Solutions to CS511 Homework 10

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Exercise 1 Open Lecture Slides 24, "Deductive Closure and First-Order Theories", page 7: Carefully answer the question highlighted in green.

Hint: An appropriate answer should take no more than 4-5 lines. You will find it helpful to read the preceding pages in the same set of slides.

If M is a relational structure, the first-order theory of M is:

$$\text{Th}(M) \stackrel{\text{def}}{=} \{\phi \mid \phi \text{ is a first-order sentence s.t. } M \models \phi\}$$

Question: Is $\text{Th}(M)$ deductively closed?

Solution:

Yes, Th(M) is deductively closed.

To prove this:

Let $\phi \in \overline{\text{Th}(M)}$. Then $\text{Th}(M) \vdash \phi$.

By the Soundness Theorem, $\text{Th}(M) \models \phi$.

Since $M \models \text{Th}(M)$, we have $M \models \phi$.

By definition of $\text{Th}(M)$, $\phi \in \text{Th}(M)$.

Therefore, $\overline{\text{Th}(M)} \subseteq \text{Th}(M)$, so Th(M) is deductively closed.

Exercise 2. Open EML.Chapter 6.pdf : Do Exercise 109 on page 65.

Hint: There is some reading to do in this exercise, but the answers are straightforward. Correct answers for parts 1, 2, and 3, are each no more than a single line. An appropriate answer for part 4 invokes Compactness and can be written in three or four lines.

Graph Coloring

Hint 1 : Find a way to make use of the following fact: Every finite planar graph is four-colorable. (Do not try to prove this fact, which is difficult, but you are allowed to invoke it.)

Hint 2 : If M is a planar graph, then every subgraph of M is also planar. A subgraph of M is a graph whose vertices are a subset of the vertices of M and whose adjacency relation is a subset of the adjacency relation of M restricted to this subset.

In this exercise, I will demonstrate that every infinite planar graph is 4-colorable. The first-order theory of simple undirected graphs can be taken as a set Γ of two axioms over signature $\Sigma \stackrel{\text{def}}{=} \{R\}$ consisting of one binary relation symbol, namely:

$$\Gamma \stackrel{\text{def}}{=} \{\forall x.\forall y.R(x,y) \to R(y,x), \forall x.\neg R(x,x)\}.$$

We now expand the signature Σ to $\Sigma' = \Sigma \cup \{B, G, P, Y\}$ where B, G, P, and Y are unary predicate symbols (for 'blue', 'green', 'purple', and 'yellow').

Part 1

Write a first-order sentence ϕ_1 which, in any Σ'-structure M satisfying Γ (i.e., M is a simple undirected graph), asserts "every vertex has at least one of the colors: blue, green, purple, yellow".

Solution

$$\phi_1 := \forall x(B(x) \vee G(x) \vee P(x) \vee Y(x))$$

Part 2

Write a first-order sentence ϕ_2 which, in any Σ'-structure M satisfying Γ, asserts "every vertex has at most one color".

Solution

$$\phi_2 := \forall x((B(x) \to \neg G(x) \wedge \neg P(x) \wedge \neg Y(x)) \wedge (G(x) \to \neg P(x) \wedge \neg Y(x)) \wedge (P(x) \to \neg Y(x)))$$

Part 3

Write a first-order sentence ϕ_3 which, in any Σ'-structure M satisfying Γ, asserts "no two adjacent vertices have the same color".

Solution

$\phi_3 := \forall x \forall y (R(x,y) \to \neg(B(x) \land B(y)) \land \neg(G(x) \land G(y)) \land \neg(P(x) \land P(y)) \land \neg(Y(x) \land Y(y)))$

Part 4

Show that if M is an infinite planar graph, i.e.,

- M is a Σ-structure satisfying Γ,
- the domain of M is infinite, and
- M is planar as a graph,

then there is a Σ'-structure M', which expands M with four unary relations $B^{M'}, G^{M'}, P^{M'}, and Y^{M'}$, and which satisfies $\phi_1 \wedge \phi_2 \wedge \phi_3$, i.e., M' is four-colorable and, thus, M is also four-colorable.

Solution

Let $\Delta = \Gamma \cup \{\phi_1, \phi_2, \phi_3\}$.

Every finite substructure of M is a finite planar graph, hence 4-colorable.

Thus, every finite subset of Δ is satisfiable. By Compactness, Δ is satisfiable.

Therefore, there exists a Σ'-structure M' expanding M that satisfies Δ, making M four-colorable.

PROBLEM 1 Open EML.Chapter 6.pdf : Do part 1 and part 2 only of Exercise 108 on page 64. You do not need to do part 3 of that exercise.

Part 1

Question: Give a precise argument in about 5-10 lines for how to systematically generate the countably infinite sequence of $K_{3,3}$ and all its subdivisions, call it $G \stackrel{\text{def}}{=} (G_i \mid i \in \mathbb{N})$, and the countably infinite sequence of K_5 and all its subdivisions, call it $G' \stackrel{\text{def}}{=} (G'_i \mid i \in \mathbb{N})$. The first entries in those two sequences are $K_{3,3}$ and K_5, i.e., $G_0 \stackrel{\text{def}}{=} K_{3,3}$ and $G'_0 \stackrel{\text{def}}{=} K_5$. It is also useful to define the sequence G so that if $i < j$ then $G_i \leq G_j$, and similarly for the sequence G', i.e., successive entries in G and G' are in order of non-decreasing sizes.

Hint 1: It suffices to give an answer for one of the two sequences, say G, and to conclude by saying "G' is generated similarly."

Hint 2: In the two sequences there are many (though a finite number) subdivisions of the same size. And for the same size, it is possible but quite difficult to omit isomorphic copies; it is much easier to allow isomorphic copies in the two sequences.

Solution:

My precise argument is that, to generate the sequence G:

1. We start with $G_0 = K_{3,3}$.
2. For each $i \geq 1$:
   - We consider all possible ways to subdivide one edge of G_{i-1}.
   - Add each resulting graph as the next element in the sequence.
   - If multiple subdivisions result in graphs of the same size, include all of them.
3. Repeat step 2 indefinitely, ensuring that graphs are added in order of non-decreasing size.
4. If at any step, all possible single-edge subdivisions of previous graphs have been included, start subdividing two edges, then three, and so on.

This process generates all possible subdivisions of $K_{3,3}$ in a systematic way, allowing for isomorphic copies. G' is generated similarly, starting with $G'_0 = K_5$.

Part 2

Question: Let $M \stackrel{\text{def}}{=} (M, R^M)$ be an arbitrary simple graph, $G_i \stackrel{\text{def}}{=} (V_i, E_i)$ an arbitrary subdivision of $K_{3,3}$, and $G'_j \stackrel{\text{def}}{=} (V'_j, E'_j)$ an arbitrary subdivision of K_5. Those two subdivisions are entries in the sequences G and G' defined in the preceding part. Write first-order sentences ϕ_i and ϕ'_j such that if $M \models \phi_i$ (resp. $M \models \phi'_j$), then G_i is a subgraph of M (resp. G'_j is a subgraph of M).

Hint: You will find it convenient to name the vertices of G_i with an initial segment of the positive integers, i.e., $V_i = \{1, 2, \ldots, n_i\}$ where n_i is the size of G_i, and similarly for G'_j.

Solution:

Let $V_i = \{1, 2, \ldots, n_i\}$ be the vertices of G_i and E_i its set of edges. We can write ϕ_i as:

$$\phi_i \stackrel{\text{def}}{=} \exists x_1 \ldots \exists x_{n_i} \left(\bigwedge_{1 \leq k < l \leq n_i} x_k \not\approx x_l \wedge \bigwedge_{(k,l) \in E_i} R(x_k, x_l) \right)$$

Similarly, for G'_j with vertices $V'_j = \{1, 2, \ldots, n'_j\}$ and edges E'_j, we can write ϕ'_j as:

$$\phi'_j \stackrel{\text{def}}{=} \exists x_1 \ldots \exists x_{n'_j} \left(\bigwedge_{1 \leq k < l \leq n'_j} x_k \not\approx x_l \wedge \bigwedge_{(k,l) \in E'_j} R(x_k, x_l) \right)$$

The two first-order sentences assert the existence of distinct vertices in M that are connected according to the edge structure of G_i or G'_j, respectively. If M satisfies either of these sentences, it contains the corresponding subdivision as a subgraph.

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw10/hw10_nicholas_ikechukwu.lean

Exercise 3. From Macbeth's book:

Solutions

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw10/hw10_nicholas_ikechukwu.lean

Exercise 4. From Macbeth's book

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PROBLEM 2. From Macbeth's book

Solution

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