Appendix A

Syntax of Well-Formed Formulas

This appendix is a compendium of syntactic conventions we use in the main body of these lecture notes. It is intended as a handy reference, which can be quickly consulted whenever you need clarification on notations in the main body.

We first cover the syntax of *well-formed formulas* (wff's) of the following: *propositional logic*, the *quantified Boolean logic*, and *first-order logic*. We thus define the sets $\mathsf{WFF}_{\mathsf{PL}}$, $\mathsf{WFF}_{\mathsf{QPL}}$ and $\mathsf{WFF}_{\mathsf{FOL}}$ first. The syntax of $\mathsf{WFF}_{\mathsf{FOL}}$ includes that of *zeroth-order logic*, *equality logic*, *equational logic*, and *quasi-equational logic*, as four special cases which are therefore left to the end of this appendix. The resulting sets of wff's are denoted $\mathsf{WFF}_{\mathsf{ZOL}}$, $\mathsf{WFF}_{\mathsf{eL}}$, $\mathsf{WFF}_{\mathsf{EL}}$, and $\mathsf{WFF}_{\mathsf{QEL}}$.

Throughout, we use lower-case Greek letters from the end of the alphabet (mostly φ and ψ) and occasionally from the beginning of the alphabet (α, β, and γ) as metavariables denoting well-formed formulas (wff's). We use upper-case Greek letters $\Gamma, \Delta, \ldots$ as metavariables denoting sets of wff's.

A.1 Well-Formed Formulas of PL

The syntax of *propositional logic* (PL) is built up from a set $\mathcal{P}$ of variables and a few logical connectives:

- $\mathcal{P} = \{p_0, p_1, \ldots\}$ is a countably infinite set of *propositional variables* (also called *propositional* or *Boolean atoms*). We use p and lower-case Roman letters nearby $\{q, r, s, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{P}$.
- The set of *logical connectives* is $\{\neg, \wedge, \vee, \rightarrow\}$. We use the symbol "$\diamond$" as a metavariable ranging over the binary connectives $\wedge$, $\vee$, and $\rightarrow$.

The set $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ of well-formed propositional formulas over $\mathcal{P}$ is the least set such that:

$$\begin{aligned}\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \;\supseteq\; \mathcal{P} \;\cup\; \{\bot, \top\} \;\cup\; &\Big\{\, (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \,\Big\} \\ \cup\; &\Big\{\, (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \text{ and } \diamond \in \{\wedge, \vee, \rightarrow\} \,\Big\}.\end{aligned}$$

It is customary to omit parentheses whenever possible, using the following precedences:

- $\{\neg\}$ binds more tightly than binary connectives $\{\wedge, \vee, \rightarrow\}$,
  e.g., $\neg\varphi_1 \wedge \varphi_2$ means $((\neg\varphi_1) \wedge \varphi_2)$.
- binary connectives $\{\wedge, \vee\}$ associate to the left,
  e.g., $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ means $((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$.

- the binary connective $\{\to\}$ associates to the right,
  e.g., $\varphi_1 \to \varphi_2 \to \varphi_3$ means $(\varphi_1 \to (\varphi_2 \to \varphi_3))$.
- $\{\wedge, \vee\}$ have higher precedence than $\{\to\}$,
  e.g., $\varphi_1 \to \varphi_2 \wedge \varphi_3$ means $(\varphi_1 \to (\varphi_2 \wedge \varphi_3))$.

Whenever in doubt about the conventions, insert matching parentheses to disambiguate wff's. Also, to break precedences of logical connectives, insert parentheses; for example, if the intended wff is $((\varphi_1 \to \varphi_2) \to \varphi_3)$, we can omit the outer matching parentheses as in $(\varphi_1 \to \varphi_2) \to \varphi_3$, but not the inner matching parentheses as in $\varphi_1 \to \varphi_2 \to \varphi_3$, otherwise the wff is understood to mean $(\varphi_1 \to (\varphi_2 \to \varphi_3))$.

Exercise 124. There is an implicit induction in our definitions of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ above. Make this induction explicit in an alternative definition of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, using BNF or extended BNF notation. $\square$

A.2 Well-Formed Formulas of QPL

Quantified Boolean logic (QPL) extends PL by introducing the quantifiers $\{\forall, \exists\}$.[36] The set $\mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P})$ of wff's of QPL over $\mathcal{P}$ is the least set such that:

$$\begin{aligned}\mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P}) \;\supseteq\; & \mathcal{P} \;\cup\; \{\bot, \top\} \;\cup\; \Big\{ (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P}) \Big\} \\ & \cup\; \Big\{ (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P}) \text{ and } \diamond \in \{\wedge, \vee, \to\} \Big\} \\ & \cup\; \Big\{ (\forall p\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P}) \text{ and } p \in \mathcal{P} \Big\} \;\cup\; \Big\{ (\exists p\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P}) \text{ and } p \in \mathcal{P} \Big\}.\end{aligned}$$

To omit parentheses for better readability, we use the same precedences as in PL, in addition to the following conventions for quantifiers:

- $\forall p.\, \varphi$ means $(\forall p\, \varphi)$ and $\exists p.\, \varphi$ means $(\exists p\, \varphi)$.
- $\forall p\, q.\, \varphi$ means $(\forall p\, (\forall q\, \varphi))$ and $\exists p\, q.\, \varphi$ means $(\exists p\, (\exists q\, \varphi))$.

A.3 Well-Formed Formulas of FOL

Let $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order signature, where:

- $\mathcal{R} = \{R_1, R_2, \ldots\}$ is a countable set, possibly infinite, of relation symbols, each with an arity $\geqslant 1$. We use R and upper-case Roman letters nearby $\{P, Q, S, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{R}$.[37]
- $\mathcal{F} = \{f_1, f_2, \ldots\}$ is a countable set, possibly infinite, of function symbols, each with an arity $\geqslant 1$. We use f and lower-case Roman letters nearby $\{f, g, h, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{F}$.
- $\mathcal{C} = \{c_1, c_2, \ldots\}$ is a countable set, possible infinite, of constant symbols. We use c and lower-case Roman letters nearby $\{d, e, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{C}$.

Following most presentations of *first-order logic* (FOL), wff's may include a symbol for the equality relation, which is here denoted "$\approx$" and used in infix position, as in "$t_1 \approx t_2$". We consider the

[36] QPL is often called "quantified Boolean logic" or the logic of "quantified Boolean formulas".

[37] Some authors prefer the words "predicate" and "predicate symbol" to what we here call "relation" and "relation symbol".

symbol "$\approx$" to be outside the signature Σ. Some accounts of FOL allow Σ to be an uncountable set of symbols; our standing assumption throughout is that Σ is always countable, finite or infinite.

Besides symbols from the signature, wff's of FOL may contain variables:

- $X = \{x_0, x_1, x_2, \ldots\}$ is a countably infinite set of variables. We use letters from the end of the Roman alphabet $\{x, y, z, \ldots\}$, possibly decorated, as metavariables ranging over X.

We build up wff's gradually, starting with the set of terms $\mathsf{Terms}(\Sigma, X)$, followed by the set of atomic formulas $\mathsf{Atoms}(\Sigma, X)$, followed by the full set $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ of wff's. These are the three stages:

1. $\mathsf{Terms}(\Sigma, X)$ is the least set satisfying the condition:

$$\mathsf{Terms}(\Sigma, X) \;\supseteq\; \mathcal{C} \cup X \cup \Big\{\, f(t_1, \ldots, t_n) \;\Big|\; f \in \mathcal{F} \text{ has arity } n \geqslant 1,\ t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \,\Big\}.$$

   Since there are no relation symbols in terms, we may write $\mathsf{Terms}(\mathcal{F} \cup \mathcal{C}, X)$ instead of $\mathsf{Terms}(\Sigma, X)$.

2. $\mathsf{Atoms}(\Sigma, X)$ is the set defined by:

$$\mathsf{Atoms}(\Sigma, X) \;\stackrel{\text{def}}{=}\; \{\,\bot, \top\,\} \cup \Big\{\, R(t_1, \ldots, t_n) \;\Big|\; R \in \mathcal{R} \text{ has arity } n \geqslant 0,\ t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \,\Big\}.$$

3. $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ is the least set satisfying the condition:

$$\begin{aligned}
\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \;\supseteq\; \mathsf{Atoms}(\Sigma, X) \;&\cup\; \Big\{\, (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \,\Big\} \\
&\cup\; \Big\{\, (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \text{ and } \diamond \in \{\wedge, \vee, \to\} \,\Big\} \\
&\cup\; \Big\{\, (\forall x\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \text{ and } x \in X \,\Big\} \\
&\cup\; \Big\{\, (\exists x\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \text{ and } x \in X \,\Big\}.
\end{aligned}$$

We follow standard practice of omitting parentheses whenever possible, using the following conventions, which extend those already mentioned for PL and QPL:

- $\{\neg\}$ binds more tightly than binary connectives $\{\wedge, \vee, \to\}$, *e.g.*, $\neg\varphi_1 \wedge \varphi_2$ means $(\neg\varphi_1) \wedge \varphi_2$.
- binary connectives $\{\wedge, \vee\}$ associate to the left, *e.g.*, $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ means $(\varphi_1 \wedge \varphi_2) \wedge \varphi_3$.
- binary connective $\{\to\}$ associates to the right, *e.g.*, $\varphi_1 \to \varphi_2 \to \varphi_3$ means $\varphi_1 \to (\varphi_2 \to \varphi_3)$.
- $\{\wedge, \vee\}$ have higher precedence than $\{\to\}$, *e.g.*, $\varphi_1 \to \varphi_2 \wedge \varphi_3$ means $\varphi_1 \to (\varphi_2 \wedge \varphi_3)$.
- $\forall x.\, \varphi$ means $(\forall x\, \varphi)$ and $\exists x.\, \varphi$ means $(\exists x\, \varphi)$.
- $\forall x\, y.\, \varphi$ means $(\forall x\, (\forall y\, \varphi))$ and $\exists x\, y.\, \varphi$ means $(\exists x\, (\exists y\, \varphi))$.

When in doubt about conventions, insert matching parentheses to disambiguate the formula.

Exercise 125. There is an implicit induction in our definitions of $\mathsf{Terms}(\Sigma, X)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ above. Make this induction explicit in alternative definitions of $\mathsf{Terms}(\Sigma, X)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$, using BNF or extended BNF notation. $\square$

If we expand the signature Σ with fresh function symbols or relation symbols, then the sets $\mathsf{Terms}(\Sigma, X)$, $\mathsf{Atoms}(\Sigma, X)$, and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ are extended in the obvious way. For example, If we introduce a new relation symbol $R \notin \mathcal{R}$ of some arity $n \geqslant 1$, the set $\mathsf{Atoms}(\Sigma, X)$ is extended as follows:

$$\mathsf{Atoms}\big(\Sigma \cup \{R\}, X\big) \;\stackrel{\text{def}}{=}\; \mathsf{Atoms}(\Sigma, X) \;\cup\; \{\, R(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \,\}.$$

and the set $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ is extended to $\mathsf{WFF}_{\mathsf{FOL}}\big(\Sigma \cup \{R\}, X\big)$, by substituting $\mathsf{Atoms}\big(\Sigma \cup \{R\}, X\big)$ for $\mathsf{Atoms}(\Sigma, X)$ in the definition of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$.

An important instance of the preceding is when we allow the equality symbol "$\approx$" to occur in wff's. In this case the set of atomic wff's is now denoted $\mathsf{Atoms}\big(\Sigma \cup \{\approx\}, X\big)$ and the corresponding $\mathsf{WFF}_{\mathsf{FOL}}\big(\Sigma \cup \{\approx\}, X\big)$ is obtained by substituting $\mathsf{Atoms}\big(\Sigma \cup \{\approx\}, X\big)$ for $\mathsf{Atoms}(\Sigma, X)$ in the definition of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$.

A.4 Well-Formed Formulas of ZOL

We write $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ for the set of wff's of *zeroth-order logic* (ZOL), a proper subset of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ that mention no variables in X and no quantifiers in $\{\forall, \exists\}$. The definition is in three stages:

1. $\mathsf{Terms}(\Sigma, \varnothing)$ is the same as the set of variable-free terms of FOL:

$$\mathsf{Terms}(\Sigma, \varnothing) \;\stackrel{\text{def}}{=}\; \Big\{\, t \;\Big|\; t \in \mathsf{Terms}(\Sigma, X) \text{ and } \mathrm{FV}(t) = \varnothing \,\Big\}.$$

   Since there are no relation symbols in terms, we may write $\mathsf{Terms}(\mathcal{F} \cup \mathcal{C}, \varnothing)$ instead of $\mathsf{Terms}(\Sigma, \varnothing)$.

2. $\mathsf{Atoms}(\Sigma, \varnothing)$ is the same as the set of variable-free atomic formulas of FOL:

$$\mathsf{Atoms}(\Sigma, \varnothing) \;\stackrel{\text{def}}{=}\; \{\bot, \top\} \;\cup\; \Big\{\, \varphi \;\Big|\; \varphi \in \mathsf{Atoms}(\Sigma, X) \text{ and } \mathrm{FV}(\varphi) = \varnothing \,\Big\}.$$

3. $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ is the least set satisfying the condition:

$$\begin{aligned}\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing) \;\supseteq\; & \mathsf{Atoms}(\Sigma, \varnothing) \;\cup\; \Big\{\, (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing) \,\Big\} \\ & \cup\; \Big\{\, (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing) \text{ and } \diamond \in \{\wedge, \vee, \to\} \,\Big\}.\end{aligned}$$

In words, $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ is the set of all variable-free and quantifier-free formulas of *first-order logic* over signature Σ.

It is possible to define a *zeroth-order logic* which allows variables but disallows quantifiers. The set of wff's of such a logic is $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, X)$, indicated by the second argument $X \neq \varnothing$. $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, X)$ is intermediate between $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$, more expressive than $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ but less expressive than $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$. For all of its interesting properties, we do not examine $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, X)$ in these lecture notes.

Of particular interest for our presentation is the extension $\mathsf{WFF}_{\mathsf{ZOL}}\big(\Sigma \cup \{\approx\}, \varnothing\big)$ which allows the equality symbol "$\approx$" to occur in wff's, but still precludes variables and quantifiers. See our examination in Chapter 4.

A.5 Well-Formed Formulas of eL, EL, and QEL

We write:

- $\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big)$ for the set of wff's of *equality logic* (eL),
- $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ for the set of wff's of *equational logic* (EL),
- $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ for the set of wff's of *quasi-equational logic* (QEL),

which are all proper subsets of $\mathsf{WFF}_{\mathsf{FOL}}\big(\Sigma \cup \{\approx\}, X\big)$, the set of first-order wff's where the equality symbol $\approx$ may occur. In these three logics, relation symbols are precluded from wff's.[38]

[38] The distinction between "*equality*" and "*equation*" is a little confusing and does not conform to how we use the same words in other contexts. We use two different words here so that we can name differently two distinct logics, eL and EL. To confuse the matter a little more, what we here call *equations* and *quasi-equations* are called elsewhere *identities* and *quasi-identities*.

Starting with $\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big)$, the wff's of *equality logic*, the definition is once more in three stages:

1. $\mathsf{Terms}\big(\varnothing, X\big)$ is simply the set of all variables, as no symbols from the signature are allowed:

$$\mathsf{Terms}\big(\varnothing, X\big) \;\stackrel{\text{def}}{=}\; X.$$

2. $\mathsf{Atoms}\big(\{\approx\}, X\big)$ is restricted to the equality symbol "$\approx$" and its members are called *equalities* (between first-order variables):

$$\mathsf{Atoms}\big(\{\approx\}, X\big) \;\stackrel{\text{def}}{=}\; \{\bot, \top\} \;\cup\; \Big\{ (x \approx y) \;\Big|\; x, y \in X \Big\}.$$

3. $\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big)$ is the least set satisfying the condition:

$$\begin{aligned} \mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big) \;\supseteq\; & \mathsf{Atoms}(\{\approx\}, X) \;\cup\; \Big\{ (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X) \Big\} \\ & \cup\; \Big\{ (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X) \text{ and } \diamond \in \{\wedge, \vee, \to\} \Big\}. \end{aligned}$$

In words, $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ is the set of all (quantifier-free) propositional combinations of equalities between variables.

We define $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ and $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ simultaneously in three stages:

1. $\mathsf{Terms}\big(\mathcal{F} \cup \mathcal{C}, X\big)$ is the same as the set of terms of FOL, the least satisfying the condition:

$$\mathsf{Terms}(\mathcal{F} \cup \mathcal{C}, X) \;\supseteq\; \mathcal{C} \cup X \cup \Big\{ f(t_1, \ldots, t_n) \;\Big|\; f \in \mathcal{F} \text{ has arity } n \geqslant 1,\; t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \Big\}.$$

2. $\mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ is restricted to the equality symbol "$\approx$" and its members are called *equations* (between terms):

$$\mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \;\stackrel{\text{def}}{=}\; \{\bot, \top\} \;\cup\; \Big\{ (t_1 \approx t_2) \;\Big|\; t_1, t_2 \in \mathsf{Terms}\big(\mathcal{F} \cup \mathcal{C}, X\big) \Big\}.$$

3. $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ and $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ are defined by:

$$\begin{aligned} \mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \;&\stackrel{\text{def}}{=}\; \mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \\ \mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \;&\stackrel{\text{def}}{=}\; \Big\{ (\varphi_1 \wedge \cdots \wedge \varphi_k \to \psi) \;\Big| \\ &\qquad \varphi_1, \ldots, \varphi_k, \psi \in \mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \text{ and } k \geqslant 0 \Big\}. \end{aligned}$$

The wff's in $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ and $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ are called *equations* and *quasi-equations*, respectively. Every equation is a quasi-equation, but not conversely. It follows that $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ is a proper subset of $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$.

In these lecture notes, every *equality* is an *equation*, but not vice-versa. Moreover, although every equality is an equation, it is not the case that every wff of *equality logic* (eL) is a wff of *equational logic* (EL); wff's of eL can include all the logical connectives in $\{\neg, \wedge, \vee, \to\}$, wff's of EL do not mention any of these logical connectives.

Appendix B

Semantics of Well-Formed Formulas

We try as much as possible to give a uniform presentation of the semantics of all the formal logics considered in these lecture notes. We start with the semantics of *propositional logic*, then follow with the semantics of *quantified Boolean logic*, and then with the semantics of *first-order logic*. The semantics of the other logics are special cases of the semantics of *first-order logic* and do not need a separate treatment.

B.1 Semantics of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$

We interpret the wff's of *propositional logic* in a 2-element Boolean algebra $\mathcal{B}$, which we can take in the form:

$$\mathcal{B} \stackrel{\text{def}}{=} (B, \mathsf{Not}, \mathsf{And}, \mathsf{Or}, \mathsf{Implies}, \mathit{false}, \mathit{true}) \qquad \text{with} \quad B \stackrel{\text{def}}{=} \{\mathit{false}, \mathit{true}\},$$

where Not is a unary operation, and each of the operations in $\{\mathsf{And}, \mathsf{Or}, \mathsf{Implies}\}$ is binary. Since the domain B is finite, these operations can be conveniently defined in tabular forms as follows:[39]

	Not
false	*true*
true	*false*

And	*false*	*true*
false	*false*	*false*
true	*false*	*true*

Or	*false*	*true*
false	*false*	*true*
true	*true*	*true*

Implies	*false*	*true*
false	*true*	*true*
true	*false*	*true*

A *truth assignment* for the set $\mathcal{P}$ of propositional variables is any map $\sigma : \mathcal{P} \to \{\mathit{false}, \mathit{true}\}$. Having fixed the interpretations of the symbols $\{\neg, \wedge, \vee, \to\}$ as the operations $\{\mathsf{Not}, \mathsf{And}, \mathsf{Or}, \mathsf{Implies}\}$ of the Boolean algebra $\mathcal{B}$, the satisfaction of a propositional wff φ, *i.e.*, the truth value of φ, depends only on the assignment σ.

We next lift the truth assignment σ to all propositional wff's. We use a notation favored by computer scientists: the meaning of a syntactic object φ is denoted by inserting it between

[39] These are not what are usually called *truth-tables* of Boolean operations, which are typically written as:

p	$\neg p$
false	*true*
true	*false*

p	q	$p \wedge q$
false	*false*	*false*
false	*true*	*false*
true	*false*	*false*
true	*true*	*true*

p	q	$p \vee q$
false	*false*	*false*
false	*true*	*true*
true	*false*	*true*
true	*true*	*true*

p	q	$p \to q$
false	*false*	*true*
false	*true*	*true*
true	*false*	*false*
true	*true*	*true*

Our tabular forms for the Boolean operations here is the same tabular forms we use elsewhere in these notes whenever we deal with unary and binary operations over finite domains. In particular for binary operations, they are more compact than truth-tables, but their generalization to higher-arity functions lose their graphical appeal and are practically useless.

double brackets, as in "$[\![\varphi]\!]$" or, more precisely here, "$[\![\varphi]\!]_\sigma$" since it depends on σ. The definition of $[\![\varphi]\!]_\sigma$ is by *structural induction*, *i.e.*, on the "shape" of φ:[40]

$$
\begin{aligned}
[\![p]\!]_\sigma &\stackrel{\text{def}}{=} \sigma(p)\\
[\![\bot]\!]_\sigma &\stackrel{\text{def}}{=} \mathit{false}\\
[\![\top]\!]_\sigma &\stackrel{\text{def}}{=} \mathit{true}\\
[\![\neg\varphi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{Not}\big([\![\varphi]\!]_\sigma\big)\\
[\![\varphi\wedge\psi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{And}\big([\![\varphi]\!]_\sigma,[\![\psi]\!]_\sigma\big)\\
[\![\varphi\vee\psi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{Or}\big([\![\varphi]\!]_\sigma,[\![\psi]\!]_\sigma\big)\\
[\![\varphi\to\psi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{Implies}\big([\![\varphi]\!]_\sigma,[\![\psi]\!]_\sigma\big)
\end{aligned}
$$

Following convention:

- we write $\sigma \models \varphi$ and say σ *satisfies* φ iff $[\![\varphi]\!]_\sigma = \mathit{true}$,
- we write $\sigma \not\models \varphi$ and say σ *does not satisfy* φ iff $[\![\varphi]\!]_\sigma = \mathit{false}$,
- if for every assignment σ we have $\sigma \models \varphi$, we may write $\models \varphi$ and say φ is *valid* or is a *tautology*,
- if for some assignment σ we have $\sigma \not\models \varphi$, we may write $\not\models \varphi$ and say φ is *falsifiable*,
- if for every assignment σ we have $\sigma \not\models \varphi$, we may say φ is *unsatisfiable* or is a *contradiction*.

It is worth noting that the double-bracket notation is a convenient visual aid to separate syntax from semantics: everything inside the pair "$[\![$" and "$]\!]$" is a piece of syntax, and everything outside is about its semantics. Similarly, "$\models$" conveniently separates syntax from semantics: what is to the right of "$\models$" is a piece of syntax and what is to the left of "$\models$" is something that determines the semantics of the former. These are by now firmly established notational conventions.[41]

B.2 Semantics of $\mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P})$

Consider the definition of $[\![\varphi]\!]_\sigma$ by *structural induction* when $\varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$. We want to extend it to wff's in $\mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P})$ which mention the quantifiers $\forall$ and $\exists$. There are two steps that are missing for this extension and, before supplying them, we agree on how to write a *(one-point) adjustment* of a truth assignment σ. If $p \in \mathcal{P}$, an adjustment of σ at p is denoted "$\sigma[p \mapsto \mathit{false}]$" or "$\sigma[p \mapsto \mathit{true}]$". The precise definition is, for all $q \in \mathcal{P}$:

$$
\sigma[p \mapsto \mathit{false}]\ (q) \stackrel{\text{def}}{=} \begin{cases} \mathit{false} & \text{if } p = q,\\ \sigma(q) & \text{if } p \neq q,\end{cases}
$$

$$
\sigma[p \mapsto \mathit{true}]\ (q) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } p = q,\\ \sigma(q) & \text{if } p \neq q.\end{cases}
$$

Now for the two missing steps in the structural induction:

$$
[\![\forall p\,\varphi]\!]_\sigma \stackrel{\text{def}}{=} \mathsf{And}\left([\![\varphi]\!]_{\sigma[p\mapsto \mathit{false}]}\,,[\![\varphi]\!]_{\sigma[p\mapsto \mathit{true}]}\right)
$$

[40] Or, as computer scientists often like to say, the definition is *syntax-directed*.

[41] The double-bracket notation "$[\![\ldots]\!]$" was probably first used by computer scientists working on the denotational semantics of programming languages in the early 1970's. The double-turnstile notation "$\models$" was first introduced by mathematical logicians at least a decade earlier. The symbol "$\models$" appears throughout the classic book *Model Theory* by C.C. Chang and H. Jerome Keisler [2]; the authors point out, in the Preface of the first edition, that their book grew out of lecture notes in circulation since the early 1960's.

$$[\![\exists p\,\varphi]\!]_\sigma \;\stackrel{\text{def}}{=}\; \mathsf{Or}\left([\![\varphi]\!]_{\sigma[p\mapsto false]}\,,[\![\varphi]\!]_{\sigma[p\mapsto true]}\right)$$

Exercise 126. Based on the preceding definition, show that the following are equivalent assertions:

- $\sigma \;\models\; \forall p\,\varphi$,
- $\sigma \;\models\; \varphi[p := \bot] \;\wedge\; \varphi[p := \top]$.

And similarly, show that the following are equivalent assertions:

- $\sigma \;\models\; \exists p\,\varphi$,
- $\sigma \;\models\; \varphi[p := \bot] \;\vee\; \varphi[p := \top]$.

We write $\varphi[p := \bot]$ and $\varphi[p := \top]$ to denote the substitution of $\bot$ and $\top$ for every free occurrence of p in φ. □

Exercise 127. A wff $\varphi \in \mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P})$ is *closed* if $\mathrm{FV}(\varphi) = \varnothing$, *i.e.*, every occurrence of a variable p in φ falls in the scope of some "$\forall p$" or "$\exists p$". Use structural induction to show that, if φ is closed, then for every assignment σ it holds that either $\sigma \models \varphi$ or $\sigma \not\models \varphi$.

In words, if φ is closed, then φ is either a tautology or a contradiction.

Hint: This is subtle. In the structural induction, keep track of variables that occur free in a wff, there are finitely many of them in any wff. Formalize the idea that only a finite part of an assignment σ is relevant for the truth-value returned by $[\![\varphi]\!]_\sigma$, namely, the part that assigns a truth value to a variable occurring free in φ. □

B.3 Semantics of the Other Logics

All the other logics considered in these lecture notes are: *equality logic*, *zeroth-ary logic*, *equational logic*, *quasi-equational logic*, and *first-order logic*. The first four in this list are fragments of the last one, *first-order logic*. So, we restrict attention to the semantics of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$.

Given signature $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, a Σ-*structure* has the form:

$$\mathcal{A} \stackrel{\text{def}}{=} (A, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}}) \quad \text{where}$$

A is a non-empty set, called the *universe* or *domain* of $\mathcal{A}$, which we sometimes denote $|\mathcal{A}|$,

$$\mathcal{R}^{\mathcal{A}} \;\stackrel{\text{def}}{=}\; \{\, R^{\mathcal{A}} \subseteq A^n \mid \text{relation symbol } R \in \mathcal{R} \text{ has arity } n \geqslant 1 \,\},$$

$$\mathcal{F}^{\mathcal{A}} \;\stackrel{\text{def}}{=}\; \{\, f^{\mathcal{A}} : A^n \to A \mid \text{function symbol } f \in \mathcal{F} \text{ has arity } n \geqslant 1 \,\},$$

$$\mathcal{C}^{\mathcal{A}} \;\stackrel{\text{def}}{=}\; \{\, c^{\mathcal{A}} \in A \mid \text{constant symbol } c \in \mathcal{C} \,\},$$

$$\text{where} \quad A^n \;\stackrel{\text{def}}{=}\; \underbrace{A \times \cdots \times A}_{n}.$$

In words, a Σ-*structure* $\mathcal{A}$ assigns an interpretation to every symbol in Σ over some set of elements A. If the equality symbol $\approx$ occurs in wff's, we need to expand $\mathcal{A}$ to include an interpretation for it and write:

$$\mathcal{A} \stackrel{\text{def}}{=} (A, \approx^{\mathcal{A}}, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}}) \;\text{ or also }\; (A, =, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}})$$

since $\approx^{\mathcal{A}}$ is always interpreted as the equality "=" on the universe A.

A *valuation* for the set X of variables in the Σ-*structure* $\mathcal{A}$ is a map $\sigma : X \to A$. Note that σ maps every member of X, which is an infinite set, to an element of A. In case A is finite, σ necessarily maps many members of X to the same element of A.[42] A *(one-point) adjustment* of σ at variable x is a new valuation denoted $\sigma[x \mapsto a]$ where $a \in A$:

$$\sigma[x \mapsto a]\ (y) \ \stackrel{\text{def}}{=}\ \begin{cases} a & \text{if } x = y, \\ \sigma(y) & \text{if } x \neq y. \end{cases}$$

A Σ-structure $\mathcal{A}$ together with a valuation σ gives a meaning to every $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$. Below is the *structural induction* we use to interpret every such wff, it includes some of the steps already used for wff's in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ and $\mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P})$. Following the three stages in the definition of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$, we start with the interpretation of *terms*, continue with the interpretation of *atomic wff's*, and conclude with the interpretation of first-order wff's in general:

1. Interpretation of terms:

$$[\![x]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \sigma(x)$$

$$[\![c]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ c^{\mathcal{A}} \qquad \text{for every } c \in \mathcal{C}$$

$$[\![f(t_1,\ldots,t_n)]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ f^{\mathcal{A}}\big([\![t_1]\!]_{\mathcal{A},\sigma},\ldots,[\![t_n]\!]_{\mathcal{A},\sigma}\big) \qquad \text{for every } f \in \mathcal{F} \text{ of arity } n \geqslant 1$$

2. Interpretation of atomic wff's:

$$[\![\bot]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \mathit{false}$$

$$[\![\top]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \mathit{true}$$

$$[\![R(t_1,\ldots,t_n)]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ R^{\mathcal{A}}\big([\![t_1]\!]_{\mathcal{A},\sigma},\ldots,[\![t_n]\!]_{\mathcal{A},\sigma}\big) \qquad \text{for every } R \in \mathcal{R} \text{ of arity } n \geqslant 1$$

$$[\![t_1 \approx t_n]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \begin{cases} \mathit{false} & \text{if } [\![t_1]\!]_{\mathcal{A},\sigma} \neq [\![t_2]\!]_{\mathcal{A},\sigma} \\ \mathit{true} & \text{if } [\![t_1]\!]_{\mathcal{A},\sigma} = [\![t_2]\!]_{\mathcal{A},\sigma} \end{cases}$$

3. Interpretation of first-order wff's in general:

$$[\![\neg\varphi]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \mathsf{Not}\big([\![\varphi]\!]_{\mathcal{A},\sigma}\big)$$

$$[\![\varphi \wedge \psi]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \mathsf{And}\big([\![\varphi]\!]_{\mathcal{A},\sigma}, [\![\psi]\!]_{\mathcal{A},\sigma}\big)$$

$$[\![\varphi \vee \psi]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \mathsf{Or}\big([\![\varphi]\!]_{\mathcal{A},\sigma}, [\![\psi]\!]_{\mathcal{A},\sigma}\big)$$

$$[\![\varphi \to \psi]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \mathsf{Implies}\big([\![\varphi]\!]_{\mathcal{A},\sigma}, [\![\psi]\!]_{\mathcal{A},\sigma}\big)$$

$$[\![\forall x\, \varphi]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \begin{cases} \mathit{false} & \text{if } [\![\varphi]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \mathit{false} \quad \text{for some } a \in A \\ \mathit{true} & \text{if } [\![\varphi]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \mathit{true} \quad \text{for every } a \in A \end{cases}$$

$$[\![\exists x\, \varphi]\!]_{\mathcal{A},\sigma} \ \stackrel{\text{def}}{=}\ \begin{cases} \mathit{false} & \text{if } [\![\varphi]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \mathit{false} \quad \text{for every } a \in A \\ \mathit{true} & \text{if } [\![\varphi]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \mathit{true} \quad \text{for some } a \in A \end{cases}$$

We can view $[\![\ldots]\!]_{\mathcal{A},\sigma}$ as a two-sorted function, from a two-sorted domain $\{\text{terms}\} \cup \{\text{wff's}\}$ to a two-sorted co-domain $A \cup \{\mathit{false}, \mathit{true}\}$: It maps every term t to an element in the universe A, and every wff φ to a truth value.

Following convention:

[42] Some authors call a valuation an *assignment* or also an *environment*. We reserve the word "assignment" to appear in the expessions "truth assignment" and "assignment of truth values", a way of distinguishing it from what the word "valuation" is used for.

- we write $\mathcal{A}, \sigma \models \varphi$ and say $(\mathcal{A}, \sigma)$ *satisfies* φ iff $[\![\varphi]\!]_{\mathcal{A},\sigma} = \mathit{true}$,
- we write $\mathcal{A}, \sigma \not\models \varphi$ and say $(\mathcal{A}, \sigma)$ *does not satisfy* φ iff $[\![\varphi]\!]_{\mathcal{A},\sigma} = \mathit{false}$,
- if for every valuation σ we have $\mathcal{A}, \sigma \models \varphi$, we write $\mathcal{A} \models \varphi$ and say $\mathcal{A}$ *satisfies* φ or φ is *true in* $\mathcal{A}$,
- if for every Σ-structure $\mathcal{A}$ and valuation σ we have $\mathcal{A}, \sigma \models \varphi$, we write $\models \varphi$ and say φ is *valid*.

Exercise 128. This continues Exercise 127. Let $\mathcal{A}$ be a fixed Σ-structure. Use structural induction to show that, if $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ is closed, then for every valuation σ it holds that either $\mathcal{A}, \sigma \models \varphi$ or $\mathcal{A}, \sigma \not\models \varphi$.

Hence, when φ is closed, we can ignore σ and write $\mathcal{A} \models \varphi$ ("φ is true in $\mathcal{A}$") or $\mathcal{A} \not\models \varphi$ ("φ is false in $\mathcal{A}$"). $\square$

Exercise 129. Let $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and $\mathrm{FV}(\varphi) = \{x_1, \ldots, x_n\}$. The *existential closure* of φ is the closed wff $\exists x_1 \cdots \exists x_n.\, \varphi$ and the *universal closure* of φ is the closed wff $\forall x_1 \cdots \forall x_n.\, \varphi$.

1. Show that φ is satisfiable iff the existential closure of φ is satisfiable.
2. Show that φ is valid iff the universal closure of φ is valid. $\square$

Let Γ and Δ be sets, possibly infinite, of wff's in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$.

- We say a Σ-structure $\mathcal{A}$ is a *model* of Γ iff for every $\varphi \in \Gamma$ it holds that $\mathcal{A} \models \varphi$, in which case we may also write $\mathcal{A} \models \Gamma$.
- We say Γ *semantically entails* or *implies* (others say *logically entails* or *implies*) Δ iff every model of Γ is a model of Δ (but not necessarily the converse), and we may write $\mathsf{models}(\Gamma) \subseteq \mathsf{models}(\Delta)$.

Sometimes we may want to make explicit the signature Σ of a model $\mathcal{A}$ of Γ, in which case we may say that $\mathcal{A}$ is a Σ-model. If the equality symbol $\approx$ occurs in Γ, we may say $\mathcal{A}$ is a $(\Sigma \cup \{\approx\})$-model.

Let $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and let $\mathrm{FV}(\varphi) = \{x_1, \ldots, x_n\}$. By a slight abuse of notation, we may write $\varphi(x_1, \ldots, x_n)$ to indicate which variables have free occurrences in φ. Let $(\mathcal{A}, \sigma)$ be an interpretation for φ, consisting of a Σ-structure $\mathcal{A}$ and a valuation $\sigma : X \to A$, and let:

$$\sigma(x_1) = a_1, \ldots, \sigma(x_n) = a_n.$$

Then, instead of writing $\mathcal{A}, \sigma \models \varphi$, we may write:

$$\mathcal{A}, a_1, \ldots, a_n \models \varphi \quad \text{or also} \quad \mathcal{A} \models \varphi[a_1, \ldots, a_n]$$

with the understanding that the elements $a_1, \ldots, a_n \in A$ are substituted for the variables $x_1, \ldots, x_n$ (in the same order) in φ. Note the additional abuse of notation when we write "$\varphi[a_1, \ldots, a_n]$" (what is it?).

Appendix C

Systems of Formal Proofs

The syntax and semantics of all the formal logics in these lecture notes are basically the same that you will find elsewhere in the published literature. The differences are unessential, mostly in the notation, in the presentation, and sometimes in the terminology.

This is no longer the case when we consider their proof systems. For each of our formal logics, there are many proof systems, and each system has its own advantages and disadvantages. Given a particular formal logic $\mathcal{L}$ with its already-defined syntax and semantics, we expect all proof systems for $\mathcal{L}$ to be equivalent, in the sense that they all fulfill the requirement of the same Completeness Theorem, which asserts, in a nutshell: "whatever is true according to the semantics is also formally provable" – assuming that a Completeness Theorem is possible for $\mathcal{L}$. But that is not the only requirement by which we choose a proof system for $\mathcal{L}$, and there are indeed other requirements fulfilled by some but not all proof systems.

The profusion of proof systems for the same formal logic $\mathcal{L}$ is always a little bewildering for newcomers to this material. It takes time and effort to understand and appreciate the reasons for their differences, all related to different aspects of formal proofs (*e.g.*, the efficient implementation of procedures for deciding validity, or an examination of what is called *cut-elimination* and its implications regarding consistency, or the existence of what are called *interpolation theorems*, and other proof-theoretic matters). These are all important topics beyond the scope of these lecture notes.

But we still have to select at least one proof system to round off our presentation. We choose here a particular way of setting it up, so-called *natural deduction*, and a particular way of defining its formal rules and organizing its formal derivations. There is no overarching reason to choose *natural deduction* over the many other alternatives, except that it is a little easier to present and seems to be favored by computer scientists, particularly by researchers in areas related to automated theorem provers and interactive proof assistants.[43]

In each of our logics, if we can formally derive a wff ψ (the *conclusion*, also called *consequent*) from a finite set of wff's $\{\varphi_1, \ldots, \varphi_n\}$ (the *premises*, also called *antecedents* or *hypotheses* or *global premises*) according to the rules of natural deduction, we can assert this fact by writing:

$$\varphi_1, \ldots, \varphi_n \ \vdash \ \psi.$$

If we want to make explicit the logic we use, we may add a subscript "$\mathcal{L}$":

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{L}} \psi,$$

[43] *Natural deduction* in fact refers to a family of proof systems, not to just one. In these notes we choose a particular way of setting up *natural deduction*. Differences are minor with other presentations in books and articles elsewhere. A very readable account of the history of *natural deduction*, and the different ways in which it can be formulated, is a book chapter by F.J. Pelletier and A.P. Hazen [17].

where $\mathcal{L} \in \{\mathsf{PL}, \mathsf{QPL}, \mathsf{eL}, \mathsf{ZOL}, \mathsf{EL}, \mathsf{QEL}, \mathsf{FOL}\}$. Such an expression is called a *judgment* in these lecture notes, even though the word is not used by all authors and with the same intention. In our setup, the symbol "$\vdash$" is outside the formalism of natural deduction, in contrast to other proof systems that are called *sequent calculi*. It is only after a natural-deduction proof is completed that we use "$\vdash$", which is a symbol for us at the meta level, to separate wff's that are assumed to hold with no justification necessary (these are the global premises) from the wff appearing on the last line (the conclusion).[44] Several examples for how to use the proof rules are in Appendix D and Appendix E.

C.1 Rules for PL

Following tradition, rules are given suggestive names. For the logical connectives, here limited to $\{\wedge, \vee, \rightarrow, \neg\}$, rules come in pairs. Each pair has one *introduction* rule and one *elimination* rule, indicated by the letters "I" and "E", respectively. Sometimes the *introduction* rule has two parts, *e.g.*, rule ($\vee$I) has two parts: ($\vee\mathrm{I}_1$) and ($\vee\mathrm{I}_2$); and sometimes the *elimination* rule has two parts, *e.g.*, rule ($\wedge$E) has two parts: ($\wedge\mathrm{E}_1$) and ($\wedge\mathrm{E}_2$).

Introduction **and** ***elimination*** **rules for each of $\{\wedge, \vee, \rightarrow, \neg\}$:**

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge\mathrm{I}) \qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge\mathrm{E}_1) \qquad \frac{\varphi \wedge \psi}{\psi}\ (\wedge\mathrm{E}_2)$$

$$\frac{\varphi}{\varphi \vee \psi}\ (\vee\mathrm{I}_1) \qquad \frac{\psi}{\varphi \vee \psi}\ (\vee\mathrm{I}_2) \qquad \frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{\theta}\ (\vee\mathrm{E})$$

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \rightarrow \psi}\ (\rightarrow\mathrm{I}) \qquad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\ (\rightarrow\mathrm{E})$$

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ (\neg\mathrm{I}) \qquad \frac{\varphi \qquad \neg\varphi}{\bot}\ (\neg\mathrm{E})$$

Remark: In the rules with boxes $\{\,(\rightarrow\mathrm{I}), (\vee\mathrm{E}), (\neg\mathrm{I})\,\}$, the ellipsis points '. . .' may be empty. Thus, in ($\rightarrow$I), it is possible that $\varphi = \psi$, and similarly in the rule ($\vee$E), it is possible that $\varphi = \theta$ or $\psi = \theta$ (or both). In the rule ($\neg$I), if the ellipsis points are empty, then $\varphi = \bot$, in which case the conclusion of the rule is $\neg\varphi = \neg\bot$.

[44] The symbol "$\vdash$" is often called *turnstile* because of its resemblance to a typical turnstile if viewed from above. You may read it as "formally yields", or "formally proves", or "formally derives". According to Wikipedia, "$\vdash$" was first introduced towards the end of the 19th Century, by the mathemaical logician Gottlob Frege in 1879. It thus preceded its companion "$\models$" by many decades, which also reflects how concerns of mathematical logicians evolved over time, initially focusing on proof-theoretic issues and subsequently adding model-theoretic issues. See footnote 41 for comments on "$\models$".

An *elimination* rule for $\bot$, an *introduction* rule for $\top$:

$$\frac{\bot}{\varphi}\quad(\bot\mathrm{E})$$ (also called *ex falso quodlibet* or just *ex falso*)

$$\frac{}{\top}\quad(\top\mathrm{I})$$

There is no introduction rule for $\bot$ and no elimination rule for $\top$. So far, there are 12 rules, not all of equal importance: You will be right in guessing that you get more traction from ($\to$I) and ($\to$E) in our proof system than from the other rules, while ($\top$I) is useless (why?). But this question ("given a subset of the rules, what can be said about the wff's that are in its deductive closure?") is for another study outside these lecture notes.

Rules (LEM), (PBC), ($\neg\neg$E), and (Peirce's) :

These four rules have a special status. Without any of these four, the preceding rules define a proof system for what is called *intuitionistic propositional logic*; such a proof system is complete for a semantics of *propositional logic* based on what are called *Heyting algebras*, which include as a special case the familiar *Boolean algebras*. This is another matter outside the scope of these notes.

Adding anyone of the four rules in $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$ augments the deductive power of the proof system and makes it complete for the semantics we use for *propositional logic* in these notes, one based on *Boolean algebras*. The system so augmented is sometimes called *classical propositional logic*, the qualifier "classical" being use to make explicit the distinction with "intuitionistic".

LEM is a shorthand for *Law of Excluded Middle*. PBC is a shorthand for *Proof by Contradiction*. As its name indicates, $\neg\neg$E means *elimination of double negation*. Peirce's stands for *Peirce's Law* and is named for the 19th Century mathematical logician Charles Sanders Peirce. Here are the precise formulations of the four rules:

$$\frac{}{\varphi\vee\neg\varphi}\quad(\mathrm{LEM})\qquad\qquad \frac{\boxed{\begin{array}{c}\neg\varphi\\ \vdots\\ \bot\end{array}}}{\varphi}\quad(\mathrm{PBC})$$

$$\frac{\neg\neg\varphi}{\varphi}\quad(\neg\neg\mathrm{E})\qquad\qquad \frac{}{((\varphi\to\psi)\to\varphi)\to\varphi}\quad(\mathrm{Peirce's})$$

Exercise 130. Show that any two of the four rules $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$ are inter-derivable. In fact, they are inter-derivable using only two rules, ($\to$I) and ($\to$E).

Hint: One way is to consider $\binom{4}{2} = 6$ cases, one for each pair from the set of four rules, with each pair involving two derivations, for a total of 12 derivations. A much simpler approach requires only 4 derivations:

(a) (Peirce's) is derivable from (PBC),

(b) (LEM) is derivable from (Peirce's),

(c) ($\neg\neg$E) is derivable from (LEM),

(d) (PBC) is derivable from ($\neg\neg$E).

Schematically, you have to show that (PBC) $\Rightarrow$ (Peirce's) $\Rightarrow$ (LEM) $\Rightarrow$ ($\neg\neg$E) $\Rightarrow$ (PBC). $\square$

Exercise 131. In some accounts of natural deduction, the rule for *disjunction elimination* is given as ($\vee$E*):

$$\frac{\boxed{\begin{matrix}\varphi\\ \vdots\\ \theta\end{matrix}} \quad \boxed{\begin{matrix}\psi\\ \vdots\\ \theta\end{matrix}}}{(\varphi \vee \psi) \to \theta} \quad (\vee\mathrm{E}^*)$$

which is often more convenient to use than the standard ($\vee$E). You have to show that ($\vee$E) and ($\vee$E*) are inter-derivable. Specifically, there are two parts:

1. Show ($\vee$E) is derivable from ($\vee$E*) and ($\to$E).
2. Show ($\vee$E*) is derivable from ($\vee$E) and ($\to$I). $\square$

Exercise 132 (*Hilbert-styles vs. Natural Deduction*)**.** A *Hilbert-style proof system* is an alternative to a *natural-deduction proof system*. A Hilbert-style proof system for the propositional logic can be formulated by specifying only three *axiom schemes*:

A1: $(\varphi \to (\psi \to \varphi))$

A2: $((\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta)))$

A3: $((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$

together with a single *rule of inference*, called **modus ponens** (MP), specified as follows:

$$\frac{\varphi \qquad (\varphi \to \psi)}{\psi}$$

The MP rule is the same as the *arrow-elimination* rule ($\to$E) of natural deduction. A *Hilbert-style proof system* and a *natural-deduction proof system* have equal deductive power. There are two parts in this exercise, where Γ is an arbitrary set of propositional wff's and φ is an arbitray propositional wff. To simplify a little in both parts, assume that all wff's are written with only two logical connectives $\{\neg, \to\}$, *i.e.*, wff's mention none of the symbols in $\{\wedge, \vee, \bot, \top\}$:

1. Show that if $\Gamma \vdash_{\mathrm{H}} \varphi$, then $\Gamma \vdash_{\mathrm{ND}} \varphi$.
   Hint 1: Start by showing that any wff which is an instance of A1, or A2, or A3, is deducible in natural deduction, *i.e.*, for arbitrary wff's φ, ψ, and θ, it is the case that:
   (a) $\vdash_{\mathrm{ND}} \ (\varphi \to (\psi \to \varphi))$,
   (b) $\vdash_{\mathrm{ND}} \ ((\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta)))$,
   (c) $\vdash_{\mathrm{ND}} \ ((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$.
   Hint 2: Proceed by induction on the number of wff's, one per line, in the formal proof of $\Gamma \vdash_{\mathrm{H}} \varphi$.
2. Show that if $\Gamma \vdash_{\mathrm{ND}} \varphi$, then $\Gamma \vdash_{\mathrm{H}} \varphi$.
   Hint 1: This part is a little more involved than the first part. Start by proving some facts about Hilbert-style derivations, including the following:
   (a) $\vdash_{\mathrm{H}} \varphi \to \varphi$,
   (b) $\Gamma \cup \{\varphi\} \ \vdash_{\mathrm{H}} \psi$ iff $\Gamma \ \vdash_{\mathrm{H}} \varphi \to \psi$,
   (c) if $\vdash_{\mathrm{H}} \ \varphi \to \psi$ and $\vdash_{\mathrm{H}} \psi \to \theta$, then $\vdash_{\mathrm{H}} \varphi \to \theta$,
   (d) $\vdash_{\mathrm{H}} \neg\neg\varphi \to \varphi$.

A convenient approach to proving each fact in $\{(b), (c), (d)\}$ is to invoke the fact (or facts) preceding it.

Hint 2: Proceed by *course-of-values* induction on the number of times that rules in $\{(\to\mathrm{E}), (\to\mathrm{I}), (\neg\neg\mathrm{E})\}$ are applied to obtain $\Gamma \vdash_{\mathrm{ND}} \varphi$, which are the only three rules used when $\{\neg, \to\}$ are the only connectives in wff's. □

Exercise 133 (*Hilbert-styles vs. Natural Deduction*)**.** Read Exercise 132, without necessarily solving it, before you attempt this one. The preceding exercise proposes a strictly proof-theoretic approach to showing that a *Hilbert-style proof system* and a *natural-deduction proof system* have equal deductive power. In this exercise we consider an alternative semantic approach, which invokes Soundness and Completeness for both proof systems. Specifically, each of the two systems as here presented is *sound* and *complete* relative to the standard (classical) semantics of propositional logic based on Boolean algebras. There are three parts in this exercise, the first two of which are just plans in outline to prove the equivalence of the two systems:

1. If $\Gamma \vdash_{\mathrm{H}} \varphi$, then $\Gamma \models \varphi$ by Soundness of the *Hilbert system*. By Completeness of the *natural deduction system*, it follows that $\Gamma \vdash_{\mathrm{ND}} \varphi$.
2. If $\Gamma \vdash_{\mathrm{ND}} \varphi$, then $\Gamma \models \varphi$ by Soundness of the *natural deduction system*. By Completeness of the *Hilbert system*, it follows that $\Gamma \vdash_{\mathrm{H}} \varphi$.

Provide the details of the two preceding parts, given only in outline here, pointing out missing prerequisites (*e.g.*, we do not prove Soundness and Completeness for the *Hilbert system* in these notes) and propose ways of filling the gaps and how to prove them. (We do not add subscripts "H" and "ND" to "$\models$", because the two systems are *sound* and *complete* relative to the same semantics.)

3. Compare and discuss the pros and the cons of the proof-theoretic approach in Exercise 132 and the approach in this Exercise 133 which makes a detour through semantics. □

C.2 Rules for QPL

All the rules in Subsection C.1 for *propositional logic* can be used, in addition to *introduction* and *elimination* rules for the quantifiers, namely, two rules $\{(\forall\mathrm{I}), (\forall\mathrm{E})\}$ for "$\forall$" and two rules $\{(\exists\mathrm{I}), (\exists\mathrm{E})\}$ for "$\exists$":

$$\dfrac{\boxed{\begin{array}{ll} q & \text{fresh variable} \\ \vdots & \\ \varphi[p := q] & \end{array}}}{\forall p\ \varphi}\ (\forall\mathrm{I}) \qquad\qquad \dfrac{\forall p\ \varphi}{\varphi[p := t]}\ (\forall\mathrm{E})$$

$$\dfrac{\varphi[p := t]}{\exists p\ \varphi}\ (\exists\mathrm{I}) \qquad\qquad \dfrac{\exists p\ \varphi \qquad \boxed{\begin{array}{ll} q & \text{fresh variable} \\ \varphi[p := q] & \text{local premise} \\ \vdots & \\ \psi & \end{array}}}{\psi}\ (\exists\mathrm{E})$$

The rules for quantifiers must be used with extra care, in a way to respect the following side conditions:

- "$\varphi[p := q]$" (resp. "$\varphi[p := t]$") means that q (resp. t) is substituted for every *free occurrence* of the propositional variable p in φ.

- In the rules ($\forall$E) and ($\exists$I), we use t as a metavariable ranging over $\{\bot, \top\} \cup \mathcal{P}$. Moreover, if t is the propositional variable $q \in \mathcal{P}$, then q must be *substitutable for* p in φ, *i.e.*, every occurrence of the substituted q must be outside the scope of a pre-existing binding quantifier, "$\forall q$" or "$\exists q$", in φ.

It takes some practice to use the quantifier rules without tripping on many pattern-matching complications. It is helpful to keep in mind informal justifications for the rules, at least the two rules with boxes:

- Informal justification for ($\forall$I): *If we can derive $\varphi[p := q]$ with a fresh propositional variable q substituted for p in φ, then we can derive $(\forall p\, \varphi)$. The crucial qualification is that q is* fresh*, i.e., it does not occur anywhere outside the box. Thus, since we assume nothing about this q, the derivation works for any propositional variable substituted for q.*
- Informal justification for ($\exists$E): *If we can derive $(\exists p\, \varphi)$, then φ must hold for at least one value. We then proceed by case analysis over possible values, writing q for a generic value representing them all. If we can derive ψ, which does not mention q, from the local premise $\varphi[p := q]$, then ψ must hold regardless of the value q stands for.*

C.3 Rules for **FOL**

All the rules in Subsection C.1 for *propositional logic* can be used, in addition to the four rules for quantifiers below. In fact, these four have the same exact form as the four quantifier rules in Subsection C.2, except that now the quantification is over first-order variables rather than propositional variables. The context making clear which are intended, I choose to identify them with the same four names $\{(\forall \mathrm{I}), (\forall \mathrm{E}), (\exists \mathrm{I}), (\exists \mathrm{E})\}$:

$$\dfrac{\boxed{\begin{array}{ll} y & \qquad\qquad \text{fresh variable} \\ \vdots & \\ \varphi[x := y] & \end{array}}}{\forall x\ \varphi}\ (\forall \mathrm{I}) \qquad\qquad \dfrac{\forall x\ \varphi}{\varphi[x := t]}\ (\forall \mathrm{E})$$

$$\dfrac{\varphi[x := t]}{\exists x\ \varphi}\ (\exists \mathrm{I}) \qquad\qquad \dfrac{\exists x\ \varphi \qquad \boxed{\begin{array}{ll} y & \qquad\qquad \text{fresh variable} \\ \varphi[x := y] & \qquad\qquad \text{local premise} \\ \vdots & \\ \psi & \end{array}}}{\psi}\ (\exists \mathrm{E})$$

The side conditions for the rules of first-order quantifiers here are nearly the same as those in Subsection C.2:

- "$\varphi[x := y]$" (resp. "$\varphi[x := t]$") means that first-order variable y (resp. term t) is substituted for every *free occurrence* of the first-order variable x in φ.
- In rules ($\forall$E) and ($\exists$I), term t must be *substitutable for* x in φ, *i.e.*, every variable $y \in \mathrm{FV}(t)$ must be outside the scope of a pre-existing binding quantifier, "$\forall y$" or "$\exists y$", in φ.

If we allow the equality symbol "$\approx$" in the syntax of first-order logic, then we need two additional rules: rule ($\approx$I) which introduces one occurrence of $\approx$, and rule ($\approx$E) which eliminates one occurrence of $\approx$. They read as follows:

$$\dfrac{}{t \approx t}\ (\approx \mathrm{I}) \qquad\qquad \dfrac{t_1 \approx t_2 \qquad \varphi[x := t_1]}{\varphi[x := t_2]}\ (\approx \mathrm{E})$$

subject to the following side conditions:

- t, t_1, t_2 range over the set of first-order terms.
- In the rule ($\approx$E), terms t_1 and t_2 must be *substitutable for* x, *i.e.*, every variable $y \in \mathrm{FV}(t_1) \cup \mathrm{FV}(t_2)$ must be outside the scope of a pre-existing binding quantifier, "$\forall y$" or "$\exists y$", in φ.

The rule ($\approx$I) guarantees that $\approx$ is reflexive. The other usual properties of equality, *symmetry* and *transitivity*, follow from ($\approx$I) and ($\approx$E), as shown in the next exercise.

Exercise 134. Alternative suggestive names for ($\approx$I) and ($\approx$E) are ($\approx$*reflexive*) and ($\approx$*congruent*), respectively. Show that both of the following rules:

$$\frac{t_1 \approx t_2}{t_2 \approx t_1} \qquad (\approx \textit{symmetric})$$

$$\frac{t_1 \approx t_2 \qquad t_2 \approx t_3}{t_1 \approx t_3} \qquad (\approx \textit{transitive})$$

are derivable from ($\approx$I) and ($\approx$E). □

Exercise 135. Show that the following rule is derivable from ($\approx$E):

$$\frac{t_1 \approx u_1 \quad \cdots \quad t_n \approx u_n \qquad \varphi[x_1 := t_1, \ldots, x_n := t_n]}{\varphi[x_1 := u_1, \ldots, x_n := u_n]}$$

Particular cases of ($\approx$E*) is when φ is $R(x_1, \ldots, x_n)$ where R is a n-ary relation symbol or when φ is $f(x_1, \ldots, x_n) \approx y$ where f is a n-ary function symbol. □

C.4 Rules for eL

The set of wff's of *equality logic* is $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. These wff's do not include quantifiers, which implies that the rules in $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E})\}$ do not apply to them.

The rules of natural deduction for *equality logic* are therefore all the rules in Subsection C.1 for *propositional logic* in addition to the rules ($\approx$I) and ($\approx$E) in Subsection C.3 and the rules derived from the preceding, namely: ($\approx$ *symmetric*), ($\approx$ *transitive*), and ($\approx$E*). See Exercises 134 and 135. Keep in mind that t, t_i, and u_i, in these rules are limited to range over variables in X when used for *equality logic*.

C.5 Rules for ZOL

The set of wff's of *zeroth-order logic* is $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ or, when $\approx$ is allowed, $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$. These wff's do not include quantifiers, which implies that the rules in $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E})\}$ do not apply to them.

The rules of natural deduction for *zeroth-order logic* are therefore all the rules in Subsection C.1 for *propositional logic* in addition to the rules ($\approx$I) and ($\approx$E) in Subsection C.3 and the rules derived from the preceding, namely: ($\approx$ *symmetric*), ($\approx$ *transitive*), and ($\approx$E*). Keep in mind that t, t_i, and u_i, in these rules are limited to range over variable-free first-order terms, *i.e.*, over the set $\mathsf{Atoms}(\Sigma, \varnothing)$, when used for *zeroth-order logic*.

C.6 Rules for EL and QEL

(not yet completed)

C.7 Soundness

Soundness is a minimal requirement for any system of formal proofs: it means that formal *deducibility* (others say *derivability*) implies *semantic validity* (others say *logical validity* or also *truth*). We want a proof system to be as strong as possible, *i.e.*, to formally deduce as many semantically valid wff's as possible, without deriving a contradiction.

Theorem 136 (Soundness)**.** *For any of the logics $\mathcal{L} \in \{\mathsf{PL}, \mathsf{QPL}, \mathsf{eL}, \mathsf{ZOL}, \mathsf{EL}, \mathsf{QEL}, \mathsf{FOL}\}$ defined in these lecture notes, and for any set $\Gamma \cup \{\varphi\}$ of wff's in $\mathcal{L}$, it holds that:*

$$\Gamma \vdash_{\mathcal{L}} \varphi \quad \textit{implies} \quad \Gamma \models_{\mathcal{L}} \varphi.$$

In words, if φ is formally deducible from Γ, then φ is semantically entailed/implied by Γ. The proof system for $\mathcal{L}$ is thus not too strong.

Proof. For every logic $\mathcal{L}$ in these notes, the proof of soundness follows the same pattern: a straightforward (though somewhat laborious) induction on the length $\geqslant 1$ of the natural deduction for the judgment $\Gamma \vdash_{\mathcal{L}} \varphi$. We take the length of a natural deduction to be the number of lines containing a wff, with one wff per line. We do *not* include in the count a line which introduces a fresh variable in the rules ($\forall$I) and ($\forall$E); so, whenever we say "line" in this proof, we mean "wff" and we view a natural deduction as written top-down as a sequence of wff's.

We restrict attention to the case when $\mathcal{L}$ is FOL, which is the most involved logic in these notes. We thus omit the subscript "$\mathcal{L}$" on "$\vdash$" and "$\models$" in the rest of the proof and omit many of the obvious details. Given a natural deduction $\mathcal{D}$ for the judgment $\Gamma \vdash \varphi$, we say that $\mathcal{D}$ *satisfies the conclusion of the theorem* iff $\Gamma \models \varphi$.

The basis of the induction is when the natural deduction $\mathcal{D}$ for $\Gamma \vdash \varphi$ consists of a single line, *i.e.*, the single wff φ. In this case, $\varphi \in \Gamma$, which also implies that $\Gamma \models \varphi$, *i.e.*, $\mathcal{D}$ satisfies the conclusion of the theorem.

The induction proceeds by considering a natural deduction $\mathcal{D}$ with $k+1$ lines, with the induction hypothesis being that every natural deduction $\mathcal{D}'$ with at most k lines satisfies the conclusion of the theorem, where $k \geqslant 1$. If the natural deduction $\mathcal{D}$ has $k+1$ lines, we consider the proof rule according to which the last line in $\mathcal{D}$ is obtained – for each of the possible proof rules. There are the proof rules of PL, which can be used again in FOL, and there are the proof rules specifically belonging to FOL, namely, $\{(\forall \mathrm{I}), (\forall \mathrm{E}), (\exists \mathrm{I}), (\exists \mathrm{E}), (\approx \mathrm{I}), (\approx \mathrm{E})\}$.

Consider the rules inherited from PL first. So, suppose the last line in the natural deduction $\mathcal{D}$ with $k+1$ lines is obtained by applying the rule ($\wedge$I). Thus, $\mathcal{D}$ is a natural deduction for a judgment of the form $\Gamma \vdash (\varphi_1 \wedge \varphi_2)$. This implies there are two natural deductions $\mathcal{D}_1$ and $\mathcal{D}_2$ whose respective last lines are φ_1 and φ_2. Let the respective sets of premises in $\mathcal{D}_1$ and $\mathcal{D}_2$ be Γ_1 and Γ_2, so that $\mathcal{D}_1$ and $\mathcal{D}_2$ are natural deductions for the judgments $\Gamma_1 \vdash \varphi_1$ and $\Gamma_2 \vdash \varphi_2$. It also follows that $\Gamma \supseteq \Gamma_1 \cup \Gamma_2$.

By the induction hypothesis, we have $\Gamma_1 \models \varphi_1$ and $\Gamma_2 \models \varphi_2$. Let $(\mathcal{A}, \sigma)$ be an interpretation such that $\mathcal{A}, \sigma \models \Gamma$, which implies that both $\mathcal{A}, \sigma \models \Gamma_1$ and $\mathcal{A}, \sigma \models \Gamma_2$, because Γ_1 and Γ_2 are subsets of Γ. Hence, both $\mathcal{A}, \sigma \models \varphi_1$ and $\mathcal{A}, \sigma \models \varphi_2$, because $\Gamma_1 \models \varphi_1$ and $\Gamma_2 \models \varphi_2$. Hence, $\mathcal{A}, \sigma \models (\varphi_1 \wedge \varphi_2)$. Hence, $\Gamma \models (\varphi_1 \wedge \varphi_2)$, as desired.

We omit the cases when the last line in the natural deduction $\mathcal{D}$ with $k+1$ lines is a wff of the form $(\varphi_1 \vee \varphi_2)$ or $(\varphi_1 \to \varphi_2)$ or $(\neg \varphi)$, which are totally similar to the case when the last line is $(\varphi_1 \wedge \varphi_2)$.

For one more rule inherited from PL, consider the case when the last line in $\mathcal{D}$, which has $k+1$ lines, is obtained by applying the rule (PBC) and showing that a judgment $\Gamma \vdash \varphi$ holds. Thus, just before the last line in $\mathcal{D}$, there is a closed box, call it B, whose first line is $\neg\varphi$ (it is a "local premise" or "local hypothesis") and whose last line is $\perp$. The rule (PCB) is invoked to close B and to write the last line of $\mathcal{D}$ which is φ. If we add $\neg\varphi$ as a premise to the entire deduction, we can open the box B (*i.e.*, remove the frame of B but not its contents!) and obtain a natural deduction with k lines for the judgment $\Gamma \cup \{\neg\varphi\} \vdash \bot$. By the induction hypothesis, $\Gamma \cup \{\neg\varphi\} \models \bot$. Since for all interpretations $(\mathcal{A}, \sigma)$ we have that $\mathcal{A}, \sigma \not\models \bot$, it follows that for all interpretations $(\mathcal{A}, \sigma)$ we also have $\mathcal{A}, \sigma \not\models \Gamma \cup \{\neg\varphi\}$, *i.e.*, $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable. Hence, $\Gamma \models \varphi$, as desired.[45]

We next consider proof rules that are specific to FOL, $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E}), (\approx\mathrm{I}), (\approx\mathrm{E})\}$, and examine a natural deduction $\mathcal{D}$ with $k+1$ lines whose last line is obtained by applying one of those rules. We limit our examination to the case of rules $(\forall\mathrm{I})$ and $(\forall\mathrm{E})$, the case of the other rules being totally similar.

So, suppose the last line in $\mathcal{D}$ is obtained by applying rule $(\forall\mathrm{E})$ to show $\Gamma \vdash \varphi[x := t]$. Thus, the last line is the wff $\varphi[x := t]$, and the last but one line is $(\forall x\ \varphi)$. Let $\mathcal{D}'$ be the natural deduction consisting of the first k lines in $\mathcal{D}$, which establishes the judgment $\Gamma \vdash (\forall x\ \varphi)$. By the induction hypothesis, it holds that $\Gamma \models (\forall x\ \varphi)$. This means that for all interpretations $(\mathcal{A}, \sigma)$, if $\mathcal{A}, \sigma \models \Gamma$ then $\mathcal{A}, \sigma \models (\forall x\ \varphi)$, which in turn implies that $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi$ for all $a \in A$ where A is the universe of $\mathcal{A}$. In the notation of Appendix B.3, this is the same as $[\![\varphi]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \mathit{true}$ for all $a \in A$. Now observe that:

$$\Big\{\, [\![t]\!]_{\mathcal{A},\sigma[x\mapsto a]} \;\Big|\; a \in A \Big\} \;\subseteq\; \Big\{\, [\![x]\!]_{\mathcal{A},\sigma[x\mapsto a]} \;\Big|\; a \in A \Big\} \;=\; A.$$

Hence, $[\![\varphi[x := t]]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \mathit{true}$ for all $a \in A$. Hence, $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi[x := t]$, for every interpretation $(\mathcal{A}, \sigma)$ and every $a \in A$. This in turn implies $\mathcal{A}, \sigma \models \varphi[x := t]$ for every interpretation $(\mathcal{A}, \sigma)$, as desired.

Finally, consider the case when the last line in $\mathcal{D}$ is obtained by applying rule $(\forall\mathrm{I})$ to show $\Gamma \vdash (\forall x\ \varphi)$. Thus, the last line is the wff $(\forall x\ \varphi)$. We invoke $(\forall\mathrm{I})$ to close a box, call it B. B starts with a fresh variable y (which does not count as a separate line in $\mathcal{D}$) and ends with the wff $\varphi[x := y]$ where y occurs free. We therefore have $\Gamma \vdash \varphi[x := y]$. By the induction hypothesis, $\Gamma \models \varphi[x := y]$, *i.e.*, for every interpretation $(\mathcal{A}, \sigma)$ it holds that if $\mathcal{A}, \sigma \models \Gamma$ then $\mathcal{A}, \sigma \models \varphi[x := y]$. Equivalently, for every $a \in A$, if $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \Gamma$ then $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi[x := y]$. Since y does not occur in Γ, we also have for every $(\mathcal{A}, \sigma)$, if $\mathcal{A}, \sigma \models \Gamma$ then for every $a \in A$, it holds that $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi[x := t]$. Hence, $\Gamma \models (\forall x\ \varphi)$, as desired. $\square$

[45] For additional details for this last step, see Lemma 6 and its proof.

Appendix D

De Morgan's Laws: Semantically and Proof-Theoretically

De Morgan's Laws can be asserted as four semantically valid wff's:

1. $\models \neg(p \vee q) \quad \rightarrow \; (\neg p \wedge \neg q)$
2. $\models (\neg p \wedge \neg q) \; \rightarrow \; \neg(p \vee q)$
3. $\models (\neg p \vee \neg q) \; \rightarrow \; \neg(p \wedge q)$
4. $\models \neg(p \wedge q) \quad \rightarrow \; (\neg p \vee \neg q)$

Their semantic validity can be established using *truth tables*. For example, for the first and fourth laws we can write the following tables:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p \wedge \neg q)$	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$
false	*false*	*false*	*true*	*true*	*true*	*true*	*true*
false	*true*	*true*	*false*	*true*	*false*	*false*	*true*
true	*false*	*true*	*false*	*false*	*true*	*false*	*true*
true	*true*	*true*	*false*	*false*	*false*	*false*	*true*

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$
false	*false*	*false*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*true*	*true*	*false*	*true*	*true*
true	*false*	*false*	*true*	*false*	*true*	*true*	*true*
true	*true*	*true*	*false*	*false*	*false*	*false*	*true*

The two leftmost columns in the two tables list all possible truth assignments for the pair (p, q). The rightmost column in the first table assigns a truth-value to the wff $\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$ for each of the assignments of (p, q), and since every entry in the rightmost column is *true*, the wff is semantically valid. And similarly for the rightmost column in the second table, which establishes the semantic validity of the wff $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$.

Exercise 137. Write the truth tables for the remaining *de Morgan's Laws*: 2 and 3, to show that they are all semantically valid. $\square$

De Morgan's Laws can also be asserted in the form of four formally deducible judgments according to the proof rules in Section C.1:

1. $\vdash \neg(p \vee q) \quad \rightarrow \; (\neg p \wedge \neg q)$
2. $\vdash (\neg p \wedge \neg q) \; \rightarrow \; \neg(p \vee q)$

3. $\vdash\ (\neg p \vee \neg q)\ \rightarrow\ \neg(p \wedge q)$
4. $\vdash\ \neg(p \wedge q)\ \rightarrow\ (\neg p \vee \neg q)$

Below are natural-deduction formal proofs for the first and fourth *de Morgan's Laws*:

1	$\neg(p \vee q)$	local prem
2	p	local prem
3	$p \vee q$	$\vee\mathrm{I}_1$ 2
4	$\perp$	$\neg$E 1, 3
5	$\neg p$	$\neg$ I 2–4
6	q	local prem
7	$p \vee q$	$\vee\mathrm{I}_2$ 6
8	$\perp$	$\neg$E 1, 7
9	$\neg q$	$\neg$I 6–8
10	$\neg p \wedge \neg q$	$\wedge$I 5, 9
11	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$	$\rightarrow$I 1–10

1	$\neg(p \wedge q)$	local prem
2	$\neg(\neg p \vee \neg q)$	local prem
3	$\neg p$	local prem
4	$\neg p \vee \neg q$	$\vee\mathrm{I}_1$ 3
5	$\perp$	$\neg$E 2, 4
6	$\neg\neg p$	$\neg$ I 3–5
7	p	$\neg\neg$E 6
8	$\neg q$	local prem
9	$\neg p \vee \neg q$	$\vee\mathrm{I}_2$ 8
10	$\perp$	$\neg$E 2, 9
11	$\neg\neg q$	$\neg$I 8–10
12	q	$\neg\neg$E 11
13	$p \wedge q$	$\wedge$I 7, 12
14	$\perp$	$\neg$E 1, 13
15	$\neg\neg(\neg p \vee \neg q)$	$\neg$I 2–14
16	$(\neg p \vee \neg q)$	$\neg\neg$E 15
17	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$	$\rightarrow$I 1–16

Remark: Our proof of *de Morgan's Law* 1 avoids all the rules in $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$, in contrast to our proof of *de Morgan's Law* 4, which uses ($\neg\neg$E) three times, in lines [7], [12], and [16]. It turns out that any formal proof of *de Morgan's Law* 4 must use one of the rules in $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$ at least once; this is not a trivial result and requires a deeper examination of formal-proof systems (not in these lecture notes).

The four rules in $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$ are related, in that it suffices to use only one of them, possibly more than once, in the same formal proof. See Section C.1 and Exercise 130 for an explanation of how they are related.

Exercise 138. Write natural-deduction proofs for the remaining *de Morgan's Laws*: 2 and 3, to show that they are all formally derivable. For full credit, avoid using any of the four rules in $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$. *Hint*: For *de Morgan's Laws* 2 and 3 this is possible, though it may require a little more care. $\square$

From the preceding exercise and earlier remark, *de Morgan's Law* 4 has a special status: the first three *de Morgan's Laws* are valid intuitionistically, while *de Morgan's Law* 4 is not.

Whereas the semantic validity of a propositional wff φ is determined uniquely by means of a truth-table, in whatever shape it is written, there is in general more than one formal proof of the judgment $\vdash \varphi$, whether within the same proof system or across different proof systems for *propositional logic*.

To illustrate this last point, we revisit the proof-theoretic validation of *de Morgan's Law* 4. This is also an opportunity to compare it with a mechanized version produced with the proof assistant Lean 4. Below is another natural-deduction formal proof for *de Morgan's Law* 4, in which we invoke (LEM) instead of $(\neg\neg\mathrm{E})$ from the set $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\mathrm{Peirce's})\}$, and which we invoke only once in line [12] instead of three times.

1	$\neg(p \wedge q)$	local prem
2	p	local prem
3	q	local prem
4	$p \wedge q$	$\wedge$I 2, 3
5	$\bot$	$\neg$E 4, 1
6	$\neg q$	$\neg$I 3–5
7	$\neg p \vee \neg q$	$\vee\mathrm{I}_2$ 6
8	$p \to (\neg p \vee \neg q)$	$\to$I 2–7
9	$\neg p$	local prem
10	$\neg p \vee \neg q$	$\vee\mathrm{I}_1$ 9
11	$\neg p \to (\neg p \vee \neg q)$	$\to$I 9–10
12	$p \vee \neg p$	LEM
13	$(\neg p \vee \neg q)$	$\vee$E 12, 8, 11
14	$\neg(p \wedge q) \to (\neg p \vee \neg q)$	$\to$I 1–13

Below is the translation of the preceding natural-deduction proof into the syntax of Lean 4 where, for emphasis, we write *keywords* and what are called *tactics* in red. A phrase preceded by "`--`" in the rightmost column is a comment (ignored by the Lean 4 proof checker). For example, the comment "`obtained by closing box [2-7]`" has no effect on proof-checking and indicates that the wff $p \to (\neg p \vee \neg q)$, which is on line [8] of the natural deduction, is the result of closing the box spanning from line [2] to line [7]. It is customary to identify premises, global or local, with labels starting with letter "`h`" (for "hypothesis"); labels make premises available for later invocation in the proof. The labels here are `h`, `h1`, `h2`, `h_p`, `h_not_p`, `h_q`, and `h_not_q`. The word "`False`" below is the translation of "$\bot$" on line [5] of the natural deduction.

```
lemma de_morgan_4 {p q : Prop} : ¬(p∧q) → (¬p∨¬q) := by
                                              -- obtained by closing box [1-13]
    intro h
    have  h1 : p  → (¬p∨¬q) := by              -- obtained by closing box [2-7]
          {  intro h_p
             show ¬p∨¬q  ;
               have h_not_q : ¬q := by          -- obtained by closing box [3-5]
                  { intro h_q
                    show False ; apply h (And.intro h_p h_q) } ;
               apply Or.intro_right (¬p) h_not_q  }
    have  h2 : ¬p  →  (¬p∨¬q) := by             -- obtained by closing box [9-10]
          {  intro h_not_p
             show  ¬p∨¬q ; apply Or.intro_left (¬q) h_not_p  }
    apply Or.elim (Classical.em p) h1 h2
```

Exercise 139. Using as a guide the preceding translation of a natural-deduction proof for *de Morgan's Law* 4 into LEAN 4, do the following:

1. Translate into the syntax of LEAN 4 the natural-deduction proof for *de Morgan's Law* 1 shown after Exercise 137.
2. Translate into the syntax of LEAN 4 the natural-deduction proof for *de Morgan's Law* 4 shown after Exercise 137. □

It should be clear by now that writing formal proofs *top-down* is a tedious task. It generally requires many failed attempts to add a new line in the deduction and causing as many backtrackings. The search for a legal *top-down* (or *forward*) deduction is therefore a process of repeated backtrackings in general. If this process does not terminate, it may be because our starting global premises, or our local premises, do not in fact imply our final conjectured conclusion – not because we have not tried hard enough.

But the preceding suggests an alternative approach, which may prove more efficient and less dependent on backtrackings. This consists in building a formal deduction in *bottom-up* (or *backward*) fashion, by starting from the conjectured final conclusion and moving upward to discover the required premises at the top. This alternative approach produces successive subgoals moving upward from goals below them, which is in fact the preferred approach of many interactive proof assistants.

We leave it to you to select a few (simple) propositional wff's to test and experiment with the two methods just described for building natural-deduction formal proofs. For each propositional wff φ that you select, you should first show that $\models \varphi$, *i.e.*, φ is semantically valid, which guarantees the existence of a natural-deduction proof for the judgment $\vdash \varphi$ by completeness.

More examples and exercises of natural-deduction proofs are in Appendix E, which can be used to test and compare the two methods of building the same natural-deduction proof, *top-down* (or *forward*) and *bottom-up* (or *backward*).

Appendix E

Prenex Form and Skolemization

The process of transforming a wff with quantifiers into its *prenex form*, and the additional process of *Skolemizing* it, applies equally well to quantified Boolean wff's and first-order wff's. The transformation of the two kinds of wff's being entirely similar, we restrict our presentation to first-order wff's.

E.1 Prenex Form

A first-order wff $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ is in *prenex form* (or *prenex normal form*) iff φ consists of a (possibly empty) string of quantifiers followed by a quantifier-free wff. The string of quantifiers in φ is its *prefix* and the quantifier-free subformual of φ is its *matrix*.

We call our transformation "$\boxed{\mathsf{prenex}}$", and the result of applying it to an arbitrary wff φ is denoted "$\boxed{\mathsf{prenex}}(\varphi)$". In what follows, we use Q, possibly subscripted, to range over $\{\forall, \exists\}$. Moreover, if Q is $\forall$ (resp. $\exists$), then $\overline{Q}$ is $\exists$ (resp. $\forall$), *i.e.*, $\overline{\forall}$ denotes $\exists$ and $\overline{\exists}$ denotes $\forall$. For an arbitrary first-order wff φ, the wff $\boxed{\mathsf{prenex}}(\varphi)$ is therefore of the form $(Q_1 x_1 \cdots Q_n x_n .\, \psi)$ where $Q_1, \ldots, Q_n \in \{\forall, \exists\}$ and ψ is quantifier-free.

The definition of $\boxed{\mathsf{prenex}}(\varphi)$ is by structural induction on φ. This can be done in one of two ways: *top-bottom* (as in the definition of $\boxed{\mathsf{QPL} \mapsto \mathsf{PL}}$ in the proof of Lemma 36), or *bottom-up*. In *top-bottom*, we start with φ fully given and we think of $\boxed{\mathsf{prenex}}$ as being pushed down recursively through the sub-wff's of φ. In *bottom-up*, we define $\boxed{\mathsf{prenex}}(\varphi)$ simultaneously with φ, as the latter is being built up inductively. Our induction here is *bottom-up*, which is a bit simpler:

1. If φ is quantifier-free, then $\boxed{\mathsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \varphi$.

2. If $\varphi \stackrel{\text{def}}{=} (\neg\psi)$ and $\boxed{\mathsf{prenex}}(\psi) = (Q_1 x_1 \cdots Q_n x_n .\, \theta)$ where θ is quantifier-free, then:

$$\boxed{\mathsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \big(\overline{Q}_1 x_1 \cdots \overline{Q}_n x_n .\, \neg\theta\big).$$

In the next two cases, let:

$$\boxed{\mathsf{prenex}}(\varphi_1) = (Q_1 y_1 \cdots Q_n y_n .\, \theta_1),$$

$$\boxed{\mathsf{prenex}}(\varphi_2) = (Q_1 z_1 \cdots Q_p z_p .\, \theta_2),$$

where θ_1 and θ_2 are quantifier-free. By renaming bound variables in $\boxed{\mathsf{prenex}}(\varphi_1)$ and $\boxed{\mathsf{prenex}}(\varphi_2)$, we can assume the variables in $\{y_1, \ldots, y_n, z_1, \ldots, z_p\}$ are all distinct and that:

$$\{y_1, \ldots, y_n\} \cap \mathrm{FV}\big(\boxed{\mathsf{prenex}}(\varphi_2)\big) = \varnothing \quad \text{and} \quad \{z_1, \ldots, z_p\} \cap \mathrm{FV}\big(\boxed{\mathsf{prenex}}(\varphi_1)\big) = \varnothing.$$

3. If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \diamond \varphi_2)$ where $\diamond \in \{\wedge, \vee\}$, then:

$$\boxed{\textsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \big(Q_1 y_1 \cdots Q_n y_n\ Q_1 z_1 \cdots Q_p z_p.\, (\theta_1 \diamond \theta_2)\big).$$

4. If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \to \varphi_2)$, then:

$$\boxed{\textsf{prenex}}(\varphi)\ \stackrel{\text{def}}{=} \big(\ \overline{Q}_1 y_1 \cdots \overline{Q}_n y_n\ Q_1 z_1 \cdots Q_p z_p.\ (\theta_1 \to \theta_2)\ \big)$$

5. If $\varphi \stackrel{\text{def}}{=} (Qx.\,\psi)$ where $Q \in \{\forall, \exists\}$, then:

$$\boxed{\textsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \big(Qx.\,\boxed{\textsf{prenex}}(\psi)\big).$$

We can show, for an arbitrary first-order wff φ, that φ and $\boxed{\textsf{prenex}}(\varphi)$ are equivalent in one of two ways:

- semantically, *i.e.*, $\models\ \big(\varphi \to \boxed{\textsf{prenex}}(\varphi)\big)\ \wedge\ \big(\boxed{\textsf{prenex}}(\varphi) \to \varphi\big)$, or
- proof-theoretically, *i.e.*, $\vdash\ \big(\varphi \to \boxed{\textsf{prenex}}(\varphi)\big)\ \wedge\ \big(\boxed{\textsf{prenex}}(\varphi) \to \varphi\big)$.

Either way, we can follow the bottom-up induction which we used to define φ and $\boxed{\textsf{prenex}}(\varphi)$ simultaneously. All we need for this are Lemma 140 and Lemma 147.

Lemma 140. *Let φ be an arbitrary first-order wff. Then:*

1. $\neg(\exists x.\,\varphi)$ *and* $\big(\forall x.\,\neg\varphi\big)$ *are equivalent wff's.*
2. $\neg(\forall x.\,\varphi)$ *and* $\big(\exists x.\,\neg\varphi\big)$ *are equivalent wff's.*

Proof. We give formal natural-deduction proofs, and we ask you to give (much easier) semantic proofs in Exercise 141. For part 1, it suffices to show (why?):

$\neg\exists x.\ \varphi(x)\ \vdash\ \forall x.\ \neg\varphi(x)$ instead of $\vdash\ \big(\neg\exists x.\ \varphi(x)\big) \to \big(\forall x.\ \neg\varphi(x)\big)$, and
$\forall x.\ \neg\varphi(x)\ \vdash\ \neg\exists x.\ \varphi(x)$ instead of $\vdash \forall x.\ \neg\varphi(x)\ \to\ \neg\exists x.\ \varphi(x)$.

1	$\neg\exists x\ \varphi(x)$	premise
2	y	fresh variable
3	$\varphi(y)$	local premise
4	$\exists x\ \varphi(x)$	$\exists$I 3
5	$\bot$	$\neg$E 1, 4
6	$\neg\varphi(y)$	$\neg$I 3–5
7	$\forall x\ \neg\varphi(x)$	$\forall$I 2–6

1	$\forall x\ \neg\varphi(x)$	premise
2	$\exists x\ \varphi(x)$	local premise
3	y	fresh variable
4	$\varphi(y)$	local premise
5	$\neg\varphi(y)$	$\forall$E 1
6	$\bot$	$\neg$E 4,5
7	$\bot$	$\exists$E 2, 3–6
8	$\neg\exists x\ \varphi(x)$	PBC 2–7

The natural deduction on the left says "$\neg\exists x.\ \varphi(x)\ \vdash\ \forall x.\ \neg\varphi(x)$", and the natural deduction on the right says "$\forall x.\ \neg\varphi(x)\ \vdash\ \neg\exists x.\ \varphi(x)$". For part 2 of the lemma, we can write the following:

1	$\exists x\ \neg\varphi(x)$	premise
2	y	fresh variable
3	$\neg\varphi(y)$	local premise
4	$\forall x\ \varphi(x)$	local premise
5	$\varphi(y)$	$\forall$E 4
6	$\bot$	$\neg$E 3, 5
7	$\neg\forall x\ \varphi(x)$	$\neg$I 4–6
8	$\neg\forall x\ \varphi(x)$	$\exists$E 1, 2–7

1	$\neg\forall x\ \varphi(x)$	premise
2	$\neg\exists x\ \neg\varphi(x)$	local premise
3	y	fresh variable
4	$\neg\varphi(y)$	local premise
5	$\exists x\ \neg\varphi(x)$	$\exists$I 4
6	$\bot$	$\neg$E 2, 5
7	$\varphi(y)$	PBC 4–6
8	$\forall x\ \varphi(x)$	$\forall$I 3–7
9	$\bot$	$\neg$E 1, 8
10	$\exists x\ \neg\varphi(x)$	PBC 2–9

From the formal proof on the left, we conclude $\big(\exists x.\, \neg\varphi(x)\big) \vdash \neg\big(\forall x.\, \varphi(x)\big)$, and from the right, we conclude $\neg\big(\forall x.\, \varphi(x)\big) \vdash \big(\exists x.\, \neg\varphi(x)\big)$. □

Exercise 141. Write semantic proofs for the two equivalences in Lemma 140, noting that each equivalence consists of two implications. For example, for the first equivalence, you have to show both of the following:

- $\mathcal{A} \models \neg(\exists x.\, \varphi) \to \big(\forall x.\, \neg\varphi\big)$,
- $\mathcal{A} \models \big(\forall x.\, \neg\varphi\big) \to \neg(\exists x.\, \varphi)$,

where $\mathcal{A}$ is an arbitrary Σ-structure and, for simplicity, you can assume $\mathrm{FV}(\varphi) = \{x\}$. Do the same for the second equivalence in Lemma 140. □

As a warm-up for the proof of Lemma 147 and the exercises following it, you may try the following examples. They all involve natural deductions showing that a judgment of the form $\varphi \vdash \psi$ holds; we omit the (typically easier) proof of the corresponding semantic validity $\varphi \models \psi$.

Example 142. We write natural deductions for $\neg\varphi \vee \psi \vdash \varphi \to \psi$ and $\varphi \to \psi \vdash \neg\varphi \vee \psi$.

1	$\neg\varphi \vee \psi$	premise
2	$\neg\varphi$	local premise
3	φ	local premise
4	$\bot$	$\neg$E 2, 3
5	ψ	$\bot$E 4
6	$\varphi \to \psi$	$\to$I 3–5
7	ψ	local premise
8	φ	local premise
9	ψ	repeat 7
10	$\varphi \to \psi$	$\to$I 8–9
11	$\varphi \to \psi$	$\vee$E 1, 2–6, 7–10

1	$\varphi \to \psi$	premise
2	$\varphi \vee \neg\varphi$	LEM
3	$\neg\varphi$	local premise
4	$\neg\varphi \vee \psi$	$\vee\mathrm{I}_1$ 3
5	φ	local premise
6	ψ	$\to$E 1, 5
7	$\neg\varphi \vee \psi$	$\vee\mathrm{I}_2$ 6
8	$\neg\varphi \vee \psi$	$\vee$E 2, 3–4, 5–7

The two preceding natural deductions show that $(\varphi \to \psi)$ and $(\neg\varphi \vee \psi)$ are equivalent wff's. Note that the deduction on the left does not use any of the rules in $\{(\mathrm{LEM}), (\mathrm{PBC}), (\neg\neg\mathrm{E}), (\text{Peirce's})\}$ which is therefore legal intuitionistically, whereas the deduction on the right uses (LEM). It can be shown (not in these notes) that it is not possible to write a deduction for $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$ without invoking one of those four rules. □

Example 143. Permuting two adjacent universal quantifiers does not change the meaning of a wff, *i.e.*, the following judgment holds: $\forall x\ \forall y\ \varphi(x, y)\ \vdash\ \forall y\ \forall x\ \varphi(x, y)$, as confirmed by the following natural deduction.

1	$\forall x\ \forall y\ \varphi(x, y)$	premise
2	y_0	fresh y_0
3	x_0	fresh x_0
4	$\forall y\ \varphi(x_0, y)$	$\forall$E 1
5	$\varphi(x_0, y_0)$	$\forall$E 4
6	$\forall x\ \varphi(x, y_0)$	$\forall$I 3–5
7	$\forall y\ \forall x\ \varphi(x, y)$	$\forall$I 2–6

where only the rules ($\forall$E) and ($\forall$I) are used. □

Example 144. An existential quantifier can be distributed over a logical or "$\vee$", *i.e.*, the following judgment holds: $\exists x\ (\varphi(x) \vee \psi(x))\ \vdash\ \exists x\ \varphi(x)\ \vee\ \exists x\ \psi(x)$, as confirmed by the following natural deduction:

1	$\exists x\ (\varphi(x) \vee \psi(x))$	premise
2	x_0	fresh variable
3	$\varphi(x_0) \vee \psi(x_0)$	local premise
4	$\varphi(x_0)$	local premise
5	$\exists x\ \varphi(x)$	$\exists$I 4
6	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\vee$I 5
7	$\psi(x_0)$	local premise
8	$\exists x\ \psi(x)$	$\exists$I 7
9	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\vee$I 8
10	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\vee$E 3, 4–6, 7–9
11	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\exists$E 1, 2–10

where only rules for '$\vee$' and '$\exists$' are used, both for introduction and elimination. □

Exercise 145. Write a natural deduction to establish the converse of the judgment in Example 144, to formally prove that the following judgment holds: $\exists x\ \varphi(x) \ \vee\ \exists x\ \psi(x) \ \vdash\ \exists x\ (\varphi(x) \vee \psi(x))$. This shows that we can "push" to the "left" existential quantifiers out of the scope of a "$\vee$" immediately preceding them. □

Exercise 146. Read Example 144 and do Exercise 145 before attempting this exercise. Write natural deductions to establish the two following judgments:

- $\forall x\ \varphi(x) \ \wedge\ \forall x\ \psi(x) \ \vdash\ \forall x\ (\varphi(x) \wedge \psi(x))$.
- $\forall x\ (\varphi(x) \wedge \psi(x)) \ \vdash\ \forall x\ \varphi(x) \ \wedge\ \forall x\ \psi(x)$.

We can "push" to the "left" universal quantifiers out of the scope of a "$\wedge$" immediately preceding them. □

Lemma 147. *Let φ and ψ be arbitrary first-order wff's, such that $x \notin \mathrm{FV}(\psi)$. Then:*

1. $\big((\forall x\, \varphi) \wedge \psi\big)$ *and* $\big(\forall x\, (\varphi \wedge \psi)\big)$ *are equivalent wff's.*
2. $\big((\exists x\, \varphi) \wedge \psi\big)$ *and* $\big(\exists x\, (\varphi \wedge \psi)\big)$ *are equivalent wff's.*
3. $\big((\forall x\, \varphi) \vee \psi\big)$ *and* $\big(\forall x\, (\varphi \vee \psi)\big)$ *are equivalent wff's.*
4. $\big((\exists x\, \varphi) \vee \psi\big)$ *and* $\big(\exists x\, (\varphi \vee \psi)\big)$ *are equivalent wff's.*
5. $\big((\forall x\, \varphi) \to \psi\big)$ *and* $\big(\exists x\, (\varphi \to \psi)\big)$ *are equivalent wff's.*
6. $\big((\exists x\, \varphi) \to \psi\big)$ *and* $\big(\forall x\, (\varphi \to \psi)\big)$ *are equivalent wff's.*
7. $\big(\psi \to (\forall x\, \varphi)\big)$ *and* $\big(\forall x\, (\psi \to \varphi)\big)$ *are equivalent wff's.*
8. $\big(\psi \to (\exists x\, \varphi)\big)$ *and* $\big(\exists x\, (\psi \to \varphi)\big)$ *are equivalent wff's.*

In parts $\{1, 2, 3, 4\}$, we omit the cases when the two components of the logical connectives are permuted, as in $\big(\psi \wedge (\exists x\, \varphi)\big)$ instead of $\big((\exists x\, \varphi) \wedge \psi\big)$, because "$\wedge$" and "$\vee$" are commutative binary connectives.

Proof. Left as an exercise, which should be straightforward after studying Examples 142, 143, and 144, and doing Exercises 145 and 146. □

Exercise 148. Prove each of the four odd-numbered (or the four even-numbered) equivalences in Lemma 147 twice: once by writing natural deductions (more tedious), and once by providing rigorous semantic arguments (simpler and easier). □

Proposition 149. *Let φ be an arbitrary first-order wff and $\psi \stackrel{\text{def}}{=} \boxed{\mathsf{prenex}}(\varphi)$. Then φ and ψ are equivalent wff's.*

Proof. We repeat the *bottom-up* induction that defines φ and $\boxed{\mathsf{prenex}}(\varphi)$ simultaneously. But now, at every step of the induction, we also show that φ and $\boxed{\mathsf{prenex}}(\varphi)$ are equivalent wff's. At step 2 of the induction, you need to us Lemma 140 repeatedly to "move" quantifiers to the left past the logical negation "$\neg$". At steps 3 and 4, you need to use Lemma 147 to "move" quantifiers outside the logical connectives "$\wedge$", "$\vee$", and "$\to$". All obvious details omitted. $\square$

E.2 Skolem Form

Let ψ be a first-order wff in prenex form. Again here, as in Section E.1, ψ may be a quantified Boolean wff or a first-order wff. We limit our examination to the first-order case, the case of quantified Boolean wff's being totally similar.

The *Skolemization* of ψ produces another first-order wff, call it θ, in prenex form where the prefix of quantifiers mentions only the universal "$\forall$". Our name for the transformation from ψ to θ is "$\boxed{\mathsf{skolem}}$". The wff θ is obtained by initially setting θ to ψ and then repeatedly applying the following three-step sequence to it:[46]

1. Find the leftmost $\exists$ in the quantifier prefix of ψ, which binds a variable x and appears as "$\exists\, x$",
2. Introduce a fresh function symbol f_x of arity equal to the number of $\forall$'s to the left of "$\exists\, x$",
3. If the $\forall$'s to the left of "$\exists\, x$" are "$\forall y_1 \cdots \forall y_n$", then cross out "$\exists\, x$" from the quantifier prefix and replace all occurrences of x in the matrix of ψ by the term $f_x(y_1, \ldots, y_n)$.

This process is bound to terminate because the initial prefix of quantifiers in ψ has finite length. We denote the resulting θ by writing "$\boxed{\mathsf{skolem}}(\psi)$", and refer to it as the *Skolem form* of ψ.

Note that there are as many new fresh function symbols "f_x" in θ as there are existential quantifiers "$\exists\, x$" in the prefix of the initial wff ψ in prenex form. These fresh function symbols are called *Skolem functions.* Note also that if the leftmost "$\exists\, x$" in the initial ψ is not preceded by any $\forall$, the associated Skolem function f_x has arity $= 0$, *i.e.*, f_x is a constant symbol.

If φ is an arbitrary first-order wff, not necessarily in prenex form, then we write $\boxed{\mathsf{sko,pre}}(\varphi)$ to denote the two-stage transformation of φ – first, into prenex form and, second, into Skolemized form – and we also call $\boxed{\mathsf{sko,pre}}(\varphi)$ the Skolemization of φ.

While φ and $\boxed{\mathsf{prenex}}(\varphi)$ are logically equivalent ("they say the same thing"), it does not make sense to talk about the equivalence (or non-equivalence) of φ and $\boxed{\mathsf{sko,pre}}(\varphi)$ because the signature of the latter is different from the signature of φ. Nevertheless, we have the following result. Recall that a *sentence* φ is a closed formula, *i.e.*, $\mathrm{FV}(\varphi) = \varnothing$.

Proposition 150. *Let φ and Γ be an arbitrary first-order sentence and set of first-order sentences. We then have:*

1. *φ is satisfiable iff $\boxed{\mathsf{sko,pre}}(\varphi)$,*
2. *Γ is satisfiable iff $\boxed{\mathsf{sko,pre}}(\Gamma)$.*

In Part 2, we have to be careful that, when we Skolemize distinct wff's φ_1 and φ_2 of Γ, we introduce distinct Skolem functions for each wff, i.e., the Skolem functions of φ_1 do not interfere with the Skolem functions of φ_2.

[46] The words *Skolemize* and *Skolemization* are derived from the name of the mathematical logician Thoralf Skolem. If you want to find out more about the many uses of *Skolemization*, click here.

Proof. We leave the proof of Part 2 as an easy exercise implied by Part 1. For Part 1, we can assume that φ is already in prenex form, by Proposition 149. It suffices to show how the elimination of the leftmost existential quantifier from the prefix of φ produces another prenex form, say θ, which is equisatisfiable with φ, and then the same process can be repeated for the elimination of all the other existential quantifers in the prefix of φ. Let then φ be of the form:

$$\varphi \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n \exists y\ \varphi_0$$

where $n \geqslant 0$ and φ_0 is a a prenex form such that $\mathrm{FV}(\varphi_0) \subseteq \mathrm{FV}(\varphi) \cup \{x_1, \ldots, x_n, y\}$ and, because φ is closed, in fact $\mathrm{FV}(\varphi_0) = \{x_1, \ldots, x_n, y\}$. According to the Skolemization process, θ is of the form:

$$\theta \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n\ \big(\varphi_0[y := f_y(x_1, \ldots, x_n)]\big)$$

where f_y is a fresh n-ary function symbol. Σ and $\Sigma' \stackrel{\text{def}}{=} \Sigma \cup \{f_y\}$ are the signatures of φ and θ, respectively.

Let $\mathcal{A}$ be a Σ-structure. The expansion $\mathcal{A}' \stackrel{\text{def}}{=} (\mathcal{A}, f_y^{\mathcal{A}'})$ of $\mathcal{A}$ is a Σ'-structure. Let A be the universe of $\mathcal{A}$, which is also the universe of $\mathcal{A}'$. If $\mathcal{A}' \models \theta$, it is easy to check that $\mathcal{A} \models \varphi$. Hence, if θ is satisfiable, so is φ.

Conversely, let $\mathcal{A} \models \varphi$ and let $\sigma : X \to A$ be an arbitrary valuation where A is the universe of $\mathcal{A}$. We construct a Σ'-structure $\mathcal{A}'$ by expanding $\mathcal{A}$ so that for every $a_1, \ldots, a_n \in A$, the function $f_y^{\mathcal{A}'}$ maps $(a_1, \ldots, a_n)$ to b where:[47]

$$\mathcal{A},\ \big(\sigma[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n, y \mapsto b]\big)\ \models\ \ \varphi_0.$$

We choose the interpretation $f_y^{\mathcal{A}'}$ of the Skolem function f_y precisely so that the preceding satisfaction holds. It is now easy to check that $\mathcal{A}' \models \theta$. Hence, if φ is satisfiable, then so is θ. □

Exercise 151. What goes wrong in the proof of Proposition 150 if φ is an open wff?

Hint: Try the open wff $\varphi(y) \stackrel{\text{def}}{=} \exists! v \forall w \big(R(a, w) \wedge R(v, w)\big) \to \exists x \big(R(a, y) \wedge R(x, y)\big)$, where "$\exists!$" means "there exists exactly one", R is a binary relation symbol and a is a constant symbol. Show that $\models \varphi(y)$, but the construction in the proof of Proposition 150 produces an open wff $\theta(y)$ not satisfied by any structure $\mathcal{A}$, unless we introduce additional constraints at the meta-level on $\mathcal{A}$. □

Exercise 152. Let R be a binary relation symbol and f a unary function symbol.

1. Show that the sentence $\varphi \stackrel{\text{def}}{=} \forall x\, R\big(x, f(x)\big) \to \forall x \exists y\, R(x, y)$ is valid. Do it in two different ways:

   (a) proof-theoretically, $\vdash \varphi$, using natural deduction, and

   (b) semantically, $\models \varphi$.

2. Show that the sentence $\psi \stackrel{\text{def}}{=} \forall x \exists y\, R(x, y) \to \forall x\, R\big(x, f(x)\big)$ is not valid. Note that ψ is just the converse implication of φ.

   Hint: Try a semantic approach, *i.e.*, show $\not\models \psi$. You need to define a structure $\mathcal{A}$ so that the left-hand side of "$\to$" in ψ is true in $\mathcal{A}$ but the right-hand side of "$\to$" is false in $\mathcal{A}$.

3. Conclude that $\forall x \exists y\, R(x, y)$ and $\forall x\, R\big(x, f(x)\big)$ are not equivalent first-order wff's.

Remark: Despite the conclusion in part 3, Proposition 150 asserts $\forall x \exists y\, R(x, y)$ and $\forall x\, R\big(x, f(x)\big)$ are equisatisfiable, *i.e.*, if there is a model for one, then there is a model for the other, and vice-versa. □

[47] Review the definition of $\sigma[x \mapsto a]$ in Section B.3.

Appendix F

Alternative Compactness Proofs

We present two alternative proofs of Compactness for *propositional logic*. At bottom, these are not "new" proofs, but different presentations of the same fundamental idea (or topological core, if you will) underlying the proof of Theorem 2 in Section 1. This fundamental idea is what *König's Lemma* asserts. The difference here is that they make the connection with topology a little more explicit by naming and presenting the same key concepts differently. One can read the first alternative proof as an elaboration of the proof in Section 1, and the second alternative proof as an elaboration of the first.[48]

Lemma 153 (König's Lemma)**.** *Every infinite, finitely branching, tree $\mathcal{T}$ has an infinite path.*

Proof. Using induction, we define an infinite sequence of nodes $\alpha_0, \alpha_1, \ldots$, forming an infinite path in $\mathcal{T}$. At stage 0 of the induction, let α_0 be the root node of $\mathcal{T}$, which has infinitely many successors by the hypothesis that $\mathcal{T}$ is infinite. At every stage $n \geqslant 1$, assume we have already selected nodes $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ so far, forming a path of length $(n-1)$, such that α_{n-1} has infinitely many successors. By hypothesis, $\mathcal{T}$ is finitely branching, which implies α_{n-1} has only finitely many immediate successors. Hence, one of the immediate successors of α_{n-1}, say β, must have infinitely many successors. Define α_n to be β, which has infinitely many successors in $\mathcal{T}$, and proceed to stage $n+1$ of the induction. $\square$

The preceding proof is not constructive: We do not have an algorithm to select the next node β at stage n after having already selected nodes $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. We only know that one of the immediate successors of α_{n-1} is the root node of an infinite subtree; we know that it exists, but we do not know which one it is. So, at every stage we invoke what is called the Axiom of Choice to select the next node β.

For the next alternative proof of Compactness for PL, we specialize *König's Lemma* (KL) to the case of binary trees, where every node has exactly two successors. The *full binary tree* is a tree without leaf nodes, which therefore has $2^{\aleph_0}$ distinct infinite paths. Every binary tree can be viewed as an initial fragment of the full binary tree, *i.e.*, by inserting a copy of the full binary tree at every leaf node of the former.

Another way of stating KL relative to binary trees is to say: *If a binary tree has arbitrarily long full finite paths, then it has an infinite path*, which is the form we use in the next proof. By a "full finite path" we mean a path that starts at the root node and ends at a leaf node. This form of KL specialized to binary trees is sometimes called *Weak König's Lemma* (WKL).

The next exercise is a little application of WKL, which has a distinctly topological flavor.

[48] And there are still other proofs with a decidedly algebraic or topological content. A particular construction nicely complementing the material in this appendix is Łoś's Theorem which proves Compactness using what are called *ultrafilters* and *ultraproducts*. Search the Web for "propositional compactness via ultraproducts" and "first-order compactness via ultraproducts".

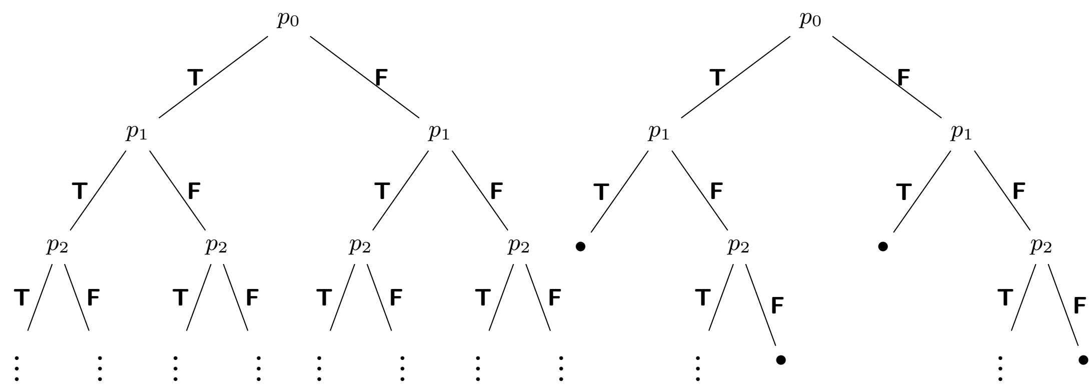

Figure F.1: *On the left*: The top three levels of the full binary tree $\mathcal{T}_{\text{full}}$, with each left/right edge labelled T/F. *On the right*: An example of how $\mathcal{T}_{\text{full}}$ is pruned if p_0 occurs nowhere in Γ (and therefore has no effect on the satisfiability of Γ) and Γ contains the wff $(\neg p_1 \wedge p_2)$. Starting from the root, if a path reaches a leaf node "•", the corresponding truth assignment is doomed to falsify Γ.

Exercise 154 (*Sequential Compactness*)**.** We write $A = [0,1]$ for the closed interval of all real numbers between 0 and 1. Let $\boldsymbol{a} \overset{\text{def}}{=} (a_n \,|\, n \in \mathbb{N})$ be an infinite sequence of elements in A. Give a precise argument for the fact that there is an infinite subsequence $\boldsymbol{a}'$ of $\boldsymbol{a}$, say $\boldsymbol{a}' \overset{\text{def}}{=} (a_k \,|\, k \in K)$ where $K \subseteq \mathbb{N}$, such that $\boldsymbol{a}'$ converges to an element $b \in A$. (Take the elements of the sequence $\boldsymbol{a}$ and subsequence $\boldsymbol{a}'$ to be listed in order of increasing indices.) This property is called the Sequential Compactness of the set of reals; in general, this concept neither implies nor is implied by Compactness as defined in topology, but for some topological spaces (*e.g.*, the reals), the two are equivalent.

Alternative Proof I of Theorem 2 (Compactness for Propositional Logic). As in the earlier proof of Theorem 2 in Section 1, we only need to consider the non-trivial implication "$\Leftarrow$": If Γ is finitely satisfiable, then Γ is satisfiable.

The set $\mathcal{P}$ of propositional variables is countably infinite: $\{p_0, p_1, p_2, \ldots\}$. Assume a fixed ordering of $\mathcal{P}$, in the order of their indices $0, 1, 2, \ldots$. We view a truth assignment $\sigma : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ as defining an infinite path in the full binary tree, call it $\mathcal{T}_{\text{full}}$, by using T as the label of every left edge and F as the label of every right edge. See left of Figure F.1 for a partial graphic representation of $\mathcal{T}_{\text{full}}$.

From $\mathcal{T}_{\text{full}}$ we define another binary tree $\mathcal{T}(\Gamma)$ by pruning some of the infinite paths as follows: Given infinite path $\sigma \overset{\text{def}}{=} t_0 t_1 t_2 \cdots t_n \cdots$ in $\mathcal{T}_{\text{full}}$ where $t_n \in \{\mathsf{T}, \mathsf{F}\}$ for every $n \geqslant 0$, let k be the smallest integer (if any) such the truth assignment corresponding to σ falsifies some wff in Γ; if such a k exists, delete from $\mathcal{T}_{\text{full}}$ all paths extending the finite path $t_0 t_1 \cdots t_k$. At the node where $\mathcal{T}_{\text{full}}$ is pruned, we replace p_{k+1} by a leaf node denoted "•". See right of Figure F.1 for an example of how $\mathcal{T}_{\text{full}}$ is pruned when p_0 occurs nowhere in Γ and Γ includes the wff $(\neg p_1 \wedge p_2)$. By this definition of $\mathcal{T}(\Gamma)$, note that $\mathcal{T}_{\text{full}}$ is none other than $\mathcal{T}(\varnothing)$, which is $\mathcal{T}_{\text{full}}$ without any pruning.

The resulting $\mathcal{T}(\Gamma)$ contains some full finite paths (possibly none) and some infinite paths (possibly none). Γ is satisfiable iff $\mathcal{T}(\Gamma)$ contains an infinite path, so that also Γ is not satisfiable iff $\mathcal{T}(\Gamma)$ does not contain an infinite path. By WKL, if $\mathcal{T}(\Gamma)$ does not contain an infinite path, then $\mathcal{T}(\Gamma)$ does not contain arbitrarily long full finite paths, *i.e.*, there is a finite bound $k \geqslant 1$ such that all full finite paths have length $\leqslant k$. But this implies there is a finite subset of Γ which is not satisfiable. $\square$

Exercise 155. This exercise is couched in a language which is a little more familiar to computer scientists. Given a set $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, the binary tree $\mathcal{T}(\Gamma)$ induced by Γ is defined in the proof above. In $\mathcal{T}(\Gamma)$, every finite path is a finite sequence, and every infinite path is an infinite sequence, where all the entries are in $\{\mathsf{T},\mathsf{F}\}$. The set of all such finite sequences is denoted $\{\mathsf{T},\mathsf{F}\}^*$, and the set of all such infinite sequences is denoted $\{\mathsf{T},\mathsf{F}\}^\omega$. The root node of $\mathcal{T}(\Gamma)$ is represented by the empty sequence ε, every leaf node is connected to the root node by a finite path, and every infinite path is one that can be traversed without ever reaching a leaf node.

An arbitrary binary tree $\mathcal{U}$ can therefore be represented by a subset of $\{\mathsf{T},\mathsf{F}\}^*\cup\{\mathsf{T},\mathsf{F}\}^\omega$, satisfying two conditions:

- $\mathcal{U}$ is prefix-closed, *i.e.*, for every (possibly infinite) path $\pi_1 \in \{\mathsf{T},\mathsf{F}\}^* \cup \{\mathsf{T},\mathsf{F}\}^\omega$ and every finite path $\pi_2 \in \{\mathsf{T},\mathsf{F}\}^*$, if $\pi_1 \in \mathcal{U}$ and π_2 is a prefix of π_1, then $\pi_2 \in \mathcal{U}$.
- For every finite path $\pi \in \{\mathsf{T},\mathsf{F}\}^*$, it holds that $\pi\mathsf{T} \in \mathcal{U}$ iff $\pi\mathsf{F} \in \mathcal{U}$, *i.e.*, every non-leaf node has two immediate successors.

Given an arbitrary $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, let $\mathcal{T}_\omega(\Gamma)$ be the subset of infinite paths in $\mathcal{T}(\Gamma)$:

$$\mathcal{T}_\omega(\Gamma) \overset{\text{def}}{=} \mathcal{T}(\Gamma) \cap \{\mathsf{T},\mathsf{F}\}^\omega.$$

By the preceding proof, $\mathcal{T}_\omega(\Gamma) \neq \varnothing$ iff the set Γ is satisfiable. Let Γ_0, Γ_1, and Γ_2, be defined as follows:

$$\Gamma_0 \overset{\text{def}}{=} \{\neg p_1 \wedge p_2\},$$
$$\Gamma_1 \overset{\text{def}}{=} \Big\{\, p_1 \to (\neg p_2 \wedge \cdots \wedge \neg p_k) \,\Big|\, k \geqslant 2 \Big\},$$
$$\Gamma_2 \overset{\text{def}}{=} \Big\{\, \neg p_1 \to (\neg p_2 \wedge \cdots \wedge \neg p_k) \,\Big|\, k \geqslant 2 \Big\}.$$

There are two parts in this exercise:

1. Define the sets of infinite paths $\mathcal{T}_\omega(\Gamma_0)$, $\mathcal{T}_\omega(\Gamma_1 \cup \Gamma_2)$, and $\mathcal{T}_\omega(\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$ as subsets of $\{\mathsf{T},\mathsf{F}\}^\omega$. In your answers, use the notational conventions of *regular languages* and *regular ω-languages* that we have used earlier in the presentation of this exercise.
2. For each of the three sets defined in part 1, explain how they indicate whether the corresponding set of wff's is satisfiable or not.

You will find it useful to consult Figure F.1. □

The next proof makes explicit reference to notions in topology. As indicated in the preceding alternative proof, a truth assignment σ can be denoted by a path in the full binary tree $\mathcal{T}_{\text{full}}$, now viewed as an ω-sequence in the product space $\{\mathsf{T},\mathsf{F}\}^\omega$. (We write ω for the first infinite ordinal, which is the set of natural numbers listed in their standard order.)

We view $\{\mathsf{T},\mathsf{F}\}^\omega$ as the underlying space of a product topology $\big(\{\mathsf{T},\mathsf{F}\}^\omega, \mathcal{O}\big)$, thus making every truth assignment a "point" in that topology. $\mathcal{O}$ is a family of *open sets* that define the topology, which are in this case subsets of points in $\{\mathsf{T},\mathsf{F}\}^\omega$ and satisfy the usual requirements of a topology:

- The empty set $\varnothing$ and the full space $\{\mathsf{T},\mathsf{F}\}^\omega$ are in $\mathcal{O}$,
- $\mathcal{O}$ is closed under *arbitrary* unions,
- $\mathcal{O}$ is closed under *finite* intersections.

We can define a subset $U \subseteq \{\mathsf{T},\mathsf{F}\}^\omega$ to be *open* iff there is a finite set of indices $I \subseteq \omega$ such that:

$$U \;=\; \prod \Big\{\, A_i \,\Big|\, i \in \omega \text{ and } A_i \subseteq \{\mathsf{T},\mathsf{F}\} \Big\} \quad \text{where } A_i = \{\mathsf{T},\mathsf{F}\} \text{ for every } i \in \omega - I.$$

In words, in the infinite product $U = A_0 \times A_1 \times \cdots \times A_i \times \cdots$, for all but finitely many indices i it is the case that $A_i = \{\mathsf{T},\mathsf{F}\}$. A set $U \subseteq \{\mathsf{T},\mathsf{F}\}^\omega$ is *closed* iff it is the complement of an open set. In the case of the product topology, every open subset $U \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ is also closed, and thus called *clopen*.

Let A be a subset of $\{\mathsf{T}, \mathsf{F}\}^\omega$. An *open covering of* A is a family of open sets $\{\, U_i \mid i \in I \,\} \subseteq \mathcal{O}$ such that $A \subseteq \bigcup\{\, U_i \mid i \in I \,\}$. And A is said *compact* if every open covering of A has a finite subcovering; *i.e.*, there exists a finite subfamily $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ of $\{\, U_i \mid i \in I\}$ such that $A \subseteq (U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_n})$. By Tychonoff's Theorem in topology, the product topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$ is *compact*, which means that every open covering of a subset of points $A \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ has a finite subcovering.

Alternative Proof II of Theorem 2 (Compactness for Propositional Logic). Again here, we only need to consider the non-trivial implication "$\Leftarrow$": If Γ is finitely satisfiable, then Γ is satisfiable.

For every propositional wff φ, let $A_\varphi \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ be the collection of all points/truth assignments that satisfy φ. The set A_φ is a closed (and open) subset of $\{\mathsf{T}, \mathsf{F}\}^\omega$ in the topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$, which follows from the fact that φ only mentions finitely many propositional variables. It is easy to check that, for every finite subset Δ of Γ, if Δ is satisfiable, then $\bigcap\{\, A_\varphi \mid \varphi \in \Delta \,\}$ is not empty. Hence, the family of closed sets $\{\, A_\varphi \mid \varphi \in \Gamma \,\}$ satisfies the *finite intersection property*. Moreover, the product topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$ is compact, as noted above. Hence, $\bigcap\{\, A_\varphi \mid \varphi \in \Gamma \,\}$ is not empty, which is the desired conclusion. $\square$

Appendix G

Games on Rectangular Grids

Many examples and exercises in these lecture notes have to do with some combinatorial problems on rectangular grids. One common example is the so-called *Queens Problem* which consists in placing n mutually non-attacking queens on a $n \times n$ chessboard. Many of these problems can be formally specified by wff's whose satisfiability (or unsatisfiability) corresponds to the existence (or non-existence) of a solution.

We adopt a uniform way of referring to the cells of a $m \times n$ rectangular grid where $m, n \geqslant 1$. The special case when $m = n = 8$ is the standard 8×8 square chessboard. We choose to number the rows from top to bottom with the integers $1, 2, \ldots, m$, and the columns from left to right with the integers $1, 2, \ldots, n$. With this numbering, the pair (i, j) denotes the cell located at the intersection of the i-th row and the j-th column. This is also the standard convention for identifying positions in a two-dimensional matrix. Note, however, that this is not how we usually view the coordinates of the Cartesian plane, where the first coordinate is along the horizontal axis (going rightward) and the second coordinate is along the vertical axis (going upward).

As with two-dimensional matrices, a *diagonal* is a (-45^o)-*diagonal* directed downward starting from the west edge or north edge of the grid, and an *antidiagonal* is *antidiagonal* a $(+45^o)$-*diagonal* directed upward starting from the west edge or south edge of the grid. Figure G.1 illustrates our conventions for the case of a 6×8 rectangular grid.

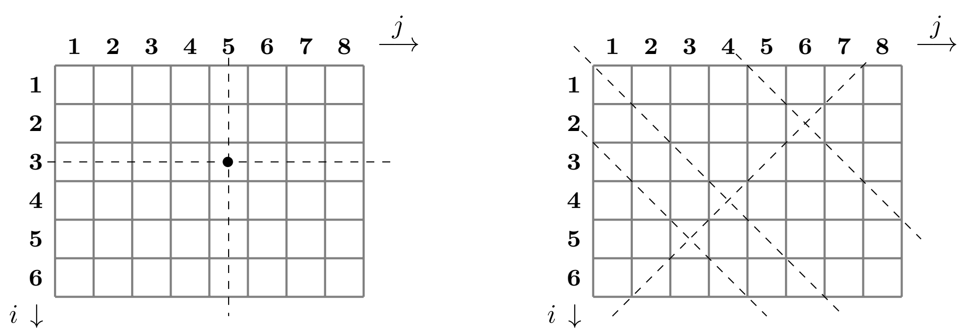

Figure G.1: Example of a 6×8 grid. In the left figure, cell $(i, j) = (3, 5)$ is located at the intersection of row $i = 3$ and column $j = 4$. The right figure shows three *diagonals* and one *antidiagonal*.

We extend our conventions for rectangular grids to semi-infinite grids. Specifically, some execises and examples consider grids with infinitely many rows in the downward direction and infinitely many columns in the rightward direction. In both directions, rows and columns are identified by the positive integers $1, 2, 3, \ldots$. The semi-infinite grid is shown in Figure G.2, where the cells are each assigned a natural number (in italics) according to a traversal that starts at the north-west corner and traverses successive antidiagonals downwards. Ignore for the moment the indexed frames $\Box_1, \Box_2, \Box_3, \ldots$, in the figure.

Below is the correspondence between the cells in the grid, each identified by an ordered pair, and the first ten natural numbers indicating the order in which the cells are visited:

$$\begin{array}{ccccccccccc} (1,1) & (1,2) & (2,1) & (1,3) & (2,2) & (3,1) & (1,4) & (2,3) & (3,2) & (4,1) & \cdots \\ \mathit{0} & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \cdots \end{array}$$

Observe that, in our semi-infinite grid, all diagonals are infinite whereas all antidiagonals are finite. Moreover, for every $k \geqslant 1$, there is exactly one antidiagonal of length k. Here are the first four antidiagonals:

$$\underbrace{\mathit{0}} \qquad \underbrace{\mathit{1} \quad \mathit{2}} \qquad \underbrace{\mathit{3} \quad \mathit{4} \quad \mathit{5}} \qquad \underbrace{\mathit{6} \quad \mathit{7} \quad \mathit{8} \quad \mathit{9}}$$

Because all the antidiagonals are traversed in sequence, from shorter to longer, all the cells in the entire grid are visited by our traversal.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	$\cdots$
1	$\boxed{\mathit{0}}_1$	*1*	*3*	*6*	*10*	*15*	*21*	*28*	*36*	*45*	$\cdots$	
2	*2*	*4*	*7*	$\boxed{\mathit{11}}_3$	*16*	*22*	*29*	*37*	*46*	$\cdots$		
3	*5*	$\boxed{\mathit{8}}_2$	*12*	*17*	*23*	*30*	*38*	*47*		$\cdots$		
4	*9*	*13*	*18*	*24*	*31*	*39*	$\boxed{\mathit{48}}_5$	$\cdots$				
5	*14*	*19*	$\boxed{\mathit{25}}_4$	*32*	*40*	*49*	$\cdots$					
6	*20*	*26*	*33*	*41*	*50*	$\cdots$						
7	*27*	*34*	*42*	*51*	$\cdots$							
8	*35*	*43*	*52*	$\cdots$								
9	*44*	*53*	$\cdots$									
10	*54*	$\cdots$										
11	$\cdots$											
$\vdots$												

Figure G.2: The semi-infinite rectangular grid covering the entire south-east quadrant of the Cartesian plane. The cells are each assigned a natural number (in italics) according to a traversal that starts at the north-west corner and traverses successive antidiagonals downwards. The indexed frames $\Box_1, \Box_2, \Box_3, \ldots$, are the positions of $Q_1, Q_2, Q_3, \ldots$ in Exercise 159 .

Exercise 156. Show that the *pairing function* $\mathsf{pair} : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$,

$$\mathsf{pair}(i,j) \stackrel{\text{def}}{=} (1/2)(i^2 + j^2 + 2 \cdot i \cdot j - i - 3 \cdot j),$$

correctly defines the numbering of the semi-infinite grid shown in Figure G.2. As usual, $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^+$ is the set of positive integers. Specifically, by induction on $k \geqslant 1$, show that the k entries of the antidiagonal:

$$(1,k) \qquad (2,k-1) \qquad (3,k-2) \qquad \cdots \qquad (k,1)$$

are assigned the natural numbers:

$$\frac{k^2-k}{2} \qquad \frac{k^2-k}{2}+1 \qquad \frac{k^2-k}{2}+2 \qquad \cdots \qquad \frac{k^2-k}{2}+(k-1)$$

in the same order, respectively. □

G.1 The Queens Problem and Variations

Whether on a square $n \times n$ grid (*i.e.*, chessboard) or on a semi-infinite grid, a solution of the *Queens Problem* must satisfy four conditions:

(a) *There is exactly one queen in every row*, *i.e.*, for every row $i \geqslant 1$ there exists only one column $j \geqslant 1$ such that cell (i, j) contains a queen.

(b) *There is exactly one queen in every column*, *i.e.*, for every column $j \geqslant 1$ there exists only one row $i \geqslant 1$ such that cell (i, j) contains a queen.

(c) *There is at most one queen in every diagonal*, *i.e.*, for every cell (i, j) and every cell (i', j'), if $(i, j) \neq (i', j')$ and $i - j = i' - j'$ then not both (i, j) and (i', j') contain a queen.

(d) *There is at most one queen in every antidiagonal*, *i.e.*, for every cell (i, j) and every cell (i', j'), if $(i, j) \neq (i', j')$ and $i + j = i' + j'$ then not both (i, j) and (i', j') contain a queen.

It is a standard exercise in an introductory course on the design of algorithms to show that, for every $n \geqslant 4$, the n-*Queens Problem* has a solution, in fact, many solutions. There is a degenerate case $n = 1$ which trivially has a solution: a single queen on a 1×1 chessboard.

For the infinite *Queens Problem* played on the semi-infinite grid, there are solutions, but more delicate to define. Among the latter, we can add one of two additional conditions, (e) or (f):

(e) *For almost every $n \geqslant 1$, the initial $n \times n$ north-west sub-grid does* not *contain a solution of the n-*Queens Problem.

(f) *For infinitely many $n \geqslant 1$, the initial $n \times n$ north-west sub-grid contains a solution of the* n-Queens Problem.

In (e), "for almost every" means for "for all but finitely many". On close examination, (e) and (f) are related; each is the negation of the other.

Condition (e) can be stated equivalently as two conditions:

(e.1) For almost every row $i \geqslant 1$, there is a column $j \leqslant i$, such that none of the cells in $\{(1, j), (2, j), \ldots, (i, j)\}$ contains a queen.

(e.2) For almost every column $j \geqslant 1$, there is a row $i \leqslant j$, such that none of the cells in $\{(i, 1), (i, 2), \ldots, (i, j)\}$ contains a queen.

In words, (e.1) says the initial $i \times i$ sub-grid does not contain a solution of the i-*Queens Problem* and (e.2) says the initial $j \times j$ sub-grid does not contain a solution of the j-*Queens Problem*.

Exercise 157. Assume that all the conditions in $\{(a), (b), (c), (d)\}$ are satisfied in a solution of the infinite *Queens Problem*. Show that condition (e.1) holds iff condition (e.2) holds. In words, not both conditions are necessary.

Hint: Use the Pigeonhole Principle. □

In the rest of this section we show how to build two qualitatively different solutions of the infinite Queens Problem. Our first solution satisfies the five conditions $\{(a), (b), (c), (d), (e)\}$. Our second solution, which is far more delicate to define, satisfies the five conditions $\{(a), (b), (c), (d), (f)\}$.

Terminology: To facilitate our presentation below, what we call a (*rectangular*) *board* is a (rectangular) grid on which queens or other pieces are placed. In particular, a grid is the same as an *empty* board.

A first solution of the infinite *Queens Problem*

Our first solution of the infinite Queens Problem satisfies $\{(a), (b), (c), (d), (e)\}$, *i.e.*, almost every initial sub-board of this infinite solution is not a solution of a finite Queens Problem. In fact, our construction shows a stronger result: Except for the degenerate 1×1 initial sub-board, every $n \times n$ initial sub-board is not a solution of the n-Queens Problem.

We have an infinite supply of queens $\{Q_1, Q_2, Q_3, \ldots\}$ which we have to place on the infinite board so that none is under attack from the other queens. Our construction proceeds in stages, with *Stage* k responsible for placing queen Q_k on the board, for every $k \geqslant 1$. Initially, the infinite sequence of unoccupied rows is $\mathsf{Rows} = (1, 2, 3, \ldots)$ and the infinite sequence of unoccupied columns is $\mathsf{Cols} = (1, 2, 3, \ldots)$, *i.e.*, there are no queens on the board. We write $\mathsf{head}(\mathsf{Rows})$ and $\mathsf{head}(\mathsf{Cols})$ to denote the first entries in those sequences:

Stage 1. Place Q_1 on cell $(1, 1)$ and update:
$\mathsf{Rows} := \mathsf{Rows} - \{1\}$ and $\mathsf{Cols} := \mathsf{Cols} - \{1\}$.

Stage k, even. Let $j := \mathsf{head}(\mathsf{Cols})$ and let i be the smallest entry in Rows such that:

- $i > k$, and
- cell (i, j) is not reachable by any of the queens in $\{Q_1, Q_2, \ldots, Q_{k-1}\}$.

Place Q_k on cell (i, j) and update
$\mathsf{Rows} := \mathsf{Rows} - \{i\}$ and $\mathsf{Cols} := \mathsf{Cols} - \{j\}$.

Stage k, odd. Let $i := \mathsf{head}(\mathsf{Rows})$ and let j be the smallest entry in Cols such that:

- $j > k$, and
- cell (i, j) is not reachable by any of the queens in $\{Q_1, Q_2, \ldots, Q_{k-1}\}$.

Place Q_k on cell (i, j) and update:
$\mathsf{Rows} := \mathsf{Rows} - \{i\}$ and $\mathsf{Cols} := \mathsf{Cols} - \{j\}$.

The purpose of adding condition $i > k$ in "*Stage k, even*" (resp., $j > k$ in "*Stage k, odd*") is to guarantee that queen Q_k is placed below (resp., right of) the initial $k \times k$ sub-board, which cannot thus be a solution of the k-Queens Problem.

Exercise 158. Give a precise argument for the correctness of the preceding procedure. You may want to consult (or do) Exercise 157 before attempting this one. □

Exercise 159. Use the traversal depicted in Figure G.2 to produce the same infinite board as that defined by the preceding procedure. The indexed frames $\square_1$, $\square_2$, $\square_3$, ... in the figure, correspond to the positions of the queens $Q_1, Q_2, Q_3, \ldots$. □

A second solution of the infinite *Queens Problem*

We first define what we call *regular boards*, which are solutions of the finite n-*Queens Problem* when n is an odd integer which is both greater than 3 and not a multiple of 3. What defines a regular $n \times n$ board are the two following conditions:

- n queens are placed on n adjacent diagonals including the main diagonal, and
- $(n - 1)$ empty diagonals are equally divided between the south-west corner and the north-east corner of the grid.

Thus, out of a total of $2n - 1$ diagonals, the n diagonals on which a queen is placed are adjacent to each other and centrally located. The first three regular boards of respective dimensions 5×5, 7×7, and 11×11 are shown in Figure G.3.

Exercise 160. A nice little combinatorial exercise: Show there is no solution of the 9-*Queens Problem* satisfying the two conditions of regular boards stated above. Extend this fact to the n-*Queens Problem* when n is an odd integer which is a multiple of 3.

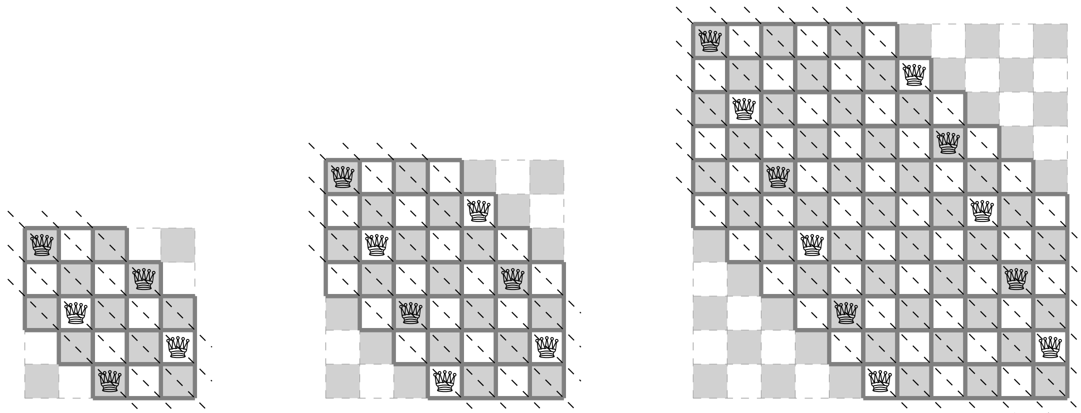

Figure G.3: Solutions for the n-Queens Problem when $n \in \{5, 7, 11\}$. For every odd n which is not a multiple of 3, we use the same pattern to generate a solution, here called a *regular board*: n queens are placed on n adjacent diagonals including the main diagonal, and $(n-1)$ empty diagonals are equally divided between the SW and NE parts of the grid. A *regular board*, as here defined, does not exist if n is even or odd multiple of 3.

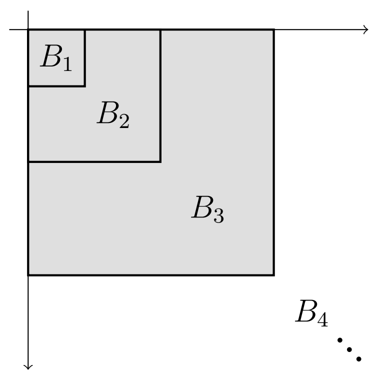

Figure G.4: Shape of the infinite board B_∞ (not to scale) over the entire south-east quadrant.

Hint: For the case $n = 9$, place 5 queens on and below the main diagonal as in Figure G.3, then use an exhaustive search to discover that it is not possible to place the remaining 4 queens above the main diagonal to produce a solution of the 9-*Queens Problem*. □

In what follows we build our second solution of the infinite *Queens Problem* by starting from the regular board of dimension 5×5, call it B_1, although the same construction works if we start from any regular board of dimension $\neq 5 \times 5$. Before we give the details of the construction, what we build schematically is an infinite sequence of nested finite boards:

$$B_1 \;\subseteq\; B_2 \;\subseteq\; B_3 \;\subseteq\; \cdots \;\subseteq B_k \;\subseteq\; \cdots$$

which is *nested* in the sense that, for every $k \geqslant 1$, board B_k is the north-west sub-board of B_{k+1}. We denote by B_∞ the limit of this infinite sequence, which is here just the union of its entries:

$$B_\infty \stackrel{\text{def}}{=} \bigcup_{k \geqslant 1} B_k.$$

B_∞ is an infinite board, covering the south-east quadrant of the Cartesian plane; its shape is depicted in Figure G.4. Below we show that every finite board B_k satisfies $\{(a), (b), (c), (d)\}$ stated in the opening paragraph of Section G.1 and is therefore a solution of the Queens Problem.

A simple way of describing the definition of board B_{k+1} from board B_k is to use the *tensor product* operation which is found in introductory books on linear algebra. Consider 0-1 square matrices where entries "1" correspond to positions of queens on the board. Using standard notation of linear algebra, given $m \times m$ matrix $A = (a_{i,j})$ and $n \times n$ matrix B, the *tensor product* $A \otimes B$ is a matrix of dimension $m \cdot n \times m \cdot n$:

$$A \otimes B \stackrel{\text{def}}{=} \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,m}B \end{pmatrix}$$

The desired sequence of boards is defined inductively:

$$B_1 \stackrel{\text{def}}{=} \text{the } 5 \times 5 \text{ regular board defined in Figure G.3,}$$

$$B_{k+1} \stackrel{\text{def}}{=} B_1 \otimes B_k \quad \text{for every } k \geqslant 1.$$

The $5^2 \times 5^2$ board B_2 is shown in Figure G.5. The pattern described for B_2 is repeated for the $5^k \times 5^k$ board B_k for every $k \geqslant 3$. By the definition of tensor product, the set of boards $\{B_1, B_2, B_3, \ldots\}$ is indeed a nested sequence, in the sense that B_k is the $5^k \times 5^k$ north-west sub-board of the $5^{k+1} \times 5^{k+1}$ board B_{k+1} for every $k \geqslant 3$. It remains to show that every B_k is a solution of the 5^k-*Queens Problem* for every $k \geqslant 1$ (left to you in the next exercise).

Exercise 161. By induction on $k \geqslant 1$, prove that the $5^k \times 5^k$ board B_k is a solution of the 5^k-*Queens Problem*.

Hint: As often in a non-trivial induction, you need to strengthen the *induction hypothesis* (IH). For the induction step in proving that "B_{k+1} is a solution of the 5^{k+1}-*Queens Problem*" from "B_k is a solution of the 5^k-*Queens Problem*", consider adding to IH the following hypotheses:

1. The 5^k occupied diagonals of B_k are all adjacent and include: the main diagonal, plus $(5^k - 1)/2$ diagonals below/west of the main, plus $(5^k - 1)/2$ diagonals above/east of the main.
2. The $(5^k - 1)$ empty diagonals of B_k are equally divided between the south-west and north-west of B_k.
3. The south-east $4 \cdot 5^{k-1} \times 4 \cdot 5^{k-1}$ sub-board of B_k is a solution of the $(4 \cdot 5^{k-1})$-Queens Problem. □

What we call a $(2,1)$-*diagonal* in the rectangular grid is a line joining two points whose coordinates are (i,j) and $(i+2, j+1)$, in contrast to a *diagonal* which joins two points (i,j) and $(i+1, j+1)$. In degrees, a $(2,1)$-*diagonal* is a (-63.44^o)-diagonal, whereas a *diagonal* is a (-45^o)-diagonal.

Similarly, what we here call a $(-1,2)$-*antidiagonal* is a line joining two points whose coordinates are (i,j) and $(i-1, j+2)$, in contrast to a *antidiagonal* which joins two points (i,j) and $(i-1, j+1)$. A $(-1,2)$-*antidiagonal* is a $(+26.56^o)$-diagonal, whereas a *antidiagonal* is a $(+45^o)$-diagonal. Figure G.6 shows examples of $(2,1)$-*diagonals* and $(-1,2)$-*antidiagonals*.

Exercise 162. We have presented two solutions of the infinite *Queens Problem*. The first solution satisfies condition (e), the second solution satisfies condition (f), where (e) and (f) are each the negation of the other and defined at the beginning of Section G.1. There are other aspects that differentiate the two infinite solutions.

1. Show that, in our second solution B_∞ satisfying (f), every diagonal is occupied by one queen. (This property is not shared with our first solution satisfying (e), where some of the diagonals are not occupied by any queen, but it is more difficult to prove.)

   Hint: For some intuition, consider Figure G.5 which shows B_1 as an initial north-west sub-board of B_2, and B_2 as an initial north-west sub-board of the infinite board B_∞.

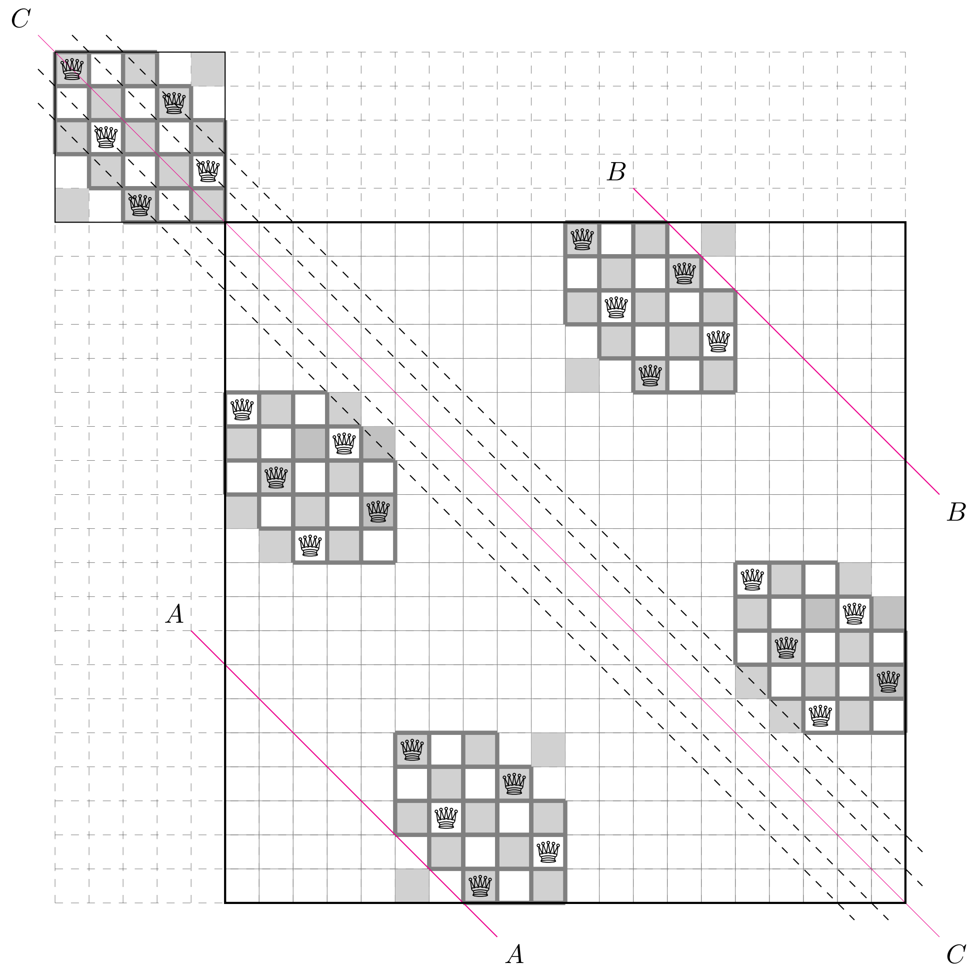

Figure G.5: The initial 5×5 board B_1 is the north-west sub-board of the $5^2 \times 5^2$ board B_2, which contains $25 = 5^2$ *occupied diagonals*, all adjacent (diagonal C, 12 below/west of C, 12 above/east of C), and $24 = 5^2 - 1$ *empty diagonals* (diagonal A plus 11 below/west of A, diagonal B plus 11 above/east of B).

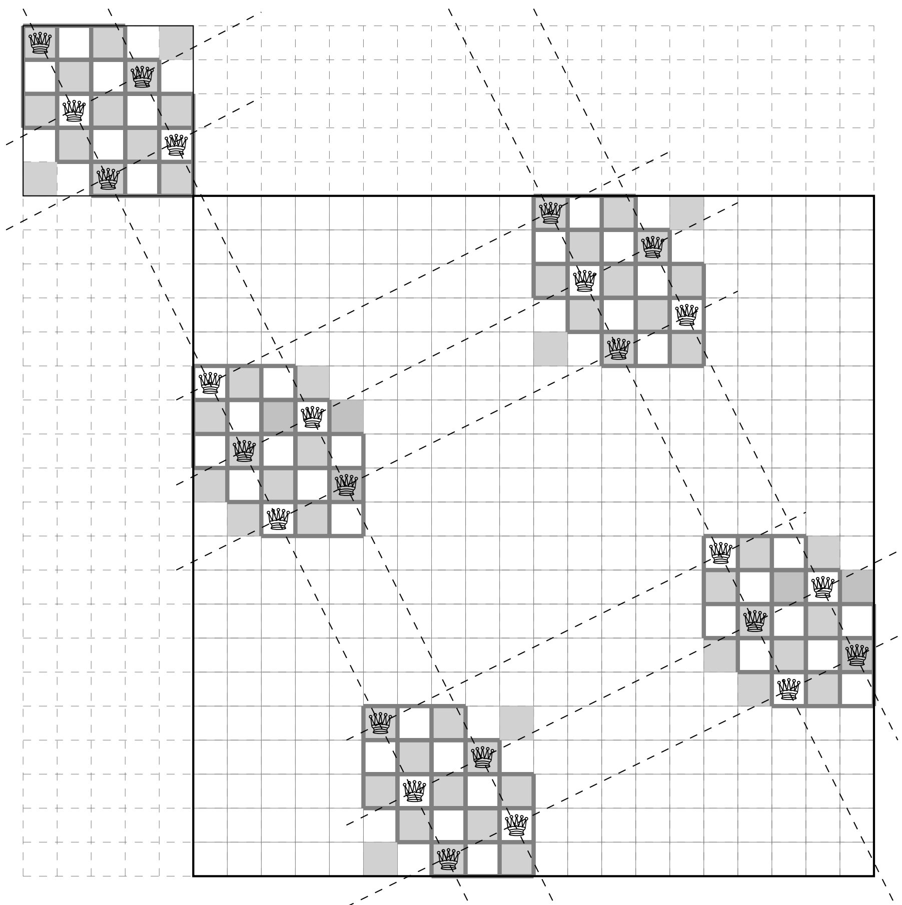

Figure G.6: Once more, the initial 5×5 board B_1 is the north-west sub-board of the $5^2 \times 5^2$ board B_2, now showing all *occupied* $(2,1)$-*diagonals* and *occupied* $(-1,2)$-*antidiagonals*.

2. The infinite board B_∞ is the limit of a nested sequence $\mathcal{B}$ of finite boards, namely:

$$\mathcal{B}:\ B_1\ \subseteq\ B_2\ \subseteq\ B_3\ \subseteq\ \cdots .$$

Let B_1^{T} be the transpose of the initial board B_1. Define a new nested sequence of finite boards $\mathcal{B}' : B_1' \subseteq B_2' \subseteq B_3' \subseteq \cdots$ where:

$$B_1' \stackrel{\text{def}}{=} B_1 \text{ or } B_1^{\mathrm{T}},$$

$$B_{k+1}' \stackrel{\text{def}}{=} (B_1 \otimes B_k') \text{ or } (B_1^{\mathrm{T}} \otimes B_k') \quad \text{for every } k \geqslant 1.$$

$\mathcal{B}'$ is not uniquely defined, since at every step we may non-deterministically use B_1 or B_1^{T}. Show that the limit B_∞' of $\mathcal{B}'$ is another solution of the infinite *Queens Problem* satisfying condition (f). You can think of $\mathcal{B}'$ as a family of infinitely many solutions satisfying (f), with B_∞ being a particular member of $\mathcal{B}'$ which is obtained when at every step we choose B_1 instead of B_1^{T}.

3. Show that, for every $k \geqslant 1$, all the queens in the initial sub-board B_k of the infinite board B_∞ are located on 2^k $(2,1)$-diagonals. More specifically, the leftmost of those 2^k occupied $(2,1)$-diagonals contain 3^k queens (the most loaded), which is also the main $(2,1)$-diagonal starting at the north-west corner of the grid, and the rightmost occupied $(2,1)$-diagonal contains 2^k queens (the least loaded).

   Hint: For some intuition, consider Figure G.6.

4. Show that for every queen in some position (i,j) of the infinite board B_∞ which is to the right/east of the main $(2,1)$-diagonal, there is a queen in position $(1+2\ell, 1+\ell)$ on the main $(2,1)$-diagonal such that $(1+2\ell, 1+\ell)$ and (i,j) are on the same $(-1,2)$-antidiagonal.

   Hint: Again, for some intuition, consider Figure G.6.

5. Show that the properties in parts 3 and 4 above, which are true of the infinite board B_∞, are not true of any of the other infinite boards $B_\infty' \neq B_\infty$ as defined in part 2.

A little further examination (omitted here) shows that the property in part 4 above uniquely specifies B_∞. The property in part 3 is not needed for this unique specification of B_∞. □

G.2 The *Queens&Rooks Problem* and Variations

In the *Queens&Rooks Problem* we have to place more than one *queen* and more than one *rook* on a chessboard so that no piece can attack another. We play it on a $n \times n$ chessboard with $p = \lceil n/3 \rceil$ queens and $(n-p)$ rooks, with $n \geqslant 3$. (There are no solutions for $n \leqslant 2$.)

This problem is inspired by a two-person game. Each player takes a turn in placing a piece from an initial supply of $p \geqslant 1$ queens and $q \geqslant 1$ rooks such that $p+q=n$. The game invariant is that it continues as long as the next-placed piece cannot attack, or be attacked by, an already-placed piece on the board. At every turn, each player tries to place a piece that will thwart the other player's ability to preserve this game invariant. The winner is the player who places the last piece that does not violate the invariant, and this last piece may or may not be the last from the initial supply of $p+q$ pieces. The game is typically played with $p = \lceil n/3 \rceil$ and $q = n-p$. For the most common and simplest version, the chessboard has dimension 6×6, with $p=2$ and $q=4$, with 1 queen and 2 rooks assigned to each player. *Fun suggestion*: Define a winning strategy for the starting player on the 6×6 board.

An example of a solution for this problem when $n = 15$, with $p = 5$ and $q = 10$, is shown in Figure G.7. A similarly configured 12×12 board is also possible, where the queens are placed in the north-west 4×4 sub-board and the rooks are placed in the south-east 8×8 sub-board. However, a similarly configured 9×9 board is not possible, because we cannot place 3 non-attacking queens on a 3×3 sub-board.

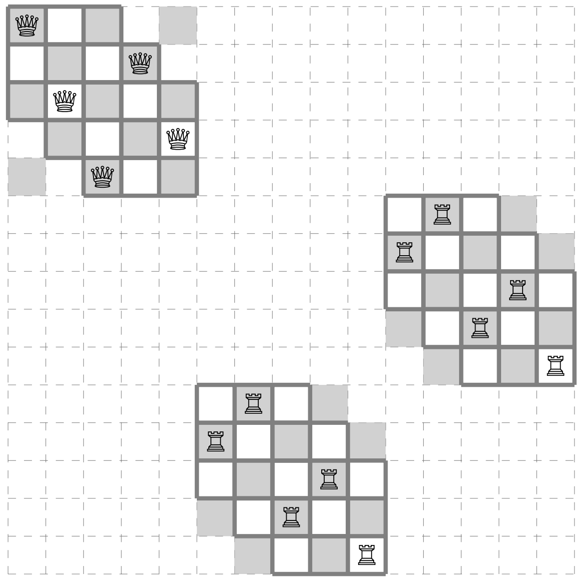

Figure G.7: A 15×15 board with 5 queens and 10 rooks, all non-attacking, which is therefore a solution of the *Queens&Rooks Problem* for $n = 15$.

The infinite *Queens&Rooks Problem*

Just as it is possible to extend the finite *Queens Problem* to an infinite *Queens Problem*, it is also possible to extend the finite *Queens&Rooks Problem* to an infinite *Queens&Rooks Problem*, both on the semi-infinite grid covering the entire south-east quadrant of the Cartesian plane. In some respects, the infinite version of *Queens&Rooks Problem* is sometimes harder and sometimes easier than the infinite version of *Queens Problem*.

G.3 The *Knight's Tour Problem* and Variations

Given a rectangular $m \times n$ grid G, a *Knight's Tour* is a sequence of knight's moves that visit every cell of G exactly once. If the first square and the last square in this sequence are one knight's move apart, the sequence is said to be a *Closed Knight's Tour*. Figure G.8 shows a *Knight's Tour* on a 3×4 grid (middle figure) and a *Closed Knight's Tour* on a 3×10 grid (right figure).

Not every rectangular grid has a *Knight's Tour* or a *Closed Knight's Tour*. To be more specific, it is known that (Schwenk's Theorem [19]): The $m \times n$ grid, where $m \leqslant n$, has a *Closed Knight's Tour* if and only if none of the following three conditions hold:

- m and n are both odd,
- $m \in \{1, 2, 4\}$,
- $m = 3$ and $n \in \{4, 6, 8\}$.

In particular, there is no *Closed Knight's Tour* on the 3×4 grid, shown in the middle of Figure G.8. If we lift the restriction that the tour must be closed, then there is a *Knight's Tour* (not necessarily closed) on every $m \times n$ grid where $m, n \geqslant 5$.

Exercise 163. By Schwenk's Theorem, for $m = 3$ and every even integer $n \geqslant 10$, there is a *Closed Knight's Tour* on the $3 \times n$ grid. Without invoking Schwenk's Theorem, show the following generalization: For infinitely many $n \geqslant 10$, there is a *Knight's Tour* (not necessarily closed) on the $3 \times n$ grid.

Hint: Consider stringing together several copies of the *Knight's Tour* on the 3×4 grid shown in the middle of Figure G.8 along a horizontal strip of height 3 and length a multiple of 4.

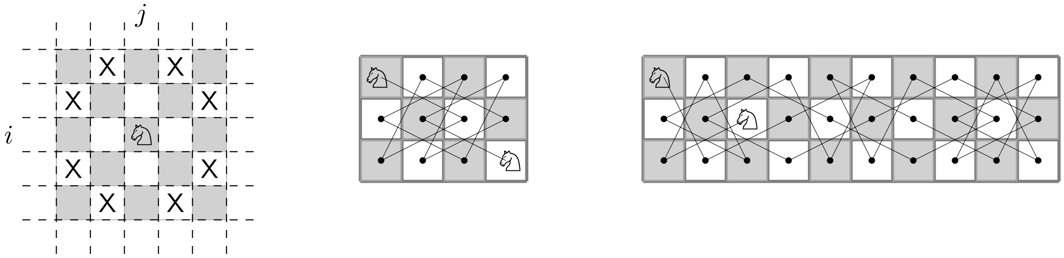

Figure G.8: To the left, a knight in position (i, j) on the grid with all the positions it can reach in one move, each marked with ‘X’. In the middle, a *Knight's Tour* on the 3×4 grid; it cannot be closed because the two end points of the tour, each marked with ‘♘’, are not one knight's move apart. To the right, a *Knight's Tour* on the 3×10 grid; it can be closed because the two end points of the tour, each marked with ‘♘’, are one knight's move apart.

G.4 The ***No-Three-In-Line Problem*** and Variations