MATH 242 - HW7

due: 03/06/2024

1. Show the integral converges via Comparison Test

$$\int_1^\infty \frac{x^2}{x^5+1}dx$$

$$\int_1^\infty \frac{x^2}{x^5+1}dx < \int_1^\infty \frac{x^2}{x^5}dx = \int_1^\infty \frac{1}{x^3}dx < \infty \quad p=3$$

make denom smaller

2. Show the integral diverges via Comparison Test

$$\int_1^\infty \frac{7x^4+2}{x^5}dx$$

$$\int_1^\infty \frac{7x^4+2}{x^5} > \int_1^\infty \frac{x^4}{x^5}dx = \int_1^\infty \frac{1}{x}dx \to \infty \quad p=1$$

make numerator smaller

3. Evaluate the Improper Integral

$$\int_8^\infty \frac{2}{x^2-4}dx$$

$\frac{2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

$2 = A(x+2) + B(x-2)$

if $x=2$

$2 = 4A \Rightarrow A = \frac{1}{2}$

if $x=-2$

$2 = -4B \Rightarrow B = -\frac{1}{2}$

$= \frac{1}{2}\left(\int_8^\infty \frac{1}{x-2} - \frac{1}{x+2}\,dx\right)$

$= \frac{1}{2}\left(\ln(x-2) - \ln(x+2)\Big|_8^\infty\right.$

$= \ln\left(\sqrt{\frac{x-2}{x+2}}\right)\Big|_8^\infty$

$= \lim_{t\to\infty} \underbrace{\ln\left(\sqrt{\frac{t-2}{t+2}}\right)}_{=0} - \ln\left(\sqrt{\frac{6}{10}}\right)$

$= \boxed{\ln\left(\sqrt{\frac{5}{3}}\right)}$

4. Evaluate the Improper Integral

$$\int_0^1 y\ln(y)dy$$

$u = \ln(y)$ $dv = y\,dy$

$du = \frac{1}{y}dy$ $v = \frac{y^2}{2}$

$= \frac{y^2}{2}\ln(y) - \frac{1}{2}\int y\,dy$

$= \frac{y^2}{2}\ln(y) - \frac{y^2}{4}$

$= \frac{y^2}{4}\left(2\ln(y) - 1\right)\Big|_0^1$

$= \boxed{-\frac{1}{4}} + \lim_{t\to 0^+} t^2\ln(t)$

$\hookrightarrow \frac{\frac{1}{t}}{\frac{2}{t^2}} = \frac{\frac{1}{t}}{-\frac{4}{t^3}} = \frac{t^2}{4} = 0$

5. State whether each sequence diverges or converges, and to what it converges to in that case.

(a) as $n \to \infty$...

$$\{\sin(\frac{n\pi}{2n+1})\}_{n=0}^{\infty}$$

$\to \sin(\frac{\pi}{2}) = \boxed{1}$

(b)

$$\{\sin(\frac{n\pi}{6})\}_{n=0}^{\infty}$$

DNE

(c)

$$-e^{-4n} < \{(-1)^n e^{-4n}\}_{n=0}^{\infty} < e^{-4n}$$

$\to 0 \qquad \to 0$

$\to \boxed{0}$

(d)

$$\lim_{n\to\infty} \{(1 - \frac{8}{n})^n\}_{n=1}^{\infty} = y$$

$\ln(y) = \lim_{n\to\infty} n \ln(1 - \frac{8}{n})$

$= \lim \frac{\ln(1-\frac{8}{n})}{\frac{1}{n}} \quad \frac{\frac{1}{1-\frac{8}{n}} \cdot \frac{8}{n^2}}{-\frac{1}{n^2}} = -8$

$\Rightarrow y = \boxed{e^{-8} = \frac{1}{e^8}}$

(e)

$$\{2 + 3^n 7^{-n}\}_{n=1}^{\infty}$$

$\rightarrow \boxed{2}$

(f)

$$\{\frac{3n + 2\sqrt{n}}{5n}\}_{n=1}^{\infty} = \frac{3}{5}$$

(g)

$$\{\cos(\frac{2 + 3n^3}{1 + 2n^2 + 4n^3})\}_{n=1}^{\infty} = \cos\left(\frac{3}{4}\right)$$