Solutions to CS511 Homework 07

Nicholas Ikechukwu - U71641768

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Exercise 1. Lecture Slides 22, FO Definability, Appendix. Ex. 3 on pg2.

Hint: Carefully use the hint, as well as the special case given in lecture on Thursday, October 10. Use ≈, instead of =, for the formal symbol whose interpretation is equality. In LaTeX, you can typeset ≈ with approx.

Every finite subset X ⊆ N is first-order definable in the structure $(\mathbb{N}; <)$.

Solutions to Ex.3 : First-Order Definability

Ex.3. Every finite subset $X \subseteq \mathbb{N}$ is first-order definable in the structure $\mathbb{N}; <$. We can use the first-order definable constant 0 and the successor function.

Let $\phi_{\{0\}}(x)$ define the constant 0 as:

$$\phi_{\{0\}}(x) \stackrel{\text{def}}{=} \forall y(x \approx y \vee x < y)$$

Let $\phi_{\text{succ}}(x, y)$ define the successor function:

$$\phi_{\text{succ}}(x, y) \stackrel{\text{def}}{=} (x < y) \wedge \forall z(x < z \rightarrow (y \approx z \vee y < z)) \wedge \forall z(z < y \rightarrow (z \approx x \vee z < x))$$

A finite subset $X = \{a_1, a_2, \ldots, a_n\}$ can be defined as follows:

$$\phi_X(x) \stackrel{\text{def}}{=} \bigvee_{i=1}^{n} \phi_{\text{succ}}^{i-1}(0, x)$$

where $\phi_{\text{succ}}^{i-1}(0, x)$ denotes applying the successor function $(i - 1)$ times starting from 0.

Exercise 2. Lecture Slides 22, FO Definability, Appendix. Ex 10 on page 3.

Hint 1 : Review how we showed, in lecture on Thursday October 10, that the unary predicate prime was first-order definable in in structure (ℕ; =, ¡, +, ·, S, 0). See pages 15-16 in Lecture Slides 21 where prime is called pr

Hint 2 : You case use any of the preceding exercises in Lecture Slides 22, FO Definability, Appendix as "lemmas" without proving them again, i.e., use the statements of the exercises as premises that you do not need to prove.

Show that the predicate prime : $\mathbb{N} \to \{\text{true}, \text{false}\}$ is first-order definable in the structure $(\mathbb{N}; |, +, 0)$.

Solution to Ex. 10

1. We define the predicate prime(x) using the following first-order formula:

$$\phi_{\text{prime}}(x) \stackrel{\text{def}}{=} \neg(x \approx 1) \wedge \forall y \forall z[(y|x \wedge z|x) \to (y \approx 1 \vee z \approx 1 \vee y \approx x \vee z \approx x)]$$

2. This formula uses only the divisibility relation $|$ and equality $\approx$. We need to show that the other components can be defined using $\{|, +, 0\}$:
3. $x \approx 1$ can be defined as $\phi_1(x) \stackrel{\text{def}}{=} \forall y(y|x)$
4. $x \approx 0$ can be defined as $\phi_0(x) \stackrel{\text{def}}{=} \forall y(x|y)$
5. Using these, we can rewrite $\phi_{\text{prime}}(x)$ as:

$$\phi_{\text{prime}}(x) \stackrel{\text{def}}{=} \neg\phi_1(x) \wedge \forall y \forall z[(y|x \wedge z|x) \to (\phi_1(y) \vee \phi_1(z) \vee y \approx x \vee z \approx x)]$$

PROBLEM 1 Open EML.Chapter 4.pdf. Do part 2 and part 3 in Exercise 71 on page 46.

Hint: You will have to do a fair amount of reading in the same chapter before doing this problem, in particular, you should carefully study the material in Example 70 right before Exercise 71. However, by the time you have finished your reading, your answers will be relatively easy to write.

Solution to Ex. 2:

We can prove that if every finite subgraph of G is k-colorable, then G is k-colorable, using the Compactness Theorem for ZOL.

1. Let $\Gamma = \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Phi_1 \cup \Phi_2 \cup \Phi_3$ as defined in the background.
2. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. This Γ_0 only mentions a finite number of vertices of G.
3. Let G_0 be the finite subgraph of G induced by these vertices.
4. By our assumption, G_0 is k-colorable.
5. This k-coloring of G_0 can be extended to a model M_0 of Γ_0.
6. Therefore, every finite subset of Γ is satisfiable.
7. By the Compactness Theorem for ZOL (Theorem 65), Γ itself is satisfiable.
8. Let M be a model of Γ. By the Basic Herbrand Theorem (Theorem 58), we can assume M is a Herbrand model.
9. The reduct $M_0 = (M, R^M)$ contains an isomorphic copy of G (by the Claim in the background).
10. The interpretation of $C_1, \ldots, C_k$ in M gives a valid k-coloring of this copy of G.
11. Therefore, G is k-colorable.

Solution to Ex. 3:

Let $G = (N, E)$ be a planar graph. We will prove that G is 4-colorable.

1. Every finite subgraph of G is planar (as planarity is a hereditary property).
2. By the Four Color Theorem, every finite planar graph is 4-colorable.
3. Therefore, every finite subgraph of G is 4-colorable.
4. By the result from Exercise 2, since every finite subgraph of G is 4-colorable, G itself is 4-colorable.

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

Exercise 3. From Macbeth's book

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

Exercise 4. From Macbeth's book

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PROBLEM 2. From Macbeth's book

Solution

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