Game Theory

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Part I

Introduction

Chapter 1

Rationality

Game theory is a branch of decision theory, that is, the theory that proposes ways to take decisions optimally. In general, decision theory provides different ways to approach the problem of deciding which choice is optimal. Each method is better suited for a different context, for instance, some methods provide frameworks to take decisions, while others apply to one rational player only.

An important thing to notice is that decision is an optimisation problem, in fact, we want to optimise the decisions (or choices) taken by one agent. Before going on, let us stop a moment on the concept of agent.

> **Definition 1.1** (Agent)**.** *An agent is an entity that is able to take decisions.*

A particular situation may involve one or more optimal agents. Usually, the higher the number of participating agents, the harder the optimisation problem is.

Game theory is just one way of optimising the decision-making process. Since we have said that each decision-making theory has its own characteristics let us define what are the main features of game theory. First of all, let us define game theory as follows.

> **Definition 1.2** (Game theory)**.** *Game theory is a decision-making process (or a decision theory) that can be applied to those situations that can be seen as games.*

An important part of Definition 1.2 is that the problems to which game theory is applied must be seen as games. This means that it's time to further specify what a game is.

> **Definition 1.3** (Game)**.** *A game is an efficient model that can be applied to real-life situations (e.g., biology, economy). A game has to satisfy the following features:*
>
> 1. *It must have* ***at least two participants****.*
> 2. *It must have an* ***initial situation****.*
> 3. *Each player of the game has an* ***optimal choice****.*
> 4. *The game is governed by* ***rules*** *that all players must follow.*

5. *A game ends with some **final outcomes**.*
6. *Each player has **preferences over the possible choices** that can be taken. One can follow a player's choices to reach the final outcome of the game for that player.*

1.1 Player

Another important concept, which appears many times in the Definition 1.3 of game, is the idea of player. A player is somehow a refinement of an agent and has two important characteristics: selfishness and rationality.

Definition 1.4 (Player)**.** *A player is a:*

- ***selfish** and*
- ***rational***

agent.

1.1.1 Selfishness

The first characteristic of a player is quite easy to analyse. A player is selfish if he/she puts his/her own preferences over the outcome of the game and doesn't care about the outcomes for the other players. Note that this definition has nothing to do with ethical considerations. It is in fact only a mathematical assumption. Moreover, a player, being selfish, could generate a higher outcome for other players, still optimising his/her outcome. Put it another way, a player wants to maximise his/her profit.

1.1.2 Rationality

Rationality is a little bit trickier to analyse. To define what rational means, let us introduce the concept of preference relation.

Definition 1.5 (Preference relation)**.** *Let $\mathcal{X}$ be a set of choices (i.e., actions a player can do). A preference relation on $\mathcal{X}$ is a binary relation $\succeq$ such that the following properties hold:*

- ***Reflexive property**. For every choice $x \in \mathcal{X}$, x is preferred, or equally preferable to itself.*
$$x \succeq x \qquad \forall x \in \mathcal{X} \tag{1.1}$$
- ***Completeness**. For every couple of choices $x, y \in \mathcal{X}$, either x is preferred to y, y is preferred to x or both options are equally preferable.*
$$x \succeq y \vee y \succeq x \vee \text{ both} \qquad \forall x, y \in \mathcal{X} \tag{1.2}$$
- ***Transitive property**. Given any three choices $x, y, z \in \mathcal{X}$, if x is preferred to y and y is preferred to z, then x is also preferred to z.*
$$x \succeq y \wedge y \succeq z \Rightarrow x \succeq z \qquad \forall x, y, z \in \mathcal{X} \tag{1.3}$$

An important thing to notice is that the preference relation isn't a quantitative relation but it only defines an order among the elements of the set $\mathcal{X}$ of all choices. In other words, we can't compare two choices x and y using the usual $>, <, =$ symbols.

Now that we have sorted out what a preference relation is, we can start defining what rational means. In particular, we will use 5 assumptions, called **rationality assumptions**, to properly define what rationality is.

First rationality assumption

Let us start by enunciating the first rationality assumption, which directly uses the preference relation.

> **Principle 1.1** (First rationality assumption)**.** *Players can define a preference relation over the outcomes of the game. Such an order has to be consistent.*

Just to be clear, let us better refine what consistency means. An order is consistent if, during the game, it isn't changed by the player. Say for instance a player prefers to eat apples (choice $a \in \mathcal{X}$) instead of bananas (choice $b \in \mathcal{X}$), i.e., $a \succeq b$. Later on in the game, it can't happen that the same player changes his/her choice preferring bananas over apples.

Second rationality assumption

As we have already said, the preference relation doesn't define a quantitative relation between choices. To consider also the value of a choice, we have to introduce a function, called utility function.

> **Definition 1.6** (Utility function)**.** *A utility function $u : \mathcal{X} \to \mathbb{R}$ is a function from the set of choices to the real numbers, that maps every choice into a real value (i.e., the value or utility of that choice).*

The utility function allows us to directly compare choices using their values (i.e., using the classical comparison with $>, <$ and $=$). An important thing to underline is that, for every couple of choices $x, y \in \mathcal{X}$, if x is preferred to y, then the utility of x has to be higher than the one of y. Formally,

> **Definition 1.7** (Utility function representing preference)**.** *Let $\succeq$ be a preference relation over a set of choices $\mathcal{X}$. A utility function $u : \mathcal{X} \to \mathbb{R}$ represents $\succeq$ if*
>
> $$\forall x, y \in \mathcal{X} \quad u(x) \geq u(y) \iff x \succeq y \tag{1.4}$$

This is quite intuitive if we remember that the utility function defines how good a choice is for a player. If a choice x is more valuable (i.e., it's better, has a higher value) than a choice y, then it must follow that x is preferred to y (and vice versa, if $x \succeq y$, then x is more valuable).

The utility function is very useful, however, it's not always possible to define one (i.e., it might not exist a function u over a set of choices $\mathcal{X}$). This doesn't, however, have to scare us since for a finite set of choices it's always possible to find a utility function u. Moreover, when we can define

u, we can also define an infinite number of equivalent utility functions simply applying a strictly increasing transformation to u (e.g., adding a positive constant, $\tilde{u} = u + C$).

Another thing we can add is that if we call $\mathcal{X}_i$ the set of choices of player i (for every player i in the set of all players), then the Cartesian product of all $\mathcal{X}_i$ is the set of choices $\mathcal{X}$ of the game.

$$\mathcal{X} = \times \mathcal{X}_i \tag{1.5}$$

Now we are all set to introduce the second rationality assumption.

Principle 1.2 (Second rationality assumption). *Every agent participating in the game (i.e., every player) is able to provide a measurement of its choices (i.e., a utility function), whenever necessary.*

Third rationality assumption

Before introducing the third rationality assumption, let us stop for a moment to analyse an example (or more properly a paradox) that shows that many players, in real life, are not rational. Consider the following choice, let us call it choice C_1, between two different scenarios:

A You gain 2500 with 0.33 probability, 2400 with 0.66 probability and 0 with 0.01 probability.

B You gain 2500 with 0 probability, 2400 with 1 probability and 0 with 0 probability.

In a practical experiment, done by Maurice Allais, most people choose option B, because of the secure win. Let us now consider the following experiment C_2, with the following scenarios:

C You gain 2500 with 0.33 probability and 0 with 0.67 probability.

D You gain 2400 with 0.34 probability and 0 with 0.66 probability.

In this case, most people chose option C, contrary to what can be seen as rational. Let us understand why this choice (or equally choice A and D) is not rational. To understand why it's not rational to choose B and C or A and D, let us compute the expected reward in both cases. If we prefer B over A, in the first experiment, then it means that the expected reward of the first option (A) is lower than the one of the second option (B). Formally, we would write

$$0 \cdot u(2500) + 1 \cdot u(2400) + 0 \cdot u(0) \geq 0.33 \cdot u(2500) + 0.66 \cdot u(2400) + 0 \cdot 0.01 \tag{1.6}$$
$$u(2400) - 0.66 \cdot u(2400) \geq 0.33 \cdot u(2500) \tag{1.7}$$
$$0.34 \cdot u(2400) \geq 0.33 \cdot u(2500) \tag{1.8}$$

In the second case, if we say that C is better than D, then we would say that

$$0.33 \cdot u(2500) + 0.67 \cdot u(0) \geq 0.34 \cdot u(2400) + 0.66 \cdot u(0) \tag{1.9}$$
$$0.33 \cdot u(2500) \geq 0.34 \cdot u(2400) \tag{1.10}$$

If we use u consistently (i.e., we give the same value to $u(2400)$ and to $u(2500)$ in both experiments), then we obtain an incoherence between the first and second experiment. In fact, the group of people involved in the experiment said that

$$0.34 \cdot u(2400) \geq 0.33 \cdot u(2500)$$

and

$$0.33 \cdot u(2500) \geq 0.34 \cdot u(2400)$$

which is clearly a contradiction. The same happens if the majority of people would have voted A and D. Note that in this experiment we are assuming that the people involved are rational and the choice chosen by the majority is the rational choice.

Now that we have explained this contradiction, we can introduce the third rationality assumption.

> **Principle 1.3** (Third rationality assumption)**.** *The players use consistently the laws of probability. In particular, they are consistent with respect to the calculation of expected utilities. They are able to update probabilities according to the Bayes rule and they are able to mix their actions.*

In other words, we can say that players use the expected value to build their utility function in presence of random events.

Fourth rationality assumption

The fourth rationality assumption states that:

> **Principle 1.4** (Fourth rationality assumption)**.** *Players understand the consequences of all their actions, the consequences of this information on any other player, the consequences of the consequences and so on.*

Let us now introduce an example to better understand the meaning of this new assumption. The following example is called *beauty contest problem*, even if the general form we will analyse has nothing to do with beauty. The game works as follows:

- Each participant has to choose a number between 1 and 100.
- Calling M, the mean of the number chosen by the players, the goal of each player is to guess a number as close as possible to qM, where q is a known number between 0 and 1.

If we consider $q = \frac{1}{2}$, there is no point in choosing a number greater than 50 since, in the worst case, everyone chooses 100, then the mean would be 100 (more or less) and the number to guess would be $\frac{1}{2} \cdot 100 = 50$. However, if all players are rational, they do the same reasoning, and they know that every other player does the same reasoning. As a consequence, there is no point in choosing a number bigger than 25 since, if everybody chooses 50 (in the worst case), the number to guess would be $\frac{1}{2} \cdot 50 = 25$. This reasoning can be iteratively repeated until all players reach the smallest number possible, i.e., 1. This example shows what happens if every player is rational and knows the consequences of his/her and others' actions.

Fifth rationality assumption

Finally, we have reached the last rational assumption.

> **Principle 1.5** (Fifth rationality assumption)**.** *The players are able to use decision theory, whenever it is possible. Namely, given a set of alternatives $\mathcal{X}$, and a utility function u on $\mathcal{X}$,*

each player seeks a $\bar{x} \in \mathcal{X}$ such that

$$u(\bar{x}) > u(x) \quad \forall x \in \mathcal{X} \tag{1.11}$$

In other words, a rational player would always use decision theory to find an action $\bar{x}$ which is optimal, hence better than any other action $x \in \mathcal{X}$ he or she can do.

1.1.3 Principle of elimination of strictly dominated strategies

An important consequence of the rationality assumptions is the following principle.

Principle 1.6 (Principle of elimination of strictly dominated strategies)**.** *A player does not take an action a if she/he has available an action b providing her/him a strictly better result, no matter what the other players do.*

In other words, if a choice a of a player is worst than another choice b, then the player will never choose a. This means that a player will always discard an action a that is worst than an action b, for every possible action of the other players.

1.2 Game representation

Now that we understood the basics of how an agent plays a game, it's time to define how a game is described. A game with two players is represented using a bimatrix in which:

- Each row represents a choice for player 1.
- Each column represents a choice for player 2.
- Each cell is a row vector containing two values. If we consider a cell in row r and column c, then the cell contains the utility for players 1 and 2, when player 1 chooses row r and player 2 chooses column c.

Note that we are describing games with two players, however, we can easily extend the description to games with n players, using an n-dimensional bimatrix where each dimension represents the set of possible choices of a player and each cell contains n utility values. Since an example is always a good thing, let us consider Game 1.12, involving two players.

$$\begin{pmatrix} (10, 11) & (2, 14) \\ (15, 3) & (5, 6) \end{pmatrix} \tag{1.12}$$

The game bimatrix can be divided into two different (simple) matrices, each representing the game value from one player's perspective. The values of players 1 and 2 are shown in Equations 1.13 and 1.14.

$$\begin{pmatrix} 10 & 2 \\ 15 & 5 \end{pmatrix} \tag{1.13}$$

$$\begin{pmatrix} 11 & 14 \\ 3 & 6 \end{pmatrix} \tag{1.14}$$

If we consider Equation 1.13, we can see that, if player 1 picks choice 1 (i.e., the first row) then it gets a reward (in the sense of the value of the final outcome) of 10 or 3, depending on the choice of player 2 (if the latter chooses the first column, the former gets 10, otherwise 3).

1.2.1 Solving a game

Having (hopefully) understood how a game is represented, we have to understand how to solve it. In particular, we have to understand how to apply Principle 1.5. The idea is that we can treat each player separately and check if a choice (i.e., a column or a row) dominates the others.

> **Definition 1.8** (Domination). *A row r_1 (or equivalently a column c_1) dominates a row r_2 (column c_2) if each value of r_1 (c_1) is bigger (or in general better) than the corresponding value of r_2 (c_2).*

If we consider Game 1.12, we can see that the second row dominates the first since 15 is bigger than 10 and 5 is bigger than 2. This means that, independently from what Player 2 plays (i.e., what column he or she chooses), it's always better for Player one to play row 2. Using the same principle, the second column dominates the first since 14 is bigger than 11 and 6 is bigger than 3. Since Player 1 picks the second choice (row) and Player 2 picks the second choice (column), then the final outcome of the game is the cell in the bottom right corner. This means that the value obtained by players 1 and 2 is 5 and 6, respectively.

Let us now consider the following Game to understand a curious concept.

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix} \tag{1.15}$$

As we can see, the values, in this case, are always smaller than the respective values (i.e., in the same position) in the previous example. However if we solve this game (Player 1 chooses row 1 because it dominates the second, Player 2 chooses column 1 because it dominates the second), we obtain $(8,8)$ as a result, which is better than the previous case, in which the optimal outcome was $(5,6)$. This means that the players might obtain a better outcome from a game, even if it has lower utility values than another game in every possible situation.

Adding choices

The games and relative bimatrices we analysed so far offered only two choices for each player since we had two rows and two columns. We can easily represent games with more choices by adding one row, for Player 1, or one column, for Player 2, for each added choice. Consider for instance Game 1.16 and 1.17 which is an extension of the former.

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix} \tag{1.16}$$

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix} \tag{1.17}$$

The optimal value of Game 1.16 is (10, 10), obtained since row and column 1 dominate row and column 2. However, if we consider Game 1.17, we can compute that the optimal value is (1, 1), obtained by choosing row and column 1. This example shows that, in contradiction with what one might think, sometimes having more choices leads to a worst result.

Applicability of the principle of domination

As for now, we have seen games in which it's always possible to apply the Principle 1.6 of elimination of strictly dominated strategies. This isn't however always true, since in some cases we can't find a choice that strictly dominates the others. Consider for instance the following game.

$$\begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}$$

For both players, no choice is strictly dominant (e.g., the first 1 in row 2, dominates the first 0 in row 1, however, the 0 in row 2 doesn't dominate the 1 in row 1). In this case, it's obvious that the best strategy leads to a reward of $(1,1)$, however, we can't define a unique set of independent choices that lead to such choices. The important thing to understand is that Player 1 and Player 2 make a choice independently, i.e., without coordination, hence they can't decide whether to play row and column 2 or row and column 1. For instance, Player 1 could decide to play row 1 and Player 2 could decide to play column 2. This leads to the outcome $(0,0)$, which is not optimal. To optimally solve this game, players should be able to coordinate themselves.

1.3 Examples

Let us now consider some practical examples of games.

1.3.1 Voting game

Let us consider a game with three players P_1, P_2 and P_3. Each player can choose among three options A, B or C and each player has a different preference relation among the options. The relation is shown above.

$$\begin{aligned} P_1 &: A \gneqq B \gneqq C \\ P_2 &: B \gneqq C \gneqq A \\ P_3 &: C \gneqq A \gneqq B \end{aligned}$$

The alternative with more votes (i.e., with at least 2 votes) wins and, in case of a draw, the choice of P_1 is the winning one.

Choice A is dominant for P_1 and, since she/he has a privileged position (in case of a draw her/his choice wins), we can say that P_1 always chooses A. On the other hand, A is dominated by other choices for P_2 and P_3. Moreover, options A and B are not advantageous for P_2 and P_3 respectively, since they are last in the ranking (i.e., are dominated by other choices), hence P_2 won't choose A and P_3 won't choose B. Let us now draw the possible outcomes of the game (Table 1.1) depending on the choices of P_2 and P_3, knowing that P_1 will choose A for sure and that P_2 and P_3 won't choose A and B respectively. From Table 1.1, it might seem that A is the winning option, however, if we check the preferences of players P_2 and P_3, we notice that in both cases C is a better option than A, hence for both players it's better to choose C, otherwise option A would be the winning one, and both players would get a worst result. Note that players P_2 and P_3 both choose C (so that the outcome is C) because they know that all the players use decision theory (Rationality Assumption 1.5) and because they know what are the consequences of the choices of the other players (Rationality Assumption 1.4).

Also notice that P_1 loses the game, even if she/he has a privileged position (she/he chooses the winning option in case of a draw).

P_3 / P_2	C	A
B	A	A
C	C	A

Table 1.1: Outcomes of the voting game when P_1 plays A, P_2 doesn't play A and P_3 doesn't play B.

1.3.2 Prisoner's dilemma

Let us now analyse a classical problem in Game Theory: the prisoner's dilemma. Two players (P_1 and P_2) are convicted to 7 years in jail. Luckily, the judge is so kind as to open to a penalty discount. In particular, the judge will apply the following rules:

- If **both players confess**, they will stay in jail for **5 years**.
- If **only one** of the two players confesses, the one that confesses is **free** while the other is convicted to 7 years.
- If **none of the players confesses**, both have to face **1 year** in jail.
- The players can't communicate.

For starters, we can draw the bimatrix of the game.

$$\begin{pmatrix} (5,5) & (0,7) \\ (7,0) & (1,1) \end{pmatrix}$$

Note that row and column 1 represent the choice of confessing, while row and column 2 represent the choice of not confessing. To check domination, we have to remember that a value is better than the other if it's lower, hence:

- **Row 1** (player 1 confesses) dominates row 2 (player 1 doesn't confess), hence player 1 confesses.
- **Column 1** (player 2 confesses) dominates column 2 (player 2 doesn't confess), hence player 2 confesses.

Following the aforementioned choices, both players are convicted to 5 years of jail each. It's interesting to note that this outcome isn't the best one for both players (it would have been for both to keep their mouths shut). To reach the most advantageous result, both players would have to trust the other not confessing. Say for instance that Player 1, contrary to the Principle 1.6 of elimination of strictly dominated strategies. Player 2 could betray player 1 and confess to being free. Basically, the best solution is obtained with coordination and trust, which we don't have in this game.

1.3.3 Chicken game

Picture the following scenario. Two cars are driving on a narrow road one facing the other. The road is too narrow for both to pass hence one has to swiftly stop at the side of the road. Given this scenario we can define the following values:

- Both players earn a value of -1 if both decide to stop since they lose time.

- If only one player stops, he/she will get a value of 1 while the other a value of 10.
- If neither of the two stops, both will get a value of -10 since they crash.

The bimatrix of the game can be written as follows.

$$\begin{pmatrix} (-1,-1) & (1,10) \\ (10,1) & (-10,-10) \end{pmatrix}$$

As we can see, in this case, we can't find a solution using the Principle 1.6 of elimination of strictly dominated strategies since we can't find a row or column in which each element dominates the respective element in the other rows or columns. It's therefore clear that Principle 1.6 of elimination of strictly dominated strategies can't always be applied and in some cases, more advanced techniques are required.

Part II

Games

Chapter 2

Extensive games

2.1 Extensive games definition

Extensive games are perfect information games, hence it's useful to define what a perfect information game is.

> **Definition 2.1** (Perfect information game)**.** *A game is said to be a perfect information game if every player knows every information about the game (i.e., what happens in the present, what happened in the future and the development of the game).*

2.1.1 Perfect politician game

To help describing extensive games, let us introduce an example: the politician game. The game works as follows:

- Three politicians have to vote a law to increase their salaries.
- The vote is public.
- The law passes (i.e., the the salary is increased) if the majority votes yes.
- The salary is increased even to those politicians that voted no, if the law passes.
- The vote proceeds in a sequential way (i.e., each politician votes after the previous one, knowing what the previous politicians have voted).

It's in the interest of politicians to increase their salaries, however they also have to consider the fact that, voting to increase their salary will decrease their popularity (the vote is public and the public opinion doesn't like politicians who want to increase their salary). Given these considerations, we can define the following values for each outcome of the game:

- If a politician votes **no** and the **law is approved**, he/she gets 4 points.
- If a politician votes **yes** and the **law is approved**, he/she gets 3 points.
- If a politician votes **no** and the **law is not approved**, he/she gets 2 points.

- If a politician votes **no** and the **law is not approved**, he/she gets 1 point.

To wrap things up, we can define extensive games as follows.

> **Definition 2.2** (Extensive game)**.** *An extensive game is a game with the following characteristics:*
>
> - *The players have to vote on a proposal.*
> - *The players vote in sequence.*
> - *The final outcome is determined by the majority of votes.*

2.1.2 Game representation

Extensive games can be represented using a tree structure where:

- Each node is a point in which a player makes a choice.
- Each arc is a choice.
- The leaves are the possible outcomes of the game and we can assign a value to each outcome.

Let's apply what we have just learned to the politician game. The tree representing the game is shown in Figure 2.1.

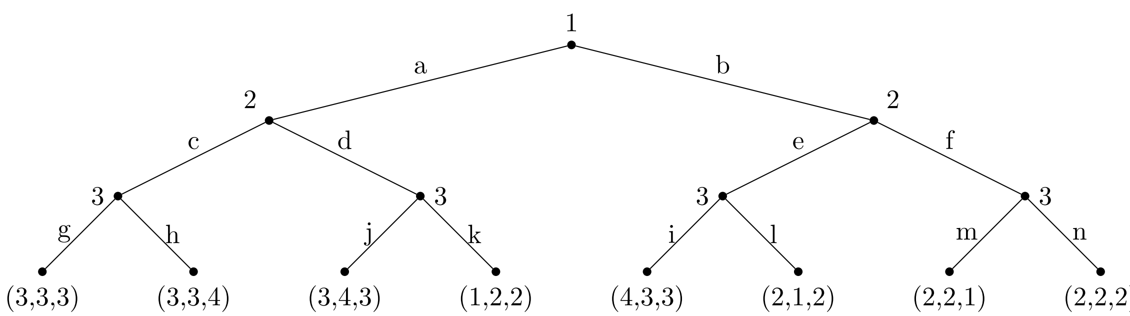


Figure 2.1: The tree representation of the politician game. A left bifurcation represents a *yes* vote, a right bifurcation represents a *no* vote.

Chance

Game trees can also include the concept of randomness. In particular, a random node is used to represent a move done because of a random event. The probability of a random outcome happening is written on the arcs. The arc is usually represented as a dashed line instead of a normal line. Consider, for instance, the following game with two players:

- Player 1 decides whether to play or not the game. If Player 1 doesn't play, both players receive 0 reward.

- Next, Player 2 decides whether to play (if Player 1 decided to play). If Player 2 doesn't play, both players receive 0 reward.
- If both players have decided to play, a coin is flipped and if the result is head (with probability 0.5), Player 1 wins, otherwise Player 2 wins.

The game we just introduced can be represented as in Figure 2.2.

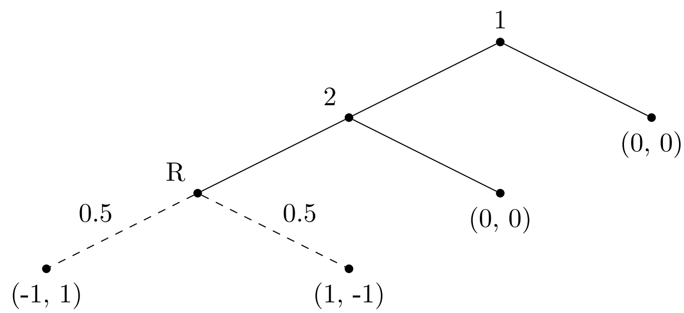


Figure 2.2: The tree representation of a random extensive game.

2.1.3 Formal definitions

After giving a general description of extensive games and their representation, let us state some basic definitions. Let us start by defining what a finite directed graph is.

Definition 2.3 (Finite directed graph). *A finite directed graph is a pair (V, E) where*

- *V is a finite set of vertices.*
- *$E \subset (V \times V)$ is a finite set of edges. In other words. E is a set of ordered pairs of vertices where the first vertex is the vertex where the edge starts and the second is the vertex where the edge ends.*

Now that we know what a finite direct graph is, we can define a path in a finite directed graph as follows.

Definition 2.4 (Path). *A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices and edges*

$$v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$$

such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$. The number k is called the length of the path.

A particular subset of directed graphs is the one of oriented graphs. An oriented graph is defined as follows.

Definition 2.5 (Oriented graph). *An oriented graph is finite directed graph (V, E) having no bi-directed edges, that is for all j, k at most one between (v_j, v_k) and (v_k, v_j) may be arrows*

> *of the graph.*

In other words, an oriented graph is a graph in which for any two vertices v_i and v_j we can't have an edge from v_i to v_j and another from v_j to v_i.

Finally, we have everything we need to define a tree, which we need to represent an extensive game.

> **Definition 2.6** (Tree)**.** *A tree is a triple (V, E, x_0) where (V, E) is an oriented graph and x_0, called root of the tree, is a vertex in V such that there is a unique path from x_0 to x, for every vertex x in V.*

We are one definition away from formalising an extensive game, in fact we have to say what a child is.

> **Definition 2.7** (Child)**.** *Given a tree (V, E, x_0), a child of a vertex $v \in V$ is any vertex $x \in V$ such that $(v, x) \in E$. A vertex is called a leaf if it has no children. We say that the vertex $x \in V$ follows the vertex v if there is a path from v to x.*

Having all the necessary pieces, we can define an extensive game as follows.

> **Definition 2.8** (Extensive form game)**.** *An extensive form game with perfect information consists of:*
>
> 1. *A finite set $N = \{1, \ldots, n\}$ of players (n being the cardinality of N).*
> 2. *A game tree (V, E, x_0).*
> 3. *A partition of the vertices that are not leaves into sets $\{P_1, P_2, \ldots, P_{n+1}\}$.*
> 4. *A probability distribution for each vertex in P_{n+1}, defined on the edges from the vertex to its children.*
> 5. *A n-dimensional vector attached to each leaf representing the list of possible outcomes.*

Let us analyse more in depth some of the points of the definition above:

- The vertices are divided in $n+1$ partitions because we have to consider also the vertices used for randomness. In particular, a partition P_i contains all the possible nodes that represent a choice for player i. The union of all partitions coincide with the set of vertices

  $$\bigcup_{i=1}^{n+1} P_i = V$$

  and no vertex is in more than one partition

  $$P_i \cap P_j = \varnothing \quad i \neq j$$

  In other words, the set P_i, for $i \leq n$, is the set of nodes $v \in V$ where Player i must choose a child of v, representing a possible move from him/her at v. In our example (Figure 2.1), P_1

contains all the vertices with number 1, P_2 all vertices with number 2, P_3 all the vertices with number 3 and the leafs contain the values for each outcome.

- The probability distribution mentioned at point 4 is used to model randomness. In particular, we need to define, for each edge exiting from a random node, the probability that a such choice is made.
- In point 5 the definition says that for each leaf we have a vector of n dimensions (where n is the number of players). Each vector contains the values of an outcome (represented as a leaf) for each player. Basically, given an outcome, the first element is the value of such outcome for Player 1, the second is the value for Player 2 and so on.
- The partition P_{n+1}, representing a random choice node, can be empty and it is when a game involves no randomness. Also note that, when P_{n+1} is empty, the players have only preferences on the leaves, hence a utility function is not required.

2.2 Solving an extensive game

2.2.1 Backward induction

Defining an extensive game is important, however we would also like to solve it. The basic idea for solving an extensive game is to start from the leafs of the tree and climb the tree to the root x_0 applying Rationality Assumptions 1.4 and 1.5. Basically:

1. The last player chooses, for each choice he/she can do, the best outcome for him/her (applying Rationality Assumption 1.5).
2. The second-last player chooses, for each choice he/she can do, the best outcome for him/her (applying Rationality Assumption 1.5), knowing that the last player has chosen the best outcomes for him/her (Rationality Assumption 1.4).
3. The algorithm goes on like this...

Now that we have a general idea of how the algorithm works, let us formally define what the length of a game is and then introduce the algorithm to solve extensive games.

Definition 2.9 (Game length)**.** *The length of a game, represented with the tree (V, E, x_0), is the longest path from the root x_0 to a leaf $v \in V$.*

The algorithm for solving extensive form games is called backward induction and recursively find a solution starting from the leaves of the game tree.

Algorithm 2.1 (Backward induction)**.** *Backward induction allows to solve games recursively:*

1. *Applying Rationality Assumption 1.5 one can solve all the games of length 1.*
2. *Applying Rationality Assumption 1.4 one can solve all the games of length $i+1$, knowing that the games of length at most i have been solved.*

Thanks to backward induction, we can solve games of any length since we can start solving all the games of length one (i.e., the choices of the last player) using the first rule and then solve the games of bigger length. Note that when we speak about games of length 1 we refer to a sub-tree of the game tree where the longest path from the root of the sub-tree to the leafs is 1. This means that backward induction can be applied when every vertex is the root of a sub-tree (i.e., of a sub-game). With this in mind, we say that we start solving each sub-tree of length 1, then we solve all the sub-trees of length 2 and so on until we reach x_0 and the whole game is solved.

The correctness of Algorithm 2.1 is stated by the following theorem.

Theorem 2.1 (First rationality theorem)**.** *The rational outcomes (solutions) of a finite, perfect information game are those given by the procedure of backward induction.*

Theorem 2.1 states that Algorithm 2.1 always returns the optimal outcome of a game, however, we must remember that such outcome might not be unique.

2.2.2 Examples

Before going on analysing extensive games, let us stop to give a look at some examples. We will first solve Game 1.1 and then we will introduce a simple game with no unique solution.

Politician game

To solve the politician game (Figure 2.1) we can apply the backward induction algorithm (Algorithm 2.1). In particular:

1. Initially, we focus on all the games of length 1, i.e., all the possible choices of Player 3.
   - If we consider the sub-tree on the left, path h is better for Player 3 (since the last value of the vector, i.e., the one related to Player 3, is higher for the leaf reached by h), hence he/she chooses **path** h.
   - If we consider the sub-tree on the middle-left, path j is better for Player 3, hence he/she chooses **path** j.
   - If we consider the sub-tree on the middle-right, path i is better for Player 3, hence he/she chooses **path** i.
   - If we consider the sub-tree on the right, path n is better for Player 3, hence he/she chooses **path** n.
2. Knowing which paths Player 3 will follow, Player 2 chooses:
   - **Path** d since it takes (considering that Player 3 has chosen path j over path k) to an outcome with value 4 (leaf $(3, 4, 3)$) which is bigger than the value obtained following $c \to h$ (3, from leaf $(3, 3, 4)$).
   - **Path** e since it takes (considering that Player 3 has chosen path i over path l) to an outcome with value 3 (leaf $(4, 3, 3)$) which is bigger than the value obtained following $f \to n$ (2, from leaf $(2, 2, 2)$).
3. Knowing which path Players 2 and 3 followed, Player 1 chooses:

- **Path** b since it takes to an outcome with value 4 (leaf $(4, 3, 3)$) which is bigger than the value obtained following $a \to c \to h$ (value 3, from leaf $(3, 3, 4)$).

The optimal solution is therefor $b \to e \to i$.

Non unique solution

Let us consider the game represented by the tree in Figure 2.3. In this game, Player 1 chooses twice, once before and once after Player 2. Let's apply backward induction (Algorithm 2.1) on this tree:

1. At the first step we can solve the sub-tree of length 1. Path f is more advantageous for him/her, hence he/she chooses **path** f.
2. At the second step, knowing what Player 1 chose, Player 2 can either choose path $c \to f$, which leads to a value of 3 for him/her (leaf $(4, 3)$) or path d, which also leads to a value of 3 (leaf $(0, 3)$). Note that in this case we have used, as we should always do, the assumption that players are selfish.
3. Not knowing how Player 2 breaks ties (i.e., how he/she chooses between two outcomes with same value), Player 1 has to analyse all possible outcomes.
   - If Player 2 chooses path $c \to f$, then it's better to follow arc a and get a reward of 4 (from leaf $(4, 3)$) following path $a \to c \to f$.
   - If Player 2 chooses path d, then it's better to follow arc b and get a reward of 3 (from leaf $(4, 3)$), since by choosing a, Player 1 would have gotten a reward of 0 (leaf $(0, 3)$).

As we can see, this game has two different solutions, namely $(4, 3)$ and $(3, 4)$. Another important thing to note is that even if backward induction always finds a solution (under the right assumptions), it can be applied only to small games due to the complexity of the algorithm.

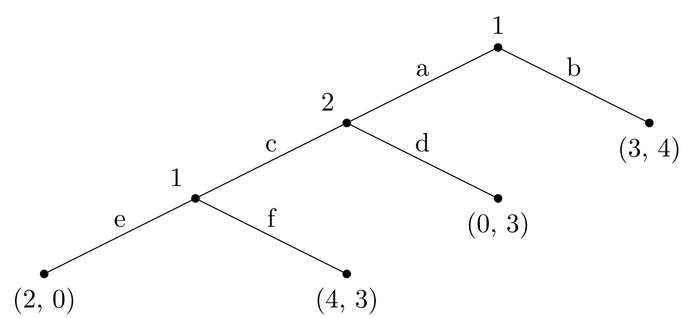


Figure 2.3: An example of game with non unique solution.

2.3 The chess theorem

An important result of Game Theory regarding extensive games is the chess theorem, stated by Von Neumann.

Theorem 2.2 (Chess theorem (Von Neumann)). *In the game of chess one and only one of the following alternatives holds:*

1. *The white has a way to win, no matter what the black does.*
2. *The black has a way to win, no matter what the white does.*
3. *The white has a way to force at least a draw, no matter what the black does, and the same holds for the black.*

Proof. Let us assume that the length of a game of chess is $2k$, hence each player has k moves (i.e., k choices, k turns to play) to do. Let us call

- a_i the move (i.e., the choice) done by the white player at turn i.
- b_i the move (i.e., the choice) done by the black player at turn i.

With this setup in mind we can write the first result of the theorem as

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \exists a_3 : \ldots \exists a_k : \forall b_k \Rightarrow \text{ white wins} \tag{2.1}$$

Basically, we are saying that it exists an initial move a_1 such that, for every possible move the black does, the white has a sequence of moves to win, independently what move is chosen by the black player at any given turn.

Say that 2.1 is wrong. Remembering that $\neg(\forall x A(x)) \equiv \exists x : \neg A(x)$ and $\neg(\exists x : A(x)) \equiv \forall x \neg A(x)$ and applying the logical negation to Proposition 2.1 we obtain

$$\begin{aligned}
\neg(\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \exists a_3 : \ldots \exists a_k : \forall b_k \Rightarrow \text{ white wins})\\
\forall a_1 \neg(\forall b_1 \exists a_2 : \forall b_2 \exists a_3 : \ldots \exists a_k : \forall b_k \Rightarrow \text{ white wins})\\
\forall a_1 \exists b_1 : \neg(\exists a_2 : \forall b_2 \exists a_3 : \ldots \exists a_k : \forall b_k \Rightarrow \text{ white wins})\\
\forall a_1 \exists b_1 : \forall a_2 \exists b_2 : \forall a_3 \exists b_3 \ldots \forall a_k \exists b_k \Rightarrow \text{ white doesn't win}
\end{aligned}$$

The last result says that, for any given starting move a_1, there exist a move b_2 such that, independently from the choices of the white player, he/she can't win and the black player can either draw or win. Basically, negating the first alternative of the chess theorem we obtained either the second or third alternative, hence proving the theorem (we can easily swap the role of black and white to prove that negating the second alternative we obtain the first or the third one). □

The chess theorem (Theorem 2.2) has a corollary that considers games in which it's impossible to reach a draw.

Theorem 2.3 (Chess theorem). *Consider a finite perfect information game with two players, where the only possible outcomes are the victory of one or the other player. Then one and only one of the following alternative holds:*

1. *The first player can win, no matter what the second one does.*
2. *The second player can win, no matter what the first one does.*

An important thing to notice about the chess theorem (Theorem 2.2) is that we know that a game such as chess ends in one of three ways, however we can't always tell in which of the three ways. In particular, we can't divide solutions into tree categories:

- **Very weak solutions**. A solution is said to be very weak when it's the rational outcome of a game (one of the three alternatives of Theorem 2.2), however we don't know which is the solution. The solution to the chess game is a very weak solution since we know, thanks to Theorem 2.2, that either one player wins or the game ends in a draw, however we don't know which is the actual solution among the three.
- **Weak solutions**. A solution is said to be weak if we know which of the three alternatives stated in Theorem 2.2 is the rational outcome of the game, however we don't know how to get there.
- **Solutions**. A solution is a rational outcome of a game and we know both the resulting outcome (out of the three of Theorem 2.2) and the algorithm to get such result.

2.4 Impartial combinatorial games

An important subset of extensive games are the impartial combinatorial games. An impartial combinatorial game is formally defined as follows.

> **Definition 2.10** (Impartial combinatorial game)**.** *An impartial combinatorial game is a game such that:*
>
> 1. *There are two players moving in alternate order.*
> 2. *There is a finite number of positions in the game.*
> 3. *The players follow the same rules.*
> 4. *The game ends when no further moves are possible.*
> 5. *The game does not involve chance.*
> 6. *In the classical version, the winner is the player leaving the other player with no available moves, in the misère version the opposite.*

To make things clearer, here is an example of impartial combinatorial game:

- A set of cards is divided into k piles of cards.
- At her/his turn each player takes as many cards as she/he wants (at least one) from one and only one pile.
- The player that can't draw any cards (i.e., the one remaining without cards) loses.

The game aforementioned has many variants, but they are all based on the same core game. In some cases the player can draw only a specific number of cards, in other not less than a fixed number.

2.4.1 Game representation

Impartial combinatorial games can be represented using a vector of positions. In particular, in the example we are considering, the game at a given stage can be represented as a vector of k non negative integers $(n_1, \ldots, n_k)$ where each element n_i represents the number of cards in deck i.

2.4.2 Solving impartial combinatorial game

Before looking into how impartial combinatorial games can be solved, we have to introduce some concepts. For starters, we can divide the set of all possible states (also called positions) $(n_1, \ldots, n_k)$ in which a game can be into two sets:

- **P-positions**. The set of P-positions contains all the loosing positions. This means that if a player finds himself/herself in a P-position, he/she is loosing.
- **N-positions**. The set of N-positions contains all the winning positions. This means that if a player finds himself/herself in a N-position, he/she is winning.

Note that, splitting the positions (i.e., states) into two sets, we highlight that we care about the state of the game and not the player moving. In other words, it is the state of the game that matters, and not who is called to move. Also notice that, when a player is in a certain position, it still has to play.

Now that we know P and N positions we can define the rules to move from one position to the other, in particular:

1. The terminal position is a P-position.
2. From a P-position, only N-positions are available. This means that, if a player is in a losing position, after a move, the other player finds itself in a winning position.
3. From a N-position it's possible to go to a P-position (but also to a N-position).

This means that the player starting in a N-position wins, in fact from a N-position it can always force a transition to a P-position (rule 3) from which it's only possible to go back to a N-position. Say for instance that Player 1 starts in a N-position. If she/he can find a way to transition to a P-position, then Player 2 can only transition to a N-position. This means that, as long as Player 1 can find a way to move from a N-position to a P-position, he/she will always be in a N-position and Player 2 will always be in a P-position, without the possibility to change this trend.

2.4.3 Nim game

In order to prove the rules to move from a position to the other (and consequently that a player that starts in an N-position always wins), we have to introduce an example of impartial combinatorial game, the Nim game, and a algebraic group. Let's start from the Nim game

> **Definition 2.11** (Nim game)**.** *The Nim game is defined as* $(n_1, ..., n_k)$ *where* n_i *is a positive integer for all* i*. At her/his turn any player is supposed to take one (and only one)* n_i *and replace it with* $\hat{n} < n$*. The winner is the player who arrives at the position* $(0, \ldots, 0)$*, i.e., that after playing reaches position* $(0, \ldots, 0)$*. Basically, the goal is to leave the other player in the position* $(0, \ldots, 0)$*.*

Next, we have to define a new operation $\oplus$ on the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of naturals numbers.

> **Definition 2.12** (Nim sum)**.** *For any two numbers* $n_1, n_2 \in \mathbb{N}$*, the sum* $\oplus$ *is computed as follows*

1. *Write n_1 and n_2 in binary. Their binary representation can be written as $[n_1]_2$ and $[n_2]_2$.*
2. *Write the sum of $[n_1]_2$ and $[n_2]_2$ without considering the carry, for each binary digit. This means that, 0+1+1, is equal to 0 and no carry is taken to the column on the left.*

To clear things up, if we want to sum $1_{10} = 01_2$, $1_{10} = 01_2$ and $2_{10} = 10_2$ we get $10_2 = 2_{10}$.

$$\begin{array}{rccc} 1_{10}= & 0 & 1 & \oplus \\ 1_{10}= & 0 & 1 & \oplus \\ 2_{10}= & 1 & 0 & = \\ \hline & 1 & 0 & =2_{10} \end{array} \tag{2.2}$$

Thanks to the new operation we have just introduced we can define a group (see Appendix A.1) on the set $\mathbb{N}$ defined as follows.

Definition 2.13 (Nim group)**.** *The set of natural numbers $\mathbb{N}$ with the operation $\oplus$ of sum without carry is an abelian group, 0 being the identity.*

$$(\mathbb{N}, \oplus, 0)$$

The Nim group we have just defined can prove itself very useful, in fact it can be used to prove the rules that regulate P and N positions. Consider the following theorem.

Theorem 2.4 (Bouton)**.** *In a Nim game, a position $(n_1, \ldots, n_k)$ is a P-position if and only if the sum of the values $n_1, \ldots, n_k$ is 0.*

$$(n_1, \ldots, n_k) \text{ P-position} \iff \bigoplus_{i=1}^{k} n_i = 0$$

Proof. To prove this theorem, and the three rules of N and P positions, we have to remember that in a group, the cancellation law holds (see A.1), hence we can say

$$n_1 \oplus n_2 = n_1 \oplus n_3 \Rightarrow n_2 = n_3$$

Let's prove the three rules one by one:

1. **The terminal position is a P-position**. The position $(0, \ldots, 0)$, which is by definition a P-position, has sum 0.
2. **From a P-position, only N-positions are available**. Say we are in a position in which $n_1 \oplus \cdots \oplus n_k = 0$. After picking a deck (say the first, without loss of generality) and removing some cards, we reach a position $(\hat{n}_1, \ldots, n_k)$ where $\hat{n}_1 < n_1$. If the sum of the elements in the new position was 0, we would have

   $$n_1 \oplus \cdots \oplus n_k = 0$$

   and

   $$\hat{n}_1 \oplus \cdots \oplus n_k = 0$$

which means that

$$n_1 \oplus \cdots \oplus n_k = 0 = \hat{n}_1 \oplus \cdots \oplus n_k$$

For the cancellation law

$$n_1 \oplus \cdots \oplus n_k = \hat{n}_1 \oplus \cdots \oplus n_k \Rightarrow n_1 = \hat{n}_1$$

which is clearly wrong because a player has to pick at least one card, hence n_1 must be strictly smaller than $\hat{n}_1$. This means that it's not possible to go from a P-position to another P-position.

3. **From a N-position it's possible to go to a P-position**. To prove this statement we only have to find a way to go to a P-position, starting from a N-position. Luckily, we have an algorithm to do such thing:

   (a) Expand all elements n_i in binary form.

   (b) Pick one element with a 1 in the leftmost column with an even number of 1s.

   (c) Put 0 in the leftmost column with an even number of 1s of the chosen deck.

   (d) Flip the bits of the chosen deck for in all those columns in which the result is 1. It's easy to demonstrate that the new number is smaller than the previous one since a 1 followed by m zeros is always bigger than a number with a 0 followed by m ones.

   To better show how the algorithm works, let us consider the game $(4, 6, 5) = (100_2, 110_2, 101_2)$. Since $100_2 \oplus 110_2 \oplus 101_2 = 111_2$ we are in a N-position.

$$\begin{array}{rccccc} 4_{10}= & 1 & 0 & 0 & \oplus \\ 6_{10}= & 1 & 1 & 0 & \oplus \\ 5_{10}= & 1 & 0 & 1 & = \\ \hline & 1 & 1 & 1 & \end{array}$$

   Every number has a 1 in the leftmost position, hence we can arbitrarily choose one of the three. Say we pick the first deck. Applying step 3 of the algorithm we change 100_2 into 011_2 since

   - The bit on the left is changed to 0 by rule.
   - The bit in the middle is flipped given that the middle bit of the result is 1.
   - The bit on the right is flipped given that the rightmost bit of the result is 1.

   After this move we reach state $(3, 6, 5) = (011_2, 110_2, 101_2)$. If we compute the sum between 011_2, 110_2 and 101_2 we get 0, hence the new state is a P-position.

$$\begin{array}{rccccc} 3_{10}= & 0 & 1 & 1 & \oplus \\ 6_{10}= & 1 & 1 & 0 & \oplus \\ 5_{10}= & 1 & 0 & 1 & = \\ \hline & 0 & 0 & 0 & \end{array}$$

   Note that, from a N-position is also possible to stay in a N-position, however, having found an algorithm to go to a P-position, a player in a N-position can for the other player to loose.

□

2.5 Strategies

When talking about games we usually associate the word strategy. Backward induction is actually defining a strategy for each player, hence it's useful to introduce a strategic representation of a game. Strategies can be divided in

- **Pure strategies**.
- **Mixed strategies**.

In particular, if we call P_i the set of all nodes where Player i has to take a choice (i.e., to make a move), we define a pure strategy as follows.

> **Definition 2.14** (Pure strategy). *A pure strategy for Player i is a function defined on the set P_i, associating to each node $v \in P_i$ a child x, or equivalently an edge (v, x).*

In other words, a pure strategy defines a specific move for each possible point where a player can make a choice. On the other hand, we can define a mixed strategy as:

> **Definition 2.15** (Mixed strategy). *A mixed strategy for Player i is a probability distribution on the set of the pure strategies for i. If Player i has n pure strategies, then her/his set of mixed strategies is*
>
> $$\sum_n = \{p = (p_1, \dots, p_n) : p_j \geq 0 \wedge \sum_j p_j = 1\} \tag{2.3}$$
>
> *where p_j is the probability to use strategy j.*

In practice a strategy is a function that associates to each node (i.e., to each possible choice or bifurcation) the best choice a player can do. Note that we aren't saying if a strategy is good or not, we are only defining a way to associating a node with one of its children. The final outcome solution depends on the mixed strategies.

2.5.1 Strategy matrix representation

Given a game tree, we can obtain a matrix containing all the possible strategies in a game, with the outcome they lead to. The strategies' matrices contains every strategy, even those that are useless, repeated or impossible. Also note that each player has its own strategy matrix. To build the matrix for Player i, we have to consider every possible combination of edges. In particular:

- On the columns we put the combination of strategies of Player i. Basically, we have to take all vertices $v \in P_i$ and consider all the possible combinations of the edges exiting from the vertices in P_i (removing combinations of edges that exit from the same vertex). Consider for instance $P_i = \{v_1, v_2\}$ where two edges a, b exit from v_1 and c, d exit from v_2. All possible combinations (which are written on the columns of the matrix) are ac, ad, bc, bd.
- On the rows we have to combine the strategies of every other player. In the simple case of games with two players, the rows contain the combination of strategies of the other player.

- Each cell is filled with the value obtained when combining the strategies on the row and column of the cell. Basically, given a combination of strategies we have to follow the path defined by the sequence of strategies and insert the outcome obtained following such path.

Note that, the matrix we obtain is not necessarily a square matrix. The matrix and tree representations are equivalent however they analyse the problem under two different lights:

- The **tree** representation is a **sequential and extensive** representation that highlights the fact that players make moves one after the other.
- The **matrix** representation is a **parallel** representation used to highlight the concept of **strategy**. Another difference is that the matrix representation is redundant since it contains more information that needed (e.g., impossible paths).

Example

Before going on, let us analyse an example to check if we fully understood how to write the matrix representation of a game, given its tree representation. Let us consider the tree in Figure 2.4.

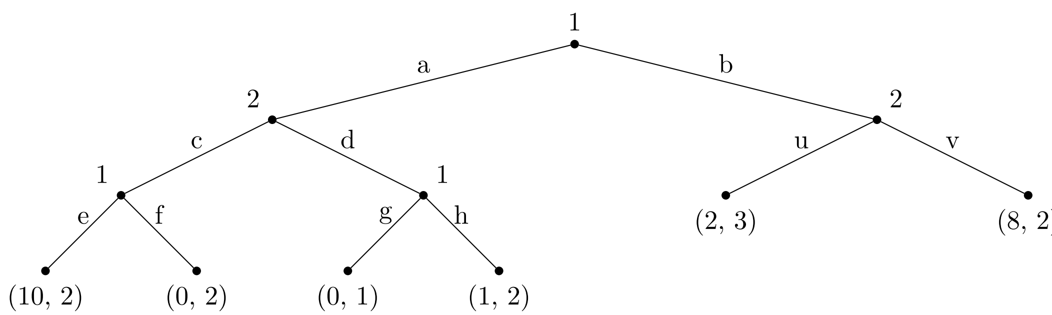


Figure 2.4: The tree representation of a game.

Say we want to build the strategy matrix from Player 2's perspective.

1. First we have to find all the combinations of edges exiting from the nodes in P_2. Since all the arcs are c, d, u and v, the combinations (which should be written on the columns) are cu, cv, du and dv.
2. After finding Player 2's combinations we have to consider Player 1's. The edges exiting from the vertices in P_1 are a, b, e, f, g, h, hence we will write on the rows aeg, aeh, afg, afh, beg, beh, bfg and bfh.

The resulting matrix is shown in Table 2.1. Note that the table we have obtained can be used to represent the game for Player 1. In particular, the table we have obtained is general and we see the columns as the strategy of a player and the rows as the strategy of the other.

2.5.2 Von Neumann theorem for strategies

Now that we know the concept of strategy we can state Theorem 2.2 using strategies.

	cu	cv	du	dv
aeg	(10, 2)	(10, 2)	(0, 1)	(0, 1)
aeh	(10, 2)	(10, 2)	(1, 2)	(1, 2)
afg	(0, 2)	(0, 2)	(0, 1)	(0, 1)
afh	(0, 2)	(0, 2)	(1, 2)	(1, 2)
beg	(2, 3)	(8, 2)	(2, 3)	(8, 2)
beh	(2, 3)	(8, 2)	(2, 3)	(8, 2)
bfg	(2, 3)	(8, 2)	(2, 3)	(8, 2)
bfh	(2, 3)	(8, 2)	(2, 3)	(8, 2)

Table 2.1: The matrix representation of the tree in Figure 2.4.

Theorem 2.5 (Chess theorem (Von Neumann)). *In the chess game one and only one of the following alternatives holds:*

1. *The white has a winning strategy.*
2. *The black has a winning strategy.*
3. *Both players have a strategy leading them at least to a tie.*

If we were to represent this theorem on a strategy matrix, we would say that either:

1. One row has all white results. This makes sense because, a row contains all possible strategies of the white player and if all results on a row are white, then the black can't win, independently from what he/she plays.
2. One column has all black results. This makes sense because, a column contains all possible strategies of the black player and if all results on a column are black, then the white can't win, independently from what he/she plays.
3. On rows, a mixture of ties and white results and on columns, a mixture of ties and black results.

The Von Neumann theorem excludes the possibility that there isn't a winning strategy for the white (a row with all white results) and there isn't a winning strategy for the black (a column with all black results).

2.5.3 Complexity

To wrap things up, let us analyse the complexity of the strategy representation. In particular, the number of strategies increases exponentially because each node's children. Given the partition of nodes for a given player $P_i = \{v_1, \dots, v_k\}$ where a node $v_j \in P_i$ has n_j children, the total number of strategies are

$$\prod_{i=1}^{k} n_i \tag{2.4}$$

Note that we are considering the product of the number of children because we have to combine all the possible strategies, i.e., all the possible edges exiting from every node. This means that the number of possible strategy is very big, even for small games.

2.6 Games with imperfect information

In some cases, players must make moves at the same time, and so they cannot have full knowledge of each other's moves. This means that we can't talk about perfect information anymore. Luckily, we can still represent such games using trees. In particular, we have to add a dashed line between the nodes in which a player might be. Say we have two nodes $v_a \in P_i$ and $v_b \in P_i$. If Player j doesn't know if Player i is in v_a or v_b then we should draw a dashed line between v_a and v_b. Basically, the dashed line marks the lack of full knowledge.

For instance, we can represent the prisoner dilemma as shown in Figure 2.5.

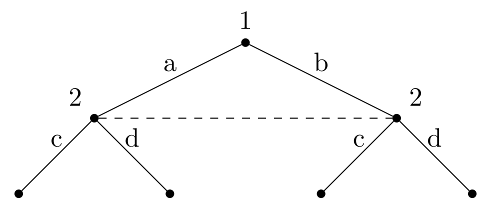


Figure 2.5: The tree representation of the prisoner dilemma.

2.6.1 Formal definition

As one might have understood, we like formal definitions, so it's time to give one for extensive games with imperfect information. However, before giving a precise definition of an extensive game with imperfect information, we have to introduce the concept of an information set.

Definition 2.16 (Information set)**.** *An information set for a Player i is a pair $(U_i, A(U_i))$ with the following properties:*

1. *$U_i \subset P_i$ is a nonempty set of vertices $v_1, \dots, v_k$.*
2. *Each vertex $v_j \in U_i$ has the same number of children.*
3. *$A_i(U_i)$ is a partition of the children of $v_1 \cup \dots \cup v_k$ with the property that each element of the partition contains exactly one child of each vertex v_j.*

The information set U_i is used to say that Player i knows he/she is in one of the vertices of U_i, however he/she doesn't know in which one. Moreover, each set in $A_i(U_i)$ represents an available move for Player i. Graphically, $A_i(U_i)$ is the same choice, i.e. an edge, coming out of the different vertices).

Now that we know the concept of information set, we can formally define an extensive game with imperfect information.

Definition 2.17 (Extensive game with imperfect information)**.** *An extensive form game with imperfect information is constituted by*

1. *A finite set $N = \{1, \dots, n\}$ of players.*

2. *A game tree* (V, E, x_0)*.*
3. *A partition made by sets* $P_1, P_2, \ldots, P_{n+1}$ *of the vertices which are not leaves.*
4. *A partition* $(U_i^j), j = 1, \ldots, k_i$ *of the set* P_i*, for all* i*, with* (U_i^j, A_j^i) *information set for all players* i *for all vertices* j *(with the same number of children).*
5. *A probability distribution, for each vertex in* P_{n+1}*, defined on the edges going from the vertex to its children.*
6. *An* n*-dimensional vector attached to each leaf.*

Note that if the partition comprises just a single vertex, a game with imperfect information becomes the same as a game with perfect information.

2.6.2 Solving imperfect information games

Knowing the formal definition of imperfect information games, we can show a way for solving them. An imperfect information extensive game can be solved similarly to a perfect information one. We only have to redefine the concept of pure (and mixed) strategy.

Definition 2.18 (Pure strategy for perfect information extensive games)**.** *A pure strategy for Player* i *in an imperfect information game is a function defined on the collection* U *of his information sets and assigning to each* $U_i \in U$ *an element of the partition* $A(U_i)$*. A mixed strategy is a probability distribution over pure strategies.*

This means that a game of perfect information is a particular game of imperfect information where all information sets of all players are singletons (i.e. consist of only one vertex).

Chapter 3

Zero-sum games

3.1 Game description

Zero-sum games are games in which everything is played simultaneously (i.e., at the same time, hence without a sequential order) and two players compete one against the other. The main feature of zero-sum games, which gives them their name, is that the reward gained by Player 1 is lost by Player 2. Formally,

> **Definition 3.1** (Zero sum game). *A two-player zero-sum game in strategic form is a triplet*
>
> $$\big(X, Y, f : X \times Y \to \mathbb{R}\big) \tag{3.1}$$
>
> *where*
>
> - *X is the strategy space of Player 1.*
> - *Y is the strategy space of Player 2.*
> - *$f(x, y)$ is what Player 1 gets from Player 2 when they play respectively $x \in X$ and $y \in Y$. Basically, f is the utility function of Player 1.*

Since f is the utility function of Player 1 and Player 2 gains what Player 1 loses, we can write the utility function $g : X \times Y \to \mathbb{R}$ of Player 2 as

$$g = -f$$

The intrinsic symmetry of zero-sum games helps us with the analysis.

3.2 Finite games

Let's start our journey through zero-sum games by analysing finite zero-sum games. In finite zero-sum games, players have a finite number of strategies n and m respectively, hence we can write the strategy sets as

$$X = \{1, 2, \dots, n\}$$

and

$$Y = \{1, 2, \ldots, m\}$$

The game can therefore be described, in strategic form, by a $n \times m$ payoff matrix, called **payoff matrix**, that has

- One row for each strategy of Player 1.
- One column for each strategy of Player 2.

$$\begin{pmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1m} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{im} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nj} & \cdots & p_{nm} \end{pmatrix} \tag{3.2}$$

The element p_{ij} is the payoff, or the payment, of Player 2 to Player 1 when they play strategies $i \in X$ and $j \in Y$. Note that the payments p_{ij} are integer numbers (i.e., in $\mathbb{Z}$), hence we can also represent the case in which Player 1 pays Player 2 using a negative value as a payoff.

3.2.1 Solving finite zero-sum games

Since Player 1 gains what Player 2 loses, the former wants to maximise the payoff (in fact, the payoff is what he/she gains). This makes sense even if we consider negative numbers, in fact, by maximising the payoff, Player 1 tries to avoid negative payoffs (i.e., he/she doesn't want to pay Player 2). On the other hand, since the payoff is what Player 2 loses, he/she wants to minimise the payoff (i.e., he/she wants to give away as few as possible). Even in this case, this reasoning makes sense for negative numbers since by minimising the payoff, Player 2 chooses negative numbers (if available), which means that he/she will receive a payment from Player 1. This leads us to say that:

- Player 1 can guarantee herself/himself to get at least

$$v_1 = \max_i \min_j p_{ji} \quad \forall i \in Y, j \in X$$

- Player 2 can guarantee himself/herself to get at least

$$v_2 = \min_j \max_i p_{ji} \quad \forall i \in Y, j \in X$$

The values v_1 and v_2 are called **conservative values** of Player 1 and Player 2, respectively, since they only identify the bounds (lower and upper, respectively) of the game solution's value. Note that v_1 is smaller or at least equal to v_2.

$$v_1 \leq v_2$$

Let's make an example to better understand how conservative values are computed. Consider the zero-sum Game 3.3.

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix} \tag{3.3}$$

First, we can compute Player 1's conservative value by considering each row and taking the minimum value for each row. Player 1's conservative value is the maximum among the values we have just found. In practice we have:

$$\begin{aligned} v_1 &= \max_i\{\min_j\{4,3,1\}, \min_j\{7,5,8\}, \min_j\{8,2,0\}\} \\ &= \max_i\{1,5,0\} \\ &= 5 \end{aligned}$$

Similarly, for Player 2 we can compute the conservative value by taking the maximum value on each column and than take the minimum among these values. In practice:

$$\begin{aligned} v_2 &= \min_j\{\max_i\{4,7,8\}, \max_i\{3,5,2\}, \max_i\{1,8,0\}\} \\ &= \min_j\{8,5,8\} \\ &= 5 \end{aligned}$$

When choosing the best strategy for each player, we always have to consider that the players are rational. If we assume that:

- $v_1 = v_2 = v$.
- $\bar{\imath}$ is the row such that $p_{\bar{\imath}j} \geq v_1 = v \quad \forall j \in Y$.
- $\bar{\jmath}$ is the column such that $p_{i\bar{\jmath}} \leq v_2 = v \quad \forall i \in X$.

Then $p_{\bar{\imath}j} = v$, $p_{i\bar{\jmath}} = v$ and v is the optimal outcome of the game. Basically, we are saying that if the upper and lower values coincide and are equal to v, then v is the optimal outcome of the game. This result is justified by the fact that:

- $\bar{\imath}$ is the optimal strategy for Player 1 since she/he can't get more than v_2, since v_2 is the conservative value of Player 2 (i.e., the highest value that Player 2 is willing to pay).
- $\bar{\jmath}$ is the optimal strategy for Player 2 since he/she can't get more than v_1, since v_1 is the conservative value of Player 1 (i.e., the lowest value that Player 1 is willing to receive).

In other words, we can say that

- $\bar{\imath}$ maximises the function $\alpha(i) = \min_j p_{ji}$.
- $\bar{\jmath}$ minimises the function $\beta(j) = \max_i p_{ji}$.

3.3 Arbitrary games

Games aren't always finite hence we have to extend the theory of finite zero-sum games to arbitrary, infinite zero-sum games. Definition 3.1 still holds, however, the strategy sets X and Y are not necessarily finite anymore. Representing the game with a matrix of payoffs p_{ij} is also not possible. We should use the function $f(x, y)$, instead.

This means that we also have to revise the conservative values of Player 1 and Player 2. In particular, we can say that:

- Player 1 can compute her/his conservative value v_1 as

$$v_1 = \sup_x \inf_y f(x, y) \tag{3.4}$$

- Player 2 can compute his/her conservative value v_2 as

$$v_2 = \inf_y \sup_x f(x, y) \tag{3.5}$$

As before, if the conservative values coincide and are equal to v, we can say that the game has a value of v.

3.3.1 Solving arbitrary zero-sum games

As for finite sum games, if we assume that:

- $v_1 = v_2 = v$.
- There exist a strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v \quad \forall y \in Y$.
- There exist a strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v \quad \forall x \in X$.

then

- $\bar{x}$ is an optimal strategy for Player 1.
- $\bar{y}$ is an optimal strategy for Player 2.
- v is the rational outcome of the game.

This result holds because Player 2 won't give away more than v, since he/she chooses $\bar{y}$ for which $f(x, \bar{y}) \leq v$, hence Player 1 has to choose a value which is equal to v, i.e., $f(\bar{x}, y)$. The same happens, with opposite roles. In other words, Player 1 can't have more than v since Player 2 won't give away more than that. More formally:

- $\bar{x}$ is the optimal for Player 1 since it maximises the function $\alpha(x) = \inf_y f(x, y)$. Basically, $\alpha(x)$ is the value of the optimal choice of Player 2 if he/she knows that Player 1 plays x. This means that Player 1 will take the best choice when Player 2 takes the best for himself/herself.
- $\bar{y}$ is the optimal for Player 2 since it minimises the function $\beta(x) = \sup_x f(x, y)$. Basically, $\beta(y)$ is the value of the optimal choice of Player 1 if he/she knows that Player 2 plays y. This means that Player 2 will take the best choice when Player 1 takes the best for herself/himself.

Different conservative values

Up until now, we have assumed that the conservative values coincide and are equal to v. This isn't however always the case, hence we have to describe a way to handle such cases. The following theorem fixes our new setting.

Theorem 3.1 (Different conservative values)**.** *Let X and Y be non-empty sets and let f :*

$X \times Y \to \mathbb{R}$ be an arbitrary real function. Then

$$v_1 = \sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y) = v_2 \tag{3.6}$$

Proof. To demonstrate Theorem 3.1, we can start by writing that, by definition it's true that, for each x, y

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus we can write

$$\inf_y f(x,y) \leq \sup_x f(x,y)$$

When discussing the optimal outcome, we have used the functions $\alpha(x) = \inf_y f(x,y)$ and $\beta(x) = \sup_x f(x,y)$, hence we can replace them in the previous equation to obtain

$$\alpha(x) \leq \beta(y)$$

Since the previous inequality is true for each x and y then it's also true that

$$\sup_x \alpha(x) \leq \inf_y \beta(y)$$

from which, replacing the definitions $v_1 = \sup_x \alpha(x) = \sup_x \inf_y f(x,y)$ and $v_2 = \inf_y \beta(y) = \inf_y \sup_x f(x,y)$, follows

$$v_1 \leq v_2$$

$\square$

Formally, when $v_1 < v_2$, given an utility $n \times m$ matrix $\mathbf{P}$, we can identify:

- The **strategy spaces** as

$$\sum_k = \{\mathbf{x} = (x_1, \ldots, x_k) : x_i \geq 0 \wedge \sum_{i=1}^{k} x_i = 1\}$$

  with $k = n$ for Player 1 and $k = m$ for Player 2. What we are saying is that the strategy space of Player a is the space of all possible probability distributions over the possible moves of that player.

- The **utility function** as

$$f(x,y) = \sum_{i=1,\ldots,n;j=1,\ldots,m} x_i y_j p_{ij} = (\mathbf{x}, \mathbf{P}\mathbf{y}) = \mathbf{x} \cdot (\mathbf{P} \cdot \mathbf{y})$$

  where p_{ij} is the element of $\mathbf{P}$ corresponding to the utility of Player 1 when she/he plays row i and Player 2 plays column j (and the reward of Player 2 is $-p_{ij}$). Basically, the utility function is the sum over the possible payoffs p_{ij} weighted with the probability that payoff p_{ij} is obtained (i.e., that strategy (i,j) is played). Function f is also called **payoff function**.

We are saying that when we don't have a strategy that leads to a unique solution with the same value for both players, we have to consider mixed strategies, which include a probability distribution. By using a mixed strategy we are saying that a player doesn't have a specific move which is its best move, but it has to play only knowing that he/she can obtain a certain outcome with a given probability. In this case, solving the game means finding a probability distribution that maximises the reward obtained by Player 1 (or minimises the payoff given away by Player 2).

In this setting, we want to find out if a rational outcome for a zero-sum game always exists. Namely, we need to prove:

- $v_1 = v_2$
- There exists an optimal strategy $\bar{x}$ for Player 1 fulfilling
$$v_1 = \inf_y f(\bar{x}, y)$$
- There exists an optimal strategy $\bar{y}$ for Player 2 fulfilling
$$v_2 = \sup_x f(x, \bar{y})$$

In the finite case, $\bar{x}$ and $\bar{y}$ always exist, thus existence is equivalent to the coincidence of the conservative values. In the general case, the existence of optimal strategies is given by the von Neumann theorem.

> **Theorem 3.2** (Existence of a rational outcome (von Neumann))**.** *There always exists a rational outcome for a finite, zero-sum game with two players, as described by a payoff matrix* $\mathbf{P}$.

This theorem assures that, either there is a solution in pure strategies or, even when there are no solutions of a finite zero-sum game in pure strategies, there always exist:

- A mixed strategy for Player 1, namely a probability distribution $x = (x_1, \ldots, x_n)$, over her/his possible pure strategies (i.e., rows), such that, for all columns j
$$(x, p_{\cdot j}) = \sum_{i=1}^{n} x_i p_{ij} \geq v \quad \forall j \in \{1, \ldots, m\}$$
- A mixed strategy for Player 2, namely a probability distribution $y = (y_1, \ldots, y_m)$, over his/her possible pure strategies (i.e., columns), such that, for all rows i
$$(y, p_{i\cdot}) = \sum_{j=1}^{m} y_j p_{ij} \leq v \quad \forall i \in \{1, \ldots, n\}$$

The common bound v is the value of the game obtained in mixed strategies and Player 1 tries to make it as big as possible whereas Player 2 as small as possible. To prove this theorem, we need two more theorems:

- The second separation result, which is used to prove the von Neumann theorem.
- The first separation result, which is used to prove the second separation result.

Let's start by stating the first separation result and proving it.

Theorem 3.3 (First separation result). *Let C be a convex proper subset of the Euclidean space $\mathbb{R}^l$ and assume $\bar{x} \in cl\ C^c$. Then there is an element $0 \neq x^*$ such that, $\forall c \in C$*

$$(x^*, c) \geq (x^*, \bar{x})$$

This result simply gives us a criterion to tell apart an external point from an external point. More precisely, the theorem tells us that, given a convex set C and a point $\bar{x}$ outside C or on its border, then there exist an point x^* such that, for each $c \in C$ we have

$$|x^*||c| \cos(\theta_{x^*,c}) \geq |\bar{x}||c| \cos(\theta_{\bar{x},c})$$

Proof. Assume that point $\bar{x}$ doesn't belong to the closure of C and let us call p its projection (B.2) on cl C. Thanks to Theorem B.1 we know that, for every point inside C,

$$\|\bar{x} - p\| < \|\bar{x} - c\| \tag{3.7}$$

or, more properly

$$(\bar{x} - p, c - p) \leq 0 \tag{3.8}$$

Let us now define $x^* \coloneqq p - \bar{x} \neq 0$. This allows us to rewrite 3.8 as

$$(\bar{x} - x^* + \bar{x}, c + x^* - \bar{x}) \leq 0 \tag{3.9}$$

and by simplifying

$$(-x^*, c - x^* - \bar{x}) \leq 0 \tag{3.10}$$

The inequality 3.10 can also be written as

$$(-x^*, -x^*) + (-x^*, c - \bar{x}) \leq 0 \tag{3.11}$$

or as

$$(x^*, x^*) - (x^*, c - \bar{x}) \leq 0 \tag{3.12}$$

But because

$$(x^*, x^*) = \|x^*\|\|x^*\| \cos\theta_{x^*,x^*} = \|x^*\|^2$$

then we obtain

$$(x^*, c - \bar{x}) \geq \|x^*\|^2 \tag{3.13}$$

Since $\|x^*\|^2 > 0$ then $(x^*, c - \bar{x})$ is surely greater or equal to 0, hence we can write.

$$(x^*, c) - (x^*, \bar{x}) \geq 0 \tag{3.14}$$

By linearity we can also write

$$(x^*, c) - (x^*, \bar{x}) \geq 0 \tag{3.15}$$

and

$$(x^*, c) \geq (x^*, \bar{x}) \tag{3.16}$$

If $\bar{x} \in$ cl $C \setminus C$ (i.e., $\bar{x}$ is on the border of C), we take a sequence $\{x_n\} \subset C^c$ such that $x_n \to \bar{x}$. From the first step of the proof we can find some norm one x_n^* such that

$$(x_n^*, c) \geq (x_n^*, x_n) \quad \forall c \in C \tag{3.17}$$

So, given that for some sub-sequence one has $x_n^* \to x^*$, taking the limit of the above inequality yields

$$(x^*, c) \geq (x^*, \bar{x}) \quad \forall c \in C \tag{3.18}$$

□

As a corollary of Theorem 3.3 we have the following theorem which gives a more practical interpretation to the first separation result.

Theorem 3.4 (First separation result (corollary 1))**.** *Let C be a closed convex set in a Euclidean space, let x be on the boundary of C. Then there is a hyperplane containing x and leaving all of C in one of the half-spaces determined by the hyperplane. Such hyperplane is said to be an* ***hyperplane supporting*** C ***at*** x.

We can visualise the result stated by Theorem 3.4 with the aid of Figure 3.1. The green line is the hyperplane supporting C at x.

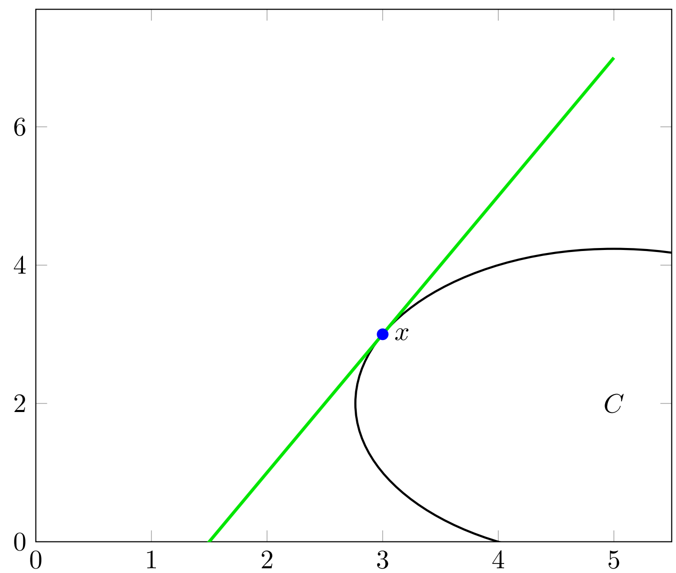


Figure 3.1: A visual representation of Theorem 3.4.

The following corollary is also a consequence of Theorem 3.3.

Theorem 3.5 (First separation result (corollary 2))**.** *Let C be a closed convex set in a Euclidean space. Then C is the intersection of all half-spaces, defined by the hyperplanes supporting C at $x, \forall x$, containing it.*

Before introducing the second separation result, let us remind that the interior of a set X, represented as $\text{int}X$, is the set of points that do not belong to the boundary of that set. Let us now move to the second separation result, which is stated by the following theorem.

Theorem 3.6 (Second separation result)**.** *Let A, C be closed convex subsets of $\mathbb{R}^l$ such that the interior $intA$ is nonempty and $intA \cap C = \emptyset$ (i.e., the interior of A and C are disjoint). Then there are $0 \neq x^*$ and $b \in \mathbb{R}$ such that, $\forall a \in A, \forall c \in C$*

$$(x^*, a) \geq b \geq (x^*, c)$$

Proof. Since $\bar{x} = 0 \in (\text{int}A - C)^c$, from Theorem 3.3 with $\bar{x} = 0$, there is $x^* \neq 0$ such that

$$(x^*, x) \geq 0 \quad \forall x \in \text{int}A - C \tag{3.19}$$

Thus for $x = a - c$ we obtain

$$(x^*, a - c) \geq 0 \quad \forall x \in \text{int}A - C \tag{3.20}$$

and by linearity

$$(x^*, a) \geq (x^*, c) \quad \forall a \in \text{int}A, \forall c in C \tag{3.21}$$

By extension this implies

$$(x^*, a) \geq (x^*, c) \quad \forall a \in \text{cl int}A, \forall c in C \tag{3.22}$$

□

Theorem 3.6 requires two convex sets A and C such that the interior of A is nonempty and its intersection with C is empty. This means that the only boundary of A can intersect C. In this context, the theorem says that we can find a point x^* not in the origin, and a real value b such that

- The scalar product between x^* and $a \in A$ is an upper bound for b.
- The scalar product between x^* and $c \in C$ is a lower bound for b.

The set H of all points that have scalar product with x^* equal to b is called separating hyperplane and A and C are contained in the two different half-spaces generated by H. In practice, the second separation result gives us a way to determine whether a point is in set A or in set C. More precisely, a point x is in A if it's in the half-space where A is contained

$$x \in A \iff (x^*, x) \geq b$$

and a point x is in C if it's in the other half-plane (i.e., where C lies),

$$x \in C \iff (x^*, x) \leq b$$

Thanks to the separation results we have just stated, we can define what is an optimal pure strategy for a player in a zero-sum game. An important result is stated by the following theorem.

Theorem 3.7 (Optimal pure strategy). *If a player knows the strategy played by the other player, she can always use a pure strategy to get the best outcome.*

Proof. Consider for instance Player 2 who knows that Player 1 plays a mixed strategy $\bar{x}$. Then Player 2 must minimise the function

$$f(\bar{x}, y) = (\bar{x}, Py)$$

over the simplex $\sum_m$ of Player 2's strategies. The maximum is reached in at leas one vertex e_j and it corresponds to a pure strategy. □

This means that, once the choice of one player is fixed, the optimisation becomes a linear problem over a simplex (recall that the utility function is bilinear). Formally, given the payoff matrix P, and

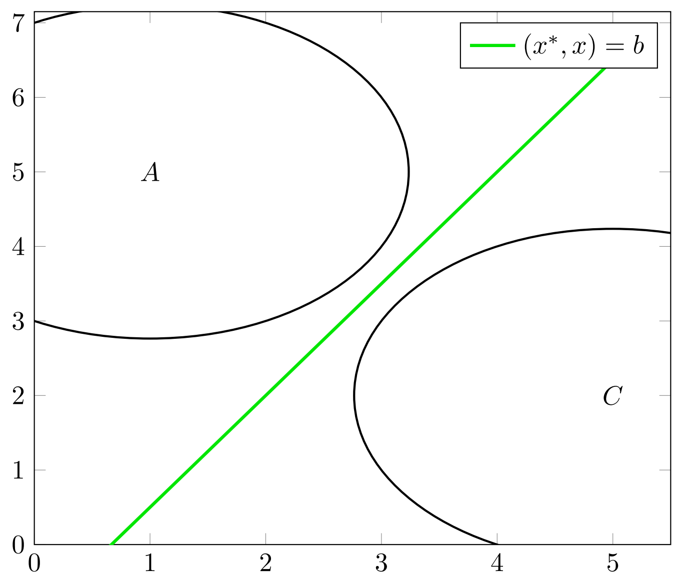


Figure 3.2: A visual representation of Theorem 3.6.

by denoting the column vector j with $p_{\cdot j}$ and the row vector i with $p_{i \cdot}$, respectively, the payoff of the first player in the mixed extension of the game is

$$f(x, y) = (x, Py)$$

Theorem 3.7 implies that, in order to verify the existence of a rational outcome for the game, we need to prove that there are mixed strategies $\bar{x}$ and $\bar{y}$, as well as a value v, such that

- $(\bar{x}, Pe_j) = \bar{x}, p_{\cdot j} \geq v$, for every column j.
- $(e_i, p_{i \cdot} \bar{y}) \leq v$ for every row i.

where e_j is the j-th strategy of the second player and e_i is the i-th strategy of the first player.

Von Neumann theorem proof

Let us now finally prove Theorem 3.2.

Proof. Let's suppose, without loss of generality that all payoffs are positive (we can add a constant to every payoff so that they are all positive). Let us now take the column vectors $p_1, \ldots, p_m$, each in $\mathbb{R}^n$ and call C their convex hull. Basically, we are taking the strategies of Player 1, interpreting them as points and taking their convex hull. Let us also define

$$Q_t = \{x \in \mathbb{R}^n : x_i \leq t\} \tag{3.23}$$

and

$$v = \sup\{t \geq 0 : Q_t \cap C \neq \varnothing\} \tag{3.24}$$

Since int $Q_v \cap C \neq \varnothing$, thanks to Theorem 3.6, the sets Q_v and C can be separated by an hyperplane. This means that there are coefficients $\bar{x}_1, \ldots, \bar{x}_n$ with some $\bar{x}_i \neq 0$ and $b \in R$ such that

$$(\bar{x}, u) = \sum_{i=1}^{n} \bar{x}_i u_i \leq b \leq \sum_{i=1}^{n} \bar{x}_i w_i = (\bar{x}, w) \qquad \forall u \in Q_v, w \in C \tag{3.25}$$

Then it follows that:

- All $\bar{x}_i$s must be non-negative and we can assume that $\sum \bar{x}_i = 1$.
- $b = v$ in fact if we define $\bar{v} := (v, \ldots, v) \in Q_v$ then we can write

  $$(\bar{x}, \bar{y}) = \sum \bar{x}_i v = v \sum \bar{x}_i = v$$

  But because $(\bar{x}, u) \leq b$, for all $u \in Q_v$, then it must also be true for v and we obtain

  $$v \leq b$$

  But if $b > v$, by taking a small $a > 0$ such that $b \geq v + a$ then we have

  $$\sup \left\{ \bar{x}_i u_i : u \in Q_{v+a} \right\} < b$$

  which implies $Q_{v+a} \cap C \neq \varnothing$, against the definition of v, hence it must be $b = v$.
- $Q_v \cap C \neq \varnothing$. Let $w \in Q_v \cap C$ so that $\bar{w} = \sum_{j=1}^{m}$, since C is convex, for some $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in \sum_m$. Since $\bar{w} \in Q_v$ it follows $\bar{w}_i \leq v$ for all i.

Now we have to prove that strategy $\bar{x}$ is optimal for Player 1, that $\bar{y}$ is optimal for Player 2 and that v is the value of the game.

- Consider Player 1. Since $(\bar{x}, w) \geq v$ for every $w \in C$, by the first separation result (3.3) and since every column $p_{\cdot j} \in C$, we have that

  $$(\bar{x}, p \cdot j) \geq v \qquad \forall j$$

- Consider $\sum_{j=1}^{m} \bar{y}_j p_j = w \in Q_v \cap C$. Then $w_i = \text{var} y p_{i \cdot}$. Since $w \in Q_v$, then $w_i \leq v$ for every i and

  $$v \geq w_i = \bar{y} p_{i \cdot}$$

□

3.3.2 Example

Let us consider an example, in which we can apply von Neumann's theorem, to clear things up. Let us consider the following game

$$P = \begin{pmatrix} 7 & 1 & 4 & 9 \\ 3 & 10 & 6 & 2 \\ 4 & 5 & 3 & 0 \end{pmatrix}$$

The first thing we should always do is to apply the rationality assumptions and eliminate any row or column dominated by another row or column, respectively. In this case we can't eliminate any row or column. We should also check if it's possible to remove a row or column because it's dominated by a convex combination of the other rows or columns. The third column is in fact dominated by a convex combination of the first two, taking as weights 0.5, hence we can remove it. What we are left with is the following game

$$P' = \begin{pmatrix} 7 & 1 & 4 & 9 \\ 3 & 10 & 6 & 2 \end{pmatrix}$$

Now we can consider Player 1 and draw the convex polygon C with vertices in

- $(p_{11}, p_{21}) = (7, 3)$
- $(p_{12}, p_{22}) = (1, 10)$
- $(p_{13}, p_{23}) = (4, 6)$
- $(p_{14}, p_{24}) = (9, 2)$

As we can see from Figure 3.3, the separating hyperplane has equation

$$p_{2j} = -p_{1j} + 10$$

which touches the convex hull C for $j = 1$ and $j = 3$. Rearranging this equation, we obtain

$$p_{1j} + p_{2j} = 10$$

or

$$\frac{1}{2}p_{1j} + \frac{1}{2}p_{2j} = 5$$

This means that Player 1 plays

$$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

and the value of the game is 5.

Since the separating hyperplane touches Player 1's hull for $j = 1$ and $j = 3$ we have to find a convex combination of columns 1 and 3 to obtain $(5, 5)$, which is the vector $v = (v, \dots, v)$. Since columns 1 and 3 are $(7, 3)$ and $(4, 6)$ we obtain the system

$$\begin{cases} 7q + 4(1-q) = 5 \\ 3q + 6(1-q) = 5 \end{cases}$$

We can therefore write

$$\begin{aligned} 7q + 4(1-q) &= 3q + 6(1-q) \\ 7q + 4 - 4q &= 3q + 6 - 6q \\ 6q &= 2 \\ q &= \frac{1}{3} \end{aligned}$$

This means that Player 2 has to play

$$\left(\frac{1}{3}, 0, \frac{2}{3}, 0\right)$$

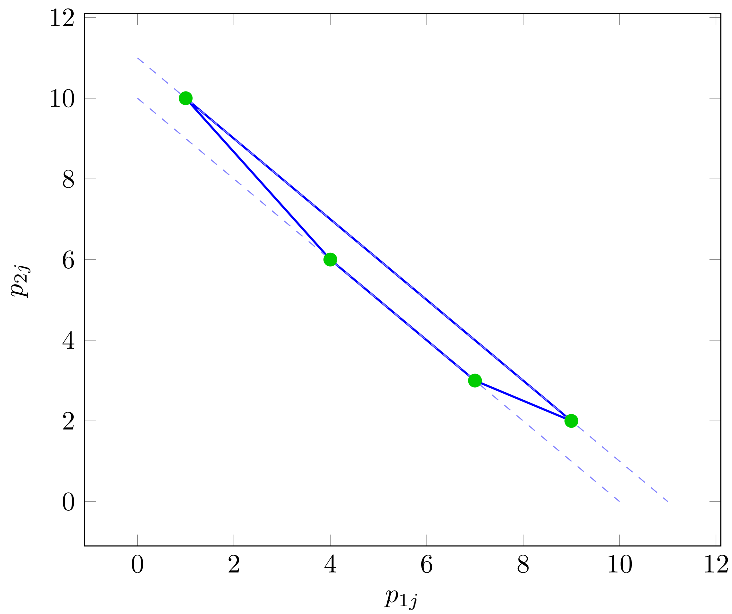


Figure 3.3: The polygon representing Player 1's strategies.

3.4 Optimal strategies in the general case

Von Neumann's theorem (3.2) can be used whenever we can reduce the number of strategies of one player to two (since we have to draw the separating hyperplane). This isn't however always possible, hence we should find other tools to solve a zero-sum game, in the most general case. In particular, we will use **linear programming**.

Let's try and formalise the solution of a zero-sum game as a linear programming problem starting from Player 1. She/He has to maximise the value v of the game using mixed strategies (which also include pure strategies if we consider all but one probabilities to 0) which have a value not smaller than v. This means that Player 1 has to solve the following problem

$$\begin{aligned} &\max_{x,v} v \\ &s.t. \ \sum_{i=1}^{n} x_i p_{ij} \geq v \quad \forall j \in \{1, \dots, m\} \\ &\qquad x_i \geq 0 \quad \forall i \in \{1, \dots, n\} \\ &\qquad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

In a more compact form we can write

$$\begin{aligned}
&\max_{x,v} v \\
&s.t.\ P^t x \geq v 1_m \\
&\qquad x \geq 0 \\
&\qquad (1, x) = 1
\end{aligned}$$

where 1_m is an m-dimensional vector of all 1s. Following the same reasoning, we can say that Player 2 wants to minimise the value of w, using mixed strategies whose value isn't bigger than v. Formally,

$$\begin{aligned}
&\min_{y,w} w \\
&s.t.\ \sum_{j=1}^{m} y_j p_{ij} \leq w \quad \forall i \in \{1, \ldots, n\} \\
&\qquad y_j \geq 0 \quad \forall j \in \{1, \ldots, m\} \\
&\qquad \sum_{j=1}^{m} x_j = 1
\end{aligned}$$

Or more compactly,

$$\begin{aligned}
&\min_{x,w} w \\
&s.t.\ Py \leq w 1_n \\
&\qquad y \geq 0 \\
&\qquad (1, y) = 1
\end{aligned}$$

These two problems resemble the second set of dual problems used in linear programming

$$\begin{aligned}
&\min\ cx &\qquad\qquad &\max\ yb \\
&\quad Ax \geq b & &\quad yA \leq c \\
&\quad x \geq 0 & &\quad y \geq 0
\end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c, x \in \mathbb{R}^n$ and $b, y \in \mathbb{R}^m$. If we can reduce our formulation to the one we have just described, we can apply the weak and strong duality theorems. In particular, the former tells us that, if the solutions $\bar{x}$ and $\bar{y}$ of the first and second problem are the same (let us call v that value), then they are optimal. Moreover, we can be in one of the following cases:

- If the primal is feasible and the dual is infeasible, then $v = -\infty$.
- If the primal is infeasible and the dual is feasible, then $v = +\infty$.

When the problem has a solution v, which is optimal (i.e., it's the same for both problems), then we can also write a set of conditions, called **complementary slackness conditions**, that allow us to find the optimal $\bar{y}$ and $\bar{x}$. The conditions are the following

$$\begin{aligned}
\bar{y}(b - A\bar{x}) = 0 \\
(c - \bar{y}A)\bar{x} = 0
\end{aligned}$$

These conditions ensure that

- If $\bar{y}_i$ is not 0, then $A_i\bar{x} - b_i$ has to be 0 (i.e. $A_i\bar{x} = b_i$).
$$\bar{y}_i \neq 0 \Rightarrow A_i\bar{x} = b_i$$
- If $A_i\bar{x} - b_i$ is not 0 (i.e. $A_i\bar{x} \neq b_i$), then $\bar{y}_i$ has to be 0.
$$A_i\bar{x} \neq b_i \Rightarrow \bar{y}_i = 0$$
- If $\bar{x}_j$ is not 0, then $c_j - \bar{y}A^j$ has to be 0 (i.e. $c_j = \bar{y}A^j$).
$$\bar{x}_j \neq 0 \Rightarrow c_j = \bar{y}A^j$$
- If $c_j - \bar{y}A^j$ is not 0 (i.e. $c_j \neq \bar{y}A^j$), then $\bar{x}_j$ has to be 0.
$$c_j \neq \bar{y}A^j \Rightarrow \bar{x}_j = 0$$

Now that we have all the tools to solve a linear programming problem, we simply have to reduce our problem to the form we use in linear programming. Let us consider the linear programming problems

$$\begin{aligned} &\min c\alpha \\ &\quad A\alpha \geq b \\ &\quad \alpha \geq 0 \end{aligned} \qquad\qquad \begin{aligned} &\max b\beta \\ &\quad A^t\beta \leq c \\ &\quad \beta \geq 0 \end{aligned}$$

Without loss of generality, we can assume that every element of the payoff matrix P is positive. If this is not true, we can add $k = -\min p_{ij}$ to every value of P. If every payoff is positive, then the value of the game is necessarily positive. To obtain the form used in linear programming we have to change minimise in maximise and maximise in minimise. Let's start by dividing the probabilities x_i over v to obtain the elements α_i of α

$$\alpha_i = \frac{x_i}{v}$$

Now we can write the last constraint as

$$\begin{aligned} \sum x_i &= 1 \\ \sum \alpha_i v &= 1 \\ v\sum \alpha_i &= 1 \\ \sum \alpha_i &= \frac{1}{v} \end{aligned}$$

This means that, instead of maximising v, we can minimise the sum of α_i, which means minimising $\frac{1}{v}$, which is equivalent to maximising v. The same reasoning can be applied to the second problem applying the transformation

$$\beta_j = \frac{y_j}{w}$$

In conclusion, to solve a zero-sum game we have to solve the problems

$$\begin{aligned} &\max 1_n\alpha \\ &\quad A^t\alpha \geq b \\ &\quad \alpha \geq 0 \end{aligned} \qquad\qquad \begin{aligned} &\min 1_m\beta \\ &\quad A\beta \leq c \\ &\quad \beta \geq 0 \end{aligned}$$

which coincide with the second couple of linear programming's dual problems and obtain the mixed strategies of Players 1 and 2 as

- $x = v\alpha$
- $y = v\beta$

Finally, we have successfully found a linear programming problem that has the same shape of the linear programming problem we've used for solving a zero-sum. We can therefore write the complementary slackness conditions that can be used to solve the game. In particular, the complementary slackness related to the problems

$$\begin{aligned} &\max_{x,v}\ v \\ &s.t.\ P^t x \geq v 1_m \\ &\quad\ x \geq 0 \\ &\quad\ (1,x) = 1 \end{aligned} \qquad\qquad \begin{aligned} &\min_{x,w}\ w \\ &s.t.\ Py \leq w 1_n \\ &\quad\ y \geq 0 \\ &\quad\ (1,y) = 1 \end{aligned}$$

are

$$\sum_{k=1}^{m} p_{ik}\bar{y}_k = v \qquad \forall i = \{1,\ldots,n\} : \bar{x}_i > 0$$

and

$$\sum_{k=1}^{n} p_{kj}\bar{x}_k = v \qquad \forall j = \{1,\ldots,m\} : \bar{y}_j > 0$$

These equations represent the fact that

- Since $\bar{y}$ is optimal for Player 2, one has $\sum_{k=1}^{m} p_{ik}\bar{y}_k = v$ for all i and hence $x_i > 0$ implies that row i is optimal for Player 1.
- Since $\bar{x}$ is optimal for Player 2, one has $\sum_{k=1}^{n} p_{kj}\bar{x}_k = v$ for all j and hence $y_j > 0$ implies that column j is optimal for Player 2.

3.5 Symmetric games

Symmetric games are a particular case of zero-sum games. In order to build a symmetric game, we need the definition of antisymmetric matrix and of fair game.

> **Definition 3.2** (Antisymmetric matrix)**.** *An $n \times n$ matrix P with elements p_{ij} is said to be antisymmetric if*
>
> $$p_{ij} = -p_{ji} \qquad \forall i,j \in \{1,\ldots,n\}$$

An example of antisymmetric matrix is shown in Equation 3.26. Note that a matrix must have 0s on its diagonal to be antisymmetric.

$$\begin{pmatrix} 0 & -1 & 7 \\ 1 & 0 & 5 \\ -7 & -5 & 0 \end{pmatrix} \tag{3.26}$$

> **Definition 3.3** (Fair game)**.** *A finite zero sum game is fair if the associated payoff matrix is **antisymmetric**.*

In fair games there is no favourite player, hence the roles of Player 1 and Player 2 can be swapped, without modifying the outcome of the game. Basically, this means that the payoff matrix, with opposite sign, $-P$ corresponds to its transpose P^t (which is in fact an equivalent definition of antisymmetric matrix).

$$-P \equiv P^t$$

Everything we've said can be summed in the following proposition.

Proposition 3.1 (Conservative value for fair games)**.** *If the payoff matrix $P = (p_{ij})$ is antisymmetric, the conservative value is $v = 0$ and $\bar{x}$ is an optimal strategy for player 1 if and only if it is optimal for Player 2.*

Proof. From the definition of fair game we have

$$P^t = -P$$

thus we can write

$$(Px, x) = (x, P^t x) = -(x, Px) = -(Px, x)$$

Because, by definition, an antisymmetric matrix has all zeros on the diagonal, then it must hold

$$f(x, x) = 0 \qquad \forall x \in X$$

This implies that

- $v_1 \leq 0$
- $v_2 \geq 0$

because, for sure, the minimum on each row is at most 0 (but could be smaller), and the maximum on each column is at least 0 (but could be bigger). But v_1 is a lower bound whilst v_2 is an upper bound, hence it must be $v = 0$ (which is the only value that satisfies both inequalities). Let's now prove that a strategy $\bar{x}$ is optimal if it's optimal for both players. Having found v, we can say that $\bar{x}$ is optimal for Player 1 if

$$(\bar{x}, Py) \geq 0 \qquad \forall y \in Y$$

or equivalently, if

$$(P^t \bar{x}, y) \geq 0 \qquad \forall y \in Y$$

But because P is antisymmetric, the inequality becomes

$$(P\bar{x}, y) \leq 0 \qquad \forall y \in Y$$

Therefore $\bar{x}$ is optimal also for the second player. □

If a strategy $\bar{x}$ optimal for Player 1 is optimal also for Player 2, then we can focus only on one player only. From Proposition 3.1 we have that, to solve a symmetric game we have to solve the system of inequalities

$$\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\
&\vdots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\
&\vdots \\
&x_1 p_{1n} + \cdots + x_n p_{nn} \geq 0
\end{aligned}$$

where we have always to add the constraints

$$x_i \geq 0 \ \forall i \in \{1, \ldots, n\} \qquad \sum_{i=1}^{n} x_i = 1$$

3.5.1 Principle of indifference

Let us consider the following system

$$\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\
&\vdots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\
&\vdots \\
&x_1 p_{1n} + \cdots + x_n p_{nn} \geq v
\end{aligned}$$

with v unknown. Given this system we want to know when it's possible to have a strict inequality. Say that $\bar{x}$ is optimal for Player 1 and the strict inequality holds, namely

$$\bar{x_1} p_{1j} + \cdots + \bar{x_n} p_{nj} > v \tag{3.27}$$

Then Player 2 never plays column j since Player 1 would be able to get a payoff greater than v. In other words, in his/her mixed strategy, Player 2 assigns 0 to column j. But every strategy to which Player 2 assigns positive probability gives the same value to Player 1, hence, Player 1 is indifferent to all such strategies used by Player 2 at equilibrium.

Part III

The Nash model

Chapter 4

Non-cooperative games

4.1 Nash equilibrium

4.1.1 Non-cooperative games

Before introducing the Nash model, we have to define the concept of non-cooperative game.

> **Definition 4.1** (Non-cooperative game). *A two-player game whose strategic form is defined as*
> $$(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$$
> *is called* ***non-cooperative game****. In the non-cooperative game above:*
>
> - *X is the strategy set of Player 1,*
> - *Y is the strategy set of Player 2,*
> - *f is the utility function of Player 1,*
> - *g is the utility function of Player 2,*
>
> *and the utility functions f, g are uncorrelated.*

Note that a non-cooperative game can be seen as an extension of a zero-sum game as we can see a zero-sum game as a non-cooperative game in which $g = -f$.

4.1.2 Nash equilibrium profile

Given a non-cooperative game, we can define a Nash equilibrium profile as follows.

> **Definition 4.2** (Nash equilibrium profile). *Let $G = (X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ be a non-cooperative game. A Nash equilibrium profile for G is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that:*

> 1. *$\bar{x}$ is the best strategy for Player 1 when Player 2 plays his/her optimal strategy $\bar{y}$, i.e.,*
>
> $$f(\bar{x},\bar{y}) \geq f(x,\bar{y}) \quad \forall x \in X$$
>
> 2. *$\bar{y}$ is the best strategy for Player 2 when Player 1 plays her/his optimal strategy $\bar{x}$, i.e.,*
>
> $$g(\bar{x},\bar{y}) \geq g(\bar{x},y) \quad \forall y \in Y$$

A Nash equilibrium profile (NEp) is a joint combination of strategies (i.e, $\bar{x}$ and $\bar{y}$), which is stable with respect to unilateral deviations of any individual player. We say that a NEp is stable because both players will stick with the profile, in fact, in both cases, the utilities are maximised, therefore it wouldn't make sense for the players to change their strategies since it would lead to a worse outcome. As always, the underlying assumption is that both players are rational, therefore they will stick to their optimal strategy and they know the other player will do the same. Namely, at equilibrium, neither player can improve her/his utilities by changing strategy. In fact, it is not even convenient for the players to change, given that each one takes for granted that the other one will play the selected strategy.

Examples

Prisoners' dilemma To better understand the idea of Nash equilibrium profile, let us consider the prisoner's dilemma. The game can be represented with the following payoff matrix

$$\begin{pmatrix} (5,5) & (0,7) \\ (7,0) & (1,1) \end{pmatrix}$$

Remember that in this case, higher utility values are worst, since the utility represents the years of jail. This means that instead of maximising the utility function, each player has to minimise it (he/she has to minimise the number of years in jail). By the principle of dominating strategies, we know that the outcome of the game is $(5,5)$ (row 1 dominates row 2, column 1 dominates column 2), however, we can also use the Nash model to find the Nash equilibrium. The best strategy is $\bar{x} = x_1$ (first row) and $\bar{y} = y_1$ (first column) hence we obtain:

- $f(\bar{x},\bar{y}) = f(x_1,y_1) = 5 \leq f(x_2,\bar{y}) = f(x_2,y_1) = 7$
- $g(\bar{x},\bar{y}) = g(x_1,y_1) = 5 \leq f(\bar{x},y_2) = f(x_1,y_2) = 7$

The Nash conditions are met, hence (x_1,y_1) is a Nash equilibrium profile which, in this case, is also unique.

Another example Let us now consider a more complex example in the form of the following payoff matrix.

$$P = \begin{pmatrix} (3,3) & (6,1) & (1,3) \\ (1,6) & (1,1) & (6,4) \\ (2,1) & (4,6) & (5,5) \end{pmatrix}$$

Let us also switch back to a maximisation problem (the higher the utility function, the better the outcome). If we consider a tentative equilibrium (x_1,y_1), we can check if it's an actual Nash equilibrium profile by directly applying Definition 4.2. In particular, we can verify that:

- $f(\bar{x}, \bar{y}) = f(x_1, y_1) > f(x_3, \bar{y}) > f(x_2, \bar{y})$
- $g(\bar{x}, \bar{y}) = g(x_1, y_1) \geq g(\bar{x}, y_3) > g(\bar{x}, y_2)$

hence (x_1, y_1) is a Nash equilibrium. In general, we can find a Nash equilibrium using the following algorithm (4.1).

Algorithm 4.1 (Iterative Nash equilibrium)**.** *Iteratively:*

1. *Fix* $\bar{x} = x_i$.
2. *Check on row* i *the Nash equilibrium conditions (using both* f *and* g*) for all possible columns* y_j*. If the conditions are met for a certain* y_j*, we found an equilibrium point* (x_i, y_j)*, otherwise, we discard* $x = x_i$.
3. *Restart from point 1 with* $\bar{x} = x_{i+1}$.

The same algorithm can be executed symmetrically on the columns. Namely, we should try all values of $\bar{y}$ and for each value of $\bar{y}$, iterate on the values of x. The set of all points (x_i, y_j) that satisfy the Nash equilibrium conditions is the set of Nash equilibrium points.

Note that, despite the principle of dominated strategies, the Nash equilibrium is always applicable since it doesn't depend on multiple conditions and it's enough to check the single values one against the others.

4.1.3 Multiple players

The Nash model can be applied even to games that involve more than two players. In particular, the principle of the Nash equilibrium, and the way of reasoning remains unchanged. The only difference is that the notation is more complex since it has to take into account a list of utility functions and outcomes. In particular, let us consider a n-player game with strategy sets X_i for each player and payoffs $u_i : X \to \mathbb{R}$ with $X = \prod_{i=1}^{n} X_i$. In this setting, a strategy profile is represented as

$$x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

and x_{-i} denotes the strategy profile x without the element x_i

$$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

We can also write

$$x = (x_i, x_{-i})$$

to emphasise the role of x_i. Thanks to this notation we can say that

$$\overline{x} = (\overline{x}_i)_{i=1}^{n} = (\overline{x}_1, \ldots, \overline{x}_{i-1}, \overline{x}_i, \overline{x}_{i+1}, \ldots, \overline{x}_n)$$

is a Nash equilibrium profile for the game if

$$u_i(\overline{x}) \geq u_i(x, \overline{x}_{-i}) \quad \forall i \in \{1, \ldots, n\}, \forall x \in X_i$$

4.1.4 Nash rationality paradigm

The Nash equilibrium provides a new definition of rationality. In particular, we can redefine the notion of dominant strategies as follows.

Definition 4.3 (Dominant strategy)**.** *A strategy $\overline{x}$ is a dominant strategy for Player 1 if*

$$f(\overline{x}, y) \geq f(x, y) \quad \forall x, y$$

If the equality holds for some y then the strategy is said to be weakly dominant.

Dominant strategies can be used to compute the Nash equilibrium profile. In particular, the following proposition holds.

Proposition 4.1. *Let $\bar{x} \in X$ be a (weakly) dominant strategy for Player 1. If $\bar{y}$ maximises the function $y \mapsto g(\bar{x}, y)$ for Player 2, then $(\bar{x}, \bar{y})$ is a Nash equilibrium profile.*

Proof. Let us prove this statement. For $\bar{x}$ weakly dominant, it's true that

$$f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$$

hence the first Nash condition is satisfied. Moreover, since we maximised $y \mapsto g(\bar{x}, y)$, then it holds

$$g(\bar{x}, \bar{y}) \geq g(\bar{x}, y) \quad \forall y$$

since $\bar{y}$ is the maximum. This means that the second Nash condition is satisfied. □

Note that the weakly dominance of a strategy influences the uniqueness of the Nash equilibrium. In particular:

Proposition 4.2 (Uniqueness of the Nash equilibrium)**.** *Given a strategy $\bar{x} \in X$ for Player 1:*

- *If $\bar{x}$ is weakly dominant, there could be more than one Nash equilibrium profile.*
- *If $\bar{x}$ is strictly dominant, there exists one and only one Nash equilibrium profile.*

Proof. Let's assume there exists a second Nash equilibrium (x_i, y_j) other than $(\bar{x}, \bar{y})$. Now:

- If $\bar{x}$ is weakly dominant we have
$$f(x_i, y_j) \geq f(\bar{x}, y_j)$$
by definition of weak dominance. Since $(\bar{x}, \bar{y})$ is a Nash equilibrium, by definition, we have that
$$f(\bar{x}, y) \geq f(x, y) \quad \forall x \in X, y \in Y$$
The inequation is true for all ys, then it's true also for $y = y_j$ and $x = x_i$ and we have that
$$f(\bar{x}, y_j) \geq f(x_1, y_j)$$
Since we have considered a weak dominance, both the initial assumption and the inequation we've just written can be true. In particular, we have that
$$f(\bar{x}, y_j) = f(x_i, y_j)$$
and (x_i, y_j) is (another) Nash equilibrium.

- If x_i is strictly dominant we have

$$f(x_i, y_j) > f(\bar{x}, y_j)$$

by definition of dominance. Since $(\bar{x}, \bar{y})$ is a Nash equilibrium, by definition, we have that

$$f(\bar{x}, y) > f(x, y) \quad \forall x \in X, y \in Y$$

This must be true for all x, y, hence also for $x = x_i$ and $y = y_j$ and we can write

$$f(\bar{x}, y_j) > f(x_i, y_j)$$

This is however in contradiction with our initial assumption, hence (x_i, y_j) isn't a Nash equilibrium.

□

4.2 Nash equilibrium in perfect information games

The Nash equilibrium method can be applied to a game with perfect information and we can find a Nash equilibrium profile using backward induction. In particular, for perfect information games, we have the following result.

Proposition 4.3 (Nash equilibrium on perfect information games)**.** *Backward induction provides a Nash equilibrium profile for a game of perfect information since players systematically make an optimal choice in every part of the tree of the game.*

Note that, the backward induction method provides one of the possible Nash equilibrium profiles and we don't even know if it's unique. Backward induction doesn't even ensure finding all of the Nash equilibrium profiles.

4.3 Nash equilibrium in zero-sum games

The Nash model can be applied to zero-sum games, too, remembering that $g = -f$. This result is summed up in the following theorem.

Theorem 4.1 (Nash equilibrium in zero-sum games)**.** *Let X, Y be nonempty strategy sets, $f : X \times Y \to \mathbb{R}$ and $g = -f$ the utility functions of Players 1 and 2, respectively. Then the following statements are equivalent:*

1. *The pair $(\bar{x}, \bar{y})$ fulfils:*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \forall y \in Y$$

2. *The following conditions are satisfied:*
   (a) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$
   (b) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$
   (c) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$

In other words,

1. The first condition says that the equilibrium $(\bar{x}, \bar{y})$ yields conservative values for both players since its value is smaller than the conservative value $f(\bar{x}, y)$ of Player 2 and it's bigger than the conservative value $f(x, \bar{y})$ of Player 1.

2. The second condition says that the conservative values for the players are the same (condition (a)) and the players have to solve two independent problems (conditions (b) and (c)).

Proof. We can start by showing that statement 1 implies statement 2. In particular, remembering that

- $f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y)$, since Player 2 chooses the smallest value available when Player 1 plays her/his best strategy.

- $f(\bar{x}, \bar{y}) = \sup_x f(x, \bar{y})$, since Player 1 chooses the highest value available when Player 2 plays his/her best strategy.

and from the inequality in statement 1, we get

$$\sup_x f(x, \bar{y}) = f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y)$$
$$v_2 = \inf_y \sup_x f(x, y) \leq \sup_x f(x, \bar{y}) = f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y) \leq \sup_x \inf_y f(x, y) = v_1$$

Since it always holds $v_1 \leq v_2$ (Player 1's conservative value is a lower bound and Player 2's is an upper bound), then the inequalities in the equation above are actually equalities (because $v_1 \leq v_2$ and $v_2 \leq v_1$ imply $v_1 = v_2$), hence we can write

$$v_2 = \inf_y \sup_x f(x, y) = \sup_x f(x, \bar{y}) = f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y) = v_1$$

which contains the equalities of statement 2.

Let us now prove statement 1 from statement 2. Thanks to the equalities in statement 2 we can write

$$\inf_y \sup_x f(x, y) = \sup_x f(x, \bar{y}) \geq f(\bar{x}, \bar{y}) \geq \inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$$

But for the first condition in statement 2 $\inf_y \sup_x f(x, y)$ must be equal to $\sup_x \inf_y f(x, y)$, hence all inequalities must be equalities and:

- If $f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y)$, then it must be $f(\bar{x}, y) \geq f(\bar{x}, \bar{y})$

- If $f(\bar{x}, \bar{y}) = \sup_x f(x, \bar{y})$, then it must be $f(x, \bar{y}) \leq f(\bar{x}, \bar{y})$

□

4.4 Nash theorem

4.4.1 Existence of the Nash equilibria

As for now, we've said that a game might have one or more Nash equilibrium profiles. Still, we haven't talked about the existence of such a profile. To talk about the existence of Nash equilibrium profiles we need a couple of definitions.

Definition 4.4 (Multifunction)**.** *Let A and B be two sets. A function*

$$f : A \to 2^B \tag{4.1}$$

where 2^B is the powerset (i.e., the set of partitions) of B, is called multifunction.

In other words, a multifunction from A to B returns a set of values of B instead of a single value.

Definition 4.5 (Best response)**.** *Given the multifunctions $BR_1 : Y \to 2^X$ and $BR_2 : X \to 2^Y$, defined as*

$$BR_1(y) = \arg\max\{f(\cdot, y)\}$$

and

$$BR_2(x) = \arg\max\{g(x, \cdot)\}$$

We call best response multifunction $BR : X \times Y \to 2^X \times 2^Y$ the function

$$BR(x, y) = \big(BR_1(y), BR_2(x)\big)$$

Note that:

- The function BR_1 returns the set of xs that maximise $f(\cdot, y)$.
- The function BR_2 returns the set of ys that maximise $g(x, \cdot)$

Thanks to these definitions, we can state the following result.

Proposition 4.4. *An outcome $(\bar{x}, \bar{y})$ is a Nash equilibrium for a game if and only if*

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y})$$

Namely, $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if it's a fixed point for the multifunction BR (i.e., a point that, passed to the function, allows to obtain the point itself, or in this case such that it belongs to the set returned by the function). Another important result is Kakutani's theorem.

Theorem 4.2 (Kakutani's)**.** *Let Z be a compact convex subset of an Euclidean space, and let $F : Z \to 2^Z$ such that $F(z)$ is a nonempty closed convex set for all $z \in Z$. Suppose also F has a closed graph. Then F has a fixed point, namely there is a $\bar{z} \in Z$ such that $\bar{z} \in F(\bar{z})$.*

Remember that a closed graph means that

$$y_n \in F(z_n) \forall n, y_n \to y, z_n \to z \Rightarrow y \in F(z)$$

4.4.2 Nash theorem

Before stating the Nash theorem we need the following definition:

Definition 4.6 (Quasi-concave function). *A function $f : A \to \mathbb{R}$ is said to be quasi-concave if the sets*

$$f_a = \{z : f(z) \geq a\}$$

are convex (possibly empty) for all $a \in \mathbb{R}$.

Finally, we can state the Nash theorem.

Theorem 4.3 (Nash). *Given the game*

$$(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$$

if:

1. *X and Y are compact convex subsets of some Euclidean space.*
2. *f, g are continuous.*
3. *$x \mapsto f(x, y)$ is quasi-concave for all $y \in Y$.*
4. *$y \mapsto g(x, y)$ is quasi-concave for all $x \in X$.*

then the game has an equilibrium.

Proof. Let us take $BR_1(y)$ and $BR_2(x)$ nonempty (X and Y are compact), closed (f and g are continuous), and convex valued (f and g are quasi concave). In this scenario, we have to prove that $BR(x, y)$ has a closed graph, since Kakutani's theorem (4.2) ensures that in such case $BR(x, y)$ has a fixed point. Let us suppose

- $(u_n, v_n) \in BR(x_n, y_n)$
- $(u_n, v_n) \to (u, v)$.
- $(x_n, y_n) \to (x, y)$.

To prove that $(u, v) \in BR(x, y)$ we can write

$$f(u_n, y_n) \geq f(z, y_n), \quad g(x_n, v_n) \geq g(x_n, t) \quad \forall z \in X, \forall t \in Y$$

If we consider the limits for the inequalities above we obtain

$$f(u, y) \geq f(z, y), \quad g(x, v) \geq g(x, t) \quad \forall z \in X, \forall t \in Y$$

□

From a practical point of view, the Nash theorem provides the conditions for the existence of the Nash equilibrium. Namely, it's enough to check the quasi-concave conditions (i.e., number 3 and 4) to find a Nash equilibrium. As a corollary of Nash's theorem, we have the following theorem.

Theorem 4.4 (Nash (corollary)). *Let (A, B) be the representation of a game where A is the payoff matrix of Player 1 and B is the payoff matrix for Player 2. The game (A, B) always* *admits a Nash equilibrium profile in mixed strategies.*

An example

Let us consider the following example to understand how we can compute a Nash equilibrium. Let us consider the game

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}$$

Since neither of the players has a dominant strategy, we have to look for mixed strategies. In particular:

- Player 1 plays strategy $(p, 1-p)$.
- Player 2 plays strategy $(q, 1-q)$.

This means that we can compute Player 1's payoff as

$$\begin{aligned} f(p,q) &= p \cdot (1 \cdot q + 0 \cdot (1-q)) + (1-p) \cdot (0 \cdot q + 1 \cdot (1-q)) \\ &= 1 \cdot p \cdot q + 0 \cdot p \cdot (1-q) + 0 \cdot (1-p) \cdot q + 1 \cdot (1-p) \cdot (1-q) \\ &= pq + (1-p)(1-q) \\ &= pq + 1 - q - p + pq \\ &= p(2q-1) + 1 - q \end{aligned}$$

We can do the same for Player 2 and obtain

$$\begin{aligned} g(p,q) &= q \cdot (0 \cdot p + 2 \cdot (1-p)) + (1-q) \cdot (3 \cdot p + 0 \cdot (1-p)) \\ &= 0 \cdot p \cdot q + 3 \cdot p \cdot (1-q) + 2 \cdot (1-p) \cdot q + 0 \cdot (1-p) \cdot (1-q) \\ &= 3p(1-q) + 2q(1-p) \\ &= 3p - 3pq + 2q - 2pq \\ &= 3p - 5pq + 2q \\ &= q(2-5p) + 3p \end{aligned}$$

Note that the payoff is written in terms of p and q because we don't have a pure strategy as a solution for either of the players. As next step, we need to define the multifunctions

$$BR_1(q) = \arg\max\{f(\cdot, q)\}$$

and

$$BR_2(p) = \arg\max\{g(p, \cdot)\}$$

To achieve this goal we have to find the equilibrium points of the utility functions. Let's start with f and compute its derivative with respect to p.

$$\frac{df(p,q)}{p} = \frac{d}{dp}\big[p(2q-1) + 1 - q\big] = 2q - 1$$

If we put the derivative to 0, we get the equilibrium point.

$$2q - 1 = 0 \iff q = \frac{1}{2}$$

This means that, if $q = \frac{1}{2}$, any choice of p is acceptable (if we consider $p \in [0,1]$). The value of q might still be different from $\frac{1}{2}$, then

- If $0 \leq q < \frac{1}{2}$, $f(p,q)$ is maximised for $p = 0$, which is the only possible solution.
- If $q > \frac{1}{2}$, $f(p,q)$ is maximised for $p = 1$.

This means that the $BR_1(q)$ function is

$$BR_1(q) = \begin{cases} p = 0 & \text{if } 0 \leq q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

Applying the same reasoning to $g(p,q)$ we obtain

$$\frac{dg(p,q)}{dq} = 2 - 5p = 0 \iff p = \frac{2}{5}$$

Therefore the function $BR_2(p)$ is

$$BR_2(p) = \begin{cases} q = 1 & \text{if } 0 \leq q < \frac{2}{5} \\ q \in [0,1] & \text{if } q = \frac{2}{5} \\ q = 0 & \text{if } q > \frac{2}{5} \end{cases}$$

The intersection between $BR_1(q)$ and $BR_2(p)$, as shown in Figure 4.1 is $p = \frac{2}{5}$ and $q = \frac{1}{2}$ and allows us to obtain the optimal strategy, which is

$$BR = \left\{ \left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$$

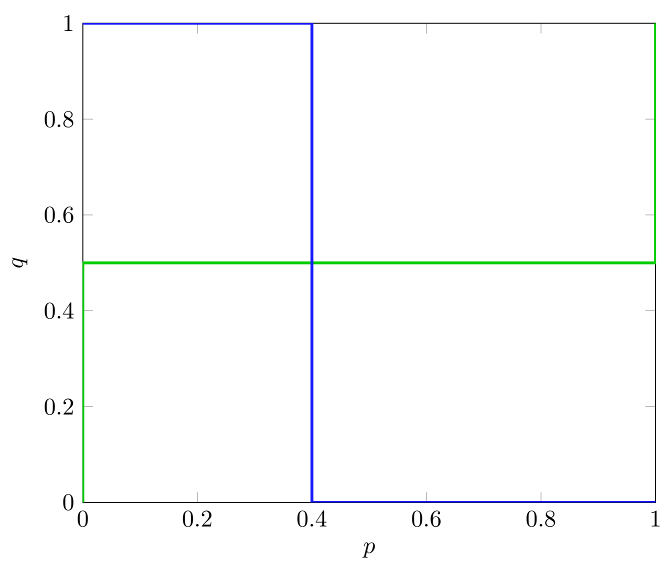


Figure 4.1: A best response function.

4.4.3 Best reaction in pure strategies

Let us highlight one important thing. The utility function of any player is linear in its own variable, therefore there exist some pure strategies which are optimal responses to any mixed strategy of the opponent. This works for both players. Namely, for every (mixed) strategy y of the second player, $BR_1(y)$ contains at least a pure strategy since Player 1 maximises a linear function over a simplex (in fact, such a strategy is a vertex of the simplex).

Before going on, let us define the support of a vector $\bar{x}$ as follows:

Definition 4.7 (Support). *Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ be a mixed strategy. The support $spt\bar{x}$ of $\bar{x}$ is the set of values of the vector which are positive, namely*

$$spt\bar{x} = \{i : \bar{x}_i > 0\}$$

In other words, the support is the set of rows or columns which are considered in the mixed strategy, i.e., for which the probability is not 0.

Now, given a Nash equilibrium $(\bar{x}, \bar{y})$ in mixed strategies, given the supports $\text{spt}\bar{x} = \{1, \dots, k\}$ and $\text{spt}\bar{y} = \{1, \dots, l\}$ and assuming $f(\bar{x}, \bar{y}) = v$, then:

- For $i \in \arg\{\bar{x}\}$ and $j \in \arg\{\bar{y}\}$, is going to provide v as an outcome, i.e., the one of the equilibrium. Namely
$$\begin{cases} a_{11}\bar{y}_1 + a_{12}\bar{y}_2 \cdots + a_{1l}\bar{y}_l = v \\ \vdots \\ a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 \cdots + a_{kl}\bar{y}_l = v \end{cases}$$

- Any other combination of strategies is going to provide a final outcome less than or equal to v, which means they are not optimal.
$$\begin{cases} a_{(k+1)1}\bar{y}_1 + a_{(k+1)2}\bar{y}_2 \cdots + a_{(k+1)l}\bar{y}_l \leq v \\ \vdots \\ a_{n1}\bar{y}_1 + a_{n2}\bar{y}_2 \cdots + a_{nl}\bar{y}_l \leq v \end{cases}$$

The same reasoning can be applied to Player 2 for which we can write:

$$\begin{cases} b_{11}\bar{x}_1 + b_{21}\bar{x}_2 \cdots + b_{k1}\bar{x}_k = w \\ \vdots \\ b_{1l}\bar{x}_1 + b_{2l}\bar{x}_2 \cdots + b_{kl}\bar{x}_k = w \\ b_{1(l+1)}\bar{x}_1 + b_{2(l+1)}\bar{x}_2 \cdots + b_{k(l+1)}\bar{x}_k \leq w \\ \vdots \\ b_{1n}\bar{x}_1 + b_{2n}\bar{x}_2 \cdots + b_{kn}\bar{x}_k \leq w \end{cases}$$

Moreover, the principle of indifference still applies since the final outcome is (v, w) despite what the other plays, respecting the assumptions.

Example

Let us consider the following game, as an example, to show what we have just explained.

$$\begin{pmatrix} (2,2) & (a,3) & (3,3) \\ (4,0) & (3,4) & (5,b) \\ (2,3) & (5,2) & (4,26) \end{pmatrix}$$

The goal is to find the values of a and b such that there is a Nash equilibrium with support in rows 1 and 2 for the first player and columns 2 and 3 for Player 2. If the support vector for Player 1 is

$$\text{spt}\bar{x} = \{1,2\}$$

then Player 1's mixed strategy is

$$\bar{x} = (p, 1-p, 0)$$

Equivalently, if Player 2's support vector is

$$\text{spt}\bar{y} = \{2,3\}$$

then his/her mixed strategy is

$$\bar{y} = (0, q, 1-q)$$

Let's consider Player 1 first. We can write the system

$$\begin{cases} aq + 3(1-q) = v \\ 3q + 5(1-q) = v \\ 5q + 4(1-q) \leq v \end{cases}$$

which can be written in short as

$$\begin{cases} aq + 3(1-q) = 3q + 5(1-q) \\ 5q + 4(1-q) \leq 3q + 5(1-q) \end{cases}$$

Solving this system we get

$$\begin{cases} q = \frac{2}{a-1} \\ q \leq \frac{1}{3} \end{cases}$$

Replacing q in the second inequality we obtain

$$a \geq 7$$

Now we can repeat the same process for Player 2 for which we obtain:

$$2p + 0(1-p) \leq 3p + 4(1-p)3p + 4(1-p) = 3p + b(1-p)$$

from which we can compute $b = 4$.

4.4.4 Full support

Say we are now looking for fully mixed Nash equilibria, that is all rows and columns are played with positive (non-null) probabilities. If $(\bar{x}, \bar{y})$ is a fully mixed Nash equilibrium, then we can write

$$a_{i1}\bar{y}_1 + a_{i2}\bar{y}_2 + \dots a_{im}\bar{y}_m = a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \dots a_{km}\bar{y}_m$$

for all rows $i, k = 1, \dots, n$ and similarly

$$b_{1r}\bar{x}_1 + b_{2r}\bar{x}_2 + \dots b_{nr}\bar{x}_n = b_{1s}\bar{x}_1 + b_{2s}\bar{x}_2 + \dots b_{ns}\bar{x}_n$$

for all columns $r, s = 1, \dots, m$ with the conditions $\sum_i p_i = 1$ and $\sum_i q_i = 1$. This means that we can equate any mixed strategy. Also, note that the indifference principle still applies and we have to use it to choose a strategy.

Brute force algorithm

One way of finding a Nash equilibrium in pure strategies is using brute force.

Algorithm 4.2 (Bruteforce Nash equilibrium)**.** *The algorithm works as follows:*

1. *Guess the support spt*($\bar{x}$) *and spt*($\bar{y}$) *of the equilibria.*
2. *Focus on the equalities (since inequalities will lead to sub-optimal solutions) adding normalisation conditions (the sum of probabilities has to be 1) and the constraints to meet the support's hypothesis.*
3. *Solve the system of inequalities*

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \\ \sum_{j=1}^{m} a_{ij} y_j = v & \forall i \in spt(\bar{x}) \\ x_i = 0 & \forall i \notin spt(\bar{x}) \end{cases}$$

and

$$\begin{cases} \sum_{j=1}^{m} y_j = 1 & \\ \sum_{i=1}^{n} b_{ij} x_i = w & \forall j \in spt(\bar{y}) \\ y_j = 0 & \forall j \notin spt(\bar{y}) \end{cases}$$

4. *Check the ignored inequalities. If the inequalities hold (i.e.,* $\sum a_{ij} y_j \leq v$ *and* $\sum b_{ij} x_i \leq w$*) the algorithm ends, otherwise, the algorithm is repeated with different supports.*

The problem with this algorithm is the high number of possible combinations we might have to try, which are $(2^n - 1)(2^m - 1)$.

4.5 Examples

4.5.1 Braes paradox

4.5.2 Duopoly, monopoly and leader models

Let's consider a game in which the players are two firms Firm 1 and Firm 2. The firms choose quantities q_1 for Firm 1 and q_2 for Firm 2 of a good to produce. For both companies, the unit cost of the good is $c > 0$. Moreover, a quantity a saturates the market, hence if the companies produce more than a in total (summed) then it's not convenient to produce anymore. The price of a good depends on the quantity on the market and it is given by

$$p(q_1, q_2) = \max\{a - (q_1 + q_2), 0\}$$

This means that, if the companies produce more than what the market can absorb (i.e., a), then they can't have a profit (the price is 0). In this context, the payoff of Firm 1 is

$$\begin{aligned} u_1(q_1, q_2) &= q_1 \cdot p(q_1, p_1) - c \cdot q_1 \\ &= q_1[a - (q_1 + q_2)] - cq_1 \\ &= -q_1^2 + q_1(a - c - q_2) \end{aligned}$$

On the other hand, the payoff of Firm 2 is

$$\begin{aligned} u_2(q_1, q_2) &= q_2 \cdot p(q_1, p_1) - c \cdot q_2 \\ &= q_2[a - (q_1 + q_2)] - cq_2 \\ &= -q_2^2 + q_2(a - c - q_1) \end{aligned}$$

This is the general setting in which we can define the monopoly, duopoly and leader models.

Monopoly model

In the general setting, if $q_2 = 0$, then Firm 2 is out of the market and Firm 1 has a monopoly. This means that Firm 1's utility is

$$u_1(q_1, q_2) = -q_1^2 + q_1(a - c)$$

We can now maximise this function by computing the derivative with respect to q_1 to obtain

$$\frac{du_1}{dq_1} = \frac{d}{dq_1}(-q_1^2 + q_1(a - c)) = -2q_1 + a - c$$

By putting the derivative to 0 we obtain

$$\begin{aligned} -2q_1 + a - c &= 0 \\ -2q_1 &= -a + c \\ q_1 &= \frac{a - c}{2} \end{aligned}$$

Hence the equilibrium is

$$(\bar{q}_1, \bar{q}_2) = \left(\frac{a - c}{2}, 0\right)$$

We can use this information to compute the price to which Firm 1 should be selling the good, which is

$$p(q_1, q_2) = \max\left\{a - \left(\frac{a-c}{2} + 0\right), 0\right\} = \frac{a+c}{2}$$

We can also compute the payoff for Firm 1, which is

$$\begin{aligned} u_1(q_1, q_2) &= -q_1^2 + q_1(a - c - q_2) \\ &= -\left(\frac{a-c}{2}\right)^2 + \frac{a-c}{2}(a-c) \\ &= -\frac{(a-c)^2}{4} + \frac{(a-c)^2}{2} = \frac{(a-c)^2}{4} \end{aligned}$$

4.5.3 Duopoly model

In a duopoly, Firm 1 and Firm 2 share the market, hence we have $q_1 \neq 0$ and $q_2 \neq 0$. Let's start by looking at the equilibrium for Firm 1. As always we want to compute the best response, hence we have to find the maximum for q_1 by computing the derivative and putting it to 0. The derivative is

$$\begin{aligned} \frac{du_1}{dq_1} &= \frac{d}{dq_1}(-q_1^2 + q_1(a - c - q_2)) \\ &= -2q_1 + a - c - q_2 \end{aligned}$$

By putting this derivative to 0 we obtain

$$\bar{q}_1 = \frac{a - c - \bar{q}_2}{2} \tag{4.2}$$

Repeating the same reasoning for Firm 2 we get

$$\begin{aligned} \frac{du_2}{dq_2} &= \frac{d}{dq_2}(-q_2^2 + q_2(a - c - q_1)) \\ &= -2q_2 + a - c - q_1 \end{aligned}$$

By putting this derivative to 0 we obtain

$$\bar{q}_2 = \frac{a - c - \bar{q}_1}{2}$$

We can now rewrite this last result as

$$\bar{q}_1 = a - c - 2\bar{q}_2 \tag{4.3}$$

We can now replace this result in 4.2 to obtain

$$a - c - 2\bar{q}_2 = \frac{a - c - \bar{q}_2}{2} \tag{4.4}$$

$$2a - 2c - 4\bar{q}_2 = a - c - \bar{q}_2 \tag{4.5}$$

$$2a - 2c - a + c = -\bar{q}_2 + 4\bar{q}_2 \tag{4.6}$$

$$a - c = 3\bar{q}_2 \tag{4.7}$$

$$\frac{a-c}{3} = \bar{q}_2 \tag{4.8}$$

We can now replace this value in 4.2 to obtain

$$\bar{q}_1 = \frac{a - c - \frac{a-c}{3}}{2} \tag{4.9}$$
$$= \frac{3a - 3c - a + c}{6} \tag{4.10}$$
$$= \frac{2a - 2c}{3} \tag{4.11}$$
$$= \frac{a - c}{3} \tag{4.12}$$

The Nash equilibrium profile of a duopoly is therefore

$$(\bar{q}_1, \bar{q}_2) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right) \tag{4.13}$$

The equilibrium can be used to compute the optimal price of the good

$$p(\bar{q}_1, \bar{q}_2) = a - (\bar{q}_1 + \bar{q}_2) = a - \left(\frac{a-c}{3} + \frac{a-c}{3}\right) = a - \frac{2}{3}(a-c) \tag{4.14}$$

and the payoff, which is the same for both, since $\hat{q}_1 = \hat{q}_2$

$$u_1(\hat{q}_1, \hat{q}_2) = u_2(\hat{q}_1, \hat{q}_2) = -\frac{(a-c)^2}{9} + \frac{a-c}{3}\left(a - c - \frac{a-c}{3}\right) = \frac{(a-c)^2}{9} \tag{4.15}$$

Leader model

In the leader model, a firm, say Firm 1, announces its strategy and Firm 2 reacts accordingly.

Chapter 5

Potential games

5.1 Potential games

Let's consider finite games with common payoff, which means that all the players share the same payoff function

$$p : X \to \mathbb{R}$$

valid for all utility functions. This means we can write, for every player i

$$u_i(x_1, \dots, x_n) = p(x_1, \dots, x_n)$$

If we consider a strategy profile

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$$

such that

$$p(\bar{x}) \geq p(x) \qquad \forall x \in X$$

then $\bar{x}$ is a Nash equilibrium in pure strategies. Note that, there might be other Nash equilibria in pure or mixed strategies. However, $\bar{x}$ is the best strategy for all players.

5.1.1 Best response dynamics

Now we want to define a procedure to find the Nash equilibrium in pure strategies $\bar{x}$. In particular, we are going to use an algorithm called **payoff improving procedure**.

> **Algorithm 5.1** (Payoff improving procedure)**.** *The payoff-improving procedure works as follows:*
>
> 1. *Start from an arbitrary strategy profile* $(x_1, \dots, x_n) \in X$.
> 2. *Ask if any player has a better strategy* x_i' *that strictly increases his/her payoff. Namely, we want to find if it exists a* i *such that*
>
> $$u_i(x_i', x_{-i}) > u_i(x_i, x_{-i})$$
>
> *If* x_i' *exists, we replace* x_i *with* x_i' *and we repeat the procedure, otherwise, we stop since we have found the Nash equilibrium* $\bar{x}$.

Each iteration of the procedure strictly increases the value $p(x)$, so that no strategy profile $x \in X$ can be visited twice. Since X is a finite set, the procedure must reach a pure Nash equilibrium after at most $|X|$ steps. In particular, the following result holds.

Proposition 5.1 (Payoff improving procedure completeness)**.** *The payoff-improving procedure is guaranteed to reach a global minimum $\bar{x}$.*

5.1.2 Payoff equivalence

Let us now consider an arbitrary finite game with payoffs

$$u_i : X \to \mathbb{R}$$

and let us build another payoff function obtained by adding a term c_i to u_i, namely

$$\tilde{u}_i(x_1, \dots, x_n) = u_i(x_1, \dots, x_n) + c_i$$

If c_i is a constant or a term that depends on the x_{-i} and not x_i then the best responses and the equilibria remain the same. In particular,

Definition 5.1 (Diff-equivalent payoffs)**.** *The payoffs $\tilde{u}_i$ and u_i are said to be **diff-equivalent** if the difference*

$$\tilde{u}_i(x_1, \dots, x_n) - u_i(x_1, \dots, x_n) = c_i(x_{-i})$$

does not depend on Player i's decision x_i but only on the strategies of the other players.

This means that for all strategies $x_i', x_i \in X_i$, the difference between the payoff $\tilde{u}_i$ and u_i are the same:

$$\tilde{u}_i(x_i', x_{-i}) - u_i(x_i', x_{-i}) = \tilde{u}_i(x_i, x_{-i}) - u_i(x_i, x_{-i}) = c_i(x_{-i})$$

Using the notation

$$\Delta f(x_i', x_i, x_{-i}) = f(x_i', x_{-i}) - f(x_i, x_{-i})$$

then the equality can be written as

$$\Delta \tilde{u}_i(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i})$$

Let us now state the results obtained so that we can prove them.

Theorem 5.1 (Diff-equivalent payoffs)**.** *Finite games with diff-equivalent payoffs have the same pure Nash equilibria.*

Proof. The best reaction multifunction, for every player i, is the same when considering two diff-equivalent payoffs u_i and $\tilde{u}_i$, no matter how different from each other the latter functions are. □

5.1.3 Potential game

Definition 5.2 (Potential game)**.** *A finite game with strategy sets X_i and payoffs $u_i : X \to R$ is called a potential game if it is diff-equivalent to a game with common payoffs. That is, there exists a potential function $p : X \to \mathbb{R}$ such that for each i, for every $x_{-i} \in X_{-i}$, and all $x_i', x_i \in X_i$ we have*

$$\Delta u_i(x_i', x_i, x_{-i}) = \Delta p(x_i', x_i, x_{-i})$$

As a result:

- Every finite potential game has at least one pure Nash equilibrium (because of Theorem 5.1).
- In a finite potential game every best response iteration reaches a pure Nash equilibrium in finitely many steps (because of Algorithm 5.1).

Example

Let us consider the following game as an example to understand what's the meaning of potential.

$$\begin{pmatrix} (10,10) & (0,11) \\ (11,0) & (1,1) \end{pmatrix}$$

A potential for this game is

$$p(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Let's consider Player 1 first. We have $x_{-1} = x_2$ and

- When the first column is fixed, moving from the first strategy (row) to the second strategy (row) gives $\Delta u_1 = 11 - 10 = 1$, which is equal to $\Delta p = 1 - 0$.
- When the second column is fixed, moving from the first strategy to the second strategy gives $\Delta u_1 = 1 - 0 = 1$, which is equal to $\Delta p = 2 - 1$.

Moving to Player 2 we have $x_{-2} = x_1$ and

- When the first row is fixed, moving from the first strategy (column) to the second strategy (column) gives $\Delta u_2 = 11 - 10 = 1$, which is equal to $\Delta p = 1 - 0$.
- When the second row is fixed, moving from the first strategy to the second strategy gives $\Delta u_2 = 1 - 0 = 1$, which is equal to $\Delta p = 2 - 1$.

5.1.4 Finding a potential

The distinctive feature of a potential game is its potential p, hence we want to find a way to compute it. A potential

$$p : X \to \mathbb{R}$$

which is characterised by

$$\Delta u_i(x_i', x_i, x_{-i}) = \Delta p(x_i', x_i, x_{-i})$$

If we add a constant to the potential p, we obtain a new potential. This fact can be used to compute a potential. If we fix an arbitrary profile

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$$

and set

$$p(\bar{x}) = 0$$

then, step by step, we can move from $p(x)$ to $p(\bar{x})$ by replacing one variable at a time considering the differences that have to be maintained. In fact, we can write:

$$\begin{aligned}
p(x_1, x_2, \dots, x_n) - p(\bar{x}_1, x_2, \dots, x_n) &= u_1(x_1, x_2, \dots, x_n) - u_1(\bar{x}_1, x_2, \dots, x_n) \\
p(\bar{x}_1, x_2, \dots, x_n) - p(\bar{x}_1, \bar{x}_2, \dots, x_n) &= u_2(\bar{x}_1, x_2, \dots, x_n) - u_2(\bar{x}_1, \bar{x}_2, \dots, x_n) \\
&\vdots \\
p(\bar{x}_1, \bar{x}_2, \dots, x_n) - p(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) &= u_n(\bar{x}_1, \bar{x}_2, \dots, x_n) - u_n(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)
\end{aligned}$$

If we add the lines above, we obtain

$$p(x_1, x_2, \dots, x_n) - p(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = \sum_{i=1}^{n} \left[u_i(\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \dots, x_n) - u_i(\bar{x}_1, \dots, \bar{x}_{i-1}, \bar{x}_i, \dots, x_n)\right]$$

but since we have set $p(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = 0$, then we get

$$p(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \left[u_i(\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \dots, x_n) - u_i(\bar{x}_1, \dots, \bar{x}_{i-1}, \bar{x}_i, \dots, x_n)\right]$$

hence we have found a way to compute the potential. If the game admits a potential the sum on the right-hand side of the equation above is independent of the particular order used. The converse is also true. However, checking that all these orders yield the same answer is impractical for more than 2 or 3 players. Let us consider a game with two players to better understand how potential is built. If we consider Player 1 and a potential exists, we can fix a column (i.e., a strategy for Player 2) and check the difference between the utility function when playing a row or another. Namely, for any row, we can compute the difference with the other rows. This process can be repeated for any column. We can do the same for Player 2 fixing a row.

Example

Let us consider an example to practically understand how to compute a potential. The game

$$\begin{pmatrix}
(2,5) & (2,6) & (3,7) & (8,9) & (5,7) \\
(1,4) & (1,5) & (3,7) & (2,3) & (0,2) \\
(6,5) & (2,2) & (0,0) & (6,3) & (3,1)
\end{pmatrix}$$

To compute the potential we can start by putting the top left corner of the potential to 0 and by moving along the columns (using the first value of each cell) and the rows (using the second value of each cell) so that all values are coherent. What we obtain is:

$$\begin{pmatrix}
0 & 1 & 2 & 4 & 2 \\
-1 & 0 & 2 & -2 & -3 \\
4 & 1 & -1 & 2 & 0
\end{pmatrix}$$

5.2 Social cost and efficiency

Definition 5.3 (Pareto efficient equilibrium)**.** *An equilibrium is Pareto efficient if it is not possible to increase the utility of a player without decreasing the utility of some other player.*

Nash equilibria don't need be Pareto efficient and in fact, they can be bad for all the players, like in Braess's paradox or in the Prisoner's dilemma. Hence, we want to weigh how good a solution is using a cost function instead of utilities. In particular, we introduce a function, called social cost

$$C : X \to \mathbb{R}_+$$

that assigns a cost to a strategy profile $x = (x_1, \dots, x_n)$:

$$x \mapsto C(x)$$

The lower the cost, the better the strategy is. The benchmark we will use to understand if a strategy is good with respect to all the players is the minimum value that a benevolent social planner could achieve, hence

$$\text{Opt} = \min_{x \in X} C(x)$$

Now we can classify solutions $x \in X$ using the ratio $\frac{C(x)}{\text{Opt}}$ which measures how far an outcome x is from being optimal. In particular:

- A large value implies a big loss in terms of social welfare since the outcome (the nominator) is far from the optimal outcome.
- A quotient close to 1 implies that x is almost as efficient as an optimal solution.

5.2.1 Price-of-Anarchy and Price-of-Stability

Definition 5.4 (Price-of-Anarchy)**.** *Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The Price-of-Anarchy is defined as*

$$PoA = \max_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt}$$

The Price-of-Anarchy is the ratio of the worst social cost of an equilibrium and the optimal social cost.

Definition 5.5 (Price-of-Stability)**.** *Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The Price-of-Stability is defined by*

$$PoS = \min_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt}$$

The Price-of-Stability is the ratio of the best social cost of an equilibrium and optimal social cost. Note that the Price-of-Anarchy is bigger than the Price-of-Stability, which is always bigger than 1, namely

$$1 \le PoS \le PoA$$

In particular:

- $PoA \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{x})$ is no worse than $\alpha \cdot \text{Opt}$.
- $PoS \leq \alpha$ means that there exists some equilibrium with social cost at most $\alpha \cdot \text{Opt}$.

Price-of-Stability estimation

The Price-of-Stability can be estimated. For doing so, we need the following proposition.

Proposition 5.2. *Consider a cost minimisation finite potential game with potential $p : X \to \mathbb{R}$ and suppose that there exist $\alpha, \beta > 0$ such that*

$$\frac{1}{\alpha} C(x) \leq p(x) \leq \beta C(x) \qquad \forall x \in X$$

Then $PoS \leq \alpha\beta$.

Proof. Let $\bar{x}$ be a minimum of $p(\cdot)$ so that $\bar{x}$ is a Nash equilibrium. Then it holds:

$$\frac{1}{\alpha} C(\bar{x}) \leq p(\bar{x}) \leq p(x) \leq \beta C(x) \qquad \forall x \in X$$

Since this is true for all x, we can choose the strategy x, yielding the optimal outcome $\text{Opt} = \min_{x \in X} C(x)$ and hence it follows that $C(\bar{x}) \leq \alpha\beta\text{Opt}$. □

5.2.2 Price-of-Anarchy and Price-of-Stability with utilities

In case a game deals with utilities rather than costs, we define

$$\text{Opt} = \max_{x \in X} U(x)$$

where $U(x)$ is some fixed social utility function. Since we are using utilities, we can define PoS and PoA maximising U. In particular, we have

Definition 5.6 (Price-of-Anarchy with utilities)**.** *Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The Price-of-Anarchy is defined as*

$$PoA = \max_{\bar{x} \in NE} \frac{Opt}{U(\bar{x})} = \frac{Opt}{\min_{\bar{x} \in NE}(U(\bar{x}))}$$

The Price-of-Anarchy is the ratio of the worst social cost of an equilibrium and the optimal social cost.

Definition 5.7 (Price-of-Stability with utilities)**.** *Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The Price-of-Stability is defined by*

$$PoS = \min_{\bar{x} \in NE} \frac{Opt}{U(\bar{x})} = \frac{Opt}{\max_{\bar{x} \in NE}(U(\bar{x}))}$$

Chapter 6

Repeated games

6.1 Repeated games

Let us consider the following game

$$\begin{pmatrix} (6,6) & (0,10) & (-2,-2) \\ (10,0) & (1,1) & (-1,-1) \\ (-2,-2) & (-1,-1) & (-2,-2) \end{pmatrix}$$

which has an equilibrium $(1,1)$ in strictly dominated strategies. Now suppose that the game is repeated N times (e.g., is played for N days), hence it's a **repeated game**, or **stage game**. The equilibrium $(1,1)$ we have found is still an equilibrium when repeating the game, however, we would like to find out if there is a way for both players to obtain a better outcome than $(1,1)$, like $(6,6)$. Namely, we want to find out if the repeated game has other Nash equilibria with a better outcome. Formally, we have to show that, for every $a > 0$, if N is sufficiently big, each player can get at least $6 - a$ on average. Let us consider the game above repeated N times and the following strategy to get to such a result:

1. Player one (two) plays the first row (column) the first $N - k$ days and the second row (column) the last k days if the second (first) player uses the same strategy.

2. If one day the second (first) deviates from the strategy at point 1, from that stage on player one (two) plays the third row (column).

In other words, the players collaborate for $N - k$ times to get the best outcome (i.e., $(6,6)$) and then they play the strictly dominant strategy for the remaining k days. As soon as one player stops collaborating, i.e. it doesn't follow the aforementioned strategy, the other player chooses the last row or column that forces both players to get a negative outcome. This should be enough to ensure that both players stick with the proposed strategy. Let us formally show that in fact, it's better to stick to the strategy. When following this strategy, each player receives

$$\frac{(N-k) \cdot 6 + k \cdot 1}{N}$$

However, if one player deviates at time $N - k$ (i.e., when it's more convenient since it can't get 6 as a reward anymore), it will get

$$\frac{(N-k-1) \cdot 6 + 10 + k \cdot -1}{N}$$

since the player:

1. Plays row or column 1 until time $N-k-1$.
2. At the time $N-k$ plays row or column 2 to get a higher reward.
3. Is forced to play column 2 to get the best reward possible, which is however negative.

We want the first strategy to be better than the second, hence we want the first outcome to be not smaller than the second:

$$\begin{aligned}
\frac{(N-k)\cdot 6+k\cdot 1}{N} &\geq \frac{(N-k-1)\cdot 6+10+k\cdot -1}{N} \\
(N-k)\cdot 6+k\cdot 1 &\geq (N-k-1)\cdot 6+10-k\cdot 1 \\
(N-k)\cdot 6+2k &\geq (N-k)\cdot 6-6+10 \\
(N-k)\cdot 6+2k &\geq (N-k)\cdot 6+4 \\
2k &\geq 4 \\
k &\geq 2
\end{aligned}$$

This means that the former strategy profile is better than the latter when the players have to play the strictly dominated strategy (i.e., $(1,1)$) at least twice.

If we now compute the limit for $N \to \infty$, that is to say, we compute the value obtained by each player when playing an infinite amount of times we get

$$\lim_{N\to\infty} \frac{(N-k)\cdot 6+k\cdot 1}{N} = 6$$

which shows that on average the players can get at least $6-a$ each per day (with a being a small number) if they play a sufficiently large number of days. As a result, we have shown that the strategy above is in fact a Nash equilibrium.

Note that when a game is repeated many times, the collaboration between the players, even if dominated in the one-shot game, can be based on rationality, as we have shown above. However, the cooperative strategy of the NEp has a weakness because it's based on a mutual threat of the players, which is not completely credible since by punishing the player who deviates from the agreement the other will also damage herself. In general, the number of the NEp in the repetition of the game is very large.

6.1.1 Infinite repetitions of the game

In the previous example, we analysed an example of a repeated game. Let us now generalise the concept considering games with infinite repetitions (i.e., $N \to \infty$). If we want to define such a game, we have to state:

- The strategies used by each player.
- The payoffs received by each player.

Strategy

At each stage τ, the players know which outcome has been selected at stage $\tau - 1$, thus a player's strategy s is

$$s = (s(\tau))_{\tau=0,\dots,N} = (s(0), s(1), \dots, s(N))$$

where, for each τ, $s(\tau)$ is a specification of moves of the game, which is in general a function of the past choices of the players. In the prisoner dilemma, a possible strategy could be

$$s = (s(0), s(1))$$

where

1. $s(0)$ is *do not confess.*
2. $s(1)$ is *confess if the other player did not confess at stage zero, otherwise confess.*

Payoff

Since we are considering infinite repetitions of a game, it's impossible to sum up the payoff at each repetition since we will obtain, in any case, an infinite value. We can use discount factors to obtain a meaningful payoff function. More precisely, we introduce a discount factor δ between 0 and 1, namely

$$0 < \delta < 1$$

and we define the payoff of player i as

$$u_i(s,t) = (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau u_i(s(\tau), t(\tau)) \tag{6.1}$$

where

- s and t are the strategies of Player 1 and Player 2, respectively.
- $u_i(s(\tau), t(\tau))$ is the stage-game payoff of player i at time τ given strategy profile $(s(\tau), t(\tau))$.
- $(1-\delta)$ is the normalising factor. This is needed because if the payoff at time τ is always equal to a, independently from τ, then we get

$$\begin{aligned}
u_i(s,t) &= (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau u_i(s(\tau), t(\tau)) \\
&= (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau a \\
&= a(1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau \\
&= a(1-\delta)\frac{1}{1-\delta} && \text{Geometric series} \\
&= a
\end{aligned}$$

instead of $\frac{a}{1-\delta}$.

6.1.2 Threat values

One important factor of a repeated game is the fact that the strategy works because each player threatens the other to obtain a bad payoff. Let us analyse this aspect a bit deeper.

Definition 6.1 (Threat values)**.** *For the bimatrix game* (A, B) *representing a stage game*

$$\underline{v}_1 = \min_j \max_i a_{ij}$$

and

$$\underline{v}_2 = \min_i \max_j b_{ij}$$

are called the threat values of Player 1 and Player 2, respectively.

For instance, say $\underline{v}_1$ is obtained with $j = \bar{j}$. If Player 2 wants to punish Player 1, $\underline{v}_1$ is the highest utility Player 1 can get if Player 2 uses $\bar{j}$.

Note that threat values $\underline{v}_1$ and $\underline{v}_2$ are not the conservative values of the two players. If we consider the example we did before, namely

$$\begin{pmatrix} (6,6) & (0,10) & (-2,-2) \\ (10,0) & (1,1) & (-1,-1) \\ (-2,-2) & (-1,-1) & (-2,-2) \end{pmatrix}$$

we obtain

$$\begin{aligned} \underline{v}_1 &= \min_j \max_i a_{ij} \\ &= \min_j \{10, 6, -1\} \\ &= -1 \end{aligned}$$

which we have seen in the example is the best that Player 2 can make Player 1 pay in case he/she stops following the coordinated rules.

Theorem 6.1 (Folk theorem)**.** *Let* (A, B) *be a bimatrix game. For every feasible payoff vector* $v = (v_1, v_2) = (a_{\bar{i}\bar{j}}, b_{\bar{i}\bar{j}})$ *such that* $v_i > \underline{v}_i$*, there exists* $\bar{\delta} < 1$ *such that for all* $\delta > \bar{\delta}$ *there is a Nash equilibrium of the repeated game with discounting factor* δ*, which yields payoffs* v*.*

Proof. Let's consider a payoff vector

$$v = (v_1, v_2) = (a_{\bar{i}\bar{j}}, b_{\bar{i}\bar{j}})$$

such that $v_i > \underline{v}_i$. Let us now define the following strategy s: *play the strategy yielding* v *at any stage, unless the opponent deviates at time* t*. In this case, play the thread strategy from the stage t+1 onward.* Now we have to show that

- s provides the utility vector v.
- s is a Nash equilibrium for all δ close to 1.

At time t Player i can gain at most $\max_{i,j} a_{ij}$, which is not what he/she gains from following the coordinated strategy but the maximum value he/she can get. If Player 1 deviates at time t, he/she will gain at most $\underline{v}_1$ from $t+1$ on by using strategy s_t, which is the strategy used when deviating at time t. Player 1's payoff would be:

$$\begin{aligned}
u_1(s_t) &\leq (1-\delta)\Bigg(\sum_{\tau=0}^{t-1} \delta^\tau v_1 + \delta^t \max_{i,j} a_{ij} + \sum_{\tau=t+1}^{\infty} \delta^\tau \underline{v}_1\Bigg) && (6.2)\\
&\leq (1-\delta)\Bigg(v_1 \sum_{\tau=0}^{t-1} \delta^\tau + \delta^t \max_{i,j} a_{ij} + \underline{v}_1 \sum_{\tau=t+1}^{\infty} \delta^\tau\Bigg) && (6.3)\\
&\leq (1-\delta)\Bigg(v_1 \sum_{\tau=0}^{t-1} \delta^\tau + \delta^t \max_{i,j} a_{ij} + \underline{v}_1 \sum_{\tau=0}^{\infty} \delta^\tau - \underline{v_1} \sum_{\tau=0}^{t} \delta^\tau\Bigg) && (6.4)\\
&\leq (1-\delta)\Bigg(v_1 \frac{1-\delta^t}{1-\delta} + \delta^t \max_{i,j} a_{ij} + \underline{v}_1 \frac{1}{1-\delta} - \underline{v_1} \frac{1-\delta^{t+1}}{1-\delta}\Bigg) && (6.5)\\
&\leq (1-\delta)\Bigg(v_1 \frac{1-\delta^t}{1-\delta} + \delta^t \max_{i,j} a_{ij} + \underline{v}_1 \frac{1-1+\delta^{t+1}}{1-\delta}\Bigg) && (6.6)\\
&\leq v_1(1-\delta^t) + (1-\delta)\delta^t \max_{i,j} a_{ij} + \delta^{t+1}\underline{v}_1 && (6.7)
\end{aligned}$$

Note that this is an upper bound for the value of $u_1(s_t)$ since Player 1 changes strategy but we don't know what Player 2 would play at time t. If Player 1 doesn't change his/her strategy, then the payoff is

$$u_1(s) = (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau v_1 = v_1 \tag{6.8}$$

Sticking to the coordinated strategy is better if

$$\begin{aligned}
u_1(s_t) &\leq u_1(s) && (6.9)\\
v_1(1-\delta^t) + (1-\delta)\delta^t \max_{i,j} a_{ij} + \delta^{t+1}\underline{v}_1 &\leq v_1 && (6.10)\\
(1-\delta)\delta^t \max_{i,j} a_{ij} + \delta^{t+1}\underline{v}_1 &\leq v_1 - v_1(1-\delta^t) && (6.11)\\
(1-\delta)\delta^t \max_{i,j} a_{ij} + \delta^t\delta\underline{v}_1 &\leq v_1\delta^t && (6.12)\\
(1-\delta) \max_{i,j} a_{ij} + \delta\underline{v}_1 &\leq v_1 && (6.13)\\
-\delta \max_{i,j} a_{ij} + \delta\underline{v}_1 &\leq v_1 - \max_{i,j} a_{ij} && (6.14)\\
\delta(\underline{v}_1 - \max_{i,j} a_{ij}) &\leq v_1 - \max_{i,j} a_{ij} && (6.15)\\
\delta(\max_{i,j} a_{ij} - \underline{v}_1) &\geq \max_{i,j} a_{ij} - v_1 && (6.16)
\end{aligned}$$

By properly setting $\underline{\delta}_1 = \frac{\max_{i,j} a_{ij} - v_1}{\max_{i,j} a_{ij} - \underline{v}_1}$ we can say that

$$0 < \underline{\delta}_1 < 1 \tag{6.17}$$

hence by setting

$$\underline{\delta}_1 = \frac{\max_{i,j} a_{ij} - v_1}{\max_{i,j} a_{ij} - \underline{v}_1} \tag{6.18}$$

and

$$\underline{\delta}_2 = \frac{\max_{i,j} a_{ij} - v_2}{\max_{i,j} a_{ij} - \underline{v}_2} \tag{6.19}$$

the theorem is proved with

$$\underline{\delta} = \max_{i=1,2} \underline{\delta}_i \tag{6.20}$$

□

6.2 Correlated equilibria

Let us now consider the following game:

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

This game has three equilibria, which are:

- $[(1,0),(0,1)]$ with outcome $(2,7)$.
- $[(0,1),(1,0)]$ with outcome $(7,2)$.
- $[(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3})]$ with outcome $(\frac{14}{3},\frac{14}{3})$.

If we consider the following distribution over the outcomes,

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

we get an outcome

$$\frac{1}{3}\cdot 6 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 7 = \frac{15}{3}$$

for both players, which is better than the outcome obtained with the mixed strategy, i.e., $\frac{14}{3}$. This strategy isn't however a Nash equilibrium, hence we have to find ways to convince players to adhere to it.

The idea is to introduce an external arbitrator which makes a random choice on the outcome using the probabilities in the outcomes distribution matrix. The arbitrator then tells privately each player what strategy to play. The players, not knowing what the other player will play, don't have an incentive on changing move, hence they will stick to the assigned strategy.

Let us consider, for instance, the following cases to understand why players are not interested in changing their strategy:

1. The arbitrator selects outcome $(7,2)$. Player 1 is told to play the second row, Player 2 first column. Player 1 now knows that Player 2 is told to play the first column since the second one has probability 0: he does not deviate since the outcome is a Nash equilibrium. Player 2 knows that the probability Player 1 is told to play the first row is $\frac{1}{2}$ (since the probability of playing either one or the other row is the same) since both rows have the same probability of $\frac{1}{3}$ to be chosen. Player 2's expected value is then $\frac{1}{2}(6+2) = 4$. If he deviates his expected value is $\frac{1}{2}(7+0) = 3.5$: hence no interest to deviate for both.

2. The random choice selects outcome $(6,6)$. Player 1 is told to play the first row, Player 2 first column. Both players know that the other player will play the two strategies with the same probability. Thus the expected value following the suggestion is $\frac{1}{2}(6+2)$. If the player deviates his expected value is $\frac{1}{2}(7+0)$: hence no interest in deviating for both.

3. The random choice selects outcome $(2,7)$. The result is the same as in case 1, but with switched roles.

As we can see, the players can use the outcome distribution, which they know, to understand what result will they get if they changed their outcome. Basically, they are updating the outcome distribution based on the strategy they received.

Let us now formalise what we have seen as an example. In particular, we want to give a definition of correlated equilibrium.

Definition 6.2 (Correlated equilibrium)**.** *Given the game $(A,B) = (a_{ij}, b_{ij})$ with $i = 1, \dots, n$ and $j = 1, \dots, m$, let $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$ and $X = I \times J$. A correlated equilibrium is a probability distribution $p = (p_{ij})$ on X such that*

$$\sum_{j=1}^{m} p_{\bar{i}j} a_{\bar{i}j} \geq \sum_{j=1}^{m} p_{\bar{i}j} a_{ij} \quad \forall i \in I \quad \forall \bar{i} \in I$$

and

$$\sum_{i=1}^{m} p_{i\bar{j}} b_{i\bar{j}} \geq \sum_{i=1}^{m} p_{i\bar{j}} b_{ij} \quad \forall j \in J \quad \forall \bar{j} \in J$$

Practically, a correlated equilibrium is a probability distribution over the outcomes such that if we choose any strategy for a player, say strategy $\bar{i}$ for Player 1, then the expected outcome obtained when playing that strategy (i.e., $\sum_{j=1}^{m} p_{\bar{i}j} a_{\bar{i}j}$) is greater than the one obtained by with another strategy but using the same probability values (i.e., $\sum_{j=1}^{m} p_{\bar{i}j} a_{ij}$). In our example, we get:

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$$

which turns into:

$$\begin{cases} 6p_1 + 2p_2 \geq 7p_1 + 0p_2 \\ 7p_3 + 0p_4 \geq 6p_3 + 2p_4 \\ 6p_1 + 2p_3 \geq 7p_1 + 0p_3 \\ 7p_2 + 0p_4 \geq 6p_2 + 2p_4 \\ p_1 + p_2 + p_3 + p_4 = 1 \\ p_i \geq 0 \end{cases}$$

Now that we know what a correlated equilibrium is, we will like to understand if we can always find one. The following theorem states an important result in this sense.

Theorem 6.2 (Existence of correlated equilibria)**.** *The set of the correlated equilibria of a finite game is nonempty. A NEp profile generates a correlated equilibrium. Moreover, given* *the Nash equilibrium $(\bar{x}, \bar{y})$, the probability distribution on the outcome matrix is*

$$p = (p_{ij})$$

where each element is

$$p_{ij} = \bar{x}_i \bar{y}_j$$

Proof. We have to prove that

$$\sum_{j=1}^{m} \bar{x}_{\bar{i}} \bar{y}_j a_{\bar{i}j} \geq \sum_{j=1}^{m} \bar{x}_{\bar{i}} \bar{y}_j a_{ij} \quad \forall i \in I$$

This is true for $\bar{x}_{\bar{i}} = 0$, hence we have to prove it for $0 < \bar{x}_{\bar{i}} < 1$. We have

$$\sum_{j=1}^{m} \bar{y}_j a_{\bar{i}j} \geq \sum_{j=1}^{m} \bar{y}_j a_{ij} \quad \forall i \in I$$

In this inequality, given that Player 2 plays his equilibrium strategy $\bar{y}$:

- The left-hand side is Player 1's expected utility if he/she chooses row $\hat{\imath}$.
- The right-hand side is Player 1's expected utility if he/she chooses row i.

The inequality holds since the pure strategy is played with positive probability hence $\bar{\imath}$ must be one of the best reactions to $\bar{y}$. □

Another important result is the following.

Theorem 6.3 (Existence of correlated equilibria)**.** *The set of the correlated equilibria of a finite game is a nonempty convex polytope.*

Proof. Remember that a convex polytope is a closed bounded convex set which is the smallest convex set containing a finite number of points. The set of the correlated equilibria is the solution set of a system of $n^2 + m^2$ linear inequalities (where n, m are the number of the pure strategies of the players), called incentive constraints, plus the conditions of being a probability distribution, that is

$$p_{ij} \geq 0$$

with

$$\sum p_{ij} = 1$$

□

Proposition 6.1. *If a row $\bar{\imath}$ is strictly dominated, then $p_{\bar{i}j} = 0$ for every j.*

Proof. Let's suppose $\bar{\imath}$ is strictly dominated by i. Since

$$\sum_{j=1}^{m} p_{\bar{i}j}(a_{\bar{i}j} - a_{ij}) \geq 0$$

then it must be $p_{\bar{i}j} = 0$ for every j. □

The most important conclusion we can draw is that there is essentially a unique rationality paradigm in the whole theory: the idea of best reaction. As we have seen, even though the idea of NE is not always fully convincing, it still remains the foundation of rationality in non-cooperative game theory.

Chapter 7

Cooperative games

7.1 Cooperative games

We want to analyse cooperative games, namely games in which players can cooperate. First, given a set of players, let us define a coalition as follows.

Definition 7.1 (Coalition)**.** *Given a set of players N, a coalition*

$$A \in \wp(N)$$

is a subset of N.

For instance, given a set of players $N = \{1, 2, 3, 4\}$, a coalition could be $A = \{1, 4\}$. Now we can define a cooperative game.

Definition 7.2 (Cooperative game)**.** *A cooperative game is a couple (N, V) where*

- *N is the set of all players, containing n players (i.e., $|N| = n$).*
- *V is a utility multifunction*
$$V : \wp(N) \to \mathbb{R}^n$$
such that $V(A) \subseteq \mathbb{R}^{|A|}$ for any coalition $A \in \wp(N)$ formed by a subset of all the players.

The utility multifunction of a cooperative game associates each player of a coalition A to the game's aggregate utilities such that

$$x = (x_i)_{i \in A} \in V(A)$$

if the players in A can guarantee utility x_i to every player $i \in A$ by acting by themselves in the game. In other words, $V(A)$ is the set of $|A|$ utilities, one for each player in the coalition A. The i-th utility is the one that Player i can guarantee to every player in A by participating in the coalition.

Definition 7.3 (Transferable unit game)**.** *A transferable unit game is a function*

$$v : \wp(N) \to \mathbb{R}$$

such that $v(\varnothing) = 0$.

The fact that the utility v, also called side-payment function, takes on real values reflects the idea that the aggregated utility $v(A)$ can be freely divided among the members of the coalition A. The idea is that when players win as a coalition, they obtain a certain reward, which is split among the players in the coalition.

Also note that every TU game is also a cooperative game, in fact, the value of $v(A)$ can be replaced by a subset $V(A)$ of $\mathbb{R}^{|A|}$ such that

$$V(A) = \left\{ (x_i)_{i \in A} : \sum_{i \in A} x_i \leq v(A) \right\}$$

7.1.1 An example

Let us analyse the following example to better understand cooperative games. N players have a glove each, some of them have only a right glove, the others only a left globe. Since gloves can be sold only in pairs, the aim of the game is to form pairs. If we assume a game where Player 1 and Player 2 have a right glove whereas Player 3 has a left glove. The TU game of the game we have just described is

$$\begin{cases} v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = 0 \\ v(\{1,3\}) = 1 \\ v(\{2,3\}) = 1 \\ v(N) = 1 \end{cases}$$

7.1.2 Examples of cooperative games

Sellers and buyers

A game with sellers and buyers involves some players that sell a product and other interesting in buying it. Buyers and sellers give a different value to each product. A game with three players, namely one seller and two buyers can be represented as

$$\begin{cases} v(\{1\}) = a \\ v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = b \\ v(\{1,3\}) = c \\ v(\{2,3\}) = 0 \\ v(N) = c \end{cases}$$

where a, b, c are the values given by Player 1 (the seller), Player 2 and Player 3 (the buyers) such that $a < b < c$.

Glove game

Definition 7.4 (Glove game). *A glove game is a triplet*

$$(N, \{L, R\}, v)$$

where

- $N = \{1, 2, \ldots, n\}$ *is the set of players.*
- $\{L, R\}$*is a partition of the players (i.e.,* $L \cup R = N$ *and* $L \cap R = \varnothing$*).*

The value of a coalition is defined as

$$v(S) = \min\left\{|S \cap L|, |S \cap R|\right\} \quad \forall S \subseteq N$$

Less formally, the glove game can be seen as a game in which each player has exactly one glove, which is either a right or left glove. A pair of gloves is sold for 1 euro, hence players want to collaborate to sell as many pairs of gloves as possible. The partition $\{L, R\}$ represents the players that have a left or a right glove, respectively and the value function counts how many left and right gloves a coalition has. The value is the minimum between the number of right and left gloves.

Children game

Definition 7.5 (Children game). *A children game is a triple (N, W, v) where*

- $N = \{1, 2, \ldots, n\}$ *is the set of players.*
- W *is the amount that players can win.*

The value of a coalition is:

$$v(S) = \begin{cases} W & \textit{if } |S| \geq 2 \\ 0 & \textit{otherwise} \end{cases} \quad \forall S \subseteq N$$

In practice we can see a children game as a game in which n people vote who should win the prize W. If a person receives the majority of votes, then it receives the prize W. Players can make agreements and share the prize.

Weighted majority game

Definition 7.6 (Weighted majority game). *A weighted majority game is a triplet*

$$(\{w_1, \ldots, w_n\}q, v)$$

that represents a game in which n *parties are supposed to take a decision. Party* i *has* w_i

members and for a proposal to be passed it needs at least q votes. The value is defined as

$$v(A) = \begin{cases} 1 & \text{if } \sum_{i \in A} w_i \geq q \\ 0 & \text{otherwise} \end{cases} \quad \forall S \subseteq N$$

Airport game

Definition 7.7 (Airport game)**.** *An airport game is a quadruple*

$$(N, \{N_1, N_2, \ldots, N_k\}, \{c_1, c_2, \ldots, c_k\}, v)$$

where

- *N is the set of players (planes in this case).*
- *$\{N_1, N_2, \ldots, N_k\}$ is a partition of N.*
- *$\{c_1, c_2, \ldots, c_k\}$ is a set of costs, one for each partition N_i (i.e., cost c_i is associated to partition N_i).*

The value of a coalition is defined as

$$v(S) = \max\{c_i : i \in S\} \quad \forall S \subseteq N$$

An airport game can be interpreted as follows. A collection of airlines flying N planes needs a new runway close to some city. The set of planes is partitioned into groups of similar sizes $\{N_1, N_2, \ldots, N_k\}$ so that to each N_i there corresponds the cost c_i of the runway construction. The value of a coalition is the money that the airlines have to spend to build a runway knowing that the runway has to be used by all planes in the coalition (larger planes require a longer runway, hence more building cost c_i).

Bankruptcy game

Definition 7.8 (Bankruptcy game)**.** *A bankruptcy game is a quadruple*

$$B = (N, c, E, v)$$

where

- $N = 1, \ldots, n$ *is the set of creditors.*
- $c = c_1, \ldots, c_n$ *is the set of credits. Credit* $c_i i$ *is associated with Player* i*.*
- E *is the estate.*
- v *is the value of a coalition.*

Bankruptcy is defined by

$$E \leq \sum_{i \in N} c_i = C$$

A bankruptcy game can be seen in two different was, called pessimistic and optimistic. A pessimistic view of the game imposes that the value of a coalition is

$$v_P(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \quad \forall S \subseteq N$$

The value of a coalition can be also be expressed using a more optimistic view, which is however less realistic. In this view, the value of a coalition is

$$v_O(S) = \min\left(E, \sum_{i \in S} c_i\right) \quad \forall S \subseteq N$$

Peer games

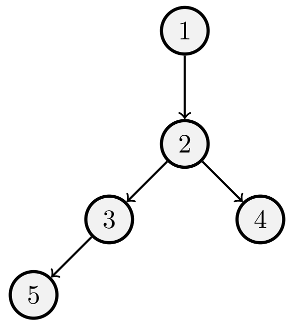
An interesting type of cooperative game is a peer game.

Definition 7.9 (Peer game)**.** *Let* $N = \{1, \ldots, n\}$ *be the set of players and* $T = (N, A)$ *a directed rooted tree. Each agent* i *has an individual potential* v_i *which represents the gain that player* i *can generate if all players at the higher levels of the hierarchy cooperate with him. The peer game is the game* v *such that*

$$v(S) = \sum_{i \in N : S(i) \subseteq S} v_i$$

where $S(i)$ *is the set of superiors of* i*, namely the set of all agents in the unique directed path connecting 1 to* i*.*

For instance, the following tree



represents a peer game with function v defined as:

- $v(A) = \begin{cases} 0 & \text{if } 1 \notin A \\ v_1 & \text{if } 2 \notin A \end{cases}$
- $v(\{1,2\}) = v(\{1,2,5\}) = v_1 + v_2$
- $v(\{1,2,4\}) = v(\{1,2,4,5\}) = v_1 + v_2 + v_4$
- $v(\{1,2,3,4\}) = v_1 + v_2 + v_3 + v_4$
- $v(\{1,2,3,5\}) = v_1 + v_2 + v_3 + v_5$
- $v(N) = v_1 + v_2 + v_3 + v_4 + v_5$

7.1.3 Properties of transferable unit games

Let us now state some properties of cooperative games. Let $\mathcal{G}(N)$ be the set of cooperative games having N as set of players, $S_1, \dots, S_{2^n-1}$ the list of coalitions, $(v_1, \dots, v_{2^n-1})$ a game with $v_i = v(S_i)$.

Proposition 7.1. *The set $\mathcal{G}(N)$ is isomorphic to $\mathbb{R}^{2^n-1}$.*

Proposition 7.2. *Given the set $\{u_A : A \subseteq N\}$ of unanimity games,*

$$u_A(T) = \begin{cases} 1 & \textit{if } A \subseteq T \\ 0 & \textit{otherwise} \end{cases}$$

is a base for the space $\mathcal{G}(N)$.

7.1.4 Additive and superadditive games

Among all transferable unit games, some are very interesting. In particular, additive and superadditive games are very interesting.

Definition 7.10 (Additive game)**.** *A game in $\mathcal{G}(N)$ is additive if*

$$v(A \cup B) = v(A) + v(B) \qquad \forall A, B \in \wp(N), A \cap B = \varnothing$$

The set of the additive games is a vector space of dimension n.

Definition 7.11 (Superadditive game)**.** *A game in $\mathcal{G}(N)$ is superadditive if*

$$v(A \cup B) \geq v(A) + v(B) \qquad \forall A, B \in \wp(N), A \cap B = \varnothing$$

7.1.5 Simple games

Another class of interesting transferable unit games is the class of simple games.

Definition 7.12 (Simple game)**.** *A game $v \in \mathcal{G}(N)$ is simple if:*

- $v(S) \in \{0, 1\}$ *for every non-empty coalition S.*
- $A \subseteq C$ *implies $v(A) \leq v(C)$.*
- $v(N) = 1$.

In other words, each coalition S can either get 0 or 1, hence each coalition can win or lose. In particular, if

- $v(A) = 1$, coalition A wins.
- $v(A) = 0$, coalition A loses.

The grand coalition (i.e., the one with every player) always wins and if a coalition is a subset of another, then the value of the former has to be smaller than the latter's.

Definition 7.13 (Minimal winning coalition)**.** *A coalition A in a simple game v is called minimal winning coalition if*

- $v(A) = 1$*, i.e. it's a winning coalition.*
- $B \subsetneq A \implies v(B) = 0$*, i.e., A is the smallest coalition with utility equal to 1 (every strictly smaller coalition is a losing one).*

7.2 Solution of cooperative games

7.2.1 Solution

The easiest concept we have to deal with when solving a cooperative game is the idea of solution.

Definition 7.14 (Cooperative game solution vector). *A solution vector for the game $v \in \mathcal{G}(N)$ is a vector*

$$(x_1, \ldots, x_n)$$

which assigns a utility x_i to each player i.

Definition 7.15 (Cooperative game solution). *A solution concept, briefly, a solution, for the game $v \in \mathcal{G}(N)$ is a multifunction*

$$S : \mathcal{G}(N) \to \mathbb{R}^n$$

that assigns a set of solution vectors, possibly empty, to a game.

7.2.2 Imputation

Among all solution vectors, we are interested in those with a specific set of properties. These solutions are called imputations.

Definition 7.16 (Imputation). *The solution*

$$I : \mathcal{G}(N) \to \mathbb{R}^n$$

such that $x \in I(v)$ is called imputation if

- $x_i \geq v\big(\{i\}\big) \qquad \forall i \in N$
- $\sum_{i=1}^{n} x_i = v(N)$

Let us analyse more in-depth the conditions under which a solution vector $x = (x_1, \ldots, x_n)$ belongs to the imputation.

- The first condition tells us that solution x is an imputation if a player i participates in the solution if the solution assigns him/her a value x_i bigger than what he/she would have obtained if by himself/herself (i.e., $v(\{i\})$).
- The second condition can be split into two conditions:
  - **Feasibility**,

    $$\sum_{i=1}^{n} x_i \leq v(N)$$

    If the grand coalition is formed the amount available to the players is $v(N)$ (namely, the sum of the utilities obtained by every player isn't more than the utility obtained by the grand coalition).
  - **Efficiency**,

    $$\sum_{i=1}^{n} x_i \geq v(N)$$

The overall amount will be effectively distributed among all the players. Efficiency is a mandatory requirement in cooperative games, in fact, it makes a real difference compared to non-cooperative games. Namely, if the players split $v(N)$, then it can't happen that a part of $v(N)$ is given to no player.

The imputation set lies in the hyperplane

$$H = \left\{x \in \mathbb{R}^n : \textstyle\sum_{i=1}^n x_i = v(N)\right\}$$

and it is bounded since, for all i, $x_i \geq v(i)$. Since it is defined by linear inequalities, it is the intersection of half-spaces. Also note that:

- If a game v satisfies
$$v(N) \geq \sum_i v(\{i\})$$
then the imputation is not empty (i.e., there exists a solution which is also an imputation). Moreover, we are sure that the imputation set is non-empty when the game is superadditive (but the opposite is not always true).

- If the game is additive, then
$$I(v) = \{(v(1), \dots, v(n))\}$$

Proposition 7.3. *The imputation set $I(v)$ is a polytope (C.1), i.e. the smallest closed convex set containing a finite number of points.*

7.2.3 Core

Now we want to further add some conditions to an imputation.

Definition 7.17 (Core)**.** *The core of a game v is the solution*

$$C : \mathcal{G}(N) \to \mathbb{R}^n$$

such that

$$C(v) = \left\{x \in \mathbb{R}^n : \textstyle\sum_{i=1}^n x_i = v(N) \wedge \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N\right\}$$

Note that the core of a game is a subset of the imputations, in fact, it imposes even stricter conditions. In particular, imputations are efficient distributions of utilities accepted by all individual players, while core vectors are efficient distributions of utilities accepted by all coalitions. Even more precisely, the core is the set of solution vectors $x \in \mathbb{R}^n$ such that:

- the player's utilities add up the value of the grand coalition $v(N)$, and
- the sum of utilities of a coalition (i.e., $\sum_{i \in S} x_i$) is not smaller than the utility of the coalition $v(S)$.

As we have done for solutions and imputations, we can state the following result.

Proposition 7.4. *The core $C(v)$ of a game v is a polytope (i.e. the smallest closed convex set containing a finite number of points).*

As for the imputation, the core reduces to the singleton $(v(1), ..., v(n))$ if v is additive, however, superadditive games can have an empty core.

Example

If we consider the glove game we have analysed before, we find out that the core of that game is

$$C(v) = (0, 0, 1)$$

and

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 0 \\ x_1 + x_3 \geq 1 \\ x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

Core in simple games

Let us now consider simple games. In these games, we can have a player called veto player. In particular,

Definition 7.18 (Veto player)**.** *In a game v, a player i is a veto player if $v(A) = 0$ for all coalitions such that $i \notin A$.*

In other words, a player i is a veto player if every coalition he/she isn't in loses.

Theorem 7.1 (Core in simple games)**.** *Let v be a simple game. Then $C(v) \neq \varnothing$ if and only if there is at least one veto player. When a veto player exists, the core is the closed convex polytope with the vectors $(0, \ldots, 1, \ldots, 0)$ as extreme points, where the 1 corresponds to the veto player.*

Proof. Say that a game v has no veto player. Then for every player i there is a coalition A_i such that $i \notin A_i$ and $v(A_i) = 1$ (since in a simple game a coalition has either value 1 or 0). In particular, $N \setminus \{i\}$ is a winning coalition for every i. Suppose now that $(x_1, x_2, \ldots, x_n) \in C(v)$, then it follows

$$\sum_{j \neq i} x_j \geq \sum_{j \in A_i} x_j = 1 \qquad i = 1, \ldots, n$$

However, by summing up the above inequalities from 1 to n we obtain

$$(n-1) \sum_{j=1}^{n} x_j = n$$

which is in contradiction with $\sum_{j=1}^{n} x_j = 1$, hence the game must have at least one veto player. $\square$

The non-emptiness of the core can be represented as a linear programming problem. In particular, we have the following result.

Theorem 7.2 (Core in simple games as linear programming problem)**.** *The linear programming problem*

$$\min \sum_{i=1}^{n} x_i$$
$$\sum_{i \in S} x_i \geq v(S) \qquad \forall S \subseteq N$$

has always a nonempty set of solutions C. The core $C(v)$ is nonempty and $C(v) = C$ if and only if the optimal value of the LP is $v(N)$.

Note that the value V of the linear programming problem is $V \geq v(N)$, due to the constraint

$$\sum_{i} x_i \geq v(N)$$

Thus, for every solution x satisfying the constraint of the problem, we have

$$\sum_{i=1}^{n} x_i \geq v(N)$$

As with every linear programming problem, we can write the dual of the problem used for analysing the core of a game. In particular, we get:

Theorem 7.3 (Core in simple games as linear programming problem)**.** *Given a game v, its core $C(v)$ is non-empty if and only if every vector*

$$(\lambda_S)_{S \subseteq N}$$

such that

$$\lambda_S \geq 0 \qquad \forall S \subseteq N$$

and

$$\sum_{S: i \in S \subseteq N} \lambda_S = 1 \qquad \forall i = 1, \ldots, n$$

satisfies

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

Note that the coefficients λ_S (one for every coalition that can be formed) can be interpreted as indicating how much, in percentage, a given coalition S represents the players. Therefore, the theorem suggests that no matter what quota the players contribute to the coalition, the weighted values must not exceed the overall amount of utility.

Proof. The linear programming problem in Theorem 7.2 has the following matrix form:

$$\begin{cases} \min\langle c, x\rangle \\ Ax \geq b \end{cases} \tag{7.1}$$

where

- $c = \mathbf{1}_n$,
- $b = (v(\{1\}), v(\{2\}), \dots, v(\{1,2\}), \dots, v(N))$,
- A is a $(2^n - 1) \times n$ matrix whose rows are coalitions and columns are players. If $A_{ij} = 1$, then Player 1 is in coalition i, otherwise.

The dual of the aforementioned problem is

$$\begin{cases} \max \sum_{S\subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \qquad \forall i \end{cases}$$

Since the primal has at least one finite solution, the fundamental duality theorem tells us that there exist also a solution for the dual, and there is no duality gap. Thus the core $C(v)$ is not empty if and only if the value V of the dual problem is such that $V \leq v(N)$. □

Balanced coalitions

Definition 7.19 (Balanced coalitions). *A family $(S_1, \dots, S_m)$ of coalitions is said to be balanced if there exists $\lambda = (\lambda_1, \dots, \lambda_m)$ such that*

$$\sum_{k:i\in S_k} \lambda_k = 1 \qquad \forall i \in N$$

and

$$\lambda_i > 0 \qquad \forall i = 1, \dots, m$$

If a family of coalitions is balanced, λ is called balancing vector.

Note that, for every player i, the coefficients λ_k indicate how much she contributes to each coalition S_k in the family so that their sum over all coalitions is 1.

In practice, the condition $\sum_{k:i\in S_k} \lambda_k = 1$ says that, for Player i, the sum of λ of all coalitions in which he/she's in is 1. For instance, say we have a set of players $N = \{1, 2, 3, 4\}$. The family $(\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\})$ of coalitions is balanced with balancing vector

$$\lambda = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$$

On the other hand, if we have $N = \{1, 2, 3\}$, the family of coalitions $(\{1, 2\}, \{1, 3\}, \{3\})$ is not balanced since player 3 is in a coalition with player 1, but also in a coalition on his/her own. Since player 2 participates only in coalition S_1, λ_1 must be 1. But this would mean that $\lambda_2 = 0$ otherwise the condition

$$\sum_{k:1\in S_k} \lambda_k = 1$$

wouldn't be true. But this is impossible since λ_i must be strictly positive.

Note that, given a balancing vector $\lambda = (\lambda_S)_{S\subseteq N}$ fulfilling the inequalities defining the dual constraint set, the positive coefficients in λ are the balancing vectors of a balanced family. This means that, if we take whatever vector λ (which satisfies the dual constraints), we can always associate to it a family of coalitions and its corresponding balancing vector. Let us consider, for instance, $N = \{1, 2, 3\}$. The vector

$$\lambda = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right)$$

can be associated with the family of coalitions

$$(\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, N)$$

If we consider

$$\lambda = \left(0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$$

instead, we get the family

$$(\{1, 3\}, \{2, 3\}, N)$$

Among all families of balanced coalitions, we are interested in finding those that are minimal.

Definition 7.20 (Minimal balancing family)**.** *A minimal balancing family is a balancing family for which there is no sub-family that is balanced.*

We can use the balancing vector to find out if a balanced family of coalitions is minimal. In particular, we have the following result:

Proposition 7.5. *A balanced family is minimal if and only if its balancing vector is unique.*

Another important result regarding balanced families of coalitions is the following.

Theorem 7.4. *The positive coefficients of the extreme points of the constraint set*

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_i = 1 \quad\quad \forall i \in N \end{cases}$$

are the balancing vectors of the minimal balanced coalitions.

This means that to find the extreme points of the dual constraint set it is enough to find balanced families with a unique balancing vector. Another important result is the following.

Proposition 7.6. *The partitions of N are minimal balanced families, in fact, the relevant condition*

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

is automatically fulfilled if the game is superadditive.

Finally, let us state some results regarding superadditive games.

Proposition 7.7. *There are superadditive games v such that $C(v) \neq \varnothing$. There are non-superadditive games vsuch that $C(v) \neq \varnothing$.*

7.2.4 The nucleolus

Definition 7.21 (Excess of a coalition)**.** *Let v be some TU game. The excess of a coalition A over the imputation x is*

$$e(A, x) = v(A) - \sum_{i \in A} x_i$$

The excess of a coalition is a measure of dissatisfaction of the coalition with respect to the assignment of the imputation x.

We can use the excess of a coalition to understand if an imputation belongs to the core. In particular, the following property holds.

Proposition 7.8. *An imputation x of the game v belongs to the core $C(v)$ if and only if*

$$e(A, x) \leq 0 \qquad \forall A \subseteq N$$

Proof. An imputation belongs to the core if the sum of the coalition's utilities is at least the value of the coalition, namely if

$$\sum_{i \in S} x_i \geq v(S) \qquad \forall S \in \wp(N)$$

By moving the sum of the utilities to the right we obtain

$$0 \geq v(S) - \sum_{i \in S} x_i \qquad \forall S \in \wp(N)$$

The difference on the right is the excess of a coalition, hence we can write

$$e(S, x) \leq 0 \qquad \forall S \in \wp(N)$$

□

Definition 7.22 (Lexicographic vector attached to an imputation)**.** *The lexicographic vector attached to the imputation x is the $2^n - 1$-th dimensional vector $\theta(x)$ such that*

> 1. $\theta_i(x) = e(A, x)$, *for some* $A \subseteq N$.
> 2. $\theta_1(x) \geq \theta_2(x) \geq \cdots \geq \theta_{2^n-1}(x)$.

In other words, the lexicographic vector arranges the excesses of the coalition over the imputation x in decreasing order.

Before defining a nucleolus solution we have to define an ordering relation between vectors in a Euclidean space. In particular, we have

Definition 7.23 (Ordering relation)**.** *Let x, y be two vectors in a Euclidean space. We write*

$$x \leq_L y$$

if

- $x = y$, *or*
- *there exists j such that*
$$x_i = y_i \quad \forall i < j$$
and
$$x_j \leq y_j$$

The relation $\leq_L$ defines a total order in any Euclidean space.

Thanks to this ordering relation we can define a nucleolus solution as follows.

Definition 7.24 (Nucleolus solution)**.** *The nucleolus solution is the solution*

$$\nu : \mathcal{G}(N) \to \mathbb{R}^n$$

such that $\nu : v \to \mathbb{R}^n$ is the set of the imputations x such that, for all imputations y of the game v it holds

$$\theta(x) \leq_L \theta(y)$$

The nucleolus is therefore the set of imputations that are lexicographically smaller than the imputations y of a game. Let us now state some interesting properties of nucleolus solutions.

Theorem 7.5 (One point solution)**.** *For every TU game v with non-empty imputation set, the nucleolus $\nu(v)$ is a singleton. Thus the nucleolus is a one-point solution.*

Proposition 7.9. *Let v be a game. If a v has non-empty core, then the nucleolus solution belongs to the core, i.e.*

$$C(v) \neq \varnothing \implies \nu(v) \in C(v)$$

Proof. For all $x \in C(v)$, by definition of core, $\theta_1(x) \leq 0$. Since the nucleolus minimises the excesses, we have $\theta(\nu(v)) \leq 0$. Then $\nu(v)$ is in the core. $\square$

7.3 The Shapley value

Given a one-point solution

$$\phi : \mathcal{G}(N) \to \mathbb{R}^n$$

we would like it to have the following properties:

- **Efficiency**. $\sum_{i \in N} \phi_i(v) = v(N)$ for all games v.
- **Symmetry**. If $v \in \mathcal{G}(N)$ is a game such that

  $$v(A \cup \{i\}) = v(A \cup \{j\})$$

  for every A not containing i, j, then

  $$\phi_i(v) = \phi_j(v)$$

  Namely, if two players i and j add the same value to a coalition, then their reward must be the same.
- **Null player property**. If a game $v \in \mathcal{G}(N)$ and a player $i \in N$ are such that

  $$v(A) = v(A \cup \{i\}) \quad \forall A$$

  then

  $$\phi_i(v) = 0$$

  This means that, if a player doesn't add any value to any coalition, then its utility should be 0. In other words, a player contributing nothing to any coalition cannot get anything.
- **Additivity**. $\phi(v + w) = \phi(v) + \phi(w)$ for every $v, w \in \mathcal{G}(N)$.

7.3.1 The Shapley theorem

Let us now introduce an important result.

Theorem 7.6 (Shapley). *Let $\sigma : \mathcal{G}(N) \to \mathbb{R}^n$ be defined by*

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

Then σ satisfies the properties of efficiency, symmetry, null player and additivity. Conversely, if $\tilde{\sigma}$ is a one-point solution satisfying the properties of efficiency, symmetry, null player and additivity, then $\tilde{\sigma} = \sigma$. We call σ the Shapley value.

In other words, there exists a unique one-point solution σ satisfying the properties of efficiency, symmetry, null player and additivity. Each element of the summation is the product of two terms:

- The term $v(S \cup \{i\} - v(S))$ is called **marginal contribution** of Player i to coalition $S \cup \{i\}$. This means that the Shapley value is the weighted sum of all marginal contributions of the players.
- The term $\frac{s!(n-s-1)!}{n!}$ is the **weight** of the marginal contribution. To give an interpretation to this term, suppose the players decide to meet, at some place and at some time, where they arrive at the meeting in random order. Choose a coalition S and a player $i \notin S$, then the ratio

$$\frac{s!(n-s-1)!}{n!}$$

is the probability that player i arrives at the meeting when exactly the s members of the coalition S (i.e. no other player) are already there.

Proof. We want to prove that the Shapley value satisfies the properties of efficiency, symmetry and additivity.

- **Efficiency**: $\sum_{i=1}^{n} \sigma_i(v) = v(N)$. Let us consider the term $v(S \cup \{i\}) - v(S)$ of the Shapley value. If we consider $S = N \setminus \{i\}$, the term $v(N)$ appears n times (once for every player i) in the sum. In these cases, since the coalition S contains all players but i, we have $s = n - 1$ and the coefficient of the sum is

$$\frac{(n-1)!(n-n)!}{n!} = \frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$$

Let us consider another coalition $T \neq N$. The term $v(T)$ appears both with positive and negative coefficients, in particular:

  - The positive coefficient $\frac{(t-1)!(n-t)!}{n!}$ appears t times, once for every player $i \in S$ when $S = T \setminus \{i\}$, hence its contribution is $\frac{t(t-1)!(n-t)!}{n!} = \frac{t!(n-t)!}{n!}$.
  - The negative coefficient $-\frac{t!(n-t-1)!}{n!}$ appears $n-t$ times, once for every player $i \notin T$ when $S = T$, hence its contribution is $-\frac{t!(n-t)!}{n!}$.

This means that in the sum

$$\sum_{i=1}^{n} \sigma_i(v) = \sum_{i=1}^{n} \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right]$$

the terms $v(T)$ cancel out and the term $v(N)$ appears with coefficient 1.

- **Symmetry**: $v(S \cup \{i\}) = v(S \cup \{j\}) \Rightarrow \sigma_i(v) = \sigma_j(v)$. To prove this property we can rewrite the Shapley value as

$$\begin{aligned}
\sigma_i(v) &= \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] \\
&= \sum_{S \in 2^{N \setminus \{i \cup\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] \\
&+ \sum_{S \in 2^{N \setminus \{i \cup\}}} \frac{(s+1)!(n-s-2)!}{n!} [v(S \cup \{i \cup j\}) - v(S \cup \{j\})]
\end{aligned}$$

Similarly, inverting the roles of i and j, we can write

$$\begin{aligned}
\sigma_j(v) &= \sum_{S\in 2^{N\setminus\{j\}}} \frac{s!(n-s-1)!}{n!}[v(S\cup\{j\})-v(S)] \\
&= \sum_{S\in 2^{N\setminus\{i\cup\}}} \frac{s!(n-s-1)!}{n!}[v(S\cup\{j\})-v(S)] \\
&+ \sum_{S\in 2^{N\setminus\{i\cup\}}} \frac{(s+1)!(n-s-2)!}{n!}[v(S\cup\{i\cup j\})-v(S\cup\{i\})]
\end{aligned}$$

If we consider symmetric players, hence for which $v(S\cup\{i\}) = v(S\cup\{j\})$, then the two values are the same and the property is proven.

- **Null player**: $v(A) = v(A\cup\{i\}) \Rightarrow \phi_i(v) = 0$. Because $v(A) = v(A\cup\{i\})$, then $v(A\cup\{i\}) - v(A) = 0$, hence every term of the sum is 0 and the Shapley value is 0.
- **Additivity**: $\phi(v+w) = \phi(v)+\phi(w)$. By rewriting the Shapley value we obtain

$$\begin{aligned}
\sigma_i(v+w) &= \sum_{S\in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!}[(v+w)(S\cup\{i\})-(v+w)(S)] \\
&= \sum_{S\in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!}[v(S\cup\{i\})+w(S\cup\{i\})-v(S)-w(S)] \\
&= \sum_{S\in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!}[v(S\cup\{i\})-v(S)] \\
&+ \sum_{S\in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!}[w(S\cup\{i\})-w(S)] \\
&= \sigma_i(v)+\sigma_i(w)
\end{aligned}$$

□

7.3.2 Simple games

Let us now discuss Shapley values in simple games (7.12). In simple games, the value of each coalition can be either 0 or 1, hence the Shapley values are

$$\sigma_i(v) = \sum_{A\in\mathcal{A}_i} \frac{a!(n-a-1)!}{n!}$$

where $\mathcal{A}_i$ is the set of coalitions A such that

- $i \notin A$.
- $v(A) = 0$, i.e., A is not winning.
- $v(A\cup\{i\}) = 1$, i.e., $A\cup\{i\}$ is winning.

Shapley values can also be written in the form

$$\sigma_i(v) = \sum_{A \in \mathcal{W}_i} \frac{(a-1)!(n-a)!}{n!}$$

where $\mathcal{W}_i$ is the set of coalitions A such that

- $i \in A$.
- $v(A) = 1$, i.e., A is winning.
- $v(A \setminus \{i\}) = 0$, i.e., $A \setminus \{i\}$ is not winning.

Example

Let us consider an example to better understand Shapley values in simple games. Let's consider the game with the following values:

$$\begin{cases} v(\{1\}) = 0 \\ v(\{2\}) = v(\{3\}) = 1 \\ v(\{1,2\}) = 4 \\ v(\{1,3\}) = 4 \\ v(\{2,3\}) = 2 \\ v(N) = 8 \end{cases}$$

This game can be represented in Table 7.1 where each row represents an order in which players can enter a coalition. Each column represents a player and cell (r, c) contains the value added by player c when joining the coalition in the order specified by row r. Consider for instance the third row in which first Player 2 enters the coalition. He/She is then followed by Player 1 and Player 3. When entering the coalition, Player 2 adds a value of 1, since the empty coalition has value 0 and Player 2 has value 1 ($v(\{2\}) = 1$). When Player 1 enters the coalition we go from $v(\{2\}) = 1$ to $v(\{1,2\}) = 4$, hence Player 1 adds a value of 3. Finally, when Player 3 enters the coalition we go from $v(\{1,2\}) = 4$ to $v(N) = 8$, hence Player 3 adds a value of 4 to the coalition $\{1,2\}$.

	1	2	3
123	0	4	4
132	0	4	4
213	3	1	4
231	6	1	1
312	3	4	1
321	6	1	1
	$\frac{18}{6}$	$\frac{15}{6}$	$\frac{15}{6}$

Table 7.1: An example of a simple cooperative game with its Shapley values.

When we have written this table, we can compute the Shapley values (which are written in the last row), either by

- **Using the values in the table**. For each column, we can sum the values of that column and divide the result by the number of combinations.

- **Applying directly the formula**. For instance, for Player 1 we obtain

$$\begin{aligned}
\sigma_1(v) &= \sum_{S \in 2^{N \setminus \{1\}}} \frac{s!(n-s-1)!}{n!}[v(S \cup \{1\} - v(S))] \\
&= \frac{s!(n-s-1)!}{n!}[v(S \cup \{1\}) - v(\{2\})] \\
&+ \frac{s!(n-s-1)!}{n!}[v(S \cup \{1\}) - v(\{3\})] \\
&+ \frac{s!(n-s-1)!}{n!}[v(S \cup \{1\}) - v(\{2,3\})] \\
&= \frac{1!(3-1-1)!}{3!}[v(\{1,2\}) - v(\{2\})] \\
&+ \frac{1!(3-1-1)!}{3!}[v(\{1,3\}) - v(\{3\})] \\
&+ \frac{2!(3-2-1)!}{3!}[v(N) - v(\{2,3\})] \\
&= \frac{1}{6}[4-1] + \frac{1}{6}[4-1] + \frac{2}{6}[8-2] \\
&= 3
\end{aligned}$$

Semivalues

In simple games, the Shapley value measures the fraction of the power of every player. In order to measure the relative power of the players in a simple game, the requirement of efficiency is not mandatory, hence coalitions could even form in a different way from the case of the Shapley value. To account for this situation, we can introduce semivalues. First, we have to state some preliminary definitions.

Definition 7.25 (Probabilistic power index)**.** *A probabilistic power index ψ on the set of simple games is a vector $\psi(v) = (\psi_i(v))$ with*

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) m_i(v, S)$$

where p_i is a probability measure on $2^{N \setminus \{i\}}$ and $m_i(v, S) = v(S \cup \{i\}) - v(S)$.

Now we can define semivalues as

Definition 7.26 (Semivalue)**.** *A probabilistic power index ψ on the set of simple games is a semivalue if there exists a vector $(p_0, \dots, p_{n-1})$ such that*

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_s m_i(v, S)$$

This means that, in a semivalue, the probabilistic power index depends only on the cardinality s of a coalition, but not on the index i of the player. Note that, since the index ψ is probabilistic, the usual probability conditions must hold, namely:

- $p_s \geq 0$.
- $\sum_{n=0}^{n-1} \binom{n-1}{s} p_s = 1$

Definition 7.27 (Regular semivalue)**.** *A semivalue ψ is said to be regular if*

$$p_s > 0 \qquad \forall s \in \{0, \dots, n-1\}$$

Some examples of semivalues are:

- The **Shapley values**.
- The **Banzhaf values**

$$\beta_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{1}{2^{n-1}} (v(S \cup \{i\}) - v(S))$$

- The **binomial values**

$$p_s = q^s (1-q)^{n-s-1}$$

  for every $0 < q < 1$.
- The **marginal value**

$$p_s = \begin{cases} 0 & \text{for } s = 0, \dots, n-2 \\ 1 & \text{for } s = n-1 \end{cases}$$

- The **dictatorial value**

$$p_s = \begin{cases} 0 & \text{for } s = 1, \dots, n-1 \\ 1 & \text{for } s = 0 \end{cases}$$

Appendix A

Algebra

A.1 Group

Definition A.1 (Group)**.** *A triplet*

$$(X, +, e)$$

where

- X *is a set.*
- $+$ *is an associative operation on* X *(not necessarily the sum).*
- $e \in X$ *is the identity of the operation* $+$*, that is an element such that*

$$e + x = x \quad \forall x \in X$$

- *All elements* x *in* X *have an inverse, i.e., an element* $x^{-1} \in X$ *such that*

$$x + x^{-1} = e$$

is called ***group****. If the operation* $+$ *is commutative, the group is said to be commutative or abelian.*

Appendix B

Operations research

B.1 Set convexity

The concept of complexity is vital for game theory.

Definition B.1 (Convex set)**.** *A set $C \in \mathbb{R}^n$ is convex if, for any couple of points $x, y \in C$, and provided $\lambda \in [0, 1]$ it holds*

$$\lambda x + (1 - \lambda) y \in C$$

We can also define the concept of convex combination of elements of a vector.

Definition B.2 (Convex combination)**.** *A convex combination of elements $x_1, \ldots, x_n$ is any vector $\mathbf{x}$ of the form*

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

with

$$\lambda_i \geq 0, \forall i \in \{1, \ldots, n\}$$

and

$$\sum_{i=1}^{n} \lambda_i = 1$$

Thanks to the concept of convex combination we can say that a set C is convex if and only if, for every $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, for every $c_1, \ldots, c_n \in C$, for all n, then

$$\sum_{i=1}^{n} \lambda_i c_i \in C$$

B.1.1 Convex hull

Definition B.3 (Convex hull). *A convex hull of a set C, denoted by co C is the union of all convex supersets of C,*

$$co\ C = \bigcap_{A \in \mathcal{C}} A$$

where $\mathcal{C}$ is the set of convex supersets of C, i.e.,

$$\mathcal{C} = \{A : C \subset A, A \text{ convex}\}$$

Alternatively, the convex hull of a set C is the smallest convex set that contains C.

Practically, we can compute the convex hull of a set C as

$$\text{co } C = \left\{\sum_{i=1}^{n} \lambda_i c_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C \forall i, n \in \mathbb{N}\right\} \tag{B.1}$$

Namely, the convex hull of any set C is the set of all convex combinations of points in C. For instance, the convex hull of the set

$$C = \{(1,1), (3,1), (5,2), (4,4), (2,5), (0,4)\}$$

is shown in Figure B.1.

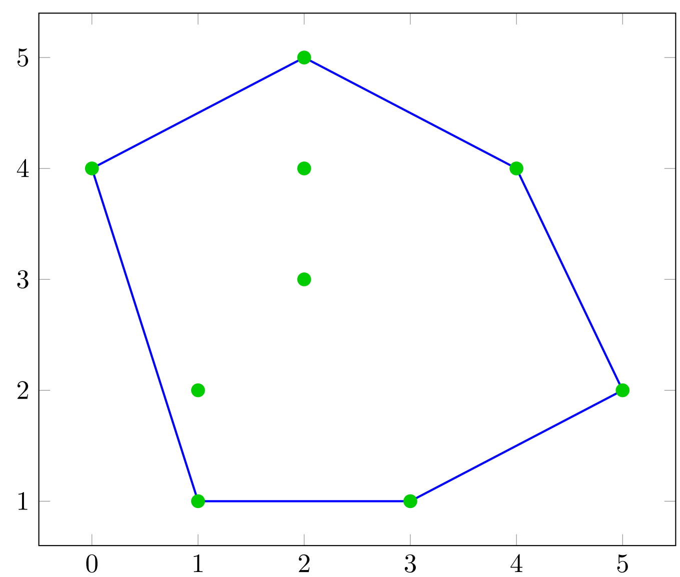


Figure B.1: A convex hull.

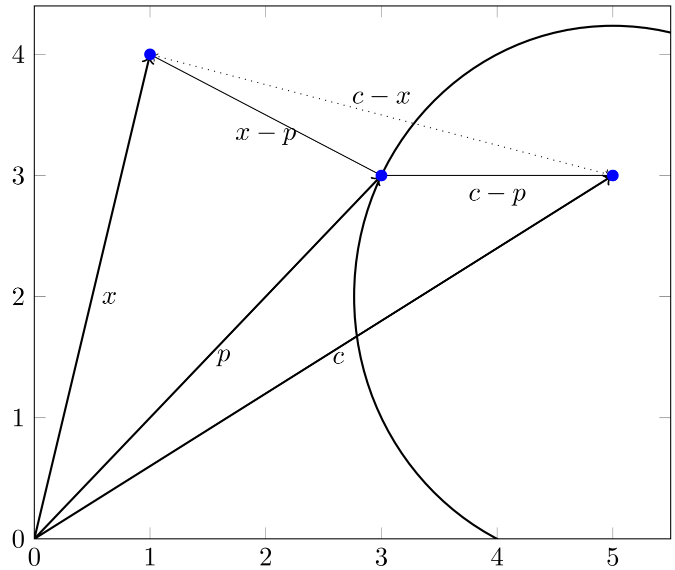


Figure B.2: The projection p.

B.1.2 Projection

Theorem B.1 (Projection)**.** *Given a closed convex set C and a point x outside C, there is a unique element $p \in C$, called projection, such that for all points in C, the distance between x and the projection is smaller than the distance between c and x.*

$$\|p - x\| \leq \|c - x\| \quad \forall c \in C$$

The projection p is characterised by the following properties:

- $p \in C$
- $(x - p, c - p) \leq 0, \quad \forall c \in C$

In other words, p is the closest point to x belonging to the set C. Moreover, by the last property, p forms an obtuse angle (if seen in two dimensions) between x and any $c \in C$. To see why this statement is true, we have to remember that the scalar product between two vectors x and y is defined as

$$x \cdot y = |x| \cdot |y| \cdot \cos\theta$$

where θ is the angle between x and y. Since the modules of x and y are always positive numbers, the sign of the scalar product depends on the cosine, which is smaller than 0 for angles between $\pi/2$ (90 degrees) and $\frac{3\pi}{2}$ (270 degrees) which are all obtuse angles. A graphical representation of this fact is shown in Figure B.2.

Appendix C

Geometry

C.1 Geometry

C.1.1 Polytope

Polytopes are the generalisation of three-dimensional polyhedra to any number of dimensions. Formally,

> **Definition C.1** (Polytope)**.** *A polytope is a geometric object with flat sides (faces).*

C.1.2 Simplex

A simplex is a generalisation of the notion of a triangle or tetrahedron to arbitrary dimensions. Formally,

> **Definition C.2** (Simplex)**.** *A k-simplex is a k-dimensional polytope which is the convex hull of its $k+1$ vertices.*
>
> $$C = \left\{ \theta_0 u_0 + \cdots + \theta_k u_\theta \,\middle|\, \textstyle\sum_{i=0}^{k} \theta_i = 1 \quad \textit{and } \theta_i \geq 0 \quad \forall i = 0, \ldots, k \ \textit{and } u_i \in \mathbb{R}^k \right\} \tag{C.1}$$

Definitions

Theorems and principles

Propositions

Algorithms