

## MASx52: Assignment 4

Solutions and discussion are written in blue. A sample mark scheme, with a total of 35 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Let  $B_t$  be a standard Brownian motion.

- (a) Write down the distribution of  $B_t$ , and write down  $\mathbb{E}[B_t]$  and  $\mathbb{E}[B_t^2]$ .
- (b) Let  $0 \leq u \leq t$ . Show that  $\mathbb{E}[(B_t - B_u)^2 \mid \mathcal{F}_u] = t - u$ .

*Solution.*

- (a)  $B_t \sim N(0, t)$ , [1] and  $\mathbb{E}[B_t] = 0$ , [1]  $\mathbb{E}[B_t^2] = t$ . [1]
- (b) We have

$$\begin{aligned}\mathbb{E}[(B_t - B_u)^2 \mid \mathcal{F}_u] &= \mathbb{E}[(B_t - B_u)^2] \\ &= t - u.\end{aligned}$$

[1] In the first line we use that, by the properties of Brownian motion,  $B_t - B_u$  is independent of  $\mathcal{F}_u$ . [1] Then, we use that  $B_t - B_u \sim N(0, t - u)$ , which is the same distribution as  $B_{t-u}$  [1], followed by the third formula in part (a) with  $t - u$  in place of  $t$ . [1]

2. Write down the following stochastic differential equations in integral form, over the time interval  $[0, t]$ .

- (a)  $dX_t = 2(X_t + 1) dt + 2B_t dB_t$ .
- (b)  $dY_t = 3Y_t dt$ .

Write down a differential equation satisfied by  $Y_t$ , and find its solution with the initial condition  $Y_0 = 1$ .

Suppose that  $X_0 = 1$ . Show that  $f(t) = \mathbb{E}[X_t]$  satisfies  $f'(t) = 2f(t) + 2$  and hence find  $f(t)$ .

*Solution.*

- (a) We have

$$X_t = X_0 + \int_0^t 2(X_u + 1) du + \int_0^t 2B_u dB_u.$$

[2]

- (b) We have

$$Y_t = Y_0 + \int_0^t 3Y_u du.$$

[2]

Differentiating (b), by the fundamental theorem of calculus we have

$$\frac{dY_t}{dt} = 3Y_t$$

[1] with solution  $Y_t = Ae^{3t}$ . Since  $Y_0 = 1$  we have  $A = 1$ . [1]

In (a), taking expectations we have

$$\begin{aligned}\mathbb{E}[X_t] - \mathbb{E}[X_0] &= \int_0^t 2\mathbb{E}[X_u] + 2 \, du + 0 \\ f(t) - f(0) &= \int_0^t 2f(u) + 2 \, du\end{aligned}$$

because Ito integrals are zero mean martingales. [1] Differentiating this equation, by the fundamental theorem of calculus we have

$$f'(t) = 2f(t) + 2$$

which has solution  $f(t) = Ce^{2t} - 1$ . [1]

Putting in  $t = 0$  gives  $f(0) = 1 = C - 1$ , so we obtain  $f(t) = 2e^{2t} - 1$ . [1]

3. Use Ito's formula to calculate the stochastic differential of  $dZ_t$  where

(a)  $Z_t = tB_t$

(b)  $Z_t = 1 + t^2X_t$  where  $dX_t = \mu dt + \sigma B_t dB_t$  and  $\mu, \sigma$  are deterministic constants.

(c)  $Z_t = e^{-2t}S_t$  where  $dS_t = 2S_t dt + 5S_t dB_t$ .

In which cases is  $Z_t$  is a martingale?

*Solution.* We have

(a)

$$\begin{aligned}dZ_t &= \{(B_t) + (0)(t) + \frac{1}{2}(1)^2(1)\} dt + (t)(1) dB_t \\ &= B_t dt + t dB_t.\end{aligned}$$

[3]

(b)

$$\begin{aligned}dZ_t &= \{2tX_t + (\mu)(t^2) + \frac{1}{2}(\sigma B_t)^2(0)\} dt + (\sigma B_t)(t^2) dB_t \\ &= (2tX_t + \mu t^2) dt + \sigma t^2 B_t dB_t.\end{aligned}$$

[4]

(c)

$$\begin{aligned}dZ_t &= \{(-2e^{-2t}S_t) + (2S_t)(e^{-2t}) + \frac{1}{2}(5S_t)^2(0)\} dt + (5S_t)(e^{-2t}) dB_t \\ &= 5e^{-2t}S_t dB_t.\end{aligned}$$

[4]

Case (c) is a martingale because here  $dZ_t$  has only a  $(\dots)dB_t$  term, and therefore  $Z_t = Z_0 + \int_0^t \dots dB_t$  is a martingale because Ito integrals are martingales. [1]

4. Let  $S_t$  be a geometric Brownian motion, with drift  $\mu \in \mathbb{R}$ , volatility  $\sigma > 0$ , and (deterministic) initial condition  $S_0$ .

- (a) Find  $\mathbb{E}[S_t]$  and deduce that  $S_t$  is not a Brownian motion when  $\mu \neq 0$ .  
(b) Is  $S_t$  a Brownian motion when  $\mu = 0$ ?

*Solution.*

- (a) The formula for geometric Brownian motion is

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right).$$

So, taking expectations, using the formula for  $\mathbb{E}[e^Z]$  where  $Z$  is normally distributed, and using that  $S_0$  is deterministic,

$$\begin{aligned} \mathbb{E}[S_t] &= S_0 e^{(\mu - \frac{\sigma^2}{2})t} \mathbb{E}[e^{\sigma B_t}] \\ &= S_0 e^{(\mu - \frac{\sigma^2}{2})t} e^{\frac{\sigma^2 t}{2}} \\ &= S_0 e^{\mu t} \end{aligned}$$

[3] A Brownian motion  $B_t$  has  $\mathbb{E}[B_t] = \mathbb{E}[B_0]$ , but for  $\mu \neq 0$  we have shown that  $\mathbb{E}[S_t]$  is non-constant, which means that  $S_t$  cannot be a Brownian motion. [2]

- (b) It remains to consider the case  $\mu = 0$ . In this case,  $S_t = S_0 e^{\sigma B_t - \frac{\sigma^2}{2}t}$ . We recall that, for a Brownian motion,  $B_t^2 - t$  is a martingale, [1] and for  $S_t$  we have  $S_t^2 - t = S_0^2 e^{2\sigma B_t - \sigma^2 t} - t$ . This gives us

$$\begin{aligned} \mathbb{E}[S_t^2 - t] &= S_0^2 \mathbb{E}[e^{2\sigma B_t}] e^{-\sigma^2 t} - t \\ &= S_0^2 e^{\frac{4\sigma^2}{2}t} e^{-\sigma^2 t} - t \\ &= S_0^2 e^{\sigma^2 t} - t \end{aligned}$$

[2] which is clearly non-constant. Hence  $S_t^2 - t$  is not a martingale, so  $S_t$  is not a Brownian motion. [1]

[Note: There are *lots* of other ways to solve this question!]