MASx52: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme, with a total of 30 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

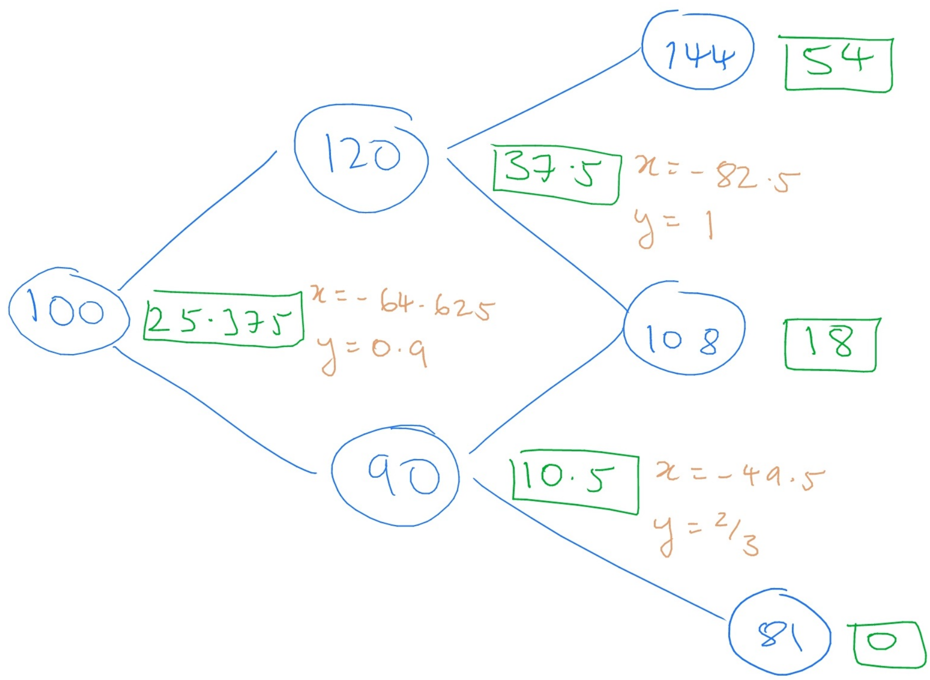
1. Consider the binomial model with $r = \frac{1}{11}$, $d = 0.9$, $u = 1.2$, $s = 100$ and time steps $t = 0, 1, 2$.

   (a) Draw a recombining tree of the stock price process, for time $t = 0, 1, 2$.

   (b) Find the value, at time $t = 0$, of a European call option that gives its holder the option to purchase one unit of stock at time $t = 2$ for a strike price $K = 90$. Write down the hedging strategy that replicates the value of this contract, at all nodes of your tree.

   You may annotate your tree from (a) to answer (b).

   Solution. As in the lecture notes, we write the value of a unit of stock (in blue) inside the nodes of the tree, to answer (a), and write the value of the contingent claim at the various nodes, in square boxes (in green), next to the nodes themselves; the answer to the first part of (b) appears at the root node. For the second part of (b), the replicating portfolios $h = (x, y)$ that would be held at each node are written (in orange) as $x = \ldots, y = \ldots$.



   (To find these numbers you will need to either solve suitable linear equations and/or use the risk neutral valuation formula – see the lecture notes for details.) [2, for (a)], [7, for (b)].

2. Let $S_n = \sum_{i=1}^{n} X_i$, be a random walk, in which $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common distribution $\mathbb{P}[X_i = \frac{1}{i^2}] = \mathbb{P}[X_i = -\frac{1}{i^2}] = \frac{1}{2}$.

(a) Show that $\mathbb{E}[|S_n|] \le \sum_{i=1}^{n} \frac{1}{i^2}$.

(b) Explain briefly why part (a) means that S_n is bounded in L^1.

(c) Show that there exists a random variable S_∞ such that $S_n \stackrel{a.s.}{\to} S_\infty$ as $n \to \infty$.

Solution.

(a) Using the triangle inequality, and monotonicity of $\mathbb{E}$, [1]

$$\mathbb{E}[|S_n|] = \mathbb{E}\left[\left|\sum_{i=1}^{n} X_i\right|\right] \le \mathbb{E}\left[\sum_{i=1}^{n} |X_i|\right] = \mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{i^2}\right] = \sum_{i=1}^{n} \frac{1}{i^2}.$$

[1]

(b) From part (a) we have $\mathbb{E}[|S_n|] \le \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$, which is finite [1] and independent of n. Hence $\sup_n \mathbb{E}[|S_n|] < \infty$. [1]

(c) We aim to use the martingale convergence theorem.[1] We must check that (S_n) is a martingale.

We use the filtration $\mathcal{F}_n = \sigma(X_i : i = 1, \dots, n)$. Since $X_i \in m\mathcal{F}_n$, we have $S_n \in m\mathcal{F}_n$.[1] We have already shown in (a) that $\mathbb{E}[|S_n|] < \infty$, so $S_n \in L^1$. [1] Lastly,

$$\begin{aligned}\mathbb{E}[S_{n+1} \,|\, \mathcal{F}_n] &= \mathbb{E}[X_{n+1} + S_n \,|\, \mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1}] + S_n \\ &= S_n.\end{aligned}$$

[1] Here we use that $S_n \in m\mathcal{F}_n$, [1] that X_{n+1} is independent of $\mathcal{F}_n$, [1] and that $\mathbb{E}[X_{n+1}] = 0$. [1]

3. (a) Let Z be a random variable taking values in $[1, \infty)$ and for $n \in \mathbb{N}$ define

$$X_n = \begin{cases} Z & \text{if } Z \in [n, n+1) \\ 0 & \text{otherwise.} \end{cases} \qquad (\star)$$

Suppose that $Z \in L^1$. Use the dominated convergence theorem to show that $\mathbb{E}[X_n] \to 0$ as $n \to \infty$.

(b) Instead, let Z be the continuous random variable with probability density function

$$f(x) = \begin{cases} x^{-2} & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

and define X_n using $(\star)$. Show that Z is not in L^1, but that $\mathbb{E}[X_n] \to 0$.

(c) Comment on what part (b) tells us about the dominated convergence theorem.

Solution.

(a) We look to use the dominated convergence theorem. For any $\omega \in \Omega$ we have $Z(\omega) < \infty$, hence for all $n \in \mathbb{N}$ such that $n > Z(\omega)$ we have $X_n(\omega) = 0$.[1] Therefore, as $n \to \infty$, $X_n(\omega) \to 0$, which means that $X_n \to 0$ almost surely. [1]

We have $|X_n| \le Z$ and $Z \in L^1$, so we can use Z are the dominating random variable. [1] Hence, by the dominated convergence theorem, $\mathbb{E}[X_n] \to \mathbb{E}[0] = 0$. [1]

(b) We have

$$\mathbb{E}[Z] = \int_1^\infty x f(x)\,dx = \int_1^\infty x^{-1}\,dx = [\log x]_1^\infty = \infty.$$

[1], which means $Z \notin L^1$ [1] and also that

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[\mathbb{1}_{\{Z\in[n,n+1)\}}Z] = \int_1^\infty \mathbb{1}_{\{x\in[n,n+1)\}} x f(x)\,dx \\ &= \int_n^{n+1} x^{-1}\,dx = [\log x]_n^{n+1} = \log(n+1) - \log n = \log\left(\frac{n+1}{n}\right).\end{aligned}$$

[1] As $n \to \infty$, we have $\frac{n+1}{n} = 1 + \frac{1}{n} \to 1$, hence (using that log is a continuous function) we have $\log(\frac{n+1}{n}) \to \log 1 = 0$. Hence, $\mathbb{E}[X_n] \to 0$. [1]

(c) Suppose that we wanted to use the DCT in (b). We still have $X_n \to 0$ almost surely, but any dominating random variable Y would have to satisfy $Y \geq |X_n|$ for all n, meaning that also $Y \geq Z$, which means that $\mathbb{E}[Y] \geq \mathbb{E}[Z] = \infty$; thus there is no dominating random variable $Y \in L^1$. [1] Therefore, we can't use the DCT here, but we have shown in (b) that the conclusion of the DCT does hold: [1] we have that $\mathbb{E}[X_n]$ does tend to zero.

We obtain that the conditions of the DCT are *sufficient* but not *necessary* for its conclusion to hold. [1]