MASx52: Assignment 5

1. (a) Within the Black-Scholes model, use the risk neutral valuation formula

$$F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid \mathcal{F}_t\right]$$

to show that price at time t of the contingent claim $\Phi(S_T) = 3S_T + 5$ is given by

$$F(t, S_t) = 3S_t + 5e^{-r(T-t)}.$$

(b) Describe a portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.

2. (a) Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and S_t be an Ito process satisfying $dS_t = \alpha S_t\,dt + \sigma S_t\,dB_t$. Let $Y_t = S_t^3$. Show that

$$dY_t = \left(3\alpha + 3\sigma^2\right) Y_t\,dt + 3\sigma Y_t\,dB_t$$

Deduce that Y_t is a geometric Brownian motion, and write down its drift and volatility.

(b) Within the Black-Scholes model, find the price $F(t, S_t)$ at time $t \in [0, T]$ of the contingent claim $\Phi(S_T) = S_T^3$.

3. Let X_t be an Ito process satisfying $dX_t = X_t^2\,dB_t$, and let $F(t, x)$ be a solution of the partial differential equation

$$\frac{\partial F}{\partial t}(t, x) + \frac{1}{2}x^4\frac{\partial^2 F}{\partial x^2}(t, x) = 0$$

with the boundary condition $F(T, x) = x$. Use Ito's formula to find $dF(t, X_t)$ and hence show that $F(t, x) = \mathbb{E}_{t,x}[X_T]$.