CS 511, Fall 2024, Lecture Slides 29, Part III

Analytic Tableaux for Classical First-Order Logic

(How to Handle Equality)

Assaf Kfoury

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How to Handle Equality with the Tableau Method

There are three basic approaches when $\approx$ is introduced in the syntax of FOL:

Approach 1: We introduce a new binary predicate symbol "eq" and substitute it for every occurrence of "$\approx$". After which, we explicitly axiomatize "eq" as a congruence relation and include the axioms in the set of premises.

- An axiomatization of "eq" as a congruence is in Definition 61, pages 41-42, in [EML.Chapter_4.pdf] (posted on Piazza).
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Approach 3: (Applicable to second tableau method, *free-variable tableaux*, only.) We take equalities and inequalities between terms into account when searching for substitutions to close paths, using what is called **E-unification** ("unification modulo an equational theory").

- The most sophisticated of the three approaches and the most efficient in implementations. Nothing about E-unification in these slides (lack of time!).

Adding Expansion and Closure Rules

Alternative ways of adding expansion/closure rules to the *Approach 2* :

- For the *ground-tableaux* method, we can add what are called ***Jeffrey's Rules*** (3 rules):

$$\frac{\begin{array}{c}\varphi(t)\\ t \approx s\end{array}}{\varphi(s)} \qquad \frac{\begin{array}{c}\varphi(t)\\ s \approx t\end{array}}{\varphi(s)} \qquad \frac{\neg(t \approx t)}{\mathsf{X}}$$

  where $\{t, s\}$ are arbitrary ground terms, and wff $\varphi(s)$ is obtained by substituting s for one occurence of t in wff $\varphi(t)$.

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- For the *ground-tableaux* method, we can alternatively add ***Reeves' Rules*** (4 rules):

$$\frac{\begin{matrix}P(t_1,\ldots,t_n)\\ \neg P(s_1,\ldots,s_n)\end{matrix}}{\neg\big((t_1 \approx s_1)\wedge\cdots\wedge(t_n \approx s_n)\big)} \qquad \frac{\neg\big(f(t_1,\ldots,t_n) \approx f(s_1,\ldots,s_n)\big)}{\neg\big((t_1 \approx s_1)\wedge\cdots\wedge(t_n \approx s_n)\big)} \qquad \frac{t \approx s}{s \approx t} \qquad \frac{\neg(t \approx t)}{\mathsf{X}}$$

  where $\{t, s, t_1, \ldots, t_n.s_1, \ldots, s_n\}$ are arbitrary ground terms, P is an arbitrary n-ary predicate symbol, and f is an arbitrary n-ary function symbol.

example for *ground-tableaux* method (first TABLEAU method)

with ***Jeffrey's Rules***: Prove validity of $\varphi \stackrel{\text{def}}{=} \exists x \forall y (x \approx y) \to \forall v \forall w (v \approx w)$

φ is equivalent to $\neg(\exists x \forall y (x \approx y)) \vee \forall v \forall w (v \approx w)$

EXPANSION RULES

$(\wedge) \quad \dfrac{\varphi \wedge \psi}{\begin{array}{l}\varphi \\ \psi\end{array}}$

$(\neg\wedge) \quad \dfrac{\neg(\varphi \wedge \psi)}{\neg\varphi \mid \neg\psi}$

$\vdots$

$(\neg\neg) \quad \dfrac{\neg\neg\varphi}{\varphi}$

$(\forall) \quad \dfrac{\forall x\, \varphi(x)}{\varphi[x := t]}$

$(\exists) \quad \dfrac{\exists x\, \varphi(x)}{\varphi[x := c]}$

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$\boxed{1:0}$	$\neg(\neg(\exists x \forall y (x \approx y)) \vee \forall v \forall w (v \approx w))$
$\boxed{2:1}$	$\neg\neg\exists x \forall y (x \approx y)$
$\boxed{3:1}$	$\neg\forall v \forall w (v \approx w)$
$\boxed{4:2}$	$\exists x \forall y (x \approx y)$
$\boxed{5:3}$	$\neg\forall w (a \approx w)$
$\boxed{6:5}$	$\neg(a \approx b)$
$\boxed{7:4}$	$\forall y (c \approx y)$
$\boxed{8:7}$	$(c \approx a)$
$\boxed{9:7}$	$(c \approx b)$
$\boxed{10:8,9}$	$(a \approx b)$
$\boxed{11:10,6}$	$\neg(a \approx a)$
	X

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$(\forall)$ $\dfrac{\forall x\, \varphi(x)}{\varphi[x := t]}$

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Adding Expansion and Closure Rules

Alternative ways of adding expansion/closure rules to the *Approach 2*:

- For the *free-variable* method, we can add what are called ***Fitting's Rules*** (3 rules):

$$1: \quad \frac{T}{\sigma(T) \oplus_\pi \varphi(s)} \qquad \pi \in \mathit{paths}(T), \quad \{\varphi(t), (t' \approx s)\} \subseteq \mathit{WFF's}(\pi), \quad \sigma = \mathit{MGU}(t, t')$$

$$2: \quad \frac{T}{\sigma(T) \oplus_\pi \varphi(s)} \qquad \pi \in \mathit{paths}(T), \quad \{\varphi(t), (s \approx t')\} \subseteq \mathit{WFF's}(\pi), \quad \sigma = \mathit{MGU}(t, t')$$

$$3: \quad \frac{T}{\sigma(T) \oplus_\pi \times} \qquad \pi \in \mathit{paths}(T), \quad \{\neg(t \approx t')\} \subseteq \mathit{WFF's}(\pi), \quad \sigma = \mathit{MGU}(t, t')$$

A simpler alternative to the third rule is:

$$3': \quad \frac{T \oplus_\pi \neg(t \approx t')}{\sigma(T) \oplus_\pi \times} \qquad \pi \in \mathit{paths}(T), \quad \sigma = \mathit{MGU}(t, t')$$

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For more details and elaboration on how to handle equality with tableaux for FOL, consult:

B. Beckert, "Semantic Tableaux with Equality," *J. of Logic and Computation*, 7(1):39-58, 1997.

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