MATH 242 - HW11

due: 04/17/2024

1. Find the Maclaurin series (the Taylor series centered at $a = 0$) for the provided function $f(x)$ and then also find its interval of convergence. (Hint: it may be easier to find the Maclaurin series of a simpler function $g(x)$ and then use a substitution.)

   (a)
   $$f(x) = e^{-2x}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Downarrow$$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \underbrace{\lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right)}_{} |x| \overset{?}{<} 1$$

$$= \lim_{n\to\infty}\left(\frac{2}{n+1}\right) = 0$$

$$\Rightarrow \boxed{(-\infty, \infty)}$$

$$\left\{ \text{or} \quad 3^x = \left(e^{\ln(3)}\right)^x = e^{\ln(3)x} \right.$$

(b)

$f(x) = 3^x$

$$f'(x) = \ln(3)3^x$$

$$f''(x) = \ln^2(3)3^x$$

$$\vdots$$

$$f^{(k)}(x) = \ln^k(3)3^x$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{\ln^k(3)3^0}{k!} x^k$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{\ln^k(3)}{k!} x^k}$$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \underbrace{\lim_{k\to\infty} \left(\frac{\ln^{k+1}(3)}{(k+1)!} \cdot \frac{k!}{\ln^k(3)}\right)}|x| \overset{?}{<} 1$$

$$= \lim_{k\to\infty} \frac{\ln(3)}{k+1} = 0$$

$$\Rightarrow \boxed{(-\infty, \infty)}$$

(c)

$$f(x) = x\cos(x)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\Rightarrow \quad x\cos(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \underbrace{\lim_{n\to\infty}\left(\frac{1}{(2n+2)!} \cdot (2n)!\right)}_{} x^2 \overset{?}{<} 1$$

$$= \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0$$

$$\Rightarrow \boxed{(-\infty, \infty)}$$

2. Find the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $a = -3$.

$$f'(x) = \frac{+1}{(1-x)^2}$$

$$f''(x) = \frac{+2}{(1-x)^3}$$

$$f'''(x) = \frac{+6}{(1-x)^4}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$$

$$= \underbrace{\frac{1}{4}}_{f(a)} + \sum_{k=1}^{\infty} \frac{f^{(k)}(-3)}{k!}(x+3)^k$$

$$= \frac{1}{4} + \sum_{k=1}^{\infty} \frac{\frac{k!}{(1+3)^{k+1}}}{k!}(x+3)^k$$

$$= \frac{1}{4} + \sum_{k=1}^{\infty} \frac{(x+3)^k}{4^{k+1}}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(x+3)^k}{4^{k+1}}}$$