MASx52: Assignment 5

Solutions and discussion are written in blue. A sample mark scheme, with a total of 45 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Let S_t be a geometric Brownian motion, with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and (deterministic) initial condition S_0.

   (a) Find $\mathbb{E}[S_t]$ and deduce that S_t is not a Brownian motion when $\mu \neq 0$.

   (b) Is S_t a Brownian motion when $\mu = 0$?

   Solution.

   (a) The formula for geometric Brownian motion is

   $$S_t = S_0 \exp\left((\mu - \tfrac{\sigma^2}{2})t + \sigma B_t\right).$$

   So, taking expectations, using the formula for $\mathbb{E}[e^Z]$ where Z is normally distributed, and using that S_0 is deterministic,

   $$\begin{aligned}\mathbb{E}[S_t] &= S_0 e^{(\mu-\frac{\sigma^2}{2})t}\mathbb{E}[e^{\sigma B_t}] \\ &= S_0 e^{(\mu-\frac{\sigma^2}{2})t} e^{\frac{\sigma^2 t}{2}} \\ &= S_0 e^{\mu t}\end{aligned}$$

   [3] A Brownian motion B_t has $\mathbb{E}[B_t] = \mathbb{E}[B_0]$, but for $\mu \neq 0$ we have shown that $\mathbb{E}[S_t]$ is non-constant, which means that S_t cannot be a Brownian motion. [2]

   (b) It remains to consider the case $\mu = 0$. In this case, $S_t = S_0 e^{\sigma B_t - \frac{\sigma^2}{2}t}$. We recall that, for a Brownian motion, $B_t^2 - t$ is a martingale, [1] and for S_t we have $S_t^2 - t = S_0^2 e^{2\sigma B_t - \sigma^2 t} - t$. This gives us

   $$\begin{aligned}\mathbb{E}[S_t^2 - t] &= S_0^2 \mathbb{E}[e^{2\sigma B_t}] e^{-\sigma^2 t} - t \\ &= S_0^2 e^{\frac{4\sigma^2}{2}} e^{-\sigma^2 t} - t \\ &= S_0^2 e^{\sigma^2 t} - t\end{aligned}$$

   [2] which is clearly non-constant. Hence $S_t^2 - t$ is not a martingale, so S_t is not a Brownian motion. [1]

   [Note: There are *lots* of other ways to solve this question!]

2. Consider the SDE

   $$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$

   (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation

   $$f'(t) = t + f(t)$$

   Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

(b) Let $Y_t = X_t^2$. Show that

$$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

(c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation

$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

Solution.

(a) Writing in integral form we have

$$X_t = X_0 + \int_0^t (u + X_u)\,du + \int_0^t 2u\,dB_u.$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t (u + X_u)\,du\right] + \mathbb{E}\left[\int_0^t 2u\,dB_u\right] \\
&= \mathbb{E}[X_0] + \int_0^t \mathbb{E}[u + X_u]\,du + 0 \\
&= \mathbb{E}[X_0] + \int_0^t u + \mathbb{E}[X_u]\,du \\
f(t) &= f(0) + \int_0^t u + f(u)\,du.
\end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$f'(t) = t + f(t).$$

If we set $f(t) = Ce^t - t - 1$ then $f'(t) = Ce^t - 1$ [1], so clearly this is a solution.

(b) Using Ito's formula [1] we have

$$\begin{aligned}
dY_t &= \left(0 + (t + X_t)(2X_t) + \frac{1}{2}(2t)^2(2)\right)\,dt + (2t)(2X_t)\,dB_t \\
&= 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t
\end{aligned}$$

[3]

(c) Writing in integral form we have

$$Y_t = Y_0 + 2\int_0^t 2u^2 + uX_u + X_u^2\,du + \int_0^t 4uX_u\,dB_u$$

[1] Taking expectation, and recalling that Ito integrals are zero mean martingales [1],

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}[Y_0] + 2\mathbb{E}\left[\int_0^t 2u^2 + uX_u + X_u^2\,du\right] + \mathbb{E}\left[\int_0^t 4uX_u\,dB_u\right] \\
&= \mathbb{E}[Y_0] + \int_0^t 2\mathbb{E}\left[2u^2 + uX_u + X_u^2\right]\,du + 0 \\
&= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + u\mathbb{E}\left[X_u\right] + \mathbb{E}\left[X_u^2\right]\,du \\
&= \mathbb{E}[Y_0] + 2\int_0^t 2u^2 + uf(u) + v(u)\,du
\end{aligned}$$

[1] Differentiating, by the fundamental theorem of calculus, [1]

$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

3. Let $T > 0$. Use the Feynman-Kac formula to find an explicit solution $F(x,t)$ to the partial differential equation

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2}(x,t) = 0$$

subject to the boundary condition $F(T,x) = x - \frac{T}{2}$.

Hint: It may help to recall that $\int_0^t B_u \, dB_u = \frac{B_t^2}{2} - \frac{t}{2}$.

Solution. From the Feynman-Kac formula, with $\alpha(t,x) = \frac{1}{2}$ and $\beta(t,x) = x$ we have that

$$F(t,x) = \mathbb{E}_{t,x}[X_T - \tfrac{T}{2}]$$

where $dX_t = \frac{1}{2}\, dt + B_t \, dB_t$. [1] Thus, in integral form, [1]

$$\begin{aligned} X_T &= X_t + \int_t^T \tfrac{1}{2}\, ds + \int_t^T X_s \, dB_s \\ &= X_t + \frac{T-t}{2} + \int_t^T X_s \, dB_s \end{aligned}$$

which gives

$$\begin{aligned} F(t,x) &= \mathbb{E}_{t,x}\left[X_t + \frac{T-t}{2} + \int_t^T X_s \, dB_s - \frac{T}{2}\right] \\ &= \mathbb{E}\left[x - \frac{t}{2} + \int_t^T X_s \, dB_s\right] \\ &= x - \frac{t}{2} \end{aligned}$$

[2] Here we use that Ito integrals are zero mean martingales. [1]

4. (a) Within the Black-Scholes model, use the risk neutral valuation formula to find the prices at time t of the contingent claims

   i. $\Phi(S_T) = 3S_T + 5$, where $0 \le t \le T$.

   ii. $\Psi(S_T) = S_1 S_T + 1$, where $1 \le t \le T$.

   (b) For a portfolio containing a single contract with contingent claim $\Phi(S_T)$:

   i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

   ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

   (c) Suggest one reason why we might want to hold a delta neutral portfolio.

Solution.

(a) i. Using the explicit formula for geometric Brownian motion (see the formula sheet) we obtain

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_T+5\,|\,\mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)}+5\,|\,\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left(3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)}\,|\,\mathcal{F}_t\right]+5\right) \\
&= e^{-r(T-t)}\left(3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)}\right]+5\right) \\
&= e^{-r(T-t)}\left(3S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)}+5\right) \\
&= e^{-r(T-t)}\left(3S_te^{r(T-t)}+5\right) \\
&= 3S_t+5e^{-r(T-t)}
\end{aligned}$$

[4] Here, we use that S_t is $\mathcal{F}_t$ measurable,[1] and that $Z=\sigma(B_T-B_t)\sim N(0,\sigma^2(T-t)$ is independent of $\mathcal{F}_t$. [1] We use the formula sheet to provide an explicit formula for $\mathbb{E}[e^Z]$.

ii. Assuming $1\leq t\leq T$, we have

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_1S_T+1\,|\,\mathcal{F}_t\right] &= e^{-r(T-t)}\left(S_1\mathbb{E}^{\mathbb{Q}}[S_T\mathcal{F}_t]+1\right) \\
&= S_1e^{rt}e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T\mathcal{F}_t]+e^{-r(T-t)} \\
&= S_1e^{rt}e^{-rt}S_t+e^{-r(T-t)} \\
&= S_1S_t+e^{-r(T-t)}.
\end{aligned}$$

[2] Here we use that $S_1\in\mathcal{F}_t$ for $t\geq 1$, [1] and the fact (from Lemma 14.4.1 in lectures) that $M_t=e^{-rt}S_t$ is a martingale in the risk-neutral world.[1]

(b) i. The value of our portfolio at time t is given by $F(t,S_t)$, where F is as in part (a). If we add an amount α of stock into our portfolio then its new value will be $V(t,S_t)=F(t,S_t)+\alpha S_t$. [1] To achieve delta neutrality, we want to choose α such that

$$0=\frac{\partial V}{\partial s}(0,S_0)=3+\alpha.$$

[1] Hence $\alpha=-3$. [1]

ii. Our new portfolio has value $V(t,S_t)=F(t,S_t)-3S_t=5e^{-r(T-t)}$, and hence $\frac{\partial V}{\partial s}=0$ for all time. Hence, in this case our portfolio will stay delta neutral for all time.

(c) A delta neutral portfolio is advantageous because its value is, typically, less sensitive so sudden changes in the stock price. [1]