

## MASx52: Assignment 2

1. Let  $(X_n)$  be a sequence of i.i.d. random variables, each with a uniform distribution on  $[-1, 1]$ . Define

$$S_n = \sum_{i=1}^n X_i,$$

where  $S_0 = 0$ . Let  $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ .

- (a) Show that  $S_n$  is a martingale, with respect to the filtration  $\mathcal{F}_n$ .  
(b) Find  $\mathbb{E}[S_3^2 | \mathcal{F}_2]$  in terms of  $X_2$  and  $X_1$ , and hence show that

$$\mathbb{E}[S_3^2 | \mathcal{F}_2] = S_2^2 + \frac{1}{3}.$$

- (c) Write down a deterministic function  $f : \mathbb{N} \rightarrow \mathbb{R}$  such that

$$M_n = S_n^2 - f(n)$$

is a martingale (justification is not required – make a guess!).

2. Consider the one-period market with  $r = \frac{1}{10}$ ,  $s = 2$ ,  $d = \frac{1}{2}$  and  $u = 3$ , in our usual notation. A contract specifies that

*The holder of the contract will sell 2 units of stock, and be paid  $K$  units of cash, at time 1.*

- (a) Explain briefly why the contingent claim of this contract is

$$\Phi(S_1) = K - 2S_1.$$

- (b) Find a replicating portfolio  $h$  for this contingent claim.  
(c) Write down the value  $V_0^h$  of  $h$  at time 0.  
(d) Find the numerical values of risk-neutral probabilities

$$q_u = \frac{(1+r) - d}{u - d} \quad \text{and} \quad q_d = \frac{u - (1+r)}{u - d}.$$

Hence, check that  $\frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[\Phi(S_1)]$  and  $V_0^h$  have the same values.

- (e) For which  $K$  does the contract have value zero at time 0?