k-likelihood regions

Suppose that we have i.i.d. data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, for which each data point is modelled as a random sample from $N(\mu, \sigma^2)$ where μ is unknown and σ^2 is known. Find the k-likelihood region R_k for the parameter μ.

First, we need to find the MLE $\hat{\mu}$ of μ. The likelihood function for our model is

$$L(\mu; \mathbf{x}) = \prod_{i=1}^{n} \phi(x_i; \mu) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right),$$

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The log likelihood is

$$\ell(\mu; \mathbf{x}) = -\frac{n}{2}\left(\log(2\pi) + \log(\sigma^2)\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2.$$

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The usual process of maximisation shows that the maximum likelihood estimator is the sample mean,

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Now we are ready to identify the k-likelihood region for μ. By definition, the k-likelihood region is

$$R_k = \left\{\mu \in \mathbb{R} \middle| l(\mu; \mathbf{x}) - l(\hat{\mu}; \mathbf{x})\right| \leq k\right\}.$$

So, $\mu \in R_k$ if and only if

$$\left| \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 \right| \leq k.$$

We can simplify this inequality, by noting that

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\mu)^2-\sum_{i=1}^{n}(x_i-\hat{\mu})^2 &= \sum_{i=1}^{n} x_i^2-2x_i\mu+\mu^2-x_i^2+2x_i\hat{\mu}-\hat{\mu}^2\\ &= n\mu^2-n\hat{\mu}^2+2(\hat{\mu}-\mu)\sum_{i=1}^{n}x_i\\ &= n\mu^2-n\hat{\mu}^2+2(\hat{\mu}-\mu)n\hat{\mu}\\ &= n(\mu^2+\hat{\mu}^2-2\mu\hat{\mu})\\ &= n(\hat{\mu}-\mu)^2.\end{aligned}$$

So, $\mu \in R_k$ if and only if

$$\frac{n}{2\sigma^2}|\hat{\mu}-\mu|^2 \leq k,$$

or in other words, $|\hat{\mu}-\mu| \leq \sigma\sqrt{\frac{2k}{n}}$, so

$$R_k = \left[\hat{\mu}-\sigma\sqrt{\frac{2k}{n}},\ \hat{\mu}+\sigma\sqrt{\frac{2k}{n}}\right].$$

k-likelihood tests

In Example 37, if we used a 2-likelihood test, would we accept the hypothesis that the radioactive decay of carbon-15 is equal to $\lambda = 0.27$?

We had found, given the data, that the likelihood function of θ was

$$L(\lambda; \mathbf{x}) = \lambda^{15} e^{-47.58\lambda}$$

and the maximum likelihood estimator of λ was $\hat{\lambda} \approx 0.32$.

The 2-likelihood region for λ is the set

$$R_2 = \left\{ \lambda > 0 L(\lambda; \mathbf{x}) \geq e^{-2} L(\hat{\lambda}; \mathbf{x}) \right\},$$

so $\lambda \in R_2$ if and only if

$$\lambda^{15} e^{-47.58\lambda} \geq e^{-2} L(0.32; \mathbf{x}) = 1.24 \times 10^{-15}.$$

Note that, unlike the previous example, we can't simplify this inequality and find a 'nice' form for the likelihood region.

Our hypothesis is that, in fact, $\lambda = 0.27$. Our 2-likelihood test will pass if $\lambda = 0.27$ is within the 2-likelihood region, and fail if not. We can evaluate (use e.g. `R`),

$$0.27^{15} e^{-47.58 \times 0.27} \approx 7.78 \times 10^{-15}$$

and note that $7.78 \times 10^{-15} \geq 1.24 \times 10^{-15}$. Hence $\lambda = 0.27$ is within the 2-likelihood region and we accept the hypothesis.