MATH 242 - HW12

due: 04/26/2024 (guaranteed corrections), 04/29/2024 (possible corrections), 05/01/2024 (last day, all HW due, no corrections on HW12)

1. Find the solution of the differential equation that satisfies the given initial condition

   (a)

$$\frac{dy}{dt} = y^2 \ln(t); \quad y(1) = -1$$

$$\int \frac{dy}{y^2} = \int \ln(t)\,dt \qquad u = \ln(t) \quad dv = dt \qquad du = \frac{1}{t}dt \quad v = t$$

$$-\frac{1}{y} = t\ln(t) - \int dt$$

$$= t\ln(t) - t + C$$

$$\frac{1}{y} = C - t\ln(t) + t$$

$$y(t) = \frac{1}{C - t\ln(t) + t}$$

$$y(1) = \frac{1}{C+1} = -1 \Rightarrow \boxed{C = -2}$$

$$\boxed{y(t) = \frac{1}{t - t\ln(t) - 2}}$$

$y(0)=0 \Rightarrow y(c) = c^2 = 0$

$c = 0$

(b)

$$\frac{dy}{dx} = \sqrt{yx}; \quad \cancel{y(1) = 2}$$

$$\int \frac{dy}{\sqrt{y}} = \int \sqrt{x}\, dx$$

$$2\sqrt{y} = \frac{2}{3}\sqrt{x^3} + C$$

$$\sqrt{y} = \frac{\sqrt{x^3}}{3} + C$$

$$y(x) = \left(\frac{\sqrt{x^3}}{3} + C\right)^2$$

$$\boxed{y(x) = \frac{x^3}{9}}$$

(c)

$$y' = xy - x; \quad y(0) = 2$$

$$\frac{dy}{dx} = x(y-1)$$

$$\int \frac{dy}{y-1} = \int x\, dx$$

$$\ln|y-1| = \frac{x^2}{2} + C$$

$$y - 1 = Ce^{x^2/2}$$

$$y(x) = Ce^{x^2/2} + 1$$

$$y(0) = C + 1 = 2$$

$$\boxed{C = 1}$$

$$\boxed{y(x) = e^{x^2/2} + 1}$$

(d)

$$x^3y' + 3x^2y = \cos(x); \quad y(\pi) = 0$$

$$\int \frac{d}{dx}\left[x^3 y\right]dx = \int \cos(x)\,dx$$

$$x^3 y = \sin(x) + C$$

$$y(x) = \frac{\sin(x) + C}{x^3}$$

$$y(\pi) = \frac{C}{\pi^3} = 0 \Rightarrow \boxed{C = 0}$$

$$\boxed{y(x) = \frac{\sin(x)}{x^3}}$$

$$(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0; \quad y(0) = 2$$

$$(x^2+1)\frac{dy}{dx} + 3xy = 3x$$

$$\frac{dy}{dx} + \left(\frac{3x}{x^2+1}\right)y = \frac{3x}{x^2+1}$$

$$(x^2+1)^{3/2}\frac{dy}{dx} + \sqrt{x^2+1}\cdot 3xy = \sqrt{x^2+1}\,3x$$

$$\int \frac{d}{dx}\left[(x^2+1)^{3/2}y\right] = \int \sqrt{x^2+1}\,3x\,dx$$

$$(x^2+1)^{3/2}y = \frac{3}{2}\int \sqrt{u}\,du \qquad u = x^2+1 \qquad \frac{3}{2}du = 3x\,dx$$

$$(x^2+1)^{3/2} + C$$

$$y(x) = 1 + \frac{C}{(x^2+1)^{3/2}}$$

$$e^{\int \frac{3x}{x^2+1}dx}$$

$$u = x^2+1$$

$$\frac{3}{2}du = 3x\,dx$$

$$e^{\frac{3}{2}\int \frac{1}{u}du}$$

$$e^{\frac{3}{2}\ln(u)}$$

$$u^{3/2}$$

$$(x^2+1)^{3/2}$$

$$y(0) = 1 + C = 2 \Rightarrow \boxed{C = 1}$$

$$\boxed{y(x) = 1 + \frac{1}{(x^2+1)^{3/2}}}$$

Let $y(t)$ volume of CO_2 @ time t $\Rightarrow$ $y(0) = \frac{.15}{100} \cdot 180$
$= .27 m^3$

2. A room with volume $180m^3$ initially contains 0.15% carbon dioxide. At one end of the room fresher air with only 0.05% carbon dioxide flows in at a rate of $2m^3$ per minute. The air gets mixed together and flows out the other room at the same rate of $2m^3$ per minute. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

flow in: $\frac{0.05}{100} \cdot 2m^3 = \frac{1}{1000} m^3$ per minute

flow out: $\frac{y(t)}{180} \cdot 2m^3 = \frac{y(t)}{90} m^3$ per minute

$\frac{dy}{dt} = \frac{1}{1000} - \frac{y}{90} = \frac{9 - 100y}{9000}$

$\int \frac{dy}{9-100y} = \int \frac{1}{9000} dt$

$-\frac{1}{100} \ln(9-100y) = \frac{t}{9000} + C$

$\ln(9-100y) = C - \frac{t}{90}$

$9 - 100y = Ce^{-t/90}$

$-100y = Ce^{-x/90} - 9$

$y(t) = \frac{9}{100} + Ce^{-t/90}$

3. A $400L$ tank is full of chlorine water with a concentration $0.05g$ of chlorine per liter of water. Fresh water without chlorine is puopmped into the tank at $4L/s$. The misture is stirred and pumped out at a rate of $10L/s$ Find the chlorine in the tank as a function of time.

$y(0) = \frac{9}{100} + C = .27$

$C = 0.18$

$\boxed{y(t) = 0.09 + 0.18e^{-\frac{t}{90}}}$