

MAS350: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme, with a total of 20 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Determine if the following functions are integrable.

(a) $f : (0, \infty) \rightarrow \mathbb{R}$ by $f(x) = 1/x^2$.

(b) $g : (0, 1) \rightarrow \mathbb{R}$ by $g(x) = \log x$

Solution.

(a) Note that $x^{-2} > 0$ for $x \in (0, \infty)$. By Riemann integration, we have

$$\int_{1/n}^n x^{-2} dx = [-x^{-1}]_{1/n}^n = -\frac{1}{n} + n.$$

[1] Note that $f_n(x) = \mathbb{1}_{\{x \in (1/n, n)\}} x^{-2}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $f(x) = x^{-2}$ for $x \in (0, \infty)$. [1] Hence, by the monotone convergence theorem we have

$$\int_0^\infty x^{-2} dx = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} + n \right) = +\infty.$$

Thus x^{-2} is not integrable on $(0, \infty)$. [1]

(b) By Riemann integration, we have

$$\int_{1/n}^1 \log x dx = [x \log x - x]_{1/n}^1 = (-1) - \left(\frac{1}{n} \log \frac{1}{n} - \frac{1}{n} \right) = \frac{1 + \log n}{n} - 1.$$

Noting that $\log x \in (-\infty, 0)$ for $x \in (0, 1)$, multiplying the above by -1 gives

$$\int_{1/n}^1 |\log x| dx = 1 - \frac{1 + \log n}{n}.$$

[1] We have that $g_n(x) = |\log x| \mathbb{1}_{x \in (1/n, 1)}$ is a monotone increasing sequence of non-negative functions, with pointwise convergence to $g(x) = |\log x|$ for $x \in (0, 1)$. [1] Hence, by the monotone convergence theorem,

$$\int_0^1 |\log x| dx = \lim_{n \rightarrow \infty} \left(1 - \frac{1 + \log n}{n} \right) = 1.$$

Thus $\log x$ is integrable on $(0, 1)$. [1]

2. In this question we work with Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be integrable. Prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f(x)| \mathbb{1}_{\{|f(x)| \geq n\}} dx = 0.$$

(b) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be integrable. It is a fact that: *for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\text{if } E \subseteq \mathbb{R} \text{ satisfies } \lambda(E) \leq \delta \text{ then } \int_E |g| dx \leq \epsilon.$$

Consider the following twelve statements.

A. Let $\epsilon > 0$.

B. Note that $|g(x)| \leq M$, for some $M \in (0, \infty)$.

C. Note that $|g(x)| = |g(x)| \mathbb{1}_{\{|g(x)| < M\}} + |g(x)| \mathbb{1}_{\{|g(x)| \geq M\}}$.

D. Note that $|g(x)| = |g(x)| \mathbb{1}_{\{|g(x)| < 1\}} + |g(x)| \mathbb{1}_{\{|g(x)| \geq 1\}}$.

E. By part (a), choose $M > 0$ such that $\int_{\mathbb{R}} |g(x)| \mathbb{1}_{\{|g(x)| \geq M\}} dx \leq \frac{\epsilon}{2}$.

F. By part (a), choose $M > 0$ such that $\int_{\mathbb{R}} |g(x)| \mathbb{1}_{\{|g(x)| \geq 1\}} dx \leq M$.

G. Take $\delta = \frac{\epsilon}{2M}$. Let $E \subseteq \mathbb{R}$ be such that $\lambda(E) \leq \delta$.

H. Let $E \subseteq \mathbb{R}$ be such that $\lambda(E) \leq \epsilon$. Take $\delta = \frac{\epsilon}{2M}$.

I. Note that $\int_E |g(x)| \mathbb{1}_{\{|g(x)| < M\}} dx \leq \int_{\mathbb{R}} |g(x)| \mathbb{1}_{\{|g(x)| < M\}} dx$.

J. Note that $\int_E |g(x)| \mathbb{1}_{\{|g(x)| \geq M\}} dx \leq \int_{\mathbb{R}} |g(x)| \mathbb{1}_{\{|g(x)| \geq M\}} dx$.

K. By monotonicity of the integral, $\int_E |g(x)| \mathbb{1}_{\{|g(x)| < M\}} dx \leq \int_E M dx = M\lambda(E)$.

L. By monotonicity of the integral, $\int_E |g(x)| \mathbb{1}_{\{|g(x)| < M\}} dx \leq \int_E |g(x)| dx \leq M$.

M. Hence $\int_E |g(x)| dx \leq \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon$

Seven of these statements, when arranged into the correct order, prove the fact stated in italics. The other five statements are not required.

Which seven statements are required?

Solution.

(a) Define $f_n(x) = |f(x)| \mathbb{1}_{|f(x)| \geq n}$.

Note that, since $f(x) \in \mathbb{R}$ we have $f_n(x) \rightarrow 0$ for all $x \in \mathbb{R}$ (because for large enough n we have $|f(x)| < n$). [1]

Also, $|f_n(x)| \leq |f(x)|$ for all $n \in \mathbb{N}$, which provides a dominating function because f is integrable. [1] Hence by the dominated convergence theorem,

$$\int_{\mathbb{R}} f_n(x) dx \rightarrow 0$$

as required. [1]

(b) Statements A, C, E, G, J, K, M are required.

[Score $\max(n - 3, 0)$ where n is the number of correct statements included, plus one mark for no other statements included. (If more than seven statements given in answer, consider only the first seven given.)]

3. (a) Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0, 1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2] = 0$.

- (b) Let $(X_n)_{n \in \mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.

Solution.

- (a) We have $\mathbb{P}[U_n \leq a] = a$. For any (deterministic) sequence (x_n) the events $\{U_n < x_n\}$ are independent, because the U_n are independent. [1] Noting that $\sum 1/n = \infty$ and $\sum 1/n^2 < \infty$, we have $\sum_n \mathbb{P}[U_n < 1/n] = \infty$ and $\sum_n \mathbb{P}[U_n < 1/n^2] < \infty$. [1] By the second Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and by the first Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$. [1]
- (b) Let $E_n = \{X_n + i = i \text{ for } i = 1, 2, 3, 4, 5, 6\}$. We have $\mathbb{P}[E_n] = (1/6)^6 > 0$. Note that E_n and E_{n+6} are independent (but E_n and E_{n+1} are not!). [1] We have $\sum_{n=1}^{\infty} \mathbb{P}[E_{6n}] = \sum_{n=1}^{\infty} (1/6)^6 = \infty$, [1] hence by the second Borel-Cantelli lemma we have $\mathbb{P}[E_{6n} \text{ i.o.}] = 1$. [1] Noting that $\{E_{6n} \text{ i.o.}\} \subseteq \{E_n \text{ i.o.}\}$, we have $\mathbb{P}[E_n \text{ i.o.}] = 1$.