

MAS350: Assignment 1

Solutions and discussion are written in blue. A sample mark scheme, with a total of 20 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Recall that the Borel σ -field $\mathcal{B}(\mathbb{R})$ is the smallest σ -field on \mathbb{R} containing all open intervals (a, b) with $-\infty < a < b < \infty$. Define

$$A = \bigcup_{n=1}^N [a_n, b_n]$$

where $a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 < \dots$ are real numbers.

- (a) Prove, starting from the definition given above, that $A \in \mathcal{B}(\mathbb{R})$.
- (b) Write down a formula for the Lebesgue measure of A , in terms of the a_i and b_i . Is your formula valid if $N = \infty$?
- (c) Consider the following claims.
 - (i) The Borel σ -field is an infinite set.
 - (ii) The Borel σ -field contains an infinite number of infinite sets.
 - (iii) All countable sets are Borel sets with zero Lebesgue measure.
 - (iv) All Borel sets with positive Lebesgue measure contain at least one open interval.
 - (v) The Cantor set is a Borel set.
 - (vi) The Cantor set has Lebesgue measure zero.

In each case (i)-(vi), state whether you believe the claim to be true or false. For claims that you believe are true, give a proof. For claims that you believe are false, give a counterexample. Use parts (a) and (b) to support your arguments.

Solution.

- (a) For $b \in \mathbb{R}$, since $(b, n) \in \mathcal{B}(\mathbb{R})$ for all $n > b$, also $\cup_n (b, n) = (b, \infty) \in \mathcal{B}(\mathbb{R})$. [1] Similarly $(-\infty, a) = \cup_n (-n, a) \in \mathcal{B}(\mathbb{R})$ for all a .
Hence, $[a, b] = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty)) \in \mathcal{B}(\mathbb{R})$. [1]
Hence also $A = \cup_{i=1}^N [a_i, b_i] \in \mathcal{B}(\mathbb{R})$. [1]

Pitfall: Note that here we are using a particular definition of the Borel sets, which doesn't immediately tell us that half-open intervals such as (a, ∞) are Borel. We can deduce it easily, however. There are many different equivalent definitions.

- (b) We have

$$\lambda(A) = \sum_{n=1}^N (b_n - a_n).$$

[1] By countable additivity of disjoint sets (from the definition of a measure) [1] this formula is valid when $N = \infty$. [1]

- (c) (i) True. For example, $\mathcal{B}(\mathbb{R})$ contains each of the sets $(x, x + 1)$, for $x \in \mathbb{R}$, and there are infinitely many of these. [1]
- (ii) True. We can use the same example as in (i), because each of the sets $(x, x + 1)$ is infinite. [1] *Pitfall:* Make sure you keep track of the difference between a set and a set of sets.
- (iii) True. [1] If a set A is countable, then we may write it in the form $A = \bigcup_{n=1}^{\infty} [a_n, a_n]$. By part (a) this means A is a countable union of Borel sets, and hence is itself Borel. Our formula from part (b) shows that A has Lebesgue measure zero. [1]
- (iv) False. [1] Recall that \mathbb{Q} is countable, and hence also a Borel set by the previous part. Hence the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are a Borel set. Since $\lambda(\mathbb{Q}) = 0$ we have $\lambda(\mathbb{R} \setminus \mathbb{Q}) = \infty$, but the irrational numbers do not contain any open intervals. [1]
- (v) True. Recall the iterative ‘middle third’ construction of the Cantor set as $C = \bigcap_n C_n$ (see lecture notes), where $C_1 = [0, 1]$ and C_{n+1} is constructed from C_n by removing the middle third of each closed interval. [1] Thus C_n is the disjoint union of 2^n closed intervals, and we can write it in the form $\bigcup_{i=1}^{2^n} [a_i, b_i]$. Thus C_n is Borel by part (a), and since σ -fields are closed under countable intersections, we have $C \in \mathcal{B}(\mathbb{R})$ too. [1]
- (vi) True. In the iterative ‘middle third’ construction of the Cantor set as $C = \bigcap_n C_n$, the n^{th} stage C_n is a union of 2^n disjoint closed intervals each with length 3^{-n} . Using part (b), the Lebesgue measure of C_n is therefore $(\frac{2}{3})^n$. [1] Since the first stage $C_1 = [0, 1]$ has finite measure, in fact $\lambda(C_1) = 1$, this means $\lambda(C) = \lim_n \lambda(C_n) = \lim_n (\frac{2}{3})^n = 0$. [1]

2. Let λ denote Lebesgue measure and let $\mathcal{B}(\mathbb{R})$ denote the Borel σ -field on \mathbb{R} . This question concerns examples of decreasing sequences of Borel sets (B_n) and measures m on $\mathcal{B}(\mathbb{R})$ such that

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) \neq \lim_{N \rightarrow \infty} m\left(\bigcap_{n=1}^N B_n\right).$$

- (a) Taking $m = \lambda$, show that $B_n = (-\infty, -n]$ is an example of this type.
- (b) Find a second example, with the additional property that $\bigcap_{n=1}^{\infty} B_n$ is non-empty.
- (c) Find a third example, with the additional property that B_1 is countable.

Solution.

- (a) We have $\lambda(B_n) = \sum_{j=n}^{\infty} \lambda((-j-1, -j]) = \infty$ and thus $\lim_n \lambda(B_n) = \infty$, [1] but $\bigcap_n (-\infty, -n] = \emptyset$ which has measure zero. [1]
- (b) Take e.g. $B_n = (-\infty, -n] \cup [0, 1]$. Then $\lambda(B_n) = \infty$ as before, but now $\bigcap_n B_n = [0, 1]$ which is non-empty with Lebesgue measure 1. [1]
- (c) Take m to be counting measure on \mathbb{N} (the σ -field can be $\mathcal{P}(\mathbb{N})$ here) and let $B_n = \{n, n+1, \dots, \infty\}$. and then $m(B_n) = \infty$ but $m(\bigcap_n B_n) = m(\emptyset) = 0$. [1]

Pitfall: Remember the conditions of the theorem! In general, $m(\bigcap_n B_n) = \lim_n m(B_n)$ for decreasing B_n *only* if $m(B_1)$ is finite. Once you remember this, you know to start by trying (any) example where $m(B_1)$ is infinite, and from there you don’t have far to go.