MATH 242 - WS6

02/22/2024

1. Evaluate the definite integral:

$$\int_0^1 (1+x^2)dx = \left(x + \frac{x^3}{3}\right\Big|_0^1 = 1 + \frac{1^3}{3} - 0$$

$$= \frac{4}{3}$$

$$= 1.\overline{3}$$

$$f(x) = 1 + x^2$$

2. Find the Trapezoid rule approximation T_5 using $n = 5$ rectangles.

$a = 0,\ b = 1,\ \Delta x = \frac{b-a}{5} = \frac{1}{5} \Rightarrow x_0 = 0, x_1 = \frac{1}{5}, x_2 = \frac{2}{5}, x_3 = \frac{3}{5}, x_4 = \frac{4}{5}, x_5 = 1$

$\Rightarrow f(x_0) = 1, f(x_1) = \frac{26}{25}, f(x_2) = \frac{29}{25}, f(x_3) = \frac{34}{25}, f(x_4) = \frac{41}{25}, f(x_5) = 2$

$$T_5 = \frac{\Delta x}{2}\left[f(x_0) + f(x_5) + 2\left(f(x_1) + f(x_2) + f(x_3) + f(x_4)\right)\right]$$

$$= \frac{1}{10}\left[1 + 2 + 2\frac{26+29+34+41}{25}\right] = \frac{3}{10} + \frac{130}{125}$$

$$= \frac{150}{500} + \frac{520}{500} = \frac{670}{500} = \frac{67}{50} = 1.34$$

3. Using the last two problems, what is the magnitude of the absolute error $|E_{T_5}| = |\int_0^1 (1+x^2)dx - T_5|$?

$$|E_{T_5}| = \frac{67}{50} - \frac{4}{3}$$

$$= \frac{201 - 200}{150} = \frac{1}{150} = 0.00\overline{6}$$

$f'(x) = 2x \quad |f''(x)| \leq 2$

4. What is the upper bound of of the error $|E_{T_5}|$ applied to this particular $n = 5$, $a = 0$, $b = 1$, and $f(x) = 1 + x^2$? Comment on how this compares to what you found in the last question.

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2}$$

$$|E_{T_5}| \leq \frac{2 \cdot 1^3}{12 \cdot 5^2} = \frac{1}{150}.$$

indeed we were (unfortunately) this bad

$f(x) = 5x^4 \quad f'(x) = 20x^3 \quad f''(x) = 60x^2 \quad f'''(x) = 120x \quad f''''(x) = 120$

5. Suppose you wish to estimate $\int_0^1 5x^4 dx$ to an accuracy having an Error of $|E| < 10^{-3}$. How large would you need to choose n for...

   (a) Simpson's Rule?

   (b) the Trapezoid Rule?

   (c) Mid-point Rule?

If $a > b > 0$

$\frac{1}{a} < \frac{1}{b}$

a) $10^{-3} > \dfrac{120 \cdot 1^5}{180 n^4} \Rightarrow \dfrac{3}{2000} > \dfrac{1}{n^4}$

$\sqrt[4]{\dfrac{2000}{3}} < n \quad \hookrightarrow \boxed{n = 6}$

b) $10^{-3} > \dfrac{60 \cdot 1^3}{12 n^2} \Rightarrow \dfrac{1}{5000} > \dfrac{1}{n^2}$

$\sqrt{5000} < n \quad \hookrightarrow \boxed{n = 71}$

c) $10^{-3} > \dfrac{60 \cdot 1^3}{24 n^2} \Rightarrow \dfrac{1}{2500} > \dfrac{1}{n^2}$

$\sqrt{2500} < n \quad \hookrightarrow \boxed{n = 51}$