Slow entropy of some skew products

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Often they will be invertible, but not always

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3. Natural problem: distinguish $S \rtimes_\tau T_1$ and $S \rtimes_\tau T_2$
4. This problem is well-studied for measure preserving systems; we study it for topological systems.
5. We propose a solution based on slow entropy, an invariant introduced by Katok and Thouvenot.

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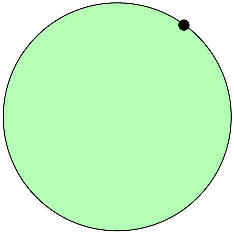

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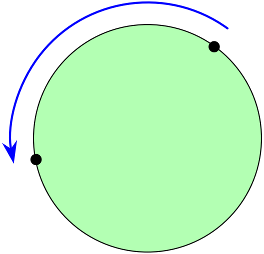

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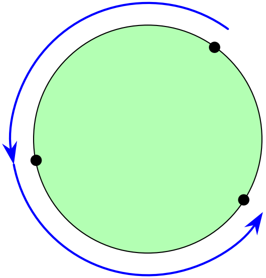

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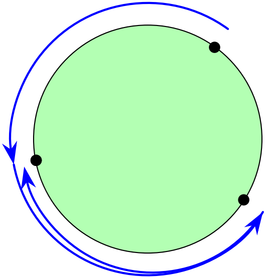

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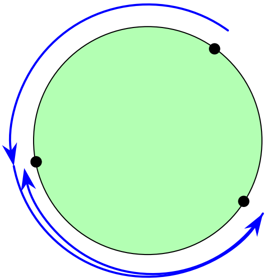

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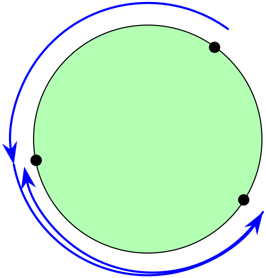

Take the space

$$\{-1, 1\}^{\mathbb{Z}} \times \mathbb{T}$$

and the transformation

$$(y, x) \mapsto (\sigma(y), T^{y(0)}(x))$$

The $[T, T^{-1}]$ system

The $[T, T^{-1}]$ system is the skew product

- Base: $\sigma \curvearrowright (\{-1, 1\}^{\mathbb{Z}}, \mu_{\frac{1}{2},\frac{1}{2}})$
- Fiber: $T \curvearrowright (X, \mathcal{X}, \mu)$

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Question: are they all isomorphic copies of the same system, or are they many different examples?

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One can prove that all of these systems have the same entropy

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Theorem (Austin 2015)

Yes. There is an invariant I with $I_{\mu_{\frac{1}{2},\frac{1}{2}}\times\mu}(\sigma \rtimes T) = h_\mu(T)$.

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The new transformation is

$$S \rtimes_\tau T \curvearrowright (Y, \mathcal{Y}, \nu) \times (X, \mathcal{X}, \mu)$$

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What happens in the general case?

For which choices of $S \curvearrowright (Y, \mathcal{Y}, \nu)$ and $\tau \colon Y \to \mathbb{Z}$ is the entropy of the fiber an isomorphism invariant in the corresponding class of skew products?

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3. A mixing SFT $S \curvearrowright (Y, \mathcal{Y}, \nu)$, ν a Gibbs measure, and $\tau\colon\ Y \to \mathbb{Z}$ satisfying a technical condition (Austin 2015)

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- What happens for topological systems?
- We propose a solution, it works for a diverse class of S and τ.

Topological skew products

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Find an invariant that captures ~~topological entropy~~ *entropy-type invariants* of T.

Intuition

The problem is interesting when the entropy of the fiber "disappears". Informally,

$$\#\text{ of }(n,\epsilon)\text{-Bowen balls needed to cover the space} \approx e^{n\cdot h_{top}(S)+f(n)h_{top}(T)}$$

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for some sublinear $f(n)$.
We want to capture the contribution of $h_{top}(T)$ in this relation.

$\#$ of (n, ϵ)-Bowen balls $:= \mathrm{spa}(T, n, \epsilon) \quad n \in \mathbb{N}, \epsilon > 0$
needed to cover the space

Topological slow entropy

Definition (Katok and Thouvenot '97)

Let $\mathbf{a} = \{a_n(t)\}_{n\in\mathbb{N},t>0}$ be a family of functions $(0,\infty) \to (0,\infty)$ with

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The upper slow entropy of (X, T) with scale $\mathbf{a}$ is

$$\overline{\mathrm{ent}}_{\mathbf{a}}(T) = \lim_{\epsilon\to 0} \overline{\mathrm{ent}}_{\mathbf{a}}(T,\epsilon)$$

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- With scale $\{n^t\}_{n\in\mathbb{N},t>0}$ we obtain the so called polynomial complexity or polynomial entropy of the system.

The main result

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Theorem (C, arXiv:2506.17932)

Let (Y, S) be a subshift and let $\tau\colon Y \to \mathbb{Z}$ be continuous. Then there is a scale $\boldsymbol{a}$ depending only on (Y, S, τ) such that for every invertible topological dynamical system (X, T) we have

$$\overline{\mathrm{ent}}_{\boldsymbol{a}}(S \rtimes_\tau T) = \underline{\mathrm{ent}}_{\boldsymbol{a}}(S \rtimes_\tau T) = h_{top}(T),$$

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A relative version of the main result

Theorem (C, arXiv:2506.17932)

Let (Y, S) be a subshift and let $\tau \colon Y \to \{-1, 0, 1\}$ be continuous. For every scale $\boldsymbol{b}$ we can find a scale $\boldsymbol{c}$ such that for every invertible topological dynamical system (X, T) we have

$$\underline{\mathrm{ent}}_{\boldsymbol{b}}(T) \leq \overline{\mathrm{ent}}_{\boldsymbol{c}}(S \rtimes_\tau T) \leq \overline{\mathrm{ent}}_{\boldsymbol{b}}(T)$$

provided that τ satisfies the condition of being λ-unbounded for some $\lambda > 0$ (defined soon).

Thus we can recover the slow entropy of T with scale b provided we restrict to systems with the property $\overline{\mathrm{ent}}_{\mathbf{b}}(T) = \underline{\mathrm{ent}}_{\mathbf{b}}(T)$.

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- For $y \in Y$ we interpret $N \to \sum_{n=0}^{N-1} \tau(S^n(y)) = y_0 + \cdots + y_{N-1}$ as a walk on $\mathbb{Z}$, and we define the *range* $R_N(y)$ as the set of places in $\mathbb{Z}$ it visits in N steps:

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Definition

Given (Y, S, τ) as before, we say that τ is λ-unbounded ($\lambda > 0$) if for all $C \in \mathbb{N}$

$$\liminf_{n\to\infty} \frac{|\{w \in L_n(Y) : |R_n(w)| \geq C\}|}{|L_n(Y)|} \geq \lambda$$

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- Also true for $Y = \{-1, 1\}^{\mathbb{Z}}$.

Examples

Lets review some choices of (Y, S) and τ that have this property.

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Example

(Y, S) is a minimal subshift and $\tau\colon Y \to \mathbb{Z}$ is not a coboundary (i.e. $\tau = g - g \circ S$ for some continuous $g\colon Y \to \mathbb{R}$).

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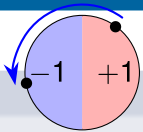
- For any T we have $h_{top}(S \rtimes_\tau T) = h_{top}(S) + \alpha h_{top}(T)$, so applying the main theorem is an overkill.
- The conclusion of the "relative main theorem" is nontrivial: if T_1 and T_2 have zero entropy but different polynomial complexity ($\underline{\mathrm{ent}}_{n^t}(T_1) < \overline{\mathrm{ent}}_{n^t}(T_2)$), one obtains a scale for slow entropy that distinguishes $S \rtimes_\tau T_1$ and $S \rtimes_\tau T_2$.

Example

The deterministic random walk (the subshift copy).



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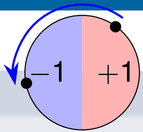

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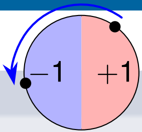

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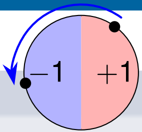

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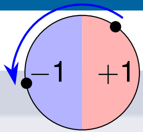

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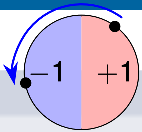

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- The same is true for the slow entropy of T.

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- Here $h_{top}(S \rtimes_\tau T) = \log(2) + f(h_{top}(T))$, for some $f\colon [0, \infty) \to [0, \infty)$ strictly increasing.
- Applying the main theorem is an overkill.
- The conclusion of the "relative main theorem" is nontrivial: if T_1 and T_2 have zero entropy but different polynomial complexity ($\underline{\mathrm{ent}}_{n^t}(T_1) < \overline{\mathrm{ent}}_{n^t}(T_2)$), one obtains a scale for slow entropy that distinguishes $S \rtimes_\tau T_1$ and $S \rtimes_\tau T_2$.

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Take an invertible system $T \curvearrowright (X, d)$.

Given $F \subset \mathbb{Z}$ finite, we write

$$d_F(x, y) = \max\{d(T^i(x), T^i(y)) : i \in F\}$$

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and

$$\mathrm{spa}(T, F, \epsilon)$$

equals the minimal number of Bowen d_F, ϵ-balls needed to cover X.

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The scale for the first main theorem is

$$a_N(t) = \sum_{w \in L_N(Y)} e^{|R_N(w)| t}$$

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We have

$$\limsup_{N \to \infty} \frac{\mathrm{spa}(S \rtimes_{\tau} T, \{0, \dots, N-1\}, \epsilon)\})}{a_N(t)}$$

is 0 for $t > h_{top}(T)$, and ∞ for $t < h_{top}(T)$. Thus $\overline{\mathrm{ent}}_{\mathbf{a}}(S \rtimes_{\tau} T) = h_{top}(T)$.

General case

If τ takes values outside $\{-1, 0, 1\}$, the sets $R_N(w)$ may have holes, and this causes extra growth of $\text{spa}(T, R_N(w), \epsilon)$.
For instance, if $R_N(w) = \{2, 4, \ldots, 2N\}$,

$$\text{spa}(T, R_N(w), \epsilon) \approx e^{\mathbf{2} \cdot h_{top}(T)|R_N(w)|}$$

It is possible to quantify this extra contribution and include it to the scale, so that they “cancels out”.

Comments and questions

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The scales constructed here don't work for these purposes (in general).

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