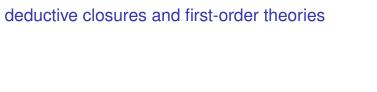
# CS 511, Fall 2024, Lecture Slides 24 Deductive Closures and First-Order Theories

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### deductive closures and first-order theories

Let  $\Gamma$  be a set of first-order sentences over signature  $\Sigma$ . The **deductive closure** of  $\Gamma$  is:

$$\overline{\Gamma} \stackrel{\mathrm{def}}{=} \{\, \varphi \mid \varphi \text{ first-order sentence s.t. } \Gamma \vdash \varphi \, \}$$

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- ightharpoonup A first-order theory  $\mathcal{T}$  over signature  $\Sigma$  consists of:
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Equivalently, a **first-order theory** is the **deductive closure** of a set of first-order sentences.

### the first-order theory of a relational structure

If  $\mathcal{M}$  is a relational structure, the **first-order theory of**  $\mathcal{M}$  is:

$$\mathsf{Th}(\mathcal{M}) \stackrel{\mathrm{def}}{=} \{\, \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{M} \models \varphi \, \}$$

**Question**: Is  $\mathsf{Th}(\mathcal{M})$  deductively closed?

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 $\textbf{Question:} \ \mathsf{Is} \ \mathsf{Th}(\mathcal{M}) \ \mathsf{deductively} \ \mathsf{closed?}$ 

Yes! Can you explain why?

# the first-order theory of $\mathcal{N}\stackrel{ ext{ iny def}}{=}(\mathbb{N},0,S)$

Consider again the structure  $\mathcal{N}\stackrel{\mathrm{def}}{=}(\mathbb{N},0,S)$  in Lecture Slides 21. The first-order theory of  $\mathcal{N}$  is:

$$\mathsf{Th}(\mathcal{N}) \stackrel{\mathrm{def}}{=} \{\, \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{N} \models \varphi \, \}$$

Some sentences that are true in  $\mathcal{N}$ :

- S1  $\forall x \neg (Sx \approx 0)$
- S2  $\forall x \, \forall y \, (Sx \approx Sy \rightarrow x \approx y)$
- S3  $\forall y (\neg (y \approx 0) \rightarrow \exists x (y \approx Sx))$
- S4.1  $\forall x \neg (Sx \approx x)$
- $S4.2 \qquad \forall x \, \neg (SSx \approx x)$ 
  - . . .
- S4.n  $\forall x \neg (\underbrace{S \cdots S}_{n} x \approx x)$

. . .

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- what can we say about the deductive closure of the set  $\Gamma$  above:  $\overline{\Gamma} = \{ \varphi \mid \varphi \text{ first-order sentence s.t. } \Gamma \vdash \varphi \}$ ?

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- ▶ clearly  $\mathcal{N} \models \varphi$  for every  $\varphi \in \Gamma$  so that  $\Gamma \subseteq \mathsf{Th}(\mathcal{N})$
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- ightharpoonup certainly  $\overline{\Gamma}\subseteq \mathsf{Th}(\mathcal{N})$ , by soundness
- in fact, the equality holds:

$$\overline{\Gamma} = \mathsf{Th}(\mathcal{N})$$
 (not shown here)

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we therefore say that  $\Gamma$  is an **axiomatization** of  $\mathsf{Th}(\mathcal{N})$  because every sentence  $\varphi$  made true by  $\mathcal{N}$  is formally deduced from  $\Gamma$ 

#### first-order theories of several structures over domain ${\mathbb N}$

#### From Lecture Slides 21:

$$\begin{split} \mathcal{N} &\stackrel{\mathrm{def}}{=} (\mathbb{N}, 0, S), \quad \mathcal{N}_1 \stackrel{\mathrm{def}}{=} (\mathbb{N}, 0, S, <), \quad \mathcal{N}_2 \stackrel{\mathrm{def}}{=} (\mathbb{N}, 0, S, <, +) \\ \mathcal{N}_3 &\stackrel{\mathrm{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot) \\ \mathcal{N}_4 \stackrel{\mathrm{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot, \mathrm{pr}) \qquad \text{where } \mathrm{pr}(x) \stackrel{\mathrm{def}}{=} \mathrm{true} \mathrm{\ iff} \ x \mathrm{\ is \ prime} \\ \mathcal{N}_5 \stackrel{\mathrm{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot, \mathrm{pr}, \uparrow) \qquad \mathrm{where} \quad x \uparrow y \stackrel{\mathrm{def}}{=} x^y \end{split}$$

#### 1. FACT

The first-order theory of each of  $\mathcal{N}$ ,  $\mathcal{N}_1$ , and  $\mathcal{N}_2$ , is **axiomatizable** and **decidable**.

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The first-order theory of each of  $\mathcal{N}$ ,  $\mathcal{N}_1$ , and  $\mathcal{N}_2$ , is **axiomatizable** and **decidable**.

#### 2. FACT

The first-order theory of each of  $\mathcal{N}_3$ ,  $\mathcal{N}_4$ , and  $\mathcal{N}_5$ , is **axiomatizable** but **not** decidable.

