# MAS223 Statistical Inference and Modelling

# Exercises

The exercises are grouped into sections, corresponding to chapters of the lecture notes. Within each section exercises are divided into warm-up questions, ordinary questions, and challenge questions. Note that there are no exercises accompanying Chapter 8.

The vast majority of exercises are ordinary questions. Ordinary questions will be used in homeworks and tutorials; they cover the material content of the course. Warm-up questions are typically easier, often nothing more than revision of relevant material from first year courses. Challenge questions are typically harder and test ingenuity.

## Contents

# 1 Univariate Distribution Theory

### Warm-up Questions

**1.1** Let $X$ be a random variable taking values in $\{1, 2, 3\}$, with $\mathbb{P}[X = 1] = \mathbb{P}[X = 2] = 0.4$. Find $\mathbb{P}[X = 3]$, and calculate both $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

**1.2** Let $Y$ be a random variable with probability density function (p.d.f.) $f(y)$ given by

$$f(y) = \begin{cases} y/2 & \text{for } 0 \le y < 2; \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that $Y$ is between $\frac{1}{2}$ and 1. Calculate $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$.

### Ordinary Questions

**1.3** Define $F : \mathbb{R} \to [0, 1]$ by

$$F(y) = \begin{cases} 0 & \text{for } y \le 0; \\ y^2 & \text{for } y \in (0, 1); \\ 1 & \text{for } y \ge 1. \end{cases}$$

(a) Sketch the function $F$, and check that it is a distribution function.

(b) If $Y$ is a random variable with distribution function $F$, calculate the p.d.f. of $Y$.

**1.4** Let $X$ be a discrete random variable, taking values in $\{0, 1, 2\}$, where $\mathbb{P}[X = n] = \frac{1}{3}$ for $n \in \{0, 1, 2\}$. Sketch the distribution function $F_X : \mathbb{R} \to \mathbb{R}$.

**1.5** Define $f : \mathbb{R} \to [0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{for } x < 0; \\ e^{-x} & \text{for } x \ge 0. \end{cases}$$

(a) Show that $f$ is a probability density function.

(b) Find the corresponding distribution function and evaluate $\mathbb{P}[1 < X < 2]$.

**1.6** Sketch graphs of each of the following two functions, and explain why each of them is not a distribution function.

(a) $F(x) = \begin{cases} 0 & \text{for } x \le 0; \\ x & \text{for } x > 0. \end{cases}$

(b) $F(x) = \begin{cases} 0 & \text{for } x < 0; \\ x + \frac{1}{4}\sin 2\pi x & \text{for } 0 \le x < 1; \\ 1 & \text{for } x \ge 1. \end{cases}$

**1.7** Let $k \in \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} k(x - x^2) & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of $k$ for which $f(x)$ is a probability density function, and calculate the probability that $X$ is greater than $\frac{1}{2}$.

**1.8** The probability density function $f(x)$ is given by

$$f(x) = \begin{cases} 1 + x & \text{for } -1 \leq x \leq 0; \\ 1 - x & \text{for } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find the corresponding distribution function $F(x)$ for all real $x$.

**1.9** Let

$$F(x) = \frac{e^x}{1 + e^x} \qquad \text{for all real } x.$$

(a) Show that $F$ is a distribution function, and find the corresponding p.d.f. $f$.

(b) Show that $f(-x) = f(x)$.

(c) If $X$ is a random variable with this distribution, evaluate $\mathbb{P}[|X| > 2]$.

**1.10** Show that $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, defined for all $x \in \mathbb{R}$, is a probability density function.

**1.11** (a) Show that

$$f(x) = \begin{cases} x^{-2} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

(b) Show that the expectation of a random variable $X$, with the probability density function $f$ given in (b), is not defined.

(c) For which values of $r \in [0, \infty)$ is $\int_1^\infty x^{-r}\,dx$ finite?

(d) Give an example of a random variable $Y$ for which $\mathbb{E}[Y] < \infty$ but $\mathbb{E}[Y^2]$ is not defined.

**1.12** The discrete random variable $X$ has the probability function

$$\mathbb{P}[X = x] = \begin{cases} \frac{1}{x(x+1)} & \text{for } x \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Use the partial fractions of $\frac{1}{x(x+1)}$ to show that $\mathbb{P}[X \leq x] = 1 - \frac{1}{x+1}$, for all $x \in \mathbb{N}$.

(b) Write down the distribution function $F(x)$ of $X$, for $x \in \mathbb{R}$. Sketch its graph. What are the values of $F(2)$ and $F(\frac{3}{2})$?

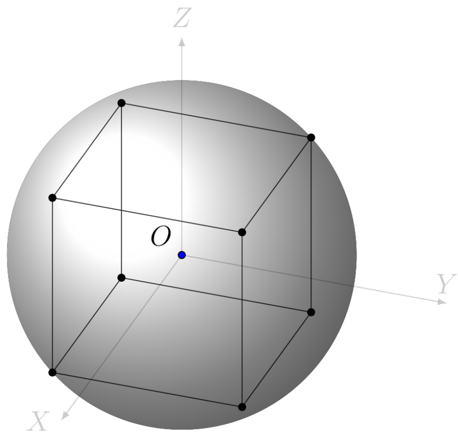
(c) Evaluate $\mathbb{P}[10 \leq X \leq 20]$.

(d) Is $\mathbb{E}[X]$ defined? If so, what is $\mathbb{E}[X]$? If not, why not?

### Challenge Questions

**1.13** Show that there is no random variable $X$, with range $\mathbb{N}$, such that $\mathbb{P}[X = n]$ is constant for all $n \in \mathbb{N}$.

**1.14** Recall the meaning of 'inscribing' a cube within a sphere: the cube sits inside of the sphere, with each vertex of the cube positioned on the surface of the sphere. It is not especially easy to illustrate this on a two dimensional page, but here is an attempt:




Suppose that ten percent of the surface of the sphere is coloured blue, and the rest of the surface is coloured red. Show that, regardless of which parts are coloured blue, it is always possible to inscribe a cube within the sphere in such a way as all vertices of the cube are red.

*Hint: A cube has eight corners. Suppose that position of the cube is sampled uniformly from the set of possible positions. What is the expected number of corners that are red?*

# 2 Standard Univariate Distributions

### Warm-up Questions

**2.1** (a) A standard fair dice is rolled 5 times. Let $X$ be the number of sixes rolled. Which distribution (and which parameters) would you use to model $X$?

(b) A fair coin is flipped until the first head is shown. Let $X$ be the total number of flips, including the final flip on which the first head appears. Which distribution (and which parameter) would you use to model $X$?

### Ordinary Questions

**2.2** Let $\lambda > 0$. Write down the p.d.f. $f$ of the random variable $X$, where $X \sim Exp(\lambda)$, and calculate its distribution function $F$. Hence, show that $\frac{f(t)}{1-F(t)}$ is constant for $t > 0$.

**2.3** Let $\lambda > 0$ and let $X$ be a random variable with $Exp(\lambda)$ distribution. Let $Z = \lfloor X \rfloor$, that is let $Z$ be $X$ rounded down to the nearest integer. Show that $Z$ is geometrically distributed with parameter $p = 1 - e^{-\lambda}$.

**2.4** Let $\mu \in \mathbb{R}$. Let $X_1$ and $X_2$ be independent random variables with distributions $N(\mu, 1)$ and $N(\mu, 4)$, respectively. Let $T_1, T_2$ and $T_3$ be defined by

$$T_1 = \frac{X_1 + X_2}{2}, \qquad T_2 = 2X_1 - X_2, \qquad T_3 = \frac{4X_1 + X_2}{5}.$$

Find the mean and variance of $T_1$, $T_2$ and $T_3$. Which of $\mathbb{E}[T_1]$, $\mathbb{E}[T_2]$ and $\mathbb{E}[T_3]$ would you prefer to use as an estimator of $\mu$?

**2.5** Let $X$ be a random variable with $Ga(\alpha, \beta)$ distribution.

(a) Let $k \in \mathbb{N}$. Show that

$$\mathbb{E}[X^k] = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}.$$

Hence, calculate $\mu = \mathbb{E}[X]$ and $\sigma^2 = \text{Var}(X)$ and verify that these formulas match the ones given in lectures.

(b) Show that $\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{2}{\sqrt{\alpha}}$.

**2.6** (a) Using `R`, you can obtain a plot of, for example, the p.d.f. of a $Ga(3, 2)$ random variable between 0 and 10 with the command

```
curve(dgamma(x,shape=3,scale=2),from=0,to=10)
```

Use `R` to investigate how the shape of the p.d.f. of a Gamma distribution varies with the different parameter values. In particular, fix a value of $\beta$, see how the shape changes as you vary $\alpha$.

(b) Investigate the effect that changing parameters values has on the shape of the p.d.f. of the Beta distribution. To produce, for example, a plot of the p.d.f. of $Be(4,5)$, use

```
curve(dbeta(x,shape1=4,shape2=5),from=-1,to=2)
```

**2.7** Suggest which standard discrete distributions (or combination of them) we should use to model the following situations.

(a) Organisms, independently, possess a given characteristic with probability $p$. A sample of $k$ organisms with the characteristic is required. How many organisms will need to be tested to achieve this sample?

(b) In Texas Hold'em Poker, players make the best hand they can by combining two cards in their hand with five 'community' cards that are placed face up on the table. At the start of the game, a player can only see their own hand. The community cards are then turned over, one by one.

A player has two hearts in her hand. Three of the community cards have been turned over, and only one of them is a heart. How many hearts will appear in the remaining two community cards?

Use a computer to find the probability of seeing $k = 0, 1, 2$ hearts.

**2.8** Let $X$ be a $N(0,1)$ random variable. Use integration by parts to show that $\mathbb{E}[X^{n+2}] = (n+1)\mathbb{E}[X^n]$ for any $n = 0, 1, 2, \ldots$. Hence, show that

$$\mathbb{E}[X^n] = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ (1)(3)(5)\ldots(n-1) & \text{if } n \text{ is even.} \end{cases}$$

**2.9** Let $X \sim N(\mu, \sigma^2)$. Show that $\mathbb{E}[e^X] = e^{\mu + \frac{\sigma^2}{2}}$.

### Challenge Questions

**2.10** Let $X$ be a random variable with a continuous distribution, and a strictly increasing distribution function $F$. Show that $F(X)$ has a uniform distribution on $(0,1)$.

Suggest how we might use this result to simulate samples from standard distributions.

**2.11** Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

# 3 Transformations of Univariate Random Variables

## Warm-up Questions

**3.1** Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^3 + 1$.

(a) Sketch the graph of $g$, show that $g$ is strictly increasing, and find its inverse function $g^{-1}$.

(b) Let $R = [0, 2]$. Find $g(R)$.

## Ordinary Questions

**3.2** Let $X$ be a random variable with p.d.f.

$$f_X(x) = \begin{cases} x^{-2} & \text{for } x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $g(x) = e^x$ and let $Y = g(X)$.

(a) Show that $g(x)$ is strictly increasing. Find its inverse function $g^{-1}(y)$, and $\frac{dg^{-1}(y)}{dy}$.

(b) Identify the set $R_X$ on which $f_X(x) > 0$. Sketch $g$ and show that $g(R_X) = (e, \infty)$.

(c) Deduce from (a) and (b) that $Y$ has p.d.f.

$$f_Y(y) = \begin{cases} (\log y)^{-2} \frac{1}{y} & \text{for } y > e \\ 0 & \text{otherwise.} \end{cases}$$

**3.3** Let $X$ be a random variable with the uniform distribution on $(0, 1)$, and let $Y = \frac{-\log X}{\lambda}$ where $\lambda > 0$. Show that $Y$ has an $Exp(\lambda)$ distribution.

**3.4** Let $\alpha, \beta > 0$.

(a) Show that $B(\alpha, \beta) = B(\beta, \alpha)$.

(b) Let $X$ be a random variable with the $Be(\alpha, \beta)$ distribution. Show that $Y = 1 - X$ has the $Be(\beta, \alpha)$ distribution.

**3.5** Let $\alpha > 0$.

(a) Show that $B(\alpha, 1) = \frac{1}{\alpha}$.

(b) Let $X \sim Be(\alpha, 1)$ distribution. Let $Y = \sqrt[r]{X}$ for some positive integer $r$. Show that $Y$ also has a Beta distribution, and find its parameters.

**3.6** Let $X$ have a uniform distribution on $[-1, 1]$. Find the p.d.f. of $|X|$ and identify the distribution of $|X|$.

**3.7** Let $\alpha, \beta > 0$ and let $X \sim Be(\alpha, \beta)$. Let $c > 0$ and set $Y = c/X$. Find the p.d.f. of $Y$.

**3.8** Let $\Theta$ be an angle chosen according to a uniform distribution on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and let $X = \tan\Theta$. Show that $X$ has the Cauchy distribution.

**3.9** Let $X$ be a random variable with the p.d.f.

$$f(x) = \begin{cases} 1+x & \text{for } -1 < x < 0; \\ 1-x & \text{for } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density functions of

(a) $Y = 5X + 3$

(b) $Z = |X|$

**3.10** Let $X$ have the uniform distribution on $[a, b]$.

(a) For $[a, b] = [-1, 1]$, find the p.d.f. of $Y = X^2$.

(b) For $[a, b] = [-1, 2]$, find the p.d.f. of $Y = |X|$.

**3.11** Let $X$ have a uniform distribution on $[-1, 1]$ and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{for } x \le 0; \\ x^2 & \text{for } x > 0. \end{cases}$$

Find the distribution function of $g(X)$.

**3.12** Let $X$ be a random variable with the Cauchy distribution. Show that $X^{-1}$ also has the Cauchy distribution.

## Challenge Questions

**3.13** If we were to pretend that $g(x) = 1/x$ was strictly monotone, we could (incorrectly) apply Lemma 3.1 and use the formula $f_Y(y) = f_X(g^{-1}(y))|\frac{dg^{-1}}{dy}|$ to solve **3.12**. We would still arrive at the correct answer. Can you explain why?

Can you construct another example of a case in which the relationship $f_Y(y) = f_X(g^{-1}(y))|\frac{dg^{-1}}{dy}|$ holds, but where the function $g$ is not monotone?

**3.14** Let $Y$ and $\alpha, \beta$ be as in Question **3.7**.

(a) If $\alpha > 1$, show that $\mathbb{E}[Y] = \frac{c(\alpha+\beta-1)}{\alpha-1}$.

(b) If $\alpha \le 1$ show that $\mathbb{E}[Y]$ is not defined.

# 4 Multivariate Distribution Theory

### Warm-up questions

**4.1** Let $T = \{(x, y) \ : \ 0 < x < y\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} e^{-2x-y} & \text{for } (x,y) \in T; \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the region $T$. Calculate $\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy$ and $\int_0^\infty \int_0^\infty f(x,y)\,dy\,dx$, and verify that they are equal.

**4.2** Sketch the following regions of $\mathbb{R}^2$.

(a) $S = \{(x, y) \ : \ x \in [0, 1],\, y \in [0, 1]\}$.

(b) $T = \{(x, y) \ : \ 0 < x < y\}$.

(c) $U = \{(x, y) \ : \ x \in [0, 1],\, y \in [0, 1],\, 2y > x\}$.

### Ordinary Questions

**4.3** Let $(X, Y)$ be a random vector with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} ke^{-(x+y)} & \text{if } 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$

(a) Using that $\mathbb{P}[(X, Y) \in \mathbb{R}^2] = 1$, find the value of $k$.

(b) For each of the regions $S, T, U$ in **4.2**, calculate the probability that $(X, Y)$ is inside the given region.

(c) Find the marginal p.d.f. of $Y$, and hence identify the distribution of $Y$.

**4.4** Let $S = [0, 1] \times [0, 1]$, and let $U$ and $V$ have joint probability density function

$$f_{U,V}(u,v) = \begin{cases} \frac{4u+2v}{3} & (u,v) \in S; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{P}[U + V \le 1]$.

(b) Find $\mathbb{P}[V \le U^2]$.

**4.5** For the random variables $U$ and $V$ in Exercise **4.4**:

(a) Find the marginal p.d.f. $f_U(u)$ of $U$.

(b) Find the marginal p.d.f. $f_V(v)$ of $V$.

(c) For $v$ such that $f_V(v) > 0$, find the conditional p.d.f. $f_{U|V=v}(u)$ of $U$ given $V = v$.

(d) Check that each of $f_U$, $f_V$ and $f_{U|V=v}$ integrate over $\mathbb{R}$ to 1.

(e) Calculate the two forms of conditional expectation, $\mathbb{E}[U|V = v]$, and $\mathbb{E}[U|V]$.

**4.6** Let $(X, Y)$ be a random vector with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{y-x}{2} & x \in [-1, 0], y \in [0, 1]; \\ \frac{x+y}{2} & x \in [0, 1], y \in [0, 1]; \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal p.d.f. of $X$. Show that the correlation coefficient $X$ and $Y$ is zero. Show also that $X$ and $Y$ are not independent.

**4.7** Let $X$ be a random variable. Let $Z$ be a random variable, independent of $X$, such that $\mathbb{P}[Z = 1] = \mathbb{P}[Z = -1] = \frac{1}{2}$. Let $Y = XZ$.

(a) Show that $X$ and $Y$ are uncorrelated.

(b) Give an example in which $X$ and $Y$ are not independent.

**4.8** Let $X$ and $Y$ be independent random variables, with $0 < \mathrm{Var}(X) = \mathrm{Var}(Y) < \infty$. Let $U = X + Y$ and $V = XY$. Show that $U$ and $V$ are uncorrelated if and only if $\mathbb{E}[X] + \mathbb{E}[Y] = 0$.

**4.9** Let $\lambda > 0$. Let $X$ have an $Exp(\lambda)$ distribution, and conditionally given $X = x$ let $U$ have a uniform distribution on $[0, x]$. Calculate $\mathbb{E}[U]$ and $\mathrm{Var}(U)$.

**4.10** Let $k \in \mathbb{R}$ and let $(X, Y)$ have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} kx \sin(xy) & \text{for } x \in (0, 1), y \in (0, \pi), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of $k$.

(b) For $x \in (0, 1)$, find the conditional probability density function of $Y$ given $X = x$.

(c) Find $\mathbb{E}[Y|X]$.

**4.11** Let $U$ have a uniform distribution on $(0, 1)$, and conditionally given $U = u$ let $X$ have a uniform distribution on $(0, u)$.

(a) Find the joint p.d.f of $(X, U)$ and the marginal p.d.f. of $X$.

(b) Show that $\mathbb{E}[U|X = x] = \frac{x-1}{\log x}$.

**4.12** Let $(X, Y)$ have a bivariate distribution with joint p.d.f. $f_{X,Y}(x, y)$. Let $y_0 \in \mathbb{R}$ be such that $f_Y(y_0) > 0$. Show that $f_{X|Y=y_0}(x)$ is a probability density function.

## Challenge Questions

**4.13** Give an example of random variables $(X, Y, Z)$ such that

$$\mathbb{P}[X < Y] = \mathbb{P}[Y < Z] = \mathbb{P}[Z < X] = \frac{2}{3}.$$

# 5 Transformations of Multivariate Distributions

## Warm-up Questions

**5.1** (a) Define $u = x$ and $v = 2y$. Sketch the images of the regions $S, T$ and $U$ from question **4.2** in the $(u, v)$ plane.

(b) Define $u = x + y$ and $v = x - y$. Sketch the images of the regions $S, T$ and $U$ from question **4.2** in the $(u, v)$ plane.

## Ordinary Questions

**5.2** The random variables $X$ and $Y$ have joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-y} & \text{if } x \in (0, 2), y \in (0, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

Define $u = u(x, y) = x + y$ and $v = v(x, y) = 2y$. Let $U = u(X, Y)$ and $V = v(X, Y)$.

(a) Find the inverse transformation $x = x(u, v)$ and $y = y(u, v)$ and calculate the value of $J = \det \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$.

(b) Sketch the set of $(x, y)$ for which $f_{X,Y}(x, y)$ is non-zero. Find the image of this set in the $(u, v)$ plane.

(c) Deduce from (a) and (b) that the joint p.d.f. of $(U, V)$ is

$$f_{U,V}(u, v) = \begin{cases} (u - \frac{v}{2})e^{-v/2} & \text{for } v > 0, u \in (2v, 2v + 4) \\ 0 & \text{otherwise.} \end{cases}$$

**5.3** The random variables $X$ and $Y$ have joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}(x + y)e^{-(x+y)} & \text{for } x, y \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $U = X + Y$ and $W = X$.

(a) Find the joint p.d.f. of $(U, W)$ and the marginal p.d.f. of $U$.

(b) Recognize $U$ as a standard distribution and, using the result of Question **2.5**(a), evaluate $\mathbb{E}[(X + Y)^5]$.

**5.4** Let $X$ and $Y$ be a pair of independent random variables, both with the standard normal distribution. Show that the joint p.d.f. of $(U, V)$ where $U = X^2$ and $V = X^2 + Y^2$ is given by

$$f_{U,V}(u, v) = \begin{cases} \frac{1}{8\pi}e^{-v/2}u^{-1/2}(v - u)^{-1/2} & \text{for } 0 \leq u \leq v \\ 0 & \text{otherwise.} \end{cases}$$

**5.5** Let $(X, Y)$ be a random vector with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & x > y > 0; \\ 0 & \text{otherwise.} \end{cases}$$

(a) If $U = X - Y$ and $V = Y/2$, find the joint p.d.f. of $(U, V)$.

(b) Show that $U$ and $V$ are independent, and recognize their (marginal) distributions as standard distributions.

**5.6** Let $X$ and $Y$ be a pair of independent and identically distributed random variables. Let $U = X + Y$ and $V = X - Y$.

(a) Show that $\operatorname{Cov}(U, V) = 0$, and give an example (with justification) to show that $U$ and $V$ are not necessarily independent.

(b) Show that $U$ and $V$ are independent in the special case where $X$ and $Y$ are standard normals.

**5.7** Let $X$ and $Y$ be independent random variables with distributions $Ga(\alpha_1, \beta)$ and $Ga(\alpha_2, \beta)$ respectively. Show that the random variables $U = \frac{X}{X+Y}$ and $V = X + Y$ are independent with distributions $Be(\alpha_1, \alpha_2)$ and $Ga(\alpha_1 + \alpha_2, \beta)$ respectively.

**5.8** As part of Question **5.7**, we showed that if $X$ and $Y$ are independent random variables with $X \sim Ga(\alpha_1, \beta)$ and $Y \sim Ga(\alpha_2, \beta)$, then $X + Y \sim Ga(\alpha_1 + \alpha_2, \beta)$.

(a) Use induction to show that for $n \geq 2$, if $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim Ga(\alpha_i, \beta)$ then

$$\sum_{i=1}^{n} X_i \sim Ga\left(\sum_{i=1}^{n} \alpha_i, \beta\right).$$

(b) Hence show that for $n \geq 1$, if $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables then

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2.$$

*You may use the result of Example 12, which showed that this was true in the case $n = 1$ (and, recall that the $\chi^2$ distribution is a special case of the Gamma distribution).*

**5.9** Let $X$ and $Y$ be a pair of independent random variables, both with the standard normal distribution. Show that $U = X/Y$ has the Cauchy distribution, with p.d.f.

$$f_U(u) = \frac{1}{\pi} \frac{1}{1 + u^2}$$

for all $u \in \mathbb{R}$.

**5.10** Let $n \in \mathbb{N}$. The $t$ distribution (often known as Student's $t$ distribution) is the univariate random variable $X$ with p.d.f.

$$f_X(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}},$$

for all $x \in \mathbb{R}$. Here, $n$ is a parameter, known as the number of degrees of freedom.

Let $Z$ be a standard normal random variable and let $W$ be a chi-squared random variable with $n$ degrees of freedom, where $Z$ and $W$ are independent. Show that

$$X = \frac{Z}{\sqrt{W/n}}$$

has the $t$ distribution with $n$ degrees of freedom.

### Challenge Questions

**5.11** The formula (5.2), from the typed lecture notes, holds whenever the transformation $u = u(x, y)$, $v = v(x, y)$ is both one-to-one and onto, and the Jacobian matrix of derivatives exists. Use this fact to provide an alternative proof of Lemma 3.1 (i.e. of the univariate transformation formula).

**5.12** Let $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$ be independent. Show that $U = \min(X, Y)$ has distribution $Exp(\lambda_1 + \lambda_2)$, and that $\mathbb{P}[\min(X, Y) = X] = \frac{\lambda_1}{\lambda_1+\lambda_2}$. Extend this result, by induction, to handle the minimum of finitely many exponential random variables.

Let $W = \max(X, Y)$. Show that $U$ and $W - U$ are independent.

# 6 Covariance Matrices and and Multivariate Normal Distributions

## Warm-up Questions

**6.1** Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma} = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

(a) Calculate $\det(\mathbf{A})$ and find both $\mathbf{A\Sigma}$ and $\mathbf{A\Sigma A}^T$.

(b) Let $\mathbf{A}$ be the vector $(2, 1)$. Show that $\mathbf{A\Sigma A}^T = 21$.

## Ordinary Questions

**6.2** Let $\mathbf{X} = (X, Y)^T$ be a random vector with

$$\mathbb{E}[\mathbf{X}] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{Cov}(\mathbf{X}) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $U = X + Y$ and $V = 2X - 2Y + 1$. Write $\mathbf{U} = (U, V)^T$.

(a) Write down a square matrix $\mathbf{A}$ and a vector $\mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{U} = \mathbf{AX} + \mathbf{b}$.

(b) Show that the mean vector and covariance matrix of $\mathbf{U}$ are given by

$$\mathbb{E}[\mathbf{U}] = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \text{Cov}(\mathbf{U}) = \begin{pmatrix} 5 & -2 \\ -2 & 4 \end{pmatrix}$$

(c) Show that the correlation coefficient $\rho$ of $U$ and $V$ is equal to $\frac{-1}{\sqrt{5}}$.

**6.3** Let $X$ and $Y$ be independent standard normal random variables.

(a) Write down the covariance matrix of the random vector $\mathbf{X} = (X, Y)^T$.

(b) Let $\mathbf{R}$ be the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that $\mathbf{RX}$ has the same covariance matrix as $\mathbf{X}$.

**6.4** Three (univariate) random variables $X$, $Y$ and $Z$ have means 3, $-4$ and 6 respectively and variances 1, 1 and 25 respectively. Further, $X$ and $Y$ are uncorrelated; the correlation coefficient between $X$ and $Z$ is $\frac{1}{5}$ and that between $Y$ and $Z$ is $-\frac{1}{5}$. Let $U = X + Y - Z$ and $W = 2X + Z - 4$ and set $\mathbf{U} = (U, W)^T$.

(a) Find the mean vector and covariance matrix of $\mathbf{X} = (X, Y, Z)^T$.

(b) Write down a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that $\mathbf{U} = \mathbf{AX} + \mathbf{b}$.

(c) Find the mean vector and covariance matrix of $\mathbf{U}$.

(d) Evaluate $\mathbb{E}[(2X + Z - 6)^2]$.

**6.5** Suppose that the random vector $\mathbf{X} = (X_1, X_2)^T$ follows the bivariate normal distribution with $\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$, $\mathrm{Var}(X_1) = 1$, $\mathrm{Cov}(X_1, X_2) = 2$ and $\mathrm{Var}(X_2) = 5$.

(a) Calculate the correlation coefficient of $X_1$ and $X_2$. Are $X_1$ and $X_2$ independent?

(b) Find the mean and the covariance matrices of

$$Y = \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{X} \qquad \text{and} \qquad \mathbf{Z} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{X}.$$

What are the distributions of $Y$ and $\mathbf{Z}$?

**6.6** Let $X_1$ and $X_2$ be bivariate normally distributed random variables each with mean 0 and variance 1, and with correlation coefficient $\rho$.

(a) By integrating out the variable $x_2$ in the joint p.d.f., verify that the marginal distribution of $X_1$ is indeed that of a standard univariate normal random variable. *Hint: Use the fact that the integral of a $N(\mu, \sigma^2)$ p.d.f. is equal to* 1.

(b) Show, using the 'usual' formula for the conditional p.d.f. that the conditional p.d.f. of $X_2$ given $X_1 = x_1$ is $N(\rho x_1, 1 - \rho^2)$.

**6.7** Let $\mathbf{X} = (X_1, X_2)^T$ have a $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$. Let

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{-\sigma_1}{\sigma_2} \\ \frac{\sigma_2}{\sigma_1} & 1 \end{pmatrix}.$$

Find the distribution of $\mathbf{Y} = \mathbf{AX}$, and deduce that any bivariate normal random vector can be transformed by a linear transformation into a vector of independent normal random variables.

**6.8** The random vector $\mathbf{X} = (X_1, X_2, X_3)^T$ has an $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution where $\boldsymbol{\mu} = (-1, 1, 2)^T$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 144 & -30 & 48 \\ -30 & 25 & 10 \\ 48 & 10 & 64 \end{pmatrix}.$$

(a) Find the correlation coefficients between $X_1$ and $X_2$, between $X_1$ and $X_3$ and between $X_2$ and $X_3$.

(b) Let $Y_1 = X_1 + X_3$ and $Y_2 = X_2 - X_1$. Find the distribution of $\mathbf{Y} = (Y_1, Y_2)^T$ and hence find the correlation coefficient between $Y_1$ and $Y_2$.

**6.9** (a) Let $X$ be a (univariate) standard normal random variable and let $Y = X$. Does the random vector $\mathbf{X} = (X, Y)^T$ have a bivariate normal distribution?

(b) Let $X$ and $Y$ be two independent (univariate) standard normal random variables, and let $Z$ be a random variable such that $\mathbb{P}[Z = 1] = \mathbb{P}[Z = -1] = \frac{1}{2}$. Are $XZ$ and $YZ$ independent, and does the random vector $\mathbf{X} = (XZ, YZ)^T$ have a bivariate normal distribution?

## Challenge Questions

**6.10** Recall that an *orthogonal matrix* $\mathbf{R}$ is one for which $\mathbf{R}^{-1} = \mathbf{R}^T$, and recall that if an $n \times n$ matrix $\mathbf{R}$ is orthogonal, $\mathbf{x}$ is an $n$-dimensional vector, and $\mathbf{y} = \mathbf{R}\mathbf{x}$ then $\sum_{i=1}^n y_i^2 = \mathbf{y} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{x} = \sum_{i=1}^n x_i^2$.

Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a vector of independent normal random variables with common mean 0 and variance $\sigma^2$. Let $\mathbf{R}$ be an orthogonal matrix and let $\mathbf{Y} = \mathbf{R}\mathbf{X}$.

(a) Show that $\mathbf{Y}$ is also a vector of independent normal random variables, with common mean 0 and variance $\sigma^2$.

(b) Suppose that all the elements in the first row of $\mathbf{R}$ are equal to $\frac{1}{\sqrt{n}}$ (you may assume that an orthogonal matrix exists with this property). Show that $Y_1 = \sqrt{n}\bar{X}$, where $\bar{X}$ is the sample mean of $\mathbf{X}$, and that

$$\sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2.$$

(c) Hence, use Question **5.8** to deduce that, if $s^2 = \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right)$ is the sample variance of $\mathbf{X}$,

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1},$$

and that it is independent of $\bar{X}$.

(d) Let $\mu \in \mathbb{R}$. Deduce that the result of part (c) also holds if the $X_i$ have mean $\mu$ (so as they are i.i.d. $N(\mu, \sigma^2)$ random variables).

**6.11** Let $Z_1, Z_2, \dots$ be independent identically distributed normal random variables with mean $\mu$ and variance $\sigma^2$. We regard $n$ samples of these as a set of data. Write $\bar{Z}$ and $s^2$ respectively for the sample mean and variance.

Combine **6.10**(d) and **5.10** to show that the statistic

$$X' = \frac{\sqrt{n}(\bar{Z} - \mu)}{s}$$

has the $t$ distribution with $n - 1$ degrees of freedom.

# 7 Likelihood and Maximum Likelihood

### Warm-up Questions

**7.1** Let $f : \mathbb{R} \to \mathbb{R}$ by $f(\theta) = e^{-\theta^2+4\theta}$.

(a) Find the first derivative of $f$, and hence identify its turning point(s).

(b) Calculate the value of the second derivative of $f$ at these turning point(s). Hence, deduce if the turning point(s) are local maxima or local minima.

**7.2** Let $(a_i)_{i=1}^n$ be a sequence with $a_i \in (0,\infty)$. Show that $\log\left(\prod_{i=1}^n a_i\right) = \sum_{i=1}^n \log a_i$.

### Ordinary Questions

**7.3** A single sample of $x = 3$ is obtained from a geometric distribution $X$ with unknown parameter $\theta$. That is, $\mathbb{P}[X = x] = \theta^x(1-\theta)$ for $x \in \{0, 1, 2, \ldots\}$.

(a) Write down the likelihood function $L(\theta; 3)$, and state its domain $\Theta$.

(b) Find the maximum likelihood estimator of $\theta$.

**7.4** Suppose that we have a biased coin, which throws a head with probability $\theta \in [0, 1]$, and a tail with probability $1 - \theta$. The coin flips are independent of each other. We toss the coin $n$ times, and record the total number of heads.

(a) Let $X$ be the total number of heads thrown after $n$ tosses. Write down the distribution of $X$.

(b) Write down the likelihood function $L(\theta; x)$ of $X$, where $x$ is the total number of heads in $n$ tosses. State the range of values taken by $\theta$.

(c) Suppose that in $n = 10$ tosses we throw $x = 7$ heads. Based on this data, find the maximum likelihood estimator of $\theta$.

**7.5** Consider a sequence of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, for some $n > 0$, in each of the following cases. The data are i.i.d. samples from:

(a) the gamma distribution $Ga(\alpha, \theta)$, where $\alpha$ is known.

(b) the beta distribution $Be(\theta, 1)$, where second parameter is known to be equal to 1.

(c) the normal distribution $N(\theta, \sigma^2)$, where $\sigma^2$ is known.

(d) the normal distribution $N(\mu, \theta)$, where $\mu$ is known.

In each case, do the following:

(i) Write down the likelihood function $L(\theta; \mathbf{x})$, and specify the range of values $\Theta$ taken by $\theta$.

(ii) Find the log likelihood $\ell(\theta; \mathbf{x})$, and simplify it as much as you can.

(iii) Find the maximum likelihood estimator $\hat{\theta}$.

**7.6** A sample $(x_1, x_2, x_3) = (4, 0, 3)$ of three independent observations is obtained from a Poisson distribution with parameter $\lambda$. Here, $\lambda$ is known to be in $\Lambda = \{1, 2, 3\}$. Find the likelihood of each of the possible values of $\lambda$, and hence find the maximum likelihood estimator.

**7.7** A sequence of i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is taken from an inverse Gaussian distribution, which has the p.d.f.

$$f(x) = \sqrt{\frac{\theta}{2\pi x^3}} \exp\left(-\frac{\theta(x-\mu)^2}{2\mu^2 x}\right) \qquad \text{when } x > 0,$$

and $f(x) = 0$ for $x \leq 0$, with parameters $\mu, \theta > 0$.

Both $\mu$ and $\theta$ are unknown. Find the maximum likelihood estimator of $(\mu, \theta)$.

**7.8** Suppose that we have a sequence of i.i.d. samples $\mathbf{x} = (x_1, \ldots, x_n)$ taken from a $N(\mu, \sigma^2)$ distribution, where $\mu$ is unknown and $\sigma^2$ is known. Recall from Example 42 that the maximum likelihood estimator is $\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$ and the $k$-likelihood region for $\mu$ is given by

$$\left[\bar{x} - \sigma\sqrt{\frac{2k}{n}},\ \bar{x} + \sigma\sqrt{\frac{2k}{n}}\right].$$

Suppose that $n = 144$, $\sigma^2 = 1$ and the data satisfies $\bar{x} = 2.05$. Does a 2-likelihood test support the hypothesis that the true mean is $\mu = 2$?

**7.9** In **Q7.4** we tossed a coin $n = 10$ times and observed $x = 7$ heads. We modelled the number of heads $X$ as a $Bi(n, \theta)$ distribution with $n = 10$ and unknown $\theta$.

(a) Use a software package of your choice to find $R_2$, the range of values for $\theta$ which the log likelihood is within 2 of its maximum value.

(b) In this situation, an approximate 95% confidence interval is given by $\hat{\theta} \pm 1.96\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$, where $\hat{\theta} = x/n$. Compare this confidence interval to your answer in (a).

(c) Repeat this analysis with $n = 100$ and $x = 70$.

(d) Consider the following statements. Which do you believe? Give arguments to support your conclusions.

(i) *"Using more samples means that we can be more confident about the accuracy of our estimators."*

(ii) *"In both cases ($n = 10$ and $n = 100$) we used a just single sample of $x$, so in both cases we should have the same level of confidence in our estimator $\hat{\theta}$."*

(iii) *"If we don't have enough samples to feel certain, then we shouldn't believe in statistics."*

**7.10** As in **Q7.5**(b), let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of i.i.d. samples from the $Beta(\theta, 1)$ distribution, where $\theta$ is unknown. Show that the $k$-likelihood region for $\theta$ is

$$R_k = \left\{\theta \in [0, \infty)\ :\ \left|\hat{\theta}(\theta - \hat{\theta}) + \log(\theta/\hat{\theta})\right| \leq \frac{k}{n}\right\},$$

where $\hat{\theta}$ denotes the maximum likelihood estimator of $\theta$.

**7.11** (a) Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of i.i.d. samples of the random variable $X$, and suppose that the distribution of $X$ has the parameters $\theta$. Let $\ell$ be the corresponding log likelihood function. Show that

$$\ell(\boldsymbol{\theta}; \mathbf{x}) = \sum_{i=1}^{n} \log f_X(x_i; \boldsymbol{\theta}).$$

(b) Write $\phi(x;\, \mu, v)$ for the probability density function of the $N(\mu, v)$ distribution, with parameters $\boldsymbol{\theta} = (\mu, v)$. Let $X \sim logN(\mu, v)$ have. Show that for $x > 0$

$$f_X(x;\, \mu, v) = \frac{1}{x}\phi(\log x;\, \mu, v).$$

Find the corresponding log-likelihood function $\ell(\mu, v;\, \mathbf{x})$, in terms of $\phi$.

(c) Without doing any further calculations, combine your results from (a) and (b) with the results of Example 40, to deduce formulae for the maximum likelihood estimator of $(\mu, v)$.

## Challenge Questions

**7.12** The Pareto distribution has parameters $\alpha > 0$ and $\beta > 0$, with p.d.f.

$$f(x; \boldsymbol{\theta}) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}} \quad \text{when } x \geq \beta$$

and $f(x; \boldsymbol{\theta}) = 0$ when $x < \beta$. A set of $n$ i.i.d. samples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are taken from a Pareto distribution, where the parameters $\boldsymbol{\theta} = (\alpha, \beta)$ are both unknown.

Find the corresponding maximum likelihood estimator of $\boldsymbol{\theta}$.