MASx52: Assignment 5

Solutions and discussion are written in blue. Some common pitfalls are indicated in teal. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

Marks are given for [A]*ccuracy,* [J]*ustification, and* [M]*ethod.*

1. Consider the SDE
$$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$

   (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation
   $$f'(t) = t + f(t)$$
   Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

   (b) Let $Y_t = X_t^2$. Show that
   $$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

   (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation
   $$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

Solution.

(a) Writing in integral form we have
$$X_t = X_0 + \int_0^t (u + X_u)\,du + \int_0^t 2u\,dB_u.$$

[1A] Taking expectation, and recalling that Ito integrals are zero mean martingales [J1],
$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t (u + X_u)\,du\right] + \mathbb{E}\left[\int_0^t 2u\,dB_u\right] \\
&= \mathbb{E}[X_0] + \int_0^t \mathbb{E}[u + X_u]\,du + 0 \\
&= \mathbb{E}[X_0] + \int_0^t u + \mathbb{E}[X_u]\,du \\
f(t) &= f(0) + \int_0^t u + f(u)\,du.
\end{aligned}$$

[1A] Differentiating, by the fundamental theorem of calculus, [1M]
$$f'(t) = t + f(t).$$

If we set $f(t) = Ce^t - t - 1$ then $f'(t) = Ce^t - 1$ [1A], so clearly this is a solution.

(b) Using Ito's formula [1M] we have

$$
\begin{aligned}
dY_t &= \left(0 + (t + X_t)(2X_t) + \frac{1}{2}(2t)^2(2)\right) dt + (2t)(2X_t)\, dB_t \\
&= 2\left(2t^2 + tX_t + X_t^2\right) dt + 4tX_t\, dB_t
\end{aligned}
$$

[2A]

(c) Writing in integral form [1M] we have

$$
Y_t = Y_0 + 2\int_0^t 2u^2 + uX_u + X_u^2\, du + \int_0^t 4uX_u\, dB_u
$$

Taking expectation, and recalling that Ito integrals are zero mean martingales [1J],

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}[Y_0] + 2\mathbb{E}\left[\int_0^t 2u^2 + uX_u + X_u^2\, du\right] + \mathbb{E}\left[\int_0^t 4uX_u\, dB_u\right] \\
&= \mathbb{E}[Y_0] + \int_0^t 2\mathbb{E}\left[2u^2 + uX_u + X_u^2\right] du + 0 \\
&= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + u\mathbb{E}\left[X_u\right] + \mathbb{E}\left[X_u^2\right] du \\
&= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + uf(u) + v(u)\, du
\end{aligned}
$$

[1A] Differentiating, by the fundamental theorem of calculus, [1M]

$$
v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).
$$

2. Let $T > 0$. Use the Feynman-Kac formula to find an explicit solution $F(x,t)$ to the partial differential equation

$$
\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2}(x,t) = 0
$$

subject to the boundary condition $F(T,x) = x - \frac{T}{2}$.

Hint: It may help to recall that $\int_0^t B_u\, dB_u = \frac{B_t^2}{2} - \frac{t}{2}$.

Solution. From the Feynman-Kac formula, with $\alpha(t,x) = \frac{1}{2}$ and $\beta(t,x) = x$ we have that

$$
F(t,x) = \mathbb{E}_{t,x}[X_T - \tfrac{T}{2}]
$$

where $dX_t = \frac{1}{2}\, dt + B_t\, dB_t$. [1A] Thus, in integral form, [1M]

$$
\begin{aligned}
X_T &= X_t + \int_t^T \tfrac{1}{2}\, ds + \int_t^T X_s\, dB_s \\
&= X_t + \frac{T-t}{2} + \int_t^T X_s\, dB_s
\end{aligned}
$$

which gives

$$
\begin{aligned}
F(t,x) &= \mathbb{E}_{t,x}\left[X_t + \frac{T-t}{2} + \int_t^T X_s\, dB_s - \frac{T}{2}\right] \\
&= \mathbb{E}\left[x - \frac{t}{2} + \int_t^T X_s\, dB_s\right] \\
&= x - \frac{t}{2}
\end{aligned}
$$

[2A] Here we use that Ito integrals are zero mean martingales. [1J]

3. (a) Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and S_t be an Ito process satisfying $dS_t = \alpha S_t\, dt + \sigma S_t\, dB_t$. Let $Y_t = S_t^3$. Show that Y_t satisfies the SDE

$$dY_t = \left(3\alpha + 3\sigma^2\right) Y_t\, dt + 3\sigma Y_t\, dB_t$$

Deduce that Y_t is a geometric Brownian motion, and write down its drift and volatility.

(b) Within the Black-Scholes model, show that the price $F(t, S_t)$ at time $t \in [0, T]$ of the contingent claim $\Phi(S_T) = S_T^3$ is given by

$$F(t, S_t) = S_t^3 e^{2r(T-t)+3\sigma^2(T-t)}.$$

(c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = S_T^3$.

i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

Solution.

(a) By Ito's formula, [1A]

$$\begin{aligned} dY_t &= \left((0) + \alpha S_t(3S_t^2) + \frac{1}{2}\sigma^2 S_t^2(6S_t)\right) dt + \sigma S_t(3S_t^2) dB_t \\ &= \left(3\alpha + 3\sigma^2\right) Y_t\, dt + 3\sigma Y_t\, dB_t. \end{aligned}$$

[2A] So, Y_t is a geometric Brownian motion with drift $3\alpha + 3\sigma^2$ and volatility 3σ. [1A]

(b) Using the explicit formula for geometric Brownian motion (see the formula sheet) with drift $3\alpha + 3\sigma^2$ and volatility 3σ, we have that

$$\begin{aligned} Y_T &= Y_t \exp\left(\left(3\alpha + 3\sigma^2 - \tfrac{9}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right) \\ &= Y_t \exp\left(\left(3\alpha - \tfrac{3}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right). \end{aligned}$$

[1A] Note that in the risk neutral world $\mathbb{Q}$ we have $\alpha = r$. [1J] Therefore, using the risk neutral valuation formula (see the question, or the formula sheet), the arbitrage free price of the contingent claim $Y_T = \Phi(S_T) = S_T^3$ at time t is

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[Y_T \mid \mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_t^3 \exp\left(\left(3\alpha - \tfrac{3}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right) \mid \mathcal{F}_t\right] \\ &= e^{-r(T-t)} S_t^3 e^{(3r - \frac{3}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T - B_t)} \mid \mathcal{F}_t\right] \\ &= e^{-r(T-t)} S_t^3 e^{(3r - \frac{3}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T - B_t)}\right] \\ &= e^{-r(T-t)} S_t^3 e^{(3r - \frac{3}{2}\sigma^2)(T-t)} e^{\frac{9}{2}\sigma^2(T-t)} \\ &= S_t^3 e^{2r(T-t)+3\sigma^2(T-t)}. \end{aligned}$$

[2A] Here, we use that S_t is $\mathcal{F}_t$ measurable. [1J] We then use the properties of Brownian motion to tell us that $3\sigma(B_T - B_t)$ is independent of $\mathcal{F}_t$ [1J] with distribution $N(0, (3\sigma)^2(T-t))$, followed by the formula sheet to explicitly evaluate $\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T - B_t)}\right]$. [1J]

(c) i. The value of our portfolio at time t is given by $F(t, S_t)$, where F is as in part (b). If we add an amount α of stock into our portfolio then its new value will be $V(t, S_t) = F(t, S_t) + \alpha S_t$. [1M] To achieve delta neutrality, we want to choose α such that

$$0 = \frac{\partial V}{\partial s}(0, S_0) = 3S_0^2 e^{2rT+3\sigma^2 T} + \alpha.$$

[1J] Hence $\alpha = -3S_0^2 e^{2rT+3\sigma^2 T}$. [1A]

ii. Our new portfolio has value $V(t, S_t) = F(t, S_t) - 3S_0^2 e^{2rT+3\sigma^2 T} S_t$, and hence

$$\begin{aligned}\frac{\partial V}{\partial s}(t, S_t) &= 3S_t^2 e^{2r(T-t)+3\sigma^2(T-t)} - 3S_0^2 e^{2rT+3\sigma^2 T} S_t \\ &= 3S_t e^{2rT+3\sigma^2 T}\left(e^{-2rt-3\sigma^2 t} - 3S_0 S_t\right).\end{aligned}$$

[2A] Therefore, $\frac{\partial V}{\partial s}$ is zero only when either $S_t = 0$ (which does occur because S_t is a geometric Brownian motion, which is never zero), or when the term in brackets is zero (which, after $t = 0$, has probability zero, because S_t has a continuous distribution). [1J] Hence, our new portfolio is not delta neutral at any time after time 0. [1J]

Total marks: 35