**MAS350: Assignment 1**

1. Recall that the Borel $\sigma$-field $\mathcal{B}(\mathbb{R})$ is the smallest $\sigma$-field on $\mathbb{R}$ containing all open intervals $(a, b)$ with $-\infty < a < b < \infty$. Define

$$A = \bigcup_{n=1}^{N} [a_n, b_n]$$

where $a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 < \ldots$ are real numbers.

   (a) Prove, starting from the definition given above, that $A \in \mathcal{B}(\mathbb{R})$.

   (b) Write down a formula for the Lebesgue measure of $A$, in terms of the $a_i$ and $b_i$. Is your formula valid if $N = \infty$?

   (c) Consider the following claims.

      (i) The Borel $\sigma$-field is an infinite set.

      (ii) The Borel $\sigma$-field contains an infinite number of infinite sets.

      (iii) All countable sets are Borel sets with zero Lebesgue measure.

      (iv) All Borel sets with positive Lebesgue measure contain at least one open interval.

      (v) The Cantor set is a Borel set.

      (vi) The Cantor set has Lebesgue measure zero.

   In each case (i)-(vi), state whether you believe the claim to be true or false. For claims that you believe are true, give a proof. For claims that you believe are false, give a counterexample. Use parts (a) and (b) to support your arguments.

2. Let $\lambda$ denote Lebesgue measure and let $\mathcal{B}(\mathbb{R})$ denote the Borel $\sigma$-field on $\mathbb{R}$. This question concerns examples of decreasing sequences of Borel sets $(B_n)$ and measures $m$ on $\mathcal{B}(\mathbb{R})$ such that

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) \neq \lim_{n\to\infty} m\left(\bigcap_{n=1}^{N} B_n\right).$$

   (a) Taking $m = \lambda$, show that $B_n = (-\infty, -n]$ is an example of this type.

   (b) Find a second example, with the additional property that $\cap_{n=1}^{\infty} B_n$ is non-empty.

   (c) Find a third example, with the additional property that $B_1$ is countable.

3. Let $S$ be a finite set and $\Sigma$ be a $\sigma$-field on $S$. Consider the set

$$\Pi = \{A \in \Sigma \; : \; \text{if } B \in \Sigma \text{ and } B \subseteq A \text{ then either } B = A \text{ or } B = \emptyset\}. \qquad (\star)$$

   (a) Show that $\Pi$ is a finite set.

   (b) Using (a), let us enumerate the elements of $\Pi$ as $\Pi = \{\Pi_1, \Pi_2, \cdots, \Pi_k\}$, where each $\Pi_i$ is distinct from the others.

      (i) Show that $\Pi_i \cap \Pi_j = \emptyset$ for $i \neq j$. *Hint: Could* $\Pi_i \cap \Pi_j$ *be an element of* $\Pi$*?*

      (ii) Show that $\cup_{i=1}^{k} \Pi_i = S$. *Hint: If* $C = S \setminus \cup_{i=1}^{k} \Pi_i$ *is non-empty, is* $C \in \Pi$*?*

      (iii) Let $A \in \Sigma$. Show that

$$A = \bigcup_{i \in I} \Pi_i$$

      where $I = \{i = 1, \ldots, k \; : \; A \cap \Pi_i \neq \emptyset\}$.