Linear Programming

CS5491: Artificial Intelligence
ZHICHAO LU

Content Credits: **Prof. Wei**'s CS4486 Course
and **Prof. Boddeti**'s AI Course

WHAT TO EAT?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Health Goals

- $2000 \leq Calories \leq 2500$
- $Sugar \leq 100g$
- $Calcium \geq 700mg$

Food	Cost	Calories	Sugar	Calcium
stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

CAN SEARCH ALGORITHMS HELP?

Essence of search algorithms:

- explore all nodes in search tree until you reach the one you want
- use additional information to avoid search all nodes

Can we adopt a similar strategy here?

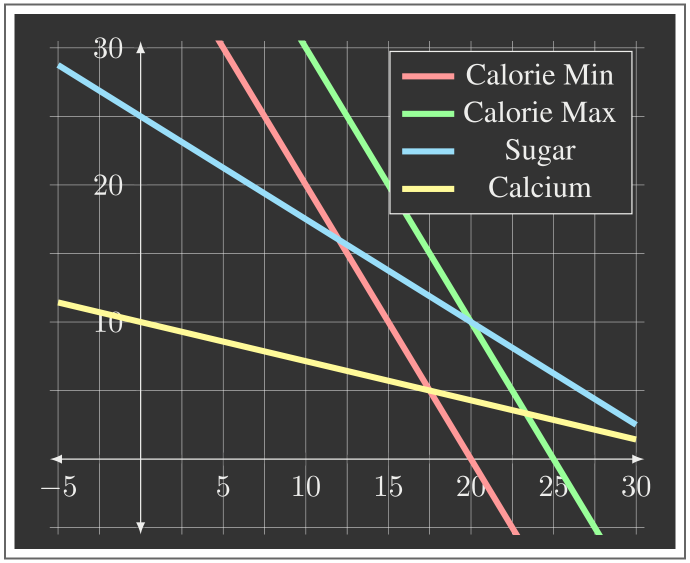
OPTIMIZATION

Problem Description

Optimization Representation

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Graphical Representation



WHAT TO EAT?: CONSTRAINTS

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CONSTRAINT SATISFACTION PROBLEMS

Map Coloring

$$Any\ \mathbf{x}$$

$$s.t.\ \mathbf{x}\ \text{satisfies constraints}$$

WHAT TO EAT?: COSTS

We are trying healthy by finding the optimal amount of food to purchase.

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OPTIMIZATION FORMULATION

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> Diet Problem
>
> $$\begin{aligned} \min_{\mathbf{x}} \quad & cost(\mathbf{x}) \\ s.t. \quad & \mathbf{x} \text{ satisfies constraints} \end{aligned}$$

OPTIMIZATION FORMULATION

Health Goals

- $2000 \leq$ Calories $\leq$ 2500
- Sugar $\leq$ 100g
- Calcium $\geq$ 700mg

Food	Cost	Calories	Sugar	Calcium
stir-fry (per oz)	1	100	3	20
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Diet Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & cost(\mathbf{x}) \\ s.t. \quad & calories(\mathbf{x}) \leq limit \\ & calories(\mathbf{x}) \geq limit \\ & sugar(\mathbf{x}) \leq limit \\ & calcium(\mathbf{x}) \geq limit \end{aligned}$$

OPTIMIZATION FORMULATION

Health Goals

- $2000 \leq Calories \leq 2500$
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- $Calcium \geq 700mg$

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stir-fry (per oz)	1	100	3	20
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> Diet Problem
>
> $$\begin{aligned} \min_{\mathbf{x}} \quad & 1x_1 + 0.5x_2 \\ s.t. \quad & 100x_1 + 50x_2 \geq 2000 \\ & 100x_1 + 50x_2 \leq 2500 \\ & 3x_1 + 4x_2 \leq 100 \\ & 20x_1 + 70x_2 \geq 700 \end{aligned}$$

OPTIMIZATION FORMULATION

Diet Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & c_1 x_1 + c_2 x_2 \\ s.t. \quad & a_{1,1}x_1 + a_{1,2}x_2 \geq b_1 \\ & a_{2,1}x_1 + a_{2,2}x_2 \leq b_2 \\ & a_{3,1}x_1 + a_{3,2}x_2 \leq b_3 \\ & a_{4,1}x_1 + a_{4,2}x_2 \geq b_4 \end{aligned}$$

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \quad b = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

OPTIMIZATION FORMULATION

Diet Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & a_{1,1}x_1 + a_{1,2}x_2 \geq b_1 \\ & a_{2,1}x_1 + a_{2,2}x_2 \leq b_2 \\ & a_{3,1}x_1 + a_{3,2}x_2 \leq b_3 \\ & a_{4,1}x_1 + a_{4,2}x_2 \geq b_4 \end{aligned}$$

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OPTIMIZATION FORMULATION

Diet Problem

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & -a_{1,1}x_1 - a_{1,2}x_2 \leq -b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 \leq b_2 \\
& a_{3,1}x_1 + a_{3,2}x_2 \leq b_3 \\
& -a_{4,1}x_1 - a_{4,2}x_2 \leq -b_4
\end{aligned}
$$

$$
A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \quad b = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}
$$

OPTIMIZATION FORMULATION

Diet Problem

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & a_{1,1}x_1 + a_{1,2}x_2 \leq b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 \leq b_2 \\
& a_{3,1}x_1 + a_{3,2}x_2 \leq b_3 \\
& a_{4,1}x_1 + a_{4,2}x_2 \leq b_4
\end{aligned}
$$

$$
A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}
$$

OPTIMIZATION FORMULATION

Diet Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

LINEAR PROGRAMMING

Linear objective with linear constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

As opposed to general optimization

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ s.t. \quad & f_i(\mathbf{x}) \leq \mathbf{0}, i = 1, \ldots, M \\ & \mathbf{a}_i^T\mathbf{x} = \mathbf{b}_i, i = 1, \ldots, P \end{aligned}$$

LINEAR PROGRAMMING

Different formulations

Inequality Form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

General Form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} + d \\ s.t. \quad & \mathbf{G}\mathbf{x} \leq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

Standard Form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

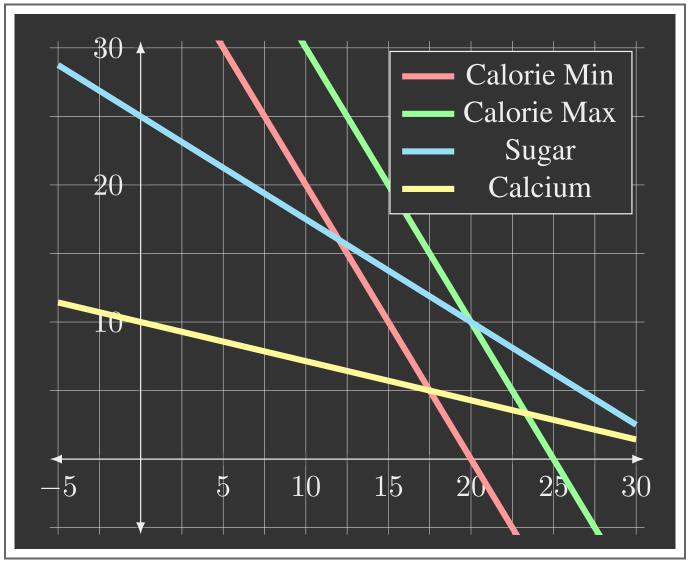
OPTIMIZATION

Problem Description

Optimization Representation

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}
\end{aligned}
$$

Graphical Representation



GRAPHICS REPRESENTATION

What shape does this inequality represent?

$$a_1x_1 + a_2x_2 \leq b_1$$

COST CONTOURS

Given the cost vector $[c_1, c_2]^T$ where will,

- $\mathbf{c}^T\mathbf{x} = 0$
- $\mathbf{c}^T\mathbf{x} = 1$
- $\mathbf{c}^T\mathbf{x} = 2$
- $\mathbf{c}^T\mathbf{x} = -1$
- $\mathbf{c}^T\mathbf{x} = -2$

LP GRAPHICAL REPRESENTATION

Inequality Form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Start with no constraints

Add one constraint at a time

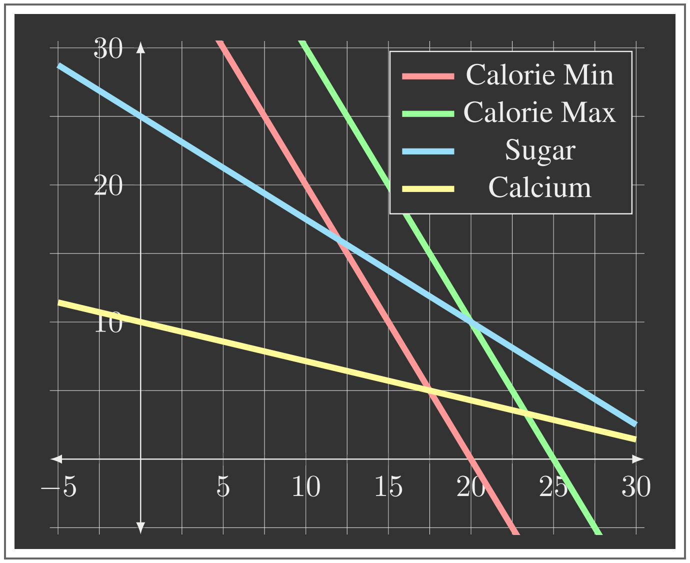
OPTIMIZATION

Problem Description

Optimization Representation

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}
\end{aligned}
$$

Graphical Representation



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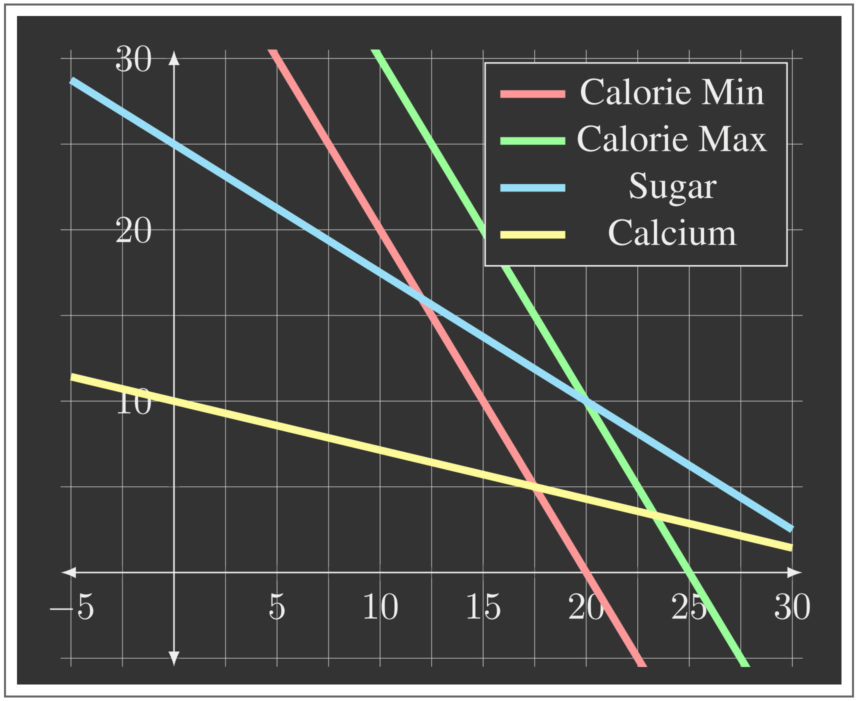
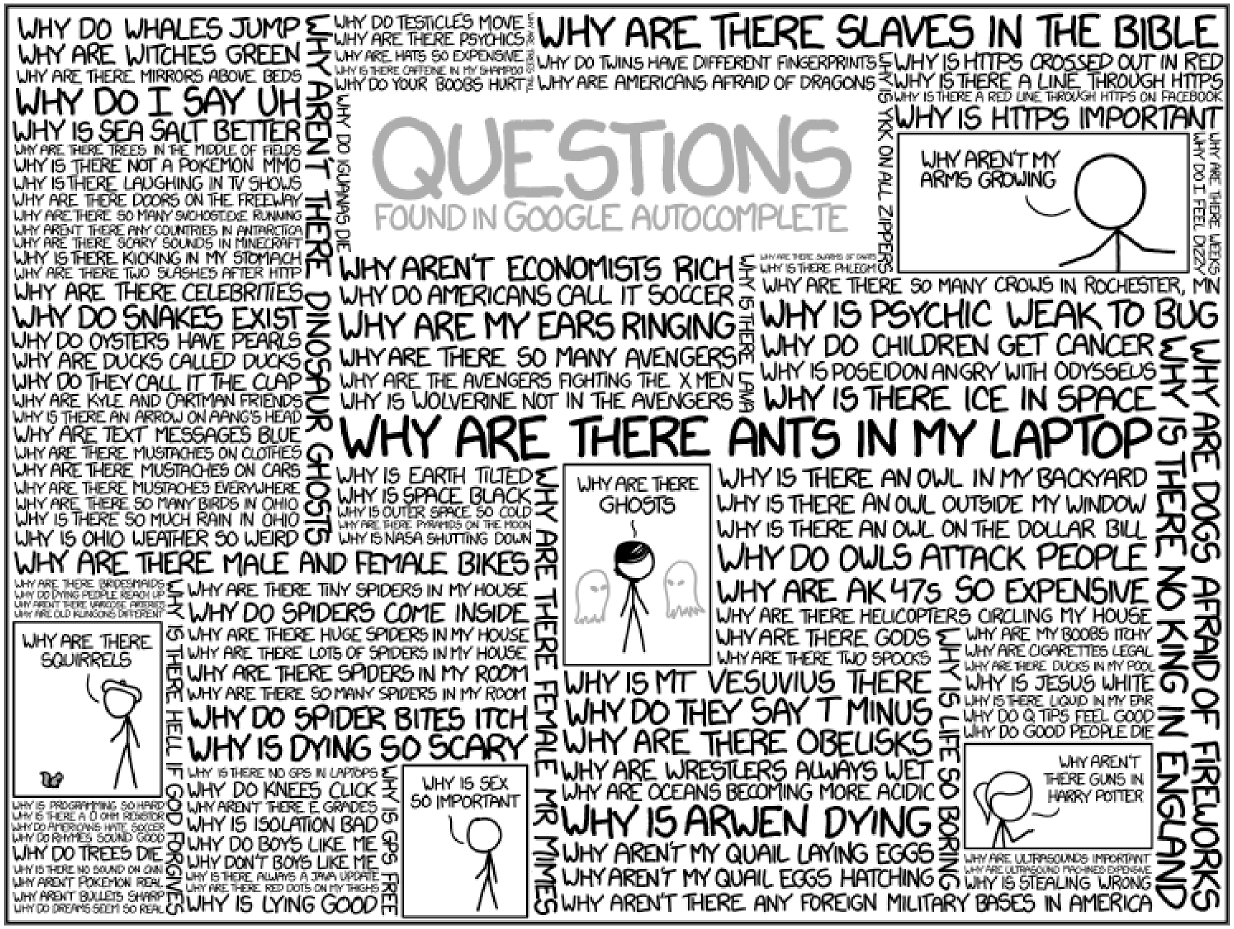
$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

SOLVING AN LP

Solutions are at feasible intersections of constraint boundaries!!



Q & A



QUESTIONS
FOUND IN GOOGLE AUTOCOMPLETE
WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS
WHY DO I SAY UH
WHY IS SEA SALT BETTER
WHY ARE THERE TREES IN THE MIDDLE OF FIELDS
WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS
WHY ARE THERE DOORS ON THE FREEWAY
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA
WHY ARE THERE SCARY SOUNDS IN MINECRAFT
WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP
WHY ARE THERE CELEBRITIES
WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS
WHY ARE DUCKS CALLED DUCKS
WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD
WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS
WHY ARE THERE MUSTACHES EVERYWHERE
WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO
WHY IS OHIO WEATHER SO WEIRD
WHY ARE THERE MALE AND FEMALE BIKES
WHY AREN'T THERE DINOSAUR GHOSTS
WHY DO IGUANAS DIE
WHY DO TESTICLES MOVE
WHY ARE THERE PSYCHICS
WHY ARE HATS SO EXPENSIVE
WHY DO YOUR BOOBS HURT
WHY ARE THERE SLAVES IN THE BIBLE
WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS
WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT
WHY IS YKK ON ALL ZIPPERS
WHY AREN'T MY ARMS GROWING
WHY DO I FEEL DIZZY
WHY ARE THERE WEEKS
WHY AREN'T ECONOMISTS RICH
WHY DO AMERICANS CALL IT SOCCER
WHY ARE MY EARS RINGING
WHY ARE THERE SO MANY AVENGERS
WHY ARE THE AVENGERS FIGHTING THE X MEN
WHY IS WOLVERINE NOT IN THE AVENGERS
WHY IS THERE LAVA
WHY IS THERE PHLEGM
WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN
WHY IS PSYCHIC WEAK TO BUG
WHY DO CHILDREN GET CANCER
WHY IS POSEIDON ANGRY WITH ODYSSEUS
WHY IS THERE ICE IN SPACE
WHY ARE THERE ANTS IN MY LAPTOP
WHY IS EARTH TILTED
WHY IS SPACE BLACK
WHY IS OUTER SPACE SO COLD
WHY ARE THERE PYRAMIDS ON THE MOON
WHY IS NASA SHUTTING DOWN
WHY ARE THERE FEMALE MR MIMES
WHY ARE THERE GHOSTS
WHY IS THERE AN OWL IN MY BACKYARD
WHY IS THERE AN OWL OUTSIDE MY WINDOW
WHY IS THERE AN OWL ON THE DOLLAR BILL
WHY DO OWLS ATTACK PEOPLE
WHY ARE AK 47s SO EXPENSIVE
WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE
WHY ARE THERE GODS
WHY ARE THERE TWO SPOCKS
WHY IS MT VESUVIUS THERE
WHY DO THEY SAY T MINUS
WHY ARE THERE OBELISKS
WHY ARE WRESTLERS ALWAYS WET
WHY ARE OCEANS BECOMING MORE ACIDIC
WHY IS ARWEN DYING
WHY AREN'T MY QUAIL LAYING EGGS
WHY AREN'T MY QUAIL EGGS HATCHING
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA
WHY IS LIFE SO BORING
WHY ARE MY BOOBS ITCHY
WHY ARE CIGARETTES LEGAL
WHY ARE THERE DUCKS IN MY POOL
WHY IS JESUS WHITE
WHY IS THERE LIQUID IN MY EAR
WHY DO Q TIPS FEEL GOOD
WHY DO GOOD PEOPLE DIE
WHY AREN'T THERE GUNS IN HARRY POTTER
WHY ARE ULTRASOUNDS IMPORTANT
WHY IS STEALING WRONG
WHY IS THERE NO KING IN ENGLAND
WHY ARE DOGS AFRAID OF FIREWORKS
WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KLINGONS DIFFERENT
WHY ARE THERE SQUIRRELS
WHY IS THERE HELL IF GOD FORGIVES
WHY ARE THERE TINY SPIDERS IN MY HOUSE
WHY DO SPIDERS COME INSIDE
WHY ARE THERE HUGE SPIDERS IN MY HOUSE
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE
WHY ARE THERE SPIDERS IN MY ROOM
WHY ARE THERE SO MANY SPIDERS IN MY ROOM
WHY DO SPIDER BITES ITCH
WHY IS DYING SO SCARY
WHY IS THERE NO GPS IN LAPTOPS
WHY DO KNEES CLICK
WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS
WHY IS LYING GOOD
WHY IS GPS FREE
WHY IS SEX SO IMPORTANT
WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR
WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD
WHY DO TREES DIE
WHY IS THERE NO SOUND ON CNN
WHY AREN'T POKEMON REAL
WHY AREN'T BULLETS SHARP
WHY DO DREAMS SEEM SO REAL

XKCD

Speaker notes