Solutions to CS511 Homework 05

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Exercise 1. [LCS, page 160]: Exercise 2.3.1, do parts (a) and (b) only

Prove the validity of the following sequents using, among others, the rules =i and =e. Make sure that you indicate for each application of =e what the rule instances ϕ, t_1 and t_2 are.

Use $\approx$, instead of =, for the formal symbol whose interpretation is equality. In LaTeX, you can typeset with "*approx*"

(a) $(y = 0) \wedge (y = x) \vdash 0 = x$

(b) $t_1 = t_2 \vdash (t + t_2) = (t + t_1)$

Solutions:

(a) $(y \approx 0) \wedge (y \approx x) \vdash 0 \approx x$

(a) $(y \approx 0) \wedge (y \approx x)$ [Premise]

(b) $y \approx 0$ [$\wedge$ elimination, 1]

(c) $y \approx x$ [$\wedge$ elimination, 1]

(d) $0 \approx y$ [=e: $\varphi(z) := (z \approx y)$, $t_1 := y$, $t_2 := 0$, from 2]

(e) $0 \approx x$ [=e: $\varphi(z) := (0 \approx z)$, $t_1 := y$, $t_2 := x$, from 4, 3]

(b) $t_1 \approx t_2 \vdash (t + t_2) \approx (t + t_1)$

(a)	$t_1 \approx t_2$	[Premise]
(b)	$(t + t_1) \approx (t + t_1)$	[=i]
(c)	$(t + t_2) \approx (t + t_1)$	[=e: $\varphi(z) := ((t + z) \approx (t + t_1))$, $t_1 := t_1$, $t_2 := t_2$, from 1, 2]

Exercise 2. LCS, page 161: Exercise 2.3.9, do parts (a) and (d) only.

Prove the validity of the following sequents in predicate logic, where F , G, P , and Q have arity 1, and S has arity 0 (a 'propositional atom'):

- (a) $\exists x(S \to Q(x)) \vdash S \to \exists x Q(x)$
- (d) $\forall x P(x) \to S \vdash \exists x(P(x) \to S)$

Solutions:

(a) $\exists x(S \to Q(x)) \vdash S \to \exists x Q(x)$

Let $\mathcal{I}$ be any interpretation in which $\exists x(S \to Q(x))$ is true.

1. There exists some element a in the domain such that $S \to Q(a)$ is true in $\mathcal{I}$.
2. To prove $S \to \exists x Q(x)$, consider two cases for S:
3. Case 1: If S is false in $\mathcal{I}$, then $S \to \exists x Q(x)$ is trivially true.
4. Case 2: If S is true in $\mathcal{I}$, then:
   (a) $Q(a)$ must be true in $\mathcal{I}$ (from steps 1 and 4).
   (b) Therefore, $\exists x Q(x)$ is true in $\mathcal{I}$.
   (c) Hence, $S \to \exists x Q(x)$ is true in $\mathcal{I}$.
5. In both cases, $S \to \exists x Q(x)$ is true in $\mathcal{I}$. Thus, whenever $\exists x(S \to Q(x))$ is true in an interpretation, $S \to \exists x Q(x)$ is also true in that interpretation, proving the validity of the sequent.

(d) $\forall x P(x) \to S \vdash \exists x (P(x) \to S)$

We prove this by contradiction:

1. Assume there exists an interpretation $\mathcal{I}$ in which $\forall x P(x) \to S$ is true but $\exists x(P(x) \to S)$ is false.
2. In $\mathcal{I}$, $\forall x \neg(P(x) \to S)$ must be true (negation of $\exists x(P(x) \to S)$).
3. This means for every element a in the domain of $\mathcal{I}$:
   (a) $P(a)$ is true and S is false.
4. Therefore, $\forall x P(x)$ is true in $\mathcal{I}$.
5. From steps 1 and 4, S must be true in $\mathcal{I}$ (by modus ponens).
6. But this contradicts step 3(a), where S is false. This contradiction shows that our assumption in step 1 must be false. Therefore, in any interpretation where $\forall x P(x) \to S$ is true, $\exists x(P(x) \to S)$ must also be true, proving the validity of the sequent.

PROBLEM 1: Let $\psi1, \psi2, and \psi3$ be the three axioms of group theory, which are written as first-order wff's on page 11 of Lecture Slides 20. Let ψ be the wff in the middle of the same page 11 of Lecture Slides 20. The wff ψ expresses the uniqueness of inverses in groups. Your task is to produce a formal proof, as a natural deduction, of the following judgment: $\psi1, \psi2, \psi3 \vdash \psi$.

Hint: Do Exercises 1 and 2 above before this problem. Also use $\approx$ for the formal symbol whose interpretation is equality, leaving $=$ for equality at the meta-level.

Solution:

Let ψ_1, ψ_2, and ψ_3 be the three axioms of group theory:

1. $\forall x(e \cdot x \approx x \wedge x \cdot e \approx x)$ (identity)
2. $\forall x \exists y(x \cdot y \approx e \wedge y \cdot x \approx e)$ (inverse)
3. $\forall x \forall y \forall z((x \cdot y) \cdot z \approx x \cdot (y \cdot z))$ (associative)

Let ϕ be the wff expressing the uniqueness of inverses in groups:

$$\phi \equiv \forall x \forall y \forall z(x \cdot y \approx e \wedge x \cdot z \approx e \to y \approx z)$$

We need to prove: $\psi_1, \psi_2, \psi_3 \vdash \phi$ Let x, y, and z be arbitrary elements of the group.

1. Assume $x \cdot y \approx e$ and $x \cdot z \approx e$.
2. From ψ_2, there exists an element x' such that $x \cdot x' \approx e$ and $x' \cdot x \approx e$.
3. By substituting e into the equation, we have $(x \cdot y) \cdot x' \approx e$.
4. By the identity axiom (ψ_1), $(x \cdot y) \cdot x' = x'$.
5. By the associative property (ψ_3), we have $x \cdot (y \cdot x') = x'$.
6. Therefore, since $e = y$, we can conclude that $y = x'$.
7. Similarly, we can show that $z = x'$.
8. Thus, we conclude that $y = z$.

Since x, y, and z were arbitrary, we have proven ϕ.

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean

Exercise 3. Hint: These should be easy if you read the book. Use existential quantifiers.

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean

Exercise 4. Hint: These use existential and universal quantifiers. The existential quantifiers are used in both context and goal, but universal quantifiers only in context.

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean

PROBLEM 2. Prove in Lean 4 the judgment for which you produced a formal proof as a natural deduction in Problem 1 above.

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw05/hw05_nicholas_ikechukwu.lean