

MASx52: Assignment 4

Solutions and discussion are written in blue. A sample mark scheme, with a total of 40 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Let B_t be a standard Brownian motion.

- (a) Write down the distribution of B_t , and write down $\mathbb{E}[B_t]$ and $\mathbb{E}[B_t^2]$.
- (b) Let $0 \leq u \leq t$. Show that $\mathbb{E}[(B_t - B_u)^2 \mid \mathcal{F}_u] = t - u$.

Solution.

- (a) $B_t \sim N(0, t)$, [1] and $\mathbb{E}[B_t] = 0$, [1] $\mathbb{E}[B_t^2] = t$. [1]
- (b) We have

$$\begin{aligned}\mathbb{E}[(B_t - B_u)^2 \mid \mathcal{F}_u] &= \mathbb{E}[(B_t - B_u)^2] \\ &= t - u.\end{aligned}$$

[1] In the first line we use that, by the properties of Brownian motion, $B_t - B_u$ is independent of \mathcal{F}_u . [1] Then, we use that $B_t - B_u \sim N(0, t - u)$, which is the same distribution as B_{t-u} [1], followed by the third formula in part (a) with $t - u$ in place of t . [1]

2. Write down the following stochastic differential equations in integral form, over the time interval $[0, t]$.

- (a) $dX_t = t dt + 2B_t dB_t$.
- (b) $dY_t = 3Y_t dt$.

Write down a differential equation satisfied by Y_t , and find its solution with the initial condition $Y_0 = 1$. Is X_t differentiable?

Solution.

- (a) We have

$$X_t = X_0 + \int_0^t u du + \int_0^t 2B_u dB_u.$$

[2]

- (b) We have

$$Y_t = Y_0 + \int_0^t 3Y_u du.$$

[2]

In (b) (using the fundamental theorem of calculus) we have

$$\frac{dY_t}{dt} = 3Y_t$$

[1] with solution $Y_t = Ae^{3t}$. Since $Y_0 = 1$ we have $A = 1$. [1]

We showed in lectures (as an example of Ito's formula) that

$$\int_0^t B_u dB_u = \frac{B_t^2}{2} - \frac{t}{2}$$

so we have

$$\begin{aligned} X_t &= X_0 + \frac{t^2}{2} + 2 \left(\frac{B_t^2}{2} - \frac{t}{2} \right) \\ &= X_0 + \frac{t^2}{2} - t + B_t^2. \end{aligned}$$

[2] This is not differentiable because the Brownian motion term B_t^2 is not differentiable (and all other terms are differentiable). [1]

[More Rigorously: If X_t was differentiable then the positive square root $Y_t = \sqrt{X_t - \frac{t^2}{2} + t} = \sqrt{B_t^2} = |B_t|$ would be differentiable on all open intervals of time for which we have $Y_t \neq 0$. Hence it would be differentiable on the (random) open intervals of time when $B_t > 0$ and $B_t < 0$. But we know that Brownian motion B_t is not differentiable, so we have a contradiction.]

3. Use Ito's formula to calculate the stochastic differential of dZ_t where

(a) $Z_t = tB_t$

(b) $Z_t = B_t^2 - t$

(c) $Z_t = 1 + t^2 X_t$ where $dX_t = \mu dt + \sigma B_t dB_t$ and μ, σ are deterministic constants.

(d) $Z_t = e^{-2t} S_t$ where $dS_t = 2S_t dt + 5S_t dB_t$.

Explain briefly why Z_t is a martingale in cases (b) and (d).

Solution. We have

(a)

$$\begin{aligned} dZ_t &= \left\{ (B_t) + (0)(t) + \frac{1}{2}(1)^2(1) \right\} dt + (t)(1) dB_t \\ &= B_t dt + t dB_t. \end{aligned}$$

[3]

(b)

$$\begin{aligned} dZ_t &= \left\{ (-1) + (0)(2B_t) + \frac{1}{2}(1)^2(2) \right\} dt + (2B_t)(1) dB_t \\ &= 2B_t dB_t. \end{aligned}$$

[3]

(c)

$$\begin{aligned} dZ_t &= \{2tX_t + (\mu)(t^2) + \frac{1}{2}(\sigma B_t)^2(0)\} dt + (\sigma B_t)(t^2) dB_t \\ &= (2tX_t + \mu t^2) dt + \sigma t^2 B_t dB_t. \end{aligned}$$

[4]

(d)

$$\begin{aligned} dZ_t &= \{(-2e^{-2t}S_t) + (2S_t)(e^{-2t}) + \frac{1}{2}(5S_t)^2(0)\} dt + (5S_t)(e^{-2t}) dB_t \\ &= 5e^{-2t}S_t dB_t. \end{aligned}$$

[4]

Cases (b) and (d) are martingales because in these cases dZ_t has only a $(\dots) dB_t$ term, and therefore $Z_t = Z_0 + \int_0^t \dots dB_t$ is a martingale because Ito integrals are martingales. [1]

4. Let S_t be a geometric Brownian motion, with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and (deterministic) initial condition S_0 .

- (a) Find $\mathbb{E}[S_t]$ and deduce that S_t is not a Brownian motion when $\mu \neq 0$.
(b) Is S_t a Brownian motion when $\mu = 0$?

Solution.

- (a) The formula for geometric Brownian motion is

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

So, taking expectations, using the formula for $\mathbb{E}[e^Z]$ where Z is normally distributed, and using that S_0 is deterministic,

$$\begin{aligned} \mathbb{E}[S_t] &= S_0 e^{(\mu - \frac{\sigma^2}{2})t} \mathbb{E}[e^{\sigma B_t}] \\ &= S_0 e^{(\mu - \frac{\sigma^2}{2})t} e^{\frac{\sigma^2 t}{2}} \\ &= S_0 e^{\mu t} \end{aligned}$$

[3] A Brownian motion B_t has $\mathbb{E}[B_t] = \mathbb{E}[B_0]$, but for $\mu \neq 0$ we have shown that $\mathbb{E}[S_t]$ is non-constant, which means that S_t cannot be a Brownian motion. [2]

- (b) It remains to consider the case $\mu = 0$. In this case, $S_t = S_0 e^{\sigma B_t - \frac{\sigma^2}{2}t}$. We recall that, for a Brownian motion, $B_t^2 - t$ is a martingale, [1] and for S_t we have $S_t^2 - t = S_0^2 e^{2\sigma B_t - \sigma^2 t} - t$. This gives us

$$\begin{aligned} \mathbb{E}[S_t^2 - t] &= S_0^2 \mathbb{E}[e^{2\sigma B_t}] e^{-\sigma^2 t} - t \\ &= S_0^2 e^{\frac{4\sigma^2}{2}t} e^{-\sigma^2 t} - t \\ &= S_0^2 e^{\sigma^2 t} - t \end{aligned}$$

[2] which is clearly non-constant. Hence $S_t^2 - t$ is not a martingale, so S_t is not a Brownian motion. [1]

[Note: There are *lots* of other ways to solve this question!]