

## MASx50: Assignment 1

Solutions and discussion are written in blue. Some common pitfalls are indicated in teal. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

Marks are given for [A]ccuracy, [J]ustification, and [M]ethod.

1. Recall that the Borel  $\sigma$ -field  $\mathcal{B}(\mathbb{R})$  is the smallest  $\sigma$ -field on  $\mathbb{R}$  containing all open intervals  $(a, b) \subseteq \mathbb{R}$ . Define

$$A = \bigcup_{n=1}^N [a_n, b_n]$$

where  $a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 < \dots$  are real numbers.

- (a) Prove, starting from the definition given above, that  $A \in \mathcal{B}(\mathbb{R})$ .
- (b) Write down a formula for the Lebesgue measure of  $A$ , in terms of the  $a_i$  and  $b_i$ . Is your formula valid if  $N = \infty$ ?
- (c) Consider the following claims.
  - (i) The Borel  $\sigma$ -field is an infinite set.
  - (ii) The Borel  $\sigma$ -field contains an infinite number of infinite sets.
  - (iii) All countable sets are Borel sets with zero Lebesgue measure.
  - (iv) All Borel sets with positive Lebesgue measure contain at least one open interval.

In each case (i)-(vi), state whether you believe the claim to be true or false. For claims that you believe are true, give a proof. For claims that you believe are false, give a counterexample. Use parts (a) and (b) to support your arguments.

*Solution.*

- (a) From the definition we have  $(b, \infty) \in \mathcal{B}(\mathbb{R})$  and  $(-\infty, a) \in \mathcal{B}(\mathbb{R})$  for all  $a$ . [1J]  
Hence,  $[a, b] = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty)) \in \mathcal{B}(\mathbb{R})$ , as  $\sigma$ -fields are closed under complements and intersections [1J]  
Hence also  $A = \bigcup_{i=1}^N [a_i, b_i] \in \mathcal{B}(\mathbb{R})$ , as  $\sigma$ -fields are closed under countable unions. [1J]
- (b) We have

$$\lambda(A) = \sum_{n=1}^N (b_n - a_n).$$

[1A] By countable additivity of disjoint sets (from the definition of a measure) this formula is valid when  $N = \infty$ . [1J]

- (c) (i) True. For example,  $\mathcal{B}(\mathbb{R})$  contains each of the sets  $(x, x+1)$ , for  $x \in \mathbb{R}$ , and there are infinitely many of these. [1J]
- (ii) True. We can use the same example as in (i), because each of the sets  $(x, x+1)$  is infinite. [1J] *Pitfall:* Make sure you keep track of the difference between a set and a set of sets.

- (iii) True. If a set  $A$  is countable, then we may write it in the form  $A = \cup_{n=1}^{\infty} [a_n, a_n]$ . By part (a) this means  $A$  is a countable union of Borel sets, and hence is itself Borel. Our formula from part (b) shows that  $A$  has Lebesgue measure zero. [1J]
- (iv) False. Recall that  $\mathbb{Q}$  is countable, and hence also a Borel set by the previous part. Hence the irrational numbers  $\mathbb{R} \setminus \mathbb{Q}$  are a Borel set. Since  $\lambda(\mathbb{Q}) = 0$  we have  $\lambda(\mathbb{R} \setminus \mathbb{Q}) = \lambda(\mathbb{R}) = \infty$ , but the irrational numbers do not contain any open intervals. [1J]

[4A for correct true/false responses]

*Pitfall:* Make sure to use (a) and (b) where they are helpful (as the question asks). In fact, you hardly need to use anything else to solve part (c).

2. Let  $\mathcal{B}(\mathbb{R})$  denote the Borel  $\sigma$ -field on  $\mathbb{R}$ . This question concerns examples of decreasing sequences of Borel sets  $(B_n)$  and measures  $m$  on  $\mathcal{B}(\mathbb{R})$  such that

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) \neq \lim_{N \rightarrow \infty} m\left(\bigcap_{n=1}^N B_n\right).$$

- (a) Let  $\lambda$  denote Lebesgue measure on  $\mathbb{R}$ . Taking  $m = \lambda$ , show that  $B_n = (-\infty, -n]$  is an example of this type.
- (b) Find a second example, with the additional property that  $\cap_{n=1}^{\infty} B_n$  is non-empty.
- (c) Find a third example, with the additional property that  $B_1$  is countable.

*Solution.*

- (a) We have  $\lambda(B_n) = \sum_{j=n}^{\infty} \lambda((-j-1, -j]) = \infty$  and thus  $\lim_n \lambda(B_n) = \infty$ , [1J] but  $\cap_n (-\infty, -n] = \emptyset$  which has measure zero. [1J]
- (b) Take e.g.  $B_n = (-\infty, -n] \cup [0, 1]$ . Then  $\lambda(B_n) = \infty$  as before, but now  $\cap_n B_n = [0, 1]$  which is non-empty with Lebesgue measure 1. [2A]
- (c) Take  $m$  to be counting measure on  $\mathbb{N}$  (the  $\sigma$ -field can be  $\mathcal{P}(\mathbb{N})$  here) and let  $B_n = \{n, n+1, \dots, \infty\}$ . and then  $m(B_n) = \infty$  but  $m(\cap_n B_n) = m(\emptyset) = 0$ . [2A]

*Pitfall:* Remember the conditions of the theorem! In general,  $m(\cap_n B_n) = \lim_n m(B_n)$  for decreasing  $B_n$  *only* if  $m(B_1)$  is finite. Once you remember this, you know to start by trying an example where  $m(B_1)$  is infinite, and from there you don't have far to go.

3. Write down the  $\liminf$  and the  $\limsup$ , as  $n \rightarrow \infty$ , of the sequence  $a_n = \frac{1+2n(-1)^n}{1+3n}$ .

*Solution.* Note that if  $n$  is even then  $a_n = \frac{1+2n}{1+3n} = \frac{1/n+2}{1/n+3}$ , whilst if  $n$  is odd then  $a_n = \frac{1-2n}{1+3n} = \frac{1/n-2}{1/n+3}$ . Hence  $\liminf_n a_n = \frac{-2}{3}$  and  $\limsup_n a_n = \frac{2}{3}$ . [2A]

4. In each of the following cases, show that the given function is measurable, from  $\mathbb{R} \rightarrow \mathbb{R}$  with the Borel  $\sigma$ -field. State clearly any results from lecture notes that you make use of.

- (a)  $f(x) = \cos x$

- (b)  $g(x) = \begin{cases} 0 & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0. \end{cases}$
- (c)  $h(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(x)}{n!}$
- (d)  $i(x) = \lfloor x \rfloor$  (i.e.  $x$  rounded down to the nearest integer)

*Solution.*

- (a) From lectures, every continuous function from  $\mathbb{R}$  to  $\mathbb{R}$  is measurable. [1J] Since  $\cos$  is continuous, it is measurable. [1J]
- (b) Let  $g_1(x) = \mathbb{1}_{[0, \infty)}(x)$  be the indicator function of  $[0, \infty)$ , which is measurable because it is the indicator function of a measurable set. [1J] Let

$$g_2(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

which is measurable because it is continuous. Then  $g(x) = g_1(x) + g_2(x)$  is measurable, because the sum of measurable functions is measurable. [1J]

- (c) First note that  $|\frac{(-1)^n x^n \cos(x)}{n!}| \leq |\frac{x^n}{n!}|$  and since the power series  $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$  converges for all  $x$ , so does the series for  $h(x)$ . [1J]

We have that  $\cos(x)$  is measurable from (a),  $f(x) = x^n$  is continuous and hence measurable, thus  $x \mapsto \frac{(-1)^n x^n \cos(x)}{n!}$  is measurable, because sums and products of measurable functions are measurable. [1J]

Since limits of measurable functions (when they exist) are measurable [1J] we have that  $h(x)$  is measurable.

- (d)  $i(x)$  is an increasing function of  $x$ , [1J] and increasing functions are measurable. [1J]  
*Alternatively:* if  $x \in [n, n+1)$  then

$$f^{-1}((x, \infty)) = \{y \in \mathbb{R} : \lfloor y \rfloor > x\} = \{y \in \mathbb{R} : \lfloor y \rfloor \geq n+1\} = [n+1, \infty)$$

is a Borel set. Here we use that  $f$  is measurable if and only if  $f^{-1}((c, \infty)) \in \mathcal{B}(\mathbb{R})$  for all  $c \in \mathbb{R}$ .

*Pitfall:* Make sure to specify which results (from lectures) you use to make your deductions.

Total marks: 30