trails on graphs satisfying $\mathscr{E}_1$ or $\mathscr{E}_2$. This computation occurs *locally*, by which we mean that at each step the algorithm works on a finite subgraph of Γ. In order to deal with algorithms on infinite graphs we rely on the notion of highly computable graph, which in simple words is an infinite graph Γ with an algorithm that can compute finite subgraphs of Γ of any desired size. The formal statement is the following.

Theorem 1.3. *If a highly computable graph satisfies $\mathscr{E}_1$ (respectively $\mathscr{E}_2$) then it admits a computable one-sided (respectively two-sided) infinite Eulerian trail.*

Thus a highly computable graph admits a one-sided (respectively two-sided) infinite Eulerian trail if and only if it admits a computable one.

This is remarkable in comparison with other results in the theory of infinite graphs for which a computable version is not possible. One such result is König's infinity lemma, which asserts that an infinite and locally finite graph has a one-sided infinite path. This result is not effective, in the sense that there are highly computable graphs satisfying the hypotheses and which admit no computable infinite path [6]. Therefore it is not possible to define a one-sided infinite path using local information in a computable manner. Other results which are not effective include Hall's matching theorem for infinite graphs [7], Ramsey's theorem [10], and 3-colorings [8]. In our result the hypothesis of highly computable graph cannot be relaxed to the weaker notion of computable graph, as we will show with an example.

There are some subtleties regarding uniformity in Theorem 1.3, related to ends. For one ended graphs satisfying $\mathscr{E}_1$ or $\mathscr{E}_2$ we can prove a stronger result, namely that it is algorithmically decidable whether a finite trail can be extended to an infinite Eulerian trail (see Proposition 4.4). A consequence of this is that the corresponding infinite trails can be computed *upon* a description of the graph, or in more informal words, a single algorithm works for every graph. For two-ended graphs satisfying $\mathscr{E}_2$ the situation is more subtle, and we only prove the *existence* of a computable two-sided infinite Eulerian trail.

We end this introduction by mentioning an application of the ideas involved in the proof presented here. A recent employment of Theorem 1.2 occurred in the paper [9], where Seward proved that every connected graph with one or two ends and with vertex degree uniformly bounded by D, has an n-th power[2] which admits a two-sided infinite Hamiltonian path. In that proof the number n depends linearly on D. This has interesting consequences in geometric group theory, it shows that every finitely generated group with one or two ends has a Cayley graph which admits a two-sided infinite Hamiltonian path. Using the same ideas presented here we were able to improve Seward's result. On the one hand we showed that the number n can be taken to be equal to 3, while on the other hand we weakened the hypothesis by allowing vertices to have unbounded but finite degree. These results and their consequences for Cayley graphs are discussed in [1].

2. Preliminaries

Throughout this paper we deal with finite and infinite undirected graphs, where two vertices may be joined by multiple edges and self loops are allowed. The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. We will assume that the edge and vertex set of a graph are disjoint.

Each edge **joins** a pair of vertices. In the case of a simple graph, we may identify edges with unordered pairs of vertices. For example, we denote by $[\![a, b]\!]$

[2]The n-th power of a graph Γ is the graph with the same set of vertices, and where a pair of vertices is connected by an edge if their distance in Γ is at most n.

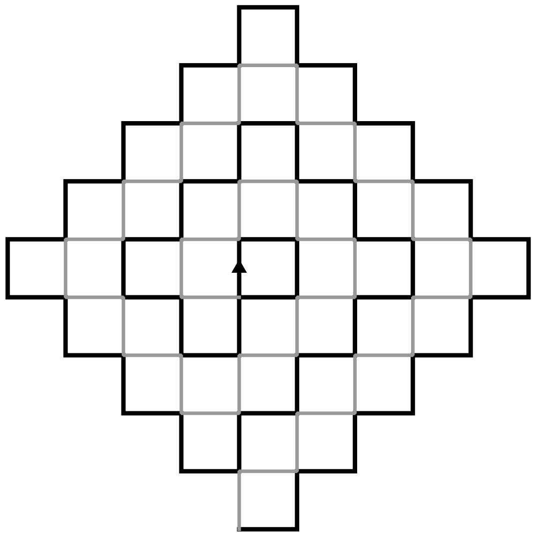

FIGURE 1. A finite portion of two-sided infinite Eulerian trail on the infinite grid graph. One edge e has been distinguished in the picture with an arrow. The edges visited by the trail after e have been colored black, while edges visited by the trail before e have been colored gray.

the graph with vertex set $\{a, \ldots, b\} \subset \mathbb{Z}$, and with edges of the form $\{c, c+1\}$ for $c \in \{a, \ldots, b-1\}$. The graphs $[\![\mathbb{N}]\!]$ and $[\![\mathbb{Z}]\!]$ are defined in a similar manner.

When two vertices x, y are joined by an edge e, they are called **adjacent** or **neighbors**, and x, e are said to be **incident**. An edge joining a vertex to itself is a **loop**. The **degree** of a vertex v in Γ, denoted $\deg_\Gamma(v)$, is the amount of edges incident to v, where loops are counted twice.

A graph homomorphism $f : \Gamma' \to \Gamma$ is a function $f : V(\Gamma) \cup E(\Gamma) \to V(\Gamma') \cup E(\Gamma')$ which sends vertices to vertices, edges to edges, and is compatible with the incidence relation.

It will be convenient for us to define trails as graph homomorphisms. A **trail** (resp. **one-sided infinite trail**, resp. **two-sided infinite trail**) on Γ is a graph homomorphism $t : [\![a, b]\!] \to \Gamma$ (resp. $t : [\![\mathbb{N}]\!] \to \Gamma$, resp. $t : [\![\mathbb{Z}]\!] \to \Gamma$) which does not repeat edges. We say that t **visits** the vertices and edges in its image, and we call it **Eulerian** when it visits every edge of Γ exactly once.

Note that a trail is finite by definition, but we may still write finite trail to emphasize this. When t is a finite trail we say that $t(a)$ is its **initial vertex** and $t(b)$ is its **final vertex**. We say that t joins $t(a)$ to $t(b)$, and we call it **closed** when $t(a) = t(b)$.

It will also be convenient to induce subgraphs from sets of edges. The **induced subgraph** by a set of edges $E \subset E(\Gamma)$ is denoted $\Gamma[E]$, its edge set is E, and its vertex set is the set of all vertices incident some edge in E. For a set of edges $E \subset E(\Gamma)$, $\Gamma - E$ denotes the subgraph induced by the set of edges $E(\Gamma) - E$. That is, we erase from Γ all edges in E, and then all vertices that were left without incident edges. Given a trail t we denote by $E(t)$ the set of edges visited by t, and we denote by $\Gamma - t$ the induced subgraph $\Gamma - E(t)$. We say that $\Gamma - t$ is obtained by **removing** t from Γ.

A graph is said to be **finite** if its edge set is finite, **even** if every vertex has even degree, **locally finite** if every vertex has finite degree, and **connected** if any two vertices are joined by a trail. A **connected component** in Γ is a connected subgraph of Γ which is maximal for the subgraph relation, and the vertex set of a connected graph is a metric space with the trail-length distance, denoted d_Γ.

3. The proof

3.1. **Some considerations.** In this subsection we introduce some convenient terminology and prove an useful lemma.

The following notion will be useful to define infinite trails from finite trails. Let $t : [\![a, b]\!] \to \Gamma$ and $s : [\![c, d]\!] \to \Gamma$ be edge disjoint trails. If the final vertex of t is also the initial vertex of s, the **concatenation of** s **at the right** of t is the trail whose domain is $[\![a, b + d - c]\!]$, whose restriction of r to $[\![a, b]\!]$ equals t, and the restriction of r to $[\![b, b + d - c]\!]$ follows the same path as s, but with the domain shifted. If the final vertex of s coincides with the initial vertex of t, we define the **concatenation of** s **at the left of** t as the trail whose domain is $[\![a - (d - c), b]\!]$, which on $[\![a - (d - c), a]\!]$ follows the route of s but with the domain shifted, and on $[\![a, b]\!]$ follows the route of t.

We say that a trail **extends** t if its restriction to the domain of t is equal to t. For example, if we concatenate a trail at the right or left of t, we obtain a trail which extends t. Finally we define the **inverse** of t, denoted $-t$, as the trail with domain $[\![-b, -a]\!]$ and which visits the vertices and edges visited by t in but in reverse order.

We now proceed to prove the following result. This will be used repeteadly along our proofs.

Lemma 3.1. *Let Γ be a connected graph and let t be a trail on Γ such that every vertex different from the initial or final vertex of t has even or infinite degree in Γ. Then there is a trail which visits all vertices and edges visited by t, with the same initial and final vertices as t, and whose remotion from Γ leaves no finite connected component.*

Proof. In this proof we only need to do some parity verifications in order to apply Euler's theorem. The only care required is because we do not assume Γ to be locally finite.

Let Γ' be the subgraph of Γ induced by the edges visited by t and the edges in finite connected components of $\Gamma - t$. We will prove that this graph admits an Eulerian trail from $t(0)$ to $t(b)$. It is clear that Γ' is connected. To see that it is finite, note that each vertex visited by t belongs to at most one connected component of $\Gamma - t$, and there is at most one finite connected component of $\Gamma - t$ for each vertex visited by t.

We now show that all vertices in Γ' different from $t(0), t(b)$ have even degree in Γ', that $t(0), t(b)$ have both even degree in Γ' if they are equal, and that $t(0), t(b)$ have both odd degree in Γ' if they are different. This proves the claim by Euler's theorem.

Let v be a vertex in Γ' not visited by t, so its degree in Γ equals its degree in Γ'. The fact that Γ' is finite shows $\deg_\Gamma v$ is finite, and then the hypothesis on Γ implies that v has even degree in Γ'. Now let v be a vertex visited by t, but different from $t(0)$ and $t(b)$. We verify that $\deg_{\Gamma'}(v)$ is finite and even. Indeed, observe that there is at most one (possibly empty) connected component of $\Gamma - t$ containing v. If this connected component is infinite, then there is no finite connected component of $\Gamma - t$ containing v, and $\deg_{\Gamma'}(v) = \deg_t(v)$, a finite and even number. If this connected component is finite then $\deg_\Gamma(v)$ is finite and then even. From this it follows that $\deg_{\Gamma'}(v)$ is also even.

We now adress $t(0)$ and $t(b)$. If our claim on the degrees of $t(0)$ and $t(b)$ fails, then Γ' would have exactly one vertex with odd degree. This contradicts the handshakes lemma, which asserts that $\sum_{v \in V(\Gamma')} \deg_{\Gamma'}(v)$ equals $2|E(\Gamma')|$, and in particular is an even number. This concludes the proof. □

3.2. **The case of one-sided infinite trails.** In this subsection we prove Theorem 1.1. We have verified in the introduction that the conditions $\mathscr{E}_1$ are necessary for a graph to admit a one-sided infinite Eulerian trail, so we need to prove that they are sufficient.

Let us define what is a **distinguished** vertex in a graph Γ satisfying $\mathscr{E}_1$. If Γ has one vertex with odd degree, then this is its only distinguished vertex. If Γ has no vertex with odd degree then all vertices with infinite degree are distinguished. Now we can give the following definition, which will be the base of our proof.

Definition 3.2. Let Γ be a graph satisfying $\mathscr{E}_1$. We say that a trail t is **right-extensible** in Γ if the following conditions hold.

(1) $\Gamma - t$ is connected.
(2) The initial vertex of t is distinguished in Γ.
(3) There is an edge e incident to the final vertex of t which was not visited by t.

A simple case by case analysis considering finitude and parity of the vertex degrees shows the following result.

Lemma 3.3. *Let Γ be a graph satisfying $\mathscr{E}_1$. If t is a right-extensible trail in Γ then $\Gamma - t$ also satisfies $\mathscr{E}_1$, and the final vertex of t is a distinguished vertex in $\Gamma - t$.*

Now the idea for proving Theorem 1.1 is very simple. In order to extend a trail t we remove it from Γ, and find a right-extensible trail in $\Gamma - t$ whose initial vertex is the final vertex of t. Then we concatenate these two trails and obtain a right-extensible trail in Γ which extends t. This process can be iterated to obtain a one-sided Eulerian infinite trail. To ensure that the obtained infinite trail is indeed Eulerian, we will show that at each step of the process we can obtain a trail that visits any edge of our choice. This is the purpose of the following two lemmas.

Lemma 3.4. *Let Γ be a graph satisfying $\mathscr{E}_1$, and let v be a distinguished vertex in Γ. Then for any edge e there is a right-extensible trail on Γ which starts at v and visits e.*

Proof. As Γ is connected, there is a trail $s : [\![0, c]\!] \to \Gamma$ with $s(0) = v$ and which visits e. We apply Lemma 3.1 to the trail s, and obtain a trail $t : [\![0, b]\!] \to \Gamma$ with the same initial vertex and which visits e, but whose deletion leaves a connected graph.

We claim that t is right-extensible. Indeed, the first and second conditions hold by our choice of t. For the third condition we separate the cases where t is closed or not. Recall that that the degree of $t(0)$ in Γ is odd or infinite because it is a distinguished vertex. If t is a closed trail then $\deg_t(t(0))$ is even, it follows that $t(b)$ has edges in Γ not visited by t. On the other hand if t is not a closed trail then $\deg_t(t(b))$ is odd while $\deg_\Gamma(t(b))$ is even or infinite, it also follows that $t(b)$ has edges in Γ not visited by t. □

Lemma 3.5. *Let Γ be a graph satisfying $\mathscr{E}_1$. Then for any right-extensible trail t and edge e there is a right-extensible trail which extends t and visits e.*

Proof. By Lemma 3.3 the graph $\Gamma - t$ satisfies $\mathscr{E}_1$ and contains the final vertex of t as a distinguished vertex. Now by Lemma 3.4 the graph $\Gamma - t$ admits a right-extensible trail $s : [\![0, c]\!] \to \Gamma - t$ which starts at the final vertex of t, and which visits e. Thus the trail $t' : [\![0, b + c]\!] \to \Gamma$ obtained by concatenating s at the right of t is right-extensible in Γ, visits e, and extends t. This concludes the argument. □

Now the proof of the following result is a straightforward application of the previous results.

Proposition 3.6. *Let Γ be a graph satisfying $\mathscr{E}_1$. Then it admits a one-sided infinite Eulerian trail.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be a numbering of the edges in $E(\Gamma)$. We define a sequence of right-extensible trails $(t_n)_{n\in\mathbb{N}}$ as follows. The first of them, t_0, is a right-extensible trail whose initial vertex is some distinguished vertex, and which visits e_0. This trail exists by Lemma 3.4. We now define $t_n, n \geq 1$ in the following recursive manner. Assume that t_{n-1} has been defined, and define t_n as a right-extensible trail which extends t_{n-1} and visits e_n. The existence of t_n is guaranteed by Lemma 3.3. Finally we define a one-sided infinite Eulerian trail $T : [\![\mathbb{N}]\!] \to \Gamma$ by setting $T(x) = t_n(x)$ and $T(\{x, x + 1\}) = t_n(\{x, x + 1\})$, for n big enough. □

Indeed, we have proved something a bit stronger.

Corollary 3.7. *Let Γ be a graph satisfying $\mathscr{E}_1$. Then a trail on Γ is right-extensible if and only if it can be extended to a one-sided infinite Eulerian trail. Moreover, a vertex is distinguished in Γ if and only if it is the initial vertex of a one-sided infinite Eulerian trail on Γ.*

Proof. We prove the first claim. If t is right-extensible, then we can replace t_0 in the proof of Proposition 3.6 by an extension of t. This is possible by Lemma 3.4, and proves the forward implication. For the remaining implication observe that the restriction of a one-sided infinite Eulerian trail to $[\![0, b]\!]$ for $b \in \mathbb{N}$ is right-extensible.

For the second claim, note that the forward implication was implicit in the proof of Proposition 3.6. For the remaining implication, it is clear that the initial vertex v of a one-sided infinite Eulerian trail must have odd or infinite degree, and that no vertex different from v can have odd degree in Γ. □

This concludes our proof of Theorem 1.1. We now proceed to consider two-sided infinite trails.

3.3. **The case of two-sided infinite trails.** In this subsection we prove Theorem 1.2. We start by verifying that the conditions $\mathscr{E}_2$ are necessary for a graph to admit a two-sided infinite Eulerian trail.

Proposition 3.8. *The conditions $\mathscr{E}_2$ are necessary for a graph to admit a two-sided infinite Eulerian trail.*

Proof. By the discussion in the introduction it only remains to verify the third condition in $\mathscr{E}_2$, for which we proceed by contradiction. Assume that Γ is a graph which admits a two-sided infinite Eulerian trail T and E is a finite set of edges which induces an even subgraph such that $\Gamma - E$ has two infinite conected components.

Let u and v be the first and last vertex in $\Gamma[E]$ visited by T, which must be different. Now let F be the set of edges visited by T after u but before v, so $E \subset F$. Observe that a restriction of T is an Eulerian trail in $\Gamma[F]$ from u to v. As $\Gamma[E]$ is an even graph, we obtain that $\Gamma[F] - E$ is a graph where u and v have odd degree, and the remaining vertices have even degree. Moreover $\Gamma[F] - E$ can not be connected, otherwise $\Gamma - E$ would be connected. Thus a connected component of $\Gamma[F] - E$

containing u is a finite and connected graph containing exactly one vertex with odd degree. This is a contradiction by the handshakes lemma. $\square$

We now proceed to prove the sufficiency of the conditions $\mathscr{E}_2$. The proof is based in the following notion.

Definition 3.9. Let Γ be a graph satisfying $\mathscr{E}_2$. We say that a trail t is **bi-extensible** in Γ if the following conditions hold.

(1) $\Gamma - t$ has no finite connected components.
(2) There is an edge e incident to the final vertex of t which was not visited by t.
(3) There is an edge $f \neq e$ incident to the initial vertex of t which was not visited by t.

If no confusion arises, we simply say that a trail is bi-extensible. Observe that when Γ has one end, the first condition in the definition simply means that $\Gamma - t$ is connected.

By the third condition in $\mathscr{E}_2$, the remotion of a closed trail from a graph satisfying $\mathscr{E}_2$ leaves a connected graph. Indeed we have the following result, whose proof is simply a verification of the conditions in $\mathscr{E}_2$.

Lemma 3.10. *Let Γ be a graph satisfying $\mathscr{E}_2$. A closed trail t is bi-extensible if and only if $\Gamma - t$ also satisfies $\mathscr{E}_2$.*

We now prove some simple facts about bi-extensible trails. As we did with right-extensible trails before, we will show that bi-extensible trails exist, and can be extended to larger bi-extensible trails.

Lemma 3.11. *Let Γ be a graph satisfying $\mathscr{E}_2$. Then for any vertex v and edge e there is a bi-extensible trail which visits v and e.*

Proof. By connectedness of Γ there is a trail $t : [\![0, b]\!] \to \Gamma$ which visits both v and e. By Lemma 3.1 we can assume that the remotion of this trail leaves no finite connected component in Γ. We now consider two cases.

If t is not closed then we claim that it is bi-extensible. Indeed, as $t(0)$ and $t(b)$ have odd degree in t, they have incident edges e and f not visited by t, and thus both vertices lie in $\Gamma - t$. Now observe that we can take $e \neq f$, as otherwise the graph $\Gamma - t$ would be forced to have a finite connected component, contradicting our choice of t.

If t is a closed trail, then we can reparametrize it in order to obtain a bi-extensible trail as follows. As Γ is connected t visits a vertex u which lies in $\Gamma - t$. Now as the degree of u in $\Gamma - t$ is even, there are at least two different edges in $\Gamma - t$ which are incident to u. This holds even if some edge incident to u in $\Gamma - t$ is a loop. We simply reparametrize t so that its initial and final vertex is u, and the trail we obtain is bi-extensible. $\square$

Lemma 3.12. *Let Γ be a graph satisfying $\mathscr{E}_2$. Then for any bi-extensible trail t and edge e there is a bi-extensible trail which extends t and visits e. Moreover, we can choose this extension of t so that its domain strictly extends the domain of t in both directions.*

Proof. We claim that given t and e as in the statement there is a bi-extensible trail s which extends t, visits e, and its domain strictly extends the domain of t in one direction, which we can choose. This claim implies the one in the statement as we can apply it twice. We proceed now to prove this claim, for which we consider three cases. Some of these cases are represented in Figure 2 on page 9.

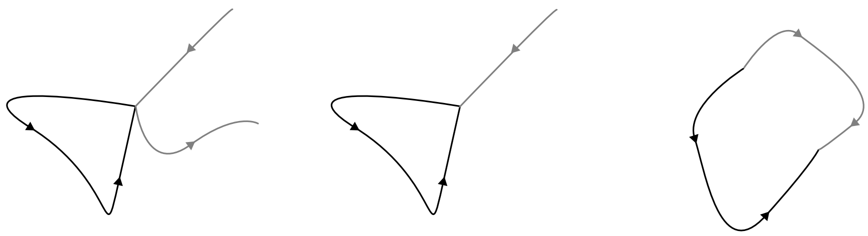

FIGURE 2. A representation of some of the cases in the proof. At the left is the first case, where t is shown in black and t_1 in gray. In the middle, the subcase of the first case in which c_1 equals b_1. At the right the second case, where t is shown in black and t_2 in gray.

In the first case $t : [\![a, b]\!] \to \Gamma$ is a closed trail. Then $\Gamma - t$ is connected by the third condition in $\mathscr{E}_2$, and the graph $\Gamma - t$ satisfies $\mathscr{E}_2$ by Lemma 3.10. We apply Lemma 3.11 to the graph $\Gamma - t$ to obtain a trail $t_1 : [\![a_1, b_1]\!] \to \Gamma - t$ which visits the vertex $t(a) = t(b)$, the edge e, and is bi-extensible on $\Gamma - t$. We split t_1 in two trails as follows. Let $c_1 \in [a_1, b_1]$ be such that $t_1(c_1) = t(a)$, and define l_1 and r_1 as the restrictions of t_1 to $[\![a_1, c_1]\!]$ and $[\![c_1, b_1]\!]$, respectively. Finally, define the trail s by concatenating l_1 to the left of t, and r_1 to its right. The fact that t_1 is bi-extensible on $\Gamma - t$ ensures that s is bi-extensible on Γ, and it visits e by construction. An alternative way to define the trail s is by concatenating $-r_1$ at the left of t, and $-l_1$ to its right.

Observe that c_1 could be equal to a_1 or b_1, in this situation the domain of s extends that of t only in one direction. We can choose this direction with the two possible definitions of s. Thus we have proved our claim in the case where t is a closed trail and $\Gamma - t$ is connected.

In the second case $t : [\![a, b]\!] \to \Gamma$ is not closed and $\Gamma - t$ is connected. We show that we can extend t to a bi-extensible closed trail s. This suffices as the new trail s belongs to the first case. As $\Gamma - t$ is connected we can take a trail $t_2 : [\![a_2, b_2]\!] \to \Gamma - t$ whose initial vertex is $t(a)$ and whose final vertex is $t(b)$. By Lemma 3.1 we can assume that $(\Gamma - t) - t_2$ has no finite connected components. We now split t_2 as follows. Let $c_2 \in [a_2, b_2]$ be such that $t_2(c_2)$ lies in $(\Gamma - t) - t_2$, that is, $t_2(c_2)$ is a vertex with incident edges not visited by t_2. By the parity of the vertex degrees there must be at least two such edges. Then define l_2 and r_2 as the restrictions of t_2 to $[\![a_2, c_2]\!]$ and $[\![c_2, b_2]\!]$, respectively. Note that the final vertex of $-l_2$ is $t(a)$, and the initial vertex of $-r_2$ is $t(b)$. Define s by concatenating $-l_2$ to the left of t, and then $-r_2$ to its right. By our choice of t_2 and c_2, s is a bi-extensible closed trail which extends t. This concludes the second case.

In the third case, t is not closed and $\Gamma - t$ is not connected. Thus $\Gamma - t$ has exactly two infinite connected components, each of which satisfies $\mathscr{E}_1$ and where the initial and final vertex of t are distinguished. We simply apply Lemma 3.4 on each of these components and then concatenate to extend t as desired. □

Now the proof of the following result is a straightforward application of the previous results.

Proposition 3.13. *Let Γ be a graph satisfying $\mathscr{E}_2$. Then it admits a two-sided infinite Eulerian trail.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be a numbering of the edges in $E(\Gamma)$. We define a sequence of bi-extensible trails $(t_n)_{n\in\mathbb{N}}$ as follows. The first of them, t_0, is a bi-extensible trail which visits e_0. This trail exists by Lemma 3.11. Now define $t_n, n \geq 1$ in the following recursive manner. Assume that t_{n-1} has been defined, and define t_n as a bi-extensible trail which extends t_{n-1}, whose domain strictly extends that of t_{n-1} in both directions, and which visits e_n. The existence of t_n is guaranteed by Lemma 3.12. Finally we define a two-sided infinite Eulerian trail $T : [\![\mathbb{Z}]\!] \to \Gamma$ by setting $T(x) = t_n(x)$ and $T(\{x, x+1\}) = t_n(\{x, x+1\})$, for n big enough. $\square$

As in the one-sided case, we have proved the following stronger result. The proof is identical to that of Corollary 3.7.

Corollary 3.14. *Let Γ be a graph satisfying $\mathscr{E}_2$. Then a trail is bi-extensible if and only if it can be extended to a two-sided infinite Eulerian trail.*

4. The computability of the proof

In this section we prove Theorem 1.3, for which we first introduce some computability notions for infinite graphs. A **computable** graph is a graph Γ whose edge and vertex sets are endowed with an indexing or numbering by decidable sets of natural numbers I and J, $E(\Gamma) = (e_i)_{i\in I}$, $V(\Gamma) = (v_j)_{j\in J}$, such that the incidence relation between edges and pairs of vertices is decidable. That is, the following is a decidable set:

$$\{(i, j, k) \mid e_i \text{ joins } v_j \text{ and } v_k\} \subset \mathbb{N}^3.$$

In a computable graph, a one-sided (resp. two-sided) infinite trail is computable if the corresponding edge and vertex functions are computable when translated to natural numbers.

The hypothesis of computable graph is rather weak. Without further hypothesis, the decidability of the incidence relation only allows us to enumerate set of neighbors of a vertex. We will need the stronger notion of **highly computable** graph, which is a computable graph for which the vertex degree function $\deg_\Gamma : V(\Gamma) \to \mathbb{N}$ is computable.

In a highly computable graph we can compute finite subgraphs of any desired size. More precisely, given a vertex v and a distance $n \in \mathbb{N}$, we can compute the finite subgraph induced by all edges incident to a vertex u with $d(u, v) \leq n$.

Some of the algorithms which we encounter are uniform in the graph, in the sense that the algorithm does not depend on the graph. A formal way to refer to this fact is with the following notion. We call a **description** of a highly computable graph the four-tuple of the algorithms which decide membership in I, J, the relation of incidence between edges and pairs of vertices, and the vertex degree function.

We proceed now to prove some lemmas about highly computable graphs.

Lemma 4.1. *There is an algorithm which on input the description of a highly computable graph Γ and a finite set of edges E in Γ, halts if and only if $\Gamma - E$ has some finite connected component.*

Proof. Fix a vertex v, and for each n compute the set of edges E_n which are incident to a vertex at distance at most n from v in the graph Γ. Then compute the induced subgraph $\Gamma_n = \Gamma[E_n] - E$. If $\Gamma - E$ has a finite connected component, we can algorithmically detect this in Γ_n for some n, and stop the procedure. $\square$

Lemma 4.2. *There is an algorithm which on input the description of a highly computable graph with one end Γ and a finite set of edges E in Γ, decides whether $\Gamma - E$ is connected.*

Proof. By Lemma 4.1, it suffices to exhibit an algorithm which on input a description of Γ and E as in the statement, halts if and only if $\Gamma - E$ has no connected component.

Given E and Γ, let V be the set of vertices in $\Gamma - E$ which are incident to some edge in E. Observe that the set V can be computed upon Γ and E. Now define a relation $R_E \subset V \times V$ as follows. A pair of vertices (u, v) lies in R_E if and only if they lie in the same connected component in $\Gamma - E$. Note that a pair (u, v) lies in R if and only if there is a trail which connects them in $\Gamma - E$, and thus such trail can be found by just performing an exhaustive search.

In summary, there is an algorithm which given the graph Γ and the set of edges E, computes the set V, and enumerates the set R_E. In order to conclude that $\Gamma - E$ is connected, we just run this algorithm, and stop the procedure once we notice that R_E has exactly one equivalence class. □

Lemma 4.3. *There is an algorithm which on input the description of a highly computable graph with two ends Γ, and a finite set of edges such that $\Gamma - E$ has two infinite connected decides whether $\Gamma - E$ has some finite connected component.*

Proof. By Lemma 4.1, it suffices to exhibit an algorithm which given Γ and E as in the statement, halts if and only if $\Gamma - E$ has no finite connected component. Indeed, we can repeat the process described in Lemma 4.2, with the only difference that we must stop the algorithm once we have found two equivalence classes in the associated relation instead of one. □

Using these lemmas, the proof of the following result is straightforward.

Proposition 4.4. *Let Γ be a highly computable graph.*

(1) If Γ satisfies $\mathscr{E}_1$, then it is algorithmically decidable whether a trail is right-extensible.
(2) If Γ satisfies $\mathscr{E}_2$ and has one end, then it is algorithmically decidable whether a trail is bi-extensible.
(3) If Γ satisfies $\mathscr{E}_2$ and has two ends, then it is algorithmically decidable whether a trail whose remotion leaves two infinite connected components is bi-extensible.

Proof. Observe that the second and third conditions in Definition 3.2 and Definition 3.9 are clearly decidable, so it only remains to verify the decidability first one. For one ended graphs satisfying $\mathscr{E}_1$ and $\mathscr{E}_2$ this follows from Lemma 4.2, while the claim for two ended graphs follows from Lemma 4.3. □

We are now in position to prove Theorem 1.3, where we just have to review the procedures described in the previous section. Observe that a highly computable graph satisfying $\mathscr{E}_1$ or $\mathscr{E}_2$ must have infinitely many vertices and edges, and we can assume that they are indexed as $E(\Gamma) = (e_i)_{i \in \mathbb{N}}$ and $V(\Gamma) = (v_i)_{i \in \mathbb{N}}$.

Proof of Theorem 1.3. We start with the $\mathscr{E}_1$ case. A procedure to obtain a one-sided infinite trail on Γ was described in Proposition 3.6, where we defined a sequence of trails $(t_n)_{n \in \mathbb{N}}$. We now justify that this procedure can be performed computably. Indeed, in order to compute the trail t_n for some n we just need to do an exhaustive search among all trails in an ordered manner. The conditions imposed on t_n are decidable by Proposition 4.4, and its existence is guaranteed by Lemma 3.5. We have sketched an algorithm which on input n outputs t_n, and this makes the associated infinite trail T computable.

We now consider the $\mathscr{E}_2$ case, which has only a minor difference. In the proof of Proposition 3.13 we defined a sequence of bi-extensible trails $(t_n)_{n \in \mathbb{N}}$. If Γ has two ends, then we choose t_0 to be a trail where $\Gamma - t_0$ has two infinite connected components[3]. Thus regardless the number of ends of Γ, it is decidable whether a trail extending t_0 is bi-extensible. The remaining of the argument is the same as in the case $\mathscr{E}_1$. $\square$

Let us now make a remark on uniformity. For one ended graphs satisfying $\mathscr{E}_1$ or $\mathscr{E}_2$, we have shown that the corresponding infinite trail can be computed upon a description of the graph. An alternative way to say this is that the infinite trail can be computed by an algorithm which takes as input finite subgraphs of Γ, that is, it uses the graph as *oracle.* For two ended graphs satisfying $\mathscr{E}_2$, however, we also need to hard code the trail t_0 from the previous proof inside the algorithm associated to Γ, and we do not know whether such trail can be computed upon a description of the graph. This is related to the following more basic question which we were not able to answer.

Question 4.5. *Let Γ be a highly computable graph with two ends. Is it possible to decide, given a finite set of edges E, whether $\Gamma - E$ has two infinite connected components?*

We end this section with a very simple example which shows that in Theorem 1.3 we can not relax the hypothesis from highly computable to computable.

Example 4.6. We construct a computable graph satisfying $\mathscr{E}_2$ but which does not admit a computable two-sided infinite Eulerian trail. Let $P \subset \mathbb{Z}$ be a computably enumerable and undecidable set, and let Γ be the graph whose vertex set is $\mathbb{Z} \times \{0\} \cup P \times \{1\}$, and with edge relation as follows. For each $z \in \mathbb{Z}$, there is one edge joining $(z, 0)$ and $(z + 1, 0)$. Moreover for each $z \in P$, there are two edges joining $(z, 0)$ and $(z, 1)$. Thus the vertex degree of $(z, 0)$ is 4 when $z \in P$, and 2 if $z \notin P$.

A computable numbering for Γ is obtained by taking a computable bijection $\mathbb{N} \to \mathbb{N} \times \{0, 1\}$, and then composing with two computable and surjective functions $\mathbb{N} \to \mathbb{Z}$, $\mathbb{N} \to P$. We claim that this computable graph does not admit a computable two-sided infinite Eulerian trail. It suffices to note that for each $z \in P$, such a trail must visit the vertices $(z, 0)$, then $(z, 1)$, and then $(z, 0)$ consecutively. Thus such a trail would allow us to compute the vertex degree function. This in turn would allow us to decide membership in P, and this is a contradiction.

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