Let $(U, V)$ be a pair of independent standard normals. Let

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

be a non-singular $2 \times 2$ matrix, and let $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ be a 2-vector. We now consider the random vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \mathbf{S} \begin{pmatrix} U \\ V \end{pmatrix} + \boldsymbol{\mu}.$$

We will show that $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is to be determined.

Using Lemma 6.3, we showed that

$$\mathbb{E}[\mathbf{X}] = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\mathrm{Cov}(\mathbf{X}) = \begin{pmatrix} s_{11}^2 + s_{12}^2 & s_{22}s_{12} + s_{21}s_{11} \\ s_{22}s_{12} + s_{21}s_{11} & s_{21}^2 + s_{22}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

The last line equality is the *definition* of $\mathbf{\Sigma} = (\sigma_{ij})$.

As part of this calculation, we can show that $\mathbf{\Sigma} = \mathbf{S}\mathbf{S}^T$.

We now want to show that $\mathbf{X}$ has the p.d.f. of $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
We use the method of transforming p.d.f.s from Chapter 5.

We start from the p.d.f. of $(U, V)$,

$$f_{U,V}(u, v) = \frac{1}{2\pi} \exp\left(-\frac{u^2 + v^2}{2}\right)$$

The forward transformation is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{S} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

and maps $\mathbb{R}^2$ to $\mathbb{R}^2$. Hence, the inverse transformation is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} = \frac{1}{\det \mathbf{S}} \begin{pmatrix} s_{22} & -s_{21} \\ -s_{12} & s_{11} \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

which means that

$$u = \frac{1}{\det \mathbf{S}} (s_{22}(x_1 - \mu_1) - s_{12}(x_2 - \mu_2)),$$
$$v = \frac{1}{\det \mathbf{S}} (-s_{21}(x_1 - \mu_1) + s_{11}(x_2 - \mu_2)).$$

This allows us to calculate $\frac{\partial u}{\partial x_1}$, $\frac{\partial u}{\partial x_2}$, $\frac{\partial v}{\partial x_1}$, $\frac{\partial v}{\partial x_2}$, and obtain that the Jacobian is

$$J = \frac{1}{\det S}.$$

Hence, the joint p.d.f. $f_{X_1,X_2}(x_1, x_2)$ of $X_1$ and $X_2$ is

$$\frac{1}{2\pi|\det \mathbf{S}|} \exp\left\{-\frac{\left[(s_{22}(x_1-\mu_1)-s_{12}(x_2-\mu_2))^2+(-s_{21}(x_1-\mu_1)+s_{11}(x_2-\mu_2))^2\right]}{2(\det \mathbf{S})^2}\right\},$$

Putting $\sigma_{ij}$s in for the $s_{ij}$s, we get

$$\frac{1}{2\pi|\det \mathbf{S}|} \exp\left\{-\frac{\left[\sigma_2^2(x_1-\mu_1)^2+\sigma_1^2(x_2-\mu_2)^2-2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2)\right]}{2(\det \mathbf{S})^2}\right\}.$$

We have $\det \mathbf{\Sigma} = \det(\mathbf{S}\mathbf{S}^T) = (\det \mathbf{S})^2$. Using that $\det \mathbf{\Sigma} = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$, we obtain

$$\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2-\sigma_{12}^2}} \exp\left\{-\frac{\sigma_2^2(x_1-\mu_1)^2-2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2)+\sigma_1^2(x_2-\mu_2)^2}{2(\sigma_1^2\sigma_2^2-\sigma_{12}^2)}\right\},$$

This matches the p.d.f. of a $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$.