MASx52: Assignment 5

Solutions and discussion are written in blue. A sample mark scheme, with a total of 45 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Consider the SDE
$$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$

   (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation
   $$f'(t) = t + f(t)$$
   Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

   (b) Let $Y_t = X_t^2$. Show that
   $$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

   (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation
   $$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

Solution.

(a) Writing in integral form we have
$$X_t = X_0 + \int_0^t (u + X_u)\,du + \int_0^2 2u\,dB_u.$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],
$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t (u + X_u)\,du\right] + \mathbb{E}\left[\int_0^2 2u\,dB_u\right] \\
&= \mathbb{E}[X_0] + \int_0^t \mathbb{E}[u + X_u]\,du + 0 \\
&= \mathbb{E}[X_0] + \int_0^t u + \mathbb{E}[X_u]\,du \\
f(t) &= f(0) + \int_0^t u + f(u)\,du.
\end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]
$$f'(t) = t + f(t).$$

If we set $f(t) = Ce^t - t - 1$ then $f'(t) = Ce^t - 1$ [1], so clearly this is a solution.

(b) Using Ito's formula [1] we have

$$\begin{aligned}
dY_t &= \left(0 + (t + X_t)(2X_t) + \frac{1}{2}(2t)^2(2)\right) dt + (2t)(2X_t)\, dB_t \\
&= 2\left(2t^2 + tX_t + X_t^2\right) dt + 4tX_t\, dB_t
\end{aligned}$$

[3]

(c) Writing in integral form we have

$$Y_t = Y_0 + 2\int_0^t 2u^2 + uX_u + X_u^2\, du + \int_0^t 4uX_u\, dB_u$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}[Y_0] + 2\mathbb{E}\left[\int_0^t 2u^2 + uX_u + X_u^2\, du\right] + \mathbb{E}\left[\int_0^t 4uX_u\, dB_u\right] \\
&= \mathbb{E}[Y_0] + \int_0^t 2\mathbb{E}\left[2u^2 + uX_u + X_u^2\right] du + 0 \\
&= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + u\mathbb{E}\left[X_u\right] + \mathbb{E}\left[X_u^2\right] du \\
&= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + uf(u) + v(u)\, du
\end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

2. (a) Within the Black-Scholes model, use the risk neutral valuation formula find the prices at time t of the contingent claims

   i. $\Phi(S_T) = 3S_T + 5$, where $0 \le t \le T$.

   ii. $\Psi(S_T) = S_1 S_T + 1$, where $1 \le t \le T$.

   (b) With the same contingents claims as in (a):

   i. Describe a constant portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.

   ii. Is it possible to replicate $\Psi(S_T)$ using a constant portfolio?

   (c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = 3S_T + 5$. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

Solution.

(a) i. Using the explicit formula for geometric Brownian motion (see the formula sheet) we obtain

$$
\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_T+5\,|\,\mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)}+5\,|\,\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)}\,|\,\mathcal{F}_t\right]+5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)}\right]+5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)}+5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{r(T-t)}+5\right) \\
&= 3S_t+5e^{-r(T-t)}
\end{aligned}
$$

[4] Here, we use that S_t is $\mathcal{F}_t$ measurable,[1] and that $Z=\sigma(B_T-B_t)\sim N(0,\sigma^2(T-t)$ is independent of $\mathcal{F}_t$. [1] We use the formula sheet to provide an explicit formula for $\mathbb{E}[e^Z]$.

ii. Assuming $1\le t\le T$, we have

$$
\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_1S_T+1\,|\,\mathcal{F}_t\right] &= e^{-r(T-t)}\left(S_1\mathbb{E}^{\mathbb{Q}}[S_T\mathcal{F}_t]+1\right) \\
&= S_1e^{rt}e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T\mathcal{F}_t]+e^{-r(T-t)} \\
&= S_1e^{rt}e^{-rt}S_t+e^{-r(T-t)} \\
&= S_1S_t+e^{-r(T-t)}.
\end{aligned}
$$

[2] Here we use that $S_1\in\mathcal{F}_t$ for $t\ge 1$, [1] and the fact (from Lemma 14.4.1 in lectures) that $M_t=e^{-rt}S_t$ is a martingale in the risk-neutral world.[1]

(b) i. At time 0, we buy three units of stock [1] and $5e^{-rT}$ in cash. [1] It's value at time t is then

$$3S_t+5e^{-rT}e^{rt}=\Phi(S_T).$$

Therefore, this portfolio replicates $\Phi(S_T)$ for all $t\in[0,T]$. [1]

ii. It isn't possible to replicate $\Psi(S_T)$ with a constant portfolio. [1] The replicating portfolio provided by Theorem 14.3.1 is unique, and contains a stock component $y_t=\frac{\partial F}{\partial s}(t,S_t)$ where $F(t,s)$ is the pricing formula obtained in (a.ii); in this case for $t\ge 1$ we have $F(t,s)=S_1s+e^{-r(T-t)}$ so y_t is non-constant. [1]

(c) The value of our portfolio at time t is given by $F(t,S_t)$, where F is as in part (a). If we add an amount α of stock into our portfolio then its new value will be $V(t,S_t)=F(t,S_t)+\alpha S_t$. [1] To achieve delta neutrality, we want to choose α such that

$$0=\frac{\partial V}{\partial s}(0,S_0)=3+\alpha.$$

[1] Hence $\alpha=-3$. [1]

3. **[On Semester 1]** Consider an urn, containing two colours of balls, black and red. At time $n=0$, the urn contains one black ball and one red ball. Then, at each time $n=1,2,\ldots$, we do the following:

- Draw a ball from the urn. Record the colour of this ball and place it back into the urn.
- Add two new balls to the urn, of the same colour as the drawn ball.

Therefore, at time n, the urn contains $2 + 2n$ balls. Let B_n denote the number of red balls in the urn, and let

$$M_n = \frac{B_n}{2 + 2n}.$$

(a) Show that M_n is a martingale, with respect to the filtration $\mathcal{F}_n = \sigma(B_i \,:\, i \leq n)$.

(b) Deduce that there exists a random variable M_∞ such that $M_n \stackrel{a.s.}{\to} M_\infty$.

(c) Show that $\mathbb{P}[M_n \leq \frac{1}{2}] = \mathbb{P}[M_n \geq \frac{1}{2}]$ for all n.

Solution.

(a) Since $M_n \in [0, 1]$ we have that $\mathbb{E}[|M_n|] \leq 1$, so $M_n \in L^1$. [1]
Since $B_n \in m\mathcal{F}_n$, we have $M_n \in m\mathcal{F}_n$. [1]
From the dynamics of the urn, we have

$$B_{n+1} = \mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}}(B_n + 2) + \mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}}B_n.$$

[1] We calculate

$$\begin{aligned}
\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] &= \mathbb{E}\left[\frac{B_{n+1}}{4+2n} \,\middle|\, \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}}(B_n + 2)}{4+2n} + \frac{\mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}}B_n}{4+2n} \,\middle|\, \mathcal{F}_n\right] \\
&= \frac{B_n+2}{4+2n}\mathbb{E}\left[\mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}} \,|\, \mathcal{F}_n\right] + \frac{B_n}{4+2n}\mathbb{E}\left[\mathbb{1}_{\{(n+1)^{th} \text{ draw is black}\}} \,|\, \mathcal{F}_n\right] \\
&= \frac{B_n+2}{4+2n}\frac{B_n}{2+2n} + \frac{B_n}{4+2n}\frac{2n+2-B_n}{2+2n} \\
&= \frac{B_n^2 + 2B_n + (2n+2)B_n - B_n^2}{(4+2n)(2+2n)} \\
&= \frac{(4+2n)B_n}{(4+2n)(2+2n)} \\
&= \frac{B_n}{2n+2} \\
&= M_n.
\end{aligned}$$

[3] Here we use that $B_n \in \mathcal{F}_n$ to take out what is known, [1] and to calculate the probabilities that the $(n+1)^{th}$ draw is red or black given knowledge of B_n.
Thus (M_n) is a martingale.

(b) Since $\mathbb{E}[|M_n|] \leq 1$, we have that (M_n) is bounded in L^1. [1]
Hence, the (first version of the) martingale convergence theorem applies, [1] with the consequence that there exists a random variable M_∞ such that $M_n \stackrel{a.s.}{\to} M_\infty$.

(c) The key point here is the roles of the colours red and black are symmetric: if we swapped the colours red and black (i.e. all red balls became black, and all black balls became red), then we would obtain an urn with *exactly* the same distribution as we started with. [1]
Let B'_n denote the number of black balls within the urn at time n, and write

$$M'_n = \frac{B'_n}{2+2n} = 1 - \frac{B_n}{2+2n}.$$

Note that $M_n + M_n' = 1$. [1] By the symmetry between red and black, M_n and M_n' have the same distribution. [1]

Hence,

$$\mathbb{P}[M_n \leq \tfrac{1}{2}] = \mathbb{P}[M_n' \leq \tfrac{1}{2}] = \mathbb{P}[1 - M_n \leq \tfrac{1}{2}] = \mathbb{P}[M_n \geq \tfrac{1}{2}]$$

as required. [1]