MASx52: Assignment 5

Solutions and discussion are written in blue. A sample mark scheme, with a total of 35 marks, is given in red, with each mark placed after the statement/deduction for which the mark would be given. As usual, mathematically correct solutions that follow a different method would be marked analogously.

1. (a) Within the Black-Scholes model, use the risk neutral valuation formula

$$F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \,|\, \mathcal{F}_t\right]$$

to show that price at time t of the contingent claim $\Phi(S_T) = 3S_T + 5$ is given by

$$F(t, S_t) = 3S_t + 5e^{-r(T-t)}.$$

(b) Describe a portfolio strategy that replicates $\Phi(S_T)$ during time $[0, T]$.

(c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = 3S_T + 5$.

i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

(d) Suggest one reason why we might want to hold a delta neutral portfolio.

Solution.

(a) Using the explicit formula for geometric Brownian motion (see the formula sheet) we obtain

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_T + 5 \,|\, \mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} + 5 \,|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(B_T-B_t)} \,|\, \mathcal{F}_t\right] + 5\right) \\
&= e^{-r(T-t)}\left(3S_t\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma}(B_T - B_t)\right] + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)} + 5\right) \\
&= e^{-r(T-t)}\left(3S_t e^{r(T-t)} + 5\right) \\
&= 3S_t + 5e^{-r(T-t)}
\end{aligned}$$

[4] Here, we use that S_t is $\mathcal{F}_t$ measurable,[1] and that $Z = \sigma(B_T - B_t) \sim N(0, \sigma^2(T-t)$ is independent of $\mathcal{F}_t$. [1] We use the formula sheet to provide an explicit formula for $\mathbb{E}[e^Z]$.

(b) At time 0, we buy three units of stock [1] and $5e^{-rT}$ in cash. [1] It's value at time t is then

$$3S_t + 5e^{-rT}e^{rt} = \Phi(S_T).$$

Therefore, this portfolio replicates $\Phi(S_T)$ for all $t \in [0, T]$.

(c) i. The value of our portfolio at time t is given by $F(t, S_t)$, where F is as in part (a). If we add an amount α of stock into our portfolio then its new value will be $V(t, S_t) = F(t, S_t) + \alpha S_t$. [1] To achieve delta neutrality, we want to choose α such that

$$0 = \frac{\partial V}{\partial s}(0, S_0) = 3 + \alpha.$$

[1] Hence $\alpha = -3$. [1]

ii. Our new portfolio has value $V(t, S_t) = F(t, S_t) - 3S_t = 5e^{-r(T-t)}$, and hence $\frac{\partial V}{\partial s} = 0$ for all time. Hence, in this case our portfolio will stay delta neutral for all time.

(d) A delta neutral portfolio is advantageous because its value is, typically, less sensitive so sudden changes in the stock price. [1]

2. (a) Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and S_t be an Ito process satisfying $dS_t = \alpha S_t\,dt + \sigma S_t\,dB_t$. Let $Y_t = S_t^3$. Show that Y_t satisfies the SDE

$$dY_t = \left(3\alpha + 3\sigma^2\right) Y_t\,dt + 3\sigma Y_t\,dB_t$$

Deduce that Y_t is a geometric Brownian motion, and write down its drift and volatility.

(b) Within the Black-Scholes model, show that the price $F(t, S_t)$ at time $t \in [0, T]$ of the contingent claim $\Phi(S_T) = S_T^3$ is given by

$$F(t, S_t) = S_t^3 e^{2r(T-t)+3\sigma^2(T-t)}.$$

(c) Suppose that our portfolio at time 0 consists of a single contract with contingent claim $\Phi(S_T) = S_T^3$.

i. Calculate the amount of stock that we would need to buy/sell in order to make our portfolio delta neutral at time 0.

ii. If we did buy/sell this amount of stock at time 0, how long would our new portfolio stay delta-neutral for?

Solution.

(a) By Ito's formula,

$$\begin{aligned} dY_t &= \left((0) + \alpha S_t(3S_t^2) + \frac{1}{2}\sigma^2 S_t^2(6S_t)\right) dt + \sigma S_t(3S_t^2)dB_t \\ &= \left(3\alpha + 3\sigma^2\right) Y_t\,dt + 3\sigma Y_t\,dB_t. \end{aligned}$$

[5] So, Y_t is a geometric Brownian motion with drift $3\alpha + 3\sigma^2$ [1] and volatility 3σ. [1]

(b) Using the explicit formula for geometric Brownian motion (see the formula sheet) with drift $3\alpha + 3\sigma^2$ and volatility 3σ, we have that

$$\begin{aligned} Y_T &= Y_t \exp\left(\left(3\alpha + 3\sigma^2 - \tfrac{9}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right) \\ &= Y_t \exp\left(\left(3\alpha - \tfrac{3}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right). \end{aligned}$$

[2] Note that in the risk neutral world $\mathbb{Q}$ we have $\alpha = r$. [1] Therefore, using the risk neutral valuation formula (see the question, or the formula sheet), the arbitrage free

price of the contingent claim $Y_T = \Phi(S_T) = S_T^3$ at time t is

$$
\begin{aligned}
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[Y_T \,|\, \mathcal{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_t^3 \exp\left(\left(3\alpha - \tfrac{3}{2}\sigma^2\right)(T-t) + 3\sigma(B_T - B_t)\right) \,|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)} S_t^3 e^{(3r-\frac{3}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T-B_t)} \,|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)} S_t^3 e^{(3r-\frac{3}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T-B_t)}\right] \\
&= e^{-r(T-t)} S_t^3 e^{(3r-\frac{3}{2}\sigma^2)(T-t)} e^{\frac{9}{2}\sigma^2(T-t)} \\
&= S_t^3 e^{2r(T-t)+3\sigma^2(T-t)}.
\end{aligned}
$$

[3] Here, we use that S_t is $\mathcal{F}_t$ measurable. [1] We then use the properties of Brownian motion to tell us that $3\sigma(B_T - B_t)$ is independent of $\mathcal{F}_t$ [1] with distribution $N(0,(3\sigma)^2(T-t))$, followed by the formula sheet to explicitly evaluate $\mathbb{E}^{\mathbb{Q}}\left[e^{3\sigma(B_T-B_t)}\right]$. [1]

(c) i. The value of our portfolio at time t is given by $F(t,S_t)$, where F is as in part (b). If we add an amount α of stock into our portfolio then its new value will be $V(t,S_t) = F(t,S_t) + \alpha S_t$. [1] To achieve delta neutrality, we want to choose α such that

$$0 = \frac{\partial V}{\partial s}(0,S_0) = 3S_0^2 e^{2rT+3\sigma^2 T} + \alpha.$$

[1] Hence $\alpha = -3S_0^2 e^{2rT+3\sigma^2 T}$. [1]

ii. Our new portfolio has value $V(t,S_t) = F(t,S_t) - 3S_0^2 e^{2rT+3\sigma^2 T} S_t$, and hence

$$
\begin{aligned}
\frac{\partial V}{\partial s}(t,S_t) &= 3S_t^2 e^{2r(T-t)+3\sigma^2(T-t)} - 3S_0^2 e^{2rT+3\sigma^2 T} S_t \\
&= 3S_t e^{2rT+3\sigma^2 T}\left(e^{-2rt-3\sigma^2 t} - 3S_0 S_t\right).
\end{aligned}
$$

[2] Therefore, $\frac{\partial V}{\partial s}$ is zero only when either $S_t = 0$ (which does occur because S_t is a geometric Brownian motion, which is never zero), or when the term in brackets is zero (which, after $t = 0$, has probability zero, because S_t has a continuous distribution). [1] Hence, our new portfolio is not delta neutral at any time after time 0. [1]