MAS350: Assignment 2

1. (a) Let O_1 and O_2 be open subsets of $\mathbb{R}$. Show that $O_1 \cup O_2$ and $O_1 \cap O_2$ are also open.

   (b) For each $n \in \mathbb{N}$ let O_n be an open subset of $\mathbb{R}$. Consider the following claims:

      i. $A = \bigcup_{n\in\mathbb{N}} O_n$ is open.

      ii. $B = \bigcap_{n\in\mathbb{N}} O_n$ is open.

   Which of these claims are true? Give a proof or a counterexample in each case.

   (c) A set $C \subseteq \mathbb{R}$ is said to be *closed* if $\mathbb{R} \setminus C$ is open. Which of your results from parts (a) and (b) hold for closed sets?

2. In each of the following cases, show that the given function is measurable, from $\mathbb{R} \to \mathbb{R}$ with the Borel σ-field. State clearly any results from lectures that you make use of.

   (a) $f(x) = x$

   (b) $g(x) = \cos x$

   (c) $h(x) = \begin{cases} 0 & \text{for } x < 0 \\ x+1 & \text{for } x \geq 0. \end{cases}$

3. Let (S, Σ, m) be a measure space, and suppose that m is a probability measure.

   (a) Let $f : S \to \mathbb{R}$ be a non-negative simple function. Show that f^2 is also a non-negative simple function.

   (b) Let $f : S \to \mathbb{R}$ be a simple function. Write $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ where the A_i are pairwise disjoint and measurable and $c_i \geq 0$. Show that

$$\left(\int_S f\,dm\right)^2 \leq \int_S f^2\,dm. \tag{$\star$}$$

   Hint: You may use Titu's lemma, which states that for $u_i \geq 0$ and $v_i > 0$,

$$\frac{\left(\sum_{i=1}^n u_i\right)^2}{\sum_{i=1}^n v_i} \leq \sum_{i=1}^n \frac{u_i^2}{v_i}.$$

   (c) In this question you should give *two* different proofs that equation ($\star$) holds when f is any non-negative measurable function.

      i. Give a proof based on the definition of the Lebesgue integral for non-negative measurable functions.

      ii. Give a proof using the monotone convergence theorem.

   (d) Does ($\star$) remain true if m is not necessarily a probability measure?