

## MAS350: Assignment 2

Solutions and discussion are written in blue. A sample mark scheme, with a total of 30 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. (a) Let  $O_1$  and  $O_2$  be open subsets of  $\mathbb{R}$ . Show that  $O_1 \cup O_2$  and  $O_1 \cap O_2$  are also open.  
(b) For each  $n \in \mathbb{N}$  let  $O_n$  be an open subset of  $\mathbb{R}$ . Consider the following claims:

- i.  $A = \bigcup_{n \in \mathbb{N}} O_n$  is open.  
ii.  $B = \bigcap_{n \in \mathbb{N}} O_n$  is open.

Which of these claims are true? Give a proof or a counterexample in each case.

- (c) A set  $C \subseteq \mathbb{R}$  is said to be *closed* if  $\mathbb{R} \setminus C$  is open. Which of your results from parts (a) and (b) hold for closed sets?

*Solution.*

- (a) Let  $x \in O_1 \cup O_2$ . Consider if  $x \in O_1$ , then there is an open interval  $I_1$  containing  $x$ . Thus  $I_1$  is an open interval within  $O_1 \cup O_2$  containing  $x$ . We can do the same for  $x \in O_2$ , then with  $x \in I_2 \subseteq O_2$ , hence  $O_1 \cup O_2$  is open. [1]

Now let  $x \in O_1 \cap O_2$ . Then for each  $i = 1, 2$  we have an open interval  $I_i \subseteq O_i$  containing  $x$ . [1] Let us write  $I_1 = (a_1, b_1)$ ,  $I_2 = (a_2, b_2)$ , and  $c_1 = \max(a_1, b_1)$ ,  $c_2 = \min(a_2, b_2)$ . Then  $(c_1, c_2) = I_1 \cap I_2$ , and since  $x \in I_1 \cap I_2$  we have  $x \in (c_1, c_2)$ . In particular this means  $c_1 < c_2$ , so  $I_1 \cap I_2$  is an open interval. [1] Also  $I_1 \cap I_2 \subseteq O_1 \cap O_2$ , so  $O_1 \cap O_2$  is open.

- (b) i. This is true. We can use exactly the same method as in part (a): let  $x \in \bigcup_n O_n$ , and assume  $x \in O_1$  (or use  $O_i$  in place of  $O_1$ ), then we have an open interval  $I_1 \subseteq O_1$  containing  $x$ , then  $I_1 \subseteq \bigcup_n O_n$ , and we are done. [1]  
ii. This is false. A counterexample is given by  $O_n = (\frac{-1}{n}, 1 + \frac{1}{n})$ , for which  $\bigcap_n O_n = [0, 1]$ . [1]

- (c) Let  $(C_n)_{n \in \mathbb{N}}$  be a sequence of closed sets. Then  $\mathbb{R} \setminus C_n$  is open, for each  $n$ . Using set operations we have

$$\mathbb{R} \setminus (C_1 \cup C_2) = (\mathbb{R} \setminus C_1) \cap (\mathbb{R} \setminus C_2)$$

$$\mathbb{R} \setminus (C_1 \cap C_2) = (\mathbb{R} \setminus C_1) \cup (\mathbb{R} \setminus C_2)$$

$$\mathbb{R} \setminus \left( \bigcup_n C_n \right) = \bigcap_n (\mathbb{R} \setminus C_n)$$

$$\mathbb{R} \setminus \left( \bigcap_n C_n \right) = \bigcup_n (\mathbb{R} \setminus C_n)$$

The first two equations combined with part (a) tell us that both the results of part (a) carry over to closed sets: both  $C_1 \cap C_2$  and  $C_1 \cup C_2$  are closed. [1]

From the fourth equation, since  $\mathbb{R} \setminus C_n$  is open (for all  $n$ ), using (b)(i) we see that  $\mathbb{R} \setminus (\bigcup_n C_n)$  is also open, hence  $\bigcap_n C_n$  is closed. [1]

However, we can't do the same for the third equation, because (b)(ii) was false. [1] Instead, we can take complements of our counterexample in (b)(ii) to find a counterexample here, giving  $C_n = \mathbb{R} \setminus (\frac{-1}{n}, 1 + \frac{1}{n}) = (-\infty, \frac{-1}{n}] \cup [1 + \frac{1}{n}, \infty)$ . Then  $\cup_n C_n = (-\infty, 0) \cup (1, \infty)$  which is not closed (because its complement  $[0, 1]$  is not open). [1]

2. In each of the following cases, show that the given function is measurable, from  $\mathbb{R} \rightarrow \mathbb{R}$  with the Borel  $\sigma$ -field. State clearly any results from lectures that you make use of.

(a)  $f(x) = x$

(b)  $g(x) = \cos x$

(c)  $h(x) = \begin{cases} 0 & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0. \end{cases}$

*Solution.*

- (a) We'll use the (original) definition of a measurable function. [1] With  $f(x) = x$ ,  $f^{-1}((c, \infty)) = (c, \infty)$ , which is a measurable set, so  $f$  is a measurable function. [1]  
(b) From lectures, every continuous function from  $\mathbb{R}$  to  $\mathbb{R}$  is measurable. [1] Since  $\cos$  is continuous, it is measurable. [1]  
(c) Let  $g_1(x) = \mathbb{1}_{[0, \infty)}(x)$  be the indicator function of  $[0, \infty)$ , which is measurable because it is the indicator function of a measurable set. [1] Let

$$g_2(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

which is measurable because it is continuous. Then  $g(x) = g_1(x) + g_2(x)$  is measurable, because the sum of measurable functions is measurable. [1]

*Pitfall:* Make sure to specify which results (from lectures) you use to make your deductions.

3. Let  $(S, \Sigma, m)$  be a measure space, and suppose that  $m$  is a probability measure.
- (a) Let  $f : S \rightarrow \mathbb{R}$  be a non-negative simple function. Show that  $f^2$  is also a non-negative simple function.
- (b) Let  $f : S \rightarrow \mathbb{R}$  be a simple function. Write  $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$  where the  $A_i$  are pairwise disjoint and measurable and  $c_i \geq 0$ . Show that

$$\left( \int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \quad (\star)$$

*Hint:* You may use Titu's lemma, which states that for  $u_i \geq 0$  and  $v_i > 0$ ,

$$\frac{(\sum_{i=1}^n u_i)^2}{\sum_{i=1}^n v_i} \leq \sum_{i=1}^n \frac{u_i^2}{v_i}.$$

- (c) In this question you should give *two* different proofs that equation  $(\star)$  holds when  $f$  is any non-negative measurable function.

- i. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.
  - ii. Give a proof using the monotone convergence theorem.
- (d) Does  $(\star)$  remain true if  $m$  is not necessarily a probability measure?

*Solution.*

- (a) We have

$$f^2 = \sum_{i=1}^n \sum_{j=1}^m c_i c_j \mathbb{1}_{A_i} \mathbb{1}_{A_j} = \sum_{i=1}^n c_i^2 \mathbb{1}_{A_i}$$

where the second inequality follows by disjointness – all the cross terms are zero. [1] We have thus expressed  $f^2$  as a simple function, and since  $c_i^2$  are non-negative,  $f^2$  is also non-negative. [1]

- (b) We have

$$\left( \int f \, dm \right)^2 = \left( \sum_{i=1}^n c_i m(A_i) \right)^2,$$

$$\int f^2 \, dm = \sum_{i=1}^n c_i^2 m(A_i).$$

[2] The required inequality follows from the above and Titu's lemma, taking  $v_i = m(A_i)$  and  $u_i = c_i m(A_i)$ . [1] Note that, because  $m$  is a probability measure,  $\sum_i m(A_i) = 1$  and we may assume  $m(A_i) > 0$  (because any  $A_i$  with zero measure have no effect on the value of the integral).

*Follow-up exercise:* See if you can derive Titu's lemma from the real version of the Cauchy-Schwarz inequality.

- (c) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be non-negative and measurable.

**First proof:** Recall that the definition of the Lebesgue integral, for non-negative measurable functions, is

$$\int f \, dm = \sup \left\{ \int s \, dm : s \text{ is simple and } 0 \leq s \leq f \right\}.$$

Hence

$$\begin{aligned} \left( \int f \, dm \right)^2 &= \left( \sup \left\{ \int s \, dm : s \text{ is simple and } 0 \leq s \leq f \right\} \right)^2 \\ &= \sup \left\{ \left( \int s \, dm \right)^2 : s \text{ is simple and } 0 \leq s \leq f \right\} \\ &\leq \sup \left\{ \int s^2 \, dm : s \text{ is simple and } 0 \leq s \leq f \right\} \\ &= \sup \left\{ \int r \, dm : r \text{ is simple and } 0 \leq r \leq f^2 \right\} \\ &= \int f^2 \, dm \end{aligned}$$

Here, the second line follows because  $\int s \, dm \geq 0$ , so the square can pass inside of the sup. [1] The third line then follows by part (b). [1] Let us now justify the fourth line.

We have shown in (a) that if  $s$  is a non-negative simple function then so is  $r = s^2$ , and clearly if  $s \leq f$  then  $s^2 \leq f^2$  (i.e. pointwise). [1] Also, if  $r$  is a non-negative simple function such that  $0 \leq r \leq f^2$ , then if we define  $s = \sqrt{r}$ , we can show (in similar style to part (a)) that  $s$  is a non-negative simple function such that  $0 \leq s \leq f$ . Here, if  $r = \sum_i c_i \mathbb{1}_{A_i}$  we would have  $s = \sum_i \sqrt{c_i} \mathbb{1}_{A_i}$ . So, the two sups in the third and fourth lines are equal using the correspondence  $r = s^2$ . [1]

**Second proof:** Now we will allow ourselves to use the monotone convergence theorem. From lectures (see the section on simple functions) there exists a sequence  $(s_n)$  of non-negative simple functions such that  $0 \leq s_n \leq s_{n+1} \leq f$  such that  $s_n \rightarrow f$  pointwise. [1] Thus, by the monotone convergence theorem, as  $n \rightarrow \infty$ ,

$$\int s_n dm \rightarrow \int f dm.$$

[1] By part (a),  $(s_n^2)$  is also a sequence of simple functions. [1] We have  $0 \leq s_n^2 \leq s_{n+1}^2 \leq f^2$ , also  $s_n^2 \rightarrow f^2$  pointwise. So by another application of the monotone convergence theorem we have

$$\int s_n^2 dm \rightarrow \int f^2 dm.$$

[1] From part (b) we have

$$\left( \int s_n dm \right)^2 \leq \int s_n^2 dm$$

for all  $n$ . Since limits preserve weak inequalities, [1] we have that

$$\left( \int f dm \right)^2 \leq \int f^2 dm$$

as required.

- (d) In general  $(\star)$  fails when  $m$  is not a probability measure. For example, take  $f(x) = x$  and let  $m$  be Lebesgue measure on  $[0, 2]$ . Then  $\int_0^2 x dx = 2$  and  $\int_0^2 x^2 dx = \frac{8}{3}$ , but  $2^2 > \frac{8}{3}$ . [1]