CS 511, Fall 2024, Lecture Slides 24

Deductive Closures and First-Order Theories

Assaf Kfoury

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deductive closures and first-order theories

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- Let Γ be a set of first-order sentences over signature Σ.
  The **deductive closure** of Γ is:

$$\overline{\Gamma} \stackrel{\text{def}}{=} \{\, \varphi \mid \varphi \text{ first-order sentence s.t. } \Gamma \vdash \varphi \,\}$$

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- A **first-order theory** $\mathcal{T}$ over signature Σ consists of:
  - a set $\mathcal{A}$ of **axioms**, which are first-order sentences over Σ,
  - together with all first-order sentences deducible from $\mathcal{A}$.

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  Equivalently, a **first-order theory**
  is the **deductive closure** of a set of first-order sentences.

the first-order theory of a relational structure

- If $\mathcal{M}$ is a relational structure, the **first-order theory of $\mathcal{M}$** is:

$$\mathsf{Th}(\mathcal{M}) \stackrel{\text{def}}{=} \{\, \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{M} \models \varphi \,\}$$

Question: Is $\mathsf{Th}(\mathcal{M})$ deductively closed?

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Question: Is $\mathsf{Th}(\mathcal{M})$ deductively closed?

- Yes! Can you explain why?

the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

Consider again the structure $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$ in Lecture Slides 21.
The first-order theory of $\mathcal{N}$ is:

$$\mathsf{Th}(\mathcal{N}) \stackrel{\text{def}}{=} \{\, \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{N} \models \varphi \,\}$$

Some sentences that are true in $\mathcal{N}$:

S1 $\quad \forall x\, \neg(Sx \approx 0)$

S2 $\quad \forall x\, \forall y\, (Sx \approx Sy \to x \approx y)$

S3 $\quad \forall y\, (\neg(y \approx 0) \to \exists x\, (y \approx Sx))$

S4.1 $\quad \forall x\, \neg(Sx \approx x)$

S4.2 $\quad \forall x\, \neg(SSx \approx x)$

$\quad \ldots$

S4.n $\quad \forall x\, \neg(\underbrace{S \cdots S}_{n} x \approx x)$

$\quad \ldots$

the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

- let $\Gamma = \{$S1, S2, S3, S4.1, S4.2, S4.3, $\ldots\}$
- clearly $\mathcal{N} \models \varphi$ for every $\varphi \in \Gamma$

  so that $\Gamma \subseteq \mathsf{Th}(\mathcal{N})$

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- what can we say about the deductive closure of the set Γ above:
  $\overline{\Gamma} = \{\, \varphi \mid \varphi$ first-order sentence s.t. $\Gamma \vdash \varphi \,\}$?

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- in fact, the equality holds:

$$\overline{\Gamma} \;=\; \mathsf{Th}(\mathcal{N}) \qquad \textit{(not shown here)}$$

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(not shown here)

- we therefore say that Γ is an **axiomatization** of $\text{Th}(\mathcal{N})$ because

  every sentence φ made true by $\mathcal{N}$ is formally deduced from Γ

first-order theories of several structures over domain $\mathbb{N}$

From Lecture Slides 21:

$$\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S), \quad \mathcal{N}_1 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <), \quad \mathcal{N}_2 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +)$$

$$\mathcal{N}_3 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot)$$

$$\mathcal{N}_4 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot, \text{pr}) \qquad \text{where } \text{pr}(x) \stackrel{\text{def}}{=} \text{true iff } x \text{ is prime}$$

$$\mathcal{N}_5 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot, \text{pr}, \uparrow) \qquad \text{where } x \uparrow y \stackrel{\text{def}}{=} x^y$$

1. **FACT**
   The first-order theory of each of $\mathcal{N}$, $\mathcal{N}_1$, and $\mathcal{N}_2$, is **axiomatizable** and **decidable**.

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   The first-order theory of each of $\mathcal{N}$, $\mathcal{N}_1$, and $\mathcal{N}_2$, is **axiomatizable** and **decidable**.

2. **FACT**
   The first-order theory of each of $\mathcal{N}_3$, $\mathcal{N}_4$, and $\mathcal{N}_5$, is **axiomatizable** but **not** decidable.

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