MASx52: Assignment 1

Solutions and discussion are written in blue. A sample mark scheme, with a total of 20 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Recall the one-period market, and its parameters r, u, d, p_u, p_d and s. We assume that $d < 1 + r < u$.

   (a) At time $t = 0$ our portfolio contains 2 unit of cash and 3 units of stock. What is the value of our portfolio at time $t = 0$? If we hold this portfolio until time $t = 1$, what is its new value?

   (b) A rival investor holds a portfolio containing 3 units of cash and 2 unit of stock. Under what condition (on the parameters) can we be *certain* that our own portfolio will have a strictly greater value at time $t = 1$?

   Solution.

   (a) At time $t = 0$, our portfolio has value $2 + 3s$. [1] At time $t = 1$, our portfolio has value $2(1 + r) + 3S_1$ [1] where S_1 is a random variable with $\mathbb{P}[S_1 = su] = p_u$ and $\mathbb{P}[S_1 = sd] = p_d$. [1]

   (b) The value of the rival investors portfolio at time $t = 1$ is $3(1+r)+2S_1$. [1] This means that our own portfolio is worth strictly more when

   $$2(1 + r) + 3S_1 > 3(1 + r) + 2S_1$$

   or, equivalently, when

   $$S_1 > 1 + r.$$

   [1] To be certain that this inequality holds occurs, we must consider a 'worst case scenario' for the value of S_1. That is, we are certain that our own portfolio will have greater value if and only if

   $$sd > 1 + r.$$

   [1] (In words, this equation says that stock is certain to outperform cash.)

2. Let $\Omega = \{HH, HT, TH, TT\}$, representing two coin tosses each of which may show either H (head) or T (tail). Let $X : \Omega \to \mathbb{R}$ be the toss in which the first head occurred, or zero if no heads occurred:

   $$X = \begin{cases} 0 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT \text{ or } \omega = HH \\ 2 & \text{if } \omega = TH. \end{cases}$$

   Let Y be the total number of heads that occurred in both tosses.

   (a) Write down the sets $X^{-1}(0)$, $X^{-1}(1)$ and $X^{-1}(2)$.

   (b) State the definition of $\sigma(X)$, and list its elements.

(c) Is Y measurable with respect to $\sigma(X)$? Why, or why not?

Solution.

(a) The pre-images are $X^{-1}(0) = \{TT\}$, $X^{-1}(1) = \{HT, HH\}$ and $X^{-1}(2) = \{TH\}$. [2]

(b) The σ-field $\sigma(X)$ is the smallest σ-field with respect to which X is measurable. [1]
To construct $\sigma(X)$, we start by adding in the pre-images from (a), along with $\emptyset$ and Ω, and then include all the unions and complements that we can make from currently added subset of Ω, until we have added them all. This gives

$$\Big\{\emptyset, \Omega, \{TT\}, \{HT, HH\}, \{TH\}, \{TT, HT, HH\}, \{TT, TH\}, \{HT, HH, TH\}\Big\}.$$

[2]. We conclude that this is a σ-field and hence also equal to $\sigma(X)$. [1]

(c) Intuitively, we expect Y not to be measurable with respect to $\sigma(X)$, because the value of Y is different for HT and HH, whereas the value of X for these two outcomes is the same - so Y depends on information that X doesn't see.
To answer the question: we note that (for example) $Y^{-1}(1) = \{HT, TH\}$ [1] which is not an element of $\sigma(X)$. Hence Y is not $\sigma(X)$-measurable. [1]

3. Let X be a random variable.

(a) Show that $Y = \cos X$ is also a random variable.

(b) For which $p \in [1, \infty)$ do we have $Y \in L^p$?

Solution.

(a) By definition,

$$\cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!},$$

which is an infinite series that converges for all $x \in \mathbb{R}$.
Recall that adding together random variables gives random variables, that multiplying together random variables gives random variables [2]. Using these facts repeatedly, we have that

$$\begin{aligned} Y_n(\omega) &= \sum_{i=0}^{n} \frac{(-1)^i \big(X_n(\omega)\big)^{2i}}{(2i)!} \\ &= 1 - \frac{X_n(\omega)^2}{2} + \frac{X_n(\omega)^4}{24} - \ldots + \frac{(-1)^n \big(X_n(\omega)\big)^{2n}}{(2n)!} \end{aligned}$$

is a random variable, for each $n \in \mathbb{N}$. [1] By definition of cos, we have $Y(\omega) = \lim_{n\to\infty} Y_n(\omega)$ for all $\omega \in \Omega$. Since limits of random variables (when they exist) are also random variables, Y is a random variable. [1]

(b) Recall that $|\cos x| \leq 1$ for all $x \in \mathbb{R}$. Hence $|Y| \leq 1$, and hence by monotonicity of $\mathbb{E}$,

$$\mathbb{E}[|Y|^p] \leq \mathbb{E}[1^p] = 1.$$

[1] Therefore, $Y \in L^p$ for all $p \in [1, \infty)$. [1]