MAS350: Assignment 2

Solutions and discussion are written in blue. A sample mark scheme, with a total of 25 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. In each of the following cases, show that the given function is measurable, from $\mathbb{R} \to \mathbb{R}$ with the Borel σ-field. State clearly any results from lectures that you make use of.

   (a) $f(x) = \cos x$

   (b) $g(x) = \begin{cases} 0 & \text{for } x < 0 \\ x+1 & \text{for } x \geq 0. \end{cases}$

   (c) $h(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(x)}{n!}$

   (d) $i(x) = \lfloor x \rfloor$ (i.e. x rounded down to the nearest integer)

   Solution.

   (a) From lectures, every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is measurable. [1] Since cos is continuous, it is measurable. [1]

   (b) Let $g_1(x) = \mathbb{1}_{[0,\infty)}(x)$ be the indicator function of $[0, \infty)$, which is measurable because it is the indicator function of a measurable set. [1] Let

   $$g_2(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

   which is measurable because it is continuous. Then $g(x) = g_1(x) + g_2(x)$ is measurable, because the sum of measurable functions is measurable. [1]

   (c) First note that $|\frac{(-1)^n x^n \cos(x)}{n!}| \leq |\frac{x^n}{n!}|$ and since the power series $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all x, so does the series for $h(x)$. [1]

   We have that $\cos(x)$ is measurable from (a), $f(x) = x^n$ is continuous and hence measurable, thus $x \mapsto \frac{(-1)^n x^n \cos(x)}{n!}$ is measurable, because sums and products of measurable functions are measurable. [1]

   Since limits of measurable functions (when they exist) are measurable [1] we have that $h(x)$ is measurable.

   (d) $i(x)$ is an increasing function of x, [1] and increasing functions are measurable. [1]

   Alternatively: if $x \in [n, n+1)$ then

   $$f^{-1}((x, \infty)) = \{y \in \mathbb{R} \, : \, \lfloor y \rfloor > x\} = \{y \in \mathbb{R} \, : \, \lfloor y \rfloor \geq n+1\} = [n+1, \infty)$$

   is a Borel set. Here we use that f is measurable if and only if $f^{-1}((c, \infty)) \in \mathcal{B}(\mathbb{R})$ for all $c \in \mathbb{R}$.

   Pitfall: Make sure to specify which results (from lectures) you use to make your deductions.

2. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

   (a) Let $f : S \to \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.

   (b) Let $f : S \to \mathbb{R}$ be a simple function. Write $f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$ where the A_i are pairwise disjoint and measurable and $c_i \geq 0$. Show that

$$\left(\int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \qquad (\star)$$

   Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

$$\frac{\left(\sum_{i=1}^{n} u_i\right)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}.$$

   (c) In this question you should give *two* different proofs that equation $(\star)$ holds when f is any non-negative measurable function. You may use your results from part (b) in both proofs.

      i. Give a proof using the monotone convergence theorem.

      ii. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.

   (d) Does $(\star)$ remain true if m is not necessarily a probability measure?

Solution.

(a) We have

$$f^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i c_j \mathbb{1}_{A_i} \mathbb{1}_{A_j} = \sum_{i=1}^{n} c_i^2 \mathbb{1}_{A_i}$$

where the second inequality follows by disjointness – all the cross terms (when $i \neq j$) are zero. [1] We have thus expressed f^2 as a simple function, and since c_i^2 are non-negative, f^2 is also non-negative. [1]

(b) We have

$$\left(\int f \, dm \right)^2 = \left(\sum_{i=1}^{n} c_i m(A_i) \right)^2,$$

$$\int f^2 \, dm = \sum_{i=1}^{n} c_i^2 m(A_i).$$

[2] The required inequality follows from the above and Titu's lemma, taking $v_i = m(A_i)$ and $u_i = c_i m(A_i)$. [1] Note that, because m is a probability measure, $\sum_i m(A_i) = 1$ and we may assume $m(A_i) > 0$ (because any A_i with zero measure will have no effect on the value of the integral).

Follow-up challenge exercise: See if you can derive Titu's lemma from the real version of the Cauchy-Schwarz inequality.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be non-negative and measurable.

First proof (using the monotone convergence theorem): From lectures (see the section on simple functions) there exists a sequence (s_n) of non-negative simple functions such that $0 \le s_n \le s_{n+1} \le f$ such that $s_n \to f$ pointwise. [1] Thus, by the monotone convergence theorem, as $n \to \infty$,

$$\int s_n \, dm \to \int f \, dm.$$

[1] By part (a), (s_n^2) is also a sequence of simple functions. [1] We have $0 \le s_n^2 \le s_{n+1}^2 \le f^2$, also $s_n^2 \to f^2$ pointwise. So by another application of the monotone convergence theorem we have

$$\int s_n^2 \, dm \to \int f^2 \, dm.$$

[1] From part (b) we have

$$\left(\int s_n \, dm \right)^2 \le \int s_n^2 \, dm$$

for all n. Since limits preserve weak inequalities, [1] we have that

$$\left(\int f \, dm \right)^2 \le \int f^2 \, dm$$

as required.

Second proof (using the definition of the integral): Recall that the definition of the Lebesgue integral, for non-negative measurable functions, is

$$\int f \, dm = \sup \left\{ \int s \, dm \ : \ s \text{ is simple and } 0 \le s \le f \right\}.$$

Hence

$$\begin{aligned}
\left(\int f \, dm \right)^2 &= \left(\sup \left\{ \int s \, dm \ : \ s \text{ is simple and } 0 \le s \le f \right\} \right)^2 \\
&= \sup \left\{ \left(\int s \, dm \right)^2 \ : \ s \text{ is simple and } 0 \le s \le f \right\} \\
&\le \sup \left\{ \int s^2 \, dm \ : \ s \text{ is simple and } 0 \le s \le f \right\} \\
&= \sup \left\{ \int r \, dm \ : \ r \text{ is simple and } 0 \le r \le f^2 \right\} \\
&= \int f^2 \, dm
\end{aligned}$$

Here, the second line follows because $\int s \, dm \ge 0$, so the square can pass inside of the sup. [1] The third line then follows by part (b). [1] Let us now justify the fourth line. We have shown in (a) that if s is a non-negative simple function then so is $r = s^2$, and clearly if $s \le f$ then $s^2 \le f^2$ (i.e. pointwise). [1] Also, if r is a non-negative simple function such that $0 \le r \le f^2$, then if we define $s = \sqrt{r}$, we can show (in similar style to part (a)) that s is a non-negative simple function such that $0 \le s \le f$. Here, if $r = \sum_i c_i \mathbb{1}_{A_i}$ we would have $s = \sum_i \sqrt{c_i} \mathbb{1}_{A_i}$. So, the two sups in the third and fourth lines are equal using the correspondence $r = s^2$. [1]

(d) In general $(\star)$ fails when m is not a probability measure. For example, take $f(x) = x$ and let m be Lebesgue measure on $[0, 2]$. Then $\int_0^2 x \, dx = 2$ and $\int_0^2 x^2 \, dx = \frac{8}{3}$, but $2^2 > \frac{8}{3}$. [1]