MAS223 Statistical Modelling and Inference

Exercises

Revision of Level 1 material

1. Let X be a random variable with range space $\{1, 2, 3\}$ with $P(X = 1) = P(X = 2) = 0.4$.
   (a) What is $P(X = 3)$?
   (b) Find the mean of X, $E(X)$.
   (c) Find the variance of X.
2. Let Y be a random variable with probability density function (p.d.f.) $f(y)$ given by
   $$f(y) = \begin{cases} y/2 & 0 \leq y < 2 \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Find the probability that Y is between $1/2$ and 1.
   (b) Find the distribution function of Y, $F(y)$.
   (c) Find the mean and variance of Y.
3. A standard fair dice is rolled 8 times. Let X be the number of sixes rolled.
   (a) Which standard distribution (and which parameters) would you expect X to have?
   (b) Under your assumption, what is $P(X = 2)$?
4. Let $f(\theta) = e^{-\theta^2+4\theta}$. Calculate the first and second derivatives of $f(\theta)$ with respect to θ, and hence find the value of θ which maximises $f(\theta)$.

5. Let $f(x, y) = e^{-(x+y)}$. Find
   (a) $\int_{x=0}^{1} \int_{y=0}^{1} f(x, y)\, dy\, dx$;
   (b) $\int_{x=0}^{\infty} f(x, y)\, dx$

6. Let
$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \text{ and } \mathbf{\Sigma} = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$
   Find $\mathbf{A\Sigma A}^T$.

7. X_1 and X_2 are independent random variables with $N(\mu, 1)$ and $N(\mu, 4)$ distributions respectively, where μ is an unknown parameter. Let T_1, T_2 and T_3 be defined by
$$\begin{aligned} T_1 &= \frac{X_1 + X_2}{2} \\ T_2 &= 2X_1 - X_2 \\ T_3 &= \frac{4X_1 + X_2}{5}. \end{aligned}$$
   Find the mean and variance of T_1, T_2 and T_3. Which would you prefer to use as an estimator of μ, and why?

Univariate distribution theory

8. The discrete random variable X has the probability function
$$p(x) = \begin{cases} \frac{1}{x(x+1)} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Find, using partial fractions, the distribution function F of X, and sketch its graph.
   (b) Evaluate $P(10 \le X \le 20)$.

(c) Find the smallest x such that $P(X \geq x) \leq 0.01$.

(d) What happens if you attempt to calculate $E(X)$?

9. Let
$$F(x) = \frac{e^x}{1+e^x} \qquad \text{for all real } x.$$

(a) Show that F satisfies properties (1) and (2), from the typed lecture notes, of a distribution function. (As it is continuous, it also satisfies (3).)

(b) Show that the corresponding probability density function f is symmetrical about zero, in the sense that $f(-x) = f(x)$ for all x.

(c) If X is a random variable with this distribution, evaluate $P(|X| > 2)$.

10. Sketch graphs of each of the following two functions, and explain why each of them is not a distribution function.

(a) $F(x) = \begin{cases} 0 & \text{for } x \leq 0; \\ x & \text{for } x > 0. \end{cases}$

(b) $F(x) = \begin{cases} 0 & \text{for } x < 0; \\ x + \frac{1}{4}\sin 2\pi x & \text{for } 0 \leq x < 1; \\ 1 & \text{for } x \geq 1. \end{cases}$

11. The probability density function $f(x)$ is given by
$$f(x) = \begin{cases} 1+x & \text{for } -1 \leq x < 0; \\ 1-x & \text{for } 0 \leq x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find the corresponding distribution function $F(x)$ for all real x.

12. Find the coefficients of skewness for the random variables X and Y described in questions 1 and 2.

13. X is a random variable with hypergeometric distribution, parameters $N = 10, N^* = 7, n = 5$. What are the values that X can take? Find the probabilities that X takes each of these values. Find the mean and variance of X.

14. Let F be the d.f. of a life length and suppose it possesses a p.d.f. f. The *hazard function* at age t is defined as

$$h(t) = \frac{f(t)}{1 - F(t)} \quad \text{for } t \geq 0.$$

[This may be interpreted as an instantaneous failure rate at age t.]

   (a) Given that the exponential distribution has p.d.f. $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$ and parameter $\lambda > 0$, show that the exponential distribution has constant hazard function.

   (b) Show that the distribution given by

$$F(t) = \frac{t}{1+t} \quad \text{for } t \geq 0$$

   has hazard function which decreases with age.

   (c) Show that the distribution given by

$$F(t) = 1 - e^{-t^2} \quad \text{for } t \geq 0$$

   has hazard function which increases with age.

15. If X is a random variable with $Ga(\alpha, \beta)$ distribution, and k is a positive integer, show that

$$E(X^k) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}.$$

Hence confirm that the mean and variance of $Ga(\alpha, \beta)$ are as given on Example 6, and find the coefficient of skewness.

16. Let X be a random variable following the beta distribution $Be(\alpha, \beta)$.
    (a) For any $c > 0$ find the p.d.f. of the random variable $Y = c/X$.
    (b) Calculate the mean of Y, stating any restrictions on the parameter space of α or β.

17. Use `R` to investigate how the shape of the p.d.f. of a Gamma distribution varies with the different parameter values. In particular, fix a value of β, see how the shape changes as you vary α, and see how this relates to the value for the coefficient of skewness that you found in question 15.

    Note that in `R` you can obtain a plot of, for example, the p.d.f. of a $Ga(3, 2)$ random variable between 0 and 10 with
    `curve(dgamma(x,shape=3,scale=2),from=0,to=10)`.

18. Let X be a random variable with the p.d.f. $f(x)$ defined in question 11. Find the p.d.f.s of
    (a) $Y = 5X + 3$;
    (b) $Z = |X|$ (hint: first find $F_Z(z) = P(Z \leq z)$).

19. Let X have a $Be(\alpha, 1)$ distribution, and let $Y = \sqrt[r]{X}$ for some positive integer r. Show that Y also has a Beta distribution, and find the parameters.

20. Let Θ be an angle chosen according to a uniform distribution on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and let $X = \tan\Theta$. Use the theory on transformations of random variables in section 1.6 to show that X has the Cauchy distribution, as defined in section 1.3.1.

Multivariate distribution theory

21. Let S be the square $\{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$, and let U and V have joint probability density function

$$f_{U,V}(u, v) = \begin{cases} \frac{4u+2v}{3} & (u, v) \in S \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(U + V \leq 1)$.

(b) Find the marginal p.d.f. of U.

(c) Find the marginal p.d.f. of V.

In each case check that your answer really is a p.d.f.

22. Let (X, Y) be a random vector with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} ke^{-(x+y)} & x > y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of k.

(b) Find the marginal p.d.f. of Y, and hence describe the distribution of Y as a standard distribution.

23. For the random variables U and V in Exercise 21:

(a) Find the conditional p.d.f. of U given $V = v$.

(b) If the conditional expectation $E(U|V)$ is written in the form $g(V)$, find the function g.

24. Let (X, Y) be a random vector with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{x+y}{2} & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ \frac{y-x}{2} & -1 \leq x \leq 0, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that X and Y are not independent but that they have correlation coefficient zero.

25. In a particular location the amount of sunshine S (in hours) in the growing season is distributed as $S \sim Ga(\alpha, \beta)$ and, conditional on S, the change in weight, C of a certain plant during the growing season is distributed as $C \sim N(kS, \sigma^2)$ (for some constant k). Find the mean and variance of C (in terms of α, β, σ^2 and k).

26. The random variables X and Y have joint p.d.f. given by

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2}(x+y)e^{-(x+y)} & \text{for } x, y \geq 0; \\ 0 & \text{elsewhere.} \end{cases}$$

Find, using a suitable bivariate transformation, the p.d.f. of the random variable $U = X + Y$. Hence evaluate $E[(X+Y)^5]$.

27. Let (X, Y) be a random vector with joint p.d.f. $f(x, y)$ as in question 22.

(a) If $U = X - Y$ and $V = Y/2$, find the joint p.d.f. of (U, V).

(b) Are U and V independent?

(c) Describe the distributions of U and V as standard distributions.

28. Let X and Y be independent random variables with distributions $Ga(\alpha_1, \beta)$ and $Ga(\alpha_2, \beta)$ respectively. Show that the random variables $U = \frac{X}{X+Y}$ and $V = X + Y$ are independent with distributions $Be(\alpha_1, \alpha_2)$ and $Ga(\alpha_1 + \alpha_2, \beta)$ respectively.

29. Question 28 showed that if X and Y are independent random variables with $X \sim Ga(\alpha_1, \beta)$ and $Y \sim Ga(\alpha_2, \beta)$, then $X + Y \sim Ga(\alpha_1 + \alpha_2, \beta)$.

(a) Use induction to show that for $n \geq 2$, if $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim Ga(\alpha_i, \beta)$ then

$$\sum_{i=1}^{n} X_i \sim Ga\left(\sum_{i=1}^{n} \alpha_i, \beta\right).$$

(b) Hence show that for $n \geq 1$, if $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables then

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2.$$

(Recall that Example 9 showed that this was true in the case $n = 1$, and recall that the χ^2 distribution is a special case of the Gamma.)

30. Random variables X, Y and Z have means 3, -4 and 6 respectively and variances 1, 1 and 25 respectively. X and Y are uncorrelated; the correlation coefficient between X and Z is $\frac{1}{5}$ and that between Y and Z is $-\frac{1}{5}$. Let $U = X + Y - Z$ and $W = 2X + Z - 4$.

    (a) Write down the mean vector and covariance matrix of $(X, Y, Z)^T$.

    (b) Find the mean vector and covariance matrix of $(U, W)^T$.

    (c) Evaluate $E(2X + Z - 6)^2$.

31. Let X and Y be independent standard normal random variables.

    (a) Write down the covariance matrix of the random vector $\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$.

    (b) Let R be the rotation matrix

    $$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

    and show that $R\mathbf{X}$ has the same covariance matrix as $\mathbf{X}$.

32. Suppose that the random vector $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ follows the bivariate normal distribution with $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = 1$, $\sigma_{12} = 2$ and $\sigma_2^2 = 5$.

    (a) Calculate the correlation coefficient of X_1 and X_2. Are X_1 and X_2 independent?

(b) Find the mean and the covariance matrices of

$$Y = \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{X}$$

and

$$\mathbf{Z} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{X}.$$

What are the distributions of Y and $\mathbf{Z}$?

33. Let X_1 and X_2 be bivariate normally distributed random variables each with mean 0 and variance 1, and with correlation coefficient ρ.

(a) By integrating out the variable x_2 in the joint p.d.f., verify that the marginal distribution of X_1 is indeed that of a standard univariate normal random variable. (HINT: the idea for the integral is similar to that in Example 5.)

(b) Show by direct derivation of the conditional p.d.f. that the conditional distribution of X_2 given $X_1 = x_1$ is $N(\rho x_1, 1 - \rho^2)$.

34. Let $\mathbf{X} = (X_1, X_2)^T$ have a $N_2(\boldsymbol{\mu}, \Sigma)$ distribution with $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 1 & -\frac{\sigma_1}{\sigma_2} \\ \frac{\sigma_2}{\sigma_1} & 1 \end{pmatrix}.$$

Find the distribution of $\mathbf{Y} = A\mathbf{X}$, and deduce that any bivariate normal random vector can be transformed by a linear transformation into a vector of independent normal random variables.

35. The Maple code

```
f:=1/(2*Pi*(sqrt(sigma1^2*sigma2^2-sigma12^2)))*exp(-(sigma2^2*(x-mu1)^2
-2*sigma12*(x-mu1)*(y-mu2)
+sigma1^2*(y-mu2)^2)/(2*(sigma1^2*sigma2^2-sigma12^2)));
```

defines `f` as the joint p.d.f. of the bivariate normal distribution

$$N_2\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right).$$

You can then set the values of μ_1, μ_2, σ_1, σ_2 and σ_{12} by using a command like

```
fnew:=eval(f,[sigma1=2,sigma2=5,sigma12=8,mu1=3,mu2=2]);
```

and you can then produce a contour plot in Maple using (for example)

```
plots[contourplot](fnew,x=-10..16,y=-15..19,grid=[100,100]);
```

and a 3D plot using

```
plot3d(fnew,x=-10..16,y=-15..19);
```

Use this code to investigate the shape of the bivariate normal distribution. In particular plot a contour plot and a 3D plot of the p.d.f. of the bivariate normal distribution

$$N_2\left(\begin{pmatrix}3\\ 2\end{pmatrix}, \begin{pmatrix}4 & 8\\ 8 & 25\end{pmatrix}\right)$$

and the bivariate normal distribution obtained by applying the transformation described in question 34 to this distribution, and comment on your plots.

(A Maple worksheet with the above Maple commands will be made available from the course website.)

36. The random vector $\mathbf{X} = (X_1, X_2, X_3)^T$ has an $N_3(\boldsymbol{\mu}, \Sigma)$ distribution where $\boldsymbol{\mu} = (-1, 1, 2)^T$ and

$$\Sigma = \begin{pmatrix}144 & -30 & 48\\ -30 & 25 & 10\\ 48 & 10 & 64\end{pmatrix}.$$

(a) Find the correlation coefficients between X_1 and X_2, between X_1 and X_3 and between X_2 and X_3.

(b) Let $Y_1 = X_1 + X_3$ and $Y_2 = X_2 - X_1$. Find the distribution of $\mathbf{Y} = (Y_1, Y_2)^T$ and hence find the correlation coefficient between Y_1 and Y_2.

37. Recall that an *orthogonal matrix* R is one for which $R^{-1} = R^T$, and recall that if an $n \times n$ matrix R is orthogonal, $\mathbf{x}$ is an n-dimensional vector, and $\mathbf{y} = R\mathbf{x}$ then $\sum_{i=1}^n y_i^2 = \mathbf{y} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{x} = \sum_{i=1}^n x_i^2$.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a vector of independent normal random variables with common mean 0 and variance σ^2.

(a) Using the theory of the multivariate normal distribution, show that if R is an orthogonal matrix, then $\mathbf{Y} = R\mathbf{X}$ is also a vector of independent normal random variables with common mean 0 and variance σ^2. (This is a generalisation of question 31.)

(b) Let R be an orthogonal matrix where the elements in the first row are all $\frac{1}{\sqrt{n}}$. (It can be shown using linear algebra that this is possible.) If $\mathbf{Y} = R\mathbf{X}$, show that $Y_1 = \sqrt{n}\bar{X}$, where $\bar{X}$ is the sample mean of $\mathbf{X}$, and that

$$\sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2.$$

(c) Hence, applying the result of question 29 to $\sum_{i=2}^n \left(\frac{Y_i}{\sigma}\right)^2$, deduce that, if S^2 is the sample variance of $\mathbf{X}$,

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right) \sim \chi^2_{n-1},$$

and that it is independent of $\bar{X}$, as claimed in section 2.4.1.

Likelihood

38. An observation from a $Bi(n, \theta)$ distribution gives the value 4.

    (a) Assuming that n is known to be 9 and that θ is unknown, write down the likelihood function of θ (for $0 \le \theta \le 1$) and plot a graph of it. (You may use `R` or Maple to do the plot.)

    (b) Assuming that θ is known to be $\frac{3}{4}$ and that n is unknown, write down a formula for the likelihood function of n (for integers $n \ge 4$) and calculate its values for $n = 4, 5, 6$ and 7.

39. Given the set of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, for some $n > 0$, write down the likelihood function $L(\theta; \mathbf{x})$, in each of the following cases. In each case you should give both the function and the parameter set Θ.

    (a) The data are i.i.d. samples from the exponential distribution $Exp(\lambda)$ with $\lambda = 1/\theta$.

    (b) The data are i.i.d. samples from the binomial $Bi(m, \theta)$ distribution, where m is known.

    (c) The data are i.i.d. samples from the normal $N(\mu, \theta)$, where μ is known.

40. Repeat question 39 for the following:

    (a) The data are i.i.d. samples from the gamma distribution $Ga(\theta, 4)$.

    (b) The data are i.i.d. samples from the inverted gamma distribution $IGa(1, \theta)$. [The $IGa(\alpha, \beta)$ distribution is the distribution of $1/U$ if $U \sim Ga(\alpha, \beta)$.]

    (c) The data are i.i.d. samples from the beta distribution $Be(\theta, \theta)$.

41. For each of the cases of question 39, find the log likelihood, and simplify it as much as you can.

42. A random sample $(x_1, x_2, x_3)^T$ of three observations from a Poisson distribution with parameter λ, where λ is known to be in $\Lambda = \{1, 2, 3\}$, gives the values $x_1 = 4, x_2 = 0, x_3 = 3$. Find the likelihood of each of the possible values of λ, and hence find the maximum likelihood estimate.

43. As in question 38, an observation from a $Bi(n, \theta)$ distribution gives the value 4.

    (a) Assuming that n is known to be 9 and that θ is unknown, find the maximum likelihood estimate of θ.

    (b) Assuming that θ is known to be $\frac{3}{4}$ and that n is unknown (but is known to be an integer satisfying $n \geq 4$), find the maximum likelihood estimate of n.

44. Find the maximum likelihood estimate of θ when the data $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are i.i.d. samples from the binomial $Bi(m, \theta)$ distribution, where m is known, as in question 39(b).

45. Find the maximum likelihood estimate of θ when the data $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are i.i.d. samples from the normal $N(\mu, \theta)$, where μ is known, as in question 39(c). Show that this estimator is unbiased.

46. Find the maximum likelihood estimate when the data $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are i.i.d. samples from the gamma distribution $Ga(3, \theta)$. If $\bar{x} = 3$, calculate the maximum likelihood estimate $\hat{\theta}$ and show that it does not depend on the sample size n.

47. A set of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population with a Pareto distribution with parameters α and β so that

$$f(x_i|\boldsymbol{\theta}) = \frac{\alpha\beta^\alpha}{x_i^{\alpha+1}}, \quad x_i \geq \beta > 0, \quad \alpha > 0,$$

    where $\boldsymbol{\theta} = (\alpha, \beta)^T$.

    Find the maximum likelihood estimate of $\boldsymbol{\theta}$ based on the above sample.

48. A set of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population with a logarithmic distribution with parameters α and β so that

$$f(x_i|\boldsymbol{\theta}) = \frac{\log x_i}{\beta(\log\beta - 1) - \alpha(\log\alpha - 1)}, \quad 1 \leq \alpha \leq x_i \leq \beta,$$

where $\boldsymbol{\theta} = (\alpha, \beta)^T$.

Find the maximum likelihood estimate of $\boldsymbol{\theta}$ based on the above sample.

49. A set of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population with a geometric distribution with parameter θ, so that the probability function of X_i is

$$p(x_i) = P(X_i = x_i) = (1-\theta)^{x_i}\theta, \quad x_i = 0, 1, 2, \ldots, \quad 0 < \theta < 1.$$

Find the maximum likelihood estimate of θ based on the above sample.

50. A set of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population with an inverse Gaussian distribution (known also as the Wald distribution), with p.d.f.

$$f(x_i) = \sqrt{\frac{\theta}{2\pi x_i^3}} \exp\left(-\frac{\theta(x_i - \mu)^2}{2\mu^2 x_i}\right), \quad x_i > 0, \quad \mu, \theta > 0.$$

(a) Assuming that μ is known, find the maximum likelihood estimate of θ based on the above sample.

(b) If both μ and θ are unknown, find the maximum likelihood estimate of (μ, θ) based on the above sample.

51. (a) As in question 38(a) an observation from a $Bi(n, \theta)$ distribution with $n = 9$ gives the value $x = 4$.

i. Find the approximate range of values for which the log likelihood is within 2 of its maximum value. [You may do this either by inspection of a plot, or by using a computer package to solve the inequality numerically.]

ii. A traditional approximate 95% confidence interval here would be of the form $\hat{\theta} \pm 1.96\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$, where $\hat{\theta} = x/n$. Compare your answer to (a) to what this would give.

(b) Repeat (a) for $n = 90$ and $x = 40$.

Revision questions

52. Suppose that the random variable X follows the beta distribution with parameters 2 and 3, i.e. $X \sim Be(2, 3)$ with p.d.f.

$$f_X(x) = \frac{\Gamma(5)}{\Gamma(2)\Gamma(3)} x(1-x)^2, \quad 0 < x < 1,$$

where $\Gamma(\alpha)$ denotes the gamma function with argument α.

(a) Find the p.d.f. of the variable $Y = 4/X$ and state the range of the values y of Y.

(b) Find $E(1/X)$, using only the density of X. Find the mean of Y without making any calculations with the density of Y.

53. Suppose that the random vector $\mathbf{X} = (X_1, X_2, X_3)^T$ follows the trivariate normal distribution $\mathbf{X} \sim N_3(\mu, V)$, where

$$\mu = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad V = \begin{pmatrix} 1 & \lambda_1 & \nu_1 \\ \lambda_2 & k & 1 \\ \nu_2 & 1 & 2 \end{pmatrix},$$

for some constants $\lambda_1, \lambda_2, \nu_1, \nu_2, k$.

(a) If the correlation of X_1 and X_2 is 0.4, the correlation of X_1 and X_3 is zero and the correlation of X_2 and X_3 is 0.9, then find the values of $\lambda_1, \lambda_2, \nu_1, \nu_2, k$.

(b) Give the marginal joint distribution of $(X_2, X_3)^T$.

(c) Find the distribution of the random variable

$$Y = 2X_1 - X_2 + \frac{1}{2}X_3.$$

54. Suppose that two random variables X and Y are linked through a third random variable Z by the relationship $Y = 2X + Z$, where X is independent of Z and $Z \sim N(0, 1)$. Assuming that $E(X) = 1$ and $\operatorname{Var} X = 2$, verify the relationships

$$\begin{aligned} E\{E(Y|X)\} &= E(Y) \\ E\{\operatorname{Var}(Y|X)\} + \operatorname{Var}\{E(Y|X)\} &= \operatorname{Var} Y. \end{aligned}$$

Explain how you have used the fact that X and Z are independent.

55. Suppose that a set of i.i.d samples $\mathbf{x} = (x_1, \ldots, x_n)^T$ is taken from the negative binomial distribution, so that each X_i has probability function

$$p(x_i) = P(X_i = x_i) = \begin{pmatrix} x_i + r - 1 \\ r - 1 \end{pmatrix} \theta^r (1 - \theta)^{x_i}$$

for some "success" probability θ satisfying $0 \leq \theta \leq 1$, where $x_i = 0, 1, 2, \ldots$ and r is the total number of "successes". If r is known, find the maximum likelihood estimate of θ.