

## MASx52: Assignment 2

Solutions and discussion are written in blue. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Let  $(X_n)$  be a sequence of i.i.d. random variables, each with a uniform distribution on the interval  $[-1, 1]$ . Define

$$S_n = \sum_{i=1}^n X_i,$$

where  $S_0 = 0$ . Let  $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ .

- (a) Show that  $S_n$  is a martingale, with respect to the filtration  $\mathcal{F}_n$ .  
(b) Find  $\mathbb{E}[S_3^2 | \mathcal{F}_2]$  in terms of  $X_2$  and  $X_1$ , and hence show that

$$\mathbb{E}[S_3^2 | \mathcal{F}_2] = S_2^2 + \frac{1}{3}.$$

- (c) Write down a deterministic function  $f : \mathbb{N} \rightarrow \mathbb{R}$  such that

$$M_n = S_n^2 - f(n)$$

is a martingale (justification is not required – make a guess!).

*Solution.*

- (a) Since  $X_i \in \sigma(X_i)$  we have  $X_i \in \mathcal{F}_n$  for all  $i \leq n$  [1]. Hence, since sums of  $\mathcal{F}_n$  measurable functions are measurable, we have also that  $S_n \in \mathcal{F}_n$  [1].

Since  $|X_i| \leq 1$  for all  $i$ , we have

$$|S_n| \leq |X_1| + |X_2| + \dots + |X_n| \leq n.$$

Thus  $S_n$  is a bounded random variable and hence  $S_n \in L^1$ . [1]

Lastly,

$$\begin{aligned} \mathbb{E}[S_{n+1} | \mathcal{F}_n] &= \mathbb{E}[X_{n+1} + S_n | \mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1} | \mathcal{F}_n] + \mathbb{E}[S_n | \mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1}] + S_n \\ &= S_n. \end{aligned}$$

[1] Here, we use the linearity of conditional expectation to deduce the second line, followed by using that  $X_{n+1}$  is independent of  $\mathcal{F}_n$  [1] and  $S_n \in \mathcal{F}_n$  to deduce the third line [1]. The final line follows because  $\mathbb{E}[X_i] = 0$  for all  $i$ . Hence  $S_n$  is a martingale.

*Feedback:* You should justify your use of the rules of conditional expectation.

(b) We have

$$S_n^3 = (X_1 + X_2 + X_3)^2 = X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 + 2X_2X_3 + 2X_1X_3.$$

[1] Hence,

$$\begin{aligned}\mathbb{E}[S_3^2 | \mathcal{F}_2] &= \mathbb{E}[X_1^2 | \mathcal{F}_n] + \mathbb{E}[X_2^2 | \mathcal{F}_2] + \mathbb{E}[X_3^2 | \mathcal{F}_2] \\ &\quad + 2\mathbb{E}[X_1X_2 | \mathcal{F}_2] + 2\mathbb{E}[X_2X_3 | \mathcal{F}_2] + 2\mathbb{E}[X_1X_3 | \mathcal{F}_2] \\ &= X_1^2 + X_2^2 + \mathbb{E}[X_3^2] + 2X_1X_2 + 2X_2\mathbb{E}[X_3] + 2X_1\mathbb{E}[X_3] \\ &= (X_1 + X_2)^2 + \frac{1}{3} \\ &= S_2^2 + \frac{1}{3}.\end{aligned}$$

[1]. Here, in the first line we use linearity of conditional expectation. To deduce the second line we use that  $X_3$  is independent of  $\mathcal{F}_2$  [1], and that  $X_1, X_2 \in m\mathcal{F}_2$  to ‘take out what is known’ [1]. We then use that

$$\mathbb{E}[X_3^2] = \int_{-1}^1 x^2 \frac{1}{2} dx = \frac{1}{3}$$

to deduce the final lines [1].

*Feedback:* Note that  $X_n$  has the *continuous* uniform distribution on the interval  $[-1, 1]$ .

(c) In view of (b), we take  $f(n) = \frac{n}{3}$ , so that  $M_n = S_n - \frac{n}{3}$  [2].

*To make this guess: note from (b) that  $\mathbb{E}[S_n^2]$  drifts upwards by  $\frac{1}{3}$  on each time step, so  $\mathbb{E}[S_n^2 - \frac{n}{3}]$  stays constant. On each step of time, we need to compensate by  $\frac{-1}{3}$ .*

*This is the only way to compensate for the drift in the form  $S_n - f(n)$ , and (possibly) obtain a martingale. To see that  $M_n$  really is a martingale: Since  $S_n \in \mathcal{F}_n$  we have  $M_n \in \mathcal{F}_n$ , and  $|M_n| \leq |S_n^2| + \frac{2n}{3} \leq n^2 + \frac{n}{3}$  so  $M_n \in L^1$ . A similar calculation to (b) then shows that  $\mathbb{E}[S_{n+1}^2 | \mathcal{F}_n] = S_n^2 + \frac{1}{3}$ , hence  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$ .*

2. Consider the one-period market with  $r = \frac{1}{10}$ ,  $s = 2$ ,  $d = \frac{1}{2}$  and  $u = 3$ , in our usual notation. A contract specifies that

*The holder of the contract will sell 2 units of stock, and be paid  $K$  units of cash, at time 1.*

(a) Explain briefly why the contingent claim of this contract is

$$\Phi(S_1) = K - 2S_1.$$

(b) Find a replicating portfolio  $h$  for this contingent claim.

(c) Write down the value  $V_0^h$  of  $h$  at time 0.

(d) Find the numerical values of risk-neutral probabilities

$$q_u = \frac{(1+r) - d}{u - d} \quad \text{and} \quad q_d = \frac{u - (1+r)}{u - d}.$$

Hence, check that  $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)]$  and  $V_0^h$  have the same values.

- (e) For which  $K$  does the contract have value zero at time 0?

*Solution.*

- (a) The holder will be paid  $K$  units of cash, resulting in a gain of  $K$ , and give away 2 units of stock, each of which is worth  $S_1$ , resulting in a loss of  $2S_1$ . [1] Hence

$$\Phi(S_1) = K - 2S_1.$$

*Feedback:* This is not a European put option. The holder of this contract *must* pay  $K$  units of cash and be given 2 stock.

- (b) The possible values taken by  $S_1$  are  $su = 6$  and  $sd = 1$ . A replicating portfolio  $h = (x, y)$  must satisfy  $V_1^h = \Phi(S_1)$ , [1] meaning that

$$\begin{aligned} (1 + \frac{1}{10})x + 6y &= K - 12 \\ (1 + \frac{1}{10})x + y &= K - 2 \end{aligned}$$

[2] We now solve these equations. Taking one away from the other, we obtain  $5y = -10$ , hence  $y = -2$  which gives  $x = \frac{K}{11/10} = \frac{10K}{11}$ . [1]

- (c) The value of the contract is

$$V_0^h = x + sy = \frac{10K}{11} - 4$$

[1] at time 0.

- (d) The risk-neutral probabilities are

$$\begin{aligned} q_u &= \frac{11/10 - 1/2}{3 - 1/2} = \frac{3/5}{5/2} = \frac{6}{25}, \\ q_d &= \frac{3 - 11/10}{3 - 1/2} = \frac{19/10}{5/2} = \frac{19}{25}. \end{aligned}$$

[1] This gives us

$$\begin{aligned} \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[\Phi(S_1)] &= \frac{1}{11/10} \left( \frac{6}{25}(K - 12) + \frac{19}{25}(K - 2) \right) \\ &= \frac{10}{11} \left( K - \frac{110}{25} \right) \\ &= \frac{10K}{11} - 4, \end{aligned}$$

[2] which is equal to the value of  $V_0^h$  that we found in (c).

- (e) The contract is worth zero at time 0 if  $\frac{10}{11}K - 4 = 0$ , that is if  $K = \frac{22}{5}$ . [1]

Total marks: 23