MATH 242 - HW9

due: 03/27/2024

1. Determine if the series converge or diverge.

   (a) (alternating)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$$

$$\lim_{n \to \infty} \frac{\sqrt{n}}{n+2} = \lim_{n \to \infty} \frac{\frac{1}{2\sqrt{n}}}{1} = 0 \checkmark$$

$$f(x) = \frac{\sqrt{x}}{x+2} \Rightarrow f'(x) = \frac{\frac{x+2}{2\sqrt{x}} - \sqrt{x}}{(x+2)^2}$$

$$= \frac{\frac{\sqrt{x}}{2} + \frac{1}{\sqrt{x}} - \sqrt{x}}{(x+2)^2}$$

$$= \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}\sqrt{x}}{(x+2)^2} \cdot \left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)$$

$$= \frac{2-x}{2\sqrt{x}(x+2)^2} < 0 \checkmark$$

$\boxed{\text{converges by AST}}$

(b) (alternating)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1} = 0 \checkmark$$

$$f(x) = \frac{\ln(x+1)}{x+1} \Rightarrow f'(x) = \frac{1-\ln(x+1)}{(x+1)^2} < 0$$

if $x > 1$

instead of $x \geq 1$

but thats OK

$\boxed{\text{converges by A.S.T}}$

2. Determine whether each Series converges Absolutely or Conditionally or Diverges:

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+4}$$

because

converges by A.S.T as $\lim_{n\to\infty} \frac{n}{n^2+4} = \lim_{n\to\infty} \frac{1}{2n} = 0$ ✓

and $f(x) = \frac{x}{x^2+4} \Rightarrow f'(x) = \frac{x^2+4-2x^2}{(x^2+4)^2}$

$$= \frac{4-x^2}{(x^2+4)^2} < 0 \checkmark$$

but not absolutely:

$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges! by Limit Comparison w/ $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$$

(b)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$$

Conditionally

$u = \ln(x)$ $\to \int_{\ln(2)}^{\infty} \frac{1}{u}\,du \to \infty$

Because

$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges via integral test $\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$

$f(x) = \frac{1}{x\ln(x)} \Rightarrow f'(x) = \frac{-\ln(x)-1}{(x\ln(x))^2} < 0$ decreasing ✓

is CTS, positive.

(yet $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges by A.S.T)

3. Determine if the series converge or diverge. A hint of a test is provided, but feel free to try/use others.

(a) (ratio)

$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

Converges

$$\lim_{n\to\infty} \left| \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} \right| = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$$

(b) (root)

$$\sum_{n=1}^{\infty} \left(\frac{3n^2 - 2}{2n^2 + 1} \right)^{5n}$$

$\boxed{\text{diverges}}$

$$\lim_{n\to\infty} \sqrt[n]{\left[\left(\frac{3n^2-2}{2n^2+1}\right)^5\right]^n} = \lim_{n\to\infty} \left(\frac{3n^2-2}{2n^2+1}\right)^5$$

$$= \left(\frac{3}{2}\right)^5 > 1$$

(c) (ratio)

$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

$\boxed{\text{converges}}$

$$\lim_{n\to\infty} \left| \frac{\frac{10^{n+1}}{(n+2)4^{2n+3}}}{\frac{10^n}{(n+1)4^{2n+1}}} \right| = \lim_{n\to\infty} \frac{n+1}{n+2} \cdot \frac{10}{16}$$

$$= \frac{5}{8} < 1$$

(d) (root)

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

$\boxed{\text{converges}}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{e^n}\right|}$$

$$= \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{e} = y$$

$$\ln(y) = \lim_{n\to\infty} \ln\left(\frac{n^{1/n}}{e}\right)$$

$$= \lim_{n\to\infty} \frac{\ln(n)}{n} - \ln(e)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{-1}{n^2}} - 1 = -1$$

$$\Rightarrow y = e^{-1} < 1$$