Discontinuous Galerkin Methods for Numerical Simulation

Exercise – Chapter 1

Summer 2025

Instructions:
This exercise sheet applies the knowledge of chapter 1 of the lecture notes. It is recommended to hand in a short solution or list with questions (1 page) to the teacher at `maximilian.bergbauer@tum.de` until May 8, 2025. Your result will not be graded, but it helps to set the pace in the online presentations and to define the topics for discussions. You can work in groups of up to 3 people for the exercise. The content will be discussed in the meeting on Friday May 9.

Study the lecture notes up to section 1.3. On the Moodle page, you find the following MATLAB programs for the one-dimensional linear transport (advection) equation:

- `advection_solver.m`: This is the main solver file for the DG method for the first and second chapter. It depends on three additional helper programs
  - `get_gauss_quadrature.m`: Definition of points and weights of the Gauss quadrature rule for arbitrary numbers of points $n \geq 1$
  - `get_gauss_lobatto_quadrature.m`: Definition of points and weights of Gauss–Lobatto quadrature rule for arbitrary numbers of points $n \geq 2$
  - `evaluate_lagrange_basis.m`: Evaluation of Lagrange polynomials defined on a given set of node points at a set of (possibly different) evaluation points
- `advection_solver_fd.m`: A finite difference solver using 3-point centered finite differences.
- `advection_solver_fem.m`: A continuous finite element solver of arbitrary polynomial degree for transport speeds $a > 0$.

Study the DG solver file and the main blocks in the code according to the given comments. In this exercise, we only study the case of linear polynomials $k = 1$. Compare the definition of the matrices with what is presented in section 1.2 in the lecture notes.

Tasks

1. Show a plot of the computed solution at time $T_f = 5$ for the DG solver with $n = 20$ elements, with the finite element solver with $n = 20$ elements, and the finite difference solver with 21 grid points. Additionally, include a figure with the DG solver using transport speed $a = -1$.
2. Report the numerical error in the maximum norm according to the command window output with the DG method for $n = 20, 40, 80, 160$ elements and for the finite difference method with $21, 41, 81, 161, 321$ elements at time $T_f = 5$.
3. Report the error of the DG solver with the 4th order Runge–Kutta time integrator, the backward Euler method, and the trapezoidal rule for $n = 80$ elements at times $T_f = 1$ and $T_f = 10$.
4. Compare the numerical error for the upwind flux with $\alpha = 0$ and the central flux $\alpha = 1$.

5. The Runge–Kutta time integrator is an explicit method. Try to find the maximal value of the Courant number $\mathrm{Cr} = \Delta t/h/|a|$, a scaling factor for the time step size, for which the solution is still stable (whereas it explodes for larger Courant numbers).