

MAS350: Assignment 2

1. (a) Let O_1 and O_2 be open subsets of \mathbb{R} . Show that $O_1 \cup O_2$ and $O_1 \cap O_2$ are also open.
(b) For each $n \in \mathbb{N}$ let O_n be an open subset of \mathbb{R} . Consider the following claims:

- i. $A = \bigcup_{n \in \mathbb{N}} O_n$ is open.
ii. $B = \bigcap_{n \in \mathbb{N}} O_n$ is open.

Which of these claims are true? Give a proof or a counterexample in each case.

- (c) A set $C \subseteq \mathbb{R}$ is said to be *closed* if $\mathbb{R} \setminus C$ is measurable. Which of your results from parts (a) and (b) hold for closed sets?

2. In each of the following cases, show that the given function is measurable, from $\mathbb{R} \rightarrow \mathbb{R}$ with the Borel σ -field. State clearly any results from lectures that you make use of.

(a) $f(x) = x$

(b) $g(x) = \cos x$

(c) $h(x) = \begin{cases} 0 & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0. \end{cases}$

3. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

- (a) Let $f : S \rightarrow \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.
(b) Let $f : S \rightarrow \mathbb{R}$ be a simple function. Show that

$$\left(\int_S f \, dm \right)^2 \leq \int_S f^2 \, dm. \quad (\star)$$

Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

$$\frac{(\sum_{i=1}^n u_i)^2}{\sum_{i=1}^n v_i} \leq \sum_{i=1}^n \frac{u_i^2}{v_i}.$$

- (c) In this question you should give *two* different proofs that equation (\star) holds when f is any non-negative measurable function.
i. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.
ii. Give a proof using the monotone convergence theorem.
(d) Does (\star) remain true if m is not necessarily a probability measure?