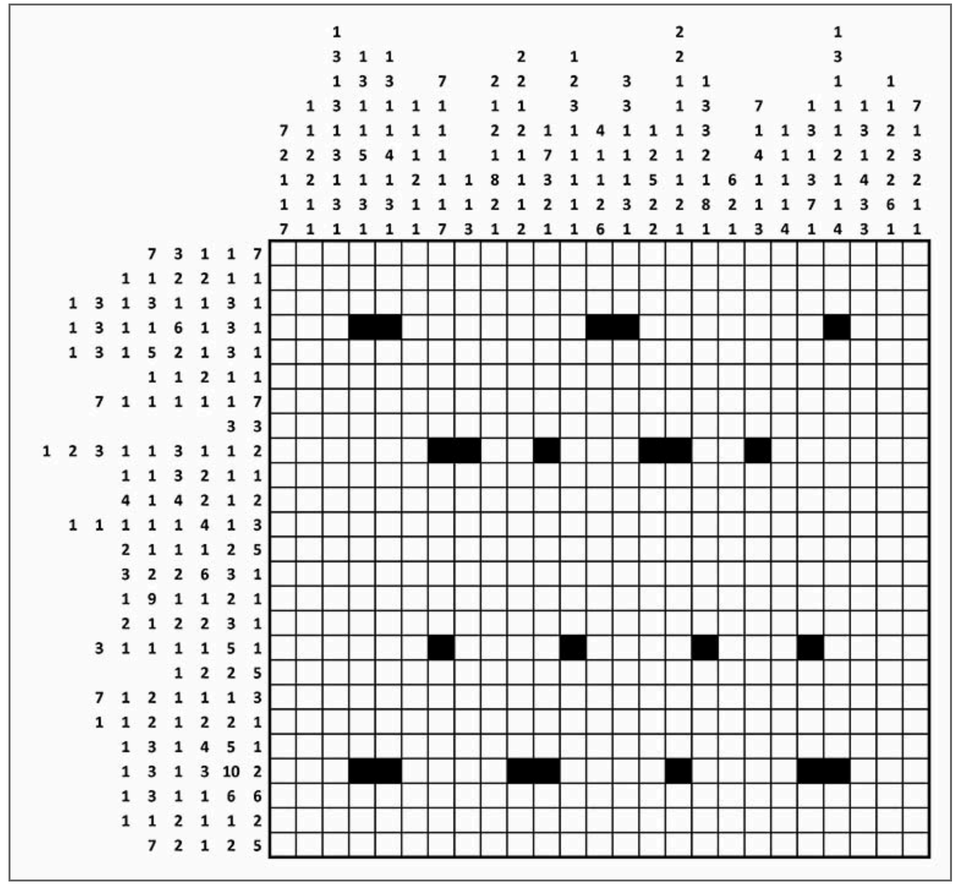
Linear Programming

CS5491: Artificial Intelligence
ZHICHAO LU

Content Credits: **Prof. Wei**'s CS4486 Course
and **Prof. Boddeti**'s AI Course

TODAY

- Solving Linear Program
- Integer Program
- Branch and Bound



GCHQ

LINEAR PROGRAMMING: WHAT TO EAT?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Health Goals

- $2000 \leq Calories \leq 2500$
- $Sugar \leq 100g$
- $Calcium \geq 700mg$

Food	Cost	Calories	Sugar	Calcium
stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

OPTIMIZATION FORMULATION

Diet Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \text{ and } \; c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

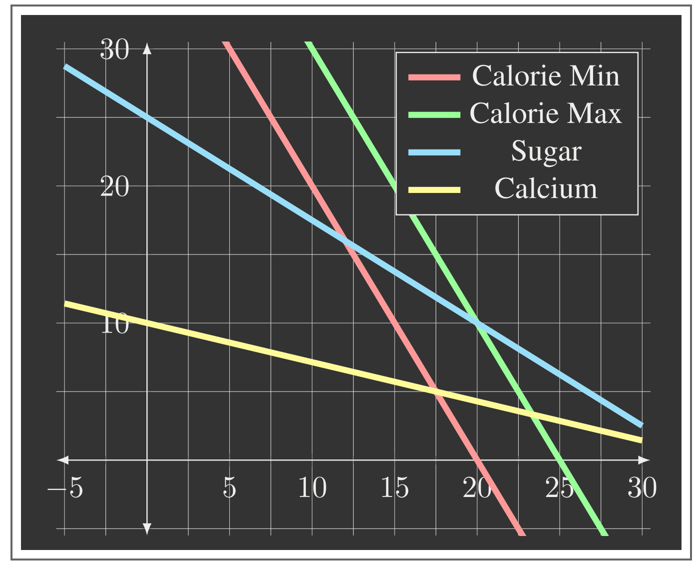
REPRESENTATION AND PROBLEM SOLVING

Problem Description

Optimization Representation

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Graphical Representation

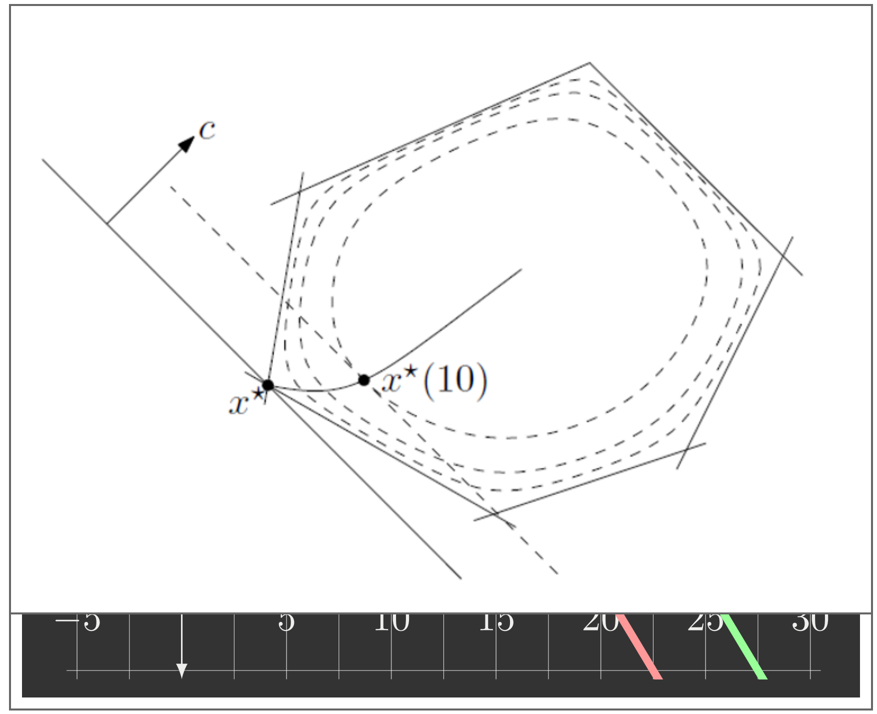


SOLVING A LP

Solutions are at feasible intersections of constraint boundaries.

Algorithms

- Check objective at all feasible intersections.
- Simplex
- Interior Point



SOLVING AN LP

But, how do we find the intersection between boundaries?

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x} \qquad s.t. \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

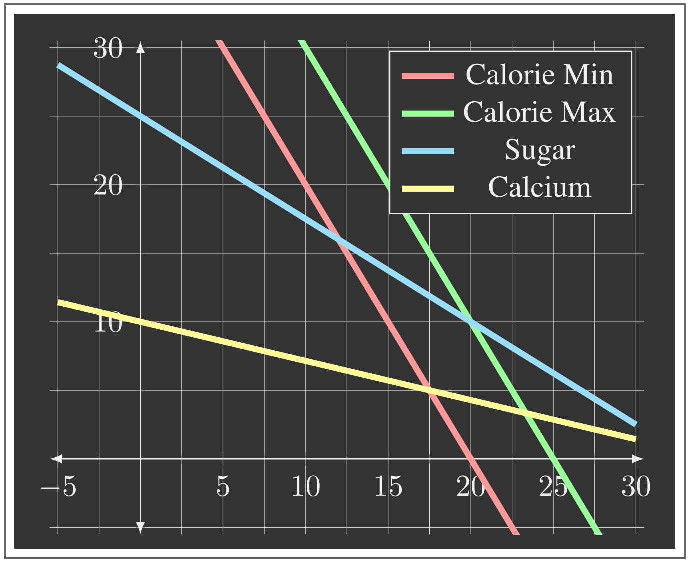
WHAT ABOUT HIGHER DIMENSIONS?

Problem Description

Optimization Representation

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Graphical Representation



SHAPES IN HIGHER DIMENSIONS

How do these linear shapes extend to 3-D, N-D?

	2-D	3-D	N-D
$a_1x_1 + a_2x_2 = b_1$	line	plane	hyperplane
$a_1x_1 + a_2x_2 \leq b_1$	halfplane	halfspace	halfspace
$a_{1,1}x_1 + a_{1,2}x_2 \leq b_1$ $a_{2,1}x_1 + a_{2,2}x_2 \leq b_2$ $a_{3,1}x_1 + a_{3,2}x_2 \leq b_3$ $a_{4,1}x_1 + a_{4,2}x_2 \leq b_4$	polygon	polyhedron	polytope

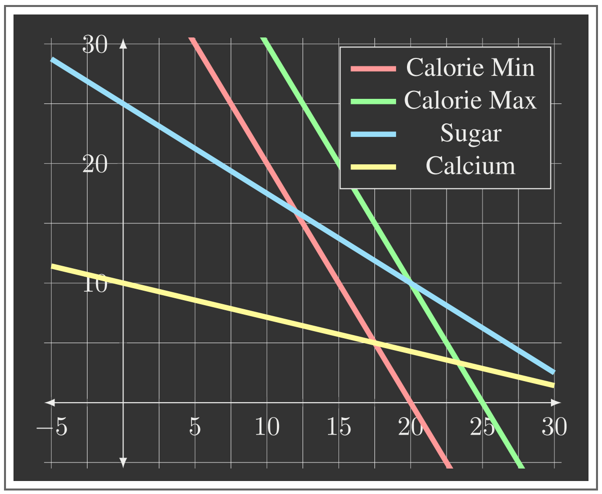
INTERSECTIONS IN HIGHER DIMENSIONS

How do these linear shapes extend to 3-D, N-D?

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



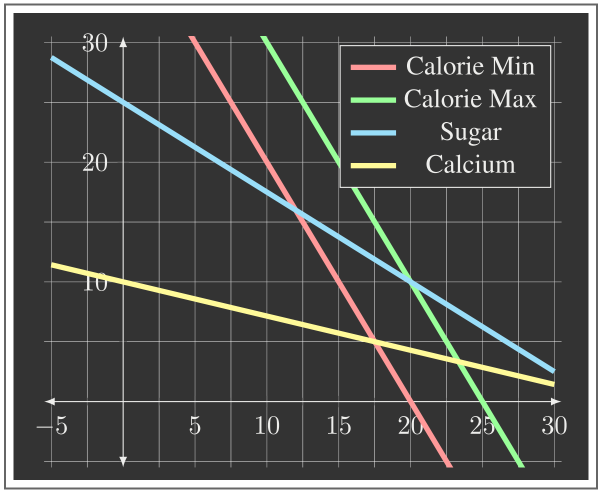
INTERSECTIONS IN HIGHER DIMENSIONS

Still looking at subsets of $\mathbf{A}$ matrix

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}
\end{aligned}
$$

$$
\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}
$$

$$
\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}
$$



LINEAR PROGRAMMING

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Health Goals

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What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

LINEAR $\longrightarrow$ INTEGER

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

Health Goals

- $2000 \leq Calories \leq 2500$
- $Sugar \leq 100g$
- $Calcium \geq 700mg$

Food	Cost	Calories	Sugar	Calcium
stir-fry (per bowl)	1	100	3	20
Boba (per glass)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (bowls) and boba (glasses) should we buy?

LINEAR VS INTEGER PROGRAMMING

Linear objective with linear constraints, but now with additional constraint that all values in x must be integers

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming
- A hybrid: Mixed Integer Linear Programming

INTEGER PROGRAMMING: GRAPHICAL REPRESENTATION

Just add a grid of integer points onto our LP representation

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

RELAXATION

Relax IP to LP by dropping integer constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

Remember Heuristics?

SOLUTION OF LINEAR VS INTEGER PROGRAM

Let y^*_{IP} be the optimal objective of an integer program P.

Let $\mathbf{x}^*_{IP}$ be the optimal point of an integer program P.

Let y^*_{LP} be the optimal objective of the LP-relaxed version of P.

Let $\mathbf{x}^*_{LP}$ be the optimal point of the LP-relaxed version of P.

Assume that P is a minimization problem.

Which of the following are true?

- $\mathbf{x}^*_{IP} = \mathbf{x}^*_{LP}$

$$\begin{aligned} y^*_{IP} = \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

$$\begin{aligned} y^*_{LP} = \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

- $y^*_{IP} \leq y^*_{LP}$
- $y^*_{IP} \geq y^*_{LP}$

SOLVING INTEGER PROGRAMS: A NAIVE SOLUTION

True/False: It is sufficient to consider the integer points around the corresponding LP solution?

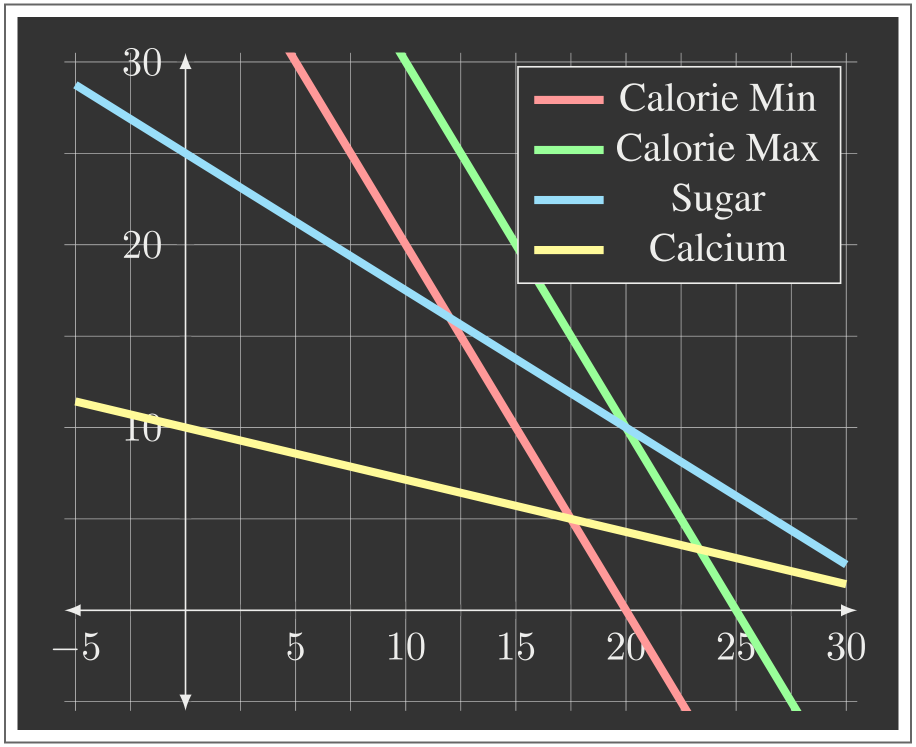
SOLVING AN IP

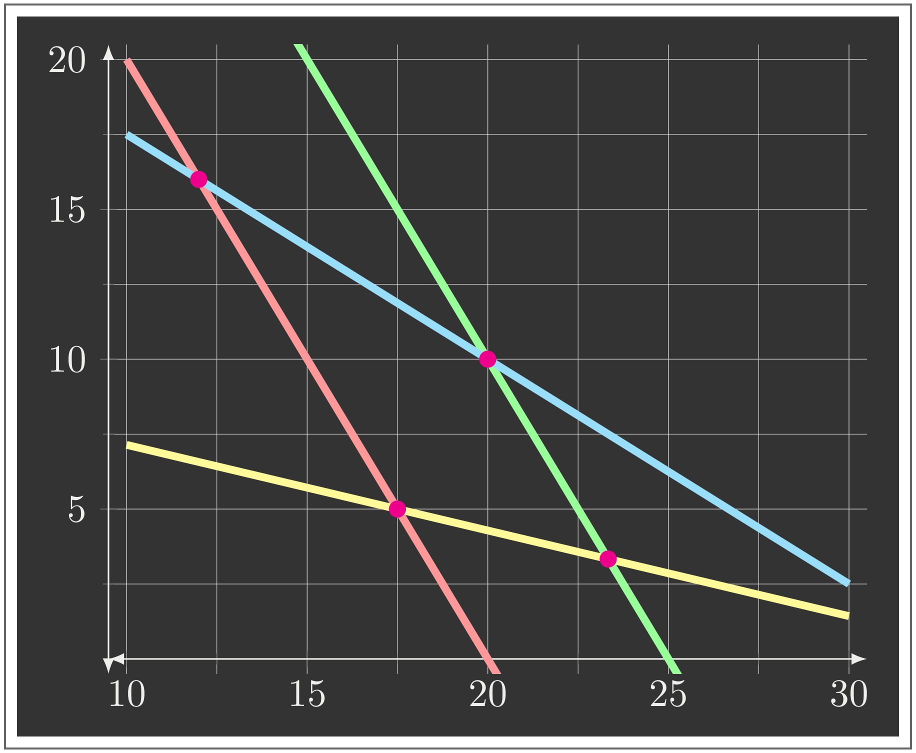
Branch and Bound algorithm

- Start with LP-relaxed version of IP
- If solution $\mathbf{x}^*_{LP}$ has non-integer value at $\mathbf{x}_i$
- Consider two branches with two different slightly more constrained LP problems:
- Left branch: Add constraint $\mathbf{x}_i \leq floor(\mathbf{x}^*_{LP})$
- Right branch: Add constraint $\mathbf{x}_i \geq ceil(\mathbf{x}^*_{LP})$
- Recursion. Stop going deeper:
- When the LP returns a worse objective than the best feasible IP objective you have seen before.
- When you hit an integer result from the LP

- When LP is infeasible

BRANCH AND BOUND EXAMPLE

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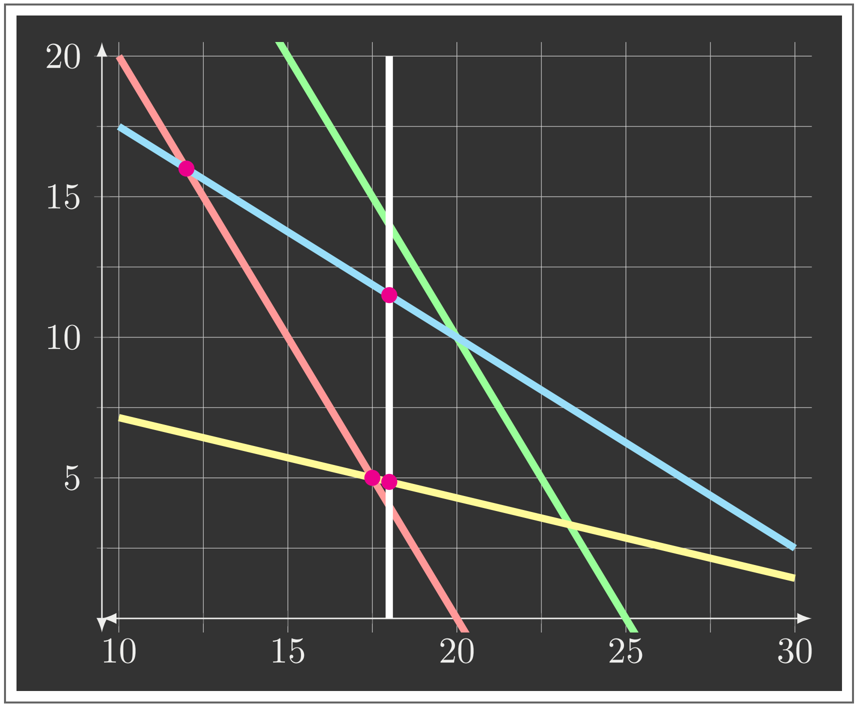
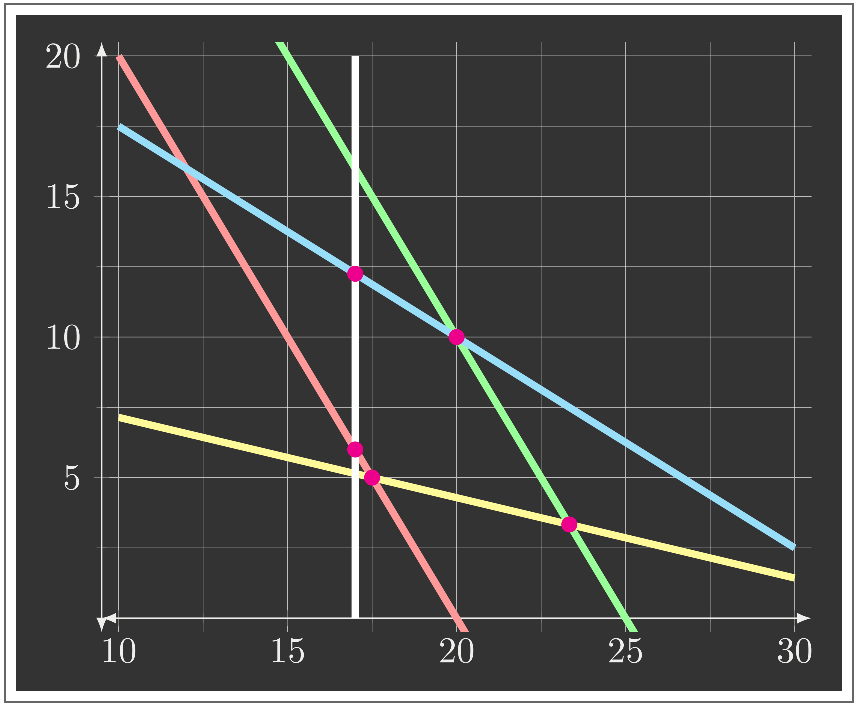
BRANCH AND BOUND EXAMPLE

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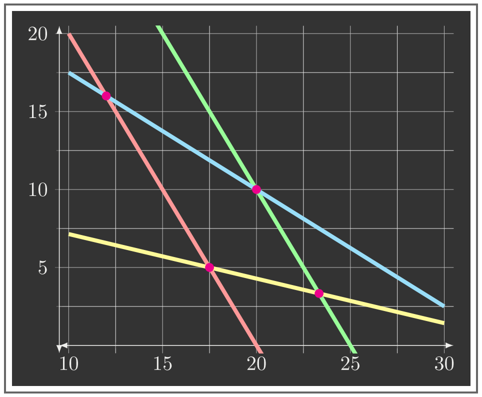
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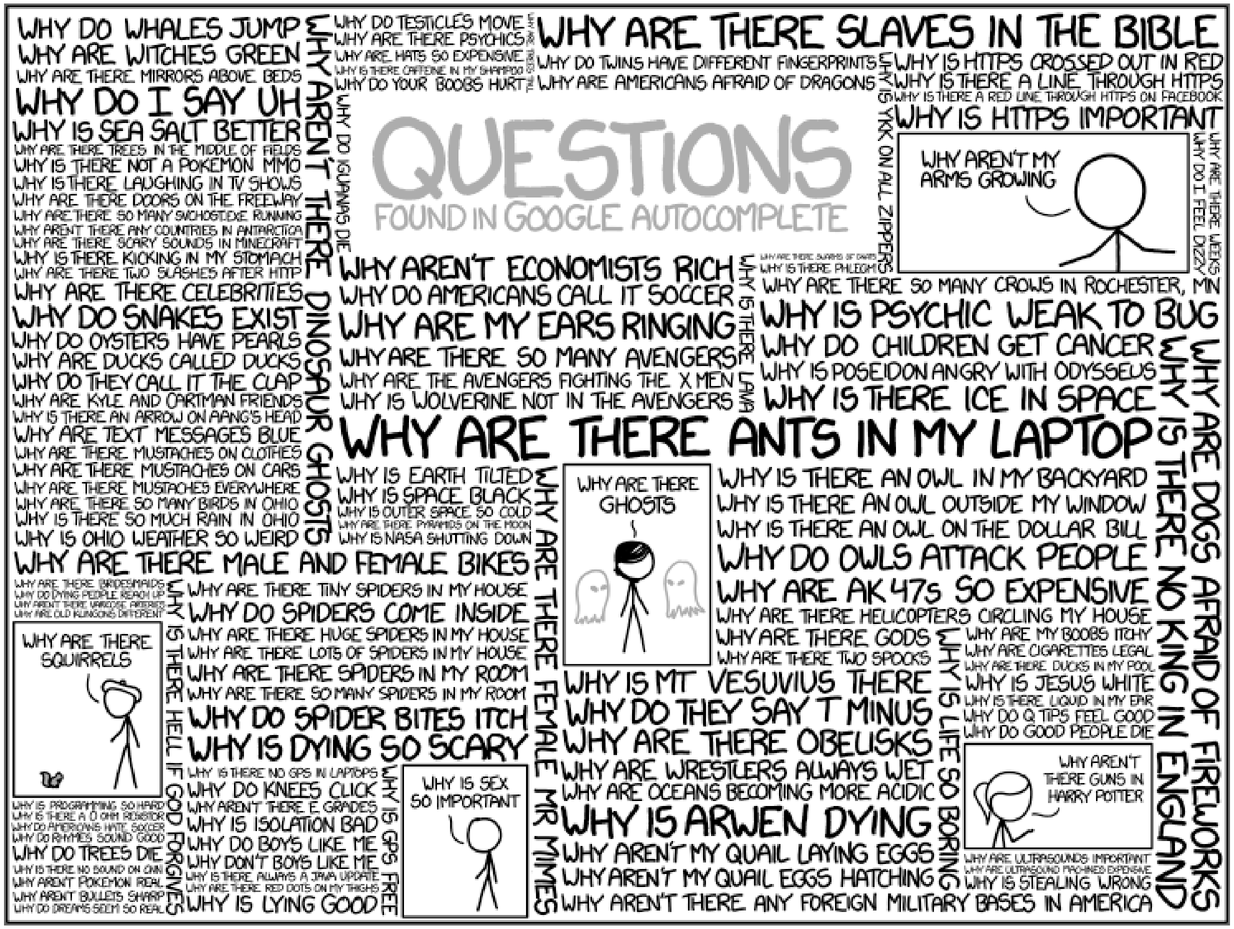
Q & A


QUESTIONS
FOUND IN GOOGLE AUTOCOMPLETE
WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS
WHY DO I SAY UH
WHY IS SEA SALT BETTER
WHY ARE THERE TREES IN THE MIDDLE OF FIELDS
WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS
WHY ARE THERE DOORS ON THE FREEWAY
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA
WHY ARE THERE SCARY SOUNDS IN MINECRAFT
WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP
WHY ARE THERE CELEBRITIES
WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS
WHY ARE DUCKS CALLED DUCKS
WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD
WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS
WHY ARE THERE MUSTACHES EVERYWHERE
WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO
WHY IS OHIO WEATHER SO WEIRD
WHY AREN'T THERE DINOSAUR GHOSTS
WHY DO IGUANAS DIE
WHY DO TESTICLES MOVE
WHY ARE THERE PSYCHICS
WHY ARE HATS SO EXPENSIVE
WHY IS THERE CAFFEINE IN MY SHAMPOO
WHY DO YOUR BOOBS HURT
WHY ARE THERE SLAVES IN THE BIBLE
WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS
WHY IS YKK ON ALL ZIPPERS
WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT
WHY AREN'T MY ARMS GROWING
WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY
WHY AREN'T ECONOMISTS RICH
WHY DO AMERICANS CALL IT SOCCER
WHY ARE MY EARS RINGING
WHY ARE THERE SO MANY AVENGERS
WHY ARE THE AVENGERS FIGHTING THE X MEN
WHY IS WOLVERINE NOT IN THE AVENGERS
WHY IS THERE LAVA
WHY IS THERE PHLEGM
WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN
WHY IS PSYCHIC WEAK TO BUG
WHY DO CHILDREN GET CANCER
WHY IS POSEIDON ANGRY WITH ODYSSEUS
WHY IS THERE ICE IN SPACE
WHY ARE THERE ANTS IN MY LAPTOP
WHY IS EARTH TILTED
WHY IS SPACE BLACK
WHY IS OUTER SPACE SO COLD
WHY ARE THERE PYRAMIDS ON THE MOON
WHY IS NASA SHUTTING DOWN
WHY ARE THERE GHOSTS
WHY IS THERE AN OWL IN MY BACKYARD
WHY IS THERE AN OWL OUTSIDE MY WINDOW
WHY IS THERE AN OWL ON THE DOLLAR BILL
WHY DO OWLS ATTACK PEOPLE
WHY ARE AK 47s SO EXPENSIVE
WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE
WHY ARE THERE GODS
WHY ARE THERE TWO SPOCKS
WHY ARE MY BOOBS ITCHY
WHY ARE CIGARETTES LEGAL
WHY ARE THERE DUCKS IN MY POOL
WHY IS JESUS WHITE
WHY IS THERE LIQUID IN MY EAR
WHY DO Q TIPS FEEL GOOD
WHY DO GOOD PEOPLE DIE
WHY IS THERE NO KING IN ENGLAND
WHY ARE DOGS AFRAID OF FIREWORKS
WHY ARE THERE MALE AND FEMALE BIKES
WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KLINGONS DIFFERENT
WHY ARE THERE SQUIRRELS
WHY IS THERE HELL IF GOD FORGIVES
WHY ARE THERE TINY SPIDERS IN MY HOUSE
WHY DO SPIDERS COME INSIDE
WHY ARE THERE HUGE SPIDERS IN MY HOUSE
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE
WHY ARE THERE SPIDERS IN MY ROOM
WHY ARE THERE SO MANY SPIDERS IN MY ROOM
WHY DO SPIDER BITES ITCH
WHY IS DYING SO SCARY
WHY ARE THERE FEMALE MR MIMES
WHY IS MT VESUVIUS THERE
WHY DO THEY SAY T MINUS
WHY ARE THERE OBELISKS
WHY ARE WRESTLERS ALWAYS WET
WHY ARE OCEANS BECOMING MORE ACIDIC
WHY IS ARWEN DYING
WHY AREN'T MY QUAIL LAYING EGGS
WHY AREN'T MY QUAIL EGGS HATCHING
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA
WHY IS LIFE SO BORING
WHY AREN'T THERE GUNS IN HARRY POTTER
WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
WHY IS STEALING WRONG
WHY IS THERE NO GPS IN LAPTOPS
WHY DO KNEES CLICK
WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS
WHY IS LYING GOOD
WHY IS GPS FREE
WHY IS SEX SO IMPORTANT
WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR
WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD
WHY DO TREES DIE
WHY IS THERE NO SOUND ON CNN
WHY AREN'T POKEMON REAL
WHY AREN'T BULLETS SHARP
WHY DO DREAMS SEEM SO REAL

XKCD

Speaker notes