MASx52: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme, with a total of *** marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

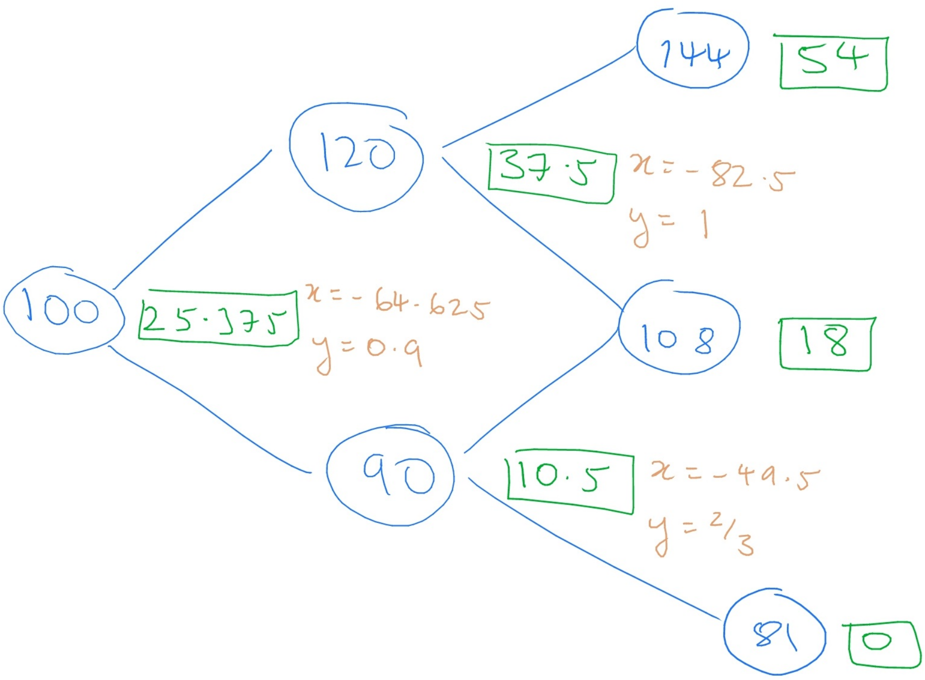
1. Consider the binomial model with $r = \frac{1}{11}$, $d = 0.9$, $u = 1.2$, $s = 100$ and time steps $t = 0, 1, 2$.

   (a) Draw a recombining tree of the stock price process, for time $t = 0, 1, 2$.

   (b) Find the value, at time $t = 0$, of a European call option that gives its holder the option to purchase one unit of stock at time $t = 2$ for a strike price $K = 90$. Write down the hedging strategy that replicates the value of this contract, at all nodes of your tree.

   You may annotate your tree from (a) to answer (b).

   Solution. As in the lecture notes, we write the value of a unit of stock (in blue) inside the nodes of the tree, to answer (a), and write the value of the contingent claim at the various nodes, in square boxes (in green), next to the nodes themselves; the answer to the first part of (b) appears at the root node. For the second part of (b), the replicating portfolios $h = (x, y)$ that would be held at each node are written (in orange) as $x = \ldots, y = \ldots$.



   [2, for (a)], [8, for (b)].

2. Let $S_n = \sum_{i=1}^{n} X_i$, be the simple symmetric random walk, in which $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common distribution $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}$.

   Let $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$, and note that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a filtration. Which of the following stochastic processes are previsible? Which are adapted? (Justification is not required.)

(a) $A_n = \mathbb{1}\{S_n \geq 0\}$

(b) $B_n = \mathbb{E}[S_n \,|\, \mathcal{F}_{n-1}]$

(c) $C_n = \min\{S_1, S_2, \ldots, S_{n-1}\}$

(d) $D_n = \max\{S_1, S_2, \ldots, S_{n+1}\}$

Solution.

(a) A_n depends only on S_n, and S_n is adapted but not previsible, so (A_n) is adapted [1] but not previsible. [1]

(b) B_n is $\mathcal{F}_{n-1}$ measurable (by definition of conditional expectation) and hence (B_n) is both adapted [1] and previsible. [1]

(c) C_n depends only on $S_1, S_2, \ldots, S_{n-1}$, so $C_n \in m\mathcal{F}_{n-1}$ and hence (C_n) is both adapted [1] and previsible. [1]

(d) D_n depends on $S_1, S_2, \ldots, S_{n+1}$, so D_n is not $\mathcal{F}_n$ measurable. Hence (D_n) is not adapted [1] or previsible. [1]

3. Let Z be a random variable taking values in $[1, \infty)$ and for $n \in \mathbb{N}$ define

$$X_n = \begin{cases} Z & \text{if } Z \in [n, n+1) \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that, if $Z \in L^1$, then $\mathbb{E}[X_n] \to 0$ as $n \to \infty$.

(b) Let Z be the continuous random variable with probability density function

$$f(x) = \begin{cases} x^{-2} & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that Z is not in L^1, but that $\mathbb{E}[X_n] \to 0$.

(c) Comment on what (a) and (b) tell us about the dominated convergence theorem.

Solution.

(a) We look to use the dominated convergence theorem. For any $\omega \in \Omega$ we have $Z(\omega) < \infty$, hence for all $n \in \mathbb{N}$ such that $n > Z(\omega)$ we have $X_n(\omega) = 0$.[1] Therefore, as $n \to \infty$, $X_n(\omega) \to 0$, which means that $X_n \to 0$ almost surely. [1]

We have $|X_n| \leq Z$ and $Z \in L^1$, so we can use Z are the dominating random variable. [1] Hence, by the dominated convergence theorem, $\mathbb{E}[X_n] \to \mathbb{E}[0] = 0$. [1]

(b) We have

$$\mathbb{E}[Z] = \int_1^\infty x f(x)\, dx = \int_1^\infty x^{-1}\, dx = [\log x]_1^\infty = \infty.$$

[1], which means $Z \notin L^1$ [1] and also that

$$\mathbb{E}[X_n] = \int_n^{n+1} x^{-1}\, dx = [\log x]_n^{n+1} = \log(n+1) - \log n = \log\left(\frac{n+1}{n}\right).$$

[1] As $n \to \infty$, we have $\frac{n+1}{n} = 1 + \frac{1}{n} \to 1$, hence (using that log is a continuous function) we have $\log(\frac{n+1}{n}) \to \log 1 = 0$. Thus $\mathbb{E}[X_n] \to 0$. [1]

(c) In part (a) we see a case where $\mathbb{E}[X_n] \to 0$ and the dominated convergence theorem can be used to prove it. [1] Suppose that we want to use the DCT in (b). We still have $X_n \to 0$ almost surely, but any dominating random variable Y would have to satisfy $Y \geq |X_n|$ for all n, meaning that also $Y \geq Z$, which means that $\mathbb{E}[Y] \geq \mathbb{E}[Z] = \infty$; thus there is no dominating random variable $Y \in L^1$. Therefore, we can't use the DCT here, [1] but we have shown in (b) that the conclusion of the DCT does hold: we have that $\mathbb{E}[X_n]$ does tend to zero. [1]

We conclude that the conditions of the DCT are *sufficient* but not *necessary* for its conclusion to hold. [1]