MASx52: Assignment 5

1. (a) Within the Black-Scholes model, use the risk neutral valuation formula

$$F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid \mathcal{F}_t\right]$$

to show that price at time t of the contingent claim $\Phi(S_T) = 3S_T + 5$ is given by

$$F(t, S_t) = 3S_t + 5e^{-r(T-t)}.$$

(b) Describe a portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.

(c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = 3S_T + 5$.

i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

(d) Suggest one reason why we might wish to hold a delta neutral portfolio.

2. (a) Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and S_t be an Ito process satisfying $dS_t = \alpha S_t\,dt + \sigma S_t\,dB_t$. Let $Y_t = S_t^3$. Show that Y_t satisfies the SDE

$$dY_t = \left(3\alpha + 3\sigma^2\right) Y_t\,dt + 3\sigma Y_t\,dB_t$$

Deduce that Y_t is a geometric Brownian motion, and write down its drift and volatility.

(b) Within the Black-Scholes model, show that the price $F(t, S_t)$ at time $t \in [0, T]$ of the contingent claim $\Phi(S_T) = S_T^3$ is given by

$$F(t, S_t) = S_t^3 e^{2r(T-t)+3\sigma^2(T-t)}.$$

(c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = S_T^3$.

i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?