# Skolemization in Nonclassical Logic

Computer Science Theory Seminar

Tallinn University of Technology

Zoom, 8 May 2023

Rosalie Iemhoff

Utrecht University

# Skolemization

The connection between quantifier combination $\forall\exists$ and functions appears at many places:

- axiom of choice:
  $\forall x \in a \exists y \in x \to \exists (f : a \to \bigcup a)\, \forall x \in a (fx \in x)$.
- constructive mathematics: (for quantifier-free $\varphi$)
  if $HA \vdash \forall x \exists y \varphi(x, y)$, then $HA \vdash \forall x \varphi(x, fx)$ for some computable $f$.

$$\vdots$$

Constructive interpretation of quantifiers:

$$\vdash \forall x \in A \exists y \in B\, \varphi(x, y)$$
$$\Leftrightarrow$$
$$\vdash \forall x \in A\, \varphi(x, fx) \text{ for some function } f : A \to B.$$

The study of the constructive content of such quantifier combinations is pursued in contructive mathematics and proof mining.

In classical logic the following holds for the dual quantifier combination:

$$\vdash \exists x \in A \forall y \in B\, \varphi(x, y)$$
$$\Leftrightarrow$$
$$\vdash \exists x \in A\, \varphi(x, fx) \text{ for some function } f : A \to B \text{ not in } \varphi.$$

It can be understood via countermodels:

$$M \vDash \forall x \in A \exists y \in B\, \neg\varphi(x, y) \text{ for some model } M$$
$$\Leftrightarrow$$
$$N \vDash \forall x \in A \neg\varphi(x, fx) \text{ for some model } N \text{ and function } f : A \to B \text{ not in } \varphi.$$

Or via satisfiability:

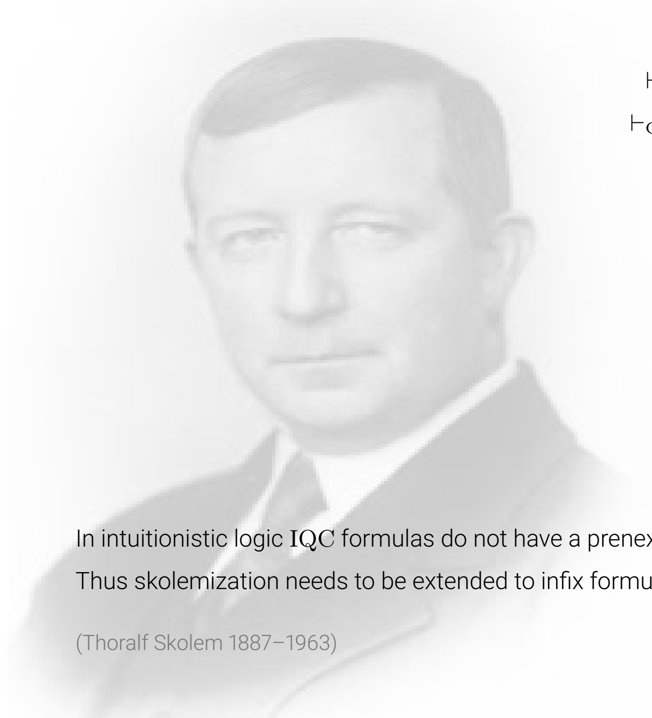
$$\forall x \in A \exists y \in B\, \neg\varphi(x, y) \text{ is satisfiable}$$
$$\Leftrightarrow$$
$$\forall x \in A \neg\varphi(x, fx) \text{ is satisfiable for some function } f : A \to B \text{ not in } \varphi.$$

The dual quantifier combination ∃∀ appears in the skolemization method for theories in classical predicate logic $\mathrm{CQC}$.

Thm For any function symbols $f, g$ not in $\varphi$ ( $\vdash$ short for $\vdash_{\mathrm{CQC}}$, $fx$ for $f(x)$ ):

$$\begin{aligned}
\vdash \exists x \forall y \varphi(x,y) \quad &\Leftrightarrow \quad \vdash \exists x \varphi(x, fx)\\
\vdash \exists x \forall y \exists u \forall v \varphi(x,y,u,v) \quad &\Leftrightarrow \quad \vdash \exists x \exists u \varphi(x, f(x), u, g(x,u))\\
\vdash \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \varphi(\bar{x}, \bar{y}) \quad &\Leftrightarrow \quad \vdash \exists x_1 \ldots \exists x_n \varphi\big(\bar{x}, f_1(x_1), \ldots, f_n(x_1, \ldots, x_n)\big).
\end{aligned}$$

This talk: Skolemization in intermediate logics.



Ex In $\mathrm{CQC}$:

$$\begin{aligned}
\vdash_{\mathrm{CQC}} \neg\exists x \forall y \varphi(x,y) \ &\Leftrightarrow \\
\vdash_{\mathrm{CQC}} \forall x \exists y \neg\varphi(x,fx) \ &\Leftrightarrow \\
\vdash_{\mathrm{CQC}} \forall x \neg\varphi(x,fx) \ &\Leftrightarrow \\
\vdash_{\mathrm{CQC}} \neg\exists x \varphi(x,fx)&.
\end{aligned}$$

In intuitionistic logic $\mathrm{IQC}$ formulas do not have a prenex normal form.

Thus skolemization needs to be extended to infix formulas.

(Thoralf Skolem 1887–1963)

Def The *strong* quantifier occurrences are positive occurrences of $\forall$ and negative occurrences of $\exists$. All other quantifier occurrences are *weak*.

Strong quantifiers become universal under prenexification, and weak quantifiers become existential.

Ex Strong quantifier occurrences are red and weak occurrences are green:

$$\forall x \varphi(x) \vee \exists y \psi(y) \to \forall x \psi(x) \vee \exists y \varphi(y)$$
$$(\forall x \varphi(x) \to \forall y \psi(y)) \to \neg\neg\forall x \psi(x) \wedge \neg\exists y \varphi(y)$$

The skolemization of infix formulas replaces the strong quantifiers on the spot (def next slide).

Ex

| formula | skolemization |
|---|---|
| $\forall x \varphi(x) \to \forall x \varphi(x)$ | $\forall x \varphi(x) \to \varphi(c)$ |
| $\exists x \big(\exists u \varphi(x,u) \to \exists y \forall v \psi(x,v)\big)$ | $\exists x \big(\varphi(x,fx) \to \exists y \psi(x, g(x,y))\big)$. |

Def The *skolemization* $\varphi^s$ of $\varphi$ is the result of replacing (from inside out) the occurrences of strong quantifiers $Qy\psi(\bar{x},y)$ in $\varphi$ by $\psi(\bar{x},f(\bar{x}))$, where $f$ is fresh and $\bar{x}$ are the variables of the weak quantifiers in which scope $Qy\psi(\bar{x},y)$ occurs.

A logic $\mathrm{L}$ *admits* skolemization if for all formulas $\varphi$: $\vdash_{\mathrm{L}} \varphi$ if and only if $\vdash_{\mathrm{L}} \varphi^s$

Thm $\mathrm{CQC}$ admits skolemization: $\vdash_{\mathrm{CQC}} \varphi$ if and only if $\vdash_{\mathrm{CQC}} \varphi^s$.

Def The *skolem Class* of a theory $\mathrm{T}$: $SC(\mathrm{T}) \equiv_{df} \{\varphi \mid \vdash_{\mathrm{T}} \varphi \Leftrightarrow \vdash_{\mathrm{T}} \varphi^s\}$.

Equivalently:

$$SC(\mathrm{T}) = \{\varphi \mid \vdash_{\mathrm{T}} \varphi \text{ and } \vdash_{\mathrm{T}} \varphi^s, \text{ or } \nvdash_{\mathrm{T}} \varphi \text{ and } \nvdash_{\mathrm{T}} \varphi^s,\}.$$

Note SC($\mathrm{CQC}$) consists of all formulas. SC($\mathrm{IQC}$) does not.

In $\mathrm{CQC}$ we have:

Thm $\vdash \varphi \Leftrightarrow \vdash \varphi^s$.

Thm (Herbrand's Theorem) $\vdash \exists x \varphi(x) \Leftrightarrow \vdash \bigvee_{i=1}^n \varphi(t_i)$ for some terms $t_i$.

In combination: $\vdash \exists x \forall y \varphi(x, y) \Leftrightarrow \vdash \bigvee_{i=1}^k \varphi\big(x, f(t_i)\big)$ for some terms $t_i$.

Likewise for longer prefixes of quantifiers.

Applications: Computational content of proofs, automated theorem proving, connection propositional and predicate logic.

Question: How about other intermediate predicate logics/theories?

Def An intermediate predicate logic/theory is a logic/theory that is an extension of intuitionistic predicate logic $\mathrm{IQC}$.

# Nonclassical theories

For many intermediate predicate theories $\mathrm{T}$ no skolemization theorem:

$$\vdash_{\mathrm{T}} \varphi \;\Rightarrow\; \vdash_{\mathrm{T}} \varphi^s \qquad \vdash_{\mathrm{T}} \varphi \;\cancel{\Leftarrow}\; \vdash_{\mathrm{T}} \varphi^s.$$

Ex The $\mathrm{CD}$ formula in $\mathrm{IQC}$:

$$\nvdash_{\mathrm{IQC}} \forall x(\varphi(x) \vee \psi) \to \forall x \varphi(x) \vee \psi \qquad \vdash_{\mathrm{IQC}} \forall x(\varphi(x) \vee \psi) \to \varphi(c) \vee \psi$$

Thus $\mathrm{CD} \notin SC(\mathrm{IQC})$.

In many intermediate predicate logics the (generalized) Herbrand theorem holds:

For any formula $\varphi$ with only weak quantifiers there exists an Herbrand expansion $\varphi^H$ of $\varphi$ that is quantifier free and $\vdash_{\mathrm{T}} \varphi \;\Leftrightarrow\; \vdash_{\mathrm{T}} \varphi^H$.

- For which (intermediate) logics is skolemization complete?
- What is the skolem Class of a given logic?

Mostly partial answers.

# Results

Last two decades: various results on skolemization in nonclassical logics:

- A sufficient condition on formulas for belonging to the skolem Class of $\mathrm{IQC}$. (Mints 1970s)
- The prenex fragment belongs to the skolem Class of wide range of first-order fuzzy logics. (Baaz & Ciabattoni & Fermüller 2001), (Baaz & Metcalfe 2010)
- First-order Łukasiewicz logic admits skolemization. (Baaz & Metcalfe 2010)

  Certain formula classes belong to the the skolem Class of first-order substructural logics. (Cintula & Metcalfe 2013)
- IQCE admits an alternative skolemization method. (Baaz & Iemhoff 2006).
- There is a labelled version of $\mathrm{IQC}$ that admits skolemization. (Baaz & Iemhoff 2008).
- and many many more …

Rosalie to Matthias in 2017:
All prenex formulas are in the skolem Class of $\mathrm{IQC}$, trivial.


First part right, second part wrong.

Thm For any well-founded tree-complete intermediate logic, any prenex formula belongs to the skolem Class of the logic. Proof Nontrivial.

Question: For which propositional formulas $A(p_1, \ldots, p_n)$ does $A(\varphi_1, \ldots, \varphi_n)$ belong to the skolem Class of a logic for any prenex formulas $\varphi_1, \ldots, \varphi_n$?

Ex For $\mathrm{IQC}$:

Positive answer: $p_1$ and $p_1 \wedge p_2$.

Negative anwer: $p_1 \to \neg\neg p_2$
$\exists x \neg\neg\varphi(x) \to \neg\neg\exists x \varphi(x)$ not in $SC(\mathrm{IQC})$ for quantifier free $\varphi$.

Negative answer: $p_1 \to p_2 \vee p_3$
$\forall x(\varphi(x) \vee \psi) \to (\forall x \varphi(x) \vee \psi)$ not in $SC(\mathrm{IQC})$ for quantifier free $\varphi, \psi$.

Note $A, B, C$ for propositional formulas, $\varphi, \psi, \chi$ for predicate formulas.

Def An intermediate logic is *well-founded tree-complete* if it is complete with respect to a class of well-founded trees.

Ex Any intermediate logics with the finite frame property is well-founded tree-complete. $\mathrm{IQC}$, $\mathrm{KC}$, $\mathrm{LC}$ are examples.

Def $A$ is *rigid*, i.e. no atom occurs twice in $A$. $A$ is a *nni* formula: no implication occurs negatively in it.

Ex $\bigvee_i \bigwedge_j (\bigwedge_k p_{ijk} \to \bigvee_l q_{ijl})$ is rigid.

Def $\varphi_1, \ldots, \varphi_n$ independent if no predicate occurs in more than one $\varphi_i$.

Thm (I. 2018) In any well-founded tree-complete intermediate logic $\mathrm{L}$, for any rigid nni formula $A(p_1, \ldots, p_n)$, for all independent prenex formulas $\varphi_1, \ldots, \varphi_n$: $A(\varphi_1, \ldots, \varphi_n) \in SC(\mathrm{L})$.

The proof uses skolemization and its dual, i.e. skolemization for derivability and satisfiability.

Thm (I. 2018) In any well-founded tree-complete intermediate logic $\mathrm{L}$, for any rigid nni formula $A(p_1, \dots, p_n)$, for all independent prenex formulas $\varphi_1, \dots, \varphi_n$: $A(\varphi_1, \dots, \varphi_n) \in SC(\mathrm{L})$.

Ex For all independent $\varphi_1, \dots, \varphi_n$: $(\varphi_1 \wedge \varphi_2 \to \varphi_3 \vee \varphi_4) \in SC(\mathrm{IQC})$.

Note Recall that

$$\forall x(\varphi(x) \vee \psi) \to \forall x \varphi(x) \vee \psi \notin SC(\mathrm{IQC}).$$

That $p_1 \to p_2 \vee p_3$ is a rigid nni formula does not contradict the theorem, as $\forall x(\varphi(x) \vee \psi), \forall x \varphi(x), \psi$ are not independent.

Whether the theorem holds for $A(p_1, p_2, p_3) = (p_1 \to p_2 \vee p_3)$: open.

Proof Let $\big(A(\varphi_1,\ldots,\varphi_n)\big)^s = A(\psi_1,\ldots,\psi_n)$. Given $M \not\Vdash A(\varphi_1,\ldots,\varphi_n)$, create $N \not\Vdash \big(A(\varphi_1,\ldots,\varphi_n)\big)^s$.

$$
\begin{tikzcd}
\varphi_1 & & \bullet & \\
& \varphi_3 \arrow[ul] \arrow[ur] & & \varphi_2 \\
& M & \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\psi_1 & & \bullet & \\
& \psi_3 \arrow[ul] \arrow[ur] & & \psi_2 \\
& N & \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$
\psi_i = \begin{cases} (\varphi^i)^s & \text{if } p_i \text{ occurs positively in } A \\ (\overline{\varphi}^i)^s & \text{if } p_i \text{ occurs negatively in } A, \end{cases}
$$

$\overline{\varphi}$ results from swapping the quantifiers of $\varphi$.

We have seen some partial answers (from the literature and this talk) to the questions:

- For which (intermediate) logics is skolemization complete?
- What is the skolem Class of a given logic?

# Alternative methods

Def An *alternative skolemization method* is a computable total translation $(\cdot)^a$ from formulas to formulas such that for all formulas $\varphi$, $\varphi^a$ does not contain strong quantifiers. A logic $\mathrm{L}$ *admits* the alternative skolemization method if

$$\vdash_{\mathrm{L}} \varphi \;\Leftrightarrow\; \vdash_{\mathrm{L}} \varphi^a. \tag{1}$$

The method is *strict* if for all Kripke models $K$ of $\mathrm{L}$ and all formulas $\varphi$:

$$K \nVdash \varphi^a \;\Rightarrow\; K \nVdash \varphi. \tag{2}$$

Motivation: Alternative skolemization methods preserve the connection between predicate and propositional logic in combination with Herbrand's theorem.

Note In many intermediate predicate logics the (generalized) Herbrand theorem holds: For any formula $\varphi$ with only weak quantifiers there exists an Herbrand expansion $\varphi^H$ of $\varphi$ that is quantifier free and $\vdash_{\mathrm{T}} \varphi \;\Leftrightarrow\; \vdash_{\mathrm{T}} \varphi^H$.

Def A logic has *width* $n$ if it is complete with resprect to a class of models that have no anti-chains of lentgh $> n$.

Def Given a number $n$, *parallel skolemization* replaces strong quantifiers $\exists x \psi(x, \bar{y})$ and $\forall x \psi(x, \bar{y})$ by, respectively,

$$\bigvee_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}) \text{ and } \bigwedge_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}).$$

Thm (Baaz&Iemhoff 2016) Any intermediate logic of finite width with constant domains admits parallel skolemization.

Cor Any tabular constant domain logic admits parallel skolemization.

From later work: it probably holds for intermediate logics with the fmp and constant domains.

Thm (I. 2017) Except for $\mathrm{CQC}$, there is no Kripke complete intermediate logic that admits a strict alternative skolemization method.

Cor The intermediate logics $\mathrm{IQC}$,

- QDn (the logic of frames of branching at most $n$),
- QKC (the logic of frames with one maximal node),
- QLC (the logic of linear frames),

and all tabular logics, do not admit any strict, alternative skolemization method.

Def The *strong quantifier free fragment (sqff)* of a logic consists of those theorems of the logic that do not contain strong quantifiers, and likewise for weak quantifiers.

Thm (I. 2017) Except for $\mathrm{CQC}$, there is no intermediate logic that is complete with respect to a class of frames and admits a strict alternative skolemization method.

Proof idea Let $\mathrm{L}$ be an intermediate logic that is complete with respect to a class of frames and let $\mathcal{K}$ be the class of models based on these frames. Let $(\cdot)^a$ be the alternative skolemization method.

Claim $\mathrm{L}$ is sound and complete with respect to $\mathcal{K}_{cd}$ the class of all models in $\mathcal{K}$ with constant domain.

Proof If $\not\vdash \varphi$, then $\not\vdash \varphi^a$. Thus $K \not\Vdash \varphi^a$ for some $K \in \mathcal{K}$. Because $\varphi^a$ no strong qfs, $K^\downarrow \not\Vdash \varphi^a$, which is $K$ in which every domain is replaced by that at the root.

$K^\downarrow \in \mathcal{K}_{cd}$, which proves the claim. $\dashv$

So $\mathrm{L}$ derives the constant domain formula $(\mathrm{CD})$ $\forall x(\varphi(x) \vee \psi) \rightarrow \forall x \varphi(x) \vee \psi$.

As $\mathrm{L} \neq \mathrm{CQC}$, there is at least one frame of $\mathrm{L}$ of depth $> 1$. On such a frame $\mathrm{CD}$ can be refuted. A contradiction.

$\dashv$

- Different work skolemization e.g. Avigad 2003 on shortening proofs.
- With Raheleh Jalali: connection between quantifier shifts and skolemization.
- Do requirements on skolem functions lead to more logics admitting the method?
- Are there useful alternatives to alternative skolemization methods?

$\vdots$

Finis