

MASx50: Assignment 3

Solutions and discussion are written in blue. A sample mark scheme is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. Let $f_n, f : [0, 1] \rightarrow \mathbb{R}$. In each of the following cases, explain whether the Monotone and/or Dominated Convergence Theorems can be used to prove that $\int_0^1 f_n(x) dx \rightarrow \int_0^1 f(x) dx$.
 - (a) $f_n(x) = \cos(\frac{x}{n}) + \sin(\frac{x}{n})$ and $f(x) = 1$.
 - (b) $f_n(x) = \mathbb{1}_{[\frac{1}{n}, 1]}(x) x^{-1}$ and $f(x) = \mathbb{1}_{(0, 1]} x^{-1}$.
 - (c) $f_n(x) = \mathbb{1}_{[0, \frac{1}{n}]}(x) n$ and $f(x) = 0$.

Solution.

- (a) DCT only (the MCT can't be used here because $f_n \leq f_{n+1}$ doesn't hold). [2]
 - (b) MCT only (the DCT can't be used here because $\int_0^1 f(x) dx = \infty$). [2]
 - (c) Neither, in this case $\int_0^1 f_n(x) dx = 1$ and $\int_0^1 f(x) dx = 0$. [2]
2. Let (S, Σ, m) be a measure space. Let $f : S \rightarrow \mathbb{R}$ be measurable and let $c > 0$. Consider the following two facts, which were stated (and proved) within the lecture notes:
 - (a) $\left| \int_S f dm \right| \leq \int_S |f| dm$,
 - (b) $m(\{x \in S : |f(x)| \geq c\}) \leq \frac{1}{c} \int_S |f| dm$.

You do *not* need to prove these facts here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \rightarrow \mathbb{R}$ be a random variable. Let \mathbb{E} denote expectation with respect to \mathbb{P} . Use this notation to write down probabilistic versions of statements (a) and (b).

Solution.

- (a) $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$. [2]
 - (b) $\mathbb{P}[X \geq c] \leq \frac{1}{c} \mathbb{E}[X]$. [2]
3. Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ where λ denotes the restriction of Lebesgue measure to the Borel σ -field $\mathcal{B}([0, 1])$ on $[0, 1]$.

$$\text{Let } X_n(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ \omega n^{3/2} & \text{if } \omega \in (0, \frac{1}{n}] \\ 0 & \text{if } \omega \in (\frac{1}{n}, 1]. \end{cases}$$

Determine in which modes of convergence we have $X_n \rightarrow 0$.

Solution.

We first check almost sure convergence. Fix some $\omega \in [0, 1]$. If $\omega \in (0, 1]$ then for all large enough n we have $\frac{1}{n} < \omega$. For such n we have $X_n(\omega) = 0$, [1] which means $X_n(\omega) \rightarrow 0$. We thus obtain that $\{X_n(\omega) \rightarrow 0\} \subseteq (0, 1]$ [1] so

$$\lambda(\{X_n(\omega) \rightarrow 0\}) \geq \lambda((0, 1]) = 1,$$

which means that $X_n \xrightarrow{a.s.} 0$. [1] It follows that $X_n \xrightarrow{\mathbb{P}} 0$ [1] and also that $X_n \xrightarrow{d} 0$. [1]

Lastly, the expectation of $|X_n|^p = X_n^p$ is given by

$$\begin{aligned} \mathbb{E}[X_n^p] &= \int_0^1 X_n(\omega)^p d\lambda(\omega) \\ &= \int_0^{\frac{1}{n}} \omega^p n^{3p/2} d\lambda(\omega) \\ &= n^{3p/2} \left[\frac{\omega^{p+1}}{p+1} \right]_0^{\frac{1}{n}} \\ &= n^{3p/2} \frac{(1/n)^{p+1}}{p+1} \\ &= n^{p/2-1}. \end{aligned}$$

[1] Here we use that $\{0\}$ is a λ -null subset of $[0, 1]$ (so values of X_n here have no effect on the integral) [1] and that $X_n(\omega) = 0$ when $\omega > \frac{1}{n}$. [1]

Noting that $n^{p/2-1} \rightarrow 0$ if and only if $p < 2$, we have that $X_n \xrightarrow{L^p} 0$ if and only if $p < 2$. [1]

For the last part, I would accept “ $p = 1$ works but $p = 2, 3, 4, \dots$ does not”

4. (a) Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of independent, identically distributed uniform random variables on $(0, 1)$. Prove that, $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$.
- (b) Let $(X_n)_{n \in \mathbb{N}}$ be the sequence of results obtained from infinitely many rolls of a fair six sided dice. Prove that the (consecutive) pattern 123456 will occur infinitely often.

Solution.

- (a) We have $\mathbb{P}[U_n \leq a] = a$. For any (deterministic) sequence (x_n) the events $\{U_n < x_n\}$ are independent, because the U_n are independent. [1]

Noting that $\sum 1/n = \infty$ and $\sum 1/n^2 < \infty$, we have $\sum_n \mathbb{P}[U_n < 1/n] = \infty$ and $\sum_n \mathbb{P}[U_n < 1/n^2] < \infty$. [1]

By the second Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n \text{ i.o.}] = 1$ and by the first Borel-Cantelli lemma $\mathbb{P}[U_n < 1/n^2 \text{ i.o.}] = 0$. [1]

- (b) Let $E_n = \{X_n + i = i \text{ for } i = 1, 2, 3, 4, 5, 6\}$. We have $\mathbb{P}[E_n] = (1/6)^6 > 0$. Note that E_n and E_{n+6} are independent (but E_n and E_{n+1} are not!). [1]

We have $\sum_{n=1}^{\infty} \mathbb{P}[E_{6n}] = \sum_{n=1}^{\infty} (1/6)^6 = \infty$, [1] hence by the second Borel-Cantelli lemma we have $\mathbb{P}[E_{6n} \text{ i.o.}] = 1$. [1]

Noting that $\{E_{6n} \text{ i.o.}\} \subseteq \{E_n \text{ i.o.}\}$, we have $\mathbb{P}[E_n \text{ i.o.}] = 1$.

Total marks: 25