MASx52: Assignment 2

Solutions and discussion are written in blue. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Let (X_n) be a sequence of i.i.d. random variables, each with a uniform distribution on the interval $[-1, 1]$. Define

$$S_n = \sum_{i=1}^{n} X_i,$$

where $S_0 = 0$. Let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

(a) Show that S_n is a martingale, with respect to the filtration $\mathcal{F}_n$.

(b) Find $\mathbb{E}[S_3^2 \,|\, \mathcal{F}_2]$ in terms of X_2 and X_1, and hence show that

$$\mathbb{E}[S_3^2 \,|\, \mathcal{F}_2] = S_2^2 + \frac{1}{3}.$$

(c) Write down a deterministic function $f : \mathbb{N} \to \mathbb{R}$ such that

$$M_n = S_n^2 - f(n)$$

is a martingale (justification is not required – make a guess!).

Solution.

(a) Since $X_i \in \sigma(X_i)$ we have $X_i \in \mathcal{F}_n$ for all $i \le n$. Hence, since sums of $\mathcal{F}_n$ measurable functions are measurable, we have also that $S_n \in \mathcal{F}_n$ [1].
Since $|X_i| \le 1$ for all i, we have

$$|S_n| \le |X_1| + |X_2| + \ldots + |X_n| \le n.$$

Thus S_n is a bounded random variable and hence $S_n \in L^1$. [1]
Lastly,

$$\begin{aligned}
\mathbb{E}[S_{n+1} \,|\, \mathcal{F}_n] &= \mathbb{E}\left[X_{n+1} + S_n \,|\, \mathcal{F}_n\right] \\
&= \mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] + \mathbb{E}[S_n \,|\, \mathcal{F}_n] \\
&= \mathbb{E}[X_{n+1}] + S_n \\
&= S_n.
\end{aligned}$$

[1] Here, we use the linearity of conditional expectation to deduce the second line, followed by using that X_{n+1} is independent of $\mathcal{F}_n$ [1] and $S_n \in \mathcal{F}_n$ to deduce the third line [1]. The final line follows because $\mathbb{E}[X_i] = 0$ for all i. Hence S_n is a martingale.

Pitfall: You should justify your use of the rules of conditional expectation.

(b) We have

$$S_n^3 = (X_1 + X_2 + X_3)^2 = X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 + 2X_2X_3 + 2X_1X_3.$$

[1] Hence,

$$\begin{aligned}
\mathbb{E}[S_3^2 \,|\, \mathcal{F}_2] &= \mathbb{E}[X_1^2 \,|\, \mathcal{F}_n] + \mathbb{E}[X_2^2 \,|\, \mathcal{F}_2] + \mathbb{E}[X_3^2 \,|\, \mathcal{F}_2] \\
&\qquad + 2\mathbb{E}[X_1X_2 \,|\, \mathcal{F}_2] + 2\mathbb{E}[X_2X_3 \,|\, \mathcal{F}_2] + 2\mathbb{E}[X_1X_3 \,|\, \mathcal{F}_2] \\
&= X_1^2 + X_2^2 + \mathbb{E}[X_3^2] + 2X_1X_2 + 2X_2\mathbb{E}[X_3] + 2X_1\mathbb{E}[X_3] \\
&= (X_1 + X_2)^2 + \frac{1}{3} \\
&= S_2^2 + \frac{1}{3}.
\end{aligned}$$

[1]. Here, in the first line we use linearity of conditional expectation. To deduce the second line we use that X_3 is independent of $\mathcal{F}_2$ [1], and that $X_1, X_2 \in m\mathcal{F}_2$ to 'take out what is known'[1]. We then use that

$$\mathbb{E}[X_3^2] = \int_{-1}^{1} x^2 \frac{1}{2}\, dx = \frac{1}{3}$$

to deduce the final lines [1].

Pitfall: Note that X_n has the *continuous* uniform distribution on the interval $[-1, 1]$.

(c) In view of (b), we take $f(n) = \frac{n}{3}$, so that $M_n = S_n - \frac{n}{3}$ [2].

To make this guess: note from (b) that $\mathbb{E}[S_n^2]$ drifts upwards by $\frac{1}{3}$ on each time step, so $\mathbb{E}[S_n^2 - \frac{n}{3}]$ stays constant. On each step of time, we need to compensate by $\frac{-1}{3}$.

This is the only way to compensate for the drift in the form $S_n - f(n)$, and (possibly) obtain a martingale. To see that M_n really is a martingale: Since $S_n \in \mathcal{F}_n$ we have $M_n \in \mathcal{F}_n$, and $|M_n| \leq |S_n^2| + \frac{2n}{3} \leq n^2 + \frac{n}{3}$ so $M_n \in L^1$. A similar calculation to (b) then shows that $\mathbb{E}[S_{n+1}^2 \,|\, \mathcal{F}_n] = S_n^2 + \frac{1}{3}$, hence $\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] = M_n$.

2. Consider the one-period market with $r = \frac{1}{10}$, $s = 2$, $d = \frac{1}{2}$ and $u = 3$, in our usual notation. A contract specifies that

   The holder of the contract will sell 2 units of stock, and be paid K units of cash, at time 1.

   (a) Explain briefly why the contingent claim of this contract is

   $$\Phi(S_1) = K - 2S_1.$$

   (b) Find a replicating portfolio h for this contingent claim.

   (c) Write down the value V_0^h of h at time 0.

   (d) Find the numerical values of risk-neutral probabilities

   $$q_u = \frac{(1 + r) - d}{u - d} \qquad \text{and} \qquad q_d = \frac{u - (1 + r)}{u - d}.$$

   Hence, check that $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)]$ and V_0^h have the same values.

(e) For which K does the contract have value zero at time 0?

Solution.

(a) The holder will be paid K units of cash, resulting in a gain of K, and give away 2 units of stock, each of which is worth S_1, resulting in a loss of $2S_1$. [1] Hence

$$\Phi(S_1) = K - 2S_1.$$

Pitfall: This is not a European put option. The holder of this contract *must* pay K units of cash and be given 2 stock.

(b) The possible values taken by S_1 are $su = 6$ and $sd = 1$. A replicating portfolio $h = (x, y)$ must satisfy $V_1^h = \Phi(S_1)$, [1] meaning that

$$(1 + \tfrac{1}{10})x + 6y = K - 12$$
$$(1 + \tfrac{1}{10})x + y = K - 2$$

[2] We now solve these equations. Taking one away from the other, we obtain $5y = -10$, hence $y = -2$ which gives $x = \frac{K}{11/10} = \frac{10K}{11}$. [1]

(c) The value of the contract is

$$V_0^h = x + sy = \frac{10K}{11} - 4$$

[1] at time 0.

(d) The risk-neutral probabilities are

$$q_u = \frac{11/10 - 1/2}{3 - 1/2} = \frac{3/5}{5/2} = \frac{6}{25},$$
$$q_d = \frac{3 - 11/10}{3 - 1/2} = \frac{19/10}{5/2} = \frac{19}{25}.$$

[1] This gives us

$$\begin{aligned}\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)] &= \frac{1}{11/10}\left(\frac{6}{25}(K-12) + \frac{19}{25}(K-2)\right)\\ &= \frac{10}{11}\left(K - \frac{110}{25}\right)\\ &= \frac{10K}{11} - 4,\end{aligned}$$

[2] which is equal to the value of V_0^h that we found in (c).

(e) The contract is worth zero at time 0 if $\frac{10}{11}K - 4 = 0$, that is if $K = \frac{22}{5}$. [1]

Total marks: 22