MASx52: Assignment 1

1. Recall the one-period market, and its parameters r, u, d, p_u, p_d and s. We assume that $d < 1 + r < u$.

   (a) At time $t = 0$ our portfolio contains 2 unit of cash and 3 units of stock. What is the value of our portfolio at time $t = 0$? If we hold this portfolio until time $t = 1$, what is its new value?

   (b) A rival investor holds a portfolio containing 3 units of cash and 2 unit of stock. Under what condition (on the parameters) can we be *certain* that our own portfolio will have a strictly greater value at time $t = 1$?

2. Let $\Omega = \{HH, HT, TH, TT\}$, representing two coin tosses each of which may show either H (head) or T (tail). Let $X : \Omega \to \mathbb{R}$ be the toss in which the first head occurred, or zero if no heads occurred:

$$X = \begin{cases} 0 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT \text{ or } \omega = HH \\ 2 & \text{if } \omega = TH. \end{cases}$$

   Let Y be the total number of heads that occurred in both tosses.

   (a) Write down the sets $X^{-1}(0)$, $X^{-1}(1)$ and $X^{-1}(2)$.

   (b) State the definition of $\sigma(X)$, and list its elements.

   (c) Is Y measurable with respect to $\sigma(X)$? Why, or why not?

3. Let $\Omega = \{1, 2, 3, 4, 5\}$, representing one roll of a five sided dice. Let $\mathcal{F}$ be the set of all subsets of Ω. Describe, in words, the information contained within the following sub-σ-fields of $\mathcal{F}$.

   (a) $\mathcal{G}_1 = \big\{\emptyset, \Omega, \{1\}, \{2, 3, 4, 5\}\big\}$

   (b) $\mathcal{G}_2 = \big\{\emptyset, \Omega, \{1, 3, 5\}, \{2, 4\}\big\}$

   (c) $\mathcal{G}_3 = \big\{\emptyset, \Omega, \{1\}, \{2, 3, 4\}, \{5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 5\}\big\}$

4. Let X be a random variable.

   (a) Show that $Y = \cos X$ is also a random variable.

   (b) For which $p \in [1, \infty)$ do we have $Y \in L^p$?