Solutions to CS511 Homework 05

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Exercise 1 [LCS, page 160]: Exercise 2.3.1, do parts (a) and (b) only

Prove the validity of the following sequents using, among others, the rules =i and =e. Make sure that you indicate for each application of =e what the rule instances ϕ, t_1 and t_2 are.

Use $\approx$, instead of =, for the formal symbol whose interpretation is equality. In LaTeX, you can typeset with "*approx*"

(a) $(y = 0) \land (y = x) \vdash 0 = x$

(b) $t_1 = t_2 \vdash (t + t_2) = (t + t_1)$

Solutions:

(a): $(y \approx 0) \land (y \approx x) \vdash 0 \approx x$

1. $(y \approx 0) \land (y \approx x)$	[Premise]
2. $y \approx 0$	[$\land$-elim, 1]
3. $y \approx x$	[$\land$-elim, 1]
4. $0 \approx y$	[$\approx$e, $\phi : z \approx z$, $t_1 : 0$, $t_2 : y$, 2]
5. $0 \approx x$	[$\approx$e, $\phi : 0 \approx z$, $t_1 : y$, $t_2 : x$, 4, 3]

(b): $t_1 \approx t_2 \vdash (t + t_2) \approx (t + t_1)$

1.	$t_1 \approx t_2$	[Premise]
2.	$(t + t_1) \approx (t + t_1)$	[$\approx$i]
3.	$(t + t_2) \approx (t + t_1)$	[$\approx$e, $\phi : (t + z) \approx (t + t_1)$, $t_1 : t_1$, $t_2 : t_2$, 2, 1]

Exercise 2 [Lecture Slides 13, page 19]: Do the exercise on that page

Formalized Simplified Definition of Substitution

Solution:

BNF Definition (Modified) with Free Variables:

```
<wff> ::= T | F | <var> | ~<wff> | (<wff> & <wff>) | (<wff> | <wff>) |
          (<wff> -> <wff>) | A<var><wff> | E<var><wff>
```

Where:

- For $\phi = \top$ or $\bot$, $FV(\phi) = \emptyset$
- For $\phi = x$ (a variable), $FV(\phi) = \{x\}$
- For $\phi = \neg\psi$, $FV(\phi) = FV(\psi)$
- For $\phi = (\psi_1 \star \psi_2)$ where $\star \in \{\wedge, \vee, \rightarrow\}$, $FV(\phi) = FV(\psi_1) \cup FV(\psi_2)$
- For $\phi = Qx\psi$ where $Q \in \{\forall, \exists\}$, $FV(\phi) = FV(\psi) \setminus \{x\}$

Some additional constraints:

1. In any wff, there is at most one binding occurrence for each variable.
2. If $x \in FV(\phi)$, then x does not occur bound in ϕ.

Simplified Substitution Definition:

Given the constraints, we can simplify the substitution definition as follows:

$$\phi[u/x] = \begin{cases} \phi & \text{if } \phi = \top \text{ or } \bot \\ u & \text{if } \phi = x \\ \phi & \text{if } \phi = y \text{ and } x \neq y \\ \neg(\psi[u/x]) & \text{if } \phi = \neg\psi \\ \psi_1[u/x] \star \psi_2[u/x] & \text{if } \phi = \psi_1 \star \psi_2 \text{ and } \star \in \{\wedge, \vee, \rightarrow\} \\ Qy(\psi[u/x]) & \text{if } \phi = Qy\psi, Q \in \{\forall, \exists\}, \text{ and } x \neq y \\ \phi & \text{if } \phi = Qx\psi \text{ and } Q \in \{\forall, \exists\} \end{cases}$$

Where:

- u is $\top$, $\bot$, or a variable
- The substitution is only defined if u is substitutable for x in ϕ, which is always true under our constraints.

Justification for the Simplification above:

- The case "ϕ if $\phi = Qy\phi', Q \in \{\forall, \exists\}, x = y$" is now covered by the last case "ϕ if $\phi = Qx\psi$ and $Q \in \{\forall, \exists\}$" because there's at most one binding occurrence for each variable.
- We don't need to check if u is substitutable for x in ϕ in the quantifier case because:
  1. There's at most one binding occurrence for each variable.
  2. A variable cannot have both free and bound occurrences.

  These conditions ensure that there's no risk of variable capture during substitution.
- The case for variables is simplified because we don't need to distinguish between free and bound occurrences of variables anymore.

PROBLEM 1 [EML.Chapter 1.pdf, page 15-16]: Do parts 2, 3, and 4 of Exercise 27. .

n-Queens Problem (continued)

(2). Infinite Chessboard Argument

The argument for the Infinite Queens Problem is flawed for the following reason:

While $\Gamma = \{\psi_n | n \geq 4\}$ is indeed finitely satisfiable (as any finite subset corresponds to a finite n-Queens problem which has a solution), the satisfaction of all ψ_n for $n \geq 4$ does not necessarily imply a solution to the Infinite Queens Problem.

The key issue is that each ψ_n only constrains a finite $n \times n$ portion of the infinite board. A model satisfying all ψ_n might place queens in a way that satisfies each finite constraint, but could potentially place infinitely many queens in some rows, columns, or diagonals when considering the entire infinite board.

In other words, the conditions (a), (b), (c), and (d) for all $n \geq 4$ do not fully capture the constraints of the Infinite Queens Problem, which requires that no two queens attack each other on the entire infinite board.

(3). Defining Θ

Let's define $\Theta = \{\theta_k | k \geq 1\}$ as follows:

$$\theta_k = \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{k} (q_{i,j} \leftrightarrow (i + j \equiv k + 1 \pmod{k})) \\ \wedge \bigwedge_{i=k+1}^{\infty} \bigwedge_{j=k+1}^{\infty} \neg q_{i,j}$$

This definition satisfies the required conditions:

(a) Each θ_k defines a solution to the k-Queens Problem by placing queens on the $k \times k$ board in positions where $i + j \equiv k + 1 \pmod{k}$, and no queens elsewhere.

(b) For $k' > k$, $\theta_{k'}$ defines a solution to a larger Queens Problem than θ_k.

(c) Any finite subset of Θ is satisfiable, as each θ_k defines a distinct, valid queen placement on a finite portion of the board.

(4). Solution to the Infinite Queens Problem

Using the Compactness Theorem for Propositional Logic, we can prove that the Infinite Queens Problem has a solution:

a. We have shown that every finite subset of Θ is satisfiable.

b. By the Compactness Theorem, Θ itself is satisfiable.

c. Let v be a truth assignment that satisfies all wffs in Θ.

d. Define a queen placement on the infinite board as follows: Place a queen at position (i, j) if and only if $v(q_{i,j}) = \text{true}$.

e. This placement is a valid solution to the Infinite Queens Problem because:

- For each k, the placement satisfies θ_k, ensuring a valid k-Queens solution in the $k \times k$ subboard.
- As k increases, larger portions of the board are covered with valid queen placements.
- In the limit, this results in a valid placement for the entire infinite board.

Therefore, we have rigorously shown that the Infinite Queens Problem has a solution.

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean

Exercise 3. Hint: These should be easy if you read the book. Use existential quantifiers.

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean

Exercise 4. Hint: These use existential and universal quantifiers. The existential quantifiers are used in both context and goal, but universal quantifiers only in context.

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean

PROBLEM 2. From Macbeth's book: Hint: The first will be hard unless you use the lemma listed in the book. The other two involve some computation, but they should be easy if you make use of scratch paper while solving. The very last one shows that order matters when rewriting and is similar otherwise.

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean