Chapter 3

Equality Logic (eL)

Equality logic (eL) is a very restrictive sublogic of *first-order logic* (FOL). We assume eL and FOL are over the same infinite set X of first-order variables. In eL, all atomic wff's are of the form $(x_i \approx x_j)$ for some $x_i, x_j \in X$. Though quite drastic, the restriction of FOL to eL is still capable of expressing non-trivial properties of first-order models.

The set of all eL wff's is denoted $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. Our first task is to translate any set Γ of wff's in eL, *i.e.*, $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$, to a set $\Gamma' \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ where $\mathcal{Q}$ is a set of doubly-indexed propositional variables:

$$\mathcal{Q} \stackrel{\text{def}}{=} \{\, q_{i,j} \mid i, j \in \mathbb{N} \,\}.$$

We do this in Lemma 49. Our propositional wff's here are not over the set $\mathcal{P}$ of singly-indexed variables in Section 1. This is a convenience to make the translation in the lemma a little easier and more transparent.

The semantics of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ requires a truth assignment $\sigma : \mathcal{Q} \to \{\mathit{true}, \mathit{false}\}$, the semantics of $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ requires a structure $\mathcal{A} \stackrel{\text{def}}{=} (A, =, \ldots)$ together with a valuation $\tau : X \to A$. The details are in Appendix B.3. We use propositional variable $q_{i,j}$ to represent the equality $(x_i \approx x_j)$ between first-order variables x_i and x_j. Thus, we want that $\sigma(q_{i,j}) = \mathit{true}$ iff $\tau(x_i) = \tau(x_j)$ in $\mathcal{A}$. To that end, we define a set of propositional wff's $\Delta(S)$ relative to a set $S \subseteq \mathbb{N}$ of indices as follows:

$$\begin{aligned}
\Delta(S) \stackrel{\text{def}}{=} & \{\, (\top \to q_{i,i}) \mid i \in S \,\} && \text{(“equality is reflexive”)} \\
\cup\ & \{\, (q_{i,j} \to q_{j,i}) \mid i, j \in S \,\} && \text{(“equality is symmetric”)} \\
\cup\ & \{\, (q_{i,j} \wedge q_{j,k} \to q_{i,k}) \mid i, j, k \in S \,\} && \text{(“equality is transitive”)}
\end{aligned}$$

Note how we use indices to model the properties of equality: reflexivity, symmetry, and transitivity.

Lemma 49. *Let Γ be a subset, finite or infinite, of $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. We can construct a set Γ' of propositional formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ such that:*

1. *Γ is finitely satisfiable iff Γ' is finitely satisfiable.*
2. *Γ is satisfiable iff Γ' is satisfiable.*

Proof. Let $S \stackrel{\text{def}}{=} \{\, i \in \mathbb{N} \mid x_i \in \mathrm{FV}(\Gamma) \,\}$. In words, S collects all the indices of first-order variables occurring in Γ. The set S may be finite or infinite.

The translation from Γ to Γ' is in two parts. We first transform each member of Γ using a function named $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}$. The desired Γ' is $\Delta(S) \cup \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$. The definition of $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}$ is by structural induction, similar to that of $\boxed{\mathsf{QPL} \mapsto \mathsf{PL}}$ in the proof of Lemma 36:

1. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(x_i \approx x_j) \stackrel{\text{def}}{=} q_{i,j}$ (for every $(x_i \approx x_j)$ in Γ)
2. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\neg\varphi) \stackrel{\text{def}}{=} \neg\, \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi)$
3. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi \wedge \psi) \stackrel{\text{def}}{=} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi) \wedge \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\psi)$
4. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi \vee \psi) \stackrel{\text{def}}{=} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi) \vee \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\psi)$
5. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi \to \psi) \stackrel{\text{def}}{=} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi) \to \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\psi)$

In words, all that $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}$ does is to replace every atomic wff of the form $(x_i \approx x_j)$ by the variable $q_{i,j}$. □

Exercise 50. Define a small subset $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ – no more than two or three wff's – such that:

1. Γ is not satisfiable, *i.e.*, for every interpretation $(\mathcal{A}, \tau)$ we have $\mathcal{A}, \tau \not\models_{\mathsf{eL}} \Gamma$.
2. However, $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$ is satisfiable, *i.e.*, there is a truth assignment σ such that $\sigma \models_{\mathsf{PL}} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$.
3. But, as predicted by the preceding lemma, $\Delta(S) \cup \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$ is not satisfiable, *i.e.*, for every truth assignment σ we have $\sigma \not\models_{\mathsf{PL}} \Delta(S) \cup \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$, where S is the set of variable indices occurring in Γ.

This shows that, in the proof of Lemma 49, we cannot omit $\Delta(S)$ in the definition of Γ'. □

3.1 Compactness and Completeness in eL

We follow the same sequence as in Section 2.1.

Theorem 51 (Compactness for Equality Logic, Version I)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. *It then holds that* Γ *is satisfiable iff* Γ *is finitely satisfiable.*

Proof. This proof is identical to the proof of Theorem 40, after replacing Lemma 36 by Lemma 49. □

For the next lemma and its corollary, review the formal semantics of *equality logic* in Appendix B.

Lemma 52. *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ *and* $\varphi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$*, both arbitrary. It then holds that* $\Gamma \models \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is unsatisfiable – or, equivalently,* $\Gamma \not\models \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is satisfiable.*

Proof. Identical to the proof of Lemma 6, except that here $\Gamma \cup \{\varphi\}$ is a set of wff's in *equality logic*. □

Corollary 53 (Compactness for Equality Logic, Version II)**.** *Let* $\Gamma \cup \{\varphi\} \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ *with* Γ *being possibly infinite. It then holds that* $\Gamma \models \varphi$ *iff there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \models \varphi$.

Proof. Identical to the proof of Corollary 7, except that here $\Gamma \cup \{\varphi\}$ is a set of wff's in *equality logic*. Moreover, here we invoke Lemma 52 instead of Lemma 6, and Theorem 51 instead of Theorem 2. □

The statements of Lemma 10 (the *Deduction Theorem*) and Lemma 12, as well as their respective proofs, hold verbatim for *equality logic* – except that "$\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$" has to be replaced by "$\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$" throughout.

Theorem 54 (Completeness for Equality Logic)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ and $\psi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$, both arbitrary, with Γ being possibly infinite. If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. Identical to the proof of Theorem 13, except that all the wff's are now wff's of *equality logic*. □

3.2 Applications and Exercises

In many ways, eL is a very weak logic. Nonetheless, it can still express non-trivial properties of its models.

Example 55 (*Infiniteness is* eL*-Expressible*)**.** Can we write a set Γ of wff's in eL such that every interpretation $(\mathcal{A}, \sigma)$ satisfying Γ is infinite? The following Γ will do:

$$\Gamma \stackrel{\text{def}}{=} \{\, \neg(x_i \approx x_j) \mid i, j \in \mathbb{N} \text{ and } i \neq j \,\}.$$

The justification is very simple. Take an arbitrary $(\mathcal{A}, \sigma)$ where the universe of $\mathcal{A}$ is a set A. If $\mathcal{A}, \sigma \models \Gamma$, then the valuation $\sigma : X \to A$ must assign a distinct element of A to every variable in X. Since X is infinite, we necessarily have that $\sigma(x_i) \neq \sigma(x_j)$ for all $i \neq j$, and the desired conclusion follows.

Remark 56. We have not invoked Compactness in Example 55, because it does not give us as much as we want. It is possible to invoke it by stating: *Every finite subset of* Γ *is satisfiable and therefore* Γ *is satisfiable*. But the conclusion that Γ is satisfiable only means that *there exists* an interpretation $(\mathcal{A}, \sigma)$ for Γ, not that *every* interpretation $(\mathcal{A}, \sigma)$ satisfies Γ. So, if we want to show that Γ is a formal specification of *all* infinite models, it does not help to invoke Compactness.

Can Γ in Example 55 distinguish between different infinite models? For example, can Γ distinguish between a model whose universe is $\mathbb{N}$ and another model whose universe is $\mathbb{R}$? No, it cannot. For our Γ here, all infinite cardinalities are the same. We consider this question again in later sections of these notes. □

Exercise 57 (*Finiteness is* eL*-Ineffable*)**.** In contrast to Example 55, we have the following facts:

1. There does not exist a set Δ of wff's in eL such that, for every interpretation $(\mathcal{A}, \sigma)$, we have $\mathcal{A}, \sigma \models \Delta$ iff $\mathcal{A}$ is finite.

   Hint: Assume otherwise and invoke Compactness to get a contradiction. You may want to use Γ from Example 55.

2. Let $n \geqslant 1$, a fixed positive integer. Show there exists a set Δ_n of wff's in eL such that, for every interpretation $(\mathcal{A}, \sigma)$, if we have $\mathcal{A}, \sigma \models \Delta_n$ then the universe A of $\mathcal{A}$ has n elements.

   Conversely, show that if the universe A of $\mathcal{A}$ has n elements, then there is a valuation $\sigma : X \to A$ such that $\mathcal{A}, \sigma \models \Delta_n$.

Give a precise argument for each of the two preceding facts. Thus, while finiteness in general is inexpressible in eL, finiteness of a fixed cardinality n is. □

(MORE TO COME)