MASx52: Assignment 5

1. Consider the SDE
$$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$
   (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation
$$f'(t) = t + f(t)$$
   Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

   (b) Let $Y_t = X_t^2$. Show that
$$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

   (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation
$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

2. Let $T > 0$. Use the Feynman-Kac formula to find an explicit solution $F(x,t)$ to the partial differential equation
$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2}(x,t) = 0$$
subject to the boundary condition $F(T,x) = x - \frac{T}{2}$.

   Hint: It may help to recall that $\int_0^t B_u\,dB_u = \frac{B_t^2}{2} - \frac{t}{2}$.

3. (a) Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and S_t be an Ito process satisfying $dS_t = \alpha S_t\,dt + \sigma S_t\,dB_t$. Let $Y_t = S_t^3$. Show that Y_t satisfies the SDE
$$dY_t = \left(3\alpha + 3\sigma^2\right)Y_t\,dt + 3\sigma Y_t\,dB_t$$
   Deduce that Y_t is a geometric Brownian motion, and write down its drift and volatility.

   (b) Within the Black-Scholes model, show that the price $F(t, S_t)$ at time $t \in [0,T]$ of the contingent claim $\Phi(S_T) = S_T^3$ is given by
$$F(t, S_t) = S_t^3 e^{2r(T-t)+3\sigma^2(T-t)}.$$

   (c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = S_T^3$.

      i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

      ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?