Chapter 6

First-Order Logic (FOL)

For our plan to reduce Compactness for FOL to Compactness for PL, we develop *Herbrand theory* further – beyond our presentation in Section 4.1, which you now need to review before you go on to the next section. Throughout, we assume the signature $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ is arbitrary, with the proviso that at least one of its three parts is not empty; if the set of constant symbols $\mathcal{C}$ is empty, we add a fresh constant symbol to it in order to be able to build a non-empty set of ground terms, *i.e.*, so that $\mathsf{Terms}(\Sigma, \varnothing) \neq \varnothing$.

6.1 Herbrand Theory

We take advantage of the transformations $\boxed{\mathsf{prenex}}$ and $\boxed{\mathsf{skolem}}$, both defined in Appendix E. Given an arbitrary first-order wff φ, we denote the application of those two transformations to φ in sequence by defining:

$$\boxed{\mathsf{sko,pre}}(\varphi) \stackrel{\text{def}}{=} \boxed{\mathsf{skolem}}\big(\boxed{\mathsf{prenex}}(\varphi)\big).$$

We present Herbrand's theorem (Theorem 84) gradually. We start with a lemma which proves the theorem for a single first-order sentence φ with the restriction that it does not contain "$\approx$".

Lemma 77. *Let φ be a first-order sentence which does not contain any subformula of the form $(t_1 \approx t_2)$. Then φ is satisfiable iff $\boxed{\mathsf{sko,pre}}(\varphi)$ has a Herbrand model.*

Proof. Let $\psi \stackrel{\text{def}}{=} \boxed{\mathsf{sko,pre}}(\varphi)$. If ψ has a model, Herbrand or not, then ψ is satisfiable. By Lemma 150, if ψ is satisfiable, then φ is satisfiable. The converse is more delicate to prove.

Suppose φ is satisfiable. By Lemma 150 again, ψ is satisfiable. Let $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be the signature of ψ, which in general expands the signature of φ. Hence, there is a structure $\mathcal{M}$ with signature Σ satisfying ψ, *i.e.*, $\mathcal{M} \models \psi$. We need to show there is a Herbrand structure satisfying ψ, *i.e.*, ψ has a Herbrand model. We proceed by first specifying a Herbrand structure $\mathcal{H}$ with signature Σ, and then by showing that $\mathcal{H}$ satisfies ψ. By definition, the signature Σ of $\mathcal{H}$ is is also the signature of $\mathcal{M}$. By definition again, the universe of $\mathcal{H}$ and the interpretations of every $f \in \mathcal{F}$ and every $c \in \mathcal{C}$ are already fixed, namely:

- the universe of $\mathcal{H}$ is $\mathsf{Terms}(\Sigma, \varnothing)$, which is the set of ground terms over Σ,
- $f^{\mathcal{H}}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} f(t_1, \ldots, t_n)$ for every n-ary $f \in \mathcal{F}$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$,
- $c^{\mathcal{H}} \stackrel{\text{def}}{=} c$ for every $c \in \mathcal{C}$.

Only the interpretation of the relation symbols in $\mathcal{R}$ need to be specified, which we set as follows:

- $(t_1, \ldots, t_n) \in R^{\mathcal{H}}$ iff $(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in R^{\mathcal{M}}$
  for every n-ary $R \in \mathcal{R}$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$.

To conclude the proof, we prove a stronger assertion, namely: *For every sentence α in Skolem form over the signature Σ which does not mention the symbol "$\approx$", it holds that if $\mathcal{M} \models \alpha$ then $\mathcal{H} \models \alpha$*, which we prove by induction on the number $k \geqslant 0$ of universal quantifiers in α:

1. *Basis step*: $k = 0$, in which case α has no quantifiers, *i.e.*, α is a propositional combination of elements in $\mathsf{Atoms}(\Sigma, \varnothing)$, which is the set of ground atoms. For this basis step, we proceed by induction on the number of propositional connectives in $\{\neg, \wedge, \vee, \to\}$ occurring in α. Remaining details of this induction are straightforward and left to you.
2. *Induction hypothesis*: The assertion holds for every sentence α in Skolem form with k universal quantifiers, for some $k \geqslant 0$.
3. *Induction step*: Let $\beta \stackrel{\text{def}}{=} \forall x\, \alpha(x)$ be an arbitrary Skolem form where $\alpha(x)$ has one free variable x and $k \geqslant 0$ universal quantifiers, and β has $k+1$ universal quantifiers.

We prove the *induction step* by a sequence of implications. Let U be the universe of $\mathcal{M}$. We write "$[x \mapsto u]$" to denote the part of a valuation that maps the free variable x to the element $u \in U$:

$$\begin{array}{ll}
 & \mathcal{M} \models \forall x\, \alpha(x) \\
\Rightarrow & \text{for all } u \in U\text{, it holds that } \mathcal{M}, [x \mapsto u] \models \alpha \\
\Rightarrow & \text{for all } u \in U \text{ of the form } u = t^{\mathcal{M}} \text{ where } t \in \mathsf{Terms}(\Sigma, \varnothing)\text{, it holds that } \mathcal{M}, [x \mapsto u] \models \alpha \\
\Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing)\text{, it holds that } \mathcal{M}, [x \mapsto t^{\mathcal{M}}] \models \alpha \\
\Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing)\text{, it holds that } \mathcal{M} \models \alpha[x := t] \quad (\alpha[x := t] \text{ is a sentence}) \\
\Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing)\text{, it holds that } \mathcal{H} \models \alpha[x := t] \quad (\text{by the } \textit{induction hypothesis}) \\
\Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing)\text{, it holds that } \mathcal{H}, [x \mapsto t^{\mathcal{H}}] \models \alpha \\
\Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing)\text{, it holds that } \mathcal{H}, [x \mapsto t] \models \alpha \quad (\mathcal{H} \text{ is a Herbrand structure}) \\
\Rightarrow & \mathcal{H} \models \forall x\, \alpha
\end{array}$$

This completes the induction and the proof of the lemma. □

Exercise 78. Two parts, for a better understanding of the preceding proof:

1. What goes wrong in the proof of Lemma 77 if φ (and therefore ψ too) contains free variables?
2. And what goes wrong if ψ is not in Skolem form?

Hint for 1: As a warm-up, try Exercise 151 first, which may be a little easier.

Hint for 2: Consider the sentence $\varphi \stackrel{\text{def}}{=} R(a) \wedge \exists x.\neg R(x)$ which is not in Skolem form, where R is a unary relation symbol and a is a constant symbol. Show there is a structure $\mathcal{M}$ satisfying φ but that $\mathcal{M}$ cannot be a Herbrand structure. □

The following theorem is more general than Lemma 77 but still restricted to first-order sentences without "$\approx$".

Theorem 79 (Herbrand)**.** *Let Γ be a set of first-order sentences, none containing a subformula of the form $(t_1 \approx t_2)$, and $\Gamma' \stackrel{\text{def}}{=} \left\{ \boxed{\mathsf{sko,pre}}(\varphi) \;\middle|\; \varphi \in \Gamma \right\}$. Then Γ is satisfiable iff Γ' has a Herbrand model.*

Proof Sketch. This is a simple variation on the proof of Lemma 77. We should be careful in making the Skolem functions distinct for each sentence $\varphi \in \Gamma$: Specifically, every time we introduce a Skolem function symbol for φ, we have to make it distinct from all Skolem function symbols generated before, whether for φ or for all other sentences in Γ. Hence, if Γ is infinite, so is the set of generated Skolem functions infinite. We omit all the straightforward details. $\square$

Theorem 84 below is a stronger version of Theorem 79: In Theorem 84, first-order sentences are unrestricted and may contain subformulas of the form $(t_1 \approx t_2)$. Before we do this, we need to take a closer look at the presence of the equality relation in Herbrand structures.

Definition 80 (*Enforcing* $\mathsf{eq}^{\mathcal{H}}$ *as a Congruence Relation*)**.** We adjust the set Δ_{eq} in Definition 61, which enforces that a Herbrand structure $\mathcal{H}$ be *well-behaved*, *i.e.*, that $\mathsf{eq}^{\mathcal{H}}$ is a congruence relation. We now write $\Delta_{\mathsf{eq}}(\Sigma)$ to make explicit the signature over which it is written, as we need to define Δ_{eq} over different signatures later in this chapter. $\Delta_{\mathsf{eq}}(\Sigma)$ is the following set of axioms (closed universal wff's) over the signature $\Sigma \cup \{\mathsf{eq}\}$:

1. $\forall x.\ \mathsf{eq}(x, x)$ (*reflexivity*)
2. $\forall x\ \forall y.\ \mathsf{eq}(x, y) \to \mathsf{eq}(y, x)$ (*symmetry*)
3. $\forall x\ \forall y\ \forall z.\ \mathsf{eq}(x, y) \wedge \mathsf{eq}(y, z) \to \mathsf{eq}(x, z)$ (*transitivity*)
4. $\forall x_1 \cdots x_n\ \forall y_1 \cdots y_n.\ \mathsf{eq}(x_1, y_1) \wedge \cdots \wedge \mathsf{eq}(x_n, y_n) \to \mathsf{eq}(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))$
   (*congruence*, one such axiom for every function symbol $f \in \mathcal{F}$ of arity $n \geqslant 1$)
5. $\forall x_1 \cdots x_n\ \forall y_1 \cdots y_n.\ \mathsf{eq}(x_1, y_1) \wedge \cdots \wedge \mathsf{eq}(x_n, y_n) \to (R(x_1, \ldots, x_n) \to R(y_1, \ldots, y_n))$
   (*congruence*, one such axiom for every relation symbol $R \in \mathcal{R}$ of arity $n \geqslant 1$)

Note the difference with the earlier Definition 61, where quantifiers are not available and cannot be used. The five parts here corresponds to the five parts in the earlier definition. The first three axioms make eq an *equivalence relation*, and the last two turn this equivalence into a *congruence relation*. $\square$

All the axioms in $\Delta_{\mathsf{eq}}(\Sigma)$ are already universal first-order sentences in prenex form and, therefore, do not need to be Skolemized. Hence, $\Delta_{\mathsf{eq}}(\Sigma) = \boxed{\mathsf{sko,pre}}(\Delta_{\mathsf{eq}}(\Sigma))$.

Let $\mathcal{M}$ be a structure for the signature Σ whose universe is M. If we take the interpretation of eq in $\mathcal{M}$ to be the equality relation on the universe M, *i.e.* if we interpret $\mathsf{eq}^{\mathcal{M}}$ as just "=", then it is easy to check that $\mathcal{M} \models \Delta_{\mathsf{eq}}(\Sigma)$. (We write eq in prefix position, whereas $=$ is used in infix position; this is a minor adjustment in the syntax which causes no problem.) However, it is important to note there are other models $\mathcal{M}'$ of $\Delta_{\mathsf{eq}}(\Sigma)$, with signature $\Sigma \cup \{\mathsf{eq}\}$ such that $\mathsf{eq}^{\mathcal{M}'}$ is not as restrictive as the equality relation $=$. The exercises to follow should develop your intuitive understanding of such situations.

Exercise 81. Consider the structure $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, =, \times, +, 0, 1)$. We define an infinite family of binary relations on $\mathbb{N}$, one for each $k \geqslant 1$, denoted Eq_k. For all $i, j \in \mathbb{N}$:

$$\mathsf{Eq}_k(i, j) \ \stackrel{\text{def}}{=}\ \big(i \ \mathsf{mod}\ 2^k \ = \ j \ \mathsf{mod}\ 2^k\big)$$

In particular, $\mathsf{Eq}_1(i, j) = \textit{true}$ iff both i and j are even or both are odd. Thus, Eq_1 partitions $\mathbb{N}$ into two disjoint subsets, the subset of *even* numbers and the subset of *odd* numbers. Similarly, Eq_2 partitions $\mathbb{N}$ into four disjoint subsets:

$$A_0 \stackrel{\text{def}}{=} \{2{\times}n \,|\, n \in \mathbb{N}\},\ \ A_1 \stackrel{\text{def}}{=} \{1{+}2{\times}n \,|\, n \in \mathbb{N}\},\ \ A_2 \stackrel{\text{def}}{=} \{2{+}2{\times}n \,|\, n \in \mathbb{N}\},\ \ A_3 \stackrel{\text{def}}{=} \{3{+}2{\times}n \,|\, n \in \mathbb{N}\},$$

with $(A_0 \cup A_2)$ and $(A_1 \cup A_3)$ being all the *even* numbers and all the *odd* numbers, respectively. It turns out that the relation Eq_k is first-order definable in $\mathcal{N}$ using the equality relation $=$ (see Exercise 82). In this exercise, you show that the converse does not hold: For every $k \geqslant 1$, the equality $=$ is not first-order definable in the structure $(\mathbb{N}, \mathsf{Eq}_k, \times, +, 0, 1)$ where we substitute Eq_k for $=$. There are three parts for you to do:

1. Show that, for every $k \geqslant 1$, the binary relation Eq_k is a congruence relation.
   Hint: For all $a, b, c \in \mathbb{N}$, it holds that $(a + b) \,\mathsf{mod}\, c = \big((a \,\mathsf{mod}\, c) + (b \,\mathsf{mod}\, c)\big) \,\mathsf{mod}\, c$ and $(a \times b) \,\mathsf{mod}\, c = \big((a \,\mathsf{mod}\, c) \times (b \,\mathsf{mod}\, c)\big) \,\mathsf{mod}\, c$.

2. Show that, for every $k \geqslant 1$, every Eq_k-congruence class is the union of two (necessarily disjoint) Eq_{k+1}-congruence classes.

The binary relation Eq_{k+1} is therefore a *finer* congruence than the binary relation Eq_k, and Eq_k is a *coarser* congruence than Eq_{k+1}, which we can express in symbols as $\mathsf{Eq}_k \sqsupset \mathsf{Eq}_{k+1}$. We thus have a nested chain of congruences:

$$\mathsf{Eq}_1 \;\sqsupset\; \mathsf{Eq}_2 \;\sqsupset\; \cdots \;\sqsupset\; \mathsf{Eq}_k \;\sqsupset\; \mathsf{Eq}_{k+1} \;\sqsupset\; \cdots$$

where each successive congruence is closer to the equality relation $=$ or, more informally, a "better" approximation of $=$.

3. Let $k \geqslant 1$ be a fixed number. We want to specify conditions under which Eq_k coincides with $=$. Specifically, if φ is a wff involving '$\approx$' and $\sigma : X \to \mathbb{N}$ is a valuation of all the variables, your task is to indicate how the universally and existentially quantified variables in φ can be bounded (to obtain an adjustment, call it ψ, of φ), and how the valuation σ can be restricted so that $\mathcal{N}, \sigma \models \psi$ iff $\mathcal{N}, \sigma \models \boxed{\approx \mapsto \mathsf{Eq}_k}\psi$, where we use $\boxed{\approx \mapsto \mathsf{Eq}_k}\psi$ for the translation that replaces every '$\approx$' by 'Eq_k' in ψ.

   Hint: For all $i, j, k \in \mathbb{N}$, if $i, j < 2^k$ then $\mathsf{Eq}_k(i,j)$ iff $i = j$. □

Exercise 82. This is an appendix to Exercise 81 and uses the same definitions. Show that the binary relation Eq_k is first-order definable in $\mathcal{N}$ using the equality relation $=$.

Hint: First, define a wff with three free variables $\varphi_{\mathrm{mod}}(x, d, y)$ such that:

$$\Big\{ (a, n, b) \in \mathbb{N}^3 \;\Big|\; a \;\mathsf{mod}\; n = b \Big\} \;=\; \Big\{ (a, n, b) \in \mathbb{N}^3 \;\Big|\; \mathcal{N} \models \varphi_{\mathrm{mod}}[a, n, b] \Big\}$$

which shows that mod is first-order definable in $\mathcal{N}$ with the equality relation $=$. Second, for a fixed $k \geqslant 1$, use $\varphi_{\mathrm{mod}}(x, d, y)$ to define $\mathsf{Eq}_k(x_1, x_2)$. □

Exercise 83. Let Σ be a signature and let $\mathcal{M} \overset{\text{def}}{=} (M, \ldots)$ be a Σ-structure with universe M. There are two parts in this exercise:

1. Starting from $\mathcal{M}$, construct a new structure $\mathcal{M}'$ for the expanded signature $\Sigma \cup \{\mathsf{eq}\}$ such that $\mathcal{M}' \models \Delta_{\mathsf{eq}}(\Sigma)$ and the interpretation of eq in $\mathcal{M}'$ does not coincide with the equality relation $=$.
2. Characterize the equality relation $=$ in contrast to the relation $\mathsf{eq}^{\mathcal{M}'}$ in any structure $\mathcal{M}'$ for the signature $\Sigma \cup \{\mathsf{eq}\}$. Is one finer than the other (in the sense of *finer* given in Exercise 81) in that one distinguishes elements that the other does not?

Hint for 1: Pick an arbitrary element $m \in M$. Define the structure $\mathcal{M}' \overset{\text{def}}{=} (M \cup \{m'\}, \ldots)$ for the signature $\Sigma \cup \{\mathsf{eq}\}$ where m' is a fresh element such that $\mathcal{M}'$ acts on m' exactly like $\mathcal{M}$ on m. The elements m and m' cannot be distinguished by the relation $\mathsf{eq}^{\mathcal{M}'}$, whereas $m \neq m'$ and thus the two elements are distinguishable by the equality relation $=$.

Hint for 2: For ground terms t_1 and t_2 over signature Σ, if $\mathcal{M}' \models (t_1 \approx t_2)$ then $\mathcal{M}' \models \mathsf{eq}(t_1, t_2)$, while the converse may or may not hold. □

We use the transformation $\boxed{\approx \mapsto \mathsf{eq}}$ defined in Subsection 4.1 once more: It replaces every atom $(t_1 \approx t_2)$ for some terms t_1 and t_2 by the atom $\mathsf{eq}(t_1, t_2)$. Note that t_1 and t_2 may now contain variables, *i.e.*, they are not necessarily gound terms as in Subsection 4.1.

Theorem 84 (Herbrand Theorem)**.** *Let φ be a wff, and Γ a set of wff's, in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$. Define:*

$$\psi \;\overset{\text{def}}{=}\; \boxed{\approx \mapsto \mathsf{eq}}\big(\boxed{\mathsf{sko,pre}}(\varphi)\big),$$

$$\Delta \stackrel{\text{def}}{=} \boxed{\approx\,\mapsto \mathsf{eq}}(\boxed{\mathsf{sko,pre}}(\Gamma)).$$

Let $\Sigma' \supseteq \Sigma$ be the signature of $\boxed{\mathsf{sko,pre}}(\varphi)$ and $\boxed{\mathsf{sko,pre}}(\Gamma)$, where $\Sigma' - \Sigma$ is the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\varphi)$ and $\boxed{\mathsf{prenex}}(\Gamma)$. The signature of ψ and Δ is therefore $\Sigma' \cup \{\mathsf{eq}\}$. It then holds that:

1. *φ is satisfiable $\Leftrightarrow$ $\psi \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model.*
2. *Γ is satisfiable $\Leftrightarrow$ $\Delta \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model.*

Proof. By Proposition 150, we can assume that φ is in Skolem form and Γ is a set of wff's all in Skolem form, in which case $\varphi = \boxed{\mathsf{sko,pre}}(\varphi)$ and $\Gamma = \boxed{\mathsf{sko,pre}}(\Gamma)$. With this assumption, we have $\Sigma' = \Sigma$. All the wff's under consideration are *universal* wff's. The wff's in $\Delta_{\mathsf{eq}}(\Sigma) = \Delta_{\mathsf{eq}}(\Sigma')$ are already in Skolem form.

The rest of the proof is a straightforward and simple adjustment to the proofs of Lemma 77 and Theorem 79. We first consider the case of a single wff φ as in Lemma 77, then generalize to an arbitrary set of wff's Γ as in Theorem 79.

The adaptation of Lemma 77 for the present proof is the only part that needs some non-trivial attention. Here, $\psi = \boxed{\approx\,\mapsto \mathsf{eq}}(\varphi)$ The non-trivial part is about constructing a Herbrand model $\mathcal{H}$ for $\psi \cup \Delta_{\mathsf{eq}}(\Sigma)$ from a model $\mathcal{M}$ for φ. The universe of $\mathcal{H}$ and the interpretation of the function symbols in $\mathcal{F}$ and constant symbols in $\mathcal{C}$ is the same as in the proof of Lemma 77. For the interpretation of the relation symbols in $\mathcal{R}$ which now includes eq, we define:

- $(t_1, \ldots, t_n) \in R^{\mathcal{H}}$ iff $(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in R^{\mathcal{M}}$
  for every n-ary $R \in \mathcal{R}$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$,
- $(t_1, t_2) \in \mathsf{eq}^{\mathcal{H}}$ iff $t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$
  for every $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$.

The first of these two bullet points is identical to the corresponding bullet point in the proof of Lemma 77; the second bullet point is new. The rest of the proof proceeds as the proof of Lemma 77, and for the case of a set of wff's Γ, as the proof of Theorem 79. This establishes the "$\Rightarrow$" implications in Part 1 and Part 2 in the theorem statement. We leave the "$\Leftarrow$" implications as a straightforward exercise. □

Exercise 85. Consider the wff's and sets of wff's φ, ψ, Γ, and Δ, and the signature Σ and its expansion $\Sigma' \supseteq \Sigma$, all as defined in Theorem 84. Prove:

1. $\psi \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model $\Rightarrow$ φ is satisfiable.
2. $\Delta \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model $\Rightarrow$ Γ is satisfiable.

As in the proof of Theorem 84, to simplify a little, assume that φ is in Skolem form and Γ is a set of wff's all in Skolem form, so that also $\Sigma' = \Sigma$.

Hint: The only question here is how to recover a model for φ, which is a Σ-structure with "$=$" among its underlying relations, from a Herbrand model for $\psi \cup \Delta_{\mathsf{eq}}(\Sigma)$, which is a $(\Sigma \cup \{\mathsf{eq}\})$-structure without "$=$" among its underlying relations. See the proof of Theorem 62 for a similar situation. □

6.2 Compactness and Completeness in FOL

Let φ be a first-order sentence in Skolem form, $\varphi \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n.\varphi_0$ over signature Σ, where φ_0 is the quantifier-free matrix and $\mathrm{FV}(\varphi_0) \subseteq \{x_1, \ldots, x_n\}$. The *Herbrand expansion*, also called *ground expansion*, of φ is:

$$\mathsf{H\_Expansion}(\varphi) \stackrel{\text{def}}{=} \left\{ \varphi_0[x_1 := t_1] \cdots [x_n := t_n] \;\middle|\; t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing) \right\}.$$

In words, the set $\mathsf{H\_Expansion}(\varphi)$ is obtained by deleting all universal quantifiers and replacing all variables by atomic terms in all possible ways. While φ is one sentence, $\mathsf{H\_Expansion}(\varphi)$ is a set of (quantifier-free) sentences, which is infinite if $\mathsf{Terms}(\Sigma, \varnothing)$ is infinite. If Γ is a set of first-order sentences in Skolem form, then:

$$\mathsf{H\_Expansion}(\Gamma) \stackrel{\text{def}}{=} \bigcup \left\{ \mathsf{H\_Expansion}(\varphi) \;\middle|\; \varphi \in \Gamma \right\}.$$

The next lemma pursues the analysis of Theorem 84.

Lemma 86. *Let φ be a sentence (closed wff) in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and let:*

$$\psi \stackrel{\text{def}}{=} \boxed{\approx\,\mapsto \mathsf{eq}}\big(\boxed{\mathsf{sko,pre}}(\varphi)\big).$$

The signature of $\boxed{\mathsf{sko,pre}}(\varphi)$ is some $\Sigma' \supseteq \Sigma$, where $\Sigma' - \Sigma$ is the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\varphi)$, and the signature of ψ is $\Sigma' \cup \{\mathsf{eq}\}$. Then φ is satisfiable iff the Herbrand expansion $\mathsf{H\_Expansion}\big(\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')\big)$ is satisfiable.

Proof. This is a straightforward consequence of Theorem 84, according to which: φ is satisfiable iff $\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand model. The universe of the Herbrand structure $\mathcal{H}$ is $\mathsf{Terms}(\Sigma', \varnothing)$. Deletion of the universal quantifiers corresponds to replacing the variables in $\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')$ by elements of the universe $\mathsf{Terms}(\Sigma', \varnothing)$ in all possible ways. All details omitted. □

Exercise 87. Supply the missing details in the proof of Lemma 86. □

Let $\Delta \stackrel{\text{def}}{=} \mathsf{H\_Expansion}\big(\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')\big)$, the set of quantifier-free sentences over the signature $\Sigma' \cup \{\mathsf{eq}\}$ in the conclusion of Lemma 86. Every wff in Δ is a propositional combination of wff's in $\mathsf{Atoms}\big(\Sigma' \cup \{\mathsf{eq}\}, \varnothing\big)$. Proceeding as in Subsection 4.2, we introduce a set $\mathcal{Y}$ of propositional variables by:

$$\mathcal{Y} = \left\{ Y_\alpha \;\middle|\; \alpha \in \mathsf{Atoms}\big(\Sigma' \cup \{\mathsf{eq}\}, \varnothing\big) \right\}.$$

Each member of $\mathcal{Y}$ is named by the upper-case letter "Y" subscripted with a ground atom α. We can now translate the set Δ of first-order wff's into a set of propositional wff's according to the transformation:

$$\boxed{\mathsf{FOL} \mapsto \mathsf{PL}} : \mathsf{WFF}_{\mathsf{FOL}}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing) \to \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Y})$$

such that for every $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing)$:

$$\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\varphi) \stackrel{\text{def}}{=} \begin{cases} Y_\alpha & \text{if } \varphi = \alpha \in \mathsf{Atoms}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing), \\ \neg\,\big(\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\psi)\big) & \text{if } \varphi = \neg\psi, \\ \big(\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\varphi_1)\big) \diamond \big(\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\varphi_2)\big) & \text{if } \varphi = \varphi_1 \diamond \varphi_2 \text{ and } \diamond \in \{\wedge, \vee, \to\}\ . \end{cases}$$

The next lemma is a continuation of Lemma 86.

Lemma 88. *Let φ be a sentence (closed wff) in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and let:*

$$\Delta \stackrel{\text{def}}{=} \mathsf{H\_Expansion}\left(\left\{\boxed{\approx\,\mapsto \mathsf{eq}}\big(\boxed{\mathsf{sko,pre}}(\varphi)\big)\right\} \cup \Delta_{\mathsf{eq}}(\Sigma')\right)$$

The signature of $\boxed{\mathsf{sko,pre}}(\varphi)$ is some $\Sigma' \supseteq \Sigma$, with $\Sigma' - \Sigma$ being the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\varphi)$, and the signature of Δ is $\Sigma' \cup \{\mathsf{eq}\}$. Then φ is satisfiable (in the sense of FOL) iff $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in the sense of PL).

Proof. By Lemma 86, φ is satisfiable iff Δ' is satisfiable. It suffices therefore to show that: Δ' is satisfiable (in the sense of first-order logic) iff $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$ is satisfiable (in the sense of propositional logic). Keep in mind that Δ' is a set of quantifier-free sentences.

Let $\{\alpha_1, \alpha_2, \ldots\} = \mathsf{Atoms}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing)$ the countable set, finite or infinite, of ground atoms occuring in Δ', and $\{Y_{\alpha_1}, Y_{\alpha_2}, \ldots\}$ the corresponding set of propositional variables occurring in $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$. For the left-to-right implication, assume there is a first-order structure $\mathcal{M}$ such that $\mathcal{M} \models \Delta'$, and derive a truth-value assignment σ from $\mathcal{M}$ such that $\sigma \models \boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$. For the right-to-left implication, assume there is a truth-value assignment σ such that $\sigma \models \boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$, and derive a first-order structure $\mathcal{M}$ from σ such that $\mathcal{M} \models \Delta'$. All straightforward details omitted. $\square$

Exercise 89. Supply the missing details in the proof of Lemma 88. $\square$

We are now ready to state our *transfer principle* from FOL to PL.

Lemma 90 (Transfer Principle)**.** *Let Γ be a set of sentences in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$, and let:*

$$\Delta \stackrel{\text{def}}{=} \mathsf{H\_Expansion}\left(\left\{\boxed{\approx \mapsto \mathsf{eq}}(\boxed{\mathsf{sko,pre}}(\Gamma))\right\} \cup \Delta_{\mathsf{eq}}(\Sigma')\right)$$

where $\Sigma' \supseteq \Sigma$ is the signature of $\boxed{\mathsf{sko,pre}}(\Gamma)$, with $\Sigma' - \Sigma$ being the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\Gamma)$ to obtain $\boxed{\mathsf{sko,pre}}(\Gamma)$. It then holds that:

1. *Γ is satisfiable (in the sense of FOL) $\Leftrightarrow$*
   $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in the sense of PL).
2. *Γ is finitely satisfiable (in the sense of FOL) $\Leftrightarrow$*
   $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is finitely satisfiable (in the sense of PL).

Proof. Part 1 is already established, when Γ is a singleton set, in Lemma 88. For the case when Γ is not a singleton set, we need to repeat and generalize the proof of Lemma 88, as well as the proofs preceding it on which it depends. This is the same generalization that we use in going from the proof of Lemma 77 to the proof of Theorem 79.

Part 2 follows from Part 1, which covers the case when Γ is a finite set. $\square$

We first prove Compactness for first-order logic by invoking results of Herbrand theory. Then, in steps almost identical to the steps in Chapter 2, we prove Completeness as a consequence of Compactness.

Theorem 91 (Compactness for First-Order Logic, Version I)**.** *Let Γ be a set of first-order sentences. Then Γ is satisfiable iff Γ is finitely satisfiable.*

Proof. The left-to-right implication is immediate. For the converse, let Γ be finitely satisfiable. We use the *transfer principle* expressed by Lemma 90 and its notation.

If Γ is finitely satisfiable, then $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is finitely satisfiable (in PL), by Part 2 in the *transfer principle*. If $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is finitely satisfiable (in PL), then $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in PL) by Theorem 2, which is Compactness for PL. If $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in PL), then Γ is satisfiable (in FOL), by Part 1 in the *transfer principle.* $\square$

Corollary 92 (Compactness for First-Order Logic, Version II)**.** *Let Γ be a set of first-order sentences and φ an arbitrary first-order sentence. Then $\Gamma \models \varphi$ iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.*

Proof. This follows from Corollary 7, Lemma 90, and Theorem 91. Some of the details reproduce details in the proof of Corollary 7 as a consequence of Lemma 6, all left as an exercise. $\square$

Exercise 93. Supply the details in the proof of Corollary 92. □

The next lemma is a weaker form of the Completeness Theorem for first-order logic. The Completeness Theorem for first-order logic in full generality is Theorem 95.

Lemma 94. *Let $\varphi_1, \ldots, \varphi_n, \psi$ be first-order sentences. If $\varphi_1, \ldots, \varphi_n \models \psi$ then $\varphi_1, \ldots, \varphi_n \vdash \psi$.*

Proof. The book [LCS] omits this lemma and its proof, though it mentions in passing that the natural-deduction proof system is "sound and complete" with respect to the formal semantics it discusses in Section 2.4 in details.[26] The proof can be carried out along the lines of the proof of Lemma 14, although the semantics of first-order logic are more involved than the semantics of propositional logic. □

Theorem 95 (Completeness for First-Order Logic)**.** *Let Γ be a set (possibly infinite) of first-order sentences, and ψ a first-order sentence. If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. Straightforward consequence of Corollary 92 and Lemma 94. The details here are almost identical to the details in the proof of Theorem 13, except that all formulas are now first-order sentences. □

6.3 Several Basic Theorems of FOL

MORE TO COME

6.4 Applications and Exercises

We start with several common examples and exercises of *first-order logic*. Some of these show that FOL subsumes our earlier logics and some show that FOL has limitations in formally expressing common mathematical properties. We then revisit some of the earlier applications, but now using the formalism and conventions of FOL, and then add a few more that are not easily formulated in the earlier logics.

Example 96 (*Finiteness of Models: Not First-Order Definable*)**.** Let φ be a first-order sentence such that, for every integer $n \geqslant 1$, there is a model of φ with at least n elements. We show that φ has an infinite model.

This is shown by a straightforward application of Compactness. First define for every $n \geqslant 2$:

$$\psi_n \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_n. \bigwedge_{1 \leqslant i < j \leqslant n} \neg(x_i \approx x_j).$$

A model of ψ_n is any whose universe contains at least n elements. Consider the infinite set:

$$\Gamma \stackrel{\text{def}}{=} \{\varphi\} \cup \{\, \psi_n \mid n \geqslant 1 \,\}.$$

Clearly, every finite subset of Γ is satisfiable. Hence, by Compactness, Γ is satisfiable. A model of Γ is necessarily infinite. □

[26] See page 96 in Michael Huth and Mark Ryan, *Logic in Computer Science*, Second Edition, Cambridge University Press, 2004.

Exercise 97 (*Downward Löwenheim-Skolem Theorem*)**.** Our standing assumption throughout is that a signature Σ is a countable set, *i.e.*, the number of symbols in Σ is a finite integer or $\aleph_0$. Let Γ be a set of sentences (closed wff's) in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$. Show that if Γ has an infinite model, then Γ has a countably infinite model.

Informally, we can say that *first-order logic* cannot enforce that models be uncountably infinite. There are various proofs of this result in the literature. In this exercise, we want you to use Herbrand Theory explicitly.

Hint: Consider Theorem 84. $\square$

6.4.1 Embeddings of PL and QPL in FOL

The logics eL, ZOL, EL, and QEL, which are examined in earlier chapters, can be seen as direct restrictions of FOL. But this is not the case for the logics PL and QPL. A propositional variable p in the logics PL and QPL stands for an assertion which may be *true* or *false* and, as such, p by itself is already a well-formed formula of PL and QPL. By contrast, a first-order variable x is meant to range over a non-empty domain or universe of discourse, and by itself is a *term* but not a *well-formed formula* in the logic FOL and in any of its restrictions.

Nevertheless, there are different ways of showing that PL and QPL can be interpreted in FOL and are no more expressive than the latter. The next example and three exercises consider different ways of doing this.

Example 98 (*Interpretation of PL in FOL, I*)**.** We can view *propositional logic* as a particular first-order axiomatization over the signature $\Sigma \stackrel{\text{def}}{=} \{\neg, \wedge, \vee, \rightarrow, \bot, \top\}$ with only function and constant symbols, by reading formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ as quantifier-free formulas in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, \mathcal{P})$. More precisely, a formula φ in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ is satisfiable in the sense of PL iff φ as a quantifier-free formula in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, \mathcal{P})$ is satisfiable in the sense of FOL in the two-element structure:

$$\mathcal{A} \stackrel{\text{def}}{=} (\{\mathit{false}, \mathit{true}\},\ \neg^{\mathcal{A}},\ \wedge^{\mathcal{A}},\ \vee^{\mathcal{A}},\ \rightarrow^{\mathcal{A}},\ \bot^{\mathcal{A}},\ \top^{\mathcal{A}})$$

where $\bot^{\mathcal{A}} = \mathit{false}$ and $\top^{\mathcal{A}} = \mathit{true}$, and the four function symbols in the signature Σ are given their standard interpretations over the two-element domain $\{\mathit{false}, \mathit{true}\}$. Moreover, a formula φ of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ is valid in the sense of PL iff φ as a quantifier-free formula of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, \mathcal{P})$ is true in the two-element Σ-structure $\mathcal{A}$ for every valuation $\sigma : \mathcal{P} \to \{\mathit{false}, \mathit{true}\}$, *i.e.*, $\mathcal{A}, \sigma \models \varphi$.

Although this last assertion explains a way of reading a propositional wff φ as a first-order wff, it is not entirely satisfactory. As a first-order wff, φ may not be valid because it is possible to interpret it in another Σ-structure $\mathcal{B}$ which is not isomorphic to the two-element Σ-structure $\mathcal{A}$ defined above. Take, for example, $\mathcal{B}$ to be a two-element Σ-structure just like $\mathcal{A}$ except that the interpretations of $\wedge$ and $\vee$ are swapped: $\wedge^{\mathcal{B}}$ is $\vee^{\mathcal{A}}$ and $\vee^{\mathcal{B}}$ is $\wedge^{\mathcal{A}}$. In such a case, if φ is the wff $p \vee \neg p$, then $\mathcal{A}, \sigma \models \varphi$ while $\mathcal{B}, \sigma \not\models \varphi$ for every valuation σ.

What we would like is that, for every propositional wff φ, there is a first-order wff φ' such that φ is valid iff φ' is valid. This is taken up in the next exercise. $\square$

Exercise 99 (*Interpretation of PL in FOL, II*)**.** This is a continutation of Example 98. Let $\Sigma' \stackrel{\text{def}}{=} \{f, g_1, g_2, g_3, c_1, c_2\}$, where we replace the symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\bot$, and $\top$ by f, g_1, g_2, g_3, c_1, and c_2, respectively, each with the same arity. We do this in order to keep a clear separation between the logical connectives used in propositional wff's, here viewed as function symbols, from the logical connectives used in first-order wff's. If need be, use the binary function symbols $\{g_1, g_2, g_3\}$ in prefix notation for easier reading, rather than the usual infix use of $\{\wedge, \vee, \rightarrow\}$.

Show that for every φ in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, now re-written using the symbols in Σ', there is a φ' in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma' \cup \{\approx\}, \mathcal{P})$ such that φ is valid in the sense of PL iff φ' is valid in the sense of FOL. Note that φ' is not restricted to be quantifier-free and may mention the equality symbol $\approx$.

Hint: Show how to write a sentence ψ in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma' \cup \{\approx\}, \mathcal{P})$ which defines the two-element structure $\mathcal{A} \stackrel{\text{def}}{=} (\{\mathit{false}, \mathit{true}\},\ \approx^{\mathcal{A}},\ f^{\mathcal{A}},\ g_1{}^{\mathcal{A}},\ g_2{}^{\mathcal{A}},\ g_3{}^{\mathcal{A}},\ c_1{}^{\mathcal{A}},\ c_2{}^{\mathcal{A}})$ in the preceding example up to isomorphism, when $\mathcal{A}$ is expanded to include "$\approx^{\mathcal{A}}$" as an underlying relation. The desired φ' is $\varphi \wedge \psi$. □

The interpretation of PL in FOL in the preceding example and exercise may still seem a little artificial, as it distinguishes the logical connectives in PL from the logical connectives in FOL, whereas we would like to understand them as playing identical roles in both logics. In the next exercise, this distinction disappears and the logical connectives in both logics are the same.

Exercise 100 (*Interpretation of PL in FOL, III*)**.** Let R be a unary relation symbol and consider the following inductively defined translation from PL to FOL, $\langle\ \rangle : \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \to \mathsf{WFF}_{\mathsf{FOL}}(\{R\}, \mathcal{P})$:

$$
\begin{aligned}
\langle p_i \rangle &\stackrel{\text{def}}{=} R(p_i) && \text{for every } p_i \in \mathcal{P},\\
\langle \bot \rangle &\stackrel{\text{def}}{=} \bot,\\
\langle \top \rangle &\stackrel{\text{def}}{=} \top,\\
\langle \neg\varphi \rangle &\stackrel{\text{def}}{=} \neg\langle \varphi \rangle,\\
\langle \varphi \diamond \psi \rangle &\stackrel{\text{def}}{=} \langle \varphi \rangle \diamond \langle \psi \rangle && \text{where } \diamond \in \{\wedge, \vee, \to\}.
\end{aligned}
$$

Informally, the translation $\langle\ \rangle$ consists in "wrapping" every variable p_i with the unary relation symbol R, making $R(p_i)$ a well-formed first-order formula, which p_i by itself is not. Give a rigorous argument, using structural induction, to establish the following assertions:

1. φ is satisfiable in the sense of PL iff $\langle \varphi \rangle$ is satisfiable in the sense of FOL.
2. φ is valid in the sense of PL iff $\langle \varphi \rangle$ is valid in the sense of FOL.

Note that the assertions in parts 1 and 2 involve two implications because of "iff" and each of the two implications has to be proved separately.

Someone suggested the following translation $(\!|\ |\!) : \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \to \mathsf{WFF}_{\mathsf{FOL}}(\{R\}, \mathcal{P})$ instead of $\langle\ \rangle$:

$$(\!|\varphi|\!) \stackrel{\text{def}}{=} \langle \varphi \rangle \wedge \big(\exists x\ R(x)\big) \wedge \big(\exists x\ \neg R(x)\big)$$

where x is a fresh first-order variable not in $\mathcal{P}$. Although the translation $(\!|\ |\!)$ can be used instead of $\langle\ \rangle$, and was presented as being "better" than $\langle\ \rangle$, give a rigorous argument for the following:

3. The added requirement in the translation $(\!|\ |\!)$, expressed by $\big(\exists x\ R(x)\big) \wedge \big(\exists x\ \neg R(x)\big)$, is not necessary for correct proofs of the assertions in parts 1 and 2. □

Exercise 101 (*Interpretation of QPL in FOL*)**.** Do Exercise 100 before you try this one. We extend the translation function $\langle\ \rangle$ in the previous exercise to $\langle\ \rangle : \mathsf{WFF}_{\mathsf{QPL}}(\mathcal{P}) \to \mathsf{WFF}_{\mathsf{FOL}}(\{R\}, \mathcal{P})$ by adding the following cases in its inductive definition:

$$
\begin{aligned}
\langle (\exists p_i\ \varphi) \rangle &\stackrel{\text{def}}{=} (\exists p_i\ \langle \varphi \rangle),\\
\langle (\forall p_i\ \varphi) \rangle &\stackrel{\text{def}}{=} (\forall p_i\ \langle \varphi \rangle).
\end{aligned}
$$

Once again, prove parts 1 and 2 in Exercise 100. □

6.4.2 Graphs and Simple Graphs

From Example 102 to Exercise 111, there are several applications of FOL to graph problems.

Example 102 (*Cliques in Simple Graphs*)**.** Our definition and conventions for simple graphs are the same as in Example 15. We here use the set of variables $X = \{x_1, x_2, \ldots\}$ indexed with the positive integers. We take graphs to be structures of the form $\mathcal{M} \stackrel{\text{def}}{=} (M, R^{\mathcal{M}})$ where the set of vertices is M and the set of edges is $R^{\mathcal{M}} \subseteq M \times M$. In contrast to Example 15, we do not need to limit graphs to have a fixed finite size.

Here is a first-order wff $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\{R, \approx\}, X)$ such that, given a graph $\mathcal{M} \stackrel{\text{def}}{=} (M, R^{\mathcal{M}})$, finite or infinite, it holds that $\mathcal{M} \models \varphi$ *iff* $\mathcal{M}$ *contains a k-clique*:

$$\varphi \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_k. \bigwedge_{1 \leqslant i < j \leqslant k} \Big(\neg(x_i \approx x_j) \ \wedge \ R(x_i, x_j) \Big).$$

Because $\mathcal{M}$ is an undirected graph, $R^{\mathcal{M}}$ is a symmetric relation, *i.e.*, if $R^{\mathcal{M}}(a, b)$ then $R^{\mathcal{M}}(b, a)$ where $a \neq b$, and there is no need to replace "$R(x_i, x_j)$" in φ with "$R(x_i, x_j) \vee R(x_j, x_i)$".

Exercises 16 and 17 are about the limitations of propositional logic in expressing the *presence* and *absence* of k-cliques in simple graphs. These limitations do not apply to first-order logic. A single first-order wff φ expresses the *presence* of a k-clique, and its logical negation φ expresses the *absence* of a k-clique, in any graph of any size. □

Exercise 103 (*Dominating Sets in Simple Graphs*)**.** Use the notation and conventions of Example 102, and the definition of *dominating set* in Exercise 18.

Let $k \geqslant 1$ be a fixed integer. Write a single wff $\psi \in \mathsf{WFF}_{\mathsf{FOL}}(\{R, \approx\}, X)$, and justify its correctness, such that: $\mathcal{M} \models \psi$ *iff* $\mathcal{M}$ *contains a k-dominating set*, where $\mathcal{M}$ is an arbitrary simple graph, finite or infinite. □

Example 104 (*Reachability in Graphs: Not First-Order Definable, I*)**.** We take a graph to be a structure of the form $\mathcal{M} \stackrel{\text{def}}{=} (M, R^{\mathcal{M}})$ or, to make explicit that equality is among the underlying relations, $\mathcal{M} \stackrel{\text{def}}{=} (M, =, R^{\mathcal{M}})$ where $R^{\mathcal{M}} \subseteq M \times M$.

We show that there is no first-order wff $\psi(x, y)$ with two free variables x and y, over the signature $\{R, \approx\}$, such that for every graph $\mathcal{M} = (M, R^{\mathcal{M}})$ and every $a, b \in M$, it holds that:

$$\mathcal{M}, a, b \ \models \ \psi \quad \text{iff} \quad \text{there is a path from } a \text{ to } b.$$

Suppose otherwise, *i.e.*, that such a wff $\psi(x, y)$ does exist. We introduce two new constant symbols, c_1 and c_2, and define $\varphi_0 \stackrel{\text{def}}{=} (c_1 \approx c_2)$ and $\varphi_1 \stackrel{\text{def}}{=} R(c_1, c_2)$, and for every $n \geqslant 2$:

$$\varphi_n \ \stackrel{\text{def}}{=} \ \exists x_1 \cdots \exists x_{n-1}.\, R(c_1, x_1) \wedge R(x_1, x_2) \wedge \cdots \wedge R(x_{n-1}, c_2).$$

The first-order sentence φ_n asserts the existence of a path from c_1 to c_2 of length $\leqslant n$, and the logical negation $\neg\varphi_n$ asserts the opposite. Consider the infinite set:

$$\Delta \ \stackrel{\text{def}}{=} \ \{\, \psi[x := c_1][y := c_2] \,\} \ \cup \ \{\, \neg\varphi_n \mid n \geqslant 0 \,\}.$$

Δ is unsatisfiable by construction. However, every finite subset of Δ has a model (Exercise 105). Hence, Δ is finitely satisfiable, and thus satisfiable by Compactness, which is a contradiction. Hence, the posited wff ψ cannot exist. □

Exercise 105 (*Reachability in Graphs: Not First-Order Definable, II*)**.** Give a careful argument showing that the set Δ defined in Example 104 is finitely satisfiable: Consider an arbitrary finite subset $\Delta_0 \subseteq \Delta$ and give the details of a possible model for Δ_0. □

Exercise 106 (*Graph Connectivity: Not First-Order Property*)**.** We consider graphs as structures of the form $\mathcal{M} \stackrel{\text{def}}{=} (M, R^{\mathcal{M}})$ where M is the set of vertices and $R^{\mathcal{M}} \subseteq M \times M$ is the set of edges. Show there cannot exist a set Γ of first-order sentences (closed wff's) over the signature $\{R, \approx\}$ such that $\mathcal{M} \models \Gamma$ iff $\mathcal{M}$ is a connected graph.

Hint 1: An approach based on invoking the conclusion of Example 104 will not work. However, a direct proof is possible, which is an easy variation on the proof in Example 104 and Exercise 105.

Hint 2: Note that Γ is assumed to be an arbitrary set of first-order sentences, possibly infinite.
□

Exercise 107 (*Two-Colorability of Graphs: First-Order Definable*)**.** The notion of *two-colorable simple graphs* coincides with the notion of *bipartite simple graphs.* Write an infinite set $\Gamma_{\text{bipartite}}$ of first-order sentences such that, for every simple graph G, it holds that $G \models \Gamma_{\text{bipartite}}$ iff G is bipartite.

Hint: G is bipartite iff every cycle in G (possibly with repeated vertices) has even length. □

Exercise 108 (*Graph Planarity*)**.** A well-known result of graph theory is *Kuratowski's Theorem*, which asserts:

- *Let G be a simple undirected graph (no self-loops, no multi-edges). Then G is non-planar iff G contains a subgraph which is a subdivision of either $K_{3,3}$ or K_5.*

$K_{3,3}$ is a conventional name of the *complete bipartite graph on six vertices* and K_5 is that of the *complete graph on five vertices.* $K_{3,3}$ is the smallest non-planar bipartite graph, and K_5 is the smallest non-planar complete graph. A *subdivision* of a graph G is a graph resulting from subdividing edges in G: The subdivision of an edge e with endpoints $\{u, v\}$ produces a new graph containing a new vertex w and two new edges e_1 and e_2 instead of e, whose endpoints are $\{u, w\}$ and $\{w, v\}$, respectively. Figure 6.1 depicts $K_{3,3}$ and K_5 and two of their respective subdivisions.

In this exercise, you need to invoke Kuratowski's Theorem without proving it (the proof is not trivial), together with two other facts, which you can also assume without proving them (which are really easy!):

- *A simple undirected graph G is planar iff every subdivision of G is planar.*
- *A simple undirected graph G is planar iff every subgraph of G is planar.*

As usual, we can take graphs as structures of the form $\mathcal{M} \stackrel{\text{def}}{=} (M, R^{\mathcal{M}})$ where the set of vertices is M and the set of edges is $R^{\mathcal{M}} \subseteq M \times M$. In this exercise, all wff's are in $\mathsf{WFF}_{\mathsf{FOL}}(\{R, \approx\}, X)$. There are three parts:

1. Give a precise argument in about 5-10 lines for how to systematically generate the countably infinite sequence of $K_{3,3}$ and all its subdivisions, call it $\mathcal{G} \stackrel{\text{def}}{=} (\, G_i \mid i \in \mathbb{N}\,)$, and the countably infinite sequence of K_5 and all its subdivisions, call it $\mathcal{G}' \stackrel{\text{def}}{=} (\, G'_i \mid i \in \mathbb{N}\,)$. The first entries in those two sequences are $K_{3,3}$ and K_5, *i.e.*, $G_0 \stackrel{\text{def}}{=} K_{3,3}$ and $G'_0 \stackrel{\text{def}}{=} K_5$. It is also useful to define the sequence $\mathcal{G}$ so that if $i < j$ then $|G_i| \leqslant |G_j|$, and similarly for the sequence $\mathcal{G}'$, *i.e.*, successive entries in $\mathcal{G}$ and $\mathcal{G}'$ are in order of non-decreasing sizes.

   Hint 1: It suffices to give an answer for one of the two sequences, say $\mathcal{G}$, and to conclude by saying "$\mathcal{G}'$ is generated similarly."

   Hint 2: In the two sequences there are many (though a finite number) subdivisions of the same size. And for the same size, it is possible but quite difficult to omit isomorphic copies; it is much easier to allow isomorphic copies in the two sequences.

2. Let $\mathcal{M} \stackrel{\text{def}}{=} (M, R^{\mathcal{M}})$ be an arbitrary simple graph, $G_i \stackrel{\text{def}}{=} (V_i, E_i)$ an arbitrary subdivision of $K_{3,3}$, and $G'_j \stackrel{\text{def}}{=} (V'_j, E'_j)$ an arbitrary subdivision of K_5. Those two subdivisions are entries in the sequences $\mathcal{G}$ and $\mathcal{G}'$ defined in the preceding part. Write first-order sentences φ_i and φ'_j such that if $\mathcal{M} \models \varphi_i$ (resp. $\mathcal{M} \models \varphi'_j$), then G_i is a subgraph of $\mathcal{M}$ (resp. G'_j is a subgraph of $\mathcal{M}$).

   Hint: You will find it convenient to name the vertices of G_i with an initial segment of the positive integers, *i.e.*, $V_i = \{1, 2, \ldots, n_i\}$ where n_i is the size of G_i, and similarly for $\mathcal{G}'$.

3. Show there cannot exist a set Γ, possibly infinite, of first-order sentences over the signature $\{R, \approx\}$ such that $\mathcal{M} \models \Gamma$ iff $\mathcal{M}$ is a non-planar graph.

   Hint: Use the sets of sentences $\{\, \varphi_i \mid i \in \mathbb{N}\,\}$ and $\{\, \varphi'_j \mid j \in \mathbb{N}\,\}$ defined in the preceding part. Apply Compactness for FOL. □

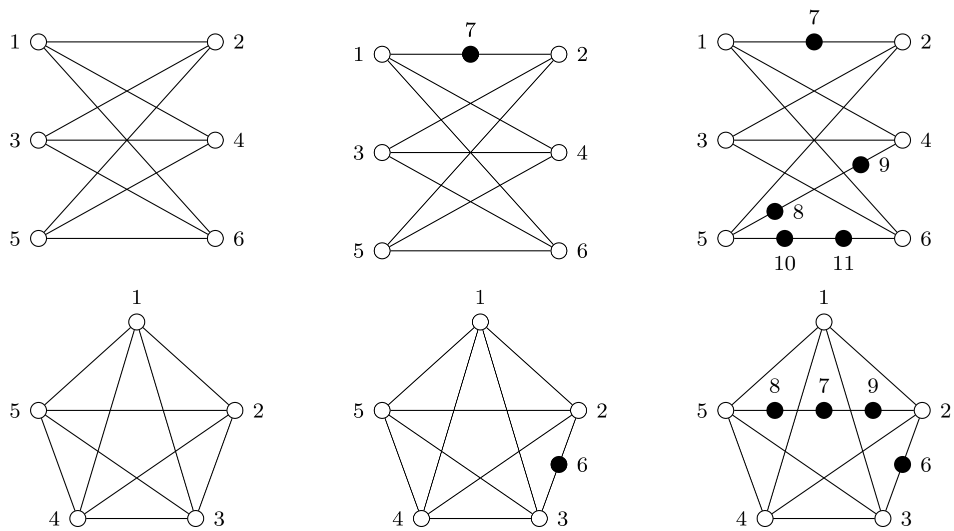

Figure 6.1: *Top row*: $K_{3,3}$ (*left*) and two of its subdivisions, one with 7 vertices (*middle*) and one with 11 vertices (*right*). *Bottom row*: K_5 (*left*) and two of its subdivisions, one with 6 vertices (*middle*) and one with 9 vertices (*right*). Black vertices are added as a result of subdividing edges.

Exercise 109 (*Graph Coloring*)**.** In this exercise we show that *every infinite planar graph is* 4-*colorable.* The background is reviewed in the introductory paragraphs of Exercise 21, and again in Example 70 and in Exercise 71. The first-order theory of simple undirected graphs can be taken as a set Γ of two axioms over signature $\Sigma \stackrel{\text{def}}{=} \{R\}$ consisting of one binary relation symbol, namely:

$$\Gamma \stackrel{\text{def}}{=} \Big\{ \forall x.\forall y.\ R(x,y) \to R(y,x), \quad \forall x.\ \neg R(x,x) \Big\}.$$

Note that, in the presence of the equality symbol $\approx$, there is some flexibility in writing Γ; for example, we may instead define Γ with one axiom only:

$$\Gamma \stackrel{\text{def}}{=} \Big\{ \forall x.\forall y.\ R(x,y) \ \to \ \neg(x \approx y) \wedge R(y,x) \Big\}.$$

We now expand the signature Σ to $\Sigma' = \Sigma \cup \{B, G, P, Y\}$ where B, G, P, and Y are unary predicate symbols (for 'blue', 'green', 'purple', and 'yellow'). There are four parts in this exercise:

1. Write a first-order sentence φ_1 which, in any Σ'-structure $\mathcal{M}$ satisfying Γ (*i.e.*, $\mathcal{M}$ is a simple undirected graph), asserts "*every vertex has at least one of the colors: blue, green, purple, yellow*".
2. Write a first-order sentence φ_2 which, in any Σ'-structure $\mathcal{M}$ satisfying Γ, asserts "*every vertex has at most one color*".
3. Write a first-order sentence φ_3 which, in any Σ'-structure $\mathcal{M}$ satisfying Γ, asserts "*no two adjacent vertices have the same color*".
4. Show that if $\mathcal{M}$ is an infinite planar graph, *i.e.*,
   - $\mathcal{M}$ is a Σ-structure satisfying Γ,
   - the domain of $\mathcal{M}$ is infinite, and
   - $\mathcal{M}$ is planar as a graph,

   then there is a Σ-structure $\mathcal{M}'$, which expands $\mathcal{M}$ with four unary relations $B^{\mathcal{M}'}$, $G^{\mathcal{M}'}$, $P^{\mathcal{M}'}$, and $Y^{\mathcal{M}'}$, and which satisfies $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$, *i.e.*, $\mathcal{M}'$ is four-colorable and, thus, $\mathcal{M}$ is also four-colorable.

Hint 1: Find a way to make use of the following fact: Every *finite* planar graph is four-colorable. (Do not try to prove this fact, which is difficult, but you are allowed to invoke it.)

Hint 2: If $\mathcal{M}$ is a planar graph, then every subgraph of $\mathcal{M}$ is also planar. A *subgraph* of $\mathcal{M}$ is a graph whose vertices are a subset of the vertices of $\mathcal{M}$ and whose adjacency relation is a subset of the adjacency relation of $\mathcal{M}$ restricted to this subset. □

Example 110 (*Topological Graphs, I*)**.** As in earlier examples and exercises, we take a simple graph $G \stackrel{\text{def}}{=} (V, E)$ to be undirected, with no self-loops and no multi-edges. The set E of edges is a collection of two-element sets where the elements are vertices in V.

A *drawing* of such a graph G, call it $\mathsf{Drw}(G)$, is a mapping of V and E into the plane such that the vertices are mapped to distinct points and each edge is mapped to a (possibly curved) line segment between two endpoints.[27] $\mathsf{Drw}(G)$ is also called a *topological graph*.[28] $\mathsf{Drw}(G)$ may subdivide an edge at *crossing points*, *i.e.*, points on the edge other than its two endpoints (which are vertices). Non-crossed pieces are called *edge segments*, whose endpoints are vertices or crossing points. $\mathsf{Drw}(G)$ partitions the plane into topologically connected regions, called *faces*. The boundary of each face consists of a cyclic sequence of edge segments. A vertex, a crossing point, or an edge, is said to be *incident* to a face if it is part of its boundary.

In this example, instead of using the interpretation $R^G = E$ of a binary relation R to specify first-order properties of graph G, we use two other binary relations, α^G and χ^G, to specify first-order properties of a drawing $\mathsf{Drw}(G)$ of G. For all drawn edges $e_1, e_2 \in \mathsf{Drw}(E)$:

$$\alpha^G(e_1, e_2) \stackrel{\text{def}}{=} \begin{cases} \textit{true}, & \text{if } e_1 \text{ and } e_2 \text{ are distinct and adjacent,} \\ & \textit{i.e.}\text{, they share a common endpoint,} \\ \textit{false}, & \text{otherwise.} \end{cases}$$

$$\chi^G(e_1, e_2) \stackrel{\text{def}}{=} \begin{cases} \textit{true}, & \text{if } e_1 \text{ and } e_2 \text{ cross each other} \\ & \text{at a point other than their endpoints,} \\ \textit{false}, & \text{otherwise.} \end{cases}$$

We can specify a drawing $\mathsf{Drw}(G)$ of G as a structure of the form:

$$\mathsf{Drw}(G) \stackrel{\text{def}}{=} (\mathsf{Drw}(V),\ \mathsf{Drw}(E),\ =,\ \alpha^G,\ \chi^G,\ \mathsf{ends}^G)$$

where $\mathsf{Drw}(V)$ and $\mathsf{Drw}(E)$ are the drawings of V and E into the plane, and ends^G is a total mapping on $\mathsf{Drw}(E)$ such that: For every drawn edge $e \in \mathsf{Drw}(E)$, there are unique $a, b \in \mathsf{Drw}(V)$, such that $\mathsf{ends}^G(e) = \{a, b\}$ are the two endpoints of the drawn edge e.

In what follows in this example and in Exercise 111, we express properties of G in terms of first-order definable properties in the reduct $(\mathsf{Drw}(E), =, \chi^G)$ of $\mathsf{Drw}(G)$. To avoid notational clutter, we call this reduct by the same name, *i.e.*, $\mathsf{Drw}(G) \stackrel{\text{def}}{=} (\mathsf{Drw}(E), =, \chi^G)$. The following are first-order definable properties, easily verified (left to you):

- Graph G is *planar* iff there is a drawing $\mathsf{Drw}(G)$ such that no two distinct drawn edges $e_1, e_2 \in \mathsf{Drw}(E)$ cross each other, *i.e.*, iff there is a drawing $\mathsf{Drw}(G)$ such that:

$$\mathsf{Drw}(G) \models\ \forall x_1 x_2.\ \neg(x_1 \approx x_2) \to \neg\chi(x_1, x_2)$$

- Graph G is *1-planar* iff there is a drawing $\mathsf{Drw}(G)$ such that every drawn edge is crossed at most once, *i.e.*, iff there is a drawing $\mathsf{Drw}(G)$ such that:

$$\mathsf{Drw}(G) \models\ \forall x_1 x_2 x_3.\ \neg(x_1 \approx x_2) \wedge \neg(x_1 \approx x_3) \wedge \neg(x_2 \approx x_3) \wedge \chi(x_1, x_2) \to \neg\chi(x_1, x_3)$$

[27] This is what is called a *Jordan arc* in analytic geometry.

[28] It is also called an *embedding* of G in the plane, which is the class of topologically equivalent drawings of G.

- Graph G is *k-planar* where $k \geqslant 2$ iff there is a drawing $\mathsf{Drw}(G)$ such that every drawn edge is crossed by at most k distinct edges, *i.e.*, iff there is a drawing $\mathsf{Drw}(G)$ such that:

$$\mathsf{Drw}(G) \models \forall x_1 \cdots x_{k+2}.\ \bigwedge \left\{ \neg(x_i \approx x_j) \;\middle|\; 1 \leqslant i < j \leqslant k+2 \right\} \wedge \bigwedge \left\{ \chi(x_1, x_j) \;\middle|\; 2 \leqslant j \leqslant k+1 \right\} \to \neg\chi(x_1, x_{k+2})$$

It should be clear that if G is *k-planar* then it is G is *k'-planar* for all $k' > k$. □

Exercise 111 (*Topological Graphs, II*)**.** You need to carefully read Example 110 before embarking on this exercise. A simple undirected graph $G = (V, E)$ is called *quasiplanar* if there is a drawing $\mathsf{Drw}(G)$ of G such that for all three distinct edges $\{e_1, e_2, e_3\} \subseteq \mathsf{Drw}(E)$ there are two edges in $\{e_1, e_2, e_3\}$ that do not cross each other. Put differently, G is *quasiplanar* if there is a drawing $\mathsf{Drw}(G)$ that does not include three mutually crossing edges. There are three parts in this exercise:

1. Write a first-order sentence φ in the signature $\{\approx, \chi\}$ such that: G is quasiplanar iff there is a drawing $\mathsf{Drw}(G) \stackrel{\text{def}}{=} (\mathsf{Drw}(E), =, \chi^G)$ of G such that $\mathsf{Drw}(G) \models \varphi$.
2. Write a first-order sentence in the signature $\{\approx, \chi\}$ to express the fact that 2-planarity implies quasiplanarity.

   Hint: This is a known result of graph theory [8] (2-planarity implies quasiplanarity) which is proved by non-trivial combinatorial reasoning. You can assume this result and do not have to prove it yourself. But now you have to express it as a first-order sentence and explain how to use it to assert that 2-planarity implies quasiplanarity.
3. (This question is more in graph theory than it is in first-order modeling.) Define an undirected simple graph G which is both quasiplanar and not 2-planar. This shows that quasiplanarity does not imply 2-planarity.

   Hint: Search the Web to find such an undirected simple graph G. □

6.4.3 Boards and Grids

Example 112 (*Queens Problem, I*)**.** Read the definition of the *Queens Problem* in Appendix G before you embark on this example. In Exercise 27 we examined the *Queens Problem* using the means available in *propositional logic*, where we had to use infinite sets of wff's and invoke Compactness. This is no longer necessary in the context of *first-order logic*, as shown next.

We define a closed first-order wff ψ which is interpreted in structures $\mathcal{M}$ of the form:

$$\mathcal{M} \stackrel{\text{def}}{=} (\mathbb{N},\ =,\ +,\ <,\ 0,\ Q)$$

where $Q \subseteq \mathbb{N} \times \mathbb{N}$ is a binary relation. Thus, ψ has to be written over the signature:[29]

$$\{\approx,\ +,\ <,\ 0,\ Q\}$$

where, for simplicity, we do not distinguish between a symbol and its interpretation in $\mathcal{M}$, except for the symbol $\approx$ and its standard interpretation as $=$. The symbols "$+$" and "$<$" have each a fixed (its standard) interpretation over $\mathbb{N}$. The only difference between two structures in the form of $\mathcal{M}$ is in the interpretation of Q. As before, $\mathbb{N}^+$ is the set of all positive integers.

Given a structure $\mathcal{M} \stackrel{\text{def}}{=} (\mathbb{N}, =, +, <, 0, Q)$ as defined above, we say the binary relation Q *represents a solution of the Queens Problem in* $\mathcal{M}$ iff three conditions are satisfied:

- $\{\,(i,j) \mid Q(i,j) = \mathit{true}\,\}$ are positions of mutually non-attacking queens in $\mathbb{N}^+ \times \mathbb{N}^+$.
- If $\{\,(i,j) \mid Q(i,j) = \mathit{true}\,\}$ is finite of size $n \geqslant 1$, then all the positions at which Q is *true* are located within the initial $n \times n$ sub-quadrant $\{1,\ldots,n\} \times \{1,\ldots,n\}$ of $\mathbb{N}^+ \times \mathbb{N}^+$.
- If $\{\,(i,j) \mid Q(i,j) = \mathit{true}\,\}$ is finite or infinite, then every *one-point adjustment* Q' of Q introduces mutually attacking queens; *i.e.*, for all $(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+$ such that $Q(i,j) = \mathit{false}$, the positions defined by the one-point adjustment $Q' \stackrel{\text{def}}{=} Q[(i,j) \mapsto \mathit{true}]$ includes mutually attacking queens.

Consistent with our conventions in Appendix G, we view $\mathbb{N}^+ \times \mathbb{N}^+$ as the south-east quadrant of the Cartesian plane: If $(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+$, then i refers to a row and j refers to a column.

We do not qualify that a solution of the *Queens Problem* must be *finite* or *infinite*. Indeed, the preceding three conditions apply, whether the solution represented by Q is finite or infinite. Our first-order sentence ψ has several parts:

$$\psi \ \stackrel{\text{def}}{=}\ \big(\psi^{\text{fin}} \vee \psi^{\text{inf}}\big) \ \wedge\ \psi^{\text{row}} \ \wedge\ \psi^{\text{col}} \ \wedge\ \psi^{\text{diag}} \ \wedge\ \psi^{\text{antidiag}}$$

where each part enforces a different aspect of a solution for the *Queens Problem*. For convenience, we use variable names $\{n, v, w, x, y, z\}$:

$$\begin{aligned}
\psi^{\text{fin}} \ &\stackrel{\text{def}}{=}\ \exists n > 0.\ \Big(\big(\forall x.\forall y.\ Q(x,y) \to 0 < x \leqslant n \wedge 0 < y \leqslant n\big) \wedge \\
&\qquad\qquad \big(\forall x.\ 0 < x \leqslant n \to \exists y.\ Q(x,y)\big) \ \wedge\ \big(\forall y.\ 0 < y \leqslant n \to \exists x.\ Q(x,y)\big)\Big) \\
&\qquad (Q \text{ represents a finite solution of the } \textit{Queens Problem}) \\
\psi^{\text{inf}} \ &\stackrel{\text{def}}{=}\ \Big(\big(\forall x > 0.\ \exists y > 0.\ Q(x,y)\big) \ \wedge\ \big(\forall y > 0.\ \exists x > 0.\ Q(x,y)\big)\Big) \\
&\qquad (Q \text{ represents an infinite solution of the } \textit{Queens Problem}) \\
\psi^{\text{row}} \ &\stackrel{\text{def}}{=}\ \big(\forall x.\forall y.\forall z.\ Q(x,y) \wedge (y \not\approx z) \to \neg Q(x,z)\big) \\
&\qquad (\text{there is at most one queen in every row})
\end{aligned}$$

[29] We include 0 and $<$ in the signature in order to simply ψ a little, but their presence is not essential: Both are first-order definable in the structure $(\mathbb{N}, =, +)$, which we leave for you to do, in the next exercise.

$$\psi^{\text{col}} \stackrel{\text{def}}{=} \big(\forall x.\forall y.\forall z.\ Q(x,y) \land (x \not\approx z) \to \neg Q(z,y)\big)$$
(there is at most one queen in every column)

$$\psi^{\text{diag}} \stackrel{\text{def}}{=} \forall x.\forall y.\forall v.\forall w.\ Q(x,y) \land ((x,y) \not\approx (v,w)) \land (x+w \approx y+v) \to \neg Q(v,w)$$
(there is at most one queen in every diagonal)

$$\psi^{\text{antidiag}} \stackrel{\text{def}}{=} \forall x.\forall y.\forall v.\forall w.\ Q(x,y) \land ((x,y) \not\approx (v,w)) \land (x+y \approx v+w) \to \neg Q(v,w)$$
(there is at most one queen in every antidiagonal)

We use several shorthands in the preceding definitions, including the following:

- $(\forall x > 0.\ \ldots)$ is shorthand for $(\forall x.\ 0 < x\ \to\ \ldots)$ and
  $(\exists x > 0.\ \ldots)$ is shorthand for $(\exists x.\ 0 < x\ \land\ \ldots)$.
- $x \leqslant y$ is shorthand for $(x < y) \lor (x \approx y)$.
- $((x,y) \approx (v,w))$ is shorthand for $(x \approx v) \land (y \approx w)$ and $(x \not\approx y)$ is shorthand for $\neg(x \approx y)$.

We omit the straightforward argument that: $\mathcal{M} \models \psi$ iff Q represents a solution, finite or infinite, of the *Queens Problem* in $\mathcal{M}$. □

Exercise 113 (*Queens Problem, II*)**.** Read and understand the formal modeling of the *Queens Problem* in Example 112. In this exercise you have to use structures $\mathcal{M}$ of the form:

$$\mathcal{M} \stackrel{\text{def}}{=} (\mathbb{N},\ =,\ +,\ <,\ 0,\ q)$$

where $q : \mathbb{N} \to \mathbb{N}$ is now a unary function. The fact that q is a unary function, rather than a binary relation Q as in Example 112, introduces some simplifications as well as some complications. Our representation in this exercise uses "0" to indicate absence of a queen; more precisely, for every $i \in \mathbb{N}$, it holds that $q(i) = 0$ iff no queen is placed in row i. A *one-point adjustment* of q is denoted $q[i \mapsto j]$ for some $i, j \in \mathbb{N}$ and defined by:

$$\big(q[i \mapsto j]\big)(x) \stackrel{\text{def}}{=} \begin{cases} q(x) & \text{if } x \neq i, \\ j & \text{if } x = i. \end{cases}$$

The function $q[i \mapsto j]$ is a *non-zero one-point adjustment* of q if $q(i) \neq j$ and $j \neq 0$. We say q *represents a solution of the Queens Problem in* $\mathcal{M}$ iff four conditions are satisfied:

- $q(0) = 0$.
- $\{\ \big(i, q(i)\big) \mid i \in \mathbb{N} \text{ and } q(i) \neq 0\ \}$ are positions of mutually non-attacking queens in $\mathbb{N}^+ \times \mathbb{N}^+$.
- If $\{\ i \in \mathbb{N} \mid q(i) \neq 0\ \}$ is finite of size $n \geqslant 1$, then $1 \leqslant q(i) \leqslant n$ for every $i \in \{1, \ldots, n\}$.
- If $\{\ i \in \mathbb{N} \mid q(i) \neq 0\ \}$ is infinite, then a *non-zero one-point adjustment* of q introduces mutually attacking queens; *i.e.*, for every $i \geqslant 1$ and every $j \geqslant 1$, if $q(i) \neq j$ the positions defined by the non-zero one-point adjustment $q[i \mapsto j]$ includes mutually attacking queens.

Your task is to define a *first-order sentence* ψ such that $\mathcal{M} \models \psi$ iff q represents a solution, finite or infinite, of the *Queens Problem* in $\mathcal{M}$. For credit, justify your answer. □

Exercise 114 (*Queens Problem, III*)**.** This is an exercise in first-order definability, specifically, definability in the structure $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, =, +)$.

1. Show that the constant 0 and the order relation $<$ are first-order definable in $\mathcal{N}$.

   Hint: Start with 0, then given that 0 is first-order definable in $\mathcal{N}$, consider $<$.
2. Based on your answer for part 1, adjust the wff ψ^{fin} in Example 112 so that it does not mention 0 and $<$.

3. Adjust one or two of the wff's which you have written for Exercise 113 so that they do not mention 0 and $<$. □

Exercise 115 (*Queens Problem, IV*)**.** Consider the formal modeling in Example 112, where we use a binary relation Q to represent a solution of the *Queens Problem* in $\mathcal{M} \stackrel{\text{def}}{=} (\mathbb{N}, =, +, <, 0, Q)$. We now limit attention to the case of an infinite solution, which is modeled by the set Δ:

$$\Delta \stackrel{\text{def}}{=} \{\psi^{\text{inf}},\ \psi^{\text{row}},\ \psi^{\text{col}},\ \psi^{\text{diag}},\ \psi^{\text{antidiag}}\}.$$

We have that $\mathcal{M} \models \Delta$ iff Q represents a solution of the infinite *Queens Problem* in $\mathcal{M}$. According to Appendix G.1, we can impose two additional conditions, (e) and (f), on a solution of the infinite *Queens Problem*, which we reproduce here:

(e) For almost every $n \geqslant 1$, the initial $n \times n$ north-west sub-grid does *not* contain a solution of the finite n-*Queens Problem*.

(f) For infinitely many $n \geqslant 1$, the initial $n \times n$ north-west sub-grid contains a solution of the finite n-*Queens Problem*.

There are eight parts in this exercise:

1. Write a first-order sentence θ_1 in the signature of $\mathcal{M}$ which expresses condition (e).
2. Write a first-order sentence θ_2 in the signature of $\mathcal{M}$ which expresses condition (f).
3. Transform the logical negation $\neg\theta_1$ by pushing "$\neg$" past all other logical connectives in θ_1 to show that $\neg\theta_1$ is equivalent θ_2. (This is another way of concluding that conditions (e) and (f) cannot be simultaneously satisfied.)

Thus, if $\mathcal{M} \models \Delta$, either $\mathcal{M} \models \Delta \cup \{\theta_1\}$ or $\mathcal{M} \models \Delta \cup \{\theta_2\}$, but $\mathcal{M} \not\models \Delta \cup \{\theta_1, \theta_2\}$. Neither of the two conditions, (e) and (f), determines a solution of the infinite *Queens Problem* up to isomorphism; that is, there are many non-isomorphic $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \models \Delta \cup \{\theta_1\}$ and $\mathcal{M}_2 \models \Delta \cup \{\theta_1\}$, and again many non-isomorphic $\mathcal{M}_3$ and $\mathcal{M}_4$ such that $\mathcal{M}_3 \models \Delta \cup \{\theta_2\}$ and $\mathcal{M}_4 \models \Delta \cup \{\theta_2\}$.

For the next four questions, assume the conclusions of Exercise 162 without solving it.

4. Write a first-order sentence ζ_1 in the signature of $\mathcal{M}$ such that $\mathcal{M} \models \Delta \cup \{\zeta_1\}$ iff Q represents a solution of the infinite *Queens Problem* where every diagonal is occupied by a queen.
5. Write a first-order sentence ζ_2 in the signature of $\mathcal{M}$ such that $\mathcal{M} \models \Delta \cup \{\zeta_2\}$ iff Q represents a solution of the infinite *Queens Problem* where the main $(2, 1)$-diagonal is occupied by infinitely many queens.
6. Write a first-order sentence ζ_3 in the signature of $\mathcal{M}$ such that $\mathcal{M} \models \Delta \cup \{\zeta_3\}$ iff Q represents a solution of the infinite *Queens Problem* where every $(2, 1)$-diagonal below/west of the main $(2, 1)$-diagonal is empty (not occupied by any queen).
7. Write a first-order sentence ζ_4 in the signature of $\mathcal{M}$ such that $\mathcal{M} \models \Delta \cup \{\zeta_4\}$ iff Q represents a solution of the infinite *Queens Problem* where, for every queen in some position (i, j), there is a queen in position (i', j') such that:
   - (i', j') is a position on the main $(2, 1)$-diagonal, and
   - there is a $(-1, 2)$-antidiagonal connecting (i', j') and (i, j).

By the analysis in Appendix G.1, if Q represents a solution of the infinite *Queens Problem* satisfying condition (f), then $\mathcal{M} \models \Delta \cup \{\theta_2\}$ which in turn implies $\mathcal{M} \models \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, because each of the sentences in $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ expresses a property of a solution satisfying condition (f). But we want more.

8. Write a first-order sentence ζ_5 in the signature of $\mathcal{M}$ such that $\Delta \cup \{\zeta_5\}$ determines a solution of the infinite *Queens Problem* up to isomorphism; that is, there is exactly one structure $\mathcal{M}$ up to isomorphism such that $\mathcal{M} \models \Delta \cup \{\zeta_5\}$.

Hint: By the results of Exercise 162, it suffices to write $\zeta_5 \stackrel{\text{def}}{=} \theta_2 \wedge \zeta_3 \wedge \zeta_4 \wedge \xi$ where ξ uniquely determines the initial 5×5 north-west board B_1 of the infinite board B_∞. $\square$

Example 116 (*Close Knight's Tour, I*)**.** Read Appendix G.3 carefully before studying this example. We only consider structures $\mathcal{M}$ of the following form in this example:

$$\mathcal{M} \stackrel{\text{def}}{=} (\mathbb{N},\ =,\ \times,\ +,\ 0,\ 1,\ <,\ Q)$$

where the underlying operations and relations other than Q have their standard interpretations on $\mathbb{N}$, and $Q \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is a ternary relation yet to be specified. We say Q represents a solution of the finite *Closed Knight's Tour* problem in $\mathcal{M}$ iff there are integers $m, n \geqslant 1$ such that:

- $A \stackrel{\text{def}}{=} \{\, (i,j) \mid Q(i,j,t) = \mathit{true}$ for some $t \in \mathbb{N}\,\}$ is the set of all the cells in an initial $m \times n$ sub-quadrant of $\mathbb{N}^+ \times \mathbb{N}^+$. Hence, there are $m \times n$ entries in A.
- $B \stackrel{\text{def}}{=} \{\, t \mid Q(i,j,t) = \mathit{true}$ for some $(i,j) \in \mathbb{N} \times \mathbb{N}\,\} = \{1,\ 2,\ \ldots\ ,\ m \times n\}$. Hence, there is a one-one correspondence between the sets A and B; the argument t is used to set a linear order on all the cells in A, with 1 referring to the first cell in the tour and $m \times n$ referring to the last cell in the tour.
- For all $(i,j), (k,\ell) \in \mathbb{N} \times \mathbb{N}$:
  - if $Q(i,j,t) = \mathit{true}$ and $Q(k,\ell,t+1) = \mathit{true}$ where $1 \leqslant t < m \times n$, or
  - if $Q(i,j,m \times n) = \mathit{true}$ and $Q(k,\ell,1) = \mathit{true}$,

  then (i,j) and (k,ℓ) are one knight's move apart.

We define a first-order sentence φ_{CKT} such that $\mathcal{M} \models \varphi_{\text{CKT}}$ iff Q represents a solution of the finite *Closed Knight's Tour* problem in $\mathcal{M}$. For convenience we use the variable names in $\{i, j, t, k, \ell, u, m, n\}$:

$$\varphi_{\text{CKT}} \stackrel{\text{def}}{=} \exists m.\exists n.\ \varphi_1 \ \wedge\ \varphi_2 \ \wedge\ \varphi_3 \ \wedge\ \varphi_4$$

where each wff φ_i models a different constraint of a finite solution:

$$\varphi_1 \stackrel{\text{def}}{=} \forall i \forall j \forall t.\ Q(i,j,t) \to (1 \leqslant i \leqslant m) \wedge (1 \leqslant j \leqslant n) \wedge (1 \leqslant t \leqslant m \times n)$$

(set the limits of the three arguments i, j, and t)

$$\varphi_2 \stackrel{\text{def}}{=} \forall i \forall j.\ (1 \leqslant i \leqslant m) \wedge (1 \leqslant j \leqslant n) \to \exists t.\ Q(i,j,t)$$

(every cell (i,j) is visited at least once)

$$\varphi_3 \stackrel{\text{def}}{=} \forall i \forall j \forall t\ \forall k \forall \ell \forall u.\ Q(i,j,t) \wedge Q(k,\ell,u) \wedge (t \not\approx u) \to (i,j) \not\approx (k,\ell)$$

(every cell (i,j) is visited at most once)

$$\begin{aligned}\varphi_4 \stackrel{\text{def}}{=} \forall i \forall j \forall t\ \forall k \forall \ell \forall u.\ \Big(\big(t+1 \approx u\big) \vee \big((t \approx m \times n) \wedge (u \approx 1)\big)\Big) \wedge Q(i,j,t) \wedge Q(k,\ell,u)\ \to \\ (k,\ell) \approx (i-2, j+1)\ \vee \\ (k,\ell) \approx (i-1, j+2)\ \vee \\ (k,\ell) \approx (i+1, j+2)\ \vee \\ (k,\ell) \approx (i+2, j+1)\ \vee \\ (k,\ell) \approx (i+2, j-1)\ \vee \\ (k,\ell) \approx (i+1, j-2)\ \vee \\ (k,\ell) \approx (i-1, j-2)\ \vee \\ (k,\ell) \approx (i-2, j-1)\end{aligned}$$

(if cell (k,ℓ) is visited right after cell (i,j), the two cells are one knight's move apart)

We use several obvious abbreviations in the definitions of the preceding wff's, which have to be expanded in order to be legal syntactically and to remain within the signature of $\mathcal{M}$:

- $i \leqslant m$ means $(i < m \vee i \approx m)$,
- $t \not\approx u$ means $\neg(t \approx u)$,
- $(k, \ell) \approx (i-2, j+1)$ means $(k+2 \approx i) \wedge (\ell \approx j+1)$,
- etc.

We omit the straightforward argument that φ_{CKT} is satisfied by $\mathcal{M}$ iff Q represents a solution of the finite *Closed Knight's Tour* problem. □

Exercise 117 (*Infinite Knight's Tour, II*)**.** Before embarking on this exercise, read Example 116 and Exercise 163 in Appendix G.3. You can assume the result of the latter without solving it. The structures $\mathcal{M}$ under consideration in this exercise are the same as in Example 116. All the first-order wff's below must be written in the signature of $\mathcal{M}$. There are four parts:

1. Define an infinite sequence of increasing integers $0 < n_1 < n_2 < \cdots < n_k < \cdots$ and an infinite sequence of first-order sentences $\psi_1, \psi_2, \ldots, \psi_k, \ldots$ such that $\mathcal{M} \models \psi_k$ iff Q represents a solution of the *Knight's Tour* (not necessarily closed) on a grid of size $\geqslant 3 \times n_k$ for every $k \geqslant 1$.
2. Define a single first-order sentence $\widetilde{\psi}$ such that $\mathcal{M} \models \widetilde{\psi}$ iff Q represents a solution of the *Knight's Tour* (not necessarily closed) on a finite grid of size $3 \times n$ for some $n > 0$.
3. Let $\Psi \stackrel{\text{def}}{=} \{\psi_k\}_{k \geqslant 1}$ be the set of closed sentences defined in part 1. Show that Ψ is consistent, *i.e.*, there is one structure $\mathcal{M}$ satisfying every member of Ψ. Use Compactness of FOL to conclude there is a *Knight's Tour* that visits the entire infinite subgrid $\{1,2,3\} \times \mathbb{N}^+$.
4. Define a single first-order sentence $\widehat{\psi}$ such that $\mathcal{M} \models \widehat{\psi}$ iff Q represents a solution of a *Knight's Tour* that visits the entire infinite subgrid $\{1,2,3\} \times \mathbb{N}^+$. Does your $\widehat{\psi}$ uniquely define the *Knight's Tour* in question? □

6.4.4 Partial Orders

Exercise 118 (*Dilworth's Theorem and its Extension to Infinite Partial Orders*)**.** Read Exercise 35 carefully before attempting this one. Your task now is to extend Dilworth's Theorem to all infinite posets using Compactness for *first-order logic*, instead of *propositional logic*. We take posets as structures of the form $\mathcal{P} \stackrel{\text{def}}{=} (A, \trianglelefteq^{\mathcal{P}})$. All wff's in this exercise are in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ for some signature $\Sigma \supseteq \{\trianglelefteq, \approx\}$.

Whereas in Exercise 35 you have to use sets of propositional wff's to express the desired conclusions for a single *given poset* of width $\leqslant 3$, not for *all posets* of width $\leqslant 3$ simultaneously, you are here asked the following:

1. Write one first-order sentence φ over an expanded signature $\Sigma \supseteq \{\trianglelefteq, \approx\}$ such that for all Σ-structures $\mathcal{M}$, finite or infinite, it holds that:

   $$\mathcal{M} \models \varphi \ \text{ iff } \ \mathcal{M}|\{\trianglelefteq\} \ \text{ is a poset of width} \leqslant 3,$$

   where "$\mathcal{M}|\{\trianglelefteq\}$" denotes the *reduct* of $\mathcal{M}$ to $\{\trianglelefteq\}$. In words, φ is now defined independently of any given poset and should assert that "any poset of width $\leqslant 3$ can be partitioned into 3 chains."

   Hint: Consider adding three unary relation symbols $\{R, S, T\}$ to the signature $\{\trianglelefteq, \approx\}$. The interpretations $\{R^{\mathcal{M}}, S^{\mathcal{M}}, T^{\mathcal{M}}\}$ in a model $\mathcal{M}$ should correspond to a three-part partition of the domain of $\mathcal{M}$.
2. Write a set Γ of first-order sentences over the signature Σ introduced in Part 1 such that (a) Γ is satisfiable, (b) $\varphi \in \Gamma$, and (c) for every Σ-structure $\mathcal{M}$ it holds that:

   $$\mathcal{M} \models \Gamma \ \text{ iff } \ \mathcal{M}|\{\trianglelefteq\} \ \text{ is an infinite poset of width} \leqslant 3.$$

   Hint: Use Dilworth's Theorem for *finite posets* (without proving it) together with Compactness.

3. It is easy to define a *finite* poset $\mathcal{P}$, together with a Σ-structure $\mathcal{M}$ as an expansion of $\mathcal{P}$, such that $\mathcal{M} \models \varphi$ in Part 1, but Part 1 also leaves open the question of whether an *infinite* such $\mathcal{P}$ exists. Part 2 settles this question, because it shows there are indeed infinite models of φ.

   Given the preceding fact, your task now is to modify φ to obtain a single first-order sentence ψ such that, for every Σ-structure $\mathcal{M}$, it holds that: $\mathcal{M} \models \psi$ iff $\mathcal{M}|\{\trianglelefteq\}$ is an *infinite* poset of width $\leqslant 3$. $\square$

6.4.5 Problems of Algebras

Exercise 119 (*Algebraic Numbers*)**.** In this exercise, you will consider the following structure (or model):

$$\mathcal{R} \stackrel{\text{def}}{=} (\mathbb{R},\ =,\ +,\ \times)$$

where $\mathbb{R}$ is the set of all real numbers, and '$+$' and '$\times$ are the usual addition and multiplication on the real numbers. We assume that the equality symbol '$\approx$' is available, so that we can also write first-order wff's that mention the symbol '$\approx$' (which is always interpreted as equality '$=$' on the real numbers), in addition to '$+$' and '$\times$'.

For ease of reading, we will not distinguish between the symbols $\{+, \times\}$ and their interpretations in the structure $\mathcal{R}$. All the questions in this problem have to do with showing whether a relation or a function over $\mathcal{R}$ is first-order definable (or can be modeled) in the structure $\mathcal{R}$. In places below, we may write '$\cdot$' instead of '$\times$' for brevity.

Remark: Consider the questions sequentially, as your answer to the next one will depend on some of the previous ones.

1. Write three first-order wff's $\varphi_{\{0\}}(x)$, $\varphi_{\{1\}}(x)$, and $\varphi_{\{2\}}(x)$, each with one free variable x, which define the singleton sets $\{0\}$, $\{1\}$, and $\{2\}$, respectively, in the structure $\mathcal{R}$.

2. Write a first-order wff $\varphi_{<}(x, y)$ with free variables x and y which defines in $\mathcal{R}$ the usual ordering "$<$" on $\mathbb{R}$, *i.e.*, for all real numbers $a, b \in \mathbb{R}$ we should have:

   $$a < b \quad \text{iff} \quad \mathcal{R} \models \varphi_{<}[a, b]$$

   Hint: $a < b$ iff there is a positive real c such that $a + c = b$.

3. A well-known result of algebra states: *Every single-variable polynomial with real coefficients and of odd degree has at least one real root.*[30]

   Your task is to show that this result of algebra can be asserted by a set Γ of first-order sentences that are interpreted in the structure $\mathcal{R}$. Put differently, the satisfaction of Γ in the structure $\mathcal{R}$ is equivalent to the forementioned result.

   Hint 1: The set Γ is necessarily infinite.

   Hint 2: A single-variable polynomial with real coefficients and of odd degree n can be written as:

   $$a_0 + a_1 \cdot x + a_2 \cdot x \cdot x + \cdots + a_n \cdot \underbrace{x \cdot x \cdots x}_{n \text{ times},\ n \text{ odd}}$$

   where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are in $\mathbb{R}$ and the degree n is an odd positive integer.

[30] This result follows from the so-called *Intermediate-Value Theorem* in algebra. Another result that helps frame this question is the so-called *Fundamental Theorem of Algebra*, which states that every degree-n single-variable polynomial $f(x)$ with complex coefficients – which include all real coefficients – has exactly n roots in the set of complex numbers, counting repeated roots. A *root* of $f(x)$ is a value a such that $f(a) = 0$.

An example of a polynomial with two distinct roots $x = 1$ and $x = -1$ is $(x^2 - 1)$, which can be factored out as $(x + 1) \cdot (x - 1)$. An example of a polynomial with one root $x = 1$ of multiplicity 2 is $(x^2 - 2x + 1)$, factored out as $(x - 1) \cdot (x - 1)$. An example of a polynomial with two distinct complex roots $x = 2i$ and $x = -2i$ is $(x^2 + 4)$, factored out as $(x + 2i) \cdot (x - 2i)$. This last polynomial is of even degree and has no real roots.

4. We expand the structure $\mathcal{R}$ by adding the sine function of trigonometry, $\sin : \mathbb{R} \to \mathbb{R}$, to obtain the expanded structure: $\mathcal{R}_{\text{sin}} \overset{\text{def}}{=} (\mathbb{R}, =, +, \times, \sin)$.

   Your task is to write a first-order wff $\varphi_{\text{pos}}(x)$ with free variable x such that $\mathcal{R}_{\text{sin}} \models \varphi_{\text{pos}}[a]$ iff a is a positive integer. The existence of such a formula $\varphi_{\text{pos}}(x)$ shows that the set $\mathbb{N} - \{0\}$ of positive integers is first-order definable in the structure $\mathcal{R}_{\text{sin}}$.

   Hint: Recall from trigonometry that $\sin(a) = 0$ iff $a \in \{\, n \times \pi \mid n \in \mathbb{Z}\,\}$ where π is the usual constant $3.14159\cdots$ and $\mathbb{Z}$ is the set of all integers. Before writing the desired $\varphi_{\text{pos}}(x)$, write the first-order wff $\varphi_{\{\pi\}}(x)$ which uniquely defines the singleton set $\{\pi\}$ in $\mathcal{R}_{\text{sin}}$.

5. An expansion of $\mathcal{R}_{\text{sin}}$ with the binary exponentiation function $\uparrow : \mathbb{R} \times (\mathbb{N} - \{0\}) \to \mathbb{R}$ is the structure $\mathcal{R}_{\text{sin},\uparrow} \overset{\text{def}}{=} (\mathbb{R}, =, +, \times, \sin, \uparrow)$. The function $\uparrow$ is defined by:

   $$x \uparrow n \ \overset{\text{def}}{=}\ x^n$$

   for all $x \in \mathbb{R}$ and $n \in (\mathbb{N} - \{0\})$. Note that the second argument of $\uparrow$ must be a positive integer.[31]

   A famous result of number theory is *Fermat's Last Theorem* (FLT): *There are no three positive integers a, b, and c that satisfy the equation $a^n + b^n = c^n$ whenever the integer exponent $n \geqslant 3$.*

   Your task is to show that FLT can be stated as a first-order sentence φ_{FLT} interpreted in $\mathcal{R}_{\text{sin},\uparrow}$.

6. An *algebraic number* is a number which is the root of a single-variable polynomial with rational coefficients. A *transcendental number* is a number which is not algebraic, which is therefore not the root of any single-variable polynomial with rational coefficients.

   Your task is to write a first-order wff $\varphi_{\text{trans}}(x)$ with free variable x which defines an infinite set of transcendental numbers in $\mathcal{R}_{\text{sin},\uparrow}$.

   Hint: If r is a transcendental number, then so is $n \cdot r$ for every integer $n \geqslant 1$. For full credit, you should prove this fact, which you can prove by contradiction, *i.e.*, you can get a contradiction by assuming that $n \cdot r$ is an algebraic number, where $n \geqslant 2$.

7. We all know that, given a quadratic polynomial $f(x) \overset{\text{def}}{=} a_2 \cdot x^2 + a_1 \cdot x + a_0$, there are two solutions (or roots) of the equation $f(x) = 0$, given by the so-called *quadratic formula*:

   $$\frac{-a_1 \pm \sqrt{a_1 \cdot a_1 - 4 \cdot a_2 \cdot a_0}}{2 \cdot a_2} \quad \text{or, if } a_2 = 1, \text{ in the simpler form:} \quad -\frac{a_1}{2} \pm \sqrt{\frac{a_1 \cdot a_1}{4} - a_0}$$

   The *quadratic formula* involves the following operations: $+$ (addition), $\times$ (multiplication), $-$ (subtraction), $\sqrt{\ }$ (square root, also called the 2-nd radical and written $\sqrt[2]{\ }$), and $\div$ (division). There is an analogous *cubic formula* for polynomials of degree 3 of the form $a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$, and again an analogous (and far more complicated) *quartic formula* for polynomials of degree 4 of the form $a_4 \cdot x^4 + a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$. These formulas define solution sets of two, three, and four roots, respectively (with the same root being counted as many times as its multiplicity).[32]

   The formulas above (*quadratic*, *cubic*, *quartic*) use the same operations $\{+, \cdot, -, \div, \sqrt[2]{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots\}$ in various combinations applied to the polynomial coefficients $\{a_0, a_1, a_2, \ldots\}$, where $\sqrt[n]{\ }$ denotes the n-th radical, which is unary operator for extracting the n-th roots of a number. We refer to this situation by saying that *these formulas define with radicals the roots of any polynomial of degree $\leqslant 4$.*

[31] It turns out that the function $\uparrow$ is first-order definable in the structure $\mathcal{R}_{\text{sin}}$. So, as far as first-order definability is concerned, there is no need to explicitly add $\uparrow$ to the underlying operations of the structure, but we added it here to simplify the problem a little.

[32] If you want to know more about these three formulas, go to the webpage The Cubic Formula.

Another famous result of algebra is the *Abel-Ruffini Theorem* (ART) which asserts: *There is no formula that defines with radicals the roots of any degree-n polynomial when $n \geqslant 5$.* More specifically, for a fixed $n \geqslant 5$, there does not exist a general formula which defines the roots of any degree-n polynomial by applying the operations in $\{+, \cdot, -, \div, \sqrt[2]{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots\}$ in some combination to the coefficients of the polynomial. For the case $n = 5$, such a general formula is called a *quintic formula*, and ART asserts that a *quintic formula* with radicals does not exist.

Your task is to show that ART for the case $n = 5$ can be expressed by an infinite set Δ_{ART} of first-order sentences interpreted in the structure $\mathcal{R}_{\sin,\uparrow}$. You can limit yourself to rational coefficients if you find it easier, but this is not necessary for a correct solution.

Hint 1: It is a fact, which you do not need to prove, that every operation in $\{-, \div, \sqrt[2]{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots\}$ is first-order definable in $\mathcal{R}_{\sin,\uparrow}$, *i.e.*, it is first-order definable using the operations in $\{+, \cdot, \sin, \uparrow\}$. So, to simplify your answer, you are allowed to assume that you can write first-order wff's over a signature containing function symbols corresponding to the operations in $\{+, \cdot, \sin, \uparrow, -, \div, \sqrt[2]{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots\}$.

Hint 2: If it existed, a *quintic formula* with radicals would be a term t built up from the operations in $\{+, \cdot, -, \div, \sqrt[2]{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots\}$ and over the variables $\{y_0, y_1, y_2, y_3, y_4, y_5\}$ such that, given a degree-5 polynomial $f(x) \stackrel{\text{def}}{=} a_5 \cdot x^5 + a_4 \cdot x^4 + a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$, where $a_0, \ldots, a_5 \in \mathbb{R}$ and $a_5 \neq 0$, the roots of $f(x)$ would be defined by:

$$u \stackrel{\text{def}}{=} t[y_0 := a_0,\ y_1 := a_1,\ y_2 := a_2,\ y_3 := a_3,\ y_4 := a_4,\ y_5 := a_5]$$

i.e., u is t after substituting the coefficients $a_0, \ldots, a_5$ for the variables $y_0, \ldots, y_5$, respectively.

(MORE TO COME)