Solutions to CS511 Homework 09

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Exercise 1. Open EML.Chapter 6.pdf: Do Exercise 107 on page 64.

Exercise 107: (Two-Colorability of Graphs: First-Order Definable). The notion of two-colorable simple graphs coincides with the notion of bipartite simple graphs. Write an infinite set $\Gamma_{bipartite}$ of first-order sentences such that, for every simple graph G, it holds that G $\models$ $\Gamma_{bipartite}$ iff G is bipartite.

Hint: G is bipartite iff every cycle in G (possibly with repeated vertices) has even length.

Solution:

Let $\Gamma_{\text{bipartite}}$ be the set of first-order sentences that express that for every cycle of length n (where n is odd), such a cycle cannot exist in the graph. For each odd $n \geq 3$, we include a sentence ϕ_n in $\Gamma_{\text{bipartite}}$:

$$\phi_n := \forall x_1 \ldots \forall x_n \left(\bigwedge_{i=1}^{n-1} E(x_i, x_{i+1}) \wedge E(x_n, x_1) \rightarrow \bigvee_{1 \leq i < j \leq n} x_i \approx x_j \right)$$

Then:

$$\Gamma_{\text{bipartite}} := \{\phi_n \mid n \geq 3 \text{ and } n \text{ is odd}\}$$

This works because:

- Each ϕ_n says "there cannot be a cycle of length n" where n is odd
- The formula enforces that if we have n vertices connected in a cycle, at least two must be the same vertex
- A graph models $\Gamma_{\text{bipartite}}$ if and only if it has no odd cycles
- By the characterization of bipartite graphs, a graph is bipartite if and only if it has no odd cycles

Therefore, $G \models \Gamma_{\text{bipartite}}$ if and only if G is bipartite.

Exercise 2. [LCS, page 163]: Do Exercise 2.4.6 (the last on that page).

Consider the three sentences:

$\phi 1 \stackrel{\text{def}}{=} \forall x P(x, x)$

$\phi 2 \stackrel{\text{def}}{=} \forall x \forall y (P(x, y) \to P(y, x))$

$\phi 3 \stackrel{\text{def}}{=} \forall x \forall y \forall z ((P(x, y) \land P(y, z) \to P(x, z)))$

which express that the binary predicate P is reflexive, symmetric and transitive, respectively. Show that none of these sentences is semantically entailed by the other ones by choosing for each pair of sentences above a model which satisfies these two, but not the third sentence – essentially, you are asked to find three binary relations, each satisfying just two of these properties.

Solution:

We can show that none of these sentences semantically entails the others, by first finding three different models:

1. A model satisfying ϕ_2 and ϕ_3 but not ϕ_1 (symmetric and transitive but not reflexive)
2. A model satisfying ϕ_1 and ϕ_3 but not ϕ_2 (reflexive and transitive but not symmetric)
3. A model satisfying ϕ_1 and ϕ_2 but not ϕ_3 (reflexive and symmetric but not transitive)

Now, we'll construct these models using simple binary relations on small sets:

Model 1: (symmetric and transitive but not reflexive)

- Domain: $A = \{1, 2\}$
- Relation: $P = \emptyset$ (empty relation)
- This is symmetric (vacuously) and transitive (vacuously) but not reflexive since $P(1, 1)$ and $P(2, 2)$ don't hold ß

Model 2: (reflexive and transitive but not symmetric)

- Domain: $A = \{1, 2\}$
- Relation: $P = \{(1, 1), (2, 2), (1, 2)\}$
- This is reflexive (all (x, x) included) and transitive, but not symmetric since $(1, 2)$ is in P but $(2, 1)$ is not

Model 3: (reflexive and symmetric but not transitive)

- Domain: $A = \{1, 2, 3\}$

- Relation: $P = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
- This is reflexive (all (x, x) included) and symmetric, but not transitive since $(1, 2)$ and $(2, 3)$ are in P but $(1, 3)$ is not

Therefore, it is clear that each sentence is independent of the others.

PROBLEM 1 Open Lecture Slides 26: Do the two parts of the exercise on page 7.

Part 1:

Exercise: Let $\phi(x, y)$ be an atomic WFF with free variables x and y, and f a unary function symbol not appearing in ϕ.

1. Show that the sentence $\forall x \phi(x, f(x)) \to \forall x \exists y \phi(x, y)$ is semantically valid, i.e., the following sequent is formally derivable:

$\vdash \forall x \phi(x, f(x)) \to \forall x \exists y \phi(x, y)$

Hint: Use any of the available methods, i.e., try to find a formal proof or try a semantic approach to show $\models \forall x \phi(x, f(x)) \to \forall x \exists y \phi(x, y)$ and then invoke the completeness of the proof rules.

Solution:

1. To show $\vdash \forall x \phi(x, f(x)) \to \forall x \exists y \phi(x, y)$

is valid, we use a semantic approach:

Suppose $M \models \forall x \phi(x, f(x))$ for some model M.

We need to show

$$M \models \forall x \exists y \phi(x, y)$$

Let a be any element in the universe of M.
We need to show $M \models \exists y \phi(a, y)$.

From $M \models \forall x \phi(x, f(x))$,

we know that $M \models \phi(a, f(a))$.

Let $b = f^M(a)$. Then $M \models \phi(a, b)$, which means $M \models \exists y \phi(a, y)$.

Since a was arbitrary, $M \models \forall x \exists y \phi(x, y)$.

Therefore, by completeness,

$$\vdash \forall x \phi(x, f(x)) \to \forall x \exists y \phi(x, y)$$

Part 2:

2. Show that the sentence $\forall x \exists y \phi(x, y) \to \forall x \phi(x, f(x))$ is NOT semanticalle valid, i.e., the following sequent is NOT derivable:

$\vdash \forall x \exists y \phi(x, y) \to \forall x \phi(x, f(x))$

Hint: Try a semantic approach, i.e., define an appropriate ϕ and a model where the left-hand side of "$\to$" is true but the right-hand side of "$\to$" is false, and then invoke the completeness of the proof rules.

Solution:

2. To show the sentence is not valid, we construct a counterexample:

Let $\phi(x, y)$ be the atomic formula $x < y$ interpreted over the real numbers $\mathbb{R}$.

Define $f(x) = x$ for all $x \in \mathbb{R}$.

Then:

- $\forall x \exists y \phi(x, y)$ is true because for any real number x, there exists a larger real number y
- However, $\forall x \phi(x, f(x))$ is false because $\phi(x, f(x))$ means $x < x$, which is false for all x

Therefore, we have found a model where $\forall x \exists y \phi(x, y)$ is true but $\forall x \phi(x, f(x))$ is false.
By completeness,

$$\nvdash \forall x \exists y \phi(x, y) \to \forall x \phi(x, f(x))$$

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw09/hw09_nicholas_ikechukwu.lean

Exercise 3. From Macbeth's book:

Solutions

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw09/hw09_nicholas_ikechukwu.lean

Exercise 4. From Macbeth's book

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw09/hw09_nicholas_ikechukwu.lean

PROBLEM 2. From Macbeth's book

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw09/hw09_nicholas_ikechukwu.lean