

MASx52: Assignment 5

1. Consider the SDE

$$dX_t = (t + X_t) dt + 2t dB_t.$$

- (a) Write this SDE in integral form, and show that $f(t) = \mathbb{E}[X_t]$ satisfies the differential equation

$$f'(t) = t + f(t)$$

Show that this equation is satisfied by $f(t) = Ce^t - t - 1$.

- (b) Let $Y_t = X_t^2$. Show that

$$dY_t = 2(2t^2 + tX_t + X_t^2) dt + 4tX_t dB_t$$

- (c) Show that $v(t) = \mathbb{E}[X_t^2]$ satisfies the differential equation

$$v'(t) = 2(2t^2 + tf(t) + v(t)).$$

2. (a) Within the Black-Scholes model, use the risk neutral valuation formula find the prices at time t of the contingent claims
- i. $\Phi(S_T) = 3S_T + 5$, where $0 \leq t \leq T$.
 - ii. $\Psi(S_T) = S_1 S_T + 1$, where $1 \leq t \leq T$.
- (b) With the same contingents claims as in (a):
- i. Describe a constant portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.
 - ii. Is it possible to replicate $\Psi(S_T)$ using a constant portfolio?
- (c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = 3S_T + 5$. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.
3. **[On Semester 1]** Consider an urn, containing two colours of balls, black and red. At time $n = 0$, the urn contains one black ball and one red ball. Then, at each time $n = 1, 2, \dots$, we do the following:

- Draw a ball from the urn. Record the colour of this ball and place it back into the urn.
- Add two new balls to the urn, of the same colour as the drawn ball.

Therefore, at time n , the urn contains $2 + 2n$ balls. Let B_n denote the number of red balls in the urn, and let

$$M_n = \frac{B_n}{2 + 2n}.$$

- (a) Show that M_n is a martingale, with respect to the filtration $\mathcal{F}_n = \sigma(B_i : i \leq n)$.
- (b) Deduce that there exists a random variable M_∞ such that $M_n \xrightarrow{a.s.} M_\infty$.
- (c) Show that $\mathbb{P}[M_n \leq \frac{1}{2}] = \mathbb{P}[M_n \geq \frac{1}{2}]$ for all n .