MATH 242 - WS8

03/07/2024

1. These series converge. Calculate their sum.

   (a)

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+2} = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \left(\frac{1}{N} - \frac{1}{N+2}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

$$\Rightarrow S = \lim_{N \to \infty} S_N = \boxed{\frac{3}{2}}$$

   (b)

$$S = \sum_{n=2}^{\infty} \frac{4}{n^2 - 1}$$

$$\frac{4}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$$

$$4 = A(n+1) + B(n-1)$$

$$A = -B$$

$$4 = A - B$$
$$= 2A$$

$$A = 2 \quad B = -2$$

$$= \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n+1} \right)$$

$$= 2 \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

put $j = n-1$

$$= 2 \underbrace{\sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+2} \right)}_{\text{Same as \#1a!}} = \boxed{3}$$

$$a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad \left(|r|<1\right)$$

(c)

$$\sum_{n=1}^{\infty} \left(\frac{2^n + 3^n}{e^{2n}} \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{e^2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{e^2}\right)^n$$

$$a = r = \frac{2}{e^2} < 1 \qquad a = r = \frac{3}{e^2} < 1$$

$$= \frac{\frac{2}{e^2}}{1 - \frac{2}{e^2}} + \frac{\frac{3}{e^2}}{1 - \frac{3}{e^2}} = \boxed{\frac{2}{e^2-2} + \frac{3}{e^2-3}}$$

(d)

[redacted] $$\sum_{n=0}^{\infty} \frac{-7}{4^{2n}}$$

$$= -7 - \frac{7}{16} - \frac{7}{256} - \cdots = \frac{-7}{1 - \frac{1}{16}}$$

$$a = -7$$

$$r = \frac{1}{16} < 1$$

$$= \frac{-112}{16-1} = \boxed{-\frac{112}{15}}$$

2. Determine if the integral test for convergence applies (the comparable continuous function $f(x)$ must be continuous, positive, and decreasing for $x \in [1, \infty)$) and then if it does, show either convergence or divergence.

(a)

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

$f(x) = x^2 e^{-x^3}$ is continuas ✓

positive ✓

$f'(x) = 2xe^{-x^3} - 3x^4 e^{-x^3}$

$= \underbrace{xe^{-x^3}}_{\oplus \text{ for } x>1} \underbrace{(2-3x^3)}_{\ominus \text{ for } x<1} < 0$ decreasing ✓

$\boxed{\text{converges}}$ since $\int_0^\infty x^2 e^{-x^3} dx$ $\quad u = x^3$

$\frac{1}{3} du = x^2 dx$

$= \int_0^\infty e^{-u} du$

$= (-e^{-u}|_0^\infty = \lim_{t\to\infty} -\frac{1}{e^t} + e^0 = 1$

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+\sqrt{n^3}}$$

$f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x^3}}$ is continuas ✓

positive ✓

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}\left(1+\sqrt{x^3}\right) - \frac{3}{2}\sqrt{x}\left(\sqrt{x}\right)}{\left(1+\sqrt{x^3}\right)^2}$$ ✓

$$= \frac{\frac{1}{2\sqrt{x}} - X}{\left(1+\sqrt{x^3}\right)^2} < 0 \text{ since } (\ )^2 > 0 \text{ and top} < 0$$

$\frac{1}{2\sqrt{x}} < X$

$\boxed{\text{diverges}}$ since $\int_1^{\infty} \frac{\sqrt{x}}{1+\sqrt{x^3}}\,dx$ $\quad u = 1+\sqrt{x^3}$

$\frac{2}{3}du = \sqrt{x}\,dx$

$$= \frac{2}{3}\int_2^{\infty} \frac{1}{u}\,du = \frac{2}{3}\left(\ln(u)\right)\Big|_2^{\infty} \longrightarrow \infty$$

$$= \frac{2}{3}\int \frac{1}{u}\,du = \frac{2}{3}\ln|u| = \frac{2}{3}\ln\left[1+\sqrt{x^2}\right|_{x=1}^{\infty}$$