MASx52: Assignment 2

1. Let (X_n) be a sequence of i.i.d. random variables, each with a uniform distribution on $[-1,1]$. Define

$$S_n = \sum_{i=1}^{n} X_i,$$

where $S_0 = 0$. Let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

   (a) Show that S_n is a martingale, with respect to the filtration $\mathcal{F}_n$.

   (b) Find $\mathbb{E}[S_3^2 \,|\, \mathcal{F}_2]$ in terms of X_2 and X_1, and hence show that

$$\mathbb{E}[S_3^2 \,|\, \mathcal{F}_2] = S_2^2 + \frac{1}{3}.$$

   (c) Write down a deterministic function $f : \mathbb{N} \to \mathbb{R}$ such that

$$M_n = S_n^2 - f(n)$$

   is a martingale (justification is not required).

2. Consider the one-period market with $r = \frac{1}{10}$, $s = 2$, $d = \frac{1}{2}$ and $u = 3$, in our usual notation. A contract specifies that

   The holder of the contract will sell 2 *units of stock, and be paid* K *units of cash, at time* 1.

   (a) Explain briefly why the contingent claim of this contract is

$$\Phi(S_1) = K - 2S_1.$$

   (b) Find a replicating portfolio h for this contingent claim.

   (c) Write down the value V_0^h of h at time 0.

   (d) Find the numerical values of risk-neutral probabilities

$$q_u = \frac{(1+r)-d}{u-d} \quad \text{and} \quad q_d = \frac{u-(1+r)}{u-d}.$$

   Hence, show that $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)] = V_0^h$.

   (e) For which K does the contract have value zero at time 0?