CS 511, Fall 2024, Lecture Slides 18

Predicate Logic (aka First-Order Logic): Proof Rules of Natural Deduction

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proof rules for equality

- equality **introduction**

$$\frac{}{t \approx t} \approx\mathrm{I}$$

- equality **elimination**

$$\frac{t_1 \approx t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]} \approx\mathrm{E}$$

formal proof for “≈” is symmetric

1	$u_1 \approx u_2$	premise
2	$u_1 \approx u_1$	≈I
3	$u_2 \approx u_1$	≈E $1, 2$

formal proof for "$\approx$" is symmetric

1	$u_1 \approx u_2$	premise
2	$u_1 \approx u_1$	$\approx$I
3	$u_2 \approx u_1$	$\approx$E $1, 2$

Question: What above corresponds to the WFF φ in the rule $\approx$E?

Answer: "$x \approx u_1$" corresponds to φ in the rule $\approx$E, so that "$u_1 \approx u_1$" corresponds to $\varphi[u_1/x]$ & "$u_2 \approx u_1$" corresponds to $\varphi[u_2/x]$

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2	$u_1 \approx u_1$	≈I
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Question: What above corresponds to the WFF φ in the rule ≈E?
Answer: "$x \approx u_1$" corresponds to φ in the rule ≈E, so that
"$u_1 \approx u_1$" corresponds to $\varphi[u_1/x]$ & "$u_2 \approx u_1$" corresponds to $\varphi[u_2/x]$

We have formally proved
$u_1 \approx u_2 \vdash u_2 \approx u_1$

and we can therefore use as a **derived proof rule**

$$\frac{t_1 \approx t_2}{t_2 \approx t_1} \quad \approx \text{ symmetric}$$

formal proof for “$\approx$” is transitive

1	$u_2 \approx u_3$	premise
2	$u_1 \approx u_2$	premise
3	$u_1 \approx u_3$	$\approx$E $1, 2$

formal proof for “≈” is transitive

1	$u_2 \approx u_3$	premise
2	$u_1 \approx u_2$	premise
3	$u_1 \approx u_3$	$\approx$E $1, 2$

Question: What above corresponds to the WFF φ in the rule $\approx$E?

Answer: “$u_1 \approx x$” corresponds to φ in the rule $\approx$E, so that

“$u_1 \approx u_3$” corresponds to $\varphi[u_3/x]$ & “$u_1 \approx u_2$” corresponds to $\varphi[u_2/x]$

formal proof for "$\approx$" is transitive

1	$u_2 \approx u_3$	premise
2	$u_1 \approx u_2$	premise
3	$u_1 \approx u_3$	$\approx$E $1, 2$

Question: What above corresponds to the WFF φ in the rule $\approx$E?

Answer: "$u_1 \approx x$" corresponds to φ in the rule $\approx$E, so that "$u_1 \approx u_3$" corresponds to $\varphi[u_3/x]$ & "$u_1 \approx u_2$" corresponds to $\varphi[u_2/x]$

We have formally proved

$u_1 \approx u_2,\ u_2 \approx u_3 \ \vdash\ u_1 \approx u_3$

and we can therefore use as a **derived proof rule**

$$\frac{t_1 \approx t_2 \qquad t_2 \approx t_3}{t_1 \approx t_3} \quad \approx \text{transitive}$$

proof rules for universal quantification

- universal quantifier **elimination**

$$\frac{\forall x\ \varphi}{\varphi[t/x]}\ \forall x\ \mathsf{E}$$

(usual assumption: t is substitutable for x)

- universal quantifier **introduction**

$$\frac{\boxed{\begin{array}{lr} x_0 & \text{fresh} \\ \vdots & \\ \varphi[x_0/x] & \end{array}}}{\forall x\ \varphi}\ \forall x\ \mathsf{I}$$

proof rules for existential quantification

- existential quantifier **introduction**

$$\frac{\varphi[t/x]}{\exists x\, \varphi}\ \exists x \text{ I}$$

- existential quantifier **elimination**

$$\frac{\exists x\,\varphi \qquad \boxed{\begin{array}{lll} x_0 & & \text{fresh} \\ & \varphi[x_0/x] & \text{assumption} \\ & \vdots & \\ & \chi & \end{array}}}{\chi}\ \exists x \text{ E}$$

(x_0 cannot occur outside its box, in particular, it cannot occur in χ)

proof rules for existential quantification

- existential quantifier **introduction**

$$\frac{\varphi[t/x]}{\exists x\ \varphi}\ \exists x\ \mathsf{I}$$

- existential quantifier **elimination**

$$\frac{\exists x\ \varphi \qquad \boxed{\begin{array}{lll} x_0 & & \text{fresh} \\ & \varphi[x_0/x] & \text{assumption} \\ & \vdots & \\ & \chi & \end{array}}}{\chi}\ \exists x\ \mathsf{E}$$

(x_0 cannot occur outside its box, in particular, it cannot occur in χ)

- **Note carefully**:
  Rule $(\exists x\ \mathsf{E})$ introduces both a **fresh** variable and an **assumption**.

example: $\forall x\, \forall y\, \varphi(x, y) \ \vdash \ \forall y\, \forall x\, \varphi(x, y)$

	1	$\forall x\, \forall y\, \varphi(x, y)$	premise
y_0	2		fresh y_0
x_0	3		fresh x_0
	4	$\forall y\, \varphi(x_0, y)$	$\forall x$ E, 1
	5	$\varphi(x_0, y_0)$	$\forall x$ E, 4
	6	$\forall x\, \varphi(x, y_0)$	$\forall x$ I, 3-5
	7	$\forall y\, \forall x\, \varphi(x, y)$	$\forall y$ I, 2-6

example: $\forall x\, (P(x) \rightarrow Q(x)),\ \forall x\, P(x) \vdash \forall x\, Q(x)$

	1	$\forall x\, (P(x) \rightarrow Q(x))$	premise
	2	$\forall x\, P(x)$	premise
x_0	3		fresh x_0
	4	$P(x_0) \rightarrow Q(x_0)$	$\forall x$ E, 1
	5	$P(x_0)$	$\forall x$ E, 2
	6	$Q(x_0)$	$\rightarrow$E, 4, 5
	7	$\forall x\, Q(x)$	$\forall x$ I, 3-6

example: $\exists x\,(\varphi(x) \vee \psi(x)) \vdash \exists x\,\varphi(x) \vee \exists x\,\psi(x)$

	1	$\exists x\,(\varphi(x) \vee \psi(x))$		premise
x_0	2			fresh x_0
	3	$\varphi(x_0) \vee \psi(x_0)$		assumption
	4	$\varphi(x_0)$	$\psi(x_0)$	assumption
	5	$\exists x\,\varphi(x)$	$\exists x\,\psi(x)$	$\exists x$ I, 4
	6	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$	$\vee$I, 5
	7	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$		$\vee$E, 3, 4-6
	8	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$		$\exists x$ E, 1, 2-7

example: $\exists x\, \varphi(x) \ \vee\ \exists x\, \psi(x) \ \vdash\ \exists x\, (\varphi(x) \vee \psi(x))$

- Yes, this is a derivable sequent – left to you.
- Hence, $\exists x\, \varphi(x) \ \vee\ \exists x\, \psi(x) \ \dashv\vdash\ \exists x\, (\varphi(x) \vee \psi(x))$

example: $\exists x\,(\varphi(x) \wedge \psi(x)) \;\vdash\; \exists x\,\varphi(x) \;\wedge\; \exists x\,\psi(x)$

- Yes, this is a derivable sequent – similar to the formal proof of $\exists x\,(\varphi(x) \vee \psi(x)) \;\vdash\; \exists x\,\varphi(x) \;\vee\; \exists x\,\psi(x)$

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- **Question:** $\boldsymbol{\exists x\, \varphi(x) \;\wedge\; \exists x\, \psi(x) \;\vdash\; \exists x\, (\varphi(x) \wedge \psi(x))}$ **??**
  No, this is not a derivable sequent

example: $\exists x\,(\varphi(x) \wedge \psi(x)) \ \vdash\ \exists x\,\varphi(x)\ \wedge\ \exists x\,\psi(x)$

- Yes, this is a derivable sequent – similar to the formal proof of $\exists x\,(\varphi(x) \vee \psi(x)) \ \vdash\ \exists x\,\varphi(x)\ \vee\ \exists x\,\psi(x)$

- **Question:** $\boldsymbol{\exists x\,\varphi(x)\ \wedge\ \exists x\,\psi(x) \ \vdash\ \exists x\,(\varphi(x) \wedge \psi(x))}$ **??**

  No, this is not a derivable sequent

  Find an interpretation (a "model") where

  $\exists x\,\varphi(x)\ \wedge\ \exists x\,\psi(x)$ is **true**, but

  $\exists x\,(\varphi(x) \wedge \psi(x))$ is **false**

- Hence, $\exists x\,(\varphi(x) \wedge \psi(x)) \ \not\dashv\vdash\ \exists x\,\varphi(x)\ \wedge\ \exists x\,\psi(x)$

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  Find an interpretation (a "model") where

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- Hence, $\exists x\,(\varphi(x) \wedge \psi(x)) \;\;\not\dashv\vdash\;\; \exists x\,\varphi(x) \;\wedge\; \exists x\,\psi(x)$

REMEMBER! To show that a WFF is **NOT** derivable, it is generally easier to find an interpretation where the WFF is not satisfiable.

example: $\exists x\, P(x), \forall x\, \forall y\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$

	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \forall y\, (P(x) \to Q(y))$	premise
y_0	3		fresh y_0
x_0	4		fresh x_0
	5	$P(x_0)$	assumption
	6	$\forall y\, (P(x_0) \to Q(y))$	$\forall x$ E, 2
	7	$P(x_0) \to Q(y_0)$	$\forall y$ E, 6
	8	$Q(y_0)$	$\to$E, 5, 7
	9	$Q(y_0)$	$\exists x$ E, 1, 4-8
	10	$\forall y\, Q(y)$	$\forall y$ I, 3-9

quantifier equivalences

Theorem

- $$\neg\forall x\, \varphi \quad \dashv\vdash \quad \exists x\, \neg\varphi$$
  $$\neg\exists x\, \varphi \quad \dashv\vdash \quad \forall x\, \neg\varphi$$
- Assume x is not free in ψ:
  $$\forall x\, \varphi \wedge \psi \quad \dashv\vdash \quad \forall x\, (\varphi \wedge \psi)$$
  $$\forall x\, \varphi \vee \psi \quad \dashv\vdash \quad \forall x\, (\varphi \vee \psi)$$
  $$\exists x\, \varphi \wedge \psi \quad \dashv\vdash \quad \exists x\, (\varphi \wedge \psi)$$
  $$\exists x\, \varphi \vee \psi \quad \dashv\vdash \quad \exists x\, (\varphi \vee \psi)$$
  $$\forall x\, (\psi \to \varphi) \quad \dashv\vdash \quad \psi \to \forall x\, \varphi$$
  $$\exists x\, (\varphi \to \psi) \quad \dashv\vdash \quad \forall x\, \varphi \to \psi$$
  $$\forall x\, (\varphi \to \psi) \quad \dashv\vdash \quad \exists x\, \varphi \to \psi$$
  $$\exists x\, (\psi \to \varphi) \quad \dashv\vdash \quad \psi \to \exists x\, \varphi$$
- $$\forall x\, \varphi \wedge \forall x\, \psi \quad \dashv\vdash \quad \forall x\, (\varphi \wedge \psi)$$
  $$\exists x\, \varphi \vee \exists x\, \psi \quad \dashv\vdash \quad \exists x\, (\varphi \vee \psi)$$

proof of only one quantifier equivalence, others in the book

- $\neg\forall x\ \varphi \vdash \exists x\ \neg\varphi$

	1	$\neg\forall x\ \varphi$	premise
	2	$\neg\exists x\ \neg\varphi$	assumption
x_0	3		fresh x_0
	4	$\neg\varphi[x_0/x]$	assumption
	5	$\exists x\ \neg\varphi$	$\exists x$ I, 4
	6	$\perp$	$\neg$E, 5, 2
	7	$\varphi[x_0/x]$	PBC, 4-6
	8	$\forall x\ \varphi$	$\forall x$ I, 3-7
	9	$\perp$	$\neg$E, 8, 1
	10	$\exists x\ \neg\varphi$	PBC, 2-9

three fundamental questions

- **Question**
  Given a WFF φ, can we automate the answer to the query "$\vdash \varphi$??"
- **Question**
  Given a WFF φ, can we automate the answer to the query "$\nvdash \varphi$??"
- **Question**
  Given a formal proof

  $$\begin{array}{ll} 1. & \varphi_1 \\ 2. & \varphi_2 \\ 3. & \vdots \\ n. & \varphi_n \end{array}$$

  can we automate the verification of the proof?

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