GLOBAL DYNAMICS

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Chapter 1

Introduction

1.1 Introduction

Sample citation of Richardson (1922).

The atmosphere is in motion and it is a continuous mixing and clashing of vortices and structures, but when it is averaged over a long period of time (Figure 1.1) it shows a remarkable simple structure. The figure shows the wind at an approximate level of about 12km, that is considered to be in the free atmosphere, far from the influence of the ground. The circulation is a large vortex around the pole that shows small oscillation in latitudes, especially pronounced over North America and the Asian Pacific Coast. The flow is therefore predominantly in the East-West direction, with a relatively small component in the meridional direction. The *circumpolar vortex* has been one of the first structures to be recognized when plentiful observations of the upper air flow became available, but it provided some of the intriguing questions that drove the development of geophysical fluid dynamics to this time. some of them have not been completely understood. What is maintaining this peculiar circulation ? Which factor determines the amplitude and location of the undulations in meridional direction ? Some of these question will be addressed in these notes

1.1.1 Coordinate systems

Spherical Coordinates

The most commonly used coordinate system for the analysis of the atmosphere and the oceans is a spherical coordinate system attached to the rotating Earth (Fig. `fig:0`). The spherical coordinates are slightly different from the usual mathematical ones as the latitude is measured from the equator and therefore it can take negative values. The longitude is running west to east.

The longitude is also known as the "zonal" direction whereas the latitude is also known as the "meridional" direction. Winds are identified by the direction they are coming from, so a "westerly" wind is coming *from* the West and an "easterly" wind is coming from the East.

This coordinate system is rotating with the Earth and therefore it generates force terms in any dynamical equation expressed in this system of coordinate, the Coriolis terms.

The Beta-plane

It is sometimes convenient to shift coordinate system if the latitudinal extension of the motion is not too great with respect to the motion parameters as they are expressed in the adimensional numbers. When this is possible, a tangent coordinate system is applied at a specific latitude ϕ_0 and the resultant Cartesian coordinates system is called the β-plane. Usually symbols (x, y) are used in this case for the zonal and meridional coordinate. In the β-plane the planetary vorticity f is linearized as $f = f_0 + \beta y$, where $\beta = \frac{\partial f}{\partial y}(\phi_0)$.

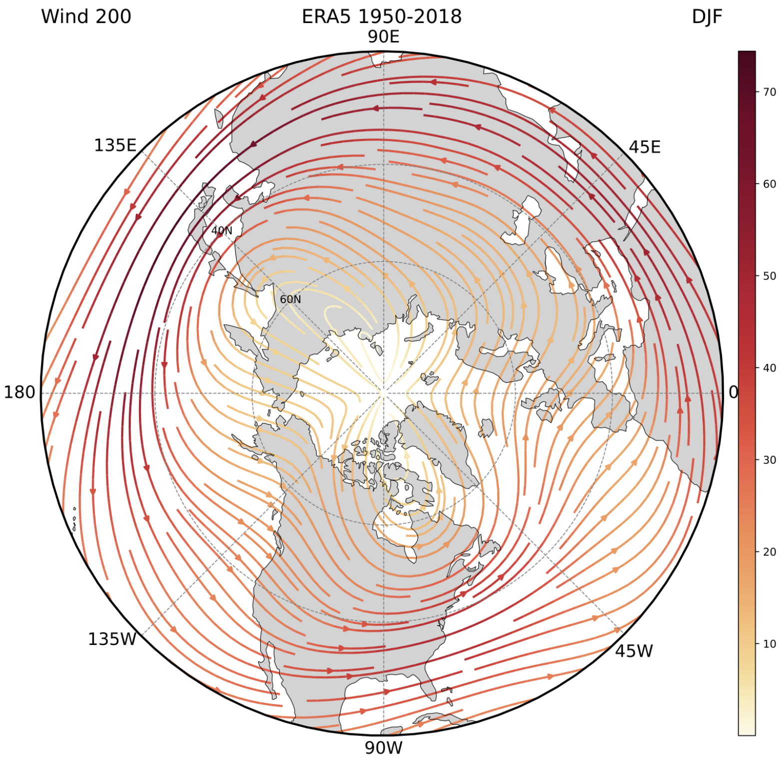

Figure 1.1:

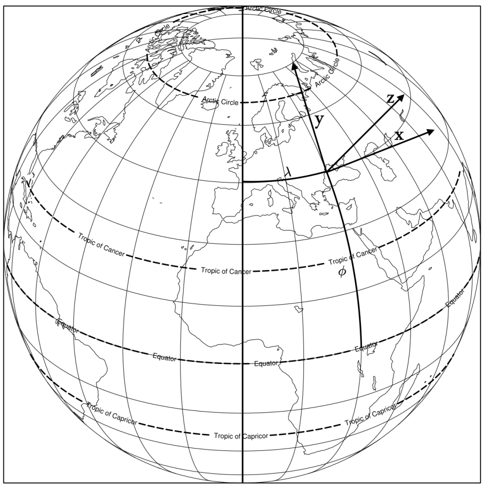

Figure 1.2: Coordinate System

1.1.2 Advective derivative

To describe the governing equation of the atmosphere and eventually of the ocean we have to understand how we write the rate of change with time of this fluid. This problem was solved by considering the fact that the rate of change in the fluid cannot be seen as arate of change with respect to a fixed system of coordinate because the system is moving with the fluid itself. Therefore, first we have to find a way to describe the change taking into account the moving system of restaurants. These can be done by using a concept developed in the 19th century by Euler that is called "advective derivative" that can be obtained from a total derivative of the property,

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial\phi}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi$$

and so it can be defined as

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\nabla\varphi$$

in this way the moving fluid can be described by derivatives with respect the "fixed" coordinate system, i.e. the Eulerian description. The alternative description of the observer moving with fluid is known as the "Lagrangian" description.

1.1.3 Primitive Equations

The equation governing the motion of the atmosphere can be written as:

$$\begin{aligned}
\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} &= -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + fv - \hat{f}w + F_\lambda \\
\frac{Dv}{Dt} - \frac{u^2\tan\phi}{r} + \frac{vw}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial\phi} - fu + F_\phi \\
\frac{Dw}{Dt} - \frac{u^2+v^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \hat{f}u + F_z
\end{aligned}$$

the $f = 2\Omega\sin\phi$ and $\hat{f} = 2\Omega\cos\phi$ terms arise from the rotating spherical coordinate system that we have chosen, other terms are generated by the spherical geometry. Some of them are small and traditionally they can be neglected, so that we arrive at the system

$$\begin{aligned}
\frac{Du}{Dt} - v\left(f + \frac{u\tan\phi}{a}\right) &= -\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} + F_\lambda \\
\frac{Dv}{Dt} + u\left(f + \frac{u\tan\phi}{a}\right) &= -\frac{1}{\rho a}\frac{\partial p}{\partial\phi} + F_\phi \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z
\end{aligned}$$

where we have also used the *Shallowness Approximation* by assuming $r = a + z \approx a$, where a is the Earth radius.

However the advective derivative must be expressed in spherical cordinates

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}$$

so that the velocity components are

$$\begin{aligned}
u &= a\cos\phi\frac{\partial\lambda}{\partial t} \\
v &= a\frac{\partial\phi}{\partial t} \\
w &= \frac{\partial z}{\partial t}
\end{aligned}$$

These equation govern the mechanical behaviour of the atmosphere, and we will see in a different form, also of the ocean. There three forces in action: pressure gradient, rotaiton via the Coriolis force and gravity.

The equation are not complete,we have three equation but five variables, so we need to find the missing relations. We are using the basic conservation principles, the latter equations describe the conservation of momentum, we can exploit the conservation of mass. The mass of the fluid must be conserved locally, because there are now sinks or sources in the atmosphere itself, so we want to write the mass of a volume of atmosphere fixed in space as

$$M = \int_V \rho \, dV$$

the mass in the volume can only change if there is a flux of mass at surface S,

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_S \rho \mathbf{v} \cdot n \, dS$$

using the divergence theorem however we have

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_V \nabla \cdot (\rho \mathbf{v}) \, dV$$

because the volume is not changing with time we can bring the derivative inside the integral and we get

$$\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \, dV = 0$$

but the volume is arbitrary, so it must be that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

is valid locally.

We have still at our disposal the conservation of thermodynamical energy and so we can also use

$$C_v \frac{DT}{Dt} = -p \frac{D}{Dt}\left(\frac{1}{\rho}\right) + Q$$

where we included the temperature and heating/cooling term Q. The state variable are then linked by the state equation

$$p = \rho R T$$

where R is the gas constant for dry air.

We can use the equation of state to write the energy equation (or the temperature equation) in a different form,

$$c_v \frac{DT}{Dt} = -p \frac{D}{Dt}\left(\frac{RT}{p}\right) + Q = -R\frac{DT}{Dt} + \frac{RT}{p}\frac{Dp}{Dt} + Q$$

yielding the alternative forms (since $c_p = c_v + R$),

$$c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = Q$$

For adiabatic processes $Q = 0$ and so

$$\begin{aligned}
c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} &= 0 \\
\frac{c_p}{T}\frac{DT}{Dt} - \frac{R}{p}\frac{Dp}{Dt} &= 0 \\
\frac{D}{Dt}\log T - \frac{R}{c_p}\frac{D}{Dt}\log p &= 0
\end{aligned}$$

integrating it we get

$$\log T/T_0 - \log\left(\frac{p}{p_0}\right)^{R/c_p} = const$$

or

$$\frac{T}{T_0}\left(\frac{p_0}{p}\right)^{R/c_p} = const$$

so the quantity, known as *potential temperature*

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

is conserved in adiabatic processes and the thermodynamics equation can be written as

$$\frac{D\theta}{Dt} = Q$$

1.1.4 Hydrostatic balance

Under the action of gravity the vertical component of the pressure gradient force balances the action of gravity, resulting in very small vertical acceleration

$$\frac{\partial p}{\partial z} = -g\rho$$

then if we take the vertical derivative of the eq. `Eq:logT`

$$\frac{1}{T_0}\frac{dT}{dz}\left(\frac{p_0}{p}\right)^{R/c_p} - \frac{p_0}{p^2}\frac{R}{c_p}\frac{T}{T_0}\left(\frac{p_0}{p}\right)^{R/c_p-1}\frac{dp}{dz} = 0$$

simplifying

$$\frac{dT}{dz} - \frac{p_0}{p^2}\frac{R}{c_p}T\left(\frac{p_0}{p}\right)^{-1}\frac{dp}{dz} = 0$$

or

$$\frac{dT}{dz} - \frac{1}{p}\frac{R}{c_p}T\frac{dp}{dz} = \frac{dT}{dz} + g\rho\frac{1}{p}\frac{R}{c_p}T = 0$$

but using the equation of state

$$\frac{dT}{dz} = -\frac{g}{c_p}$$

that gives how the temperature change with height under adiabatic conditions and when the hydrostatic balance is valid. This is known as the *adiabatic lapse rate*.

1.1.5 Summary of fundamental equations

Summarizing our discussion, the fundamental equation that describe the motion of the atmosphere then are:

$$
\begin{aligned}
&\frac{Du}{Dt} - v\left(f + \frac{u\tan\phi}{a}\right) = -\frac{1}{a\cos\phi}\frac{1}{\rho}\frac{\partial p}{\partial\lambda} + F_\lambda\\
&\frac{Dv}{Dt} + u\left(f + \frac{u\tan\phi}{a}\right) = -\frac{1}{a}\frac{1}{\rho}\frac{\partial p}{\partial\phi} + F_\phi\\
&\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z\\
&\frac{D\theta}{Dt} = Q\\
&\frac{\partial\rho}{\partial t} + \frac{1}{a\cos\phi}\left[\frac{\partial}{\partial\lambda}(\rho u) + \frac{\partial}{\partial\phi}(rv\cos\phi\right] + \frac{\partial}{\partial z}(\rho w) = 0\\
&p = \rho RT
\end{aligned}
$$

where we have used the divergence in spherical coordinates.

These equations are still not closed because we will need to express the heating/cooling term Q and the friction terms F as a function of the state variables. This will require a theory of the processes that drive them. Where $R = 287.052874J \quad kg^{-1}K^{-1}$ is the gas constant for dry air and $c_p = 1.005$ is the specific heat at constant pressure, $c_v = 0.718$ is the specific heat at constant volume, $\kappa = \frac{R}{c_p}$ and $\gamma = c_p/c_v$ is their ratio.

For theoretical and idealized studies the set of equation projected on the β-plane is also used

$$
\begin{aligned}
&\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x\\
&\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y\\
&\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z\\
&\frac{D\theta}{Dt} = Q\\
&\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\\
&p = \rho RT
\end{aligned}
$$

and the gradient operator is the cartesian operator

$$
\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
$$

and the advective derivative is then

$$
\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}
$$

Chapter 2

Circulation

2.1 Circulation theorems and Ertel's potential vorticity

The consideration of the turbolent motion of fluid had been going on for sometime. The motion of vortices that showed a clear rotation character stimulated the development of quantities that would describe the capacity of a fluid to develop rotation with some precision. Helmholz schubert2004 was the first to show that vorticity was conserved along material lines in the fluid if only conservative forces were active Thorpe2003.

Lord Kelvin introduced the concept of circulation by establishing the following integral

$$\oint_{\Omega} \mathbf{v} \cdot \mathbf{dl}$$

where the integral is along any closed curve corresponding to a material line in the fluid. A material line is a line that maintain the constituting particles along the movement of the flow. Integral of this kind can be modified using Stokes theorem to result in a simple relation between circulation and vorticity.

$$C = \oint_{\Omega} \mathbf{v} \cdot \mathbf{dl} = \int_S \nabla \times \mathbf{v} \, dS = \int_S \omega \cdot \, dS$$

where the integral is over the surface delimited by the closed circuit. The scalar product with the oriented surfae indicates that circulation is the average value of the vorticity component normal to the surface. Lord Kelvin was then able to show that for a homogenous fluid the circulation is conserved along the fluid, namely

$$\frac{DC}{Dt} = 0$$

These results were fantastic achievements for the time, but they were of limited usefulness for the atmosphere since the atmosphere is hardly an homogenous fluid. It is interesting to consider the issue if a suitable generalization of Kelvin theorem can be found for compressible flows like the atmosphere. This issue was addressed by Schutz1895 and later by silberstein1896 that asked the question of what distribution of pressure and density were needed to generate vorticity. It showed that it was related to intersecting surfaces of constant pressure and density. This fundamental result contained all the essential ingredients for the application to geophysical flows, but Silberstein considered it as purely mathematical problem. The merit to show that this ideas were enormously important for the atmosphere and the ocean has to be given to bjerknes1898 (see Thorpe2003).

To illustrate Bjerknes ideas we will consider the equations of motion for a 3-dimensional, compressible flow on the sphere

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - 2\Omega \times \mathbf{v} - \frac{1}{\rho}\nabla p - \nabla \Phi$$

where Φ is the gravitational potential, including centrifugal effects. The advection term can be transformed using the identity

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{v} + \frac{1}{2}\nabla\left(|\mathbf{v}|^2\right)$$

so

$$\frac{\partial\mathbf{v}}{\partial t} = -\omega_a\times\mathbf{v} - \frac{1}{\rho}\nabla p - \nabla\left(\Phi + \frac{1}{2}|\mathbf{v}|^2\right)$$

where $\omega = \nabla\times\mathbf{v}$ is the relative vorticity and $\omega_a = \omega + 2\mathbf{\Omega}$ is the total (relative plus planetary) vorticity. Taking the curl of eq `eq6` we get[1]

$$\frac{\partial\omega}{\partial t} = -(\mathbf{v}\cdot\nabla)\omega_a + (\omega_a\cdot\nabla)\mathbf{v} - \omega_a\nabla\cdot\mathbf{v} - \nabla\left(\frac{1}{\rho}\right)\times\nabla p$$

Now because the planetary vorticity is independent of time we can write it as

$$\frac{D\omega_a}{Dt} = (\omega_a\cdot\nabla)\mathbf{v} - \omega_a\nabla\cdot\mathbf{v} - \nabla\left(\frac{1}{\rho}\right)\times\nabla p$$

Combining (`eq7`) with the continuity equation

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{v}$$

we get

$$\frac{D\omega_a}{Dt} = (\omega_a\cdot\nabla)\mathbf{v} - \nabla\left(\frac{1}{\rho}\right)\times\nabla p + \frac{\omega_a}{\rho}\frac{D\rho}{Dt}$$

The presence of multiple totale derivative suggests that it is reasonable to try to combine them, dividing by the density ρ we can write

$$\frac{1}{\rho}\frac{D\omega_a}{Dt} = \frac{1}{\rho}(\omega_a\cdot\nabla)\mathbf{v} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p + \frac{\omega_a}{\rho^2}\frac{D\rho}{Dt}$$

or

$$\frac{D}{Dt}\left(\frac{\omega_a}{\rho}\right) = \left(\frac{\omega_a}{\rho}\cdot\nabla\right)\mathbf{v} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p$$

Assume now that a function exist that express some property of the fluid in almost conservative form

$$\frac{D\chi}{Dt} = S$$

where S are source and sinks terms for the property. Examining the term, we get

$$\frac{\omega_a}{\rho}\cdot\frac{D}{Dt}\nabla\chi = \left(\frac{\omega_a}{\rho}\cdot\nabla\right)\frac{D\chi}{Dt} - \left[\left(\frac{\omega_a}{\rho}\cdot\nabla\right)\mathbf{v}\right]\cdot\nabla\chi.$$

This equation can be proved by examining it component by component. if now take the scalar product of Eq `eq7.3` with $\nabla\chi$ we obtain

$$\nabla\chi\cdot\frac{D}{Dt}\left(\frac{\omega_a}{\rho}\right) = \left[\left(\frac{\omega_a}{\rho}\cdot\nabla\right)\mathbf{v}\right]\cdot\nabla\chi - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p\cdot\nabla\chi$$

and then summing Eq. `eq7.4` and Eq. `eq7.5` we finally obtain

$$\frac{\omega_a}{\rho}\cdot\frac{D}{Dt}\nabla\chi + \nabla\chi\cdot\frac{D}{Dt}\left(\frac{\omega_a}{\rho}\right) = \frac{\omega_a}{\rho}\nabla\cdot\frac{D\chi}{Dt} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p\cdot\nabla\chi$$

or

$$\frac{D}{Dt}\left(\frac{\omega_a\cdot\nabla\chi}{\rho}\right) = \frac{\omega_a}{\rho}\nabla\cdot\frac{D\chi}{Dt} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p\cdot\nabla\chi$$

[1] Use the vector identity $\nabla\times A\times B = A(\nabla\cdot B) - B(\nabla\cdot A) + (B\cdot\nabla)A - (A\cdot\nabla)B$ and note that in our case the term $B(\nabla\cdot A)$ correspond to $\mathbf{v}(\nabla\cdot\nabla\times\mathbf{v})$ that is identically zero.

Now if the property χ is conserved along the fluid then $\frac{D\chi}{Dt} = 0$ and it is function only of pressure and density (i.e. is a thermodynamic quantity), or the flow is barotropic ($\nabla\rho \times \nabla p = 0$) then

$$\frac{D}{Dt}\left(\frac{\omega_a \cdot \nabla\chi}{\rho}\right) = 0$$

and the quantity in bracket is conserved and it is known as Ertel'ss Potential Vorticity after Heinz Ertel Erte1942.

This quantity regulates the large scale dynamics of the atmosphere and the ocean and it is the main guiding principle for the understanding of the motions.

Going back to our original homogeneous fluid on the sphere we do have a simple conserved quantity, in fact the vertical velocity w is everywhere zero, so

$$\frac{Dz}{Dt} = w = 0$$

so by setting $\chi = z$ we also get $\nabla\chi = \hat{z} = \hat{k}$, i.e. the vertical unit vector. Therefore the Ertel'ss potential vorticity reduces to

$$\omega_a \cdot \nabla\chi = \omega_a \cdot \hat{k} = \zeta + f$$

since ζ is the vertical component of ω_a. The Ertel's potential vorticity so in this case is simply

$$\frac{D}{Dt}(\zeta + f) = 0$$

Bibliography

Richardson, L.F. (1922). *Weather Prediction by Numerical Processes*. Cambridge University Press.