CS 511, Fall 2024, Lecture Slides 22

First-Order Definability

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some notational conventions

- Throughout these lecture slides, we assume we have a fixed, but otherwise arbitrary, vocabulary/signature $\Sigma = (\mathscr{P}, \mathscr{F})$:

  $$\mathscr{P} \stackrel{\text{def}}{=} \{P_1, P_2, \ldots\} \quad \text{and} \quad \mathscr{F} \stackrel{\text{def}}{=} \{f_1, f_2, \ldots\}$$

  Even if the equality symbol $\approx$ is not mentioned in Σ, we assume it is available.
- All first-order wff's are written over the signature Σ, and therefore interpreted in Σ-structures/models.

some notational conventions

- Suppose $\mathcal{M} \stackrel{\text{def}}{=} (M, \ldots)$ is a Σ-structure with universe M, $\ell : \{\text{all variables}\} \to M$ a valuation / environment / look-up table, and φ a first-order WFF such that $\mathcal{M}, \ell \models \varphi$.

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- If φ is **closed**, we may write $\mathcal{M} \models \varphi$ instead, because ℓ plays no role in the satisfiability/unsatisfiability of φ.

  Put differently, $\mathcal{M} \models \varphi$ means $\mathcal{M}, \ell \models \varphi$ for every ℓ in this case.

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- If φ is **not closed**, *e.g.*, variables x_1, x_2, and x_3 occur free in φ, with $\ell(x_1) = a_1$, $\ell(x_2) = a_2$, and $\ell(x_3) = a_3$, with $a_1, a_2, a_3 \in M$.
  - We may write $\mathcal{M}, a_1, a_2, a_3 \models \varphi(x_1, x_2, x_3)$ instead of $\mathcal{M}, \ell \models \varphi$.

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  - We may write $\mathcal{M}, a_1, a_2, a_3 \models \varphi(x_1, x_2, x_3)$ instead of $\mathcal{M}, \ell \models \varphi$.

  - Or we may write $\mathcal{M} \models \varphi[a_1, a_2, a_3]$ instead of $\mathcal{M}, \ell \models \varphi$.

first-order definability of **relations** and **functions**

- Let $\mathcal{M} = (M; \approx^{\mathcal{M}}, P_1^{\mathcal{M}}, P_2^{\mathcal{M}}, \ldots, f_1^{\mathcal{M}}, f_2^{\mathcal{M}}, \ldots)$ be a Σ-structure.

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- Let $R \subseteq \underbrace{M \times \cdots \times M}_{k}$ be a k-ary **relation** on M for some $k \geqslant 1$.

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- Let $R \subseteq \underbrace{M \times \cdots \times M}_{k}$ be a k-ary **relation** on M for some $k \geqslant 1$.

- R is first-order definable in $\mathcal{M}$ if there is a first-order WFF $\varphi(x_1, \ldots, x_k)$ with k free variables such that:

  $$R = \left\{ (a_1, \ldots, a_k) \in M \times \cdots \times M \;\middle|\; \mathcal{M}, a_1, \ldots, a_k \models \varphi(x_1, \ldots, x_k) \right\}$$

  equivalently, using notational conventions earlier in these lecture slides:

  $$R = \left\{ (a_1, \ldots, a_k) \in M \times \cdots \times M \;\middle|\; \mathcal{M} \models \varphi[a_1, \ldots, a_k] \right\}$$

first-order definability of **relations** and **functions**

- Let $f : \underbrace{M \times \cdots \times M}_{k} \rightarrow M$ be a k-ary **function** on M.

first-order definability of **relations** and **functions**

- Let $f : \underbrace{M \times \cdots \times M}_{k} \to M$ be a k-ary **function** on M.

- f is first-order definable in $\mathcal{M}$ if the graph of f as a $(k+1)$-ary relation is first-order definable in $\mathcal{M}$.

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- **Important special case**:
  First-order definability of a subset $X \subseteq M$. View X as a unary relation.

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- **Important special case**:
  First-order definability of a subset $X \subseteq M$. View X as a unary relation.

- **Important special case**:
  First-order definability of a single element $a \in M$:
  a is first-order definable in $\mathcal{M}$ iff
  there is a first-order WFF $\varphi(x)$ s.t. $\mathcal{M}, a \models \varphi(x)$
  and $\mathcal{M}, b \not\models \varphi(x)$ for every $b \neq a$.

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