CS 511, Fall 2024, Lecture Slides 09

Do You Believe de Morgan's Laws?

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Do You Believe *de Morgan's Laws* Are Tautologies?

- Of course you believe they are!
- But now, for each, choose a most efficient procedure to confirm it!

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- Of course you believe they are!
- But now, for each, choose a most efficient procedure to confirm it!
- de Morgan's laws can be expressed as valid WFF's/tautologies:

1. $\models \neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$
2. $\models (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$
3. $\models (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$
4. $\models \neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$

or, in the form of four formally deducible sequents:

1. $\vdash \neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$
2. $\vdash (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$
3. $\vdash (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$
4. $\vdash \neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$

Available methods

Already discussed:

- Truth-tables to establish $\models \varphi$?
- Natural-deduction formal proofs to establish $\vdash \varphi$?

Yet to be discussed:

- Analytic tableaux?
- Resolution?
- BDD, OBDD, or ROBDD?
- DP or DPLL or CDCL procedures?

In this set of slides we restrict the comparison to **truth-tables** and **natural-deduction proofs**. We delay the comparaison with the other methods to later handouts.

Natural-deduction proof of de Morgan's law (1):

1	$\neg(p \vee q)$	assume
2	p	assume
3	$p \vee q$	$\vee$i 2
4	$\bot$	$\neg$e 1, 3
5	$\neg p$	$\neg$i 2-4
6	q	assume
7	$p \vee q$	$\vee$i 6
8	$\bot$	$\neg$e 1, 7
9	$\neg q$	$\neg$i 6-8
10	$\neg p \wedge \neg q$	$\wedge$i 5, 9
11	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$	$\rightarrow$i 1-10

Natural-deduction proof of de Morgan's law (2):

1	$\neg p \wedge \neg q$	assume
2	$\neg p$	$\wedge$E 1
3	$\neg q$	$\wedge$E 1
4	$p \vee q$	assume
5	p	assume
6	q	assume
7	$\neg p$	assume
8	$\bot$	$\neg$E $3, 6$
9	$\neg\neg p$	$\neg$I 7-8
10	p	$\neg\neg$E 9
11	p	$\vee$E $4, 5\text{-}5, 6\text{-}10$
12	$\bot$	$\neg$E $2, 11$
13	$\neg(p \vee q)$	$\neg$I 4-12
14	$(\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$	$\rightarrow$I 1-13

Remark: The formal proof of *de Morgan's Law* (2) above makes use of proof rule $\neg\neg$E on line 10. It is possible to write another formal proof of *de Morgan's Law* (2) which does not use $\neg\neg$E (nor any of the rules that can be derived from it, specifically, LEM and PBC).

Natural-deduction proof of de Morgan's law (3):

1	$\neg p \vee \neg q$	assume
2	$p \wedge q$	assume
3	p	$\wedge e_1$
4	q	$\wedge e_2$
5	$\neg p$	assume
6	$\neg q$	assume
7	p	assume
8	$\bot$	$\neg e \quad 4, 6$
9	$\neg p$	$\neg i \quad 7\text{-}8$
10	$\neg p$	$\vee e \quad 1, 5\text{-}5, 6\text{-}9$
11	$\bot$	$\neg e \quad 3, 10$
12	$\neg(p \wedge q)$	$\neg i \quad 2\text{-}11$
13	$(\neg p \vee \neg q) \to \neg(p \wedge q)$	$\to i \quad 1\text{-}12$

Natural-deduction proof of de Morgan's law (4):

1	$\neg(p \wedge q)$	assume
2	$\neg(\neg p \vee \neg q)$	assume
3	$\neg p$	assume
4	$(\neg p \vee \neg q)$	$\vee$i 3
5	$\perp$	$\neg$e 2, 4
6	$\neg\neg p$	$\neg$i 3-5
7	$\neg q$	assume
8	$\neg p \vee \neg q$	$\vee$i 7
9	$\perp$	$\neg$e 2, 8
10	$\neg\neg q$	$\neg$i 7-9
11	p	$\neg\neg$e 6
12	q	$\neg\neg$e 10
13	$p \wedge q$	$\wedge$i 11, 12
14	$\perp$	$\neg$e 1, 13
15	$\neg\neg(\neg p \vee \neg q)$	$\neg$i 2-14
16	$(\neg p \vee \neg q)$	$\neg\neg$e 15
17	$\neg(p \wedge q) \to (\neg p \vee \neg q)$	$\to$i 1-16

Natural-deduction proof of de Morgan's law (2), once more:

We organize the proof differently to make explicit how the rule "$\vee$E" is used on line 10; "$\vee$E" has three antecedents, two of which are boxes (here: the first box has one line, {line 5}, and the second box has five lines, {line 5, line 6, line 7, line 8, line 9}.

1	$\neg p \wedge \neg q$			assume
2	$\neg p$			$\wedge e_1$ 1
3	$\neg q$			$\wedge e_2$ 1
4	$p \vee q$			assume
5	p	assume	q	assume
6			$\neg p$	assume
7			$\bot$	$\neg$e 3, 5
8			$\neg\neg p$	$\neg$i 6-7
9			p	$\neg\neg$e 8
10	p			$\vee$e 4, 5-5, 5-9
11	$\bot$			$\neg$e 2, 10
12	$\neg(p \vee q)$			$\neg$i 4-11
13	$(\neg p \wedge \neg q) \to \neg(p \vee q)$			$\to$i 1-12

Truth-table verification of de Morgan's laws (1) and (4):

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	$\neg(p \vee q)$	$\neg(p \vee q) \to (\neg p \wedge \neg q)$
T	**T**	**F**	**F**	**T**	**F**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**F**	**F**	**T**
F	**F**	**T**	**T**	**F**	**T**	**T**	**T**

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \to (\neg p \vee \neg q)$
T	**T**	**F**	**F**	**T**	**F**	**F**	**T**
T	**F**	**F**	**T**	**F**	**T**	**T**	**T**
F	**T**	**T**	**F**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**F**	**T**	**T**	**T**

and similarly for de Morgan's laws (2) and (3)

natural-deduction proofs versus *truth-tables*

- For the four de Morgan's laws, each with two propositional variables p and q, **truth-tables** beat **natural-deduction proofs** – or do they?

natural-deduction proofs versus *truth-tables*

- For the four de Morgan's laws, each with two propositional variables p and q, **truth-tables** beat **natural-deduction proofs** – or do they?
- In these lecture notes, two natural deductions for de Morgan's laws are intuitionistically valid and two are not. The **truth tables** do not show it, the **natural-deduction proofs** show it:
  - the natural deductions for de Morgan's (2) and (4) in these lecture slides are not admissible intuitionistically (they use rule "$\neg\neg$E"),
  - the natural deductions for de Morgan's (1) and (3) in these lecture slides **are admissible intuitionistically** (they do **not** use rule "$\neg\neg$E" nor the two rules derived from it, LEM and PBC).

  - but perhaps we did not try hard enough to avoid the rule "$\neg\neg$E" in the natural deductions for (2) and (4)???
  - in fact, it is possible to write a natural deduction for de Morgan's (2) which is admissible intuitionistically.
  - however, it can be shown (not easy) that, *no matter how hard we try*, there are **no** intuitionistically admissible natural deductions for de Morgan's (4).

natural-deduction proofs versus *truth-tables*

Exercise

1. Write a **natural-deduction proof** of the following WFF:

   $$\varphi_1 \triangleq \neg(p \wedge q \wedge r) \rightarrow (\neg p \vee \neg q \vee \neg r)$$

   This is a more general version of de Morgan's law (4).

2. Write a **natural-deduction proof** of the most general de Morgan's law (4):

   $$\varphi_2 \triangleq \neg(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n)$$

   where $n \geqslant 2$.

3. Show there is a **natural-deduction proof** of the generalized de Morgan's law above φ_2 whose length (the number of lines in the proof) is $\mathcal{O}\big(n\big)$.

4. Compare the complexity of a **natural-deduction proof** of φ_2 and the complexity of a **truth-table** verification of φ_2.

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