CS 511, Fall 2024, Lecture Slides 24 – *Appendix*

Deductive Closures and First-Order Theories

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the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

Consider again the structure $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$ in Lecture Slides 21.
The first-order theory of $\mathcal{N}$ is:

$$\mathsf{Th}(\mathcal{N}) \stackrel{\text{def}}{=} \{\, \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{N} \models \varphi \,\}$$

Some sentences that are true in $\mathcal{N}$:

S1 $\quad \forall x.\, \neg(Sx \approx 0)$

S2 $\quad \forall x\, \forall y.\, (Sx \approx Sy \to x \approx y)$

S3 $\quad \forall y.\, (\neg(y \approx 0) \to \exists x\, (y \approx Sx))$

S4.1 $\quad \forall x.\, \neg(Sx \approx x)$

S4.2 $\quad \forall x.\, \neg(SSx \approx x)$

$\quad \ldots$

S4.n $\quad \forall x.\, \neg(\underbrace{S \cdots S}_{n} x \approx x)$

$\quad \ldots$

the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

(a) Graphical representation of $\mathcal{N}$, which is a model of $\{\text{S1},\ \text{S2},\ \text{S3}\}$ and infinitely many other first-order sentences:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \quad \cdots$$

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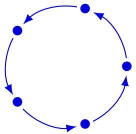
(a) Graphical representation of $\mathcal{N}$, which is a model of $\{\text{S1},\ \text{S2},\ \text{S3}\}$ and infinitely many other first-order sentences:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

(b) Another model of $\{\text{S1},\ \text{S2},\ \text{S3}\}$, different from $\mathcal{N}$:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$



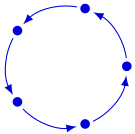
the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

(a) Graphical representation of $\mathcal{N}$, which is a model of $\{\text{S1, S2, S3}\}$ and infinitely many other first-order sentences:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

(b) Another model of $\{\text{S1, S2, S3}\}$, different from $\mathcal{N}$:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$



(c) In fact, the model in (b) satisfies $\{\text{S1, S2, S3}\} \cup \{\, \text{S4}.n \mid n \text{ not a multiple of } 5 \,\}$.

Satisfaction of $\{\, \text{S4}.n \mid n \geqslant 1 \,\}$ eliminates all cycles.

the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

(d) A model of $\{$S1, S2, S3$\} \cup \{$ S4.$n \mid n \geqslant 1 \}$ without cycles:

$$\bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \cdots$$

$$\cdots \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \bullet \xrightarrow{\;S\;} \cdots$$

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(d) A model of $\{$S1, S2, S3$\} \cup \{$ S4.$n \mid n \geqslant 1 \}$ without cycles:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

$$\cdots \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

(e) And another model of $\{$S1, S2, S3$\} \cup \{$ S4.$n \mid n \geqslant 1 \}$ without cycles:

$$\bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

$$\cdots \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

$$\cdots \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \bullet \xrightarrow{S} \cdots$$

The universe of this last model consists of one copy of $\mathbb{N}$ and two copies of $\mathbb{Z}$.

the first-order theory of $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, 0, S)$

- let $\Gamma = \{\text{S1},\ \text{S2},\ \text{S3},\ \text{S4.1},\ \text{S4.2},\ \text{S4.3}, \ldots\}$
- clearly $\mathcal{N} \models \varphi$ for every $\varphi \in \Gamma$
  so that $\Gamma \subseteq \text{Th}(\mathcal{N})$

- what can we say about the deductive closure of the set Γ above:
  $\overline{\Gamma} = \{\, \varphi \mid \varphi$ first-order sentence s.t. $\Gamma \vdash \varphi \,\}$?

- certainly $\overline{\Gamma} \subseteq \text{Th}(\mathcal{N})$, by soundness
- in fact, the equality holds:

  $$\overline{\Gamma} \;=\; \text{Th}(\mathcal{N}) \qquad \textit{(not shown here)}$$

- we therefore say that Γ is an **axiomatization** of $\text{Th}(\mathcal{N})$ because
  every sentence φ made true by $\mathcal{N}$ is formally deduced from Γ

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