MAS350: Assignment 3

1. Determine if the following functions are integrable.

   (a) $f : (0, \infty) \to \mathbb{R}$ by $f(x) = 1/x^2$.

   (b) $g : (0, 1) \to \mathbb{R}$ by $g(x) = \log x$

2. In this question we work with Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x)| \, \mathbb{1}_{\{|f(x)|\geq n\}} \, dx = 0.$$

   (b) Let $g : \mathbb{R} \to \mathbb{R}$ be integrable. It is a fact that: *for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\textit{if } E \subseteq \mathbb{R} \textit{ satisfies } \lambda(E) \leq \delta \textit{ then } \int_E |g| \, d\mu \leq \epsilon.$$

   Consider the following twelve statements.

   A. Let $\epsilon > 0$.

   B. Note that $|g(x)| \leq M$, for some $M \in (0, \infty)$.

   C. Note that $|g(x)| = |g(x)|\mathbb{1}_{\{|g(x)|<M\}} + |g(x)|\mathbb{1}_{\{|g(x)|\geq M\}}$.

   D. Note that $|g(x)| = |g(x)|\mathbb{1}_{\{|g(x)|<1\}} + |g(x)|\mathbb{1}_{\{|g(x)|\geq 1\}}$.

   E. By part (a), choose $M > 0$ such that $\int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|\geq M\}} \, dx \leq \dfrac{\epsilon}{2M}$.

   F. By part (a), choose $M > 0$ such that $\int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|\geq 1\}} \, dx \leq M$.

   G. Take $\delta = \frac{\epsilon}{2M}$. Let $E \subseteq \mathbb{R}$ be such that $\lambda(E) \leq \delta$.

   H. Let $E \subseteq \mathbb{R}$ be such that $\lambda(E) \leq \epsilon$. Take $\delta = \frac{\epsilon}{2M}$.

   I. Note that $\int_E |g(x)|\mathbb{1}_{\{|g(x)|<M\}} \, dx \leq \int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|<M\}} \, dx$.

   J. Note that $\int_E |g(x)|\mathbb{1}_{\{|g(x)|\geq M\}} \, dx \leq \int_{\mathbb{R}} |g(x)|\mathbb{1}_{\{|g(x)|\geq M\}} \, dx$.

   K. By monotonicity of the integral, $\int_E |g(x)|\mathbb{1}_{\{|g(x)|<M\}} \, dx \leq \int_E M \, dx = M\lambda(E)$.

   L. By monotonicity of the integral, $\int_E |g(x)|\mathbb{1}_{\{|g(x)|<M\}} \, dx \leq \int_E |g(x)| \, dx \leq M$.

   M. Hence $\int_E |g(x)| \, dx \leq \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon$

   Seven of these statements, when arranged into the correct order, prove the fact stated in italics. The other five statements are not required.

   Which seven statements are required?

3. (a) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0, 1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2] = 0$.

   (b) Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.