

# SOME DISCRETE DISTRIBUTIONS

Name	Parameters	Genesis / Usage	$p(x) = \mathbb{P}[X = x]$ and non-zero range	$\mathbb{E}[X]$	$\text{Var}(X)$	Comments
Uniform (discrete)	$k \in \mathbb{N}$	Set of $k$ equally likely outcomes.	$p(x) = 1/k$ $x = 1, \dots, k$	$\frac{k+1}{2}$	$\frac{k^2-1}{12}$	Fair dice roll with $k = 6$ .
Bernoulli trial	$\theta \in [0, 1]$	Experiment with two outcomes; typically, success = 1, fail = 0.	$p(x) = \theta^x(1 - \theta)^{1-x}$ $x = 0, 1$	$\theta$	$\theta(1 - \theta)$	
Binomial	$n \in \mathbb{N}$ $\theta \in [0, 1]$	Number of successes in $n$ i.i.d. Bernoulli trials.	$p(x) = \binom{n}{x}\theta^x(1 - \theta)^{n-x}$ $x = 0, 1, 2, \dots, n$	$n\theta$	$n\theta(1 - \theta)$	Often written $\text{Bin}(n, \theta)$ . $\text{Bin}(1, \theta) \sim \text{Bernoulli}(\theta)$
Geometric	$\theta \in (0, 1]$	Number of failed i.i.d. Bernoulli trials before the first success.	$p(x) = \theta(1 - \theta)^x$ $x = 0, 1, 2, \dots$	$\frac{\theta}{1-\theta}$	$\frac{\theta^2}{(1-\theta)^2}$	Alternative parametrisations: swap $\theta$ and $1 - \theta$ , or $X' = X + 1$ to include the final trial.
Negative Binomial	$k \in \mathbb{N}$ $\theta \in (0, 1]$	Number of failed i.i.d. Bernoulli trials before the $k^{\text{th}}$ success.	$p(x) = \binom{x+k-1}{x}\theta^k(1 - \theta)^x$ $x = 0, 1, 2, \dots$	$\frac{k(1-\theta)}{\theta}$	$\frac{k(1-\theta)}{\theta^2}$	Many alternative parametrisations. $\text{NegBin}(1, \theta) \sim \text{Geometric}(\theta)$ .
Hypergeometric	$N \in \mathbb{N}$ $k \in \{0, \dots, N\}$ $n \in \{0, \dots, n\}$	Number of special objects in a random sample of $n$ objects, from a population of $N$ objects with $k$ special objects.	$p(x) = \binom{k}{x}\binom{N-k}{n-x}/\binom{N}{n}$ $x = 0, \dots, n$	$\frac{nk}{N}$	$n\frac{N-n}{N-1}\frac{k}{N} \times (1 - \frac{k}{N})$	
Poisson	$\lambda \in (0, \infty)$	Counting events occurring uniformly at random within space or time.	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, 2, \dots$	$\lambda$	$\lambda$	

# SOME CONTINUOUS DISTRIBUTIONS

Name	Parameters	Genesis / Usage	$f(x)$ = p.d.f. and non-zero range	$\mathbb{E}[X]$	$\text{Var}(X)$	Comments
Uniform (continuous)	$\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$	The uniform distribution for a continuous interval.	$f(x) = \frac{1}{\beta - \alpha}$ $x \in (\alpha, \beta)$	$\frac{\alpha + \beta}{2}$	$\frac{(\beta - \alpha)^2}{12}$	
Normal	$\mu \in \mathbb{R}$ $\sigma \in (0, \infty)$	Empirically and theoretically (via CLT) a good model in many situations.	$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $x \in \mathbb{R}$	$\mu$	$\sigma^2$	Often written $N(\mu, \sigma^2)$ . Alternative parameter: $\tau = \frac{1}{\sigma^2}$ . $a N(\mu, \sigma^2) + b \sim N(a\mu + b, a^2\sigma^2)$
Exponential	$\lambda \in (0, \infty)$	Inter-arrival times of random events.	$f(x) = \lambda e^{-\lambda x}$ $x \in (0, \infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	Often written $\text{Exp}(\lambda)$ . Alternative parameter: $\theta = \frac{1}{\lambda}$ .
Gamma	$\alpha \in (0, \infty)$ $\beta \in (0, \infty)$	Lifetimes of ageing items, multi-inter-arrival times.	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ $x \in (0, \infty)$	$\frac{\alpha-1}{\beta}$ if $\alpha \geq 1$ 0 if $\alpha < 1$	$\frac{\alpha}{\beta^2}$	Often written $\Gamma(\alpha, \beta)$ . Alternative parameter: $\theta = \frac{1}{\beta}$ . $\text{Gamma}(1, \lambda) \sim \text{Exp}(\lambda)$
Beta	$\alpha \in (0, \infty)$ $\beta \in (0, \infty)$	Quantities constrained to be within intervals.	$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$ $x \in [0, 1]$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$	$\text{Beta}(1, 1) \sim \text{Uniform}(0, 1)$
Cauchy	$a \in \mathbb{R}$ $b \in (0, \infty)$	Heavy tailed, pathological examples.	$f(x) = \frac{1}{\pi b} \frac{b^2}{(x-a)^2 + b^2}$ $x \in \mathbb{R}$	undefined	undefined	
Pareto	$\alpha \in (0, \infty)$ $\beta \in (0, \infty)$	Heavy tailed quantities.	$f(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}}$ $x \in (\beta, \infty)$	$\frac{\alpha\beta}{\alpha+1}$ if $\alpha > 1$	$\frac{\alpha^2\beta}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$	Sometimes written $\text{Pareto}(\beta, \alpha)$ . $\log\left(\frac{\text{Pareto}(\alpha, \beta)}{\beta}\right) \sim \text{Exp}(\alpha)$
Weibull	$\lambda \in (0, \infty)$ $k \in (0, \infty)$	Lifetimes, extreme values.	$f(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$ $x \in (0, \infty)$	$\lambda\Gamma(1 + 1/k)$	$\lambda^2[\Gamma(1 + 2/k) + \Gamma(1 + 1/k)^2]$	$\left(\frac{\text{Weibull}(\lambda, k)}{\lambda}\right)^k \sim \text{Exp}(1)$
Log-Normal	$\mu \in \mathbb{R}$ $\sigma \in (0, \infty)$	Quantities related to exponential growth.	$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right)$ $x \in (0, \infty)$	$e^{\mu + \frac{1}{2}\sigma^2}$	$(e^{\sigma^2} - 1) \times e^{2\mu + \sigma^2}$	Often written $\text{LogN}(\mu, \sigma^2)$ . $\log(\text{LogN}(\mu, \sigma^2)) \sim N(\mu, \sigma^2)$
Chi-squared	$n \in \mathbb{N}$	Statistical testing.	$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$ $x \in (0, \infty)$	$n$	$2n$	Often written $\chi_n^2$ . $X_n^2 \sim \text{Gamma}(n/2, 1/2)$ $X_i \sim N(0, 1)$ i.i.d. $\Rightarrow \sum_1^n X_i^2 \sim \chi_n^2$
Student $t$	$n \in \mathbb{N}$	Statistical testing.	$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} (1 + \frac{x^2}{n})^{-\frac{n+1}{2}}$ $x \in \mathbb{R}$	0 if $n > 1$	$\frac{n}{n-2}$ if $n > 2$	Often written $t_n$ . Can allow $n \in (0, \infty)$ . $t_1 \equiv \text{Cauchy}(0, 1)$
Inverse Gamma	$\alpha \in (0, \infty)$ $\beta \in (0, \infty)$	Quantities related to the Gamma distribution.	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x)$ $x \in (0, \infty)$	$\frac{\beta}{\alpha-1}$ if $\alpha > 1$	$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$	Often written $\text{IGamma}(\alpha, \beta)$ . $\text{IGamma}(\alpha, \beta) \sim \frac{1}{\text{Gamma}(\alpha, \beta)}$

# SOME CONJUGATE PAIRS

Model family	Prior family	Data	Posterior parameters
Bernoulli( $\theta$ ) <sup>⊗n</sup>	$\theta \sim \text{Beta}(a, b)$	$x \in \{0, 1\}^n$	$a^* = a + \sum_1^n x_i$ $b^* = b + n - \sum_1^n x_i$
Bin( $m_1, \theta$ ) ⊗ ... ⊗ Bin( $m_n, \theta$ ) with $m_1, \dots, m_n \in \mathbb{N}$ fixed.	$\theta \sim \text{Beta}(a, b)$	$x \in \{0, 1, \dots\}^n$ where $x_i \in \{0, \dots, m_i\}$	$a^* = a + \sum_1^n x_i$ $b^* = b + \sum_1^n m_i - \sum_1^n x_i$
Geometric( $\theta$ ) <sup>⊗n</sup>	$\theta \sim \text{Beta}(a, b)$	$x \in \{0, 1, \dots\}^n$	$a^* = a + n$ $b^* = b + \sum_1^n x_i$
Poisson( $\theta$ ) <sup>⊗n</sup>	$\theta \sim \text{Gamma}(a, b)$	$x \in \{0, 1, \dots\}^n$	$a^* = a + \sum_1^n x_i$ $b^* = b + n$
Exp( $\lambda$ ) <sup>⊗n</sup>	$\lambda \sim \text{Gamma}(a, b)$	$x \in (0, \infty)^n$	$a^* = a + n$ $b^* = b + \sum_1^n x_i$
Weibull( $\theta, \beta$ ) <sup>⊗n</sup> with $\beta \in (0, \infty)$ fixed.	$\theta \sim \text{IGamma}(a, b)$	$x \in (0, \infty)^n$	$a^* = a + n$ $b^* = b + \sum_1^n x_i^\beta$
N( $\theta, \sigma^2$ ) <sup>⊗n</sup> with $\sigma \in (0, \infty)$ fixed.	$\theta \sim \text{N}(u, s^2)$	$x \in \mathbb{R}^n$	$u^* = \left( \frac{1}{\sigma^2} \sum_1^n x_i + \frac{u}{s^2} \right) / \left( \frac{n}{\sigma^2} + \frac{1}{s^2} \right)$ $(s^*)^2 = 1 / \left( \frac{n}{\sigma^2} + \frac{1}{s^2} \right)$
N( $\theta, \frac{1}{\tau}$ ) <sup>⊗n</sup> with $\tau \in (0, \infty)$ fixed.	$\theta \sim \text{N}(u, \frac{1}{t})$	$x \in \mathbb{R}^n$	$u^* = (\tau \sum_1^n x_i + ut) / (\tau n + t)$ $\frac{1}{t^*} = 1 / (\tau n + t)$
N( $\mu, \frac{1}{\tau}$ ) <sup>⊗n</sup> with $\mu \in \mathbb{R}$ fixed.	$\tau \sim \text{Gamma}(a, b)$	$x \in \mathbb{R}^n$	$a^* = a + \frac{n}{2}$ $b^* = b + \frac{1}{2} \sum_1^n (x_i - \mu)^2$
N( $\mu, \frac{1}{\tau}$ ) <sup>⊗n</sup>	$(\mu, \tau) \sim \text{NGamma}(m, p, a, b)$	$x \in \mathbb{R}^n$	$m^* = \frac{n\bar{x} + mp}{n+p}$ $p^* = n + p$ $a^* = a + \frac{n}{2}$ $b^* = b + \frac{n}{2} \left( s^2 + \frac{p}{n+p} (\bar{x} - m)^2 \right)$ where $\bar{x} = \frac{1}{n} \sum_1^n x_i$ and $s^2 = \frac{1}{n} \sum_1^n (x_i - \bar{x})^2$

See the sheet on conditional probability for the Normal-Gamma distribution.

For all other distributions, see the reference sheets of discrete and continuous distributions.

## CONDITIONAL PROBABILITY AND RELATED FORMULAE

We say that a random variable  $X$  is **discrete** if there exists a countable set  $A \subseteq \mathbb{R}^d$  such that  $\mathbb{P}[X \in A] = 1$ . In this case the function  $p_X(x) = \mathbb{P}[X = x]$ , defined for  $x \in \mathbb{R}^d$ , is known as the **probability mass function** of  $X$ . The **range** of  $X$  is the set  $R_X = \{x \in \mathbb{R}^d; \mathbb{P}[X = x] > 0\}$ .

We say that a random variable  $X$  is **continuous** if there exists a function  $f_X : \mathbb{R}^d \rightarrow [0, \infty)$  such that  $\mathbb{P}[X \in A] = \int_A f_X(x) dx$  for all  $A \subseteq \mathbb{R}^d$ . In this case  $f_X$  is known as the **probability density function** of  $X$ . The **range** of  $X$  is the set  $R_X = \{x \in \mathbb{R}^d; f_X(x) > 0\}$ .

If  $X$  and  $Y$  are discrete, and  $p_X \propto p_Y$ , then  $X \stackrel{d}{=} Y$ .

If  $X$  and  $Y$  are continuous, and  $f_X \propto f_Y$ , then  $X \stackrel{d}{=} Y$ .

If  $X$  is a random variable and  $\mathbb{P}[X \in A] > 0$  then the **conditional distribution** of  $X|_{\{X \in A\}}$  satisfies  $\mathbb{P}[X|_{\{X \in A\}} \in A] = 1$  and

$$\mathbb{P}[X|_{\{X \in A\}} \in B] = \frac{\mathbb{P}[X \in B]}{\mathbb{P}[X \in A]}$$

for all  $B \subseteq A$ .

If  $X$  and  $Y$  are random variables, with  $A \subseteq R_X$ ,  $B \subseteq R_Y$  and  $\mathbb{P}[X \in A] > 0$ , then

$$\mathbb{P}[Y|_{\{X \in A\}} \in B] = \frac{\mathbb{P}[X \in A, Y \in B]}{\mathbb{P}[X \in A]}.$$

If  $(Y, Z)$  and random variables and  $\mathbb{P}[Y = y] = 0$  then it is sometimes possible to define the conditional distribution of  $Z|_{\{Y=y\}}$  via taking the limit  $\mathbb{P}[Z|_{\{|Y-y| \leq \epsilon\}} \in A] \rightarrow \mathbb{P}[Z|_{\{Y=y\}} \in A]$  as  $\epsilon \rightarrow 0$ .

Let  $(Y, Z)$  be a pair of continuous random variables. If the conditional distribution of  $Z|_{\{Y=y\}}$  exists then it is given by

$$f_{Z|_{\{Y=y\}}}(z) = \frac{f_{Y,Z}(y, z)}{f_Y(y)}.$$

For a discrete or continuous random variable  $X$ , the **likelihood function** of  $X$  is

$$L_X(x) = \begin{cases} \mathbb{P}[X = x] & \text{if } X \text{ is discrete,} \\ f_X(X) & \text{if } X \text{ is continuous.} \end{cases}$$

The general formula for **completing the square** as a function of  $\theta \in \mathbb{R}$  is  $A\theta^2 - 2\theta B + C = A\left(\theta - \frac{B}{A}\right)^2 + C - \frac{B^2}{A}$

The **sample-mean-variance** identity states  $\sum_1^n (x_i - \mu)^2 = ns^2 + n(\bar{x} - \mu)^2$  where  $\bar{x} = \frac{1}{n} \sum_1^n x_i$  and  $s^2 = \frac{1}{n} \sum_1^n (x_i - \bar{x})^2$ .

The **Beta and Gamma functions** are given by

$$\mathcal{B}(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx, \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

They are related by  $\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ . For  $n \in \mathbb{N}$ ,  $(n-1)! = \Gamma(n)$ .

The **Normal-Gamma distribution** has p.d.f. given by

$$\begin{aligned} f_{\text{NGamma}(m, p, a, b)}(\mu, \tau) &= f_{\text{N}(m, \frac{1}{p\tau})}(\mu) f_{\text{Gamma}(a, b)}(\tau) \\ &\propto \tau^{a-\frac{1}{2}} \exp\left(-\frac{p\tau}{2}(\mu - m)^2 - b\tau\right). \end{aligned}$$

for  $\mu \in \mathbb{R}$  and  $\tau > 0$ , and zero otherwise. The parameters are  $m \in \mathbb{R}$ ,  $p \in (0, \infty)$ ,  $a \in (0, \infty)$  and  $b \in (0, \infty)$ . If  $(U, T) \sim \text{NGamma}(m, p, a, b)$  then  $T \sim \text{Gamma}(a, b)$  and  $U|_{\{T=\tau\}} \sim \text{N}(m, \frac{1}{p\lambda})$ .

## BAYESIAN MODELS AND RELATED FORMULAE

The **Bayesian model** associated to the model family  $(M_\theta)_{\theta \in \Pi}$  and prior p.d.f.  $f_\Theta(\theta)$  is the random variable  $(X, \Theta) \in \mathbb{R}^n \times \mathbb{R}^d$  with distribution given by

$$\mathbb{P}[X \in B, \Theta \in A] = \int_A \mathbb{P}[M_\theta \in B] f_\Theta(\theta) d\theta.$$

The model family satisfies  $X|_{\{\Theta=\theta\}} \stackrel{d}{=} M_\theta$ .

The distribution of  $X$  is known as the **sampling distribution**, given by

$$\begin{aligned} \mathbb{P}[X = x] &= \int_{\mathbb{R}^d} \mathbb{P}[M_\theta = x] f_\Theta(\theta) d\theta && \text{if } (M_\theta) \text{ is a discrete family,} \\ f_X(x) &= \int_{\mathbb{R}^d} f_{M_\theta}(x) f_\Theta(\theta) d\theta && \text{if } (M_\theta) \text{ is a continuous family.} \end{aligned} \quad (\star)$$

The distribution of  $\Theta|_{\{X=x\}}$  is known as the **posterior distribution** given the data  $x$ . **Bayes rule** states that

$$f_{\Theta|_{\{X=x\}}}(\theta) = \frac{1}{Z} L_{M_\theta}(x) f_\Theta(\theta)$$

where  $L_{M_\theta}$  is the likelihood function of  $M_\theta$ ; the p.d.f. in the absolutely continuous case and the p.m.f. in the discrete case. The normalizing constant  $Z$  is given by  $Z = \int_{\Pi} L_{M_\theta}(x) f_\Theta(\theta) d\theta$ , which is equal to  $\mathbb{P}[X = x]$  in the discrete case and equal to  $f_X(x)$  in the continuous case.

The **predictive distribution** is given by replacing  $f_\Theta$  in  $(\star)$  with  $f_{\Theta|_{\{X=x\}}}$ .

If  $\theta$  is a real valued parameter and  $X \sim M_\theta$ , the **reference prior**  $\Theta$  associated to the model family  $(M_\theta)$  has density function given by

$$f_\Theta(\theta) \propto \mathbb{E} \left[ \left( \frac{d}{d\theta} \log(L_{M_\theta}(X)) \right)^2 \right]^{1/2} \propto \mathbb{E} \left[ -\frac{d^2}{d\theta^2} \log(L_{M_\theta}(X)) \right]^{1/2}.$$

Consider a Bayesian model with unknown parameter  $\theta$  and data  $x$ . Let  $H_0$  be the hypothesis that  $\theta \in \Pi_0$ , and  $H_1$  be the hypothesis that  $\theta \in \Pi_1$ , where  $\Pi_0$  and  $\Pi_1$  partition the parameter space  $\Pi$ . The **prior and posterior odds ratios** of  $H_0$  against  $H_1$  are

$$\frac{\mathbb{P}[\Theta \in \Pi_0]}{\mathbb{P}[\Theta \in \Pi_1]} \quad \text{and} \quad \frac{\mathbb{P}[\Theta|_{\{X=x\}} \in \Pi_0]}{\mathbb{P}[\Theta|_{\{X=x\}} \in \Pi_1]}.$$

The **Bayes factor** is  $B = \frac{\text{posterior odds}}{\text{prior odds}}$ . The following table provides a rough guide to interpreting the Bayes factor.

Bayes factor	Interpretation: evidence in favour of $H_0$ over $H_1$
1 to 3.2	Indecisive / not worth more than a bare mention
3.2 to 10	Substantial
10 to 100	Strong
above 100	Decisive

A **high posterior density region** is a subset  $\Pi_0 \subseteq \Pi$  that is chosen to minimize the size of  $\Pi_0$  and maximize  $\mathbb{P}[\Theta|_{\{X=x\}} \in \Pi_0]$ .

If  $\Theta|_{\{X=x\}}$  has a distribution with a single peak then it is common to choose an **equally tailed** HPD region of the form  $\Pi_0 = [a, b]$  where

$$\mathbb{P}[\Theta|_{\{X=x\}} < a] = \mathbb{P}[\Theta|_{\{X=x\}} > b] = \frac{1-p}{2}$$

and some value is picked for  $p \in (0, 1)$ .

If  $Z \sim N(0, 1)$  then  $\mathbb{P}[Z \geq 1.645] \approx 0.05$ ,  $\mathbb{P}[Z \geq 1.96] \approx 0.025$  and  $\mathbb{P}[Z \geq 2.58] \approx 0.005$ .

## SOME USEFUL ALGORITHMS

The **Metropolis-Hastings** algorithm for simulating (approximate) samples from the distribution of  $Y$  is as follows. The key ingredient of the algorithm is a joint distribution  $(Y, Q)$ , where  $Q|_{\{Y=y\}}$  and  $Y|_{\{Q=y\}}$  are both well defined for all  $y \in R_Y$ , both with the same range as  $Y$ .

Let  $y_0$  be a point within  $R_Y$ . Then, given  $y_m$  we define  $y_{m+1}$  as follows.

1. Generate a *proposal point*  $\tilde{y}$  from the distribution of  $Q|_{\{Y=y_m\}}$ .
2. Calculate the value of  $\alpha = \min \left\{ 1, \frac{f_{Y|_{\{Q=\tilde{y}\}}}(y_m)f_Y(\tilde{y})}{f_{Q|_{\{Y=y_m\}}}(\tilde{y})f_Y(y_m)} \right\}$ .
3. Then, set  $y_{m+1} = \begin{cases} \tilde{y} & \text{with probability } \alpha, \\ y_m & \text{with probability } 1 - \alpha. \end{cases}$

For sufficiently large  $m$ , the distribution of  $y_m$  is approximately that of  $Y$ .

The distribution  $Q|_{\{Y=y\}}$  is called the *proposal* distribution, based on its role in steps 1 and 2. The two cases in step 3 are usually referred to as *acceptance* (when  $y_{m+1} = \tilde{y}$ ) and *rejection* (when  $y_{m+1} = y_m$ ).

The **Metropolis** algorithm is the special case

$$f_{Q|_{\{Y=y\}}}(\tilde{y}) = f_{Y|_{\{Q=\tilde{y}\}}}(y), \quad (\dagger)$$

in which case step 2 simplifies to  $\alpha = \min \left\{ 1, \frac{f_Y(\tilde{y})}{f_Y(y_m)} \right\}$ .

The **random walk Metropolis** algorithm is the choice  $Q = Y + Z$ , where  $Z$  is independent of  $Y$  and  $Q$  and satisfies  $f_Z(z) = f_Z(-z)$  for all  $z \in R_Z$ . In this case

$$Q|_{\{Y=y\}} \stackrel{d}{=} y + Z \quad \text{and} \quad Y|_{\{Q=\tilde{y}\}} \stackrel{d}{=} \tilde{y} + Z,$$

which implies  $(\dagger)$ . A common choice is  $Z \sim N(0, \sigma^2)$ .

The **random walk MCMC algorithm** is obtained by applying the random walk Metropolis algorithm to find the posterior distribution of a Bayesian model. The algorithm is as follows. We start with a (discrete or continuous) Bayesian model  $(X, \Theta)$ , where the parameter space is  $\Pi = \mathbb{R}^d$ . We want to obtain samples of  $\Theta|_{\{X=x\}}$  and we know the p.d.f.  $f_{\Theta|_{\{X=x\}}}$ .

Choose an initial point  $y_0 \in \Pi$ . Choose a continuous distribution for  $Z$  satisfying  $f_Z(z) = f_Z(-z)$  for all  $z \in \mathbb{R}$ . A common choice is  $Z \sim N(0, \sigma^2)$ .

Then, given  $y_m$ , we define  $y_{m+1}$  as follows.

1. Sample  $z$  from  $Z$  and set  $\tilde{y} = y_m + z$ .
2. Calculate  $\alpha = \min \left( 1, \frac{f_{\Theta|_{\{X=x\}}}(\tilde{y})}{f_{\Theta|_{\{X=x\}}}(y_m)} \right)$ .
3. Then, set  $y_{m+1} = \begin{cases} \tilde{y} & \text{with probability } \alpha, \\ y_m & \text{with probability } 1 - \alpha. \end{cases}$

The **Gibbs sampler** for  $\theta = (\theta_1, \dots, \theta_d)$  is as follows. We first choose an initial point  $y_0 = (\theta_1^{(0)}, \dots, \theta_d^{(0)}) \in \Pi$ . Then, for each  $i = 1, \dots, d$ , sample  $\tilde{y}$  from  $\Theta_{-i}|_{\{X=x\}}$  and set

$$y_{m+1} = (\theta_1^{(m)}, \dots, \theta_{i-1}^{(m)}, \tilde{y}, \theta_{i+1}^{(m)}, \dots, \theta_d^{(m)}).$$

Note that we increment the value of  $m$  each time that we increment  $i$ . When reach  $i = d$ , return to  $i = 1$  and repeat. For sufficiently large  $m$ , the distribution of  $y_m$  is approximately that of  $\Theta|_{\{X=x\}}$ .

The distributions of  $\Theta_i|_{\{\Theta_{-i}=\theta_{-i}, X=x\}}$ , for  $i = 1, \dots, d$ , are known as the **full conditional distributions** of  $\Theta$ . They satisfy

$$f_{\Theta_i|_{\{\Theta_{-i}=\theta_{-i}, X=x\}}}(\theta_i) \propto f_{\Theta|_{\{X=x\}}}(\theta)$$

Here  $\propto$  treats  $\theta_{-i}$  and  $x$  as constants, and the only variable is  $\theta_i$ .