

MASx52: Assignment 5

1. Consider the SDE

$$dX_t = (t + X_t) dt + 2t dB_t.$$

- (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation

$$f'(t) = t + f(t)$$

Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

- (b) Let $Y_t = X_t^2$. Show that

$$dY_t = 2(2t^2 + tX_t + X_t^2) dt + 4tX_t dB_t$$

- (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation

$$v'(t) = 2(2t^2 + tf(t) + v(t)).$$

2. Let $T > 0$. Use the Feynman-Kac formula to find an explicit solution $F(x, t)$ to the partial differential equation

$$\frac{\partial F}{\partial t}(t, x) + \frac{1}{2} \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} x^2 \frac{\partial^2 F}{\partial x^2}(x, t) = 0$$

subject to the boundary condition $F(T, x) = x - \frac{T}{2}$.

Hint: It may help to recall that $\int_0^t B_u dB_u = \frac{B_t^2}{2} - \frac{t}{2}$.

3. (a) Within the Black-Scholes model, use the risk neutral valuation formula to find the prices at time t of the contingent claims
- $\Phi(S_T) = 3S_T + 5$, where $0 \leq t \leq T$.
 - $\Psi(S_T) = S_1 S_T + 1$, where $1 \leq t \leq T$.
- (b) For a portfolio containing a single contract with contingent claim $\Phi(S_T)$:
- Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.
 - If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?
- (c) Suggest one reason why we might want to hold a delta neutral portfolio.