CS 511, Fall 2024, Lecture Slides 02

Natural Deduction and Examples of Natural Deduction in Propositional Logic

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Natural Deduction:
A Particular Proof System for Propositional Logic

- Reading: [LCS, Section 1.2]

  Remark: It is somewhat unconventional to present a proof system for a formal logic, such as *propositional logic* in [LCS, Section 1.2], before presenting its syntax in [LCS, Section 1.3].

- Reading: [EML.Appendix, pp 13-16]

from **informal/common** reasoning to **formal** reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  THEN John is late for the meeting
- John is **NOT** late for the meeting
- the train did arrive late
- **THEREFORE** there were taxis

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again symbolically:

- $(\quad P \quad \wedge \quad \neg Q \quad) \rightarrow \quad R$
- $\neg R$
- P
- $\vdash \quad Q$

more succintly:

$P \wedge \neg Q \rightarrow R, \;\; \neg R, \;\; P \;\vdash\; Q$

Formal Proof of the Judgment $* * *$

- a **judgment** (also called a **sequent**) is an expression of the form:

$$\varphi_1, \ldots, \varphi_n \;\vdash\; \psi$$

where:

1. $\varphi_1, \ldots, \varphi_n, \psi$ are **well-formed formulas** (also called **wff's**)
2. the symbol "$\vdash$" is pronounced **turnstile**
3. the wff's $\varphi_1, \ldots, \varphi_n$ to the left of "$\vdash$" are called the **premises** (also called **antecedents** or **hypotheses**)
4. the wff ψ to the right of "$\vdash$" is called the **conclusion** (also called **succedent**)

- a judgment is said to be **valid** (also **deducible** or **derivable**) if there is a **formal proof** for it

- a **formal proof** (also called **deduction** or **derivation**) is a sequence of wff's which starts with the **premises** of the judgment and finishes with the **conclusion** of the judgment:

$$
\begin{array}{ll}
\varphi_1 & \text{premise} \\
\varphi_2 & \text{premise} \\
\vdots & \\
\varphi_n & \text{premise} \\
\vdots & \\
\psi & \text{conclusion}
\end{array}
$$

  where every wff in the deduction is obtained from the wff's preceding it using a **proof rule**

Examples of Proof Rules

- $$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \quad \wedge\mathsf{I}$$
- $$\frac{\varphi \wedge \psi}{\varphi} \quad \wedge\mathsf{E}_1$$
- $$\frac{\varphi \wedge \psi}{\psi} \quad \wedge\mathsf{E}_2$$
- $$\frac{\varphi}{\neg\neg\varphi} \quad \neg\neg\mathsf{I}$$
- $$\frac{\neg\neg\varphi}{\varphi} \quad \neg\neg\mathsf{E}$$ (cannot be used in **intuitionistic logic**)

Examples of Proof Rules

- $$\frac{\varphi \qquad \varphi \to \psi}{\psi} \qquad \to\mathrm{E}$$ (or **MP** for **Modus Ponens**)
- $$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \qquad \mathbf{MT}$$ (for **Modus Tollens**)

Examples of Proof Rules

- $$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \quad \rightarrow\mathrm{E}$$ (or **MP** for **Modus Ponens**)

- $$\frac{\varphi \rightarrow \psi \qquad \neg\psi}{\neg\varphi} \quad \textbf{MT}$$ (for **Modus Tollens**)

- $$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \psi\end{array}}}{\varphi \rightarrow \psi} \quad \rightarrow\mathrm{I}$$

Examples of Proof Rules

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- $$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \quad \mathbf{MT}$$ (for **Modus Tollens**)

- $$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \to \psi} \quad \to\mathrm{I}$$

open a box when you *introduce* an **assumption** (wff φ in rule $\to$I)

close the box when you *discharge* the **assumption**

you must close every **box** and discharge every **assumption**
in order to complete a formal proof

Proof Rules Associated with Only One "¬" and with "⊥"

So far, we have an **elimination** rule and an **introduction** rule for double negation "¬¬", namely ¬¬E and ¬¬I, but not for single negation "¬". We now compensate for this lack:

- $$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg\mathrm{E}$$ (or **LNC** for **Law of Non-Contradiction**)

  where "⊥" (a single symbol) stands for "**contradiction**"

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- $$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \bot\end{array}}}{\neg\varphi}$$ ¬I

- $$\frac{\bot}{\varphi}$$ ⊥E ("if you can prove ⊥, you can prove every wff")

Two Derived Proof Rules

The two following rules are derived rules –
the first from rules $\rightarrow$I, $\neg$I, $\rightarrow$E, and $\neg\neg$E (see [LCS, pp 24-25]);
the second from rules $\vee$I, $\neg$I, $\neg$E, and $\neg\neg$E (see [LCS, pp 25-26]):

- $$\dfrac{\boxed{\begin{array}{l}\neg\varphi \\ \vdots \\ \bot\end{array}}}{\varphi}$$ **PBC** (for **Proof by Contradiction**)

- $$\dfrac{}{\varphi \vee \neg\varphi}$$ **LEM** (for **Law of Excluded Middle**)

Because $\neg\neg$E is rejected in **intuitionistic logic**, so are **PBC** and **LEM**

(a summary of all **proof rules** and some **derived rules** in [LCS, p. 27])

Examples of Natural Deductions

formal proof of the judgment $P \vdash Q \to (P \wedge Q)$:

Examples of Natural Deductions

formal proof of the judgment $P \vdash Q \to (P \land Q)$:

1	P	premise
2	Q	assume
3	$P \land Q$	$\land$I $1, 2$
4	$Q \to (P \land Q)$	$\to$I

Examples of Natural Deductions

formal proof of the judgment $P \vdash Q \to (P \land Q)$:

1	P	premise
2	Q	assume
3	$P \land Q$	$\land$I $1, 2$
4	$Q \to (P \land Q)$	$\to$I

translated into LEAN 4:

```
example {p q : Prop} (h_p : p) : q → (p ∧ q) := by
  intro h_q
  apply And.intro h_p h_q -- ‘And.intro’ is ‘∧I’ in LEAN 4
```

- *reserved words* are in **blue**
- *tactics* are in **red**

Examples of Natural Deductions

formal proof of the judgment $P \to (Q \to R) \vdash P \land Q \to R$

Examples of Natural Deductions

formal proof of the judgment $P \to (Q \to R) \vdash P \wedge Q \to R$

1	$P \to (Q \to R)$	
2	$P \wedge Q$	
3	P	$\wedge\mathsf{E}_1$ 2
4	Q	$\wedge\mathsf{E}_2$ 2
5	$Q \to R$	$\to\mathsf{E}$ $3, 1$
6	R	$\to\mathsf{E}$ $4, 5$
7	$P \wedge Q \to R$	$\to\mathsf{I}$

Examples of Natural Deductions

formal proof of the judgment $P \to (Q \to R) \vdash P \land Q \to R$

1	$P \to (Q \to R)$	
2	$P \land Q$	
3	P	$\land\text{E}_1$ 2
4	Q	$\land\text{E}_2$ 2
5	$Q \to R$	$\to\text{E}$ $3, 1$
6	R	$\to\text{E}$ $4, 5$
7	$P \land Q \to R$	$\to\text{I}$

translated into LEAN 4:

```
example {p q r : Prop} (h : p → (q → r)) : p ∧ q → r := by
  intro h_pq
  obtain ⟨ h_p , h_q ⟩ := h_pq
  have h_qr : q → r := by apply h h_p
  apply h_qr h_q
```

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formal proof of the judgment $P \wedge Q \to R \vdash P \to (Q \to R)$

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formal proof of the judgment $P \wedge Q \to R \vdash P \to (Q \to R)$

1	$P \wedge Q \to R$	
2	P	
3	Q	
4	$P \wedge Q$	$\wedge$I 2, 3
5	R	$\to$E 4, 1
6	$Q \to R$	$\to$I
7	$P \to (Q \to R)$	$\to$I

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formal proof of the judgment $P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

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formal proof of the judgment $P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

1	$P \to (Q \to R)$	
2	$P \to Q$	
3	P	
4	Q	$\to$E $3, 2$
5	$Q \to R$	$\to$E $3, 1$
6	R	$\to$E $4, 5$
7	$P \to R$	$\to$I
8	$(P \to Q) \to (P \to R)$	$\to$I

Formal Proof of the Initial Judgment:

Initial Judgment

1	$P \wedge \neg Q \rightarrow R$	premise
2	$\neg R$	premise
3	P	premise
4	$\neg Q$	assume
5	$P \wedge \neg Q$	$\wedge$I $3, 4$
6	R	$\rightarrow$E $1, 5$
7	$\perp$	$\neg$E $6, 2$
8	$\neg\neg Q$	$\neg$I 4–7
9	Q	$\neg\neg$E 8

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