Solutions to CS511 Homework 01

Nicholas Ikechukwu - U71641768

September 19, 2024

Exercise 1 Go to page 9 in Lecture Slides 06. Your task is to carefully write all the details of the proof by structural induction. These details are not included in the slides.

Solution

Concise Proof by Structural Induction

Proposition: $\forall s, t \in A^*, \text{reverse}(s \cdot t) = \text{reverse}(t) \cdot \text{reverse}(s)$
Let $P(t) := \forall s \in A^*, \text{reverse}(s \cdot t) = \text{reverse}(t) \cdot \text{reverse}(s)$
Proof by structural induction on t:

1. Base case: $t = \varepsilon$

   $\forall s \in A^*, \text{reverse}(s \cdot \varepsilon) = \text{reverse}(s) = \varepsilon \cdot \text{reverse}(s) = \text{reverse}(\varepsilon) \cdot \text{reverse}(s)$

2. Inductive step: Assume $P(t)$ holds for $t \in A^*$. Show $P(a \cdot t)$ for $a \in A$.

   $\forall s \in A^*$:

$$\begin{aligned}
\text{reverse}(s \cdot (a \cdot t)) &= \text{reverse}((s \cdot a) \cdot t) && \text{[associativity]} \\
&= \text{reverse}(t) \cdot \text{reverse}(s \cdot a) && \text{[I.H.]} \\
&= \text{reverse}(t) \cdot (\text{reverse}(a) \cdot \text{reverse}(s)) && \text{[def. of reverse]} \\
&= (\text{reverse}(t) \cdot \text{reverse}(a)) \cdot \text{reverse}(s) && \text{[associativity]} \\
&= \text{reverse}(a \cdot t) \cdot \text{reverse}(s) && \text{[def. of reverse]}
\end{aligned}$$

By structural induction, $P(t)$ holds $\forall t \in A^*$, proving the proposition.

Exercise 2 [LCS, page 87]: Exercise 1.4.15. Hint: You may find it helpful to review pages 20 and 21 in Lecture Slides 02.

Solution

Concise Proof by Mathematical Induction

Theorem: For $n \geq 1$,

$$((\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_n) \cdots) \to \psi) \to (\varphi_1 \to (\varphi_2 \to (\cdots (\varphi_n \to \psi) \cdots))))$$

Let $P(n)$ denote the theorem statement.
Proof:

1. Base case ($n = 1$):

   $P(1) : ((\varphi_1 \to \psi) \to (\varphi_1 \to \psi))$ [Trivially true]

2. Inductive step: Assume $P(k)$ holds for some $k \geq 1$. To prove $P(k+1)$:

   LHS of $P(k+1)$:

$$\begin{aligned}
&(\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_{k+1}) \cdots) \to \psi) \\
&\equiv ((\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_k) \cdots)) \wedge \varphi_{k+1} \to \psi) \\
&\equiv (\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_k) \cdots) \to (\varphi_{k+1} \to \psi)) \quad \text{[Deduction theorem]}
\end{aligned}$$

   Applying $P(k)$ to this:

   $(\varphi_1 \to (\varphi_2 \to (\cdots (\varphi_k \to (\varphi_{k+1} \to \psi)) \cdots)))$

   This is the RHS of $P(k+1)$.

Therefore, by mathematical induction, $P(n)$ holds for all $n \geq 1$.

PROBLEM 1 Show that any of the three rules (LEM),(PBC),($\neg\neg$E) are interderivable.

Solution

Interderivability of LEM, PBC, and $\neg\neg$E

We will show that the three rules Law of Excluded Middle (LEM), Proof by Contradiction (PBC), and Double Negation Elimination ($\neg\neg$E) are interderivable.

(a) (PBC) is derivable from ($\neg\neg$E)

1. $\neg\varphi \to \bot$	given
2. $\neg\varphi$	assumption
3. $\bot$	$\to$E 1, 2
4. $\neg\neg\varphi$	$\neg$I 2-3
5. φ	$\neg\neg$E 4

(b) (LEM) is derivable from (PBC)

1.	$\neg(\varphi \vee \neg\varphi)$	assumption
2.	$\quad \varphi$	assumption
3.	$\quad \varphi \vee \neg\varphi$	$\vee$I 2
4.	$\quad \bot$	$\neg$E 1, 3
5.	$\neg\varphi$	$\neg$I 2-4
6.	$\varphi \vee \neg\varphi$	$\vee$I 5
7.	$\bot$	$\neg$E 1, 6
8.	$\varphi \vee \neg\varphi$	PBC 1-7

(c) ($\neg\neg$E) is derivable from (LEM)

1.	$\neg\neg\varphi$	premise
2.	$\varphi \vee \neg\varphi$	LEM
3.	$\quad\varphi$	assumption
4.	$\quad\varphi$	reiteration 3
5.	$\quad\neg\varphi$	assumption
6.	$\quad\bot$	$\neg$E 1, 5
7.	$\quad\varphi$	$\bot$E 6
8.	φ	$\vee$E 2, 3-4, 5-7

Therefore, we have shown that ($\neg\neg$E) $\Rightarrow$ (PBC) $\Rightarrow$ (LEM) $\Rightarrow$ ($\neg\neg$E), proving that these three rules are interderivable.

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw02/hw02_nicholas_ikechukwu.lean

Exercise 3 For each of the three examples in the following three sections of Macbeth's book, your task is to remove 'sorry' and insert appropriate Lean 4 tactics

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw02/hw02_nicholas_ikechukwu.lean

Exercise 4 For each of the three examples in the following three sections of Macbeth's book, your task is to remove 'sorry' and insert appropriate Lean 4 tactics.

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw02/hw02_nicholas_ikechukwu.lean

PROBLEM 2 For each of the three examples in the following three sections of Macbeth's book, your task is to remove 'sorry' and insert appropriate Lean 4 tactics

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw02/hw02_nicholas_ikechukwu.lean