

Solutions to CS511 Homework 07

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October 24, 2024

Exercise 1. Open Lecture Slides 29, I, Analytical Tableaux for Classical First-Order Logic. Do Exercise 1 on page 14.

Hint: To show that $\{\phi_1, \phi_2, \phi_3\} \models \phi$ is equivalent to showing $\{\phi_1, \phi_2, \phi_3\} \vdash \phi$ (by completeness), which is equivalent to showing $\vdash (\phi_1 \wedge \phi_2 \wedge \phi_3) \rightarrow \phi$. These equivalences hold for formal proofs carried out according to the rules of natural deduction, and they hold again when analytic tableaux are used as a formal-proof system.

Show that $\Gamma \models \phi$

where:

$\Gamma = \{\forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow P(x, z)), \forall x \forall y (P(x, y) \rightarrow P(y, x))\}$
 $\phi = \forall x \forall y \forall z (P(x, y) \wedge P(z, y) \rightarrow P(x, z))$

Solutions to First-Order Ground Tableaux Proof:

We negate the formula we want to prove and show it leads to a contradiction:

1. $\neg(\forall x \forall y \forall z (P(x, y) \wedge P(z, y) \rightarrow P(x, z)))$
- |
2. $\exists x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge \neg P(x, z))$
- |
3. $P(x_1, y_1) \wedge P(z_1, y_1) \wedge \neg P(x_1, z_1)$
- |
4. $P(x_1, y_1)$
- |
5. $P(z_1, y_1)$
- |
6. $\neg P(x_1, z_1)$
- |
7. $P(y_1, z_1)$ (from 2, 5)
- |
8. $P(x_1, z_1)$ (from 1, 4, 7)
- |
- X

Explanation:

1. Line 1: Negation of the conclusion
2. Line 2: Converting universal quantifier to existential
3. Line 3: Instantiation with fresh variables x_1, y_1, z_1
4. Lines 4-6: Conjunction elimination
5. Line 7: From $P(z_1, y_1)$ and symmetry axiom
6. Line 8: From $P(x_1, y_1), P(y_1, z_1)$ and transitivity axiom
7. X: Contradiction between lines 6 and 8

Since we reached a contradiction, the original formula is valid.

Exercise 2. Open Lecture Slides 29, I, Analytical Tableaux for Classical First-Order Logic. Do Exercise 2 on page 14.

Hint: Review the hint in the preceding exercise

Show that $\Gamma \models \phi$

where:

$$\Gamma = \{\forall x Q(a, x, x), \forall x \forall y \forall z (Q(x, y, z) \rightarrow Q(x, s(y), s(z))), \forall x \forall y \forall z (Q(x, y, z) \rightarrow Q(y, x, z))\}$$

$$\phi = \exists x Q(s^{(2)}(a), s^{(3)}(a), x)$$

where Q is a ternary predicate symbol, s is a unary function symbol, and a is a constant symbol.

Solution using ground tableaux method: First-Order Ground Tableaux Proof

$$\begin{array}{l}
 1. \neg(\exists x Q(s^{(2)}(a), s^{(3)}(a), x)) \\
 \quad \downarrow \\
 2. \forall x Q(a, x, x) \\
 \quad \downarrow \\
 3. Q(a, a, a) \quad [\text{from 2}] \\
 \quad \downarrow \\
 4. Q(a, s(a), s(a)) \quad [\text{from 2,3}] \\
 \quad \downarrow \\
 5. Q(s(a), a, s(a)) \quad [\text{from 3,4}] \\
 \quad \downarrow \\
 6. Q(s(a), s^{(2)}(a), s^{(2)}(a)) \quad [\text{from 4,5}] \\
 \quad \downarrow \\
 7. Q(s^{(2)}(a), s(a), s^{(2)}(a)) \quad [\text{from 5,6}] \\
 \quad \downarrow \\
 8. Q(s^{(2)}(a), s^{(2)}(a), s^{(3)}(a)) \quad [\text{from 6,7}] \\
 \quad \downarrow \\
 9. Q(s^{(2)}(a), s^{(3)}(a), s^{(3)}(a)) \quad [\text{from 7,8}] \\
 \quad \downarrow \\
 10. \forall x \neg Q(s^{(2)}(a), s^{(3)}(a), x) \quad [\text{from 1}] \\
 \quad \downarrow \\
 11. \neg Q(s^{(2)}(a), s^{(3)}(a), s^{(3)}(a)) \quad [\text{from 10}] \\
 \quad \downarrow \\
 \text{X}
 \end{array}$$

The tableau closes because we derived a contradiction between lines 9 and 11, proving that $\Gamma \models \phi$.

PROBLEM 1 Open EML.Chapter 6.pdf. Do part Exercise 113 on page 69

Hint: This is a continuation of the discussion in lecture yesterday (Thursday, October 17). As suggested in lecture, read carefully and understand Example 112 on pages 68-69 before embarking on Exercise 113.

Queens Problem II

Task:

We need to construct a first-order sentence ψ that characterizes solutions of the Queens Problem using a unary function q instead of a binary relation. The structure $M \stackrel{\text{def}}{=} (\mathbb{N}, =, +, <, 0, q)$ interprets q as a function where $q(i) = 0$ means no queen in row i , and $q(i) = j$ means a queen is placed at position (i, j) .

Solution:

The first-order sentence ψ can be defined as:

$$\psi \stackrel{\text{def}}{=} \psi_{\text{base}} \wedge (\psi_{\text{fin}} \vee \psi_{\text{inf}}) \wedge \psi_{\text{valid}} \wedge \psi_{\text{noattack}}$$

where:

$$\psi_{\text{base}} \stackrel{\text{def}}{=} q(0) \approx 0$$

$$\psi_{\text{fin}} \stackrel{\text{def}}{=} \exists n > 0. (\forall i. (i > n \rightarrow q(i) \approx 0) \wedge \forall i. (0 < i \leq n \rightarrow 0 < q(i) \leq n))$$

$$\psi_{\text{inf}} \stackrel{\text{def}}{=} \forall i > 0. \exists j > 0. (q(i) \approx j)$$

$$\psi_{\text{valid}} \stackrel{\text{def}}{=} \forall i > 0. \forall j > 0. (q(i) \approx j \rightarrow \forall k > 0. (k \not\approx i \rightarrow q(k) \not\approx j))$$

$$\psi_{\text{noattack}} \stackrel{\text{def}}{=} \forall i_1 > 0. \forall i_2 > 0. (i_1 \not\approx i_2 \wedge q(i_1) \not\approx 0 \wedge q(i_2) \not\approx 0 \rightarrow |i_1 - i_2| \not\approx |q(i_1) - q(i_2)|)$$

where:

- ψ_{base} ensures $q(0) = 0$
- ψ_{fin} handles finite solutions

- ψ_{inf} handles infinite solutions
- ψ_{valid} ensures no two queens share a column
- ψ_{noattack} ensures no two queens are on the same diagonal

The absolute value notation $|x - y|$ can be expressed in first-order logic as:

$$|x - y| \approx z \stackrel{\text{def}}{=} ((x - y \approx z) \vee (y - x \approx z)) \wedge z \geq 0$$

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

Exercise 3. From Macbeth's book

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

Exercise 4. From Macbeth's book

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

PROBLEM 2. From Macbeth's book

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean