**MASx52: Assignment 5**

Solutions and discussion are written in blue. A sample mark scheme, with a total of 45 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Consider the SDE

$$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$

(a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation

$$f'(t) = t + f(t)$$

Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

(b) Let $Y_t = X_t^2$. Show that

$$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

(c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation

$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

*Solution.*

(a) Writing in integral form we have

$$X_t = X_0 + \int_0^t (u + X_u)\,du + \int_0^t 2u\,dB_u.$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t (u + X_u)\,du\right] + \mathbb{E}\left[\int_0^t 2u\,dB_u\right] \\
&= \mathbb{E}[X_0] + \int_0^t \mathbb{E}[u + X_u]\,du + 0 \\
&= \mathbb{E}[X_0] + \int_0^t u + \mathbb{E}[X_u]\,du \\
f(t) &= f(0) + \int_0^t u + f(u)\,du.
\end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$f'(t) = t + f(t).$$

If we set $f(t) = Ce^t - t - 1$ then $f'(t) = Ce^t - 1$ [1], so clearly this is a solution.

(b) Using Ito's formula [1] we have

$$\begin{aligned} dY_t &= \left(0 + (t + X_t)(2X_t) + \frac{1}{2}(2t)^2(2)\right) dt + (2t)(2X_t)\, dB_t \\ &= 2\left(2t^2 + tX_t + X_t^2\right) dt + 4tX_t\, dB_t \end{aligned}$$

[3]

(c) Writing in integral form we have

$$Y_t = Y_0 + 2\int_0^t 2u^2 + uX_u + X_u^2\, du + \int_0^t 4uX_u\, dB_u$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned} \mathbb{E}[Y_t] &= \mathbb{E}[Y_0] + 2\mathbb{E}\left[\int_0^t 2u^2 + uX_u + X_u^2\, du\right] + \mathbb{E}\left[\int_0^t 4uX_u\, dB_u\right] \\ &= \mathbb{E}[Y_0] + \int_0^t 2\mathbb{E}\left[2u^2 + uX_u + X_u^2\right] du + 0 \\ &= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + u\mathbb{E}\left[X_u\right] + \mathbb{E}\left[X_u^2\right] du \\ &= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + uf(u) + v(u)\, du \end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

2. Let $T > 0$. Use the Feynman-Kac formula to find an explicit solution $F(x,t)$ to the partial differential equation

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2}(x,t) = 0$$

subject to the boundary condition $F(T,x) = x - \frac{T}{2}$.

*Hint: It may help to recall that* $\int_0^t B_u\, dB_u = \frac{B_t^2}{2} - \frac{t}{2}$.

*Solution.* From the Feynman-Kac formula, with $\alpha(t,x) = \frac{1}{2}$ and $\beta(t,x) = x$ we have that

$$F(t,x) = \mathbb{E}_{t,x}[X_T - \tfrac{T}{2}]$$

where $dX_t = \frac{1}{2}\, dt + B_t\, dB_t$. [1] Thus, in integral form, [1]

$$\begin{aligned} X_T &= X_t + \int_t^T \tfrac{1}{2}\, ds + \int_t^T B_s\, dB_s \\ &= X_t + \frac{T-t}{2} + \frac{B_T^2 - B_t^2}{2} - \frac{T-t}{2} \\ &= X_t + \frac{B_T^2 - B_t^2}{2}. \end{aligned}$$

which gives

$$\begin{aligned} F(t,x) &= \mathbb{E}_{t,x}\left[X_t + \frac{B_T^2 - B_t^2}{2} - \frac{T}{2}\right] \\ &= \mathbb{E}\left[x + \frac{B_T^2 - B_t^2}{2} - \frac{T}{2}\right] \\ &= x - \frac{t}{2} \end{aligned}$$

[2] To deduce the final line we use that $B_t^2 - t$ is a martingale with zero mean. [1]

3. (a) Within the Black-Scholes model, use the risk neutral valuation formula to find the prices at time $t$ of the contingent claims

   i. $\Phi(S_T) = 3S_T + 5$, where $0 \le t \le T$.

   ii. $\Psi(S_T) = S_1 S_T + 1$, where $1 \le t \le T$.

   (b) For a portfolio containing a single contract with contingent claim $\Phi(S_T)$:

   i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

   ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

   (c) Suggest one reason why we might want to hold a delta neutral portfolio.

*Solution.*

(a) i. Using the explicit formula for geometric Brownian motion (see the formula sheet) we obtain

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_T + 5 \,|\, \mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} + 5 \,|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)} \,|\, \mathcal{F}_t\right] + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)}\right] + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)} + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{r(T-t)} + 5\right) \\
&= 3S_t + 5e^{-r(T-t)}
\end{aligned}$$

[4] Here, we use that $S_t$ is $\mathcal{F}_t$ measurable,[1] and that $Z = \sigma(B_T - B_t) \sim N(0, \sigma^2(T-t))$ is independent of $\mathcal{F}_t$. [1] We use the formula sheet to provide an explicit formula for $\mathbb{E}[e^Z]$.

ii. Assuming $1 \le t \le T$, we have

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_1 S_T + 1 \,|\, \mathcal{F}_t\right] &= e^{-r(T-t)}\left(S_1\mathbb{E}^{\mathbb{Q}}[S_T \mathcal{F}_t] + 1\right) \\
&= S_1 e^{rt}e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T \mathcal{F}_t] + e^{-r(T-t)} \\
&= S_1 e^{rt}e^{-rt}S_t + e^{-r(T-t)} \\
&= S_1 S_t + e^{-r(T-t)}.
\end{aligned}$$

[2] Here we use that $S_1 \in \mathcal{F}_t$ for $t \ge 1$, [1] and the fact (from Lemma 14.4.1 in lectures) that $M_t = e^{-rt}S_t$ is a martingale in the risk-neutral world.[1]

(b) i. The value of our portfolio at time $t$ is given by $F(t, S_t)$, where $F$ is as in part (a). If we add an amount $\alpha$ of stock into our portfolio then its new value will be $V(t, S_t) = F(t, S_t) + \alpha S_t$. [1] To achieve delta neutrality, we want to choose $\alpha$ such that

$$0 = \frac{\partial V}{\partial s}(0, S_0) = 3 + \alpha.$$

[1] Hence $\alpha = -3$. [1]

ii. Our new portfolio has value $V(t, S_t) = F(t, S_t) - 3S_t = 5e^{-r(T-t)}$, and hence $\frac{\partial V}{\partial s} = 0$ for all time. Hence, in this case our portfolio will stay delta neutral for all time.

(c) A delta neutral portfolio is advantageous because its value is, typically, less sensitive so sudden changes in the stock price. [1]