CS 511, Fall 2024, Lecture Slides 26

First-Order Logic: Prenex Normal Form and Skolemization

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more on quantifier equivalences

Lemma. For any string of quantifiers

$$\overrightarrow{Qx} \stackrel{\text{def}}{=} Q_1 x_1 Q_2 x_2 \cdots Q_n x_n$$

where $Q_1, Q_2, \ldots, Q_n \in \{\forall, \exists\}$ with $n \geqslant 0$, and for any WFF's φ and ψ:

- $\vdash \quad \overrightarrow{Qx}\ \neg\forall y\ \varphi \leftrightarrow \overrightarrow{Qx}\ \exists y\, \neg\ \varphi$
- $\vdash \quad \overrightarrow{Qx}\ \neg\exists y\ \varphi \leftrightarrow \overrightarrow{Qx}\ \forall y\, \neg\ \varphi$
- $\vdash \quad \overrightarrow{Qx}\,(\forall y\, \varphi \vee \psi) \leftrightarrow \overrightarrow{Qx}\, \forall z\,(\varphi\, [y := z] \vee \psi)$
- $\vdash \quad \overrightarrow{Qx}\,(\varphi \vee \forall y\, \psi) \leftrightarrow \overrightarrow{Qx}\, \forall z\,(\varphi \vee \psi\, [y := z])$
- $\vdash \quad \overrightarrow{Qx}\,(\exists y\, \varphi \vee \psi) \leftrightarrow \overrightarrow{Qx}\, \exists z\,(\varphi\, [y := z] \vee \psi)$
- $\vdash \quad \overrightarrow{Qx}\,(\varphi \vee \exists y\, \psi) \leftrightarrow \overrightarrow{Qx}\, \exists z\,(\varphi \vee \psi\, [y := z])$

where z is a fresh variable occurring nowhere else.

Proof. Similar to proof of Theorem 2.13 in [LCS], page 117. See also [EML.Appendix.pdf, Section E.1], pages 27-31.

prenex normal form

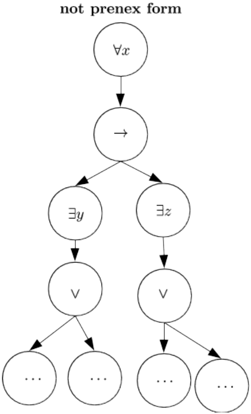
Theorem.
For every WFF φ there is an equivalent WFF ψ with the same free variables where all quantifiers appear at the beginning.

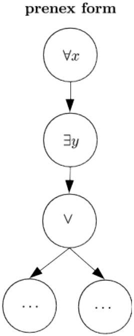
ψ is called the **prenex normal form** of φ.

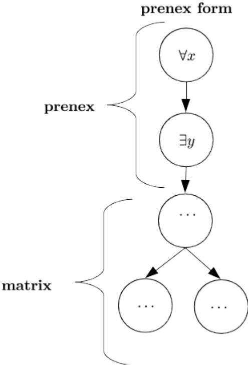
Proof. By induction on the structure of φ.

- If φ is atomic, then $\psi \stackrel{\text{def}}{=} \varphi$.
- If φ is $Qx\,\varphi_0$ where $Q \in \{\forall, \exists\}$ and ψ_0 is a PNF of φ_0, then $\psi \stackrel{\text{def}}{=} Qx\,\psi_0$.
- If φ is $\neg\varphi_0$ and ψ_0 is a PNF of φ_0, then use the two first cases in the **lemma** (on preceding slide) repeatedly, to obtain ψ.
- If φ is $\varphi_0 \vee \varphi_1$, and ψ_0 and ψ_1 are PNF's of φ_0 and φ_1, then use the four last cases in the **lemma** repeatedly, to obtain ψ.

prenex normal form (continued)







$\forall x\Big(\exists y(\neg x \vee y \vee \neg v) \rightarrow \exists z((x \rightarrow z) \vee \neg v)\Big)$

$\forall x\Big(\exists y(\neg x \vee y \vee \neg v)\Big)$

skolemization

Reference: [EML.Appendix.pdf, Section E.2], pages 31-32.

Lemma. A first-order sentence φ of the form

$$\varphi \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n \exists y\, \psi$$

over vocabulary/signature Σ is equisatisfiable with the sentence φ'

$$\varphi' \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n\, \psi[y := f(x_1, \ldots, x_n)]$$

where f is a fresh n-ary function symbol not in Σ.

Proof.

Let $\mathcal{M}$ be a model for Σ and $\mathcal{M}' \stackrel{\text{def}}{=} (\mathcal{M}, f^{\mathcal{M}'})$ a model for $\Sigma \cup \{f\}$. If $\mathcal{M}' \models \varphi'$ then $\mathcal{M} \models \varphi$. Hence, if φ' is satisfiable, then so is φ.

Conversely, let $\mathcal{M} \models \varphi$. Construct a model $\mathcal{M}'$ for $\Sigma \cup \{f\}$ by expanding $\mathcal{M}$ so that for every $a_1, \ldots, a_n \in A$, the function $f^{\mathcal{M}'}$ maps $(a_1, \ldots, a_n)$ to b where $\mathcal{M}, a_1, \ldots, a_n, b \models \psi$. Hence, $\mathcal{M}' \models \varphi'$. Hence, if φ is satisfiable, then so is φ'.

skolemization (continued)

Theorem.
If φ is a first-order sentence over the vocabulary/signature Σ,
then there is a **universal** first-order sentence φ' over an expanded
vocabulary/signature Σ' obtained by adding new function symbols
such that φ and φ' are equisatisfiable.

Proof. By repeated use of the **lemma** (on the preceding slide).

Remark. The theorem does NOT claim that φ and φ' are equivalent,
only that they are equisatisfiable.

However, it will be always the case that $\vdash \varphi' \to \varphi$, but not always that $\vdash \varphi \to \varphi'$.

exercise on skolemization

Exercise:

Let $\varphi(x, y)$ be an atomic WFF with free variables x and y, and f a unary function symbol not appearing in φ.

1. Show that the sentence $\forall x\, \varphi(x, f(x)) \to \forall x \exists y\, \varphi(x, y)$ is semantically valid, *i.e.*, the following sequent is formally derivable:

   $$\vdash\ \forall x\, \varphi(x, f(x)) \to \forall x \exists y\, \varphi(x, y)$$

   Hint: Use any of the available methods, *i.e.*, try to find a formal proof or try a semantic approach to show $\models \forall x\, \varphi(x, f(x)) \to \forall x \exists y\, \varphi(x, y)$ and then invoke the completeness of the proof rules.
2. Show that the sentence $\forall x \exists y\, \varphi(x, y) \to \forall x\, \varphi(x, f(x))$ is NOT semanticalle valid, *i.e.*, the following sequent is NOT derivable:

   $$\vdash\ \forall x \exists y\, \varphi(x, y) \to \forall x\, \varphi(x, f(x))$$

   Hint: Try a semantic approach, *i.e.*, define an appropriate φ and a model where the left-hand side of "$\to$" is true but the right-hand side of "$\to$" is false, and then invoke the completeness of the proof rules.

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