CS 511, Fall 2024, Lecture Slides 18

Predicate Logic (aka First-Order Logic): Proof Rules of Natural Deduction

Assaf Kfoury

1 October 2024

proof rules for equality

equality introduction

$$----\approx$$
I

equality elimination

$$\frac{t_1 \approx t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]} \approx \! \mathsf{E}$$

formal proof for " \approx " is symmetric

 $u_1 \approx u_2$ premise $u_1 \approx u_1$ $\approx I$ $\sim I$ \sim

formal proof for " \approx " is symmetric

$$u_1 pprox u_2$$
 premise $u_1 pprox u_1 pprox u_1$ $pprox I$ $pprox I$ $a_2 pprox u_1$ $a_2 pprox u_1$ $a_3 \prox u_2 pprox u_1$

Question: What above corresponds to the WFF φ in the rule \approx E?

Answer: " $x \approx u_1$ " corresponds to φ in the rule \approx E, so that

" $u_1 pprox u_1$ " corresponds to $\varphi[u_1/x]$ & " $u_2 pprox u_1$ " corresponds to $\varphi[u_2/x]$

formal proof for " \approx " is symmetric

$$u_1 \approx u_2$$
 premise $u_1 \approx u_1$ $\approx I$ $\approx I$ $u_2 \approx u_1$ $\approx I$ $\approx I$ 0

Question: What above corresponds to the WFF φ in the rule \approx E?

Answer: " $x \approx u_1$ " corresponds to φ in the rule \approx E, so that

"
$$u_1 pprox u_1$$
" corresponds to $\varphi[u_1/x]$ & " $u_2 pprox u_1$ " corresponds to $\varphi[u_2/x]$

We have formally proved $u_1 \approx u_2 \vdash u_2 \approx u_1$

and we can therefore use as a derived proof rule

$$\frac{t_1 \approx t_2}{t_2 \approx t_1} \approx \text{symmetric}$$

formal proof for " \approx " is transitive

 $u_2 \approx u_3$ premise $u_1 \approx u_2$ premise $u_1 \approx u_3$ $u_1 \approx u_3$ $\approx \text{E } 1, 2$

formal proof for " \approx " is transitive

$$u_2 \approx u_3$$
 premise $u_1 \approx u_2$ premise $u_1 \approx u_3$ $\approx \text{E } 1, 2$

Question: What above corresponds to the WFF φ in the rule \approx E?

Answer: " $u_1 \approx x$ " corresponds to φ in the rule \approx E, so that

" $u_1 \approx u_3$ " corresponds to $\varphi[u_3/x]$ & " $u_1 \approx u_2$ " corresponds to $\varphi[u_2/x]$

formal proof for " \approx " is transitive

$$u_2 \approx u_3$$
 premise $u_1 \approx u_2$ premise $u_1 \approx u_3$ $\approx \text{E } 1, 2$

Question: What above corresponds to the WFF φ in the rule \approx E?

Answer: " $u_1 \approx x$ " corresponds to φ in the rule \approx E, so that

"
$$u_1 \approx u_3$$
" corresponds to $\varphi[u_3/x]$ & " $u_1 \approx u_2$ " corresponds to $\varphi[u_2/x]$

We have formally proved $u_1 \approx u_2$, $u_2 \approx u_3 \vdash u_1 \approx u_3$

and we can therefore use as a derived proof rule

$$\frac{t_1 \approx t_2 \qquad t_2 \approx t_3}{t_1 \approx t_3} \quad \approx \text{transitive}$$

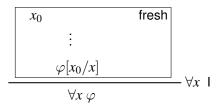
proof rules for universal quantification

universal quantifier elimination

$$\frac{\forall x \ \varphi}{\varphi[t/x]} \ \forall x \ \mathsf{E}$$

(usual assumption: t is substitutable for x)

universal quantifier introduction

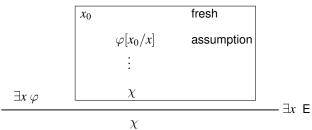


proof rules for existential quantification

existential quantifier introduction

$$\frac{\varphi[t/x]}{\exists x \ \varphi} \exists x \ \exists$$

existential quantifier elimination



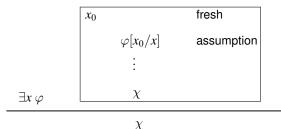
 $(x_0 \text{ cannot occur outside its box, in particular, it cannot occur in } \chi)$

proof rules for existential quantification

existential quantifier introduction

$$\frac{\varphi[t/x]}{\exists x \ \varphi} \ \exists x$$

existential quantifier elimination



 $(x_0 \text{ cannot occur outside its box, in particular, it cannot occur in } \chi)$

Note carefully:

Rule $(\exists x \; \mathsf{E})$ introduces both a **fresh** variable and an **assumption**.

example: $\forall x \ \forall y \ \varphi(x,y) \ \vdash \ \forall y \ \forall x \ \varphi(x,y)$

	1	$\forall x \ \forall y \ \varphi(x,y)$	premise
У0	2		fresh y_0
x_0	3		fresh x_0
	4	$\forall y \ \varphi(x_0,y)$	$\forall x \; E, 1 \ \forall x \; E, 4$
	5	$\varphi(x_0, y_0)$	$\forall x \; E, 4$
	6	$\forall x \ \varphi(x, y_0)$	$\forall x \mid 3-5$
	7	$\forall y \ \forall x \ \varphi(x,y)$	∀y I, 2-6

example: $\forall x \ (P(x) \to Q(x)), \ \forall x \ P(x) \vdash \ \forall x \ Q(x)$

	1	$\forall x \ (P(x) \to Q(x))$	premise
	2	$\forall x P(x)$	premise
x_0	3		fresh x_0
	4	$P(x_0) \to Q(x_0)$	$\forall x \; E, 1$
	5	$P(x_0)$	$\forall x \; E, 2$
	6	$Q(x_0)$	\rightarrow E,4,5
	7	$\forall x O(x)$	∀x 1.3-6

example: $\exists x \ (\varphi(x) \lor \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x)$

	1	$\exists x \ (\varphi(x) \lor \psi(x))$		premise	
x_0	2			fresh x_0	
	3	$\varphi(x_0) \vee \psi(x_0)$		assumption	
	4	$\varphi(x_0)$	$\psi(x_0)$	assumption	
	5	$\exists x \ \varphi(x)$	$\exists x \ \psi(x)$	$\exists x \mid 1,4$	
	6	$\exists x \ \varphi(x) \lor \exists x \ \psi(x)$	$\exists x \ \varphi(x) \lor \exists x \ \psi(x)$	∨I, 5	
	7	$\exists x \ \varphi(x) \lor \exists x \ \psi(x)$		∨E, 3, 4-6	
	8	$\exists x \ \varphi(x) \lor \exists x \ \psi(x)$		$\exists x \; E, 1, 2 \text{-} 7$	

example: $\exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x) \ \vdash \ \exists x \ (\varphi(x) \lor \psi(x))$

- Yes, this is a derivable sequent left to you.
- ► Hence, $\exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x) \ \dashv \vdash \ \exists x \ (\varphi(x) \lor \psi(x))$

example: $\exists x \ (\varphi(x) \land \psi(x)) \vdash \exists x \ \varphi(x) \land \exists x \ \psi(x)$

Yes, this is a derivable sequent – similar to the formal proof of $\exists x \ (\varphi(x) \lor \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x)$

example:
$$\exists x \ (\varphi(x) \land \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \land \ \exists x \ \psi(x)$$

- Yes, this is a derivable sequent similar to the formal proof of $\exists x \ (\varphi(x) \lor \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x)$
- ▶ Question: $\exists x \ \varphi(x) \land \exists x \ \psi(x) \vdash \exists x \ (\varphi(x) \land \psi(x))$?? No, this is not a derivable sequent

example:
$$\exists x \ (\varphi(x) \land \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \land \ \exists x \ \psi(x)$$

- Yes, this is a derivable sequent similar to the formal proof of $\exists x \ (\varphi(x) \lor \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x)$
- No, this is not a derivable sequent Find an interpretation (a "model") where $\exists x \ \varphi(x) \land \exists x \ \psi(x) \vdash \exists x \ (\varphi(x) \land \psi(x))$??

 No, this is not a derivable sequent Find an interpretation (a "model") where $\exists x \ \varphi(x) \land \exists x \ \psi(x) \text{ is true}$, but $\exists x \ (\varphi(x) \land \psi(x)) \text{ is false}$
- ▶ Hence, $\exists x \ (\varphi(x) \land \psi(x)) \ \not\vdash \ \exists x \ \varphi(x) \ \land \ \exists x \ \psi(x)$

example:
$$\exists x \ (\varphi(x) \land \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \land \ \exists x \ \psi(x)$$

- Yes, this is a derivable sequent similar to the formal proof of $\exists x \ (\varphi(x) \lor \psi(x)) \ \vdash \ \exists x \ \varphi(x) \ \lor \ \exists x \ \psi(x)$
- ▶ Question: $\exists x \ \varphi(x) \ \land \ \exists x \ \psi(x) \ \vdash \ \exists x \ (\varphi(x) \land \psi(x))$?? No, this is not a derivable sequent

 Find an interpretation (a "model") where $\exists x \ \varphi(x) \ \land \ \exists x \ \psi(x) \text{ is true, but}$ $\exists x \ (\varphi(x) \land \psi(x)) \text{ is false}$
- ▶ Hence, $\exists x \ (\varphi(x) \land \psi(x)) \ \not\vdash \ \exists x \ \varphi(x) \ \land \ \exists x \ \psi(x)$

REMEMBER! To show that a WFF is **NOT** derivable, it is generally easier to find an interpretation where the WFF is not satisfiable.

example: $\exists x \ P(x), \forall x \ \forall y \ (P(x) \rightarrow Q(y)) \ \vdash \ \forall y \ Q(y)$

1	$\exists x \ P(x)$	premise
2	$\forall x \forall y (P(x) \to Q(y))$	premise

У0	3		fresh y_0
x_0	4		fresh x_0
	5	$P(x_0)$	assumption
	6	$\forall y \ (P(x_0) \to Q(y))$	$\forall x \; E, 2$
	7	$P(x_0) \to Q(y_0)$	∀y E,6
	8	$Q(y_0)$	\rightarrow E, 5, 7
	9	$Q(y_0)$	$\exists x \; E, 1, 4-8$
	10	$\forall y \ Q(y)$	∀y 1,3-9

quantifier equivalences

Theorem

$$\neg \forall x \varphi \quad \dashv \vdash \quad \exists x \neg \varphi$$
$$\neg \exists x \varphi \quad \dashv \vdash \quad \forall x \neg \varphi$$

Assume
$$x$$
 is not free in ψ :

$$\forall x \ \varphi \wedge \psi \quad \dashv \vdash \quad \forall x \ (\varphi \wedge \psi)$$

$$\forall x \ \varphi \lor \psi \quad \dashv \vdash \quad \forall x \ (\varphi \lor \psi)$$

$$\exists x \ \varphi \wedge \psi \quad \dashv \vdash \quad \exists x \ (\varphi \wedge \psi)$$

$$\exists x \ \varphi \lor \psi \quad \dashv \vdash \quad \exists x \ (\varphi \lor \psi)$$

$$\forall x \ (\psi \to \varphi) \quad \dashv \vdash \quad \psi \to \forall x \ \varphi$$

$$\exists x \ (\varphi \to \psi) \quad \dashv \vdash \quad \forall x \ \varphi \to \psi$$

$$\forall x \ (\varphi \to \psi) \quad \dashv \vdash \quad \exists x \ \varphi \to \psi$$
$$\exists x \ (\psi \to \varphi) \quad \dashv \vdash \quad \psi \to \exists x \ \varphi$$

$$\forall x \ \varphi \land \forall x \ \psi \quad \dashv \vdash \quad \forall x \ (\varphi \land \psi)$$

$$\exists x \ \varphi \lor \exists x \ \psi \quad \dashv \vdash \quad \exists x \ (\varphi \lor \psi)$$

proof of only one quantifier equivalence, others in the book

$$ightharpoonup \neg \forall x \varphi \vdash \exists x \neg \varphi$$

	1	$\neg \forall x \varphi$	premise
	2	$\neg \exists x \ \neg \varphi$	assumption
x_0	3		fresh x_0
	4	$\neg \varphi[x_0/x]$	assumption
	5	$\exists x \neg \varphi$	$\exists x \mid 1,4$
	6		$\neg E, 5, 2$
	7	$\varphi[x_0/x]$	PBC, 4-6
	8	$\forall x \ \varphi$	$\forall x \mid 3-7$
	9	Τ	$\neg E, 8, 1$
	10	$\exists x \neg \varphi$	PBC, 2-9

three fundamental questions

Question

Given a WFF φ , can we automate the answer to the query " $\vdash \varphi$??"

Question

Given a WFF φ , can we automate the answer to the query " $\forall \varphi$??"

Question

Given a formal proof

- 1. φ_1
- 2. φ_2
- 3. ÷
- $n. \varphi_n$

can we automate the verification of the proof?

(THIS PAGE INTENTIONALLY LEFT BLANK)