MATH 242 - WS12

04/18/2024

1. Find the general or particular solution of the separable differential equation by using the method of separation of variables. *Next, check that your answer is correct by differentiating!*

   (a) $\frac{dy}{dx} - e^{x+y} = 0$

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

$$e^{-y}\,dy = e^x\,dx$$

$$\int e^{-y}\,dy = \int e^x\,dx$$

$$-e^{-y} = e^x + C$$

$$e^{-y} = C - e^x$$

$$-y = \ln(c - e^x)$$

$$\boxed{y(x) = -\ln(c - e^x)}$$

check:

$$\frac{dy}{dx} = -\frac{1}{c - e^x} \cdot -e^x = \frac{e^x}{c - e^x} = e^x \cdot e^{-\ln(c-e^x)} = e^x \cdot e^y \checkmark$$

(b) $\frac{dy}{dx} = \frac{2x\sin(x^2)}{y}$; $\quad y(0) = -1$

$$y\frac{dy}{dx} = 2x\sin(x^2)$$

$$y\,dy = 2x\sin(x^2)\,dx$$

$$\int y\,dy = \int 2x\sin(x^2) \qquad u = x^2$$

$$= \int \sin(u)\,du \qquad du = 2x\,dx$$

$$= -\cos(u) + C$$

$$\frac{y^2}{2} = C - \cos(x^2)$$

"re-define" $C = 2C$

$$y^2 = C - 2\cos(x^2)$$

$$y = \pm\sqrt{C - 2\cos(x^2)} \quad \text{now } y(0) = -1$$

$$y(0) = \pm\sqrt{C - 2} = -1$$

(so take $-$ branch and $C = 3$)

$$\boxed{y(x) = -\sqrt{3 - 2\cos(x^2)}}$$

check

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{3-2\cos(x^2)}} \cdot 4x\sin(x^2)$$

$$= \frac{2x\sin(x^2)}{-\sqrt{3-2\cos(x^2)}} = \frac{2x\sin(x^2)}{y} \checkmark$$

check: $y'(x) = \frac{1}{2}\left(\ln(x) - \frac{1}{2}\right) + \frac{x}{2}\cdot\frac{1}{x} - \frac{1}{4x^2}$

$= \frac{1}{2}\ln(x) + \frac{1}{4} - \frac{1}{4x^2} \Rightarrow xy'(x) + y =$

$\frac{x}{2}\ln(x) + \frac{x}{4} - \frac{1}{4x}$

$+\frac{x}{2}\ln(x) - \frac{x}{4} + \frac{1}{4x}$

$= x\ln(x)$ ✓

(b) $x\frac{dy}{dx} + y = x\ln(x); \quad y(1) = 0$

$$\frac{dy}{dx} + \frac{1}{x}y = \ln(x)$$

$$e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

$$x\frac{dy}{dx} + y = x\ln(x)$$

$$\frac{d}{dx}\left[xy\right] = x\ln(x)$$

$u = \ln(x) \quad dv = x\,dx$

$du = \frac{1}{x}dy \quad v = \frac{x^2}{2}$

$$\int \frac{d}{dx}\left[xy\right]dx = \int x\ln(x)\,dx$$

$$xy = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx$$

$$xy = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$$

$$y = \frac{x}{2}\ln(x) - \frac{x}{4} + \frac{C}{x}$$

$y(1) = -\frac{1}{4} + C = 0$

$\Rightarrow C = \frac{1}{4}$

$$\boxed{y(x) = \frac{x}{2}\left(\ln(x) - \frac{1}{2}\right) + \frac{1}{4x}}$$