Standard Distributions

There are many distributions which occur in many different contexts; you will already have encountered met some of them in MAS113.

Each of these ‘distributions’ is really a family of distributions, sharing a common formula for the p.f. or p.d.f., with one or more **parameter(s)**.

For example, the binomial distribution $Bi(n, p)$ has two parameters, n, the number of trials, and p, the success probability.

Standard Distributions

Standard distributions are important either because

- they arise from simple models
  (e.g. the binomial distribution from Bernoulli trials)
- or because they have special mathematical properties
  (e.g. the normal distribution and the central limit theorem).

On the MAS223 website, you can find a handout with formulas and properties of standard distributions. This handout will also be available in the exam.

Standard discrete distributions

You will already have met some of the most important discrete distributions in first-year courses:

- The Bernoulli distribution
- The Binomial distribution
- The Geometric distribution
- The Poisson distribution

More standard discrete distributions

Example 3: the Hypergeometric distribution

Example 4: the Negative Binomial distribution

The univariate normal distribution

Again, you will have encountered the normal distribution in MAS113.

If X has a normal distribution with mean μ and variance σ^2, we write $X \sim N(\mu, \sigma^2)$, and the probability density function of X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}.$$

It can be shown by integration (see MAS113) that the mean and variance of a random variable with this p.d.f. really are μ and σ^2.

The standard normal distribution

The special case $N(0,1)$, with p.d.f.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\},$$

is referred to as the **standard normal distribution**.

Standardisation

An important property of the normal distribution family is that if $X \sim N(\mu, \sigma^2)$ and a and b are constants, then

$$aX + b \sim N(a\mu + b, a^2\sigma^2).$$

In particular X can be **standardised** by letting $Z = \frac{X-\mu}{\sigma}$, so that $Z \sim N(0, 1)$.

Sums of independent normal variables

Another important property is that if we have n **independent** normal random variables $X_1, X_2, \ldots X_n$ with $X_i \sim N(\mu_i, \sigma_i^2)$ then

$$\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

(We will look at situations where the normal random variables are not independent later in this course.)

Calculating $E(e^X)$

Example 5: Finding $E(e^X)$ where X has a Normal distribution

The gamma and beta functions

The gamma and beta functions appear in the probability density functions of certain standard distributions.

In this context they can be thought of as **normalising constants**, ensuring that the p.d.f.s integrate to 1.

Both are defined as integrals.

The gamma function

The **gamma function** can be thought of as a generalisation of the factorial: it is defined by

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du$$

for $\alpha > 0$.

Note that

$$\Gamma(1) = \int_0^\infty e^{-u}\, du = [-e^{-u}]_0^\infty = 1.$$

The gamma function and factorials

It is not hard to show using integration by parts that, for $\alpha > 1$,

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

and so if α is a positive integer then this may be iterated giving

$$\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)\Gamma(\alpha - 2) = \ldots = (\alpha - 1)!.$$

In general, the integration cannot be performed explicitly. One other specific value, which appears in some formulae for standard distributions, is $\Gamma(1/2) = \sqrt{\pi}$.

Similar integrals

Frequently in this course, we will encounter integrals of the form

$$\int_0^\infty u^{\alpha-1} e^{-\beta u}\, du.$$

These are similar to the integral defining the Gamma function above, but have an extra constant β. They can be related to the Gamma function by the following result.

Lemma

If $\beta > 0$, we have

$$\int_0^\infty u^{\alpha-1} e^{-\beta u}\, du = \frac{\Gamma(\alpha)}{\beta^\alpha}.$$

The beta function

In a similar way the **beta function** is defined by

$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, du$$

for $\alpha, \beta > 0$.

It can be shown that it can be expressed in terms of the gamma function as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The chi-squared distribution with 1 degree of freedom

Consider the distribution of X^2 when $X \sim N(0,1)$. This is a skew distribution, the **chi-squared distribution with** 1 **degree of freedom**, with p.d.f.

$$f(y) = \begin{cases} \frac{1}{\sqrt{2}\Gamma(1/2)} \frac{1}{\sqrt{y}} \exp\left(-\frac{y}{2}\right) & y > 0 \\ 0 & y < 0. \end{cases}$$

In this case we write $Y = X^2 \sim \chi_1^2$ (the chi-square distribution with 1 degree of freedom).

General chi-squared distribution

The previous distribution is a special case of a more general family of distributions, namely the **chi-square distributions with n degrees of freedom**, with p.d.f.

$$f(y) = \begin{cases} \frac{1}{\sqrt{2^n}\Gamma(n/2)} y^{n/2-1} \exp\left(-\frac{y}{2}\right) & y > 0 \\ 0 & y < 0. \end{cases}$$

This is the distribution of the sum of the squares of n independent standard normal random variables, i.e. if $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim N(0,1)$ and $Y = \sum_{i=1}^{n} X_i^2$, then $Y \sim \chi^2_n$.

The Gamma distribution: p.d.f.

The chi-squared distribution is a special case of the Gamma distribution.

Let $\alpha, \beta > 0$ and set

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

The distribution with this p.d.f. is called the **Gamma distribution** with parameters α and β; if X has this p.d.f. we write $X \sim Ga(\alpha, \beta)$.

The Gamma distribution: mean and variance

Example 6: Mean and variance of the Gamma distribution

The Gamma distribution: notes

- If $\alpha = 1$ we obtain the exponential distribution with parameter β.
- If $\alpha = \nu/2$ and $\beta = 1/2$ we obtain the chi squared distribution with ν degrees of freedom.
- If $X_1 \sim Ga(\alpha_1, \beta)$ and $X_2 \sim Ga(\alpha_2, \beta)$ and X_1 and X_2 are independent, then $X_1 + X_2 \sim Ga(\alpha_1 + \alpha_2, \beta)$. (See Example 19 later in the course.)
  It follows that a sum of independent exponential random variables with the same parameter has a Gamma distribution, and also that the sum of independent chi-squared random variables has a chi-squared distribution.

The Beta distribution: p.d.f.

Now we let

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise.

Again we start by showing that $f(x)$ is a p.d.f.

The Beta distribution

The distribution with this p.d.f. is called the **Beta distribution** with parameters α and β.

If X has this p.d.f. we can write $X \sim Be(\alpha, \beta)$.

The Beta distribution: mean and variance

Example 7: Mean and variance of the Beta distribution

The Beta distribution: notes

- The beta distribution can be useful for modelling random quantities which are naturally constrained to be in $[0, 1]$ (or, via suitable scaling, in any fixed interval).
- The $Be(1, 1)$ distribution is the same as the Uniform distribution on $[0, 1]$.

R

The aim of this section is to show how to use the computer package R to plot density and distribution functions of random variables.

Most of you will have seen R in use in Level 1 courses.

For a more detailed introduction to R, including information on how to install it on your own computer, see the separate handout "An Introduction to R".

Plotting p.d.f.s

To start with, we assume that R has been installed. Suppose we wish to plot the p.d.f. $f_X(x)$ of the random variable $X \sim N(0, 1)$.

The command we use here is `curve`, which creates a curve of a given function. The form of this command is

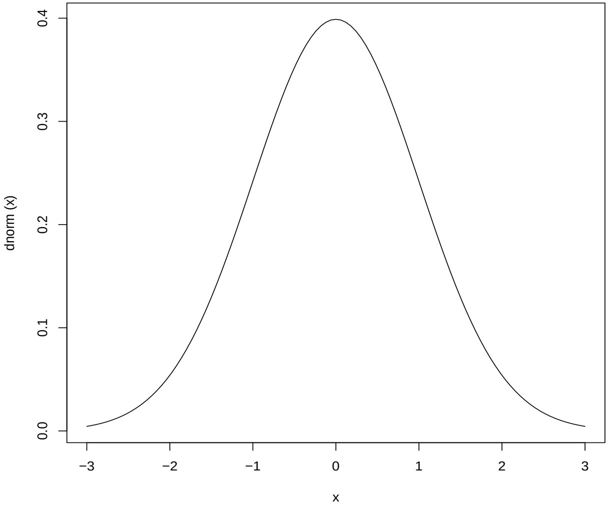
```
> curve(f(x), from="lower limit", to="upper
limit")
```

or just

```
> curve(f(x), "lower limit", "upper limit")
```

This tells R to plot a curve of a given function $y = f(x)$, where x takes values from “lower limit” to “upper limit”. If we

Results



More normal distributions

Similarly, one can produce plots of the p.d.f. of any normal variable, $X \sim N(\mu, \sigma^2)$, by using the command `dnorm(x,mean,sd)`. For example > `curve(dnorm(x,2,10),-10,14)` gives a plot of the p.d.f. of a $N(2, 100)$ variable.

More distributions

Similar plots can be obtained by finding out about the p.d.f.s of other distributions. It can be useful to use R's help system, which can be accessed with `help(topic)`.

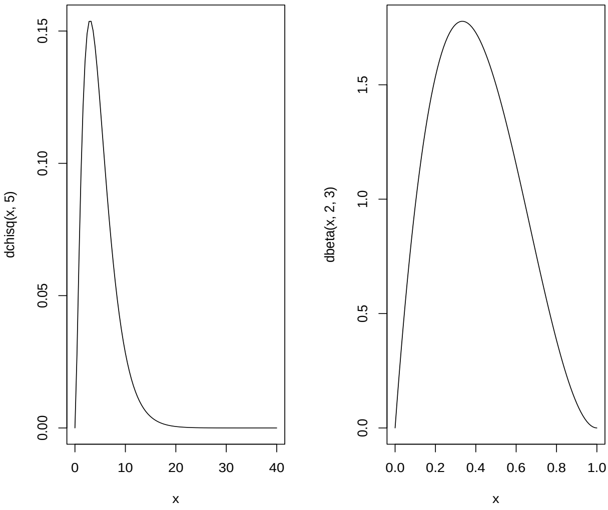
For the normal distribution use `dnorm`, for the chi-square use `dchisq`, for the Student t use `dt`, for the gamma distribution use `dgamma` (but check the definition of the p.d.f., because there are different ways of parametrising this distribution), for the beta distribution use `dbeta`, for the binomial use `dbinom`.

Example output

For example pictures of the p.d.f. of the chi-square $X \sim \chi_5$ and of the beta $Y \sim \mathrm{Beta}(2,3)$ distributions can be obtained by the following commands

```
> par(mfrow=c(1,2))
> curve(dchisq(x,5),0,40)
> curve(dbeta(x,2,3),0,1)
```

Example output



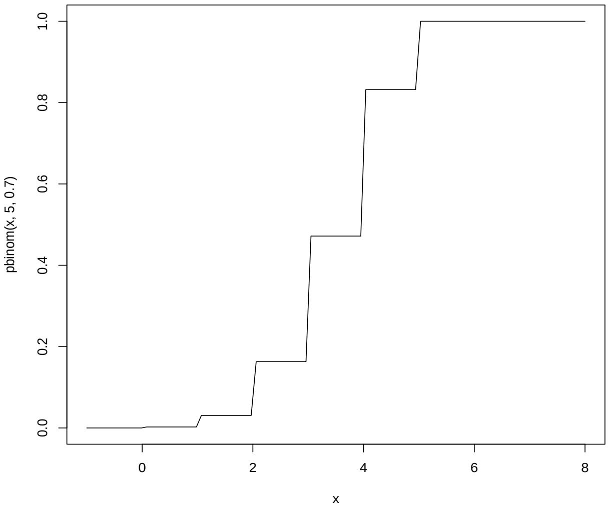
Graphs of distribution functions

For the above distributions, we can easily produce graphs of the distribution function by using `pnorm`, `pchisq`, etc. For example for the binomial, a graph of the distribution function can be obtained by the command

`> curve(pbinom(x,5,0.7),-1,8)` while the distribution function of the normal $N(0,1)$ can be produced by the command

```
> curve(pnorm,-5,5)
```

Results



Other distributions

If the distribution we wish to plot does not exist by default in `R`, then we can define it in `R` and plot it using the `curve` command as above. For more information on this, consult the on-line manuals of `R`.

Transformations

The general question here is: if we have a random variable X with a known distribution, and we have another random variable Y defined as $Y = g(X)$ for some function g, then what is the distribution of Y?

For example, earlier we claimed that if X is standard normal and $Y = X^2$ then Y has χ^2_1 distribution. Another example is that if $X \sim N(\mu, \sigma^2)$, then $Y = (X - \mu)/\sigma \sim N(0, 1)$.

In this section we will deal with transformations where both X and Y are continuous random variables.

Monotonic transformations

If g is strictly monotonic (i.e. increasing or decreasing), on the relevant range of x then there is a method operating directly with p.d.f.'s which we derive below.

Note that in this case g has an inverse function g^{-1} which is also increasing or decreasing as appropriate.

The increasing case

Firstly, in the increasing case, we may write

$$F_Y(y) = P(Y \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)).$$

Differentiating,

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y).$$

The decreasing case

Similarly in the decreasing case we have

$$F_Y(y) = P(Y \leq y) = P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)).$$

Hence

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y).$$

Combined result

To cover both cases, we write

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| .$$

Example

Example 8: Transformation of a Gamma distribution

Non-monotonic transformations

If g is not monotonic then we need to be more careful. This applies to the normal to chi squared transformation.

Example 9: Square of a standard normal