

MASx52: Assignment 5

Solutions and discussion are written in blue. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Consider the SDE

$$dX_t = (t + X_t) dt + 2t dB_t.$$

(a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation

$$f'(t) = t + f(t)$$

Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

(b) Let $Y_t = X_t^2$. Show that

$$dY_t = 2(2t^2 + tX_t + X_t^2) dt + 4tX_t dB_t$$

(c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation

$$v'(t) = 2(2t^2 + tf(t) + v(t)).$$

Solution.

(a) Writing in integral form we have

$$X_t = X_0 + \int_0^t (u + X_u) du + \int_0^t 2u dB_u.$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E} \left[\int_0^t (u + X_u) du \right] + \mathbb{E} \left[\int_0^t 2u dB_u \right] \\ &= \mathbb{E}[X_0] + \int_0^t \mathbb{E}[u + X_u] du + 0 \\ &= \mathbb{E}[X_0] + \int_0^t u + \mathbb{E}[X_u] du \\ f(t) &= f(0) + \int_0^t u + f(u) du. \end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$f'(t) = t + f(t).$$

If we set $f(t) = Ce^t - t - 1$ then $f'(t) = Ce^t - 1$ [1], so clearly this is a solution.

(b) Using Ito's formula [1] we have

$$\begin{aligned} dY_t &= \left(0 + (t + X_t)(2X_t) + \frac{1}{2}(2t)^2(2) \right) dt + (2t)(2X_t) dB_t \\ &= 2(2t^2 + tX_t + X_t^2) dt + 4tX_t dB_t \end{aligned}$$

[3]

(c) Writing in integral form we have

$$Y_t = Y_0 + 2 \int_0^t 2u^2 + uX_u + X_u^2 du + \int_0^t 4uX_u dB_u$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned} \mathbb{E}[Y_t] &= \mathbb{E}[Y_0] + 2\mathbb{E} \left[\int_0^t 2u^2 + uX_u + X_u^2 du \right] + \mathbb{E} \left[\int_0^t 4uX_u dB_u \right] \\ &= \mathbb{E}[Y_0] + \int_0^t 2\mathbb{E} [2u^2 + uX_u + X_u^2] du + 0 \\ &= \mathbb{E}[Y_0] + 2 \int_0^t 2u^2 + u\mathbb{E}[X_u] + \mathbb{E}[X_u^2] du \\ &= \mathbb{E}[Y_0] + 2 \int_0^t 2u^2 + uf(u) + v(u) du \end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$v'(t) = 2(2t^2 + tf(t) + v(t)).$$

2. (a) Within the Black-Scholes model, use the risk neutral valuation formula find the prices at time t of the contingent claims
 - i. $\Phi(S_T) = 3S_T + 5$, where $0 \leq t \leq T$.
 - ii. $\Psi(S_T) = S_1S_T + 1$, where $1 \leq t \leq T$.
- (b) With the same contingents claims as in (a):
 - i. Describe a constant portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.
 - ii. Is it possible to replicate $\Psi(S_T)$ using a constant portfolio?

Solution.

- (a) i. Using the explicit formula for geometric Brownian motion (see the formula sheet) we obtain

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[3S_T + 5 | \mathcal{F}_t] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}} \left[3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} + 5 | \mathcal{F}_t \right] \\ &= e^{-r(T-t)} \left(3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}} \left[e^{\sigma(B_T-B_t)} | \mathcal{F}_t \right] + 5 \right) \\ &= e^{-r(T-t)} \left(3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}} \left[e^{\sigma(B_T-B_t)} \right] + 5 \right) \\ &= e^{-r(T-t)} \left(3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)} + 5 \right) \\ &= e^{-r(T-t)} \left(3S_te^{r(T-t)} + 5 \right) \\ &= 3S_t + 5e^{-r(T-t)} \end{aligned}$$

[4] Here, we use that S_t is \mathcal{F}_t measurable, [1] and that $Z = \sigma(B_T - B_t) \sim N(0, \sigma^2(T-t))$ is independent of \mathcal{F}_t . [1] We use the formula sheet to provide an explicit formula for $\mathbb{E}[e^Z]$.

ii. Assuming $1 \leq t \leq T$, we have

$$\begin{aligned} e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[S_1 S_T + 1 \mid \mathcal{F}_t] &= e^{-r(T-t)} \left(S_1 \mathbb{E}^{\mathbb{Q}}[S_T \mathcal{F}_t] + 1 \right) \\ &= S_1 e^{rt} e^{-rT} \mathbb{E}^{\mathbb{Q}}[S_T \mathcal{F}_t] + e^{-r(T-t)} \\ &= S_1 e^{rt} e^{-rt} S_t + e^{-r(T-t)} \\ &= S_1 S_t + e^{-r(T-t)}. \end{aligned}$$

[2] Here we use that $S_1 \in \mathcal{F}_t$ for $t \geq 1$, [1] and the fact (from Lemma 14.4.1 in lectures) that $M_t = e^{-rt} S_t$ is a martingale in the risk-neutral world. [1]

- (b) i. At time 0, we buy three units of stock [1] and $5e^{-rT}$ in cash. [1] It's value at time t is then

$$3S_t + 5e^{-rT} e^{rt} = \Phi(S_T).$$

Therefore, this portfolio replicates $\Phi(S_T)$ for all $t \in [0, T]$. [1]

- ii. It isn't possible to replicate $\Psi(S_T)$ with a constant portfolio. [1] The replicating portfolio provided by Theorem 14.3.1 is unique, and contains a stock component $y_t = \frac{\partial F}{\partial s}(t, S_t)$ where $F(t, s)$ is the pricing formula obtained in (a.ii); in this case for $t \geq 1$ we have $F(t, s) = S_1 s + e^{-r(T-t)}$ so y_t is non-constant. [1]

3. [On Semester 1] Consider an urn, containing two colours of balls, black and red. At time $n = 0$, the urn contains one black ball and one red ball. Then, at each time $n = 1, 2, \dots$, we do the following:

- Draw a ball from the urn. Record the colour of this ball and place it back into the urn.
- Add two new balls to the urn, of the same colour as the drawn ball.

Therefore, at time n , the urn contains $2 + 2n$ balls. Let B_n denote the number of red balls in the urn, and let

$$M_n = \frac{B_n}{2 + 2n}.$$

- (a) Show that M_n is a martingale, with respect to the filtration $\mathcal{F}_n = \sigma(B_i : i \leq n)$.
(b) Show that $\mathbb{P}[M_n \leq \frac{1}{2}] = \mathbb{P}[M_n \geq \frac{1}{2}]$ for all n .

Solution.

- (a) Since $M_n \in [0, 1]$ we have that $\mathbb{E}[|M_n|] \leq 1$, so $M_n \in L^1$. [1]

Since $B_n \in m\mathcal{F}_n$, we have $M_n \in m\mathcal{F}_n$. [1]

From the dynamics of the urn, we have

$$B_{n+1} = \mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}}(B_n + 2) + \mathbb{1}_{\{(n+1)^{th} \text{ draw is not red}\}}B_n.$$

[1] We calculate

$$\begin{aligned}
\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\frac{B_{n+1}}{4+2n} \mid \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}}(B_n+2)}{4+2n} + \frac{\mathbb{1}_{\{(n+1)^{th} \text{ draw is black}\}}B_n}{4+2n} \mid \mathcal{F}_n\right] \\
&= \frac{B_n+2}{4+2n} \mathbb{E}\left[\mathbb{1}_{\{(n+1)^{th} \text{ draw is red}\}} \mid \mathcal{F}_n\right] + \frac{B_n}{4+2n} \mathbb{E}\left[\mathbb{1}_{\{(n+1)^{th} \text{ draw is black}\}} \mid \mathcal{F}_n\right] \\
&= \frac{B_n+2}{4+2n} \frac{B_n}{2+2n} + \frac{B_n}{4+2n} \frac{2n+2-B_n}{2+2n} \\
&= \frac{B_n^2 + 2B_n + (2n+2)B_n - B_n^2}{(4+2n)(2+2n)} \\
&= \frac{(4+2n)B_n}{(4+2n)(2+2n)} \\
&= \frac{B_n}{2n+2} \\
&= M_n.
\end{aligned}$$

[3] Here we use that $B_n \in \mathcal{F}_n$ to take out what is known, [1] and to calculate the probabilities that the $(n+1)^{th}$ draw is red or black given knowledge of B_n .

Thus (M_n) is a martingale.

- (b) The key point here is the roles of the colours red and black are symmetric: if we swapped the colours red and black (i.e. all red balls became black, and all black balls became red), then we would obtain an urn with *exactly* the same distribution as we started with. [1]

Let B'_n denote the number of black balls within the urn at time n , and write

$$M'_n = \frac{B'_n}{2+2n} = 1 - \frac{B_n}{2+2n}.$$

Note that $M_n + M'_n = 1$. [1] By the symmetry between red and black, M_n and M'_n have the same distribution. [1]

Hence,

$$\mathbb{P}[M_n \leq \tfrac{1}{2}] = \mathbb{P}[M'_n \leq \tfrac{1}{2}] = \mathbb{P}[1 - M_n \leq \tfrac{1}{2}] = \mathbb{P}[M_n \geq \tfrac{1}{2}]$$

as required. [1]