Taylor's Inequality: choose $K \geq |f^{(n+1)}(x)|$ for $x \in [a-d, a+d]$ $(d>0)$ to bound the remainder:

$$|R_n(x)| \leq \frac{K}{(n+1)!}|x-a|^{n+1}$$

$f(x) = T_n(x) + R_n(x)$

"remainder"

$T_n(x)$: n^{th} order Taylor polynomial

MATH 242 - WS11

04/11/2024

(use a calculator)

$e^{\pi} = 23.1406926327\ldots$

1. Approximate the numerical value of e^{π} using the 2nd order Taylor polynomial $T_2(x)$ for the function $f(x) = e^x$ centered at $a = 0$. ~~What order n has a Taylor polynomial $T_n(x)$ that would provide accuracy better than 10^{-5}?~~ changed the question

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

$$e^x \approx 1 + x + \frac{x^2}{2} \quad \text{hence}$$

$$\boxed{e^{\pi} \approx 1 + \pi + \frac{\pi^2}{2}} \quad \left(= 9.07639485\ldots \right)$$

(use a calculator)

differs about 14.06429779

For $x \in [-\pi, \pi]$, $|f^{(n+1)}(x)| = |e^x| \leq e^{\pi} = K$

$$|R_2(\pi)| \leq \frac{e^{\pi}}{(2+1)!}\pi^3 = \boxed{119.5844530S}$$

$>$

Yes did much better!

$\tan^{-1}(1.5) = 0.982794\ldots$ ✓

exact value

2. Approximate the numerical value of arctan(1.5) using the 2nd order Taylor polynomial $T_2(x)$ for the function $f(x) = \arctan(x)$ centered at $a = 1$. [redacted]

$$f(x) = \tan^{-1}(x) \Rightarrow f(a) = f(1) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(a) = f'(1) = \frac{1}{1+1^2} = \frac{1}{2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f''(a) = f''(1) = \frac{-2(1)}{(1+1^2)^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$f'''(x) = \frac{-2(1+x^2)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} = (6x^2-2)(x^2+1)$$

$$= \frac{-2-4x^2-2x^4+8x^2(1+x^2)}{(1+x^2)^4} = \frac{6x^4+4x^2-2}{(1+x^2)^4} = \frac{6x^2-2}{(1+x^2)^3}$$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$\tan^{-1}(x) \approx \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 \quad \text{hence}$$

$$\boxed{\tan^{-1}(1.5) \approx \frac{\pi}{4} + \frac{1}{4} - \frac{1}{16}} = \frac{4\pi+4-1}{16} = \frac{4\pi+3}{16}$$

$= 0.9728981633\ldots$

for $x \in [0.5, 1.5]$

$$|f'''(x)| \le \left|\frac{6x^2-2}{(1+x^2)^3}\right| \underset{@x=1}{\le} \frac{4}{8} = \frac{1}{2}$$

differs about 0.0098956...

yes did better!

$$|R_2(1.5)| \le \frac{\frac{1}{2}}{(2+1)!} \cdot \frac{1}{2}^3 = \frac{1/16}{6} = \frac{1}{96} = \boxed{0.010416}$$

* Proof $\max\limits_{x \in [0.5, 1.5]} \left| \frac{6x^2 - 2}{(1+x^2)^3} \right| = \frac{1}{2}$ @ $x = 1$

$$\frac{d}{dx}\left(\frac{6x^2-2}{(1+x^2)^3} \right) = \frac{12x(1+x^2)^3 - (6x^2-2)\cdot 3(1+x^2)^2 \cdot 2x}{(1+x^2)^6} = 0$$

$$12x(1+x^2)^3 - 6x(6x^2-2)(1+x^2)^2 = 0$$

$$12x(1+x^2) - 6x(6x^2-2) = 0$$

$$12x + 12x^3 - 36x^3 + 12x = 0$$

$$-24x^3 + 24x = 0$$

$$-24x(x^2-1) = 0$$

$$-24x(x-1)(x+1) = 0$$

$$x = -1, 0, 1 \quad \blacksquare$$