The stochastic heat equation and total positivity

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The fundamental solution $u(t,x,y)$ to the stochastic heat equation in one space dimension:

$$\partial_t u(t,x,y) = \Big(\frac{1}{2}\Delta_y + \dot{W}(t,y)\Big)u(t,x,y), \quad u(0,x,y) = \delta(x-y),$$

is a totally positive kernel, meaning $\det\{u(t,x_i,y_j)\}_{i,j=1}^n > 0$ whenever $x_1 < x_2 \ldots < x_n$ and $y_1 < y_2 \ldots < y_n$. This can be understood by interpreting such a determinant via non-crossing probabilities for a continuum random polymer. An extended version of total positivivity holds:

$$\frac{\det\{u(t,x_i,y_j)\}}{\Delta(\mathbf{x})\Delta(\mathbf{y})}$$

extends continuously to a strictly positive function on $\mathbf{R}^n \times \mathbf{R}^n$ where $\Delta(\mathbf{x}) = \prod(x_i - x_j)$. I will describe joint work with Chin Lun in which we use this property to define a dynamical version of the KPZ line ensemble.