MASx52: Assignment 1

Solutions and discussion are written in blue. A sample mark scheme, with a total of 20 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Recall the one-period market, and its parameters r, u, d, p_u, p_d and s. We assume that $d < 1 + r < u$.

   (a) At time $t = 0$ our portfolio contains 2 unit of cash and 3 units of stock. What is the value of our portfolio at time $t = 0$? If we hold this portfolio until time $t = 1$, what is its new value?

   (b) A rival investor holds a portfolio containing 3 units of cash and 2 unit of stock. Under what condition (on the parameters) can we be *certain* that our own portfolio will have a strictly greater value at time $t = 1$?

   Solution.

   (a) At time $t = 0$, our portfolio has value $2 + 3s$. [1] At time $t = 1$, our portfolio has value $2(1+r)+3S_1$ where S_1 is a random variable with $\mathbb{P}[S_1 = su] = p_u$ and $\mathbb{P}[S_1 = sd] = p_d$. [1]

   Pitfall: You are asked for the value, and not the expected value.

   (b) The value of the rival investors portfolio at time $t = 1$ is $3(1 + r) + 2S_1$. [1] This means that our own portfolio is worth strictly more when

   $$2(1 + r) + 3S_1 > 3(1 + r) + 2S_1$$

   or, equivalently, when

   $$S_1 > 1 + r.$$

   [1] To be certain that this inequality holds occurs, we must consider a 'worst case scenario' for the value of S_1. That is, we are certain that our own portfolio will have greater value if and only if

   $$sd > 1 + r.$$

   [1] (In words, this equation says that stock is certain to outperform cash.)

   Pitfall: There in only one type of stock in the one-period model, so we own the same type of stock as our rival (i.e. if theirs goes up/down, so does ours).

2. Let $\Omega = \{HH, HT, TH, TT\}$, representing two coin tosses each of which may show either H (head) or T (tail). Let $X : \Omega \to \mathbb{R}$ be the toss in which the first head occurred, or zero if no heads occurred:

   $$X = \begin{cases} 0 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT \text{ or } \omega = HH \\ 2 & \text{if } \omega = TH. \end{cases}$$

   Let Y be the total number of heads that occurred in both tosses.

(a) Write down the sets $X^{-1}(0)$, $X^{-1}(1)$ and $X^{-1}(2)$.

(b) List the elements of $\sigma(X)$.

(c) Is Y measurable with respect to $\sigma(X)$? Justify your answer.

Solution.

(a) The pre-images are $X^{-1}(0) = \{TT\}$, $X^{-1}(1) = \{HT, HH\}$ and $X^{-1}(2) = \{TH\}$. [2]

(b) $\sigma(X) = \Big\{\emptyset, \Omega, \{TT\}, \{HT, HH\}, \{TH\}, \{TT, HT, HH\}, \{TT, TH\}, \{HT, HH, TH\}\Big\}$. [2] (To construct $\sigma(X)$, we start by adding in the pre-images from (a), along with $\emptyset$ and Ω, and then include all the unions and complements that we can make from currently added subset of Ω, until we can't find any new ones.)

(c) We note that (for example) $Y^{-1}(1) = \{HT, TH\}$ [1] which is not an element of $\sigma(X)$. Hence Y is not $\sigma(X)$-measurable. [1]

3. Let $\Omega = \{1, 2, 3, 4, 5\}$, representing one roll of a five sided dice. In each case, match the σ-field to the information it contains.

(a) $\big\{\emptyset, \Omega, \{1\}, \{2, 3, 4, 5\}\big\}$

(b) $\sigma(\{1, 2, 3\}, \{3, 4, 5\})$

(c) $\big\{\emptyset, \Omega, \{1\}, \{2, 3, 4\}, \{5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 5\}\big\}$

(i) If the roll was less than or equal to 3.

(ii) If the roll was the minimum possible value, or the maximum possible value, or neither.

(iii) If the roll was equal to 3, or strictly less three, or strictly greater than 3.

(iv) If the roll was a 1 or not.

Solution. (a)-(iv), (b)-(iii), (c)-(ii) [3]

4. Let X be a random variable.

(a) Show that $Y = \cos X$ is also a random variable.

(b) For which $p \in [1, \infty)$ do we have $Y \in L^p$?

Solution.

(a) By definition,

$$\cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!},$$

which is an infinite series that converges for all $x \in \mathbb{R}$.

Recall that adding together random variables gives random variables, that multiplying together random variables gives random variables [2]. Using these facts repeatedly, we have that

$$
\begin{aligned}
Y_n(\omega) &= \sum_{i=0}^{n} \frac{(-1)^i \big(X_n(\omega)\big)^{2i}}{(2i)!} \\
&= 1 - \frac{X_n(\omega)^2}{2} + \frac{X_n(\omega)^4}{24} - \ldots + \frac{(-1)^n \big(X_n(\omega)\big)^{2n}}{(2n)!}
\end{aligned}
$$

is a random variable, for each $n \in \mathbb{N}$. [1] By definition of cos, we have $Y(\omega) = \lim_{n\to\infty} Y_n(\omega)$ for all $\omega \in \Omega$. Since limits of random variables (when they exist) are also random variables, Y is a random variable. [1]

Pitfall: The $\sum_0^\infty \ldots$ is an infinite sum – a limit of the finite sums.

(b) Recall that $|\cos x| \leq 1$ for all $x \in \mathbb{R}$. Hence $|Y| \leq 1$, and hence by monotonicity of $\mathbb{E}$,

$$\mathbb{E}[|Y|^p] \leq \mathbb{E}[1^p] = 1.$$

[1] Therefore, $Y \in L^p$ for all $p \in [1, \infty)$. [1]