Proof Theory of Partially Normal Skew Monoidal Categories

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The skew monoidal categories of Szlachányi are a weakening of monoidal categories where the three structural laws of left and right unitality and associativity are not required to be isomorphisms but merely transformations in a particular direction. In previous work, we showed that the free skew monoidal category on a set of generating objects can be concretely presented as a sequent calculus. This calculus enjoys cut elimination and admits focusing, i.e. a subsystem of canonical derivations, which solves the coherence problem for skew monoidal categories.

In this paper, we develop sequent calculi for partially normal skew monoidal categories, which are skew monoidal categories with one or more structural laws invertible. Each normality condition leads to additional inference rules and equations on them. We prove cut elimination and we show that the calculi admit focusing. The result is a family of sequent calculi between those of skew monoidal categories and (fully normal) monoidal categories. On the level of derivability, these define 8 weakenings of the $\mathsf{I}, \otimes$ fragment of intuitionistic non-commutative linear logic.

1 Introduction

Substructural logics are logical systems in which one or more structural rules are not allowed. Structural rules typically include exchange, weakening and contraction. More generally, in a sequent calculus with sequents of the form $\Gamma \longrightarrow C$, with the antecedent Γ some type of a collection of formulae, a rule is structural if it manipulates the antecedent and does not mention any connectives. Affine logics are substructural wrt. intuitionistic logics since contraction is disallowed. Linear logics are substructural wrt. affine logics since weakening is also disallowed. By dropping the exchange rule as well, we obtain ordered variants of (intuitionistic) linear logics [1], which include logical systems such as Lambek's syntactic calculus [16]. One can even identify more rudimentary structural rules, such as associativity, which is dropped in some variants of Lambek's calculus [17].

Given a sequent of the form $\Gamma \longrightarrow C$ in a certain logical system, it is natural to think of formulae in Γ as types of resources at our disposal, while the formula C is a task that needs to be fulfilled with the resources at hand. Under this interpretation, the structural rules tell us how resources can be manipulated and consumed. In intuitionistic logics, resources can be permuted, deleted and copied. In linear logics, they can be neither deleted nor copied, but they can be permuted. In non-commutative linear logics resources cannot be permuted, so they must be consumed in the order they occur in the antecedent.

In previous work [24], we started investigating the proof theory of *(left-)skew monoidal categories*, a weakening of monoidal categories introduced by Szlachányi [22] where the unitors and associator are not required to be isomorphisms but merely transformations in a particular direction. Extending Zeilberger's [27] analysis of the Tamari order (which is the free (left-)skew semigroup category) as a

sequent calculus, we introduced a sequent calculus corresponding in a precise sense to the free skew monoidal category on a set At of generating objects. This sequent calculus weakens the $I,\otimes$-fragment of intuitionistic non-commutative linear logic [1] by replacing unitality and associativity with *semi-unitality* and *semi-associativity*. This means that it is possible to derive sequents corresponding to the two unitors λ and ρ and the associator α of skew monoidal categories, but not sequents corresponding to the inverses of these structural laws. Sequents have the form $S \mid \Gamma \longrightarrow C$, where the antecedent is split into an optional formula S, called the *stoup*, and a list of formulae Γ, the *context*. The left rules apply only to the formula in the special stoup position. The tensor right rule forces the formula in the stoup of the conclusion to be the formula in the stoup of the first premise. Under the resource-as-formulae interpretation, resources are required to be consumed in the order they appear in the antecedent, with the formula in the stoup being the first. At any moment, only the formula then occupying the stoup can be decomposed on the left.

This sequent calculus enjoys cut elimination and a focused subsystem, defining a root-first proof search strategy attempting to build a derivation of a sequent. The focused calculus finds exactly one representative of each equivalence class of derivations and is thus a concrete presentation of the free skew monoidal category, as such solving the *coherence problem* for skew monoidal categories.

Skew monoidal categories differ from normal (i.e., ordinary) monoidal categories in that the two unitors and the associator are not invertible. Requiring one or more of the structural laws to be invertible, we obtain a less skew structure more like a normal monoidal category. In this paper, we perform a proof-theoretic analysis of each of the three normality conditions. The result is a family of sequent calculi between those describing the free skew monoidal category and the free monoidal category, defining altogether 8 weakenings of the $I,\otimes$ fragment of non-commutative intuitionistic linear logic.

For each of these sequent calculi, we prove cut elimination and identify a focused subsystem of canonical derivations as a concrete presentation of the free skew monoidal category of the corresponding degree of normality. We conclude by presenting a single parameterized focused sequent calculus that can handle any combination of the three normality aspects.

We fully formalized the new results presented in Section 3 in the dependently typed programming language Agda. The formalization uses Agda version 2.6.0. and it is available at `https://github.com/niccoloveltri/skewmoncats-normal`.

2   Skew Monoidal Categories

A category $\mathbb{C}$ is said to be *(left-)skew monoidal* [22] if it comes together with a distinguished object I, a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and three natural transformations λ, ρ, α typed

$$\lambda_A : I \otimes A \to A \qquad \rho_A : A \to A \otimes I \qquad \alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

satisfying the equations

(m5)
$$(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)$$

with $\alpha_{A,B,C} \otimes D$ on the left vertical (upward) and $A \otimes \alpha_{B,C,D}$ on the right vertical (downward),
$$((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A \otimes B,C,D}} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))$$

If λ, ρ or α is a natural isomorphism, we say that $(\mathbb{C}, \mathsf{I}, \otimes)$ is *left-normal*, *right-normal* resp. *associative-normal* (or *Hopf*) [14]. A *monoidal category* [4] is a fully normal skew monoidal category.

Equations (m1)–(m5) are directed versions of the original Mac Lane axioms [19] for monoidal categories. Kelly [13] observed that, in the monoidal case, equations (m1), (m3), and (m4) follow from (m2) and (m5). In the skew situation, this is not the case.

Skew monoidal categories arise naturally in many settings, for example in the study of relative monads [2] and of quantum categories [14], and have been thoroughly investigated by Street, Lack and colleagues [15, 8, 5, 6]. They, as well as Uustalu [23], present many examples. Here we show just one example where normalities also play a role.

Consider two categories $\mathbb{J}$ and $\mathbb{C}$ and a functor $J : \mathbb{J} \to \mathbb{C}$ such that the left Kan extension along J exists for every functor $\mathbb{J} \to \mathbb{C}$. The functor category $[\mathbb{J}, \mathbb{C}]$ has a skew monoidal structure given by $\mathsf{I} = J$, $F \otimes G = \mathrm{Lan}_J F \cdot G$. The unitors and associator are the canonical natural transformations $\lambda_F : \mathrm{Lan}_J J \cdot F \to F$, $\rho_F : F \to \mathrm{Lan}_J F \cdot J$, $\alpha_{F,G,H} : \mathrm{Lan}_J (\mathrm{Lan}_J F \cdot G) \cdot H \to \mathrm{Lan}_J F \cdot \mathrm{Lan}_J G \cdot H$. This category is right-normal if J is fully-faithful. It is left-normal if J is dense, which is to say that the nerve of J is fully-faithful. Finally, it is associative-normal if this nerve preserves left Kan extensions along J. This example is from the work of Altenkirch et al. [2] on relative monads. Relative monads on J are (skew) monoids in the skew monoidal category $[\mathbb{J}, \mathbb{C}]$.

2.1 The Free Skew Monoidal Category

The free skew monoidal category $\mathbf{Fsk}(\mathsf{At})$ over a set At (of atoms) can be viewed as a deductive system, which we refer to as the skew monoidal *categorical calculus*.

Objects of $\mathbf{Fsk}(\mathsf{At})$ are called *formulae*, and are inductively generated as follows: a formula is either an element X of At (an *atom*); I; or $A \otimes B$ where A, B are formulae. We write Fma for the set of formulae.

Maps between two formulae A and C are *derivations* of (singleton-antecedent, singleton-succedent) sequents $A \Longrightarrow C$, constructed using the following inference rules:

$$\frac{}{A \Longrightarrow A} \text{ id} \qquad \frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C} \text{ comp} \qquad \frac{A \Longrightarrow C \quad B \Longrightarrow D}{A \otimes B \Longrightarrow C \otimes D} \otimes$$

$$\frac{}{\mathsf{I} \otimes A \Longrightarrow A} \lambda \qquad \frac{}{A \Longrightarrow A \otimes \mathsf{I}} \rho \qquad \frac{}{(A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)} \alpha \tag{1}$$

and *identified up to* the congruence $\doteq$ induced by the equations:

(category laws)	$\mathrm{id} \circ f \doteq f \qquad f \doteq f \circ \mathrm{id} \qquad (f \circ g) \circ h \doteq f \circ (g \circ h)$	
($\otimes$ functorial)	$\mathrm{id} \otimes \mathrm{id} \doteq \mathrm{id} \qquad (h \circ f) \otimes (k \circ g) \doteq h \otimes k \circ f \otimes g$	
(λ, ρ, α nat. trans.)	$\lambda \circ \mathrm{id} \otimes f \doteq f \circ \lambda$ $\rho \circ f \doteq f \otimes \mathrm{id} \circ \rho$ $\alpha \circ (f \otimes g) \otimes h \doteq f \otimes (g \otimes h) \circ \alpha$	(2)
(m1-m5)	$\lambda \circ \rho \doteq \mathrm{id} \qquad \mathrm{id} \doteq \mathrm{id} \otimes \lambda \circ \alpha \circ \rho \otimes \mathrm{id}$ $\lambda \circ \alpha \doteq \lambda \otimes \mathrm{id} \qquad \alpha \circ \rho \doteq \mathrm{id} \otimes \rho$ $\alpha \circ \alpha \doteq \mathrm{id} \otimes \alpha \circ \alpha \circ \alpha \otimes \mathrm{id}$	

In the term notation for derivations, we write $g \circ f$ for $\mathsf{comp}\, f\, g$ to agree with the standard categorical notation.

Mac Lane's coherence theorem [19] says that, in the free monoidal category, there is exactly one map $A \Longrightarrow B$ if the formulae A and B have the same frontier of atoms and no such map otherwise.

For the free skew monoidal category, this does not hold. We have pairs of formulae that have the same frontier of atoms but no maps between them or multiple maps. There are no maps $X \Longrightarrow I \otimes X$, no maps $X \otimes I \Longrightarrow X$ and no maps $X \otimes (Y \otimes Z) \Longrightarrow (X \otimes Y) \otimes Z$. At the same time, we have two maps $\mathsf{id} \neq \alpha \circ \rho \otimes \lambda : X \otimes (I \otimes Y) \Longrightarrow X \otimes (I \otimes Y)$ and two maps $\mathsf{id} \neq \rho \otimes \lambda \circ \alpha : (X \otimes I) \otimes Y \Longrightarrow (X \otimes I) \otimes Y$.

2.2   Skew Monoidal Sequent Calculus

In [24], we showed that the free skew monoidal category $\mathbf{Fsk}(\mathsf{At})$ admits an equivalent presentation as a sequent calculus. In the latter, sequents are triples $S \mid \Gamma \longrightarrow C$. The antecedent is a pair of a *stoup S* together with a *context* Γ, while the succedent C is a single formula. A stoup is an optional formula, meaning that it can either be empty (written $-$) or contain a single formula. A context is a list of formulae. For the empty list, we usually just leave a space, but where necessary for readability, we write (). Derivations in the sequent calculus are inductively generated by these inference rules:

$$\frac{A \mid \Gamma \longrightarrow C}{- \mid A, \Gamma \longrightarrow C}\ \mathsf{pass} \qquad \frac{- \mid \Gamma \longrightarrow C}{I \mid \Gamma \longrightarrow C}\ \mathsf{IL} \qquad \frac{A \mid B, \Gamma \longrightarrow C}{A \otimes B \mid \Gamma \longrightarrow C}\ \otimes\mathsf{L}$$

$$\frac{}{A \mid\ \longrightarrow A}\ \mathsf{ax} \qquad \frac{}{- \mid\ \longrightarrow I}\ \mathsf{IR} \qquad \frac{S \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{S \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}$$

(3)

(pass for 'passivate', L, R for introduction on the left (in the stoup) resp. right) and identified up to the congruence $\overset{\circ}{=}$ induced by the equations:

(η-conversions)

$$\mathsf{ax_I} \overset{\circ}{=} \mathsf{IL\ IR} \qquad\qquad \mathsf{ax}_{A \otimes B} \overset{\circ}{=} \otimes\mathsf{L}\ (\otimes\mathsf{R}\ (\mathsf{ax}_A, \mathsf{pass}\ \mathsf{ax}_B))$$

(commutative conversions)

(4)

$$\otimes\mathsf{R}\ (\mathsf{pass}\ f, g) \overset{\circ}{=} \mathsf{pass}\ (\otimes\mathsf{R}\ (f, g)) \qquad (\text{for } f : A' \mid \Gamma \longrightarrow A,\ g : - \mid \Delta \longrightarrow B)$$

$$\otimes\mathsf{R}\ (\mathsf{IL}\ f, g) \overset{\circ}{=} \mathsf{IL}\ (\otimes\mathsf{R}\ (f, g)) \qquad (\text{for } f : - \mid \Gamma \longrightarrow A,\ g : - \mid \Delta \longrightarrow B)$$

$$\otimes\mathsf{R}\ (\otimes\mathsf{L}\ f, g) \overset{\circ}{=} \otimes\mathsf{L}\ (\otimes\mathsf{R}\ (f, g)) \qquad (\text{for } f : A' \mid B', \Gamma \longrightarrow A,\ g : - \mid \Delta \longrightarrow B)$$

Although these rules look very similar to the rules of the $I, \otimes$ fragment of intuitionistic non-commutative linear logic [1] (the sequent calculus describing the free monoidal category)—in particular, there is no left exchange rule, weakening or contraction—, there are two crucial differences.

- The left rules IL and $\otimes\mathsf{L}$ are restricted to apply only to the formula within the stoup (in the conclusion-first reading of these rules). This restriction was also present in Zeilberger's sequent calculus for the Tamari order [27]. In this calculus, it is possible to derive sequents corresponding

to the right unitor $\rho : A \Longrightarrow A \otimes I$ and the associator $\alpha : (A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)$:

$$
\cfrac{
\cfrac{}{A \mid\; \longrightarrow A}\;\text{ax} \quad \cfrac{}{-\mid\; \longrightarrow I}\;\text{IR}
}{A \mid\; \longrightarrow A \otimes I}\;\otimes\text{R}
$$

$$
\cfrac{
\cfrac{}{A \mid\; \longrightarrow A}\;\text{ax} \quad
\cfrac{
\cfrac{
\cfrac{}{B \mid\; \longrightarrow B}\;\text{ax} \quad
\cfrac{\cfrac{}{C \mid\; \longrightarrow C}\;\text{ax}}{-\mid C \longrightarrow C}\;\text{pass}
}{B \mid C \longrightarrow B \otimes C}\;\otimes\text{R}
}{-\mid B,C \longrightarrow B \otimes C}\;\text{pass}
}{
\cfrac{
\cfrac{A \mid B,C \longrightarrow A \otimes (B \otimes C)}{A \otimes B \mid C \longrightarrow A \otimes (B \otimes C)}\;\otimes\text{L}
}{(A \otimes B) \otimes C \mid\; \longrightarrow A \otimes (B \otimes C)}\;\otimes\text{L}
}\;\otimes\text{R}
$$

On the other hand, since IL and $\otimes$L only act on the formula in the stoup, there is no way of deriving sequents corresponding to inverses of ρ and α for atomic A resp. A, B, C.

- The stoup is allowed to be empty, permitting a distinction between antecedents of the form $A \mid \Gamma$ (with A inside the stoup) and antecedents of the form $- \mid A, \Gamma$ (with A outside the stoup). This distinction plays an important role in the right rule $\otimes$R, which sends the formula in the stoup, when it is present, to the first premise. In this calculus, it is possible to derive a sequent corresponding to the left unitor $\lambda : I \otimes A \Longrightarrow A$:

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{}{A \mid\; \longrightarrow A}\;\text{ax}
}{-\mid A \longrightarrow A}\;\text{pass}
}{I \mid A \longrightarrow A}\;\text{IL}
}{I \otimes A \mid\; \longrightarrow A}\;\otimes\text{L}
$$

On the other hand, since $\otimes$R always send the formula in the stoup to the first premise, it is not possible to derive a sequent corresponding to the inverse of λ for atomic A.

There are no primitive cut rules in this sequent calculus, but two forms of cut are admissible:

$$
\cfrac{S \mid \Gamma \longrightarrow A \quad A \mid \Delta \longrightarrow C}{S \mid \Gamma, \Delta \longrightarrow C}\;\text{scut}
\qquad
\cfrac{-\mid \Gamma \longrightarrow A \quad S \mid \Delta_0, A, \Delta_1 \longrightarrow C}{S \mid \Delta_0, \Gamma, \Delta_1 \longrightarrow C}\;\text{ccut}
\tag{5}
$$

Sequent calculus derivations can be turned into categorical calculus derivations by means of a function sound : $S \mid \Gamma \longrightarrow C \to [\![S \mid \Gamma]\!] \Longrightarrow C$, where the interpretation of an antecedent as a formula $[\![S \mid \Gamma]\!]$ is defined as $[\![S \mid \Gamma]\!] = [\![S \langle\!\langle\; \langle\!\langle \Gamma]\!]$ with

$$
[\![- \langle\!\langle = I \qquad [\![A \langle\!\langle = A \qquad A \langle\!\langle\;]\!] = A \qquad A \langle\!\langle B, \Gamma]\!] = (A \otimes B)\langle\!\langle \Gamma]\!]
$$

which means that $A \langle\!\langle A_1, A_2 \ldots, A_n]\!] = (\ldots (A \otimes A_1) \otimes A_2) \ldots) \otimes A_n$. The interpretation of antecedents is functorial, i.e., we have the following inference rule:

$$
\cfrac{A \Longrightarrow B}{[\![A \mid \Gamma]\!] \Longrightarrow [\![B \mid \Gamma]\!]}\;[\![\;\mid \Gamma]\!]
\tag{6}
$$

The function sound is well-defined on $\overset{\circ}{=}$-equivalence classes: given related derivations $f \overset{\circ}{=} g$, then sound $f \overset{\cdot}{=}$ sound g.

Categorical calculus derivations can be interpreted as sequent calculus derivations via a function cmplt : $[\![S \mid \Gamma]\!] \Longrightarrow C \to S \mid \Gamma \longrightarrow C$, well-defined on $\overset{\cdot}{=}$-equivalence classes: given related derivations $f \overset{\cdot}{=} g$, then cmplt $f \overset{\circ}{=}$ cmplt g. The cut rule scut is fundamental for modelling the composition operation comp of the categorical calculus.

The functions sound and cmplt establish a bijection between derivations in the categorical calculus and the sequent calculus: sound (cmplt f) $\overset{\cdot}{=} f$ and cmplt (sound g) $\overset{\circ}{=} g$, for all $f : [\![S \mid \Gamma]\!] \Longrightarrow C$ and $g : S \mid \Gamma \longrightarrow C$.

2.3 A Focused Subsystem

The congruence relation $\overset{\circ}{=}$ can be considered as a term rewrite system, by directing every equation from left to right. The resulting rewrite system is weakly confluent and strongly normalizing, hence confluent with unique normal forms.

Normal-form derivations in our sequent calculus can be described as derivations in a *focused* subsystem. In the style of Andreoli [3], we present the focused subsystem as a sequent calculus with an additional *phase annotation* on sequents. In phase L, sequents are of the form $S \mid \Gamma \longrightarrow_L C$, where S is a general stoup. In phase R, sequents take the form $T \mid \Gamma \longrightarrow_R C$, where T is an *irreducible* stoup, that is an optional atom: either empty or an atomic formula. Derivations in the focused calculus are inductively generated by the following inference rules:

$$\frac{A \mid \Gamma \longrightarrow_L C}{- \mid A,\Gamma \longrightarrow_L C} \text{ pass} \qquad \frac{- \mid \Gamma \longrightarrow_L C}{\mathsf{I} \mid \Gamma \longrightarrow_L C} \text{ IL} \qquad \frac{A \mid B,\Gamma \longrightarrow_L C}{A \otimes B \mid \Gamma \longrightarrow_L C} \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \longrightarrow_R C}{T \mid \Gamma \longrightarrow_L C} \text{ switch}$$

$$\frac{}{X \mid \ \longrightarrow_R X} \text{ ax} \qquad \frac{}{- \mid \ \longrightarrow_R \mathsf{I}} \text{ IR} \qquad \frac{T \mid \Gamma \longrightarrow_R A \quad - \mid \Delta \longrightarrow_L B}{T \mid \Gamma,\Delta \longrightarrow_R A \otimes B} \otimes\mathsf{R} \tag{7}$$

The focused rules define a sound and complete proof search strategy. The focused calculus is clearly sound: by erasing phase annotations, all of the rules are either rules of the original calculus or else (in the case of switch) have the conclusion equal to the premise. So focused calculus derivations can be embedded into sequent calculus derivations via a function $\mathsf{emb}_P : S \mid \Gamma \longrightarrow_P C \to S \mid \Gamma \longrightarrow C$, where $P \in \{\mathsf{L},\mathsf{R}\}$.

The focused calculus is also complete: we can define a normalization procedure $\mathsf{focus} : S \mid \Gamma \longrightarrow C \to S \mid \Gamma \longrightarrow_L C$ sending each derivation in the sequent calculus to a canonical representative of its $\overset{\circ}{=}$-equivalence class in the focused calculus. This means in particular that focus maps $\overset{\circ}{=}$-related derivations to *equal* focused derivations.

The functions focus and emb_L establish a bijection between derivations in the full sequent calculus (up to $\overset{\circ}{=}$) and its focused subsystem: $\mathsf{emb}_L (\mathsf{focus}\, f) \overset{\circ}{=} f$ and $\mathsf{focus} (\mathsf{emb}_L\, g) = g$, for all $f : S \mid \Gamma \longrightarrow C$ and $g : S \mid \Gamma \longrightarrow_L C$.

Putting this together with the results discussed in Section 2.2, it follows that the focused calculus is a concrete presentation of the free skew monoidal category $\mathbf{Fsk}(\mathsf{At})$. As such, the focused calculus solves the problem of characterizing the homsets of the free skew monoidal category, a.k.a. the *coherence problem*. Moreover, it solves two related algorithmic problems effectively:

- Duplicate-free enumeration of all maps $A \Longrightarrow C$ in the form of representatives of $\overset{\cdot}{=}$-equivalence classes of categorical calculus derivations: For this, find all focused derivations of $A \mid \ \longrightarrow_L C$, which is solvable by exhaustive proof search, which terminates, and translate them to the categorical calculus derivations.

- Finding whether two given maps of type $A \Longrightarrow C$, presented as categorical calculus derivations, are equal, i.e., $\overset{\cdot}{=}$-related as derivations: For this, translate them to focused derivations of $A \mid \ \longrightarrow_L C$ and check whether they are equal, which is decidable.

Different approaches to the coherence problem of skew monoidal categories have been considered before. Uustalu [23] identified a class of normal forms of objects in $\mathbf{Fsk}(\mathsf{At})$ and showed that there exists at most one map between an object and an object in normal form, and exactly one map between an object and that object's normal form. In another direction, Lack and Street [15] addressed the problem of determining equality of maps by proving that there is a faithful, structure-preserving functor $\mathbf{Fsk}(1) \to \Delta_\perp$

from the free skew monoidal category on one generating object to the category of finite non-empty ordinals and first-element-and-order-preserving functions (which is an associative-normal skew-monoidal category under the ordinal sum with unit 1). This approach was further elaborated by Bourke and Lack [5] with a more explicit description of the homsets of **Fsk**(1).

Let us use the focused calculus to analyze where multiple maps $A \Longrightarrow C$ can come from. There are two sources of non-determinism in root-first proof search in the focused calculus: (i) in phase L, when the stoup is empty, whether to apply pass or switch, and (ii) in phase R, when the succedent formula is a tensor and the rule $\otimes$R is to be applied, how to split the context into Γ and Δ. In the latter situation, only those choices where $|T|, |\Gamma| = |A|$ and $|\Delta| = |B|$ can possibly lead to a complete derivation; we write $|\ |$ for the frontier of atoms in a formula, an optional formula or a list of formulae. But there can be multiple such choices if in the middle of the context there are closed formulae (i.e., formulae made of I and $\otimes$ only): those can be freely split between Γ and Δ.

The two maps $\text{id} \neq \alpha \circ \rho \otimes \lambda : X \otimes (I \otimes Y) \Longrightarrow X \otimes (I \otimes Y)$ translate to two different focused derivations of the sequent $X \otimes (I \otimes Y) \mid \longrightarrow_L X \otimes (I \otimes Y)$ because of the non-determinism of type (i). Notice the choice between pass and switch.

$$(8)$$

The two maps $\text{id} \neq \rho \otimes \lambda \circ \alpha : (X \otimes I) \otimes Y \Longrightarrow (X \otimes I) \otimes Y$ translate to distinct focused derivations of the sequent $(X \otimes I) \otimes Y \mid \longrightarrow_L (X \otimes I) \otimes Y$ due to type-(ii) non-determinism. Here the first (from the endsequent) application of the tensor right rule $\otimes$R splits the context in two different ways: in the first derivation the unit in the context is sent to the first premise, while in the second derivation it is sent to the second premise.

$$(9)$$

3 Normality Conditions

3.1 Left-Normality

The free left-normal skew monoidal category $\mathbf{Fsk_{LN}}(\mathsf{At})$ on a set At is obtained by extending the grammar of derivations in the fully skew categorical calculus (1) with a new inference rule:

$$\frac{}{A \Longrightarrow I \otimes A} \; \lambda^{-1}$$

and extending the equivalence of derivations (2) with two new equations: $\lambda \circ \lambda^{-1} \doteq \mathsf{id}$ and $\lambda^{-1} \circ \lambda \doteq \mathsf{id}$.

An equivalent sequent calculus presentation of $\mathbf{Fsk_{LN}}(\mathsf{At})$ is obtained by adding another right rule for the tensor $\otimes$ to the fully skew sequent calculus (3), which allows one to send the formula in the stoup to the second premise, provided that all of the context is also sent to the second premise (so the antecedent of the first premise is left completely empty).

$$\frac{- \mid \; \longrightarrow A \quad A' \mid \Delta \longrightarrow B}{A' \mid \Delta \longrightarrow A \otimes B} \; \otimes\mathsf{R}_2$$

The introduction of $\otimes\mathsf{R}_2$ makes it possible to derive a sequent corresponding to $\lambda^{-1} : A \Longrightarrow I \otimes A$:

$$\frac{\dfrac{}{- \mid \; \longrightarrow I} \; \mathsf{IR} \quad \dfrac{}{A \mid \; \longrightarrow A} \; \mathsf{ax}}{A \mid \; \longrightarrow I \otimes A} \; \otimes\mathsf{R}_2$$

In particular this allows us to interpret categorical derivations into sequent calculus derivations for the left-normal case, extending the definition of the function cmplt introduced in Section 2.2.

Equivalence of derivations in the sequent calculus is the least congruence $\doteq$ induced by the equations in (4) together with the following equations:

$$\otimes\mathsf{R} \; (f, \mathsf{pass} \; g) \doteq \mathsf{pass} \; (\otimes\mathsf{R}_2 \; (f,g)) \qquad (\text{for } f : - \mid \; \longrightarrow A, \; g : A' \mid \Delta \longrightarrow B)$$

$$\otimes\mathsf{R}_2 \; (f, \mathsf{IL} \; g) \doteq \mathsf{IL} \; (\otimes\mathsf{R} \; (f,g)) \qquad (\text{for } f : - \mid \; \longrightarrow A, \; g : - \mid \Delta \longrightarrow B)$$

$$\otimes\mathsf{R}_2 \; (f, \otimes\mathsf{L} \; g) \doteq \otimes\mathsf{L} \; (\otimes\mathsf{R}_2 \; (f,g)) \qquad (\text{for } f : - \mid \; \longrightarrow A, \; g : A' \mid B', \Delta \longrightarrow B)$$

The two cut rules in (5) are also admissible in the left-normal sequent calculus. The definition of ccut uses the following rule act (for 'activate'), admissible in this sequent calculus (but not in the fully skew one) and inverting pass up to the congruence $\doteq$:

$$\frac{- \mid A, \Gamma \longrightarrow_{\mathsf{L}} C}{A \mid \Gamma \longrightarrow_{\mathsf{L}} C} \; \mathsf{act}$$

The transformation sound introduced in Section 2.2, interpreting the fully skew sequent calculus derivations as categorical calculus derivations, extends to the left-normal case. Given $f : - \mid \; \longrightarrow A$ and $g : A' \mid \Delta \longrightarrow B$, define sound $(\otimes\mathsf{R}_2(f,g))$ as:

$$\frac{[\![A' \mid \Delta]\!] \overset{\mathsf{sound}\, g}{\Longrightarrow} B \quad \dfrac{\dfrac{}{B \Longrightarrow I \otimes B} \; \lambda^{-1} \quad I \otimes B \overset{\mathsf{sound}\, f \otimes \mathsf{id}}{\Longrightarrow} A \otimes B}{B \Longrightarrow A \otimes B} \; \mathsf{comp}}{[\![A' \mid \Delta]\!] \Longrightarrow A \otimes B} \; \mathsf{comp}$$

Again, the congruence relation $\doteq$ read as a term rewrite system is weakly confluent and strongly normalizing, and normal-form derivations in the sequent calculus can be described as derivations in a

focused subsystem. Sequents in the left-normal focused calculus are annotated with two possible phase annotations, as in the fully skew focused calculus (7). The condition for switching phase is different: the formula in the stoup is still required to be irreducible, but if the stoup is empty we are allowed to switch phase only when the context is empty as well. In phase R, we also include the new tensor right rule $\otimes\mathsf{R}_2$, in which both premises are required to be R-phase derivations. Again T is an irreducible stoup: either empty or an atomic formula.

$$\dfrac{A \mid \Gamma \longrightarrow_{\mathsf{L}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{L}} C} \text{ pass} \qquad \dfrac{- \mid \Gamma \longrightarrow_{\mathsf{L}} C}{\mathsf{I} \mid \Gamma \longrightarrow_{\mathsf{L}} C} \text{ IL} \qquad \dfrac{A \mid B, \Gamma \longrightarrow_{\mathsf{L}} C}{A \otimes B \mid \Gamma \longrightarrow_{\mathsf{L}} C} \otimes\mathsf{L} \qquad \dfrac{T \mid \Gamma \longrightarrow_{\mathsf{R}} C \quad T = - \to \Gamma = ()}{T \mid \Gamma \longrightarrow_{\mathsf{L}} C} \text{ switch}$$

$$\dfrac{}{X \mid \;\longrightarrow_{\mathsf{R}} X} \text{ ax} \qquad \dfrac{}{- \mid \;\longrightarrow_{\mathsf{R}} \mathsf{I}} \text{ IR} \qquad \dfrac{T \mid \Gamma \longrightarrow_{\mathsf{R}} A \quad - \mid \Delta \longrightarrow_{\mathsf{L}} B}{T \mid \Gamma, \Delta \longrightarrow_{\mathsf{R}} A \otimes B} \otimes\mathsf{R} \qquad \dfrac{- \mid \;\longrightarrow_{\mathsf{R}} A \quad X \mid \Delta \longrightarrow_{\mathsf{R}} B}{X \mid \Delta \longrightarrow_{\mathsf{R}} A \otimes B} \otimes\mathsf{R}_2$$

All R-phase sequents in a derivation of a sequent $S \mid \Gamma \longrightarrow_{\mathsf{L}} C$ have the context empty if the stoup is empty.

The functions $\mathsf{emb}_{\mathsf{L}}$ and $\mathsf{emb}_{\mathsf{R}}$ embedding fully skew focused calculus derivations in the unfocused sequent calculus, discussed in Section 2.3, can clearly be adapted to the left-normal case. The same holds for the focus function, sending a sequent calculus derivation to its $\overset{\circ}{=}$-normal form in the focused calculus. In particular, this means that focus maps two $\overset{\circ}{=}$-equivalent derivations to the same focused derivation. Similarly to the fully skew case, focus and $\mathsf{emb}_{\mathsf{L}}$ establish a bijection between maps in the left-normal sequent calculus (up to $\overset{\circ}{=}$) and its focused subsystem.

The left-normal focused calculus has less non-determinism than the fully skew focused calculus. The non-determinism of type (i) is not there: In phase L, switch cannot be applied unless the context is empty while pass only applies if it is non-empty. The non-determinism of type (ii) remains much like in the fully skew case except that there are two tensor right-rules, $\otimes\mathsf{R}$ and $\otimes\mathsf{R}_2$. In a choice that can lead to a completed derivation, we must be able to take $|T|, |\Gamma| = |A|$ and $|\Delta| = |B|$ in $\otimes\mathsf{R}$ or $() = |A|$ and $X, |\Delta| = |B|$ in $\otimes\mathsf{R}_2$. There may be multiple such choices if in the middle of the context we have closed formulae.

In the free left-normal skew monoidal category, we have $\mathsf{id} \overset{\circ}{=} \alpha \circ \rho \otimes \lambda : X \otimes (\mathsf{I} \otimes Y) \Longrightarrow X \otimes (\mathsf{I} \otimes Y)$:

$$\mathsf{id} \overset{\circ}{=} \mathsf{id} \otimes \lambda^{-1} \circ \mathsf{id} \otimes \lambda \overset{\circ}{=} \mathsf{id} \otimes \lambda^{-1} \circ (\mathsf{id} \otimes \lambda \circ \alpha \circ \rho \otimes \mathsf{id}) \circ \mathsf{id} \otimes \lambda \overset{\circ}{=} \alpha \circ \rho \otimes \lambda$$

This collapse is reflected by there being exactly one focused derivation of the sequent $X \otimes (\mathsf{I} \otimes Y) \mid \;\longrightarrow_{\mathsf{L}} X \otimes (\mathsf{I} \otimes Y)$. Compare this with the two distinct derivations of the sequent in the fully skew focused calculus, displayed in (8). In the left-normal focused calculus, we are forced to use pass, we cannot apply switch since the stoup is empty but the context is not.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{- \mid \;\longrightarrow_{\mathsf{R}} \mathsf{I}} \text{ IR} \quad \dfrac{}{Y \mid \;\longrightarrow_{\mathsf{R}} Y} \text{ ax}}{Y \mid \;\longrightarrow_{\mathsf{R}} \mathsf{I} \otimes Y} \otimes\mathsf{R}_2}{Y \mid \;\longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ switch}}{- \mid Y \longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ pass}}{\mathsf{I} \mid Y \longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ IL}}{\mathsf{I} \otimes Y \mid \;\longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \otimes\mathsf{L}}{- \mid \mathsf{I} \otimes Y \longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ pass}}{}}{}$$

$$\dfrac{\dfrac{}{X \mid \;\longrightarrow_{\mathsf{R}} X} \text{ ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{- \mid \;\longrightarrow_{\mathsf{R}} \mathsf{I}} \text{ IR} \quad \dfrac{}{Y \mid \;\longrightarrow_{\mathsf{R}} Y} \text{ ax}}{Y \mid \;\longrightarrow_{\mathsf{R}} \mathsf{I} \otimes Y} \otimes\mathsf{R}_2}{Y \mid \;\longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ switch}}{- \mid Y \longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ pass}}{\mathsf{I} \mid Y \longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ IL}}{\mathsf{I} \otimes Y \mid \;\longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \otimes\mathsf{L}}{- \mid \mathsf{I} \otimes Y \longrightarrow_{\mathsf{L}} \mathsf{I} \otimes Y} \text{ pass}}{\dfrac{\dfrac{X \mid \mathsf{I} \otimes Y \longrightarrow_{\mathsf{R}} X \otimes (\mathsf{I} \otimes Y)}{X \mid \mathsf{I} \otimes Y \longrightarrow_{\mathsf{L}} X \otimes (\mathsf{I} \otimes Y)} \text{ switch}}{X \otimes (\mathsf{I} \otimes Y) \mid \;\longrightarrow_{\mathsf{L}} X \otimes (\mathsf{I} \otimes Y)} \otimes\mathsf{L}} \otimes\mathsf{R}}$$

The left-normal sequent calculus admits also a stoup-free presentation. This is because the rule pass is invertible. Here we only show the focused subsystem of the stoup-free variant. In phase L, sequents are of the form $\Gamma \longrightarrow_L C$, where Γ is a general list of formulae. In phase R, sequents take the form $\Lambda \longrightarrow_R C$, where Λ is an irreducible list of formulae, meaning that it is either empty or the formula in its head is an atom.

$$\frac{\Gamma \longrightarrow_L C}{I,\Gamma \longrightarrow_L C} \ IL \qquad \frac{A,B,\Gamma \longrightarrow_L C}{A \otimes B,\Gamma \longrightarrow_L C} \ \otimes L \qquad \frac{\Lambda \longrightarrow_R C}{\Lambda \longrightarrow_L C} \ \text{switch}$$

$$\frac{}{X \longrightarrow_R X} \ \text{ax} \qquad \frac{}{\longrightarrow_R I} \ IR \qquad \frac{\Lambda \longrightarrow_R A \quad \Delta \longrightarrow_L B}{\Lambda,\Delta \longrightarrow_R A \otimes B} \ \otimes R$$

Derivations of a sequent $S \mid \Gamma \longrightarrow_L C$ are in a bijection with derivations of $[\![S \langle\!\langle, \Gamma \longrightarrow_L C$ where $[\![S \langle\!\langle$ is the interpretation of a stoup as a formula introduced in Section 2.2.

3.2   Right-Normality

The free right-normal skew monoidal category $\mathbf{Fsk}_{RN}(\mathsf{At})$ on a set At is obtained by extending the grammar of derivations in the fully skew categorical calculus (1) with a new inference rule:

$$\frac{}{A \otimes I \Longrightarrow A} \ \rho^{-1}$$

and extending the equivalence of derivations (2) with two new equations: $\rho \circ \rho^{-1} \doteq \mathsf{id}$ and $\rho^{-1} \circ \rho \doteq \mathsf{id}$.

An equivalent sequent calculus presentation of $\mathbf{Fsk}_{RN}(\mathsf{At})$ is realized by adding additional left rules for I and $\otimes$ to the fully skew sequent calculus (3), relaxing the condition for deleting the unit and decomposing tensors in the antecedent:

$$\frac{S \mid \Gamma_0,\Gamma_1 \longrightarrow C}{S \mid \Gamma_0,I,\Gamma_1 \longrightarrow C} \ IC_{\Gamma_0} \qquad \frac{S \mid \Gamma_0,J,J',\Gamma_1 \longrightarrow C}{S \mid \Gamma_0,J \otimes J',\Gamma_1 \longrightarrow C} \ \otimes C^c_{\Gamma_0}$$

Here and later, J and J' stand for closed formulae. The introduction of IC makes it possible to derive a sequent corresponding to $\rho^{-1} : A \otimes I \Longrightarrow A$ for any A, including $A = X$:

$$\frac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A} \ \text{ax}}{A \mid I \longrightarrow A} \ IC_{()}}{A \otimes I \mid \ \longrightarrow A} \ \otimes L$$

The rule $\otimes C^c$ is needed since it is important to allow deletion in the context of any closed formula and not just I: we need to be able to derive, e.g., the sequent $X \mid I \otimes I \longrightarrow X$. Analogously, in the left-normal sequent calculus of Section 3.1, it was important to be able to derive the sequent $X \mid \ \longrightarrow (I \otimes I) \otimes X$, which was possible since the first premise of the inference rule $\otimes R_2$ is a sequent $- \mid \ \longrightarrow A$, which is derivable precisely when A is closed.

Equivalence of derivations in the sequent calculus is the least congruence $\overset{\circ}{\doteq}$ induced by the equations

in (4) together with the following equations:

$$\mathsf{pass}\ (\mathsf{IL}\ f) \doteq \mathsf{IC}_{()}\ f \qquad\qquad (\text{for } f: -\mid \Gamma \longrightarrow C)$$

$$\mathsf{IC}_{\Gamma_0}\ (\mathsf{IC}_{\Gamma_0,\Gamma_1}\ f) \doteq \mathsf{IC}_{\Gamma_0,\mathsf{I},\Gamma_1}\ (\mathsf{IC}_{\Gamma_0}\ f) \qquad\qquad (\text{for } f: S \mid \Gamma_0,\Gamma_1,\Gamma_2 \longrightarrow C)$$

$$\otimes\mathsf{C}^{\mathsf{c}}_{\Gamma_0}\ (\mathsf{IC}_{\Gamma_0,J,J',\Gamma_1}\ f) \doteq \mathsf{IC}_{\Gamma_0,J\otimes J',\Gamma_1}\ (\otimes\mathsf{C}^{\mathsf{c}}_{\Gamma_0}\ f) \qquad (\text{for } f: S \mid \Gamma_0,J,J',\Gamma_1,\Gamma_2 \longrightarrow C)$$

$$\mathsf{pass}\ (\mathsf{IC}_{\Gamma_0}\ f) \doteq \mathsf{IC}_{A,\Gamma_0}\ (\mathsf{pass}\ f) \qquad\qquad (\text{for } f: A \mid \Gamma_0,\Gamma_1 \longrightarrow C)$$

$$\mathsf{IL}\ (\mathsf{IC}_{\Gamma_0}\ f) \doteq \mathsf{IC}_{\Gamma_0}\ (\mathsf{IL}\ f) \qquad\qquad (\text{for } f: -\mid \Gamma_0,\Gamma_1 \longrightarrow C)$$

$$\otimes\mathsf{L}\ (\mathsf{IC}_{B,\Gamma_0}\ f) \doteq \mathsf{IC}_{\Gamma_0}\ (\otimes\mathsf{L}\ f) \qquad\qquad (\text{for } f: A \mid B,\Gamma_0,\Gamma_1 \longrightarrow C)$$

$$\otimes\mathsf{R}\ (\mathsf{IC}_{\Gamma_0}\ f, g) \doteq \mathsf{IC}_{\Gamma_0}\ (\otimes\mathsf{R}\ (f,g)) \qquad (\text{for } f: S \mid \Gamma_0,\Gamma_1 \longrightarrow A,\ g: -\mid \Delta \longrightarrow B)$$

$$\otimes\mathsf{R}\ (f, \mathsf{IC}_{\Delta_0}\ g) \doteq \mathsf{IC}_{\Gamma,\Delta_0}\ (\otimes\mathsf{R}\ (f,g)) \qquad (\text{for } f: S \mid \Gamma \longrightarrow A,\ g: -\mid \Delta_0,\Delta_1 \longrightarrow B)$$

(10)

plus the same number of similar equations for $\otimes\mathsf{C}^{\mathsf{c}}$. The only difference is in the 1st equation, with $\otimes\mathsf{L}$ instead of IL in the left-hand side, in which an extra application of pass in the right-hand side is required for the equation to be well-typed:

$$\mathsf{pass}\ (\otimes\mathsf{L}\ f) \doteq \otimes\mathsf{C}^{\mathsf{c}}_{()}\ (\mathsf{pass}\ f) \qquad\qquad (\text{for } f: J \mid J',\Gamma \longrightarrow C) \qquad\qquad (11)$$

The two cut rules in (5) are admissible in the right-normal sequent calculus. In this case, they need to be defined by mutual induction with another cut rule

$$\frac{A' \mid \Gamma \longrightarrow A \qquad S \mid \Delta_0,A,\Delta_1 \longrightarrow C}{S \mid \Delta_0,A',\Gamma,\Delta_1 \longrightarrow C}\ \mathsf{ccut}_{\mathsf{Fma}}$$

In the fully skew sequent calculus of Section 2.2, the rule $\mathsf{ccut}_{\mathsf{Fma}}$ is definable by first applying pass to the first premise and then using ccut. In the right-normal case, we have to define it simultaneously with scut and ccut because of the added cases for the added primitive rules. The definition of $\mathsf{ccut}_{\mathsf{Fma}}$ relies on a lemma, stating that, if a sequent $A \mid \Gamma \longrightarrow J$ is derivable, with J a closed formula, then both A and all the formulae in Γ are also closed.

The interpretation sound of fully skew sequent calculus derivations as categorical calculus derivations extends to the right-normal case. Given $f: S \mid \Gamma_0,\Gamma_1 \longrightarrow C$, define $\mathsf{sound}\ (\mathsf{IC}_{\Gamma_0}\ f)$ as:

$$\cfrac{\cfrac{\dfrac{\overline{[\![S\mid\Gamma_0]\!]\otimes\mathsf{I}\Longrightarrow[\![S\mid\Gamma_0]\!]}\ \rho^{-1}}{[\![[\![S\mid\Gamma_0]\!]\otimes\mathsf{I}\mid\Gamma_1]\!]\Longrightarrow[\![[\![S\mid\Gamma_0]\!]\mid\Gamma_1]\!]}\ [\![\ \mid\Gamma_1]\!] \qquad \dfrac{[\![S\mid\Gamma_0,\Gamma_1]\!]\overset{\mathsf{sound}\,f}{\Longrightarrow}C}{[\![[\![S\mid\Gamma_0]\!]\mid\Gamma_1]\!]\Longrightarrow C}}{[\![[\![S\mid\Gamma_0]\!]\otimes\mathsf{I}\mid\Gamma_1]\!]\Longrightarrow C}\ \mathsf{comp}}{[\![S\mid\Gamma_0,\mathsf{I},\Gamma_1]\!]\Longrightarrow C}$$

The double-line rules correspond to applications of the provable equality $[\![S\mid\Gamma,\Delta]\!]=[\![[\![S\mid\Gamma]\!]\mid\Delta]\!]$, while $[\![\ \mid\ ]\!]$ is the inference rule introduced in (6).

Given $f: S \mid \Gamma_0,J,J',\Gamma_1 \longrightarrow C$, define $\mathsf{sound}\ (\otimes\mathsf{C}^{\mathsf{c}}_{\Gamma_0}\ f)$ as:

$$\cfrac{\cfrac{\dfrac{\overline{[\![S\mid\Gamma_0]\!]\otimes(J\otimes J')\Longrightarrow([\![S\mid\Gamma_0]\!]\otimes J)\otimes J'}\ \alpha_{\mathsf{c}}^{-1}}{[\![[\![S\mid\Gamma_0]\!]\otimes(J\otimes J')\mid\Gamma_1]\!]\Longrightarrow[\![([\![S\mid\Gamma_0]\!]\otimes J)\otimes J'\mid\Gamma_1]\!]}\ [\![\ \mid\Gamma_1]\!] \qquad \dfrac{[\![S\mid\Gamma_0,J,J',\Gamma_1]\!]\overset{\mathsf{sound}\,f}{\Longrightarrow}C}{[\![([\![S\mid\Gamma_0]\!]\otimes J)\otimes J'\mid\Gamma_1]\!]\Longrightarrow C}}{[\![[\![S\mid\Gamma_0]\!]\otimes(J\otimes J')\mid\Gamma_1]\!]\Longrightarrow C}\ \mathsf{comp}}{[\![S\mid\Gamma_0,J\otimes J',\Gamma_1]\!]\Longrightarrow C}$$

(12)

The derivation $\alpha_c^{-1} : A \otimes (J \otimes J') \Longrightarrow (A \otimes J) \otimes J'$ is the inverse of a restricted form of the associator α in which the second and third formula are closed. It is defined and shown to invert α by induction on J' (distinguishing the two cases of J' being I and the tensor of two closed formulae).

Let us mention that, instead of IC and $\otimes C^c$, one could of course choose to work with one "big-step" inference rule

$$\frac{S \,|\, \Gamma_0, \Gamma_1 \longrightarrow C}{S \,|\, \Gamma_0, J, \Gamma_1 \longrightarrow C} \; I_{\Gamma_0}$$

but we prefer the "small-step" IC and $\otimes C^c$ especially because, in the situation of simultaneous right- and associative-normality, $\otimes C^c$ is subsumed by the $\otimes C$ rule that we will introduce in the next section.

Again, the congruence relation $\doteq$ read as a term rewrite system is weakly confluent and strongly normalizing, and normal-form derivations in the sequent calculus can be described as derivations in a focused subsystem. Sequents in the focused calculus are annotated by three possible phase annotations. Phases L and R are as in the fully skew focused calculus (7). In the new phase C, sequents are of the form $S \,|\, \Omega \,\vdots\, \Gamma \longrightarrow_C C$, where the context is split in two parts: an *anteroom* Ω and a passive context Γ. In phase C, each formula D in the anteroom is inspected, starting from the right end of the anteroom. In case D is the unit, then it is removed from the anteroom. If D is a tensor $J \otimes J'$, with J and J' closed formulae, then it is decomposed and J' is inspected next. Otherwise, D is moved to the left end of the passive context.

$$\frac{S \,|\, \Omega \,\vdots\, \Gamma \longrightarrow_C C}{S \,|\, \Omega, I \,\vdots\, \Gamma \longrightarrow_C C} \; IC \qquad \frac{S \,|\, \Omega, J, J' \,\vdots\, \Gamma \longrightarrow_C C}{S \,|\, \Omega, J \otimes J' \,\vdots\, \Gamma \longrightarrow_C C} \; \otimes C^c \qquad \frac{S \,|\, \Omega \,\vdots\, D, \Gamma \longrightarrow_C C \quad D \neq J}{S \,|\, \Omega, D \,\vdots\, \Gamma \longrightarrow_C C} \; move \qquad \frac{S \,|\, \Gamma \longrightarrow_L C}{S \,|\, \;\; \vdots\, \Gamma \longrightarrow_C C} \; switch_{LC}$$

$$\frac{A \,|\, \Gamma \longrightarrow_L C}{- \,|\, A, \Gamma \longrightarrow_L C} \; pass \qquad \frac{- \,|\, \Gamma \longrightarrow_L C}{I \,|\, \Gamma \longrightarrow_L C} \; IL \qquad \frac{A \,|\, B \,\vdots\, \Gamma \longrightarrow_C C}{A \otimes B \,|\, \Gamma \longrightarrow_L C} \; \otimes L \qquad \frac{T \,|\, \Gamma \longrightarrow_R C}{T \,|\, \Gamma \longrightarrow_L C} \; switch_{RL}$$

$$\frac{}{X \,|\, \;\; \longrightarrow_R X} \; ax \qquad \frac{}{- \,|\, \;\; \longrightarrow_R I} \; IR \qquad \frac{T \,|\, \Gamma \longrightarrow_R A \quad - \,|\, \Delta \longrightarrow_L B}{T \,|\, \Gamma, \Delta \longrightarrow_R A \otimes B} \; \otimes R$$

$$(13)$$

All C-phase, L-phase and R-phase sequents in a derivation of a sequent $S \,|\, \Omega \,\vdots\, \;\; \longrightarrow_C C$ have the (passive) context free of closed formulae.

By dropping the phase annotations (also turning $\vdots$ into a comma), we can define three functions emb_C, emb_L and emb_R embedding right-normal focused calculus derivations into the unfocused calculus. We can also define a normalization function $focus : S \,|\, \Omega \longrightarrow C \to S \,|\, \Omega \,\vdots\, \;\; \longrightarrow_C C$, which identifies $\doteq$-related derivations. The functions $focus$ and emb_C (restricted to sequents whose passive context is empty) establish a bijection between the right-normal sequent calculus and its focused subsystem.

The central design element of this focused calculus, the anteroom, together with the associating organization of the phases, is due to Chaudhuri and Pfenning [9].

In the right-normal focused calculus, the type (i) non-determinism in phase L between the $pass$ and $switch$ rules is present. But the type (ii) non-determinism in phase R concerning the split of the context at $\otimes R$ is inessential. Since the context cannot contain any closed formulae, at most one of the splits of the context into Γ, Δ can lead to a complete derivation.

In the free right-normal skew monoidal category, we have $id \doteq \rho \otimes \lambda \circ \alpha : (X \otimes I) \otimes Y \Longrightarrow (X \otimes I) \otimes Y$:

$$id \doteq \rho \otimes id \circ \rho^{-1} \otimes id \doteq \rho \otimes id \circ (id \otimes \lambda \circ \alpha \circ \rho \otimes id) \circ \rho^{-1} \otimes id \doteq (\rho \otimes \lambda) \circ \alpha$$

There is accordingly a single focused derivation of the sequent $(X \otimes I) \otimes Y \,|\, \;\; \vdots\, \;\; \longrightarrow_C (X \otimes I) \otimes Y$. Compare this with the two distinct derivations of the sequent in the fully skew focused calculus, displayed in (9). In the right-normal focused calculus, the unit is removed from the antecedent (more precisely, from the

anteroom) with an application of the IC rule, so the $\otimes$R does not have to choose in which premise to send it.

$$
\cfrac{
  \cfrac{
    \cfrac{X \mid \ \longrightarrow_R X}{X \mid \ \longrightarrow_R X \otimes I} \ \text{ax} \quad
    \cfrac{
      \cfrac{
        \cfrac{- \mid \ \longrightarrow_R I}{- \mid \ \longrightarrow_L I} \ \text{IR}
      }{- \mid \ \longrightarrow_L I} \ \text{switch}_{RL}
    }{\ } \ \otimes R \quad
    \cfrac{
      \cfrac{
        \cfrac{Y \mid \ \longrightarrow_R Y}{Y \mid \ \longrightarrow_L Y} \ \text{ax}
      }{- \mid Y \longrightarrow_L Y} \ \begin{matrix}\text{switch}_{RL}\\ \text{pass}\end{matrix}
    }{\ } \ \otimes R
  }{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{
              \cfrac{
                \cfrac{
                  \cfrac{X \mid Y \longrightarrow_R (X \otimes I) \otimes Y}{X \mid Y \longrightarrow_L (X \otimes I) \otimes Y} \ \text{switch}_{RL}
                }{X \mid \ \vdots\, Y \longrightarrow_C (X \otimes I) \otimes Y} \ \text{switch}_{LC}
              }{X \mid I \vdots\, Y \longrightarrow_C (X \otimes I) \otimes Y} \ \text{IC}
            }{X \otimes I \mid Y \longrightarrow_L (X \otimes I) \otimes Y} \ \otimes L
          }{X \otimes I \mid \ \vdots\, Y \longrightarrow_C (X \otimes I) \otimes Y} \ \text{switch}_{LC}
        }{X \otimes I \mid Y \vdots \ \longrightarrow_C (X \otimes I) \otimes Y} \ \text{move}
      }{(X \otimes I) \otimes Y \mid \ \longrightarrow_L (X \otimes I) \otimes Y} \ \otimes L
    }{(X \otimes I) \otimes Y \mid \ \vdots \ \longrightarrow_C (X \otimes I) \otimes Y} \ \text{switch}_{LC}
  }{\ }
}{\ }
$$

3.3 Associative-Normality

The free associative-normal skew monoidal category $\mathbf{Fsk}_{AN}(At)$ on a set At is obtained by extending the grammar of derivations in the fully skew categorical calculus (1) with a new inference rule:

$$
\frac{}{A \otimes (B \otimes C) \Longrightarrow (A \otimes B) \otimes C} \ \alpha^{-1}
$$

and extending the equivalence of derivations (2) with two new equations: $\alpha \circ \alpha^{-1} \doteq \text{id}$ and $\alpha^{-1} \circ \alpha \doteq \text{id}$.

The associative-normal sequent calculus is obtained by adding to (3) a new logical left rule for the tensor, relaxing the condition for decomposing a formula $A \otimes B$ in the antecedent:

$$
\frac{S \mid \Gamma_0, A, B, \Gamma_1 \longrightarrow C}{S \mid \Gamma_0, A \otimes B, \Gamma_1 \longrightarrow C} \ \otimes C_{\Gamma_0}
$$

Including the rule $\otimes C$ in the calculus makes it possible to derive the sequent corresponding to α^{-1} : $A \otimes (B \otimes C) \Longrightarrow (A \otimes B) \otimes C$:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{A \mid \ \longrightarrow A}{\ } \ \text{ax} \quad
      \cfrac{\cfrac{B \mid \ \longrightarrow B}{- \mid B \longrightarrow B} \ \begin{matrix}\text{ax}\\\text{pass}\end{matrix}}{\ }
    }{A \mid B \longrightarrow A \otimes B} \ \otimes R \quad
    \cfrac{\cfrac{C \mid \ \longrightarrow C}{- \mid C \longrightarrow C} \ \begin{matrix}\text{ax}\\\text{pass}\end{matrix}}{\ }
  }{
    \cfrac{
      \cfrac{
        \cfrac{A \mid B, C \longrightarrow (A \otimes B) \otimes C}{A \mid B \otimes C \longrightarrow (A \otimes B) \otimes C} \ \otimes C_{()}
      }{A \otimes (B \otimes C) \mid \ \longrightarrow (A \otimes B) \otimes C} \ \otimes L
    }{\ }
  }{\ } \ \otimes R
}{\ }
$$

We do not include here all the new equations that need to be added to the ones in (4) as generators of the least congruence $\overset{\circ}{=}$. They are obtained from the equalities in (10) (except for the 1st and 2rd) by replacing IC and $\otimes C^c$ with $\otimes C$ and the equation (11) by replacing $\otimes C^c$ with $\otimes C$.

The cut rules in (5) are also admissible in the associative-normal sequent calculus. In this case, they need to be defined by mutual induction with a third cut rule:

$$
\frac{S' \mid \Gamma \longrightarrow A \quad S \mid \Delta_0, A, \Delta_1 \longrightarrow C}{S \mid \Delta_0, [\![S'\langle\!\langle\, , \Gamma, \Delta_1 \longrightarrow C} \ \text{ccut}_{Stp}
$$

The definition of $\mathsf{ccut}_{\mathsf{Stp}}$ relies on an additional admissible rule that is a restricted form of the IC rule in the right-normal sequent calculus: a unit in the context can be removed, provided that the part of the context on its right is non-empty.

$$\frac{S \mid \Gamma_0, B, \Gamma_1 \longrightarrow C}{S \mid \Gamma_0, \mathsf{I}, B, \Gamma_1 \longrightarrow C} \; \mathsf{IC}^{\mathsf{c}}_{\Gamma_0}$$

The case of the function sound for the new inference rule $\otimes\mathsf{C}$ is defined as for the rule $\otimes\mathsf{C}^{\mathsf{c}}$ in the right-normal sequent calculus, given in (12), with the closed formulae J and J' replaced by arbitrary formulae, and the application α_{c}^{-1} replaced by α^{-1}.

We need not prove soundness for the rule IC^{c} since it is not primitive. But it is motivated by the presence in the fully skew categorical calculus already of a derivation $(\mathsf{id} \otimes \lambda) \circ \alpha : (A \otimes \mathsf{I}) \otimes B \Longrightarrow A \otimes B$ that, by (m2), post-inverts $\rho \otimes \mathsf{id}$.

An equivalent focused subsystem is obtained similarly to the right-normal case. It is in fact the same as the focused calculus in (13) with the rule IC removed, the rule $\otimes\mathsf{C}^{\mathsf{c}}$ replaced by $\otimes\mathsf{C}$ and the rule move modified accordingly. If the rightmost formula D in the anteroom is a tensor $A \otimes B$, then it is decomposed and B is inspected next. Otherwise, D is moved to the left end of the passive context.

$$\frac{S \mid \Omega, A, B \vdots \Gamma \longrightarrow_{\mathsf{C}} C}{S \mid \Omega, A \otimes B \vdots \Gamma \longrightarrow_{\mathsf{C}} C} \; \otimes\mathsf{C} \qquad \frac{S \mid \Omega \vdots D, \Gamma \longrightarrow_{\mathsf{C}} C \quad D \neq A \otimes B}{S \mid \Omega, D \vdots \Gamma \longrightarrow_{\mathsf{C}} C} \; \mathsf{move}$$

All C-phase, L-phase and R-phase sequents in a derivation of a sequent $S \mid \Omega \vdots \; \longrightarrow_{\mathsf{C}} C$ have the (passive) context free of formulae $A \otimes B$.

Similarly to the right-normal case discussed in Section 3.2, the associative-normal sequent calculus can be proved equivalent to its focused subsystem by means of two functions $\mathsf{emb}_{\mathsf{C}}$ and focus.

Lack and Street [15] observed that the free associative-normal skew monoidal category on one generator $\mathbf{Fsk}_{\mathsf{AN}}(1)$ is isomorphic to $\Delta_\perp$, the category of finite non-empty ordinals and first-element-and-order-preserving functions, and their main coherence theorem states that the canonical functor $\mathbf{Fsk}(1) \to \Delta_\perp$ is faithful. This seems to imply that the embedding of the fully skew sequent calculus into the associative-normal sequent calculus is faithful, but we leave a direct proof of this fact to future work.

3.4   Multiple Normality Conditions

The additional inference rules and equations for the three normality aspects can be enabled simultaneously, to yield, e.g., a sequent calculus for the free simultaneous left-normal and right-normal skew monoidal category.

The following rules define a single focused sequent calculus that can handle any combination of the three normality aspects. It is parameterized in three flags ln, rn and an for left-, right- resp. associative normality. (Recall that J, J' are metavariables for closed formulae.)

$$\frac{S \mid \Omega \vdots \Gamma \longrightarrow_{\mathsf{C}} C \quad rn}{S \mid \Omega, \mathsf{I} \vdots \Gamma \longrightarrow_{\mathsf{C}} C} \; \mathsf{IC} \qquad \frac{S \mid \Omega, A, B \vdots \Gamma \longrightarrow_{\mathsf{C}} C \quad (rn \wedge A \otimes B = J \otimes J') \vee an}{S \mid \Omega, A \otimes B \vdots \Gamma \longrightarrow_{\mathsf{C}} C} \; \otimes\mathsf{C}$$

$$\frac{S \mid \Omega \vdots D, \Gamma \longrightarrow_{\mathsf{C}} C \quad rn \to D \neq J \quad an \to D \neq A \otimes B}{S \mid \Omega, D \vdots \Gamma \longrightarrow_{\mathsf{C}} C} \; \mathsf{move} \qquad \frac{S \mid \Gamma \longrightarrow_{\mathsf{L}} C}{S \mid \; \vdots \Gamma \longrightarrow_{\mathsf{C}} C} \; \mathsf{switch}_{\mathsf{LC}}$$

$$\frac{A \mid \Gamma \longrightarrow_{\mathsf{L}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{L}} C} \; \mathsf{pass} \qquad \frac{- \mid \Gamma \longrightarrow_{\mathsf{L}} C}{\mathsf{I} \mid \Gamma \longrightarrow_{\mathsf{L}} C} \; \mathsf{IL} \qquad \frac{A \mid B \vdots \Gamma \longrightarrow_{\mathsf{C}} C}{A \otimes B \mid \Gamma \longrightarrow_{\mathsf{L}} C} \; \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} C \quad ln \wedge T = - \to \Gamma = ()}{T \mid \Gamma \longrightarrow_{\mathsf{L}} C} \; \mathsf{switch}_{\mathsf{RL}}$$

$$\frac{}{X \mid \; \longrightarrow_{\mathsf{R}} X} \; \mathsf{ax} \qquad \frac{}{- \mid \; \longrightarrow_{\mathsf{R}} \mathsf{I}} \; \mathsf{IR} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} A \quad - \mid \Delta \longrightarrow_{\mathsf{L}} B}{T \mid \Gamma, \Delta \longrightarrow_{\mathsf{R}} A \otimes B} \; \otimes\mathsf{R} \qquad \frac{- \mid \; \longrightarrow_{\mathsf{R}} A \quad X \mid \Delta \longrightarrow_{\mathsf{R}} B \quad ln}{X \mid \Delta \longrightarrow_{\mathsf{R}} A \otimes B} \; \otimes\mathsf{R}_2$$

In the case of simultaneous left- and right-normality (i.e., $ln \wedge rn$), the non-determinism of type (i) in phase L is not present and the non-determinism of type (ii) in phase R is inessential as there cannot be any closed formulae in the context. Consequently, any sequent can have at most one focused derivation: the free simultaneously left- and right-normal skew-monoidal category is thin.

Laplaza [18] proved that the free skew semigroup category is thin (and this was reproved by Zeilberger [27]). We have now seen that this remains true also when a both left-normal and right-normal unit is freely added.

4 Conclusions and Future Work

We showed that, similarly to the free skew monoidal category and the free monoidal category, the free skew monoidal categories of different degrees of partial normality can be described as sequent calculi. These sequent calculi define logics weaker than the multiplicative fragment of the intuitionistic non-commutative linear logic. They enjoy cut elimination and they also admit focusing, a deductive description of a root-first proof search strategy that finds exactly one representative from each equivalence class of derivations.

We intend to continue this study by broadening its scope to fully skew and partially normal closed and monoidal closed categories and also prounital-closed categories (where the unit is present in a non-represented way). Skew closed categories (the skew variant of closed categories of Eilenberg and Kelly [10]) were first considered by Street [21], prounital-closed categories by Shulman [20, Rev. 49]. Zeilberger [26] used a thin variant of skew closed categories, which he called imploids, in his study of the relation between typing of linear lambda terms and flows on 3-valent graphs. In the recent work [25], we dissected the Eilenberg-Kelly theorem about adjoint monoidal and closed structures on a category, revisited by Street [21] for the skew situation, to establish it in the general partially normal case. Some surprising phenomena occur around skew closed categories, e.g., the free skew closed category on a set is left-normal, but this is lost when the tensor is added. We also want to study the proof theory of skew braided monoidal categories, as recently introduced by Bourke and Lack [7].

We have in this paper explained how the free skew monoidal category of each possible degree of partial normality can be described as a sequent calculus (a "logic"). This correspondence extends to non-free partially normal monoidal categories. But in this case, rather than inductively defining the maps and their equality, the inference rules and equations of the sequent calculus merely impose closure conditions on some homset predicate and some equality relation given upfront. One could compare a non-free category to a "theory" (a set of judgments closed under some inference rules) as opposed to a "logic" (the least set of judgments closed under them, i.e., the set of derivable judgments). Cut elimination (in the sense that a set of judgments closed under the inference rules adopted minus cut would necessarily be closed also under cut) cannot be expected, neither can focusing. Bourke and Lack [6] showed that skew monoidal categories are equivalent to representable skew multicategories, a weakening of representable multicategories [11]. In our previous work [24], we showed that the map constructors and equations of a (nullary-binary) representable skew multicategory are very close to and mutually definable with those of the sequent calculus for the corresponding skew monoidal category (viewed as a deductive calculus, the representable skew multicategory uses exactly the same sequent forms, but has the basic inference rules and equations chosen differently). We expect that partially normal skew monoidal categories can be analyzed in similar terms. Specifically, we hope that the correct variations of representable skew multicategories can be systematically derived in the framework of (op)fibrations of multicategories [12], adapted for skew multicategories.

References

[1] V. M. Abrusci (1990): *Noncommutative intuitionistic linear logic*. *Math. Log. Quart.* 36(4), pp. 297–318, doi: 10.1002/malq.19900360405.

[2] T. Altenkirch, J. Chapman & T. Uustalu (2015): *Monads need not be endofunctors*. *Log. Methods Comput. Sci.* 11(1), article 3, doi: 10.2168/lmcs-11(1:3).

[3] J.-M. Andreoli (1992): *Logic programming with focusing proofs in linear logic*. *J. of Log. and Comput.* 2(3), pp. 297–347, doi: 10.1093/logcom/2.3.297.

[4] J. Bénabou (1963), *Catégories avec multiplication*. *C. R. Acad. Sci. Paris* 256, pp. 1887–1890. Available at http://gallica.bnf.fr/ark:/12148/bpt6k3208j/f1965.image.

[5] J. Bourke & S. Lack (2018): *Free skew monoidal categories*. *J. Pure Appl. Alg.* 222(10), pp. 3255–3281, doi: 10.1016/j.jpaa.2017.12.006.

[6] J. Bourke & S. Lack (2018): *Skew monoidal categories and skew multicategories*. *J. Alg.* 506, pp. 237–266, doi: 10.1016/j.jalgebra.2018.02.039.

[7] J. Bourke & S. Lack (2020): *Braided skew monoidal categories*. *Theor. Appl. Categ.*, v. 35, n. 2, pp. 19–63. Available at http://www.tac.mta.ca/tac/volumes/35/2/35-02abs.html.

[8] M. Buckley, R. Garner, S. Lack & R. Street (2014): *The Catalan simplicial set*. *Math. Proc. Cambridge Philos. Soc.* 158(12), pp. 211–222, doi: 10.1017/s0305004114000498.

[9] K. Chaudhuri & F. Pfenning (2005): *Focusing the inverse method for linear logic*. In L. Ong (ed.), *Proc. of 19th Int. Wksh. on Computer Science Logic, CSL 2005*, Lect. Notes in Comput. Sci. 3634, Springer, pp. 200–215, doi: 10.1007/11538363_15.

[10] S. Eilenberg & G. M. Kelly (1966): *Closed categories*. In S. Eilenberg, D. K. Harrison, S. Mac Lane & H. Röhl (eds.), *Proc. of Conf. on Categorical Algebra (La Jolla, 1965)*, Springer, pp. 421–562, doi: 10.1007/978-3-642-99902-4_22.

[11] C. Hermida (2000): *Representable multicategories*. *Adv. Math.* 151(2), pp. 164–225, doi: 10.1006/aima.1999.1877.

[12] C. Hermida (2004): *Fibrations for Abstract Multicategories*. In G. Janelidze, B. Pareigis & W. Tholen, eds., *Galois Theory, Hopf Algebras and Semiabelian Categories*, Fields Inst. Commun. 43, Amer. Math. Soc., pp. 281–293.

[13] G. M. Kelly (1964): *On MacLane's conditions for coherence of natural associativities, commutativities, etc.* *J. Alg.* 1(4), pp. 397–402, doi: 10.1016/0021-8693(64)90018-3.

[14] S. Lack & R. Street (2012): *Skew monoidales, skew warpings and quantum categories*. *Theor. Appl. Categ.* 26, pp. 385–402. Available at http://www.tac.mta.ca/tac/volumes/26/15/26-15abs.html.

[15] S. Lack & R. Street (2014): *Triangulations, orientals, and skew monoidal categories*. *Adv. Math.* 258, pp. 351–396, doi: 10.1016/j.aim.2014.03.003.

[16] J. Lambek (1958): *The mathematics of sentence structure*. *Amer. Math. Monthly* 65(3), pp. 154–170, doi: 10.2307/2310058.

[17] J. Lambek (1961): *On the calculus of syntactic types*. In R. Jakobson (ed.), *Structure of Language and its Mathematical Aspects* 12, Amer. Math. Soc., pp. 166–178.

[18] M. L. Laplaza (1972): *Coherence for associativity not an isomorphism*. J. Pure Appl. Alg. 2(2), pp. 107–120, doi: 10.1016/0022-4049(72)90016-3.

[19] S. Mac Lane (1963): *Natural associativity and commutativity*. Rice Univ. Stud. 49(4), pp. 28–46. Available at `http://hdl.handle.net/1911/62865`.

[20] U. Schreiber, M. Shulman et al. (2009): Closed categories. ncatlab article. (Rev. 49 was by M. Shulman, May 2018. Current version is rev. 61 from Jan. 2020) `https://ncatlab.org/nlab/show/closed+category`

[21] R. Street (2013): *Skew-closed categories*. J. Pure Appl. Alg. 217(6), pp. 973–988, doi: 10.1016/j.jpaa.2012.09.020.

[22] K. Szlachányi (2012): *Skew-monoidal categories and bialgebroids*. Adv. Math. 231(3–4), pp. 1694–1730, doi: 10.1016/j.aim.2012.06.027.

[23] T. Uustalu (2014): *Coherence for skew-monoidal categories*. In P. Levy, N. Krishnaswami (eds.), *Proc. of 5th Wksh. on Mathematically Structured Programming, MSFP 2014, Electron. Proc. in Theor. Comput. Sci.* 153, Open Publishing Assoc., pp. 68–77, doi: 10.4204/eptcs.153.5.

[24] T. Uustalu, N. Veltri & N. Zeilberger (2018): *The sequent calculus of skew monoidal categories*. Electron. Notes Theor. Comput. Sci. 341, pp. 345–370. doi: 10.1016/j.entcs.2018.11.017. (Extended version to appear in C. Casadio & P. Scott (eds.), *[Joachim Lambek Memorial Volume], Outstanding Contributions to Logic*, Springer.)

[25] T. Uustalu, N. Veltri & N. Zeilberger (to appear): *Eilenberg-Kelly reloaded*. Electron. Notes Theor. Comput. Sci.

[26] N. Zeilberger (2018): *A theory of linear typings as flows on 3-valent graphs*. In *Proc. of 33rd Ann. ACM/IEEE Symp. on Logic in Computer Science, LICS '18*, ACM, pp. 919–928, doi: 10.1145/3209108.3209121.

[27] N. Zeilberger (2019): *A sequent calculus for a semi-associative law*. Log. Methods Comput. Sci. 15(1), article 9, doi: 10.23638/lmcs-15(1:9)2019.