# MAS350: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme, with a total of 20 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Determine if the following functions are integrable.

   (a) $f : (0, \infty) \to \mathbb{R}$ by $f(x) = 1/x^2$.

   (b) $g : (0, 1) \to \mathbb{R}$ by $g(x) = \log x$

   *Solution.*

   (a) Note that $x^{-2} > 0$ for $x \in (0, \infty)$. By Riemann integration, we have

   $$\int_{1/n}^{n} x^{-2}\, dx = \left[-x^{-1}\right]_{1/n}^{n} = -\frac{1}{n} + n.$$

   [1] Note that $f_n(x) = \mathbb{1}_{\{x \in (1/n, n)\}} x^{-2}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $f(x) = x^{-2}$ for $x \in (0, \infty)$. [1] Hence, by the monotone convergence theorem we have

   $$\int_0^\infty x^{-2}\, dx = \lim_{n\to\infty} \left(-\frac{1}{n} + n\right) = +\infty.$$

   Thus $x^{-2}$ is not integrable on $(0, \infty)$. [1]

   (b) By Riemann integration, we have

   $$\int_{1/n}^{1} \log x\, dx = [x \log x - x]_{1/n}^{1} = (-1) - \left(\frac{1}{n}\log\frac{1}{n} - \frac{1}{n}\right) = \frac{1 + \log n}{n} - 1.$$

   Noting that $\log x \in (-\infty, 0)$ for $x \in (0, 1)$, multiplying the above by $-1$ gives

   $$\int_{1/n}^{1} |\log x|\, dx = 1 - \frac{1 + \log n}{n}.$$

   [1] We have that $g_n(x) = |\log x| \mathbb{1}_{x \in (1/n, 1)}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $g(x) = |\log x|$ for $x \in (0, 1)$. [1] Hence, by the monotone convergence theorem,

   $$\int_0^1 |\log x|\, dx = \lim_{n\to\infty} \left(1 - \frac{1 + \log n}{n}\right) = 1.$$

   Thus $\log x$ is integrable on $(0, 1)$. [1]

2. In this question we work with Lebesgue measure $\lambda$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Prove that

   $$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x)|\, \mathbb{1}_{\{|f(x)| \geq n\}}\, dx = 0.$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be integrable. It is a fact that: *for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\textit{if } E \subseteq \mathbb{R} \textit{ satisfies } \lambda(E) \le \delta \textit{ then } \int_E |g|\,dx \le \epsilon.$$

Consider the following twelve statements.

**A.** Let $\epsilon > 0$.

**B.** Note that $|g(x)| \le M$, for some $M \in (0, \infty)$.

**C.** Note that $|g(x)| = |g(x)|\mathbb{1}_{\{|g(x)|<M\}} + |g(x)|\mathbb{1}_{\{|g(x)|\ge M\}}$.

**D.** Note that $|g(x)| = |g(x)|\mathbb{1}_{\{|g(x)|<1\}} + |g(x)|\mathbb{1}_{\{|g(x)|\ge 1\}}$.

**E.** By part (a), choose $M > 0$ such that $\int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|\ge M\}}\,dx \le \frac{\epsilon}{2}$.

**F.** By part (a), choose $M > 0$ such that $\int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|\ge 1\}}\,dx \le M$.

**G.** Take $\delta = \frac{\epsilon}{2M}$. Let $E \subseteq \mathbb{R}$ be such that $\lambda(E) \le \delta$.

**H.** Let $E \subseteq \mathbb{R}$ be such that $\lambda(E) \le \epsilon$. Take $\delta = \frac{\epsilon}{2M}$.

**I.** Note that $\int_E |g(x)|\mathbb{1}_{\{|g(x)|<M\}}\,dx \le \int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|<M\}}\,dx$.

**J.** Note that $\int_E |g(x)|\mathbb{1}_{\{|g(x)|\ge M\}}\,dx \le \int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|\ge M\}}\,dx$.

**K.** By monotonicity of the integral, $\int_E |g(x)|\mathbb{1}_{\{|g(x)|<M\}}\,dx \le \int_E M\,dx = M\lambda(E)$.

**L.** By monotonicity of the integral, $\int_E |g(x)|\mathbb{1}_{\{|g(x)|<M\}}\,dx \le \int_E |g(x)|\,dx \le M$.

**M.** Hence $\int_E |g(x)|\,dx \le \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon$

Seven of these statements, when arranged into the correct order, prove the fact stated in italics. The other five statements are not required.

Which seven statements are required?

*Solution.*

(a) Define $f_n(x) = |f(x)|\mathbb{1}_{|f(x)|\ge n}$.
Note that, since $f(x) \in \mathbb{R}$ we have $f_n(x) \to 0$ for all $x \in \mathbb{R}$ (because for large enough $n$ we have $|f(x)| < n$). [1]
Also, $|f_n(x)| \le |f(x)|$ for all $n \in \mathbb{N}$, which provides a dominating function because $f$ is integrable. [1] Hence by the dominated convergence theorem,

$$\int_{\mathbb{R}} f_n(x)\,dx \to 0$$

as required. [1]

(b) Statements A, C, E, G, J, K, M are required.
[Score $\max(n-3, 0)$ where $n$ is the number of correct statements included, plus one mark for no other statements included. (If more than seven statements given in answer, consider only the first seven given.)]

3. (a) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0, 1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.

*Solution.*

(a) We have $\mathbb{P}[U_n \leq a] = a$. For any (deterministic) sequence $(x_n)$ the events $\{U_n < x_n\}$ are independent, because the $U_n$ are independent. [1] Noting that $\sum 1/n = \infty$ and $\sum 1/n^2 < \infty$, we have $\sum_n \mathbb{P}[U_n < 1/n] = \infty$ and $\sum_n \mathbb{P}[U_n < 1/n^2] < \infty$. [1] By the second Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and by the first Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$. [1]

(b) Let $E_n = \{X_n + i = i \text{ for } i = 1, 2, 3, 4, 5, 6\}$. We have $\mathbb{P}[E_n] = (1/6)^6 > 0$. Note that $E_n$ and $E_{n+6}$ are independent (but $E_n$ and $E_{n+1}$ are not!). [1] We have $\sum_{n=1}^{\infty} \mathbb{P}[E_{6n}] = \sum_{n=1}^{\infty} (1/6)^n = \infty$, [1] hence by the second Borel-Cantelli lemma we have $\mathbb{P}[E_{6n} \text{ i.o.}] = 1$. [1] Noting that $\{E_{6n} \text{ i.o.}\} \subseteq \{E_n \text{ i.o.}\}$, we have $\mathbb{P}[E_n \text{ i.o.}] = 1$.