Solutions to CS511 Homework 09

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Exercise 1. Open EML.Chapter 6.pdf: Do Exercise 107 on page 64.

Exercise 107: (Two-Colorability of Graphs: First-Order Definable). The notion of two-colorable simple graphs coincides with the notion of bipartite simple graphs. Write an infinite set $\Gamma_{bipartite}$ of first-order sentences such that, for every simple graph G, it holds that G $\models$ $\Gamma_{bipartite}$ iff G is bipartite.

Hint: G is bipartite iff every cycle in G (possibly with repeated vertices) has even length.

Solution:

Let $\Gamma_{\text{bipartite}}$ be the set of first-order sentences that express that for every cycle of length n (where n is odd), such a cycle cannot exist in the graph. For each odd $n \geq 3$, we include a sentence ϕ_n in $\Gamma_{\text{bipartite}}$:

$$\phi_n := \forall x_1 \ldots \forall x_n \left(\bigwedge_{i=1}^{n-1} E(x_i, x_{i+1}) \wedge E(x_n, x_1) \rightarrow \bigvee_{1 \leq i < j \leq n} x_i \approx x_j \right)$$

Then:

$$\Gamma_{\text{bipartite}} := \{\phi_n \mid n \geq 3 \text{ and } n \text{ is odd}\}$$

This works because:

- Each ϕ_n says "there cannot be a cycle of length n" where n is odd
- The formula enforces that if we have n vertices connected in a cycle, at least two must be the same vertex
- A graph models $\Gamma_{\text{bipartite}}$ if and only if it has no odd cycles
- By the characterization of bipartite graphs, a graph is bipartite if and only if it has no odd cycles

Therefore, $G \models \Gamma_{\text{bipartite}}$ if and only if G is bipartite.

Exercise 2. [LCS, page 163]: Do Exercise 2.4.6 (the last on that page).

Consider the three sentences:

$\phi 1 \stackrel{\text{def}}{=} \forall x P(x,x)$

$\phi 2 \stackrel{\text{def}}{=} \forall x \forall y (P(x,y) \implies P(y,x))$

$\phi 3 \stackrel{\text{def}}{=} \forall x \forall y \forall z ((P(x,y) \wedge P(y,z) \implies P(x,z)))$

which express that the binary predicate P is reflexive, symmetric and transitive, respectively. Show that none of these sentences is semantically entailed by the other ones by choosing for each pair of sentences above a model which satisfies these two, but not the third sentence – essentially, you are asked to find three binary relations, each satisfying just two of these properties.

Solution:

We can show that none of these sentences semantically entails the others, by first finding three different models:

1. A model satisfying ϕ_2 and ϕ_3 but not ϕ_1 (symmetric and transitive but not reflexive)
2. A model satisfying ϕ_1 and ϕ_3 but not ϕ_2 (reflexive and transitive but not symmetric)
3. A model satisfying ϕ_1 and ϕ_2 but not ϕ_3 (reflexive and symmetric but not transitive)

Now, we'll construct these models using simple binary relations on small sets:

Model 1: (symmetric and transitive but not reflexive)

- Domain: $A = \{1, 2\}$
- Relation: $P = \emptyset$ (empty relation)
- This is symmetric (vacuously) and transitive (vacuously) but not reflexive since $P(1,1)$ and $P(2,2)$ don't hold ß

Model 2: (reflexive and transitive but not symmetric)

- Domain: $A = \{1, 2\}$
- Relation: $P = \{(1,1), (2,2), (1,2)\}$
- This is reflexive (all (x,x) included) and transitive, but not symmetric since $(1,2)$ is in P but $(2,1)$ is not

Model 3: (reflexive and symmetric but not transitive)

- Domain: $A = \{1, 2, 3\}$

- Relation: $P = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$
- This is reflexive (all (x,x) included) and symmetric, but not transitive since $(1,2)$ and $(2,3)$ are in P but $(1,3)$ is not

Therefore, it is clear that each sentence is independent of the others.

PROBLEM 1 Open Lecture Slides 26: Do the two parts of the exercise on page 7.

Part 1:

Part 2:

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw09/hw09_nicholas_ikechukwu.lean

Exercise 3. From Macbeth's book:

Solutions

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw09/hw09_nicholas_ikechukwu.lean

Exercise 4. From Macbeth's book

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PROBLEM 2. From Macbeth's book

Solution

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