CS 511, Fall 2023, Lecture Slides 03

Semantics of Propositional Logic

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Semantics of Propositional Logic via Truth-Table

- Reading: [LCS, Section 1.4]

  Remark: It is also somewhat unconventional to present the semantics of a formal logic, such as *propositional logic* in [LCS, Section 1.4], after presenting its syntax in [LCS, Section 1.3] and a proof system for it in [LCS, Section 1.2].

- Reading: [EML.Appendix, pp 7-8]

some familiar truth-tables:

logical “or” ($\vee$) and logical “and” ($\wedge$)

x	y	$x \vee y$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

x	y	$x \wedge y$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

logical “implication” ($\rightarrow$)

x	y	$x \rightarrow y$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

and similarly for “negation” ($\neg$) and many other logical connectives . . . .

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \stackrel{\text{def}}{=} (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y
T	**T**
T	**F**
F	**T**
F	**F**

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x	y	$\neg x$
T	**T**	**F**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

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x	y	$\neg x$	$\neg y$
T	**T**	**F**	**F**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**F**	**T**	**T**

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x	y	$\neg x$	$\neg y$	$x \to \neg y$
T	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

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x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$
T	**T**	**F**	**F**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**

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x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$	$(x \to \neg y) \to (y \vee \neg x)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \stackrel{\text{def}}{=} (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$	$(x \to \neg y) \to (y \vee \neg x)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

- propositional wff φ is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes φ true.
- propositional wff φ is a **tautology** if **every** assignment of truth-values to the propositional atoms makes φ true.

from *propositional formulas* to *truth-tables*

consider propositional wff (**well-formed formula**): $\varphi \stackrel{\text{def}}{=} (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$	$(x \to \neg y) \to (y \vee \neg x)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

- propositional wff φ is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes φ true.
- propositional wff φ is a **tautology** if **every** assignment of truth-values to the propositional atoms makes φ true.
- $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$ is **satisfiable**, but is **not a tautology**.

Another More Complicated Truth-Table

not of a single wff, but of a judgment $(P \wedge \neg Q) \to R,\ \neg R,\ P \vdash Q$, which was shown **formally derivable** by the proof rules at the end of **Lecture Slides 02**.

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P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \wedge \neg Q) \to R$
T	**T**	**T**	**F**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**F**	**F**	**F**	**T**
F	**T**	**F**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**
F	**F**	**F**	**T**	**T**	**F**	**T**

Another More Complicated Truth-Table

not of a single wff, but of a judgment $(P \wedge \neg Q) \rightarrow R,\ \neg R,\ P \vdash Q$, which was shown **formally derivable** by the proof rules at the end of **Lecture Slides 02**.

P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \wedge \neg Q) \rightarrow R$
T	**T**	**T**	**F**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**F**	**F**	**F**	**T**
F	**T**	**F**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**
F	**F**	**F**	**T**	**T**	**F**	**T**

- when all the premises (shaded in gray) evaluate to **T**, so does the conclusion (shaded in green) – this occurs in **row 2** of the truth table,
- in such a case we write $(P \wedge \neg Q) \rightarrow R,\ \neg R,\ P \models Q$.

Relating Truth Tables and Proof Rules :

Brief overview of two fundamental concepts:
soundness and *completeness* , examined in depth later in the course.

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- If, for every interpretation/model/valuation (*i.e.*, assignment of truth values to the propositional atoms) for which all of the WFF's $\varphi_1, \varphi_2, \ldots, \varphi_n$ evaluate to **T**, it is also the case that ψ evaluates to **T**, then we write:

  $$\varphi_1,\ \varphi_2,\ \ldots\ ,\ \varphi_n\ \models\ \psi$$

  and say that "$\varphi_1, \varphi_2, \ldots, \varphi_n$ semantically entails ψ"
  or also "every model of $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a model of ψ" .

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or also "every model of $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a model of ψ".

- **Theorem (Soundness):**
If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.

- **Theorem (Completeness):**
If $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.

Relating Truth Tables and Proof Rules: *soundness* and *completeness*

- simple version of **soundness**: if $\vdash \psi$ then $\models \psi$

  Informally, "if you can prove it, then it is true".

- simple version of **completeness**: if $\models \psi$ then $\vdash \psi$

  Informally, "if it is true, then you can prove it".

- if $\models \psi$, then we say ψ is a **tautology** or a **valid formula**.

- if $\vdash \varphi$, then we say φ is **(formally) derivable** or a **(formal) theorem**.

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