MAS350: Assignment 1

Solutions and discussion are written in blue. A sample mark scheme, with a total of 30 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Recall that the Borel σ-field $\mathcal{B}(\mathbb{R})$ is the smallest σ-field on $\mathbb{R}$ containing all open intervals (a,b) with $-\infty < a < b < \infty$. Define

$$A = \bigcup_{n=1}^{N} [a_n, b_n]$$

where $a_1 \le b_1 < a_2 \le b_2 < a_3 \le b_3 < \dots$ are real numbers.

(a) Prove, starting from the definition given above, that $A \in \mathcal{B}(\mathbb{R})$.

(b) Write down a formula for the Lebesgue measure of A, in terms of the a_i and b_i. Is your formula valid if $N = \infty$?

(c) Consider the following claims.

(i) The Borel σ-field is an infinite set.

(ii) The Borel σ-field contains an infinite number of infinite sets.

(iii) All countable sets are Borel sets with zero Lebesgue measure.

(iv) All Borel sets with positive Lebesgue measure contain at least one open interval.

(v) The Cantor set is a Borel set.

(vi) The Cantor set has Lebesgue measure zero.

In each case (i)-(vi), state whether you believe the claim to be true or false. For claims that you believe are true, give a proof. For claims that you believe are false, give a counterexample. Use parts (a) and (b) to support your arguments.

Solution.

(a) For $b \in \mathbb{R}$, since $(b,n) \in \mathcal{B}(\mathbb{R})$ for all $n > b$, also $\cup_n (b,n) = (b,\infty) \in \mathcal{B}(\mathbb{R})$. [1] Similarly $(-\infty, a) = \cup_n(-n,a) \in \mathcal{B}(\mathbb{R})$ for all a.
Hence, $[a,b] = \mathbb{R} \setminus ((-\infty,a) \cup (b,\infty) \in \mathcal{B}(\mathbb{R})$. [1]
Hence also $A = \cup_{i=1}^{N}[a_i,b_i] \in \mathcal{B}(\mathbb{R})$. [1]
Pitfall: Note that here we are using a particular definition of the Borel sets, which doesn't immediately tell us that half-open intervals such as (a,∞) are Borel. We can deduce it easily, however. There are many different equivalent definitions.

(b) We have

$$\lambda(A) = \sum_{n=1}^{N}(b_n - a_n).$$

[1] By countable additivity of disjoint sets (i.e. the third property in the definition of a measure) this formula is valid when $N = \infty$. [1]

(c) (i) True. For example, $\mathcal{B}(\mathbb{R})$ contains each of the sets $(x, x+1)$, for $x \in \mathbb{R}$, and there are infinitely many of these. [1]

(ii) True. We can use the same example as in (i), because each of the sets $(x, x+1)$ is infinite. [1] *Pitfall:* Make sure you keep track of the difference between a set and a set of sets.

(iii) True. [1] If a set A is countable, the we may write it in the form $A = \cup_{n=1}^{\infty}[a_n, a_n]$. By part (a) this means A is a countable union of Borel sets, and hence is itself Borel. Our formula from part (b) shows that A has Lebesgue measure zero. [1]

(iv) False. [1] Recall that $\mathbb{Q}$ is countable, and hence also a Borel set by the previous part. Hence the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are a Borel set. Since $\lambda(\mathbb{Q}) = 0$ we have $\lambda(\mathbb{R} \setminus \mathbb{Q}) = \infty$, but the irrational numbers do not contain any open intervals. [1]

(v) True. Recall the iterative 'middle third' construction of the Cantor set as $C = \cap_n C_n$ (see lecture notes), where $C_1 = [0,1]$ and C_{n+1} is constructed from C_n by removing the middle third of each closed interval. [1] Thus C_n is the disjoint union of 2^n closed intervals, and we can write it in the form $\bigcup_{n=1}^{N}[a_n, b_n]$. Thus C_n is Borel by part (a), and since σ-fields are closed under countable intersections, we have $C \in \mathcal{B}(\mathbb{R})$ too. [1]

(vi) True. In the iterative 'middle third' construction of the Cantor set as $C = \cap_n C_n$, the n^{th} stage C_n is a union of 2^n disjoint closed intervals each with length 3^{-n}. Using part (b), the Lebesgue measure of C_n is therefore $(\frac{2}{3})^n$. [1] Since the first stage $C_1 = [0,1]$ has finite measure, in fact $\lambda(C_1) = 1$, this means $\lambda(C) = \lim_n \lambda(C_n) = \lim_n (\frac{1}{3})^n = 0$. [1]

2. Let λ denote Lebesgue measure and let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-field on $\mathbb{R}$. This question concerns examples of decreasing sequences of Borel sets (B_n) and measures m on $\mathcal{B}(\mathbb{R})$ such that

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) \neq \lim_{N\to\infty} m\left(\bigcap_{n=1}^{N} B_n\right).$$

(a) Taking $m = \lambda$, show that $B_n = (-\infty, -n]$ is an example of this type.

(b) Find a second example, with the additional property that $\cap_{n=1}^{\infty} B_n$ is non-empty.

(c) Find a third example, with the additional property that B_1 is countable.

Solution.

(a) We have $\lambda(B_n) = \sum_{j=n}^{\infty} \lambda((-j-1, -j]) = \infty$ and thus $\lim_n \lambda(B_n) = \infty$, [1] but $\cap_n(-\infty, -n] = \emptyset$ which has measure zero. [1]

(b) Take e.g. $B_n = (-\infty, -n] \cup [0,1]$. Then $\lambda(B_n) = \infty$ as before, but now $\cap_n B_n = [0,1]$ which is non-empty with Lebesgue measure 1. [1]

(c) Take m to be counting measure on $\mathbb{N}$ (the σ-field can be $\mathcal{P}(\mathbb{N})$ here) and let $B_n = \{n, n+1, \ldots, \infty\}$. and then $m(B_n) = \infty$ but $m(\cap_n B_n) = m(\emptyset) = 0$. [1]

Pitfall: Remember the conditions of the theorem! In general, $m(\cap_n B_n) = \lim_n m(B_n)$ for decreasing B_n *only* if $m(B_1)$ is finite. Once you remember this, you know to start by trying (any) example where $m(B_1)$ is infinite, and from there you don't have far to go.

3. Let S be a finite set and Σ be a σ-field on S. Consider the set

$$\Pi = \{A \in \Sigma \ : \ \text{if } B \in \Sigma \text{ and } B \subseteq A \text{ then either } B = A \text{ or } B = \emptyset\}. \qquad (\star)$$

(a) Show that Π is a finite set.

(b) Using (a), let us enumerate the elements of Π as $\Pi = \{\Pi_1, \Pi_2, \cdots, \Pi_k\}$, where each Π_i is distinct from the others.

(i) Show that $\Pi_i \cap \Pi_j = \emptyset$ for $i \neq j$. *Hint: Could $\Pi_i \cap \Pi_j$ be an element of Π?*

(ii) Show that $\cup_{i=1}^{k} \Pi_i = S$. *Hint: If $C = S \setminus \cup_{i=1}^{k} \Pi_i$ is non-empty, is $C \in \Pi$?*

(iii) Let $A \in \Sigma$. Show that

$$A = \bigcup_{i \in I} \Pi_i$$

where $I = \{i = 1, \dots, k \ : \ A \cap \Pi_i \neq \emptyset\}$.

Solution.

(a) Note that each element of Π is a subset of S. Hence Π itself is a subset of the power set $\mathcal{P}(S)$ of S. [1] Since S is a finite set, $\mathcal{P}(S)$ is also a finite set, hence Π is also finite. [1]

Pitfall: Part (b) of this question is difficult. It requires you to keep a very clear head. To solve a question like this you have to explore what you have deduce from what else, with lots of thinking 'if I knew this then I would also know that' and then trying to fit a bigger picture together, connecting your start point to your desired end point. Analysis can often be like this.

(b) i. Suppose $\Pi_i \cap \Pi_j \neq \emptyset$. Note that $\Pi_i \cap \Pi_j$ is a subset of both Π_i and Π_j. [1]

By definition of Π, any subset of Π_i is either equal to Π_i or is equal to $\emptyset$. Since we assume that $\Pi_i \cap \Pi_j \neq \emptyset$, we therefore have $\Pi_i = \Pi_i \cap \Pi_j$. [1] Similarly, $\Pi_j = \Pi_i \cap \Pi_j$.

Hence $\Pi_i = \Pi_j$, but this contradicts the fact that the Π_i are distinct from each other. [1] Thus we have a contradiction and in fact we must have $\Pi_i \cap \Pi_j = \emptyset$.

ii. By definition of Π we have $\cup_{i=1}^{k} \Pi_i \subseteq S$. Suppose $\cup_{i=1}^{k} \Pi_i \neq S$. Then $C = S \setminus \cup_{i=1}^{k} \Pi_i$ is a non-empty set in Σ.

Since C is disjoint from all the Π_i, we must have $C \notin \Pi$. [1] Noting that $C \in \Sigma$, by definition of Π this implies that there is some[1] $B_1 \subset C$ such that $B_1 \neq \emptyset$. [1]

We have that B_1 is disjoint from all the Π_i, so we must have $B_1 \notin \Pi$. Thus by the same reasoning (as we gave for C) there exists $B_2 \subset B_1$ such that $B_2 \neq \emptyset$. Iterating, we construct an infinite decreasing sequence of sets $C \supset B_1 \supset B_2 \supset B_3 \dots$ each strictly smaller than the previous one, none of which are empty. However, this is impossible because $C \subseteq S$ is a finite set. [1]

iii. Let $i \in I$. So $\Pi_i \cap A \neq \emptyset$. Noting that $\Pi_i \cap A \subseteq \Pi_i$, by definition of Π we must have $\Pi_i \cap A = \Pi_i$. That is, $\Pi_i \subseteq A$. Since we have this for all $i \in I$, we have $\cup_{i \in I} \Pi_i \subseteq A$. [1]

Now suppose that $A \setminus \cup_{i \in I} \Pi_i \neq \emptyset$. Since by (ii) we have $S = \cup_{i=1}^{k} \Pi_i$, and the union is disjoint by (i), this means that there is some Π_j with $j \notin I$ such that $A \cap \Pi_j \neq \emptyset$. [1] However $A \cap \Pi_j \subseteq \Pi_j$ so by definition of Π we must have $\Pi_j \cap A = \Pi_j$. That is $\Pi_j \subseteq A$, but then we would have $j \in I$, which is a contraction. [1]

Thus $A \setminus \cup_{i \in I} \Pi_i$ must be empty, and we conclude that $A = \cup_{i \in I} \Pi_i$.

[1] $X \subset Y$ means that $X \subseteq Y$ and $X \neq Y$ i.e. X is *strictly* smaller than the set Y