MASx52: Assignment 3

1. Consider the binomial model with $r = \frac{1}{11}$, $d = 0.9$, $u = 1.2$, $s = 100$ and time steps $t = 0, 1, 2$.

   (a) Draw a recombining tree of the stock price process, for time $t = 0, 1, 2$.

   (b) Find the value, at time $t = 0$, of a European call option that gives its holder the option to purchase one unit of stock at time $t = 2$ for a strike price $K = 90$. Write down the hedging strategy that replicates the value of this contract, at all nodes of your tree.

   You may annotate your tree from (a) to answer (b).

2. Let $S_n = \sum_{i=1}^{n} X_i$, be a random walk, in which $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common distribution $\mathbb{P}[X_i = \frac{1}{i^2}] = \mathbb{P}[X_i = -\frac{1}{i^2}] = \frac{1}{2}$.

   (a) Show that $\mathbb{E}[|S_n|] \le \sum\limits_{i=1}^{n} \frac{1}{i^2}$.

   (b) Explain briefly why part (a) means that S_n is bounded in L^1.

   (c) Show that there exists a random variable S_∞ such that $S_n \stackrel{a.s.}{\to} S_\infty$ as $n \to \infty$.

3. (a) Let Z be a random variable taking values in $[1, \infty)$ and for $n \in \mathbb{N}$ define

$$X_n = \begin{cases} Z & \text{if } Z \in [n, n+1) \\ 0 & \text{otherwise.} \end{cases} \qquad (\star)$$

   Suppose that $Z \in L^1$. Use the dominated convergence theorem to show that $\mathbb{E}[X_n] \to 0$ as $n \to \infty$.

   (b) Instead, let Z be the continuous random variable with probability density function

$$f(x) = \begin{cases} x^{-2} & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

   and define X_n using $(\star)$. Show that Z is not in L^1, but that $\mathbb{E}[X_n] \to 0$.

   (c) Comment on what part (b) tells us about the dominated convergence theorem.