MASx52: Assignment 5

1. Consider the SDE
$$dX_t = (t + X_t)\,dt + 2t\,dB_t.$$

   (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation
   $$f'(t) = t + f(t)$$
   Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

   (b) Let $Y_t = X_t^2$. Show that
   $$dY_t = 2\left(2t^2 + tX_t + X_t^2\right)\,dt + 4tX_t\,dB_t$$

   (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation
   $$v'(t) = 2\left(2t^2 + tf(t) + v(t)\right).$$

2. (a) Within the Black-Scholes model, use the risk neutral valuation formula to find the prices at time t of the contingent claims

      i. $\Phi(S_T) = 3S_T + 5$, where $0 \le t \le T$.

      ii. $\Psi(S_T) = S_1 S_T + 1$, where $1 \le t \le T$.

   (b) With the same contingents claims as in (a):

      i. Describe a constant portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.

      ii. Is it possible to replicate $\Psi(S_T)$ using a constant portfolio?

   (c) For a portfolio containing a single contract with contingent claim $\Phi(S_T)$:

      i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

      ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

   (d) Suggest one reason why we might want to hold a delta neutral portfolio.

3. **[On Semester 1]** Consider an urn, containing two colours of balls, black and red. At time $n = 0$, the urn contains one black ball and one red ball. Then, at each time $n = 1, 2, \ldots$, we do the following:

   - Draw a ball from the urn. Record the colour of this ball and place it back into the urn.
   - Add two new balls to the urn, of the same colour as the drawn ball.

   Therefore, at time n, the urn contains $2 + 2n$ balls. Let B_n denote the number of red balls in the urn, and let
   $$M_n = \frac{B_n}{2 + 2n}.$$

   (a) Show that M_n is a martingale, with respect to the filtration $\mathcal{F}_n = \sigma(B_i \,:\, i \le n)$.

   (b) Deduce that there exists a random variable M_∞ such that $M_n \stackrel{a.s.}{\to} M_\infty$.

   (c) Show that $\mathbb{P}[M_n \le \frac{1}{2}] = \mathbb{P}[M_n \ge \frac{1}{2}]$ for all n.