

CS 511, Fall 2024, Lecture Slides 20

Examples of First-Order Theories

Assaf Kfoury

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equivalence relations

1. $\forall x \quad (x \sim x)$

reflexivity

2. $\forall x \forall y \quad (x \sim y \rightarrow y \sim x)$

symmetry

3. $\forall x \forall y \forall z \quad (x \sim y \wedge y \sim z \rightarrow x \sim z)$

transitivity

equality with uninterpreted functions (EUF)

- | | |
|---|--------------|
| 1. $\forall x \ (x \approx x)$ | reflexivity |
| 2. $\forall x \ \forall y \ (x \approx y \rightarrow y \approx x)$ | symmetry |
| 3. $\forall x \ \forall y \ \forall z \ (x \approx y \wedge y \approx z \rightarrow x \approx z)$ | transitivity |

The three preceding axioms are identical to those in the theory of **equivalence relations** (preceding page).

equality with uninterpreted functions (EUF)

1. $\forall x \quad (x \approx x)$ reflexivity
2. $\forall x \forall y \quad (x \approx y \rightarrow y \approx x)$ symmetry
3. $\forall x \forall y \forall z \quad (x \approx y \wedge y \approx z \rightarrow x \approx z)$ transitivity

The three preceding axioms are identical to those in the theory of **equivalence relations** (preceding page).

4. for every function symbol $f \in \mathcal{F}$ of arity $n \geq 1$:

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n$$
$$\left(\bigwedge_{1 \leq i \leq n} x_i \approx y_i \right) \rightarrow f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n) \quad \text{congruence}$$

5. for every predicate symbol $P \in \mathcal{P}$ of arity $n \geq 1$:

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n$$
$$\left(\bigwedge_{1 \leq i \leq n} x_i \approx y_i \right) \rightarrow \left(P(x_1, \dots, x_n) \leftrightarrow P(y_1, \dots, y_n) \right) \quad \text{congruence}$$

orders

1. $\forall x \forall y \forall z \quad (x \leq y \wedge y \leq z \rightarrow x \leq z)$ transitive
2. $\forall x \quad (x \leq x)$ reflexive
3. $\forall x \forall y \quad (x \leq y \wedge y \leq x \rightarrow x \approx y)$ anti-symmetric

(1), (2) and (3) make “ \leq ” a **partial** order, which may not be **total**
(what is an example of a partial order which is not total?)

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4. $\forall x \forall y \quad (x \leq y \vee y \leq x)$ total (or linear) ordering
5. $\forall x \forall z \quad (x < z \rightarrow \exists y \quad (x < y \wedge y < z))$ dense ordering
(where " $x < y$ " abbreviates " $(x \leq y) \wedge \neg(x \approx y)$ ")

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6. $\exists x \forall y \quad (x \leq y)$ smallest element
7. $\exists x \forall y \quad (y \leq x)$ largest element

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can we express a **well-ordering** in first-order logic? i.e., “every non-empty subset has a smallest element”?

algebras with two binary operations

1. $\forall x \forall y \forall z \quad (x \oplus (y \oplus z) \approx (x \oplus y) \oplus z)$

\oplus is associative

2. $\forall x \forall y \quad (x \oplus y \approx y \oplus x)$

\oplus is commutative

3. $\forall x \forall y \forall z \quad (x \otimes (y \oplus z) \approx (x \otimes y) \oplus (x \otimes z))$

\otimes distributes over \oplus

groups

1. $\forall x \quad (e \cdot x \approx x \wedge x \cdot e \approx x)$

identity (or neutral element)

2. $\forall x \exists y \quad (x \cdot y \approx e \wedge y \cdot x \approx e)$

inverse

3. $\forall x \forall y \forall z \quad ((x \cdot y) \cdot z \approx x \cdot (y \cdot z))$

associative

groups

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2. $\forall x \exists y \quad (x \cdot y \approx e \wedge y \cdot x \approx e)$ inverse
3. $\forall x \forall y \forall z \quad ((x \cdot y) \cdot z \approx x \cdot (y \cdot z))$ associative

three preceding WFF's are true in every group,

does the following WFF φ follows from the preceding three:

$$\varphi \stackrel{\text{def}}{=} \forall x \forall y \forall z \quad (x \cdot y \approx e \wedge x \cdot z \approx e \rightarrow y \approx z) ??$$

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some special cases of groups:

4. $\forall x \forall y \quad (x \cdot y \approx y \cdot x)$ abelian group
5. $\forall x \quad (x \cdot x \approx e \rightarrow x \approx e),$ torsion-free group
 $\forall x \quad (x \cdot x \cdot x \approx e \rightarrow x \approx e),$
 $\forall x \quad (x \cdot x \cdot x \cdot x \approx e \rightarrow x \approx e), \dots$

graphs

1. $\forall x \forall y \left(R(x, y) \rightarrow R(y, x) \right)$

the graph is undirected

2. $\forall x \left(\neg R(x, x) \right)$

there are no “loops” in the graph

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assume there are two domains:

the domain V of vertices, the domain \mathbb{R} of real numbers

assume there is a capacity function: $c : V \times V \rightarrow \mathbb{R}$

a flow is a function $f : V \times V \rightarrow \mathbb{R}$

3. $\forall f \forall x \forall y \quad (f(x, y) \leq c(x, y))$

is (3) a first-order WFF?

successor function over the natural numbers

1. $\forall x \quad (\neg(Sx \approx 0))$
2. $\forall x \forall y \quad (Sx \approx Sy \rightarrow x \approx y)$
3. $\forall x \quad (\neg(x \approx 0) \rightarrow \exists y (Sy \approx x))$

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4. for every WFF $\varphi(x)$ with a single free variable x , include the axiom
 $\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall y \varphi(y)$

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with **addition**:

5. $\forall x \quad (x \dot{+} 0 \approx x)$
6. $\forall x \forall y \quad (x \dot{+} Sy \approx S(x \dot{+} y))$

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6. $\forall x \forall y \quad (x \dot{+} Sy \approx S(x \dot{+} y))$

with **addition** and **multiplication**:

7. $\forall x \quad (x \dot{\times} 0 \approx 0)$
8. $\forall x \forall y \quad (x \dot{\times} Sy \approx (x \dot{\times} y) \dot{+} x)$

linear integer arithmetic (LIA)

1. $\mathcal{P} = \{\leq\}$, $\mathcal{F} = \{+, \div\}$, $\mathcal{C} = \{0, 1\}$.
2. include all axioms for “+” and “÷”.
3. atomic WFF's are all of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \bowtie b$$

where $\bowtie \in \{\leq, <, \geq, >, \approx, \neq\}$ and a_1, \dots, a_n, b are integers.

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