

MAS350: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme, with a total of 30 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Determine if the following functions are Lebesgue integrable.

(a) $f : (0, \infty) \rightarrow \mathbb{R}$ by $f(x) = 1/x^2$.

(b) $g : (0, 1) \rightarrow \mathbb{R}$ by $g(x) = \log x$

Solution.

(a) Note that $x^{-2} > 0$ for $x \in (0, \infty)$. By Riemann integration, we have

$$\int_{1/n}^n x^{-2} dx = [-x^{-1}]_{1/n}^n = -\frac{1}{n} + n.$$

[1] Note that $f_n(x) = \mathbb{1}_{\{x \in (1/n, n)\}} x^{-2}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $f(x) = x^{-2}$ for $x \in (0, \infty)$. [1] Hence, by the monotone convergence theorem [1] we have

$$\int_0^\infty x^{-2} dx = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} + n \right) = +\infty.$$

Thus x^{-2} is not integrable on $(0, \infty)$. [1]

(b) By Riemann integration, we have

$$\int_{1/n}^1 \log x dx = [x \log x - x]_{1/n}^1 = (-1) - \left(\frac{1}{n} \log \frac{1}{n} - \frac{1}{n} \right) = \frac{1 + \log n}{n} - 1.$$

Noting that $\log x \in (-\infty, 0)$ for $x \in (0, 1)$, multiplying the above by -1 gives

$$\int_{1/n}^1 |\log x| dx = 1 - \frac{1 + \log n}{n}.$$

[1] We have that $g_n(x) = |\log x| \mathbb{1}_{x \in (1/n, 1)}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $g(x) = |\log x|$ for $x \in (0, 1)$. [1] Hence, by the monotone convergence theorem,

$$\int_0^1 |\log x| dx = \lim_{n \rightarrow \infty} \left(1 - \frac{1 + \log n}{n} \right) = 1.$$

Thus $\log x$ is integrable on $(0, 1)$. [1]

2. Let $f_n, f : [0, 1] \rightarrow \mathbb{R}$. In each of the following cases, explain whether the Monotone and/or Dominated Convergence Theorems can be used to prove that $\int_0^1 f_n(x) dx \rightarrow \int_0^1 f(x) dx$.

(a) $f_n(x) = \cos(\frac{x}{n}) + \sin(\frac{x}{n})$ and $f(x) = 1$.

- (b) $f_n(x) = \mathbb{1}_{[\frac{1}{n}, 1]}(x) x^{-1}$ and $f(x) = \mathbb{1}_{(0, 1]} x^{-1}$.
(c) $f_n(x) = \mathbb{1}_{[0, \frac{1}{n}]}(x) n$ and $f(x) = 0$.

Solution.

- (a) DCT only (the MCT can't be used here because $f_n \leq f_{n+1}$ doesn't hold). [2]
(b) MCT only (the DCT can't be used here because $\int_0^1 f(x) dx = \infty$). [2]
(c) Neither, in this case $\int_0^1 f_n(x) dx = 1$ and $\int_0^1 f(x) dx = 0$. [2]

3. Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ where λ denotes the restriction of Lebesgue measure to the Borel σ -field $\mathcal{B}([0, 1])$ on $[0, 1]$.

$$\text{Let } X_n(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ \omega n^{3/2} & \text{if } \omega \in (0, \frac{1}{n}] \\ 0 & \text{if } \omega \in (\frac{1}{n}, 1]. \end{cases}$$

Determine in which modes of convergence we have $X_n \rightarrow 0$.

Solution. For any $n \in \mathbb{N}$, we have $\{X_n \neq 0\} = \{\omega \in [0, 1] : X_n(\omega) \neq 0\} \subseteq [0, \frac{1}{n}]$. [1] Hence, for any $a > 0$,

$$\lambda(\{|X_n - 0| < a\}) \leq \lambda([0, \frac{1}{n}]) = \frac{1}{n}$$

which converges to zero as $n \rightarrow \infty$. Hence $X_n \xrightarrow{\mathbb{P}} 0$. [1] It follows that also $X_n \xrightarrow{d} 0$. [1]

Fix some $\omega \in [0, 1]$. If $\omega \in (0, 1]$ then for all large enough n we have $\frac{1}{n} < \omega$. For such n we have $X_n(\omega) = 0$, [1] which means $X_n(\omega) \rightarrow 0$. We thus obtain that $\{X_n(\omega) \rightarrow 0\} \subseteq (0, 1]$ [1] so

$$\lambda(\{X_n(\omega) \rightarrow 0\}) \geq \lambda((0, 1]) = 1,$$

which means that $X_n \xrightarrow{a.s.} 0$. [1]

Lastly, the expectation of $|X_n|$ is given by

$$\begin{aligned} \int_0^1 X_n(\omega)^p d\lambda(\omega) &= \int_0^{\frac{1}{n}} \omega^p n^{3p/2} d\lambda(\omega) \\ &= n^{3p/2} \left[\frac{\omega^{p+1}}{p+1} \right]_0^{\frac{1}{n}} \\ &= n^{3p/2} \frac{(1/n)^{p+1}}{p+1} \\ &= n^{p/2-1}. \end{aligned}$$

[2] Here we use that $\{0\}$ is a λ -null subset of $[0, 1]$ (so values of X_n here have no effect on the integral) [1] and that $X_n(\omega) = 0$ when $\omega > \frac{1}{n}$. [1]

Noting that $n^{p/2-1} \rightarrow 0$ if and only if $p < \frac{3}{2}$, we have that $X_n \xrightarrow{p} 0$ if and only if $p < \frac{3}{2}$. [1]

[I would accept “ $p = 1$ works but $p = 2, 3, 4 \dots$ does not”]

4. (a) Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0, 1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$.

- (b) Let $(X_n)_{n \in \mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.

Solution.

- (a) We have $\mathbb{P}[U_n \leq a] = a$. For any (deterministic) sequence (x_n) the events $\{U_n < x_n\}$ are independent, because the U_n are independent. [1] Noting that $\sum 1/n = \infty$ and $\sum 1/n^2 < \infty$, we have $\sum_n \mathbb{P}[U_n < 1/n] = \infty$ and $\sum_n \mathbb{P}[U_n < 1/n^2] < \infty$. [1] By the second Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and by the first Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$. [1]
- (b) Let $E_n = \{X_n + i = i \text{ for } i = 1, 2, 3, 4, 5, 6\}$. We have $\mathbb{P}[E_n] = (1/6)^6 > 0$. Note that E_n and E_{n+6} are independent (but E_n and E_{n+1} are not!). [1] We have $\sum_{n=1}^{\infty} \mathbb{P}[E_{6n}] = \sum_{n=1}^{\infty} (1/6)^6 = \infty$, [1] hence by the second Borel-Cantelli lemma we have $\mathbb{P}[E_{6n} \text{ i.o.}] = 1$. [1] Noting that $\{E_{6n} \text{ i.o.}\} \subseteq \{E_n \text{ i.o.}\}$, we have $\mathbb{P}[E_n \text{ i.o.}] = 1$.