MAS350: Assignment 2

Solutions and discussion are written in blue. A sample mark scheme, with a total of 30 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. (a) Let O_1 and O_2 be open subsets of $\mathbb{R}$. Show that $O_1 \cup O_2$ and $O_1 \cap O_2$ are also open.

   (b) For each $n \in \mathbb{N}$ let O_n be an open subset of $\mathbb{R}$. Consider the following claims:

   i. $A = \bigcup_{n\in\mathbb{N}} O_n$ is open.

   ii. $B = \bigcap_{n\in\mathbb{N}} O_n$ is open.

   Which of these claims are true? Give a proof or a counterexample in each case.

   (c) A set $C \subseteq \mathbb{R}$ is said to be *closed* if $\mathbb{R} \setminus C$ is measurable. Which of your results from parts (a) and (b) hold for closed sets?

Solution.

(a) Let $x \in O_1 \cup O_2$. Then since $x \in O_1$ there is an open interval I_1 containing x. Thus I_1 is an open interval within $O_1 \cup O_2$ containing x, so $O_1 \cup O_2$ is open. [1]

Now let $x \in O_1 \cap O_2$. Then for each $i = 1, 2$ we have an open interval $I_i \subseteq O_i$ containing x. [1] Let us write $I_1 = (a_1, b_1), I_2 = (a_2, b_2)$, and $c_1 = \max(a_1, b_1)$, $c_2 = \min(a_2, b_2)$. Then $(c_1, c_2) = I_1 \cap I_2$, and since $x \in I_1 \cap I_2$ we have $x \in (c_1, c_2)$. In particular this means $c_1 < c_2$, so $I_1 \cap I_2$ is an open interval. [1] Also $I_1 \cap I_2 \subseteq O_1 \cap O_2$, so $O_1 \cap O_2$ is open.

(b) i. This is true. We can use exactly the same method as in part (a): let $x \in \cup_n O_n$ and then since $x \in O_1$ we have an open interval $I_1 \subseteq O_1$ containing x, then $I_1 \subseteq \cup_n O_n$, and we are done. [1]

ii. This is false. A counterexample is given by $O_n = (\frac{-1}{n}, 1 + \frac{1}{n})$, for which $\cap_n O_n = [0, 1]$. [1]

(c) Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed sets. Using set operations we have

$$\begin{aligned}
\mathbb{R} \setminus (C_1 \cup C_2) &= (R \setminus C_1) \cap (\mathbb{R} \setminus C_2) \\
\mathbb{R} \setminus (C_1 \cap C_2) &= (R \setminus C_1) \cup (\mathbb{R} \setminus C_2) \\
\mathbb{R} \setminus \left(\bigcup_n C_n\right) &= \bigcap_n (\mathbb{R} \setminus C_n) \\
\mathbb{R} \setminus \left(\bigcap_n C_n\right) &= \bigcup_n (\mathbb{R} \setminus C_n)
\end{aligned}$$

The first two equations combined with part (a) tell us that both the results of part (a) carry over to closed sets: both $C_1 \cap C_2$ and $\mathbb{C}_1 \cup C_2$ are closed. [1]

From the fourth equation, since $\mathbb{R} \setminus C_n$ is open (for all n), using (b)(i) we see that $\mathbb{R} \setminus (\bigcup_n C_n)$ is also open, hence $\bigcap_n C_n$ is closed. [1]

However, we can't do the same for the third equation, because (b)(ii) was false. [1] Instead, we can take complements of our counterexample in (b)(ii) to find a counterexample here, giving $C_n = \mathbb{R} \setminus (\frac{-1}{n}, 1 + \frac{1}{n}) = (-\infty, \frac{-1}{n}] \cup [1 + \frac{1}{n}, \infty)$. Then $\cup_n C_n = (-\infty, 0) \cup (1, \infty)$ which is not closed (because its complement $[0, 1]$ is not open). [1]

2. In each of the following cases, show that the given function is measurable, from $\mathbb{R} \to \mathbb{R}$ with the Borel σ-field. State clearly any results from lectures that you make use of.

   (a) $f(x) = x$

   (b) $g(x) = \cos x$

   (c) $h(x) = \begin{cases} 0 & \text{for } x < 0 \\ x+1 & \text{for } x \geq 0. \end{cases}$

   Solution.

   (a) We'll use the (original) definition of a measurable function. [1] With $f(x) = x$, $f^{-1}((c,\infty)) = (c,\infty)$, which is a measurable set, so f is a measurable function. [1]

   (b) From lectures, every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is measurable. [1] Since cos is continuous, it is measurable. [1]

   (c) Let $g_1(x) = \mathbb{1}_{[0,\infty)}(x)$ be the indicator function of $[0,\infty)$, which is measurable because it is the indicator function of a measurable set. [1] Let

   $$g_2(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

   which is measurable because it is continuous. Then $g(x) = g_1(x) + g_2(x)$ is measurable, because the sum of measurable functions is measurable. [1]

3. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

   (a) Let $f : S \to \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.

   (b) Let $f : S \to \mathbb{R}$ be a simple function. Show that

   $$\left(\int_S f\,dm\right)^2 \leq \int_S f^2\,dm. \qquad (\star)$$

   Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

   $$\frac{\left(\sum_{i=1}^n u_i\right)^2}{\sum_{i=1}^n v_i} \leq \sum_{i=1}^n \frac{u_i^2}{v_i}.$$

   (c) In this question you should give *two* different proofs that equation $(\star)$ holds when f is any non-negative measurable function.

      i. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.

      ii. Give a proof using the monotone convergence theorem.

   (d) Does $(\star)$ remain true if m is not necessarily a probability measure?

   Solution.

(a) Since f is simple we have the representation $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ where the (A_i) are disjoint and measurable and $c_i \geq 0$. Therefore

$$f^2 = \sum_{i=1}^n \sum_{j=1}^m c_i c_j \mathbb{1}_{A_i} \mathbb{1}_{A_j} = \sum_{i=1}^n c_i^2 \mathbb{1}_{A_i}$$

where the second inequality follows by disjointness – all the cross terms are zero. [1] We have thus expressed f^2 as a simple function, and since c_i^2 are non-negative, f^2 is also non-negative. [1]

(b) We have

$$\left(\int f\,dm\right)^2 = \left(\sum_{i=1}^n c_i m(A_i)\right)^2,$$

$$\int f^2\,dm = \sum_{i=1}^n c_i^2 m(A_i).$$

[2] The required inequality follows from the above and Titu's lemma, taking $v_i = m(A_i)$ and $u_i = c_i m(A_i)$. [1] Note that, because m is a probability measure, $\sum_i m(A_i) = 1$ and we may assume $m(A_i) > 0$ (because any A_i with zero measure have no effect on the value of the integral).

Follow-up exercise: See if you can derive Titu's lemma from the real version of the Cauchy-Schwarz inequality.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be non-negative and measurable.

First proof: Recall that the definition of the Lebesgue integral, for non-negative measurable functions, is

$$\int f\,dm = \sup\left\{\int s\,dm \;:\; s \text{ is simple and } 0 \leq s \leq f\right\}.$$

Hence

$$\begin{aligned}
\left(\int f\,dm\right)^2 &= \left(\sup\left\{\int s\,dm \;:\; s \text{ is simple and } 0 \leq s \leq f\right\}\right)^2 \\
&= \sup\left\{\left(\int s\,dm\right)^2 \;:\; s \text{ is simple and } 0 \leq s \leq f\right\} \\
&\leq \sup\left\{\int s^2\,dm \;:\; s \text{ is simple and } 0 \leq s \leq f\right\} \\
&= \sup\left\{\int r\,dm \;:\; r \text{ is simple and } 0 \leq r \leq f^2\right\} \\
&= \int f^2\,dm
\end{aligned}$$

Here, the second line follows because $\int s\,dm \geq 0$, so the square can pass inside of the sup. [1] The third line then follows by part (b). [1] Let us now justify the fourth line. We have shown in (a) that if s is a non-negative simple function then so is $r = s^2$, and clearly if $s \leq f$ then $s^2 \leq f^2$ (i.e. pointwise). [1] Also, if r is a non-negative simple function such that $0 \leq r \leq f^2$, then if we define $s = \sqrt{r}$, we can show (in similar style to part (a)) that s is a non-negative simple function such that $0 \leq s \leq f$. Here, if

$r = \sum_i c_i \mathbb{1}_{A_i}$ we would have $s = \sum_i \sqrt{c_i} \mathbb{1}_{A_i}$. So, the two sups in the third and fourth lines are equal using the correspondence $r = s^2$. [1]

Second proof: Now we will allow ourselves to use the monotone convergence theorem. From lectures (see the section on simple functions) there exists a sequence (s_n) of non-negative simple functions such that $0 \le s_n \le s_{n+1} \le f$ such that $s_n \to f$ pointwise. [1] Thus, by the monotone convergence theorem, as $n \to \infty$,

$$\int s_n \, dm \to \int f \, dm.$$

[1] By part (a), (s_n^2) is also a sequence of simple functions. [1] We have $0 \le s_n^2 \le s_{n+1}^2 \le f^2$, also $s_n^2 \to f^2$ pointwise. So by another application of the monotone convergence theorem we have

$$\int s_n^2 \, dm \to \int f^2 \, dm.$$

[1] From part (b) we have

$$\left(\int s_n \, dm \right)^2 \le \int s_n^2 \, dm$$

for all n. Since limits preserve weak inequalities, [1] we have that

$$\left(\int f \, dm \right)^2 \le \int f^2 \, dm$$

as required.

(d) In general ($\star$) fails when m is not a probability measure. For example, take $f(x) = x$ and let m be Lebesgue measure on $[0, 2]$. Then $\int_0^2 x \, dx = 2$ and $\int_0^2 x^2 \, dx = \frac{8}{3}$, but $2^2 > \frac{8}{3}$. [1]