

## MASx52: Assignment 4

1. Let  $B_t$  be a standard Brownian motion.

(a) Write down the distribution of  $B_t$ , and write down  $\mathbb{E}[B_t]$  and  $\mathbb{E}[B_t^2]$ .

(b) Let  $0 \leq u \leq t$ . Show that  $\mathbb{E}[(B_t - B_u)^2 \mid \mathcal{F}_u] = t - u$ .

2. Write down the following stochastic differential equations in integral form, over the time interval  $[0, t]$ .

(a)  $dX_t = 2(X_t + 1) dt + 2B_t dB_t$ .

(b)  $dY_t = 3Y_t dt$ .

Write down a differential equation satisfied by  $Y_t$ , and find its solution with the initial condition  $Y_0 = 1$ .

Suppose that  $X_0 = 1$ . Show that  $f(t) = \mathbb{E}[X_t]$  satisfies  $f'(t) = 2f(t) + 2$  and hence find  $f(t)$ .

3. Use Ito's formula to calculate the stochastic differential of  $dZ_t$  where

(a)  $Z_t = tB_t$

(b)  $Z_t = 1 + t^2 X_t$  where  $dX_t = \mu dt + \sigma B_t dB_t$  and  $\mu, \sigma$  are deterministic constants.

(c)  $Z_t = e^{-2t} S_t$  where  $dS_t = 2S_t dt + 5S_t dB_t$ .

In which cases is  $Z_t$  a martingale?

4. Let  $S_t$  be a geometric Brownian motion, with drift  $\mu \in \mathbb{R}$ , volatility  $\sigma > 0$ , and (deterministic) initial condition  $S_0$ .

(a) Find  $\mathbb{E}[S_t]$  and deduce that  $S_t$  is not a Brownian motion when  $\mu \neq 0$ .

(b) Is  $S_t$  a Brownian motion when  $\mu = 0$ ?