CS 511, Fall 2024, Lecture Slides 21

Extended Example in First-Order Logic

Assaf Kfoury

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several structures over the domain $\mathbb{N}$ (assume "$\approx$" is available)

structures over the domain of natural numbers	vocabulary/signature: predicate symbols	function symbols
$\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$	$\mathscr{P} = \varnothing$	$\mathscr{F} = \{0, S\}$
$\mathcal{N}_1 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <)$	$\mathscr{P} = \{<\}$	$\mathscr{F} = \{0, S\}$
$\mathcal{N}_2 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +)$	$\mathscr{P} = \{<\}$	$\mathscr{F} = \{0, S, +\}$
$\mathcal{N}_3 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot)$	$\mathscr{P} = \{<\}$	$\mathscr{F} = \{0, S, +, \cdot\}$
$\mathcal{N}_4 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot, \text{pr})$ $\text{pr}(x) \stackrel{\text{def}}{=}$ true iff x is prime	$\mathscr{P} = \{<, \text{pr}\}$	$\mathscr{F} = \{0, S, +, \cdot\}$
$\mathcal{N}_5 \stackrel{\text{def}}{=} (\mathbb{N}, 0, S, <, +, \cdot, \text{pr}, \uparrow)$ $x \uparrow y \stackrel{\text{def}}{=} x^y$	$\mathscr{P} = \{<, \text{pr}\}$	$\mathscr{F} = \{0, S, +, \cdot, \uparrow\}$
$\mathcal{N}_6 \stackrel{\text{def}}{=} \ldots$		

Question: Is a new predicate (function) definable from earlier ones?

first-order definability over $\mathbb{N}$

- every number n is definable from 0 and S :

$$
\begin{aligned}
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&\;\;\cdots\\
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  which implies the graph of $S^{\mathcal{N}_2}$ is defined by $\varphi_S(x, y) = (x + 1 \approx y)$.

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  for all $m, n, p \in \mathbb{N}$, we have $m + n = p$ iff $\underbrace{S(\cdots S}_{n}(m)\cdots) = p$

  "formally": $\forall x \forall y \forall z\ [\ \underbrace{S(\cdots S}_{y}(x)\cdots) \approx z\ \leftrightarrow\ x + y \approx z\ ]$

  so perhaps $\varphi_+(x, y, z) \stackrel{\text{def}}{=} (\underbrace{S(\cdots S}_{y}(x)\cdots) \approx z\ \ \ldots)$ Not quite!

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   Hint. Use the following equivalence for all $m, n, p \in \mathbb{N}$
   $(p = 0) \vee (p = m + n)$ iff
   $(m \cdot p + 1) \cdot (n \cdot p + 1) = p^2 \cdot (m \cdot n + 1) + 1$

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  YES $m = n \uparrow p$ iff $\varphi(m, n, p)$ is true, where $\varphi(x, y, z)$ is the WFF ... *(not very difficult: try it!)*

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