

MASx50: Assignment 2

Solutions and discussion are written in blue. A sample mark scheme, with a total of 28 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Determine if the following functions are Lebesgue integrable. Use the monotone convergence theorem to justify your answers.

- (a) $f : (0, \infty) \rightarrow \mathbb{R}$ by $f(x) = 1/x^2$.
- (b) $g : (0, 1) \rightarrow \mathbb{R}$ by $g(x) = \log x$

Solution.

- (a) Note that $x^{-2} > 0$ for $x \in (0, \infty)$. By Riemann integration, we have

$$\int_{1/n}^n x^{-2} dx = [-x^{-1}]_{1/n}^n = -\frac{1}{n} + n.$$

[1] Note that $f_n(x) = \mathbb{1}_{\{x \in (1/n, n)\}} x^{-2}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $f(x) = x^{-2}$ for $x \in (0, \infty)$. [1] Hence, by the monotone convergence theorem [1] we have

$$\int_0^\infty x^{-2} dx = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} + n \right) = +\infty.$$

Thus x^{-2} is not integrable on $(0, \infty)$. [1]

- (b) By Riemann integration, we have

$$\int_{1/n}^1 \log x dx = [x \log x - x]_{1/n}^1 = (-1) - \left(\frac{1}{n} \log \frac{1}{n} - \frac{1}{n} \right) = \frac{1 + \log n}{n} - 1.$$

Noting that $\log x \in (-\infty, 0)$ for $x \in (0, 1)$, multiplying the above by -1 gives

$$\int_{1/n}^1 |\log x| dx = 1 - \frac{1 + \log n}{n}.$$

[1] We have that $g_n(x) = |\log x| \mathbb{1}_{x \in (1/n, 1)}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $g(x) = |\log x|$ for $x \in (0, 1)$. [1] Hence, by the monotone convergence theorem,

$$\int_0^1 |\log x| dx = \lim_{n \rightarrow \infty} \left(1 - \frac{1 + \log n}{n} \right) = 1.$$

Thus $\log x$ is integrable on $(0, 1)$. [1]

2. The following text describes the key steps of defining the Lebesgue integral on a measure space (S, Σ, m) . It contains *three* mistakes.

1 For indicator functions $\mathbb{1}_A$ where $A \in \Sigma$, set

$$2 \quad \int_0^\infty \int_S \mathbb{1}_A dm = m(A). \quad (\star)$$

3 For simple functions $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$, where $c_i \geq 0$ and $A_i \in \Sigma$ for all $i \in$
 4 $\{1, \dots, n\}$, extend equation (\star) by linearity to give

$$5 \quad \int_S s dm = \sum_{i=1}^n c_i m(A_i).$$

6 For non-negative measurable functions $f : S \rightarrow [0, \infty)$, define

$$7 \quad \int_S f dm = \sup \left\{ \int_S s dm : s \text{ is a continuous simple function and } 0 \leq s \leq f \right\}.$$

8 We therefore have that $\int_S f dm \in [0, \infty]$ for non-negative measurable functions f .

9 For an arbitrary measurable function $f : S \rightarrow \mathbb{R}$, write $f = f_+ - f_-$, where
 10 $f_+ = 0 \vee f$ and $f_- = -(f \wedge 0)$. Then f_+ and f_- are non-negative measurable
 11 functions. If one or both of $\int_S f_+ dm$ and $\int_S f_- dm$ is not equal to $+\infty$ then we
 12 define

$$13 \quad \int_S f dm = \int_S f_+ dm - \int_S f_- dm.$$

14 If both $\int_S f_+ dm$ and $\int_S f_- dm$ are equal to $+\infty$ then $\int_S f dm$ is ~~equal to $+\infty$~~
 15 ~~undefined~~.

Each mistake is on a distinct line. Line numbers are included for convenience and to help you reference the text.

List the line numbers containing mistakes and, for each mistake, give a corrected version.

Solution.

(a) 2, 7, 14. [3]

(b) As indicated above. [3]

3. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

- (a) Let $f : S \rightarrow \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.
- (b) Let $f : S \rightarrow \mathbb{R}$ be a simple function. Write $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ where the A_i are pairwise disjoint and measurable and $c_i \geq 0$. Show that

$$\left(\int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \quad (\star)$$

Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

$$\frac{(\sum_{i=1}^n u_i)^2}{\sum_{i=1}^n v_i} \leq \sum_{i=1}^n \frac{u_i^2}{v_i}.$$

- (c) In this question you should give *two* different proofs that equation (\star) holds when f is any non-negative measurable function. You may use your results from part (b) in both proofs.
 - i. Give a proof using the monotone convergence theorem.
 - ii. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.
- (d) Does (\star) remain true if m is not necessarily a probability measure?

Solution.

- (a) We have

$$f^2 = \sum_{i=1}^n \sum_{j=1}^m c_i c_j \mathbb{1}_{A_i} \mathbb{1}_{A_j} = \sum_{i=1}^n c_i^2 \mathbb{1}_{A_i}$$

where the second inequality follows by disjointness – all the cross terms (when $i \neq j$) are zero. [1] We have thus expressed f^2 as a simple function, and since c_i^2 are non-negative, f^2 is also non-negative. [1]

- (b) We have

$$\begin{aligned} \left(\int f \, dm \right)^2 &= \left(\sum_{i=1}^n c_i m(A_i) \right)^2, \\ \int f^2 \, dm &= \sum_{i=1}^n c_i^2 m(A_i). \end{aligned}$$

[2] The required inequality follows from the above and Titu's lemma, taking $v_i = m(A_i)$ and $u_i = c_i m(A_i)$. [1] Note that, because m is a probability measure, $\sum_i m(A_i) = 1$ and we may assume $m(A_i) > 0$ (because any A_i with zero measure will have no effect on the value of the integral).

Follow-up challenge exercise: See if you can derive Titu's lemma from the real version of the Cauchy-Schwarz inequality.

(c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be non-negative and measurable.

First proof (using the monotone convergence theorem): From lectures (see the section on simple functions) there exists a sequence (s_n) of non-negative simple functions such that $0 \leq s_n \leq s_{n+1} \leq f$ such that $s_n \rightarrow f$ pointwise. [1] Thus, by the monotone convergence theorem, as $n \rightarrow \infty$,

$$\int s_n dm \rightarrow \int f dm.$$

[1] By part (a), (s_n^2) is also a sequence of simple functions. [1] We have $0 \leq s_n^2 \leq s_{n+1}^2 \leq f^2$, also $s_n^2 \rightarrow f^2$ pointwise. So by another application of the monotone convergence theorem we have

$$\int s_n^2 dm \rightarrow \int f^2 dm.$$

[1] From part (b) we have

$$\left(\int s_n dm \right)^2 \leq \int s_n^2 dm$$

for all n . Since limits preserve weak inequalities, [1] we have that

$$\left(\int f dm \right)^2 \leq \int f^2 dm$$

as required.

Second proof (using the definition of the integral): Recall that the definition of the Lebesgue integral, for non-negative measurable functions, is

$$\int f dm = \sup \left\{ \int s dm : s \text{ is simple and } 0 \leq s \leq f \right\}.$$

Hence

$$\begin{aligned} \left(\int f dm \right)^2 &= \left(\sup \left\{ \int s dm : s \text{ is simple and } 0 \leq s \leq f \right\} \right)^2 \\ &= \sup \left\{ \left(\int s dm \right)^2 : s \text{ is simple and } 0 \leq s \leq f \right\} \\ &\leq \sup \left\{ \int s^2 dm : s \text{ is simple and } 0 \leq s \leq f \right\} \\ &= \sup \left\{ \int r dm : r \text{ is simple and } 0 \leq r \leq f^2 \right\} \\ &= \int f^2 dm \end{aligned}$$

Here, the second line follows because $\int s dm \geq 0$, so the square can pass inside of the sup. [1] The third line then follows by part (b). [1] Let us now justify the fourth line. We have shown in (a) that if s is a non-negative simple function then so is $r = s^2$, and clearly if $s \leq f$ then $s^2 \leq f^2$ (i.e. pointwise). [1] Also, if r is a non-negative simple function such that $0 \leq r \leq f^2$, then if we define $s = \sqrt{r}$, we can show (in similar style to part (a)) that s is a non-negative simple function such that $0 \leq s \leq f$. Here, if $r = \sum_i c_i \mathbb{1}_{A_i}$ we would have $s = \sum_i \sqrt{c_i} \mathbb{1}_{A_i}$. So, the two sups in the third and fourth lines are equal using the correspondence $r = s^2$. [1]

(d) In general (\star) fails when m is not a probability measure. For example, take $f(x) = x$ and let m be Lebesgue measure on $[0, 2]$. Then $\int_0^2 x dx = 2$ and $\int_0^2 x^2 dx = \frac{8}{3}$, but $2^2 > \frac{8}{3}$. [1]