MAS350: Assignment 2

1. In each of the following cases, show that the given function is measurable, from $\mathbb{R} \to \mathbb{R}$ with the Borel σ-field. State clearly any results from lectures that you make use of.

   (a) $f(x) = \cos x$

   (b) $g(x) = \begin{cases} 0 & \text{for } x < 0 \\ x+1 & \text{for } x \geq 0. \end{cases}$

   (c) $h(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(x)}{n!}$

   (d) $i(x) = \lfloor x \rfloor$ (i.e. x rounded down to the nearest integer)

2. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

   (a) Let $f : S \to \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.

   (b) Let $f : S \to \mathbb{R}$ be a simple function. Write $f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$ where the A_i are pairwise disjoint and measurable and $c_i \geq 0$. Show that

   $$\left(\int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \qquad (\star)$$

   Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

   $$\frac{\left(\sum_{i=1}^{n} u_i\right)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}.$$

   (c) In this question you should give *two* different proofs that equation $(\star)$ holds when f is any non-negative measurable function. You may use your results from part (b) in both proofs.

      i. Give a proof using the monotone convergence theorem.

      ii. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.

   (d) Does $(\star)$ remain true if m is not necessarily a probability measure?