CS 511, Fall 2024, Lecture Slides 06

Examples of Structural Induction

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Reminder: Simple Induction over the Natural Numbers

- There are different ways of showing that:

$$\sum_{1 \leqslant i \leqslant n} i \;=\; \frac{n(n+1)}{2}$$

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- There are different ways of showing that:

$$\sum_{1 \leqslant i \leqslant n} i \ = \ \frac{n(n+1)}{2}$$

- One way is by simple induction on $n \geqslant 1$. Let $S_n \overset{\text{def}}{=} \sum_{1 \leqslant i \leqslant n} i$. We want to show that $S_n = \frac{n(n+1)}{2}$ for every $n \geqslant 1$, which we can set as the **Induction Hypothesis** in this example.
  1. **Base step**: $S_1 = 1 = \frac{1(1+1)}{2}$
  2. **Inductive step**: Assume **IH** for $n-1$, when $n \geqslant 2$, and prove **IH** for n:

$$\begin{aligned} S_n &= n + S_{n-1} \\ &= n + \frac{(n-1)n}{2} \qquad \text{(by the } \textbf{IH} \text{ for } n-1) \\ &= \frac{2n + n^2 - n}{2} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

This completes the induction and the proof that $S_n = \frac{n(n+1)}{2}$ for every $n \geqslant 1$.

All finite strings/words of decimal digits

- The set of all decimal digits: $A \stackrel{\text{def}}{=} \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$

- Definition of the set A^* of all finite *strings* (or *words*) over A by induction:
  1. **Base step**: $\varepsilon \in A^*$ where ε denotes the empty string,
  2. **Inductive step**: For all $s \in A^*$ and $x \in A$, the string $s \cdot x$ is a member of A^*.

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- Alternative equivalent definition, using **BNF** notation:

$$d ::= 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$
$$s ::= \varepsilon \mid d \mid s \cdot d$$

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\begin{aligned}
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\end{aligned}
$$

For convenience, instead of $3 \cdot 8 \cdot 1 \cdot 0$, we may write: $3\ 8\ 1\ 0$.

Concatenation of two strings s and t is denoted $s \cdot t$.
For example, if $s \stackrel{\text{def}}{=} 4\,5\,6$ and $t \stackrel{\text{def}}{=} 2\,2$, then $s \cdot t = 4\,5\,6\,2\,2$.

All finite strings/words of decimal digits

- Definition of the function reverse : $A^* \to A^*$ by structural induction :
  1. **Base step**: $\underline{\text{reverse}}(\varepsilon) \stackrel{\text{def}}{=} \varepsilon$,
  2. **Inductive step**: $\underline{\text{reverse}}(s \cdot x) \stackrel{\text{def}}{=} x \cdot \underline{\text{reverse}}(s)$.

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- Definition of the function foo : $A^* \to \mathbb{R}$ by structural induction :
  1. **Base step**: $\text{foo}(\varepsilon) \stackrel{\text{def}}{=} 1.0$,
  2. **Inductive step**: $\text{foo}(s \cdot x) \stackrel{\text{def}}{=} \text{foo}(s) \ \times \ 0.5$.

Properties of the function $\underline{\text{reverse}} : A^* \to A^*$

- **Proposition**:
  For all strings $s, t \in A^*$, it holds that $\underline{\text{reverse}}(s \cdot t) = \underline{\text{reverse}}(t) \cdot \underline{\text{reverse}}(s)$.

  Proof: Use *structural induction* on $t \in A^*$ to prove the property $P(t)$ defined by:

  $$P(t) \stackrel{\text{def}}{=} \text{“for all } s \in A^* \text{ it holds that } \underline{\text{reverse}}(s \cdot t) = \underline{\text{reverse}}(t) \cdot \underline{\text{reverse}}(s)\text{”}$$

  This induction is on the structure of t as a string, there is no induction on natural numbers here.

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- **Proposition**:
  For all strings $s \in A^*$, it holds that $\boxed{\underline{\text{reverse}}(\underline{\text{reverse}}(s)) = s}$.

  Proof: By *structural induction*. (*Hint*: Use preceding proposition as a lemma.)

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  The function $\underline{\text{reverse}} : A^* \to A^*$ is one-one and onto, *i.e.*, a bijection.

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  $\underline{\text{reverse}}$ is *one-one*: for all $s_1, s_2 \in A^*$, if $\underline{\text{reverse}}(s_1) = \underline{\text{reverse}}(s_2)$ then $s_1 = s_2$.
  $\underline{\text{reverse}}$ is *onto*: for every $t \in A^*$ there is $s \in A^*$ such that $\underline{\text{reverse}}(s) = t$.

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- **Proposition**:
  For all $s \in A^*$, it holds that $\underline{\text{foo}}(\underline{\text{reverse}}(s)) = \underline{\text{foo}}(s)$.

  Proof: By *structural induction*.

Binary trees with labels in A

- Inductive definition of the set $\mathsf{BN}(A)$ of binary trees with labels in A:
  1. **Base step**: if $x \in A$, then $x \in \mathsf{BN}(A)$,
  2. **Inductive step**: if $x \in A$ and $t_1, t_2 \in \mathsf{BN}(A)$, then $\langle x, t_1, t_2 \rangle \in \mathsf{BN}(A)$.

- Different standard traversals of binary trees:
  pre_order, post_order, $\text{in\_order}^{\mathrm{L}}$ (*from the left*), $\text{in\_order}^{\mathrm{R}}$ (*from the right*), etc.

- Definition of $\boxed{\text{in\_order}^{\mathrm{L}} : \mathsf{BN}(A) \to A^*}$ by structural induction:
  1. **Base step**: $\text{in\_order}^{\mathrm{L}}(x) \stackrel{\text{def}}{=} x$,
  2. **Inductive step**: $\text{in\_order}^{\mathrm{L}}(\langle x, t_1, t_2 \rangle) \stackrel{\text{def}}{=} \text{in\_order}^{\mathrm{L}}(t_1) \cdot x \cdot \text{in\_order}^{\mathrm{L}}(t_2)$.

- Definition of $\boxed{\text{in\_order}^{\mathrm{R}} : \mathsf{BN}(A) \to A^*}$ by structural induction:
  1. **Base step**: $\text{in\_order}^{\mathrm{R}}(x) \stackrel{\text{def}}{=} x$,
  2. **Inductive step**: $\text{in\_order}^{\mathrm{R}}(\langle x, t_1, t_2 \rangle) \stackrel{\text{def}}{=} \text{in\_order}^{\mathrm{R}}(t_2) \cdot x \cdot \text{in\_order}^{\mathrm{R}}(t_1)$.

in_order$^{\mathrm{L}}$, in_order$^{\mathrm{R}}$, and reverse

A nice property which is best proved by *structural induction* . . .

- **Proposition**:

  For all binary trees $t \in \mathrm{BN}(A)$, it holds that reverse$(\text{in_order}^{\mathrm{L}}(t)) = \text{in_order}^{\mathrm{R}}(t)$.

  Proof: We prove by *structural induction* on $t \in \mathrm{BN}(A)$ the property $P(t)$ defined by:

  $$P(t) \stackrel{\text{def}}{=} \text{“for all } s \in \mathrm{BN}(A) \text{ it holds that reverse}(\text{in\_order}^{\mathrm{L}}(t)) = \text{in\_order}^{\mathrm{R}}(s)\text{”}$$

  This induction is on the structure of t as a binary tree, there is no induction on natural numbers here.

  1. **Base step**: Prove $P(t)$ when $t = x \in A$.
  2. **Inductive step**: Prove $P(t)$ when $t = \langle x, t_1, t_2 \rangle$, using the *induction hypothesis* (IH) which states that: $P(t_1)$ holds and $P(t_2)$ holds.

- **Exercise**: Fill in the missing details in the preceding proof.

  Hint: For the *inductive step*, you will need to use the definition of reverse on slide 3 and the first proposition on slide 5.

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