# CS 511, Fall 2024, Lecture Slides 28

# Sequential Compactness

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16 October 2024

# A Full Binary Tree $\mathcal{T}_{\text{full}}$

Given arbitrary $a, b \in \mathbb{Q}$ with $a \leqslant b$, we use two kinds of intervals in this presentation:

closed $\quad [a, b] \overset{\text{def}}{=} \{ r \mid a \leqslant r \leqslant b \}$

left-closed $\quad [a, b) \overset{\text{def}}{=} \{ r \mid a \leqslant r < b \}$

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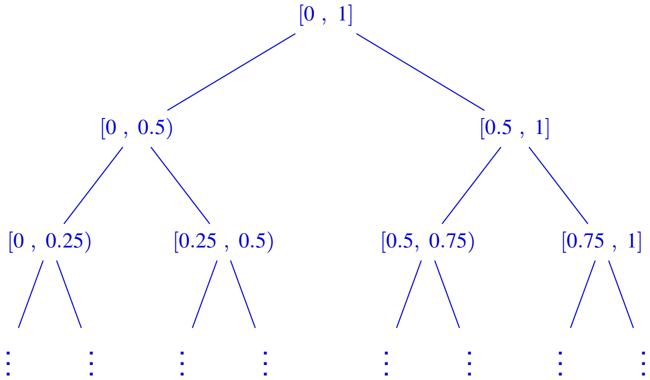
Starting with the interval $[0, 1]$ at the top, as the root node, we repeatedly divide intervals into two disjoint halves, producing an infinite full binary tree $\mathcal{T}_{\text{full}}$:



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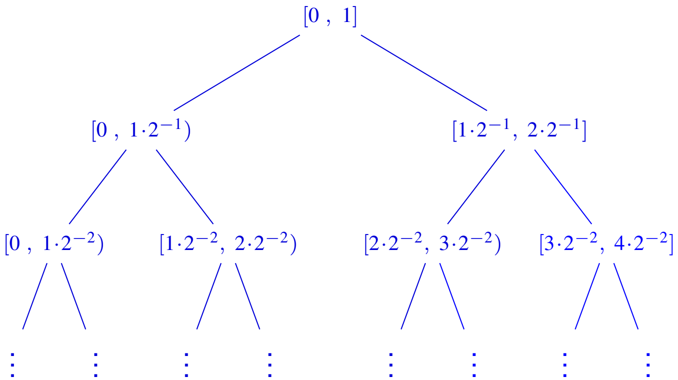
More systematically, in order to exhibit a common pattern for all the intervals in $\mathcal{T}_{\text{full}}$:



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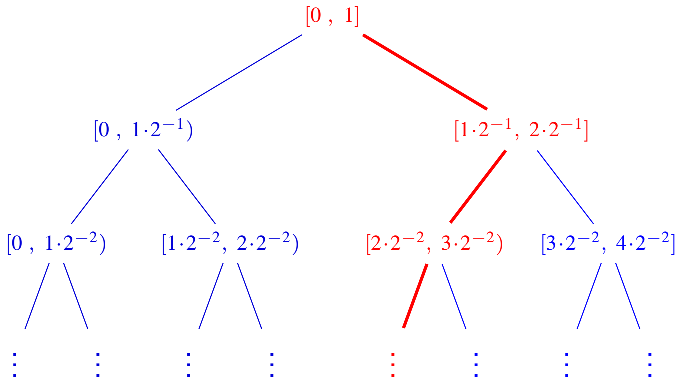



**Important observation:**

A path from the root of $\mathcal{T}_{\mathsf{full}}$ produces a strict nested sequence of intervals, *e.g.*:

$$[0,1] \supsetneq [1\cdot 2^{-1}, 2\cdot 2^{-1}] \supsetneq [2\cdot 2^{-2}, 3\cdot 2^{-2}) \supsetneq \cdots$$

This is a sequence of increasingly narrower intervals converging to a single number.

# A Full Binary Tree $\mathcal{T}_{\text{full}}$

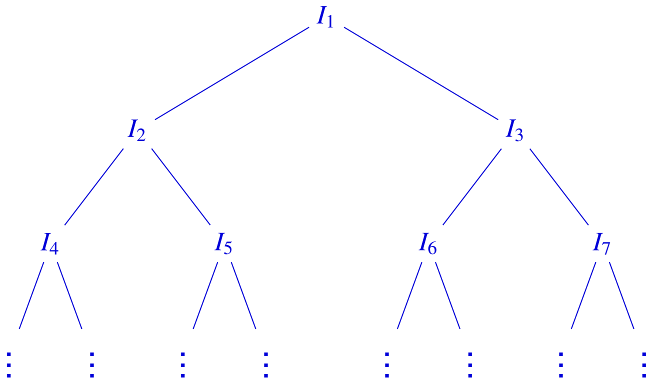
The nodes at the $k$-th level of $\mathcal{T}_{\text{full}}$ are, from left to right:

$$\underbrace{[0\cdot 2^{-k}, 1\cdot 2^{-k}) \quad [1\cdot 2^{-k}, 2\cdot 2^{-k}) \quad [2\cdot 2^{-k}, 3\cdot 2^{-k}) \quad \cdots \quad [(2^k-2)\cdot 2^{-k}, (2^k-1)\cdot 2^{-k})}_{\text{left-closed}} \quad \underbrace{[(2^k-1)\cdot 2^{-k}, 2^k\cdot 2^{-k}]}_{\text{closed}}$$

More succintly, at level $k \geqslant 1$, there are:

- $(2^k - 1)$ left-closed intervals, each of the form $[(r-1)\cdot 2^{-k},\ r\cdot 2^{-k})$ where $0 \leqslant r \leqslant 2^k - 1$,
- and one closed interval, the rightmost, $[(2^k-1)\cdot 2^{-k},\ 2^k\cdot 2^{-k}]$.

# A Full Binary Tree $\mathcal{T}_{\mathsf{full}}$ – once more



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3. Let $I_1 = I_{k_1} \supsetneq I_{k_2} \supsetneq I_{k_3} \supsetneq \cdots$ be the nested chain of intervals in the path followed using $A$, which is an infinite path by WKL (even if $A$ is finite) converging to a single number $b$ .

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4. The nested sequence $(A \cap I_{k_1}) \supsetneq (A \cap I_{k_2}) \supsetneq (A \cap I_{k_3}) \supsetneq \cdots$ consists of increasingly narrower non-empty subsets of $A$ converging to the same single point $b$.

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