SCHOOL OF MATHEMATICS AND STATISTICS **2016/17**

Stochastic Processes and Financial Mathematics **-**

This specimen paper contains 100 marks of questions for MAS352/452/6052, and 20 marks of questions exclusively for MAS452/6052. (The 'real' exam lasts 3 hours and will contain 100 marks of questions, in both cases.)

1 Let Ω be the set of pairs $\omega = (\omega_1, \omega_2)$ where $\omega_1, \omega_2 \in \{1, 2, 3\}$, representing the possible outcomes of sampling two numbers from $\{1, 2, 3\}$. Let

$$X(\omega) = \omega_1 + \omega_2$$

for each $\omega = (\omega_1, \omega_2) \in \Omega$.

(a) Which, if any, of the following statements are true? Justification is not required.

(i) The value of X is equal to the sum of the two numbers sampled.

(ii) $\sigma(X)$ is a subset of Ω.

(iii) The set $\{(1, 1), (3, 2)\}$ is an element of $\sigma(X)$.

(3 marks)

(b) Let

$$Y(\omega) = \begin{cases} 0 & \text{if } X(\omega) \text{ is even,} \\ 1 & \text{if } X(\omega) \text{ is odd.} \end{cases}$$

Write down all the elements of $\sigma(Y)$. ***(4 marks)***

2 Let X be a random variables with $0 < X < \frac{1}{2}$ and set

$$Y = \sum_{n=1}^{\infty} \frac{X^n}{n}.$$

Explain why Y is a random variable. ***(6 marks)***

You may use standard results about measurability of sums, products and limits of random variables, providing they are clearly stated.

3 Let $\alpha > 0$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, with the distribution of X_n given by $\mathbb{P}[X_n = n^\alpha] = \frac{1}{n^2}$ and $\mathbb{P}[X_n = 0] = 1 - \frac{1}{n^2}$.

(a) Show that $X_n \to 0$ in probability. ***(2 marks)***

(b) For which values of α is it true that $X_n \to 0$ in L^1? Justify your answer. ***(3 marks)***

4 [MAS452/6052 only] Let $(X_n)_{n=1}^\infty$ be a sequence of independent, identically distributed random variables with common distribution $X_n \sim Exp(1)$. That is, X_n has probability density function

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that, for any $t > 0$, we have $\mathbb{P}[X_n > t] = e^{-t}$. ***(2 marks)***

(b) Let $\alpha \in [0, \infty)$. Show that

$$\mathbb{P}[X_n > \alpha \log n \;\text{ infinitely often}] = \begin{cases} 0 & \text{if } \alpha > 1, \\ 1 & \text{if } \alpha \le 1. \end{cases}$$

(4 marks)

(c) For $n \ge 2$ let

$$S_n = \sum_{i=2}^{n} \left(\frac{X_i}{\log i} \right)^i.$$

Show that $S_n \to \infty$ almost surely as $n \to \infty$. ***(4 marks)***

5 Let $T \in \mathbb{N}$. Consider the binomial model, in discrete time, with two assets, cash and stock. Recall that, in the binomial model, we have time steps $t = 0, 1, 2, \ldots, T$ and that:

- If we hold x cash at time t, it becomes worth $x(1+r)$ at time $t+1$.
- The price of a single unit of stock at time t is S_t, where $S_0 = s$ and, independently on each time step,

$$S_{t+1} = \begin{cases} uS_t & \text{with probability } p_u, \\ dS_t & \text{with probability } p_d. \end{cases}$$

  Therefore, if we hold y units of stock at time t, they are worth yS_t.

Here, $s > 0$ is a deterministic constant,

$$d < 1 + r < u \tag{$*$}$$

are also deterministic constants, and S_1 is a random variable with $\mathbb{P}[S_1 = su] = p_u$ and $\mathbb{P}[S_1 = sd] = p_d$, where $p_u + p_d = 1$ and $p_d, p_u \in (0, 1)$.

(a) Take $T = 2$, let $p_u = p_d = 0.5$, $u = 2.0$, $d = 0.5$, $r = 0$ and $s = 4$. Consider the contingent claim

$$\Phi(S_T) = \begin{cases} 256 & \text{if } S_T = 16 \\ 16 & \text{if } S_T = 4 \\ 4 & \text{if } S_T = 1 \end{cases}$$

with exercise time $T = 2$. Draw a recombining tree of the stock price process at times $t = 0, 1, 2$. Find the arbitrage free price, at time 0, of $\Phi(S_T)$ (you may annotate your tree whilst doing so). ***(10 marks)***

(b) State one respect in which the binomial model is:

(i) A good model of a real financial market. ***(1 mark)***

(ii) A bad model of a real financial market. ***(1 mark)***

6 Consider the one-period model (that is, the binomial model described in Q5, with $T = 1$). Let $h = (x, y)$ be a portfolio, that is purchased at time 0 and held until time 1.

(a) Write down V_0^h and V_1^h, the values of h at times 0 and 1. ***(4 marks)***

(b) State what it means for $h = (x, y)$ to be an arbitrage possibility. ***(3 marks)***

(c) Consider the case when the parameters d, u and r satisfy $0 < 1 + r < d < u$ instead of $(*)$. Within this market, find a portfolio that is an arbitrage possibility. ***(3 marks)***

7 Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration, in discrete time.

(a) State the definition of a martingale M_n, with respect to $\mathcal{F}_n$. ***(4 marks)***

(b) Let $(X_i^{(n)})_{i,n\in\mathbb{N}}$ be a set of independent, identically distributed random variables, with common distribution

$$\mathbb{P}[X_i^{(n)}=1]=\tfrac{1}{2},\quad \mathbb{P}[X_i^{(n)}=3]=\tfrac{1}{2}. \qquad (\star)$$

Set Z_n by $Z_0=1$ and then, iteratively for $n=0,1,2,\ldots$ we define

$$Z_{n+1}=\begin{cases} X_1^{(n+1)}+\ldots+X_{Z_n}^{(n+1)}, & \text{if } Z_n>0\\ 0, & \text{if } Z_n=0\end{cases}$$

In words, Z_n is a Galton-Watson process with offspring distribution given by $(\star)$. Define

$$M_n=\frac{Z_n}{2^n}$$

and let $\mathcal{F}_n=\sigma(X_i^{(m)}\ :\ m=1,2,\ldots,n \text{ and } i\in\mathbb{N})$.

(i) Show that $\mathbb{E}[Z_{n+1}]=2\mathbb{E}[Z_n]$ and deduce that $\mathbb{E}[Z_n]=2^n$. ***(7 marks)***

(ii) Show that M_n is a martingale, with respect to $\mathcal{F}_n$. ***(6 marks)***

(iii) **[MAS452/6052 only]** Deduce that M_n converges almost surely as $n\to\infty$ to a real valued random variable M_∞. ***(2 marks)***

8 (a) State the definition of a standard Brownian motion B_t. ***(6 marks)***

(b) Show that B_t is a martingale, with respect to a filtration $\mathcal{F}_t$ that you should define. ***(6 marks)***

9 Let $T > 0$, let $\mu \in \mathbb{R}$ and $\sigma > 0$, and let B_t be a Brownian motion. Let X_t be a stochastic process satisfying

$$dX_t = \mu\, dt + \sigma\, dB_t. \qquad (\star)$$

For $t \in [0, T]$ and $x \in \mathbb{R}$, define $F(t, x) = \mathbb{E}_{t,x}[e^{X_T}]$, where $\mathbb{E}_{t,x}$ denotes the expectation of X during $[t, T]$ with initial condition $X_t = x$.

(a) Write $(\star)$ in integral form, over the time interval $[t, T]$. ***(2 marks)***

(b) Show that $F(t, x) = \exp\left\{x + (\mu + \frac{1}{2}\sigma^2)(T - t)\right\}$. ***(6 marks)***

(c) Hence, show that $F(t, x)$ solves the partial differential equation

$$\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} = 0.$$

and satisfies $F(T, x) = e^x$. ***(4 marks)***

10 (a) Let $\alpha \in \mathbb{R}$ and $\sigma > 0$, and let X_t be an Ito process satisfying $X_0 > 0$ and

$$dX_t = \alpha X_t\, dt + \sigma X_t\, dB_t.$$

Let $Z_t = \log(X_t)$. Find the stochastic differential dZ_t. ***(7 marks)***

(b) Let $T > 0$. Find the price, at time $t \in [0, T]$, within the Black-Scholes model, of the contingent claim $\Phi(S_T) = \log(S_T)$ with exercise date T. ***(7 marks)***

Standard notation, including the parameters r, μ and σ, and pricing formulae relating to the Black-Scholes model can be found on the supplementary sheet.

11 Let $T, K > 0$. Within the Black-Scholes model:

(a) We define $\Phi^{cash}(S_T) = 1$ to be the contingent claim of a single unit in cash, at time T. Write down formulae for the following contingent claims, each of which has exercise date $T > 0$.

(i) $\Phi^{stock}(S_T)$, of a single unit of stock.

(ii) $\Phi^{call}(S_T)$, of a European call option, with strike price K.

(iii) $\Phi^{put}(S_T)$, of a European put option, with strike price K.

(3 marks)

(b) With notation as in (a), prove the put-call parity relation,

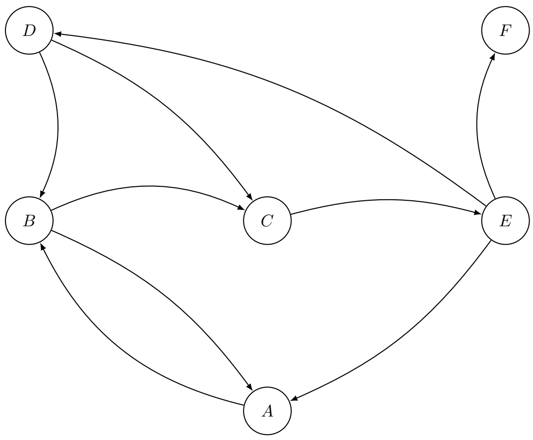
$$\Phi^{put}(S_T) = \Phi^{call}(S_T) + K\Phi^{cash}(S_T) - \Phi^{stock}(S_T).$$

(3 marks)

(c) Write down a constant portfolio, to be bought at time 0, consisting only of cash, stock, and European put options, which replicates a European call option with strike price K and exercise date T. ***(3 marks)***

12 **[MAS452/6052 only]** A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G. An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y.

The graph G has vertices and edges as shown:



Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

We define a model of debt contagion by assuming that:

(†) For any bank X, with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability

$$\eta_j = \frac{1}{1+j}$$

the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Given some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.

(a) If bank A fails, and defaults on all its loans, calculate the probability that bank E also fails. ***(3 marks)***

(b) Alternatively, if bank D fails, and defaults on all its loans, calculate the probability that bank E also fails. ***(5 marks)***

End of Question Paper