Robotics Term Project: Mathematical Report

Derivation of Equations of Motion using Lagrangian Mechanics

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Introduction

The cart-pole system, commonly referred to as an inverted pendulum on a cart, is a foundational model in robotics and control systems. This setup involves a cart of mass M moving horizontally, with a uniform rod of mass m and length $2l$ attached via an unactuated revolute joint. The primary control objective is to stabilize the rod in the upright position by applying a horizontal force F_{ext} to the cart.

This system is a classic example of an underactuated, nonlinear, and inherently unstable system. It serves as an excellent platform for understanding dynamic coupling and control strategies for systems with fewer actuators than degrees of freedom.

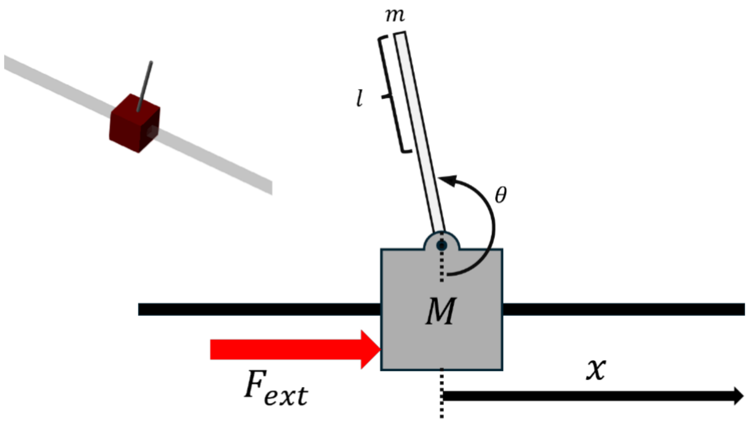

Figure 1: 2-DOF System

System Description

The physical parameters are defined as follows. The mass of the cart is $M = 2\,\text{kg}$, and the mass of the uniform rod is $m = 0.5\,\text{kg}$. The rod has a total length of $2l = 0.5\,\text{m}$, so each half-length is $l = 0.25\,\text{m}$. Gravity is taken as $g = 9.81\,\text{m/s}^2$, and the moment of inertia of the rod about its center of mass is $I = \frac{1}{3}ml^2$.

The generalized coordinates used to describe the configuration of the system are: x, the horizontal position of the cart, and θ, the angle of the rod with respect to the vertical upward direction, measured counterclockwise. With this convention, $\theta = 0$ corresponds to the upright position.

Lagrangian Approach

Lagrangian mechanics offers a systematic method for deriving equations of motion, especially for multi-body systems involving rotational dynamics. It avoids the need to resolve individual constraint forces and instead relies on energy expressions, providing a powerful tool for modeling systems with complex interactions.

The Lagrangian is given by $L = T - V$, where T is the total kinetic energy and V is the potential energy. The equations of motion follow from the Euler-Lagrange equations applied to each generalized coordinate.

Kinetic Energy

The kinetic energy of the cart is purely translational and given by $T_{cart} = \frac{1}{2}M\dot{x}^2$.

The rod exhibits both translational motion (from being attached to the moving cart) and rotational motion about its center of mass. The center of mass of the rod, located at a distance l from the pivot, has coordinates $x_{cm} = x + l\sin\theta$ and $y_{cm} = l\cos\theta$. Differentiating yields the velocities $\dot{x}_{cm} = \dot{x} + l\dot{\theta}\cos\theta$ and $\dot{y}_{cm} = -l\dot{\theta}\sin\theta$. The translational kinetic energy of the rod becomes:

$$T_{rod,trans} = \frac{1}{2}m\left[\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right]$$

The rotational kinetic energy of the rod about its center is:

$$T_{rod,rot} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{6}ml^2\dot{\theta}^2$$

Adding all components, the total kinetic energy of the system is:

$$T = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{2}{3}ml^2\dot{\theta}^2$$

Potential Energy

The potential energy arises solely from the gravitational energy of the rod, whose center of mass is at a height $l\cos\theta$. Taking the cart level as the zero reference:

$$V = mgl\cos\theta$$

Lagrangian

Substituting the expressions for T and V, we obtain the Lagrangian:

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{2}{3}ml^2\dot{\theta}^2 - mgl\cos\theta$$

Equations of Motion

Applying the Euler-Lagrange equation to x yields:

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F_{ext}$$

Applying the Euler-Lagrange equation to θ yields:

$$\frac{4}{3}ml^2\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$$

These coupled nonlinear differential equations describe the complete dynamics of the cart-pole system.

The derivation above demonstrates how Lagrangian mechanics provides a structured and powerful approach to modeling underactuated systems like the cart-pole. The resulting equations reveal the dynamic coupling between linear and angular motion, and form the foundation for control and simulation tasks in subsequent sections of the project.