MAS350: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme, with a total of 30 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Determine if the following functions are Lebesgue integrable.

   (a) $f : (0,\infty) \to \mathbb{R}$ by $f(x) = 1/x^2$.

   (b) $g : (0,1) \to \mathbb{R}$ by $g(x) = \log x$

   Solution.

   (a) Note that $x^{-2} > 0$ for $x \in (0,\infty)$. By Riemann integration, we have

   $$\int_{1/n}^{n} x^{-2}\,dx = \left[-x^{-1}\right]_{1/n}^{n} = -\frac{1}{n} + n.$$

   [1] Note that $f_n(x) = \mathbb{1}_{\{x\in(1/n,n)\}}x^{-2}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $f(x) = x^{-2}$ for $x \in (0,\infty)$. [1] Hence, by the monotone convergence theorem [1] we have

   $$\int_0^\infty x^{-2}\,dx = \lim_{n\to\infty}\left(-\frac{1}{n}+n\right) = +\infty.$$

   Thus x^{-2} is not integrable on $(0,\infty)$. [1]

   (b) By Riemann integration, we have

   $$\int_{1/n}^{1} \log x\,dx = [x\log x - x]_{1/n}^{1} = (-1) - \left(\frac{1}{n}\log\frac{1}{n} - \frac{1}{n}\right) = \frac{1+\log n}{n} - 1.$$

   Noting that $\log x \in (-\infty, 0)$ for $x \in (0,1)$, multiplying the above by -1 gives

   $$\int_{1/n}^{1} |\log x|\,dx = 1 - \frac{1+\log n}{n}.$$

   [1] We have that $g_n(x) = |\log x|\mathbb{1}_{x\in(1/n,1)}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $g(x) = |\log x|$ for $x \in (0,1)$. [1] Hence, by the monotone convergence theorem,

   $$\int_0^1 |\log x|\,dx = \lim_{n\to\infty}\left(1 - \frac{1+\log n}{n}\right) = 1.$$

   Thus $\log x$ is integrable on $(0,1)$. [1]

2. Let $f_n, f : [0,1] \to \mathbb{R}$. In each of the following cases, explain whether the Monotone and/or Dominated Convergence Theorems can be used to prove that $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$.

   (a) $f_n(x) = \cos(\frac{x}{n}) + \sin(\frac{x}{n})$ and $f(x) = 1$.

(b) $f_n(x) = \mathbb{1}_{[\frac{1}{n},1]}(x)\,x^{-1}$ and $f(x) = \mathbb{1}_{(0,1]}x^{-1}$.

(c) $f_n(x) = \mathbb{1}_{[0,\frac{1}{n}]}(x)\,n$ and $f(x) = 0$.

Solution.

(a) DCT only (the MCT can't be used here because $f_n \le f_{n+1}$ doesn't hold). [2]

(b) MCT only (the DCT can't be used here because $\int_0^1 f(x)\,dx = \infty$). [2]

(c) Neither, in this case $\int_0^1 f_n(x)\,dx = 1$ and $\int_0^1 f(x)\,dx = 0$. [2]

3. Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ denotes the restriction of Lebesgue measure to the Borel σ-field $\mathcal{B}([0,1])$ on $[0,1]$.

Let $X_n(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ \omega n^{3/2} & \text{if } \omega \in (0, \frac{1}{n}] \\ 0 & \text{if } \omega \in (\frac{1}{n}, 1]. \end{cases}$

Determine in which modes of convergence we have $X_n \to 0$.

Solution. For any $n \in \mathbb{N}$, we have $\{X_n \neq 0\} = \{\omega \in [0,1] \,:\, X_n(\omega) \neq 0\} \subseteq [0, \frac{1}{n}]$. [1] Hence, for any $a > 0$,

$$\lambda(\{|X_n - 0| < a\}) \le \lambda([0, \tfrac{1}{n}]) = \frac{1}{n}$$

which converges to zero as $n \to \infty$. Hence $X_n \xrightarrow{\mathbb{P}} 0$. [1] It follows that also $X_n \xrightarrow{d} 0$. [1]

Fix some $\omega \in [0,1]$. If $\omega \in (0,1]$ then for all large enough n we have $\frac{1}{n} < \omega$. For such n we have $X_n(\omega) = 0$, [1] which means $X_n(\omega) \to 0$. We thus obtain that $\{X_n(\omega) \to 0\} \subseteq (0,1]$ [1] so

$$\lambda(\{X_n(\omega) \to 0\}) \ge \lambda((0,1]) = 1,$$

which means that $X_n \xrightarrow{a.s.} 0$. [1]

Lastly, the expectation of $|X_n|^p = X_n^p$ is given by

$$\begin{aligned} \mathbb{E}[X_n^p] &= \int_0^1 X_n(\omega)^p \, d\lambda(\omega) \\ &= \int_0^{\frac{1}{n}} \omega^p n^{3p/2} \, d\lambda(\omega) \\ &= n^{3p/2} \left[\frac{\omega^{p+1}}{p+1}\right]_0^{\frac{1}{n}} \\ &= n^{3p/2} \frac{(1/n)^{p+1}}{p+2} \\ &= n^{p/2-1}. \end{aligned}$$

[2] Here we use that $\{0\}$ is a λ-null subset of $[0,1]$ (so values of X_n here have no effect on the integral) [1] and that $X_n(\omega) = 0$ when $\omega > \frac{1}{n}$. [1]

Noting that $n^{p/2-1} \to 0$ if and only if $p < 2$, we have that $X_n \xrightarrow{p} 0$ if and only if $p < 2$. [1]

[I would accept "$p = 1$ works but $p = 2, 3, 4 \dots$ does not"]

4. (a) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0,1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$.

   (b) Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.

Solution.

(a) We have $\mathbb{P}[U_n \le a] = a$. For any (deterministic) sequence (x_n) the events $\{U_n < x_n\}$ are independent, because the U_n are independent. [1] Noting that $\sum 1/n = \infty$ and $\sum 1/n^2 < \infty$, we have $\sum_n \mathbb{P}[U_n < 1/n] = \infty$ and $\sum_n \mathbb{P}[U_n < 1/n^2] < \infty$. [1] By the second Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and by the first Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$. [1]

(b) Let $E_n = \{X_n + i = i \text{ for } i = 1,2,3,4,5,6\}$. We have $\mathbb{P}[E_n] = (1/6)^6 > 0$. Note that E_n and E_{n+6} are independent (but E_n and E_{n+1} are not!). [1] We have $\sum_{n=1}^{\infty} \mathbb{P}[E_{6n}] = \sum_{n=1}^{\infty}(1/6)^n = \infty$, [1] hence by the second Borel-Cantelli lemma we have $\mathbb{P}[E_{6n} \text{ i.o.}] = 1$. [1] Noting that $\{E_{6n} \text{ i.o.}\} \subseteq \{E_n \text{ i.o.}\}$, we have $\mathbb{P}[E_n \text{ i.o.}] = 1$.