Optimization

CS5491: Artificial Intelligence
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Content Credits: **Prof. Wei**'s CS4486 Course
and **Prof. Boddeti**'s AI Course

TODAY

Optimization Setup

OPTIMIZATION PROBLEM: DEFINITION

Optimization Problem: Determine value of optimization variable within feasible region/set to optimize optimization objective

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{1}$$

- Optimization variable $\mathbf{x} \in \mathbb{R}^n$
- Feasible region/set $\mathcal{F} \subseteq \mathbb{R}^n$
- Optimization objective: $f : \mathcal{F} \to \mathbb{R}$

Optimal solution: $\mathbf{x}^* = \underset{\mathbf{x} \in \mathcal{F}}{\operatorname{argmin}} f(\mathbf{x})$

Optimal objective value $f^* = \min_{\mathbf{x} \in \mathcal{F}} f(\mathbf{x}) = f(\mathbf{x}^*)$

OPTIMIZATION PROBLEM: DEFINITION

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{2}$$

Optimization variable $\mathbf{x} \in \mathbb{R}^n$

- Discrete variables: Combinatorial optimization
- Continuous variables: Continuous optimization
- Mixed: Some variables are discrete, and some are continuous

OPTIMIZATION PROBLEM: DEFINITION

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{3}$$

Feasible region/set $\mathcal{F} \subseteq \mathbb{R}^n$

- Unconstrained optimization: $\mathcal{F} = \mathbb{R}^n$
- Constrained optimization: $\mathcal{F} \subset \mathbb{R}^n$
- Finding a feasible point $\mathbf{x} \in \mathcal{F}$ can already be difficult

OPTIMIZATION PROBLEM: DEFINITION

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{4}$$

Optimization objective $f : \mathcal{F} \to \mathbb{R}$

- $f(\mathbf{x}) = 1$: Feasibility problem
- Simple functions
- Linear function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x}$
- Convex function (next lecture)
- Complicated functions

- Can be implicitly represented through an algorithm which takes $\mathbf{x} \in \mathcal{F}$ as input, and outputs a value

OPTIMIZATION PROBLEM: DEFINITION

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{5}$$

Minimization can be converted to maximization (and vice versa)

$$\begin{aligned} \max_{\mathbf{x}} \quad & g(\mathbf{x}) = -f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned} \tag{6}$$

Same optimal solution and optimal objective value $g^* = -f^*$

OPTIMIZATION PROBLEM: EXAMPLE

Example: Traveling Salesman Problem (TSP)

- Problem: n cities, distance from city i to city j is $d(i, j)$, find a tour (a closed path that visits every city exactly once) with minimal total distance.
- Variable $\mathbf{x}$: ordered list of cities being visited
- x_i is the index of the i-th city being visited
- Feasible set $\mathcal{F} = \{\mathbf{x} : \text{“each city visited exactly once”}\}$

$$\mathcal{F} = \{\mathbf{x} : \mathbf{x} \in \{1, \ldots, n\}^n; \sum_k \mathbb{I}(x_k = i) = 1, \forall i \in \{1, \ldots, n\}\} \tag{7}$$

- Objective function $f(\mathbf{x}) = \text{“total distance when following } \mathbf{x}\text{”}$

$$f(\mathbf{x}) = d(x_n, x_1) + \sum_{k=1}^{n-1} d(x_k, x_{k+1}) \tag{8}$$

OPTIMIZATION PROBLEM: EXAMPLE

$$\min_{\mathbf{x}} \; f(\mathbf{x})$$
$$\text{s.t.} \; \mathbf{x} \in \mathcal{F}$$

Example: 8-Queens Problem (Solution 1)

- Variable $\mathbf{x}$: location of the queen in each column
- x_i is the row index of the queen in i-th column
- Feasible set $\mathcal{F} = \{\mathbf{x} : \text{“no queens in the row, col, diag”}\}$

$$\mathcal{F} = \{\mathbf{x} : \mathbf{x} \in \{1, \ldots, 8\}^8; x_i \neq x_j, |x_i - x_j| \neq |i - j|, \forall i, j \in \{1, \ldots, 8\}\}$$

- Objective function $f(\mathbf{x}) = 1$ (dummy)

OPTIMIZATION PROBLEM: EXAMPLE

$$\min_{\mathbf{x}} \ f(\mathbf{x})$$
$$\text{s.t.} \ \mathbf{x} \in \mathcal{F}$$

Example: 8-Queens Problem (Solution 2)

- Variable x_i, y_i: index of row and column of the i-th queen
- Feasible set $F = \{x, y : \text{ ``no queens in the row, col, diag''}\}$

$$F = \{x, y : x, y \in \{1, \dots, 8\}^8; \sum_i \mathbb{I}(x_i = k) = 1, \forall k \in \{1, \dots, 8\};$$
$$\sum_i \mathbb{I}(y_i = k) = 1, \forall k \in \{1, \dots, 8\}; |x_i - x_j| \neq |y_i - y_j|, \forall i, j \in \{1, \dots, 8\}$$

- Objective function $f(x) = 1$ (dummy)

OPTIMIZATION PROBLEM: EXAMPLE

x_i	1.0	2.0	3.0
y_i	2.1	3.98	7.0

Example: Linear Regression

- Problem: Find a such that $y_i \approx ax_i, \forall i = \{1, 2, 3\}$
- Variable a
- Feasible region $\mathbb{R}$
- Objective function $f(a)$?

$$\begin{aligned} \min_{a} \quad & \sum_{i=1}^{3} |y_i - ax_i| \\ \text{s.t.} \quad & a \in \mathbb{R} \end{aligned} \tag{9}$$

$$\begin{aligned} \min_{a} \quad & \sum_{i=1}^{3} (y_i - ax_i)^2 \\ \text{s.t.} \quad & a \in \mathbb{R} \end{aligned} \tag{10}$$

OPTIMIZATION PROBLEM: HOW TO SOLVE?

No general way to solve

Many algorithms developed for special classes of optimization problems (i.e., when $f(x)$ and $\mathcal{F}$ satisfy certain constraints)

- Convex optimization problem (CO)
- Linear Program (LP)
- (Mixed) Integer Linear Program (MILP)
- Quadratic program (QP), (Mixed) Integer Quadratic program (MIQP), Semidefinite program (SDP), Second-order cone program (SOCP), . . .

Existing solvers and code packages for these problems

- Cplex (LP, MILP, QP), Gurobi (LP, MILP, MIQP), GLPK (LP, MILP), Cvxopt (CO), DSDP5 (SDP), MOSEK (QP, SOCP), Yalmip (SDP), . . .

OPTIMIZATION PROBLEM: WHY USEFUL?

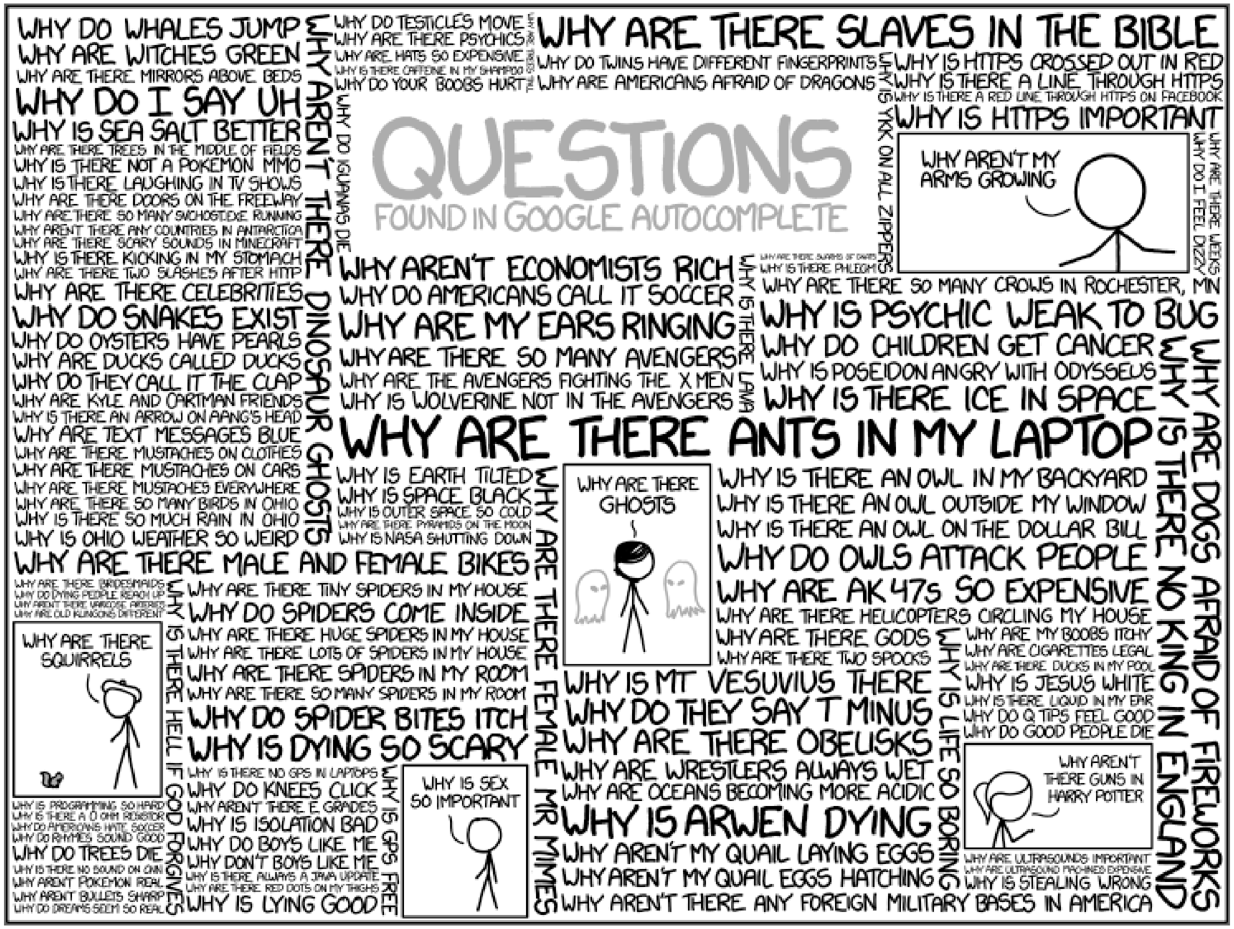
Why formulate problems as optimization problems?

- For many class of optimization problems, algorithms or algorithmic frameworks have been developed
- Decouple ”representation” and ”problem solving”

Lazy mode

- Formulate a problem as an optimization problem
- Identify which class the formulation belongs to
- Call the corresponding solver
- Done !!

Q & A


QUESTIONS
FOUND IN GOOGLE AUTOCOMPLETE
WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS
WHY DO I SAY UH
WHY IS SEA SALT BETTER
WHY ARE THERE TREES IN THE MIDDLE OF FIELDS
WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS
WHY ARE THERE DOORS ON THE FREEWAY
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA
WHY ARE THERE SCARY SOUNDS IN MINECRAFT
WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP
WHY ARE THERE CELEBRITIES
WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS
WHY ARE DUCKS CALLED DUCKS
WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD
WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS
WHY ARE THERE MUSTACHES EVERYWHERE
WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO
WHY IS OHIO WEATHER SO WEIRD
WHY AREN'T THERE DINOSAUR GHOSTS
WHY DO IGUANAS DIE
WHY DO TESTICLES MOVE
WHY ARE THERE PSYCHICS
WHY ARE HATS SO EXPENSIVE
WHY IS THERE CAFFEINE IN MY SHAMPOO
WHY DO YOUR BOOBS HURT
WHY ARE THERE SLAVES IN THE BIBLE
WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS
WHY IS YKK ON ALL ZIPPERS
WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT
WHY AREN'T MY ARMS GROWING
WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY
WHY AREN'T ECONOMISTS RICH
WHY DO AMERICANS CALL IT SOCCER
WHY ARE MY EARS RINGING
WHY ARE THERE SO MANY AVENGERS
WHY ARE THE AVENGERS FIGHTING THE X MEN
WHY IS WOLVERINE NOT IN THE AVENGERS
WHY IS THERE LAVA
WHY IS THERE PHLEGM
WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN
WHY IS PSYCHIC WEAK TO BUG
WHY DO CHILDREN GET CANCER
WHY IS POSEIDON ANGRY WITH ODYSSEUS
WHY IS THERE ICE IN SPACE
WHY ARE THERE ANTS IN MY LAPTOP
WHY IS EARTH TILTED
WHY IS SPACE BLACK
WHY IS OUTER SPACE SO COLD
WHY ARE THERE PYRAMIDS ON THE MOON
WHY IS NASA SHUTTING DOWN
WHY ARE THERE GHOSTS
WHY IS THERE AN OWL IN MY BACKYARD
WHY IS THERE AN OWL OUTSIDE MY WINDOW
WHY IS THERE AN OWL ON THE DOLLAR BILL
WHY DO OWLS ATTACK PEOPLE
WHY ARE AK 47s SO EXPENSIVE
WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE
WHY ARE THERE GODS
WHY ARE THERE TWO SPOCKS
WHY ARE MY BOOBS ITCHY
WHY ARE CIGARETTES LEGAL
WHY ARE THERE DUCKS IN MY POOL
WHY IS JESUS WHITE
WHY IS THERE LIQUID IN MY EAR
WHY DO Q TIPS FEEL GOOD
WHY DO GOOD PEOPLE DIE
WHY IS THERE NO KING IN ENGLAND
WHY ARE DOGS AFRAID OF FIREWORKS
WHY ARE THERE MALE AND FEMALE BIKES
WHY ARE THERE FEMALE MR MIMES
WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KLINGONS DIFFERENT
WHY ARE THERE SQUIRRELS
WHY IS THERE HELL IF GOD FORGIVES
WHY ARE THERE TINY SPIDERS IN MY HOUSE
WHY DO SPIDERS COME INSIDE
WHY ARE THERE HUGE SPIDERS IN MY HOUSE
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE
WHY ARE THERE SPIDERS IN MY ROOM
WHY ARE THERE SO MANY SPIDERS IN MY ROOM
WHY DO SPIDER BITES ITCH
WHY IS DYING SO SCARY
WHY IS THERE NO GPS IN LAPTOPS
WHY DO KNEES CLICK
WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS
WHY IS LYING GOOD
WHY IS GPS FREE
WHY IS SEX SO IMPORTANT
WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR
WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD
WHY DO TREES DIE
WHY IS THERE NO SOUND ON CNN
WHY AREN'T POKEMON REAL
WHY AREN'T BULLETS SHARP
WHY DO DREAMS SEEM SO REAL
WHY IS MT VESUVIUS THERE
WHY DO THEY SAY T MINUS
WHY ARE THERE OBELISKS
WHY ARE WRESTLERS ALWAYS WET
WHY ARE OCEANS BECOMING MORE ACIDIC
WHY IS ARWEN DYING
WHY AREN'T MY QUAIL LAYING EGGS
WHY AREN'T MY QUAIL EGGS HATCHING
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA
WHY IS LIFE SO BORING
WHY AREN'T THERE GUNS IN HARRY POTTER
WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
WHY IS STEALING WRONG

XKCD

Speaker notes