MASx52: Assignment 5

1. Let S_t be a geometric Brownian motion, with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and (deterministic) initial condition S_0.

   (a) Find $\mathbb{E}[S_t]$ and deduce that S_t is not a Brownian motion when $\mu \neq 0$.

   (b) Is S_t a Brownian motion when $\mu = 0$?

2. Consider the SDE
$$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$

   (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation
$$f'(t) = t + f(t)$$
   Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

   (b) Let $Y_t = X_t^2$. Show that
$$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

   (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation
$$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

3. Let $T > 0$. Use the Feynman-Kac formula to find an explicit solution $F(x,t)$ to the partial differential equation
$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2}(x,t) = 0$$
   subject to the boundary condition $F(T,x) = x - \frac{T}{2}$.

   Hint: It may help to recall that $\int_0^t B_u\,dB_u = \frac{B_t^2}{2} - \frac{t}{2}$.

4. (a) Within the Black-Scholes model, use the risk neutral valuation formula to find the prices at time t of the contingent claims

      i. $\Phi(S_T) = 3S_T + 5$, where $0 \leq t \leq T$.

      ii. $\Psi(S_T) = S_1 S_T + 1$, where $1 \leq t \leq T$.

   (b) For a portfolio containing a single contract with contingent claim $\Phi(S_T)$:

      i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

      ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

   (c) Suggest one reason why we might want to hold a delta neutral portfolio.