MASx52: Assignment 5

Solutions and discussion are written in blue. A sample mark scheme, with a total of 35 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. (a) Within the Black-Scholes model, use the risk neutral valuation formula

$$F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid \mathcal{F}_t\right]$$

to show that price at time t of the contingent claim $\Phi(S_T) = 3S_T + 5$ is given by

$$F(t, S_t) = 3S_t + 5e^{-r(T-t)}.$$

(b) Describe a portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.

Solution.

(a) Using the explicit formula for geometric Brownian motion (see the formula sheet) we obtain

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_T + 5 \mid \mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} + 5 \mid \mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)} \mid \mathcal{F}_t\right] + 5\right) \\
&= e^{-r(T-t)}\left(3S_t\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma}(B_T - B_t)\right] + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)} + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{r(T-t)} + 5\right) \\
&= 3S_t + 5e^{-r(T-t)}
\end{aligned}$$

[4] Here, we use that S_t is $\mathcal{F}_t$ measurable,[1] and that $Z = \sigma(B_T - B_t) \sim N(0, \sigma^2(T-t)$ is independent of $\mathcal{F}_t$. [1] We use the formula sheet to provide an explicit formula for $\mathbb{E}[e^Z]$.

(b) At time 0, we buy three units of stock [1] and $5e^{-rT}$ in cash. [1] It's value at time t is then

$$3S_t + 5e^{-rT}e^{rt} = \Phi(S_T).$$

Therefore, this portfolio replicates $\Phi(S_T)$ for all $t \in [0, T]$.

2. (a) Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and S_t be an Ito process satisfying $dS_t = \alpha S_t\,dt + \sigma S_t\,dB_t$. Let $Y_t = S_t^3$. Show that

$$dY_t = \left(3\alpha + 3\sigma^2\right)Y_t\,dt + 3\sigma Y_t\,dB_t$$

Deduce that Y_t is a geometric Brownian motion, and write down its drift and volatility.

(b) Within the Black-Scholes model, find the price $F(t, S_t)$ at time $t \in [0, T]$ of the contingent claim $\Phi(S_T) = S_T^3$.

Solution.

(a) By Ito's formula,

$$dY_t = \left((0) + \alpha S_t(3S_t^2) + \frac{1}{2}\sigma^2 S_t^2(6S_t) \right) dt + \sigma S_t(3S_t^2) dB_t$$
$$= \left(3\alpha + 3\sigma^2\right) Y_t \, dt + 3\sigma Y_t \, dB_t.$$

[5] So, Y_t is a geometric Brownian motion with drift $3\alpha + 3\sigma^2$ [1] and volatility 3σ. [1]

(b) Using the explicit formula for geometric Brownian motion (see the formula sheet) with drift $3\alpha + 3\sigma^2$ and volatility 3σ, we have that

$$Y_T = Y_t \exp\left(\left(3\alpha + 3\sigma^2 - \tfrac{9}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t) \right)$$
$$= Y_t \exp\left(\left(3\alpha - \tfrac{3}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t) \right).$$

[2] Note that in the risk neutral world $\mathbb{Q}$ we have $\alpha = r$. [1] Therefore, using the risk neutral valuation formula (see the question, or the formula sheet), the arbitrage free price of the contingent claim $Y_T = \Phi(S_T) = S_T^3$ at time t is

$$e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[Y_T \,|\, \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_t^3 \exp\left(\left(3\alpha - \tfrac{3}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right) \,|\, \mathcal{F}_t\right]$$
$$= e^{-r(T-t)} S_t^3 e^{(3r - \frac{3}{2}\sigma^2)(T-t)} \mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T - B_t)} \,|\, \mathcal{F}_t\right]$$
$$= e^{-r(T-t)} S_t^3 e^{(3r - \frac{3}{2}\sigma^2)(T-t)} \mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T - B_t)}\right]$$
$$= e^{-r(T-t)} S_t^3 e^{(3r - \frac{3}{2}\sigma^2)(T-t)} e^{\frac{9}{2}\sigma^2(T-t)}$$
$$= S_t^3 e^{2r(T-t) + 3\sigma^2(T-t)}.$$

[3] Here, we use that S_t is $\mathcal{F}_t$ measurable. [1] We then use the properties of Brownian motion to tell us that $3\sigma(B_T - B_t)$ is independent of $\mathcal{F}_t$ [1] with distribution $N(0, (3\sigma)^2(T-t))$, followed by the formula sheet to explicitly evaluate $\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T - B_t)}\right]$. [1]

3. Let X_t be an Ito process satisfying $dX_t = X_t^2 \, dB_t$, and let $F(t,x)$ be a solution of the partial differential equation

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}x^4 \frac{\partial^2 F}{\partial x^2}(t,x) = 0$$

with the boundary condition $F(T,x) = x$. Use Ito's formula to find $dF(t, X_t)$ and hence show that $F(t,x) = \mathbb{E}_{t,x}[X_T]$.

Solution. Using Ito's formula, we have

$$dF(t, S_t) = \left(\frac{\partial F}{\partial t} + (0) + \frac{1}{2}(X_t^2)^2 \frac{\partial^2 F}{\partial x^2} \right) dt + X_t^2 \left(\frac{\partial F}{\partial x} \right) dB_t$$
$$= X_t^2 \frac{\partial F}{\partial x} \, dB_t$$

[6] Writing in integral form, over $[t, T]$, we obtain

$$F(T, S_T) = F(t, S_t) + \int_t^T X_u^2 \frac{\partial F}{\partial x} \, dB_u.$$

[1] Taking expectations $\mathbb{E}_{t,x}$ (which denotes that X runs during time $[t,T]$ and has initial state $X_t = x$), we obtain

$$\mathbb{E}_{t,x}[F(T, X_T)] = \mathbb{E}_{t,x}[F(t, X_t)] + 0$$

because Ito integrals are martingales. [1] On the right hand side, since $\mathbb{E}_{t,x}$ specifies that $X_t = x$, we have $F(t, X_t) = F(t, x)$, which is deterministic [1]. On the left hand side, $F(T, X_T) = \Phi(X_T) = X_T$ [1], so we obtain

$$\mathbb{E}_{t,x}[\Phi(X_T)] = F(t, x)$$

as required. [1]