Data provided: formula sheet

MASx52 specimen paper

SCHOOL OF MATHEMATICS AND STATISTICS **2016/17**

Stochastic Processes and Financial Mathematics **-**

*This specimen paper contains 100 marks of questions for MAS352/452/6052,
and 20 marks of questions exclusively for MAS452/6052.
(The 'real' exam lasts 3 hours and will contain 100 marks of questions, in both cases.)*

1 Let Ω be the set of pairs $\omega = (\omega_1, \omega_2)$ where $\omega_1, \omega_2 \in \{1, 2, 3\}$, representing the possible outcomes of sampling two numbers from $\{1, 2, 3\}$. Let

$$X(\omega) = \omega_1 + \omega_2$$

for each $\omega = (\omega_1, \omega_2) \in \Omega$.

(a) Which, if any, of the following statements are true? Justification is not required.

(i) The value of X is equal to the sum of the two numbers sampled.

(ii) $\sigma(X)$ is a subset of Ω.

(iii) The set $\{(1, 1), (3, 2)\}$ is an element of $\sigma(X)$.

(3 marks)

(b) Let

$$Y(\omega) = \begin{cases} 0 & \text{if } X(\omega) \text{ is even,} \\ 1 & \text{if } X(\omega) \text{ is odd.} \end{cases}$$

Write down all the elements of $\sigma(Y)$. ***(4 marks)***

Solution. Marks are split into A (accuracy) and M (method). In all cases, alternative mathematically correct solutions receive full marks, and alternative partially correct solutions are marked analogously.

(a) (i) is true, but (ii) and (iii) are false. [3A, one for each part]

(b) We have

$$\begin{aligned} Y^{-1}(0) &= \{(1, 1), (1, 3), (3, 1), (2, 2), (3, 3)\} \\ Y^{-1}(1) &= \{(1, 2), (2, 1), (2, 3), (3, 2)\}. \end{aligned}$$

Since these two sets are disjoint and $\Omega \setminus Y^{-1}(0) = Y^{-1}(1)$ we have that

$$\sigma(Y) = \Big\{\emptyset,\, Y^{-1}(0),\, Y^{-1}(1),\, \Omega\Big\}.$$

[5A; one for each correct element, plus one for no incorrect elements]

2 Let X be a random variables with $0 < X < \frac{1}{2}$ and set

$$Y = \sum_{n=1}^{\infty} \frac{X^n}{n}.$$

Explain why Y is a random variable. ***(6 marks)***

You may use standard results about measurability of sums, products and limits of random variables, providing they are clearly stated.

Solution. Products of RVs are RVs, hence for each $N \in \mathbb{N}$, X^n is a RV, and so is $\frac{X^n}{n}$. [1M] The sum of RVs is a RV, hence $\sum_{n=1}^{N} \frac{X^n}{n}$ a RV. [1M]

Lastly, limits of RVs, when they exist, are RVs. [1M] Since $0 < X < \frac{1}{2}$ we have that $a_N = \sum_{n=1}^{N} \frac{X^n}{n}$ is a (random) increasing sequence that is bounded above by $\sum_{n=1}^{\infty} \frac{(1/2)^n}{n} \leq \sum_{n=1}^{\infty} (1/2)^n = 1$. Hence the series converges, so Y is a random variable. [2M]

3 Let $\alpha > 0$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, with the distribution of X_n given by $\mathbb{P}[X_n = n^\alpha] = \frac{1}{n^2}$ and $\mathbb{P}[X_n = 0] = 1 - \frac{1}{n^2}$.

(a) Show that $X_n \to 0$ in probability. ***(2 marks)***

(b) For which values of α is it true that $X_n \to 0$ in L^1? Justify your answer. ***(3 marks)***

Solution.

(a) For any $a > 0$ we have $\mathbb{P}[|X_n - 0| > a] \leq \mathbb{P}[X_n \neq 0] = \mathbb{P}[X_n = n^\alpha] = \frac{1}{n^2}$, [1M] which tends to zero as $n \to \infty$. [1M]

(b) We have

$$\begin{aligned}\mathbb{E}[|X_n - 0|] &= \mathbb{E}[X_n] \\ &= \tfrac{1}{n^2} n^\alpha + (1 - \tfrac{1}{n^2})(0) \\ &= n^{\alpha-2}.\end{aligned}$$

[1M] As $n \to \infty$, this tends to zero if and only if $\alpha < 2$. [1M] Hence $X_n \to 0$ in L^1 if and only if $\alpha < 2$. [1A]

4 **[MAS452/6052 only]** Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with common distribution $X_n \sim Exp(1)$. That is, X_n has probability density function

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that, for any $t > 0$, we have $\mathbb{P}[X_n > t] = e^{-t}$. ***(2 marks)***

(b) Let $\alpha \in [0, \infty)$. Show that

$$\mathbb{P}[X_n > \alpha \log n \;\text{ infinitely often}] = \begin{cases} 0 & \text{if } \alpha > 1, \\ 1 & \text{if } \alpha \leq 1. \end{cases}$$

(4 marks)

(c) For $n \geq 2$ let

$$S_n = \sum_{i=2}^{n} \left(\frac{X_i}{\log i} \right)^i.$$

Show that $S_n \to \infty$ almost surely as $n \to \infty$. ***(4 marks)***

Solution. **[MAS452/6052 only]**

(a) We have $\mathbb{P}[X_n > t] = \int_t^{\infty} f(x)\,dx = [-e^{-x}]_{x=t}^{\infty} = e^{-t}$. [2M]

(b) From part (a) we have

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n > \alpha \log n] = \sum_{n=1}^{\infty} e^{-\alpha \log n} = \sum_{n=1}^{\infty} n^{-\alpha}.$$

[1M] This sum is finite when $\alpha > 1$ and infinite when $\alpha \leq 1$. [1A]

Hence, the first Borel-Cantelli lemma tells us that $\mathbb{P}[X_n > \alpha \log n \text{ i.o.}] = 0$ when $\alpha > 1$ and, since the X_n are independent, the second Borel-Cantelli lemma tells us that $\mathbb{P}[X_n > \alpha \log n \text{ i.o.}] = 1$ when $\alpha \leq 1$. [2M + 1A]

(c) Taking $\alpha = 1$ [1M] we have $\mathbb{P}[X_n \geq \log n \text{ i.o.}] = 1$, which means that, almost surely, for infinitely many n we have $\frac{X_n}{\log n} \geq 1$ and these n we have $\left(\frac{X_n}{\log n}\right)^n \geq 1$. [1M] Hence, almost surely,

$$\sum_{n=2}^{\infty} \left(\frac{X_n}{\log n} \right)^n \geq 1 + 1 + 1 + \ldots$$

[1M] Since S_n is a (random) increasing sequence, it converges to the infinite series above, and thus $\mathbb{P}[S_n \to \infty] = 1$. [1M]

5 Let $T \in \mathbb{N}$. Consider the binomial model, in discrete time, with two assets, cash and stock. Recall that, in the binomial model, we have time steps $t = 0, 1, 2, \dots, T$ and that:

- If we hold x cash at time t, it becomes worth $x(1+r)$ at time $t+1$.
- The price of a single unit of stock at time t is S_t, where $S_0 = s$ and, independently on each time step,

$$S_{t+1} = \begin{cases} uS_t & \text{with probability } p_u, \\ dS_t & \text{with probability } p_d. \end{cases}$$

  Therefore, if we hold y units of stock at time t, they are worth yS_t.

Here, $s > 0$ is a deterministic constant,

$$d < 1 + r < u \qquad (*)$$

are also deterministic constants, and S_1 is a random variable with $\mathbb{P}[S_1 = su] = p_u$ and $\mathbb{P}[S_1 = sd] = p_d$, where $p_u + p_d = 1$ and $p_d, p_u \in (0, 1)$.

(a) Take $T = 2$, let $p_u = p_d = 0.5$, $u = 2.0$, $d = 0.5$, $r = 0$ and $s = 4$. Consider the contingent claim

$$\Phi(S_T) = \begin{cases} 256 & \text{if } S_T = 16 \\ 16 & \text{if } S_T = 4 \\ 4 & \text{if } S_T = 1 \end{cases}$$

with exercise time $T = 2$. Draw a recombining tree of the stock price process at times $t = 0, 1, 2$. Find the arbitrage free price, at time 0, of $\Phi(S_T)$ (you may annotate your tree whilst doing so). ***(10 marks)***

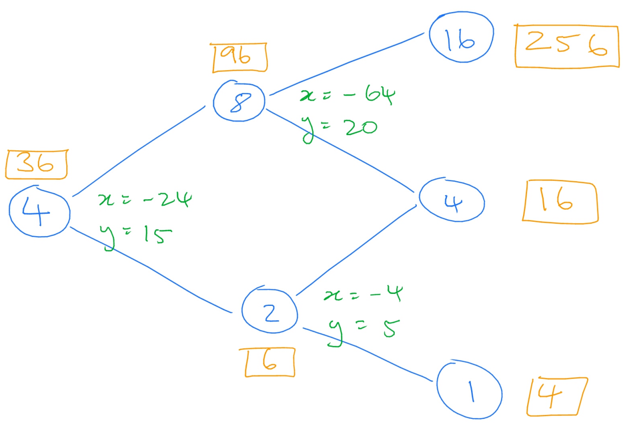
(b) State one respect in which the binomial model is:

(i) A good model of a real financial market. ***(1 mark)***

(ii) A bad model of a real financial market. ***(1 mark)***

Solution.

(a) A recombining tree of the stock price process for $t = 0, 1, 2$ looks like



The stock price process is shown in blue, with the value of the contingent claim shown in yellow and the hedging portfolios (x, y) shown in green.

 Question 5 continued on next page

5(continued)

The hedging portfolios are found by solving linear equations of the form $x+dSy = \widetilde{\Phi}(dS)$, $x+uSy = \widetilde{\Phi}(uD)$, where $\widetilde{\Phi}$ is the contingent claim and S the initial stock price of the one period model associated to the given node. The value of the contingent claim at each node is then inferred from the hedging portfolio as $V = x + Sy$.

[6A + 4M]

[Solutions based on iterating the risk neutral valuation formula for the one-period model, without computing the replicating portfolios, are equally valid and will receive full marks if correct.]

(b) (i) It incorporates the fact that cash tends to change value in a predictable way, but stocks do not. [1A; any reasonable answer accepted]

(ii) The stock price process is unrealistic in only allowing two options for a change in value at each new time step. [1A; any reasonable answer accepted]

6 Consider the one-period model (that is, the binomial model described in Q5, with $T = 1$). Let $h = (x, y)$ be a portfolio, that is purchased at time 0 and held until time 1.

(a) Write down V_0^h and V_1^h, the values of h at times 0 and 1. ***(4 marks)***

(b) State what it means for $h = (x, y)$ to be an arbitrage possibility. ***(3 marks)***

(c) Consider the case when the parameters d, u and r satisfy $0 < 1+r < d < u$ instead of $(*)$. Within this market, find a portfolio that is an arbitrage possibility. ***(3 marks)***

Solution.

(a) We have $V_0^h = x + sy$ and $V_1^h = x(1 + r) + yS_1$. [4A]

(b) h is an arbitrage possibility (in the one period model) if h it satisfies

$$V_0^h = 0$$
$$\mathbb{P}[V_1^h \geq 0] = 1$$
$$\mathbb{P}[V_1^h > 0] > 0.$$

[3A]

(c) $0 < 1+r < d < u$ tells us that, between times 0 and 1, stock always increases in value more than cash does. So (for example) at time 0 we could borrow s cash and buy a unit of stock: $h = (-s, 1)$. [1A] Then at time 1 we have

$$V_1^h = -s(1 + r) + S_1$$

[1M] and since $S_1 \geq sd > s(1 + r) > 0$, our portfolio is an arbitrage. [1M]

7 Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration, in discrete time.

(a) State the definition of a martingale M_n, with respect to $\mathcal{F}_n$. ***(4 marks)***

(b) Let $(X_i^{(n)})_{i,n\in\mathbb{N}}$ be a set of independent, identically distributed random variables, with common distribution

$$\mathbb{P}[X_i^{(n)}=1]=\tfrac{1}{2},\quad \mathbb{P}[X_i^{(n)}=3]=\tfrac{1}{2}. \tag{$\star$}$$

Set Z_n by $Z_0=1$ and then, iteratively for $n=0,1,2,\ldots$ we define

$$Z_{n+1}=\begin{cases} X_1^{(n+1)}+\ldots+X_{Z_n}^{(n+1)}, & \text{if } Z_n>0\\ 0, & \text{if } Z_n=0\end{cases}$$

In words, Z_n is a Galton-Watson process with offspring distribution given by ($\star$). Define

$$M_n=\frac{Z_n}{2^n}$$

and let $\mathcal{F}_n=\sigma(X_i^{(m)}\ :\ m=1,2,\ldots,n \text{ and } i\in\mathbb{N})$.

(i) Show that $\mathbb{E}[Z_{n+1}]=2\mathbb{E}[Z_n]$ and deduce that $\mathbb{E}[Z_n]=2^n$. ***(7 marks)***

(ii) Show that M_n is a martingale, with respect to $\mathcal{F}_n$. ***(6 marks)***

(iii) **[MAS452/6052 only]** Deduce that M_n converges almost surely as $n\to\infty$ to a real valued random variable M_∞. ***(2 marks)***

Solution.

(a) Note that $\mathbb{E}\left[X_1^{n+1}\right]=3\frac{1}{2}+1\frac{1}{2}=2$. [1A] We have

$$\begin{aligned}\mathbb{E}[Z_{n+1}]&=\mathbb{E}\left[X_1^{n+1}+\ldots+X_{Z_n}^{n+1}\right]\\
&=\sum_{k=1}^{\infty}\mathbb{E}\left[\left(X_1^{n+1}+\ldots+X_k^{n+1}\right)\mathbb{1}\{Z_n=k\}\right]\\
&=\sum_{k=1}^{\infty}\mathbb{E}\left[\left(X_1^{n+1}+\ldots+X_k^{n+1}\right)\right]\mathbb{E}\left[\mathbb{1}\{Z_n=k\}\right]\\
&=\sum_{k=1}^{\infty}\left(\mathbb{E}\left[X_1^{n+1}\right]+\ldots+\mathbb{E}\left[X_k^{n+1}\right]\right)\mathbb{P}\left[Z_n=k\right]\\
&=\sum_{k=1}^{\infty}2k\mathbb{P}[Z_n=k]\\
&=2\sum_{k=1}^{\infty}k\mathbb{P}[Z_n=k]\\
&=2\,\mathbb{E}[Z_n]. \end{aligned}\tag{1}$$

[4M] By a trivial induction, since $\mathbb{E}[Z_0]=1$, we have $\mathbb{E}[Z_n]=2^n\mathbb{E}[Z_0]=2^n$. [2M]

 Question 7 continued on next page

7(continued)

(b) We have $Z_0 = 1$, which is deterministic, and if $Z_n \in \mathcal{F}_n$ then the definition of Z_n tells us that $Z_{n+1} \in m\mathcal{F}_{n+1}$. Hence, by induction $Z_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Hence also $M_n = \frac{Z_n}{2^n} \in \mathcal{F}_n$ for all n. [2M]

From (1), we have $\mathbb{E}[Z_{n+1}] = 2\mathbb{E}[Z_n]$ so as $\mathbb{E}[Z_n] = 2^n$ for all n. Hence $\mathbb{E}[M_n] = 1$ and $M_n \in L^1$ for all n. [1M]

Lastly,

$$\begin{aligned}
\mathbb{E}[Z_{n+1} \,|\, \mathcal{F}_n] &= \sum_{k=1}^{\infty} \mathbb{E}[Z_{n+1}\mathbb{1}\{Z_n = k\} \,|\, \mathcal{F}_n] \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left[\left(X_1^{n+1} + \ldots + X_k^{n+1}\right)\mathbb{1}\{Z_n = k\} \,|\, \mathcal{F}_n\right] \\
&= \sum_{k=1}^{\infty} \mathbb{1}\{Z_n = k\}\mathbb{E}\left[X_1^{n+1} + \ldots + X_k^{n+1} \,|\, \mathcal{F}_n\right] \\
&= \sum_{k=1}^{\infty} \mathbb{1}\{Z_n = k\}(2k) \\
&= 2Z_n.
\end{aligned}$$

Here we use that Z_n is $\mathcal{F}_n$ measurable to take out what is known, and then use that X_i^{n+1} is independent of $\mathcal{F}_n$. Hence, $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$, as required. [3M]

(c) **[MAS452/6052 only]** We have shown that M_n is a martingale, and that $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = 1$. Hence M_n is bounded in L^1 and the martingale convergence theorem applies. By the martingale convergence theorem, M_n has an almost sure limit as $n \to \infty$. [2M]

8 (a) State the definition of a standard Brownian motion B_t. ***(6 marks)***

(b) Show that B_t is a martingale, with respect to a filtration $\mathcal{F}_t$ that you should define. ***(6 marks)***

Solution.

(a) B_t is a Brownian motion if:

(i) The paths of (B_t) are continuous. (Or: B_t is a random continuous function.) [1A]

(ii) For any $0 \leq u \leq t$, the random variable $B_t - B_u$ is independent of $\sigma(B_v : v \leq u)$. [2A]

(iii) For any $0 \leq u \leq t$, the random variable $B_t - B_u$ has distribution $N(0, t-u)$. [2A]

We say that Brownian motion B_t is standard if $B_0 = 0$. [1A]

(b) We define the filtration $\mathcal{F}_t = \sigma(B_u \,:\, u \leq t)$. [1A]

Since $B_t \sim N(0,t)$ we have $\mathrm{Var}(B_t) < \infty$, which implies that $B_t \in L^1$. [1M]

Since the filtration $(\mathcal{F}_t)$ is the generated filtration of B_t, is immediate that B_t is adapted. [1M]

Lastly, for any $0 \leq u \leq t$ we have

$$\begin{aligned}\mathbb{E}[B_t \,|\, \mathcal{F}_u] &= \mathbb{E}[B_t - B_u \,|\, \mathcal{F}_u] + \mathbb{E}[B_u \,|\, \mathcal{F}_u] \\ &= \mathbb{E}[B_t - B_u] + B_u \\ &= B_u.\end{aligned}$$

Here, we use the properties of Brownian motion: $B_t - B_s$ is independent of $\mathcal{F}_u$ and $\mathbb{E}[B_t] = \mathbb{E}[B_u] = 0$. [3M]

9 Let $T > 0$, let $\mu \in \mathbb{R}$ and $\sigma > 0$, and let B_t be a Brownian motion. Let X_t be a stochastic process satisfying

$$dX_t = \mu\, dt + \sigma\, dB_t. \qquad (\star)$$

For $t \in [0,T]$ and $x \in \mathbb{R}$, define $F(t,x) = \mathbb{E}_{t,x}[e^{X_T}]$, where $\mathbb{E}_{t,x}$ denotes the expectation of X during $[t,T]$ with initial condition $X_t = x$.

(a) Write $(\star)$ in integral form, over the time interval $[t,T]$. ***(2 marks)***

(b) Show that $F(t,x) = \exp\left\{x + (\mu + \frac{1}{2}\sigma^2)(T-t)\right\}$. ***(6 marks)***

(c) Hence, show that $F(t,x)$ solves the partial differential equation

$$\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2} = 0.$$

and satisfies $F(T,x) = e^x$. ***(4 marks)***

Solution.

(a) We have $X_T = X_t + \int_t^T \mu\, du + \int_t^T \sigma dB_u$. [1M + 1A]

(b) By part (a) we have $X_T = X_t + \mu(T-t) + \sigma(B_T - B_t)$. [2A] Hence,

$$\begin{aligned}
F(t,x) &= \mathbb{E}_{t,x}[e^{X_T}] \\
&= \mathbb{E}_{t,x}\left[e^{X_t + \mu(T-t) + \sigma(B_T - B_t)}\right] \\
&= \mathbb{E}\left[e^{x + \mu(T-t) + \sigma(B_T - B_t)}\right] \\
&= e^{x+\mu(T-t)}\mathbb{E}[e^{\sigma(B_T - B_t)}] \\
&= e^{x+\mu(T-t)}e^{\frac{1}{2}\sigma^2(T-t)} \\
&= \exp\left\{x + (\mu + \tfrac{1}{2}\sigma^2)(T-t)\right\}.
\end{aligned}$$

[2A] Here we use the definition of $\mathbb{E}_{t,x}$ and the formula, along with using the scaling properties of normal random variables to deduce that $\sigma(B_T - B_t) \sim N(0, \sigma^2(T-t)$ and the formula (from the supplementary sheet) for $\mathbb{E}[e^{N(\mu,\sigma)}]$. [2M]

(c) We have

$$\begin{aligned}
\frac{\partial F}{\partial t} &= -(\mu + \sigma^2)(T-t) \\
\frac{\partial F}{\partial x} &= F(t,x) \\
\frac{\partial^2 F}{\partial x^2} &= F(t,x)
\end{aligned}$$

[3A] so as $\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2} = 0$, and also

$$F(T,x) = \exp\left\{x + (\mu + \tfrac{1}{2}\sigma^2)(T-T)\right\} = e^x.$$

[1A]

10 (a) Let $\alpha \in \mathbb{R}$ and $\sigma > 0$, and let X_t be an Ito process satisfying $X_0 > 0$ and

$$dX_t = \alpha X_t\,dt + \sigma X_t\,dB_t.$$

Let $Z_t = \log(X_t)$. Find the stochastic differential dZ_t. ***(7 marks)***

(b) Let $T > 0$. Find the price, at time $t \in [0, T]$, within the Black-Scholes model, of the contingent claim $\Phi(S_T) = \log(S_T)$ with exercise date T.
(7 marks)

Standard notation, including the parameters r, μ and σ, and pricing formulae relating to the Black-Scholes model can be found on the supplementary sheet.

Solution.

(a) Using Ito's formula [1M] we have

$$\begin{aligned} dZ_t &= \left\{(0) + (\alpha X_t)\left(\frac{1}{X_t}\right) + \frac{1}{2}(\sigma X_t)^2)\left(\frac{-1}{X_t^2}\right)\right\} dt + (\sigma X_t)\left(\frac{1}{X_t}\right) dB_t \\ &= (\alpha - \tfrac{\sigma^2}{2})\,dt + \sigma\,dB_t. \end{aligned}$$

[6A; five for terms of Ito's formula, one for cancellations]

(b) If we set $\alpha = r$ [1M] then under $\mathbb{Q}$ we can set $S_t = X_t$ and hence also $\Phi(S_T) = Z_T$. Therefore (using the pricing formulae on the supplementary sheet) the price of the contingent claim at time t is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\Phi(S_T)\,|\,\mathcal{F}_t] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[Z_T\,|\,\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[Z_t + (r - \tfrac{\sigma^2}{2})(T-t) + \sigma(B_T - B_t)\,|\,\mathcal{F}_t] \\ &= e^{-r(T-t)}\left(Z_t + (r - \tfrac{\sigma^2}{2})(T-t) + \sigma\mathbb{E}^{\mathbb{Q}}[B_T - B_t\,|\,\mathcal{F}_t]\right) \\ &= e^{-r(T-t)}\left(Z_t + (r - \tfrac{\sigma^2}{2})(T-t)\right) \\ &= e^{-r(T-t)}\left(\log(S_t) + (r - \tfrac{\sigma^2}{2})(T-t)\right). \end{aligned}$$

[3A] Here, we use part (a) to get a formula for Z_T, that $Z_t \in \mathcal{F}_t$, along with the fact that Brownian motion is a martingale. [3M]

11 Let $T, K > 0$. Within the Black-Scholes model:

(a) We define $\Phi^{cash}(S_T) = 1$ to be the contingent claim of a single unit in cash, at time T. Write down formulae for the following contingent claims, each of which has exercise date $T > 0$.

(i) $\Phi^{stock}(S_T)$, of a single unit of stock.

(ii) $\Phi^{call}(S_T)$, of a European call option, with strike price K.

(iii) $\Phi^{put}(S_T)$, of a European put option, with strike price K.

(3 marks)

(b) With notation as in (a), prove the put-call parity relation,

$$\Phi^{put}(S_T) = \Phi^{call}(S_T) + K\Phi^{cash}(S_T) - \Phi^{stock}(S_T).$$

(3 marks)

(c) Write down a constant portfolio, to be bought at time 0, consisting only of cash, stock, and European put options, which replicates a European call option with strike price K and exercise date T. ***(3 marks)***

Solution.

(a) (i) $\Phi^{stock}(S_T) = S_T$. [1A]

(ii) $\Phi^{call}(S_T) = \max(S_T - K, 0)$. [1A]

(iii) $\Phi^{put}(S_T) = \max(K - S_T, 0)$. [1A]

(b) If $S_T \geq K$ then the put call-parity relation reads $0 = (S_T) - K + K - S_T$, which is true. If $S_T \leq K$ then the put-call parity relation reads $K - S_T = 0 + K - S_T$, which is true. [1M+2A]

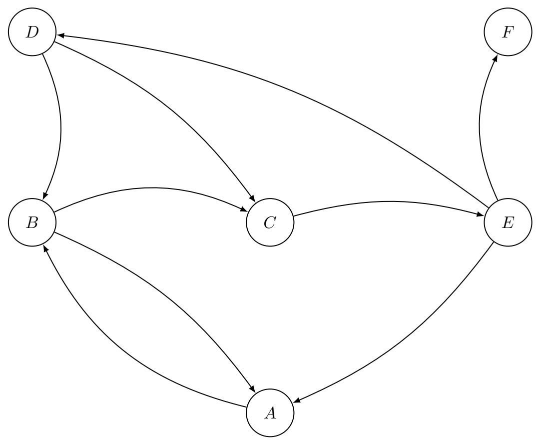
(c) Rearranging the put call parity relation, we have

$$\Phi^{call}(S_T) = \Phi^{put}(S_T) + \Phi^{stock}(S_T) - K\Phi^{cash}(S_T).$$

[1M] Therefore, the value of a call option at time T is equal to the value of one put option (with strike K and exercise T), plus one unit of stock, plus $-K$ units of cash. This is replicated by purchasing, at time 0, a portfolio of one put option, one unit of stock, and $-Ke^{-rT}$ in cash. [2A]

12 **[MAS452/6052 only]** A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G. An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y.

The graph G has vertices and edges as shown:



Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

We define a model of debt contagion by assuming that:

(†) For any bank X, with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability

$$\eta_j = \frac{1}{1+j}$$

the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Given some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.

(a) If bank A fails, and defaults on all its loans, calculate the probability that bank E also fails. ***(3 marks)***

(b) Alternatively, if bank D fails, and defaults on all its loans, calculate the probability that bank E also fails. ***(5 marks)***

Solution.

(a) If A fails, and defaults on its loan to B, then then chance that B also fails is $\frac{1}{3}$. Given that B fails, B defaults on its loan to C, and the chance that C fails is $\frac{1}{2}$. Given that C fails, C defaults on its loan to D and the chance that E fails is $\frac{1}{2}$. [1M]

The failure of $A \to B \to\to C \to E$ is the only way in which the failure of A can result in the failure of E. [1M]

12(continued)

Thus, by independence, the chance that E fails as a result of A failing is $\frac{1}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{12}$. [1A]

(b) If D fails, then the chance that C fails as result of Ds loan defaulting is $\frac{1}{3}$. The chance that B fails as a result of Ds loan defaulting is also $\frac{1}{3}$; and given that B fails the chance of C failing is $\frac{1}{3}$. So, the chance that C fails as a result of D failing is

$$\frac{1}{3} + \left(1 - \frac{1}{3}\right)\frac{1}{3}\frac{1}{3} = \frac{11}{27}.$$

[3M + 1A] Given that C fails, the chance that E fails is $\frac{1}{2}$, so the chance that E fails as a consequence of D failing is

$$\frac{11}{27}\frac{1}{2} = \frac{11}{54}.$$

[1A]

End of Question Paper