Solutions to CS511 Homework 03

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Exercise 1 Go to page 13 in Lecture Slides 09. Your task is to carefully do part 1 of the exercise on that page.

Prove by Natural Deduction: $\neg(p \land q \land r) \to (\neg p \lor \neg q \lor \neg r)$

Solution:

1. $\neg(p \land q \land r)$ [Assumption]
2. $\neg(\neg p \lor \neg q \lor \neg r)$ [Assumption for contradiction]
3. $p \land q \land r$ [De Morgan's Law applied to line 2]
4. Contradiction [Lines 1 and 3 contradict]
5. $\neg\neg(\neg p \lor \neg q \lor \neg r)$ [Negation Introduction, 2-4]
6. $\neg p \lor \neg q \lor \neg r$ [Double Negation Elimination, 5]
7. $\neg(p \land q \land r) \to (\neg p \lor \neg q \lor \neg r)$ [Conditional Proof, 1-6]

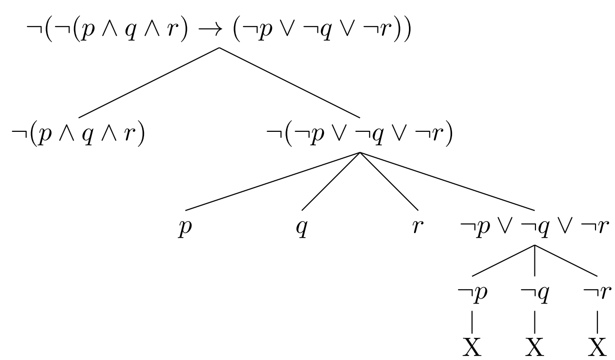
Exercise 2 Go to page 18 in Lecture Slides 10. Your task is to carefully do part 1 of the exercise on that page.

Use the tableaux method to show the validity of the following more general version of de Morgan's law (4):

$\phi_1 \equiv \neg(p \wedge q \wedge r) \to (\neg p \vee \neg q \vee \neg r)$

Solution:

1.	$\neg(\neg(p \wedge q \wedge r) \to (\neg p \vee \neg q \vee \neg r))$
2.	$p \wedge q \wedge r$
3.	$\neg(\neg p \vee \neg q \vee \neg r)$
4.	p
5.	q
6.	r
7.	$\neg p$
8.	$\neg q$
9.	$\neg r$
10.	Contradiction (from 4 and 7)
11.	Contradiction (from 5 and 8)
12.	Contradiction (from 6 and 9)



PROBLEM 1 There are do parts: (a) Go to page 13 in Lecture Slides 09 once more. Your task is to carefully do parts 2, 3, and 4 of the exercise on that page. (b) Go to page 18 in Lecture Slides 10 once more. Your task is to carefully do parts 2, 3, and 4 of the exercise on that page.

(a)

2. Natural-deduction proof of the most general de Morgan's law

For $\phi_2 \equiv \neg(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n)$, where $n \geq 2$:

(a)	$\neg(p_1 \wedge \cdots \wedge p_n)$	(Assumption)
(b)	$\neg(\neg p_1 \vee \cdots \vee \neg p_n)$	(Assumption for contradiction)
(c)	$p_1 \wedge \cdots \wedge p_n$	(From 2, by De Morgan's law)
(d)	Contradiction	(From 1 and 3)
(e)	$\neg p_1 \vee \cdots \vee \neg p_n$	(From 2-4, by Reductio ad Absurdum)
(f)	$\neg(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n)$	(From 1-5, by Conditional Proof)

3. Proof length is O(n)

The natural-deduction proof of ϕ_2 has a constant number of steps regardless of n. The only part that depends on n is the length of the formulas themselves. Therefore, the proof length is O(n).

4. Complexity comparison: Natural-deduction vs. Truth-table

- **Natural-deduction proof:** As shown above, the proof length is O(n), where n is the number of propositions.
- **Truth-table verification:** A truth table for n propositions has 2^n rows. Each row requires O(n) operations to compute.

  Total complexity: $\mathrm{O}(n \cdot 2^n)$

Comparison: The natural-deduction proof is significantly more efficient, with linear complexity O(n) compared to the exponential complexity $\mathrm{O}(n \cdot 2^n)$ of the truth-table method.

(b)

2. Use the tableaux method to show the validity of de Morgan's law (4) in general:

$\phi_2 \equiv \neg(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n)$ where $n \geq 2$.

1.	$\neg(\neg(p_1 \wedge \cdots \wedge p_n) \rightarrow (\neg p_1 \vee \cdots \vee \neg p_n))$
2.	$\neg(p_1 \wedge \cdots \wedge p_n)$
3.	$\neg(\neg p_1 \vee \cdots \vee \neg p_n)$
4.	p_1
5.	p_2
	$\vdots$
n+3.	p_n
n+4.	$\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n$
n+5.	$\neg p_1$ X
n+6.	$\neg p_2$ X
	$\vdots$
2n+4.	$\neg p_n$ X

3. Compute the precise size of the tableau (i.e., the number of nodes in the tree underlying the tableau), in Part 2 above, as a function of n (the number of variables).

The tableau has:

- $n + 4$ nodes before branching
- n branches, each with 1 node

Total number of nodes $= (n + 4) + n = 2n + 4$

4. Compare the complexity of the tableau proof for ϕ_2 in Part 2 above with the complexity of the natural-deduction proof of ϕ_2 and that of the truth-table verification of ϕ_2. For the latter two procedures, consult Lecture Slides 09.

a) Tableau method: $O(n)$ nodes and steps

b) Natural-deduction proof: $O(n)$ steps (as shown in previous lectures)

c) Truth-table verification: $O(2^n)$ rows to check all possible combinations

Comparison:

- The tableau and natural-deduction proofs have linear complexity $O(n)$.
- The truth-table verification has exponential complexity $O(2^n)$.

For large n, the tableau and natural-deduction proofs are significantly more efficient than the truth-table method. The tableau method is comparable in efficiency to the natural-deduction proof for this particular formula.

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw03/hw03_nicholas_ikechukwu.lean

Exercise 3 For each of the three examples in the following three sections of Macbeth's book, your task is to remove 'sorry' and insert appropriate Lean 4 tactics

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw03/hw03_nicholas_ikechukwu.lean

Exercise 4 For each of the three examples in the following three sections of Macbeth's book, your task is to remove 'sorry' and insert appropriate Lean 4 tactics.

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw03/hw03_nicholas_ikechukwu.lean

PROBLEM 2 For each of the three examples in the following three sections of Macbeth's book, your task is to remove 'sorry' and insert appropriate Lean 4 tactics

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw03/hw03_nicholas_ikechukwu.lean