MPS350: Exercises for computer classes – extension

⋆⋆⋆ 1. The following result comes from Exercise 6.6, which covers the case of two probability density functions. We extend it here to three. You will need to use this result as part of the question. You may also find the example code given in Exercises 5.2 and 7.4 helpful.

Let f_1, f_2 and f_3 be probability density functions with range Π. Then

$$f(\theta) = \tfrac{1}{3}f_1(\theta) + \tfrac{1}{3}f_2(\theta) + \tfrac{1}{3}f_3(\theta) \tag{$\star$}$$

is a probability density function with range Π. We now define four Bayesian models, all with the same parameter space Π and range R, but with different priors:

- *(X, Θ) is a Bayesian model with prior density f,*
- *for $i = 1, 2, 3$, let (X_i, Θ_i) be a Bayesian model with prior density f_i.*

Then

$$f_{\Theta|\{X=x\}}(\theta) = \sum_{i=1}^{3} \frac{Z_i}{Z_1 + Z_2 + Z_3} f_{\Theta_i|\{X_i=x\}}(\theta). \tag{$\dagger$}$$

where $Z_i = L_{X_i}(x)$ is the normalizing constant in Bayes theorem for the model (X_i, Θ_i).

In medicine, the term Adverse Drug Event (ADE) refers to an unwanted factor affecting a patient as a consequence of their prescribed medication. They are commonly known as side-effects.

An elicitation exercise is carried out with 3 clinical experts, to construct a prior distribution for the percentage of patients who experience an ADE within 28 days of a particular medication. The medication in its trial stages and there is no pre-existing data. The experts were asked to provide estimates of (1) the plausible range; (2) the interquartile range; and (3) the median from which a best-fitting probability distribution is derived. The results of the elicitation exercise are summarised as follows.

elicitee	plausible range	interquartile range	median
A	0 – 20%	2 – 7%	5%
B	10 – 50%	20 – 33%	25%
C	2 – 25%	4 – 12%	7%

(a) For each of the three elicitees, use the interquartile range to construct a prior $\text{Beta}(a_i, b_i)$ representing their beliefs about the proportion p of patients experiencing an ADE within 28 days. Comment on whether the elicitees beliefs are self-consistent, and consistent with each other.

Combine these three priors into a single prior, as in ($\star$), attaching equal weight to each elicitees beliefs.

Draw a plot that shows each elicitees prior, as well as the combined prior.

(b) A small trial with $n = 5$ patients is carried out, $x = 1$ of whom experience an ADE within 28 days. Carry out a Bayesian update numerically, using a suitable model, to obtain a posterior distribution for p.

(c) Repeat part (b) using the reference prior for your chosen model, instead of the elicitees. Plot the posterior obtained using a reference prior on the same graph as that obtained with the help of the elicitees.

Suppose that you were to show 95% HPD intervals for each of the two posteriors. How would they differ?

(d) A colleague asks you to explain the role of the factor $\frac{Z_i}{Z_1+Z_2+Z_3}$ in (†) in your analysis. What would you say to them?

(e) We could conduct our analysis here in different ways to the strategy used above. For example:

i. We could find the posterior $\Theta_i|_{\{X_i=x}$ distribution from each experts prior and then combine the posteriors in the style of ($\star$), giving a posterior $\frac{1}{3}\sum_1^3 f_{\Theta_i|_{\{X_i=x\}}}(\theta)$.

ii. We could have got our three elicitees together in the same room, then asked them to discuss their thoughts and agree on a (single) prior.

What do you think of these approaches by comparison to each other, and to the approach in part (b)?