Chapter 4

Zeroth-Order Logic (ZOL)

The *zeroth-order logic* (ZOL) is a fragment of *first-order logic* that mentions no quantifiers and no variables.[18] It is a restricted fragment, but which makes the transition to full first-order logic a little more gradual. The means to reduce Compactness for ZOL to Compactness for PL is what is known as *Herbrand theory*. In this chapter we need a limited version of *Herbrand theory*, the full version is used when we consider *first-order logic* with no restrictions.[19]

Given a first-order signature $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, the set of wff's of ZOL over Σ is denoted $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ when the symbol "$\approx$" for equality does not occur in wff's, and $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$ when "$\approx$" is allowed to occur in wff's. Precise definitions of the syntax of ZOL is in Appendix A.4, the semantics of ZOL is in Appendix B.3, and a proof system for ZOL is in Appendix C.5.

Though far more limited than full *first-order logic*, the expressive power of *zeroth-order logic* is not trivial, as demonstrated by the examples and exercises in Section 4.3.[20]

4.1 Intermediate Herbrand Theory[21]

Let $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order signature, as specified in Appendix A.3. $\mathsf{Terms}(\Sigma, \varnothing)$ is the set of variable-free terms over Σ, also called *ground terms* over Σ. $\mathsf{Atoms}(\Sigma, \varnothing)$ is the set of variable-free atomic formulas over Σ, also called *ground atoms* over Σ, none mentioning the equality symbol "$\approx$".

In general, Σ is not empty, even though one or two of its three parts – $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{C}$ – may be empty. If $\mathcal{C} = \varnothing$ we add a fresh constant symbol to it in order to be able to build a non-empty set of ground terms, *i.e.*, so that $\mathsf{Terms}(\Sigma, \varnothing) \neq \varnothing$. We denote a Σ-structure $\mathcal{A}$ by writing:

$$\mathcal{A} \overset{\text{def}}{=} \left(A, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}}\right)$$

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[18] The phrase "zeroth-order logic" is not standard. If you search the Web for "zeroth-order logic", you will find that some authors have used the phrase to refer to *propositional logic*, but this is not our meaning here. I take *zeroth-order logic* in the sense defined by Terence Tao; see, for example, Tao's blog [21].

[19] Jacques Herbrand is a mathematician of the early twentieth century who laid out the foundation for this theory.

[20] And beyond the simple examples in these lecture notes, the powerful concept of the *diagram* of a Σ-structure $\mathcal{A} \overset{\text{def}}{=} (A, \ldots)$ in model theory consists of all variable-free atomic wff's and their negations satisfied by the expanded structure $\mathcal{A}' \overset{\text{def}}{=} (\mathcal{A}, a)_{a \in A}$ which adds new constant symbols to the signature Σ, one constant symbol for each element in the universe A. If Σ' is the signature of $\mathcal{A}'$, the diagram of $\mathcal{A}$ consists of all the wff's in $\mathsf{Atoms}(\Sigma' \cup \{\approx\}, \varnothing)$ and their negations, a subset of $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma' \cup \{\approx\}, \varnothing)$. In Example 70 and Exercise 71 we make a simple use of the *method of diagrams*.

[21] I call it "intermediate" because it is not yet Herbrand theory in full generality. Understanding the "intermediate" case provides good intuition for how later steps are developed in Herbrand theory.

where A is the universe of $\mathcal{A}$, always assumed not empty, and $\mathcal{R}^{\mathcal{A}}$, $\mathcal{F}^{\mathcal{A}}$, and $\mathcal{C}^{\mathcal{A}}$, are the interpretations of the symbols of Σ in $\mathcal{A}$. By definition, a set of wff's $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ is satisfiable iff it has a model; in symbols, iff there is a Σ-structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$.

In general, an arbitrary Σ-structure $\mathcal{A}$ carries plenty of extra information unrelated to the satisfaction or non-satisfaction of Γ. A more economical notion is that of a *Herbrand Σ-structure $\mathcal{H}$*, which we specify as follows:

$$\mathcal{H} \ \stackrel{\text{def}}{=} \ \big(\mathsf{Terms}(\Sigma, \varnothing),\, \mathcal{R}^{\mathcal{H}},\, \mathcal{F},\, \mathcal{C}\big)$$

Note carefully how we have written the specification of $\mathcal{H}$:

- The universe of $\mathcal{H}$ is the set $\mathsf{Terms}(\Sigma, \varnothing)$ of ground terms.
- The underlying functions and constants are the members of $\mathcal{F} \cup \mathcal{C}$, all left uninterpreted.[22]

The only part in $\mathcal{H}$ that needs to be specified further is $\mathcal{R}$. Thus, two Herbrand Σ-structures, $\mathcal{H}_1$ and $\mathcal{H}_2$, are distinguished only by the interpretation of the relations in $\mathcal{R}^{\mathcal{H}_1}$ and $\mathcal{R}^{\mathcal{H}_2}$.

Theorem 58 (Basic Herbrand Theorem)**.** *Let $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ and $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$, both arbitrary. It then holds that:*

1. *φ is satisfiable $\Leftrightarrow$ φ has a Herbrand model.*
2. *Γ is satisfiable $\Leftrightarrow$ Γ has a Herbrand model.*

Proof. The implication "$\Leftarrow$", in both parts of the theorem, is immediate: If φ has a model, Herbrand or not, then φ is satisfiable, and likewise for Γ. The implication "$\Rightarrow$" is more delicate to prove.

Suppose φ is satisfiable, *i.e.*, there is a model $\mathcal{A} = \big(A,\, \mathcal{R}^{\mathcal{A}},\, \mathcal{F}^{\mathcal{A}},\, \mathcal{C}^{\mathcal{A}}\big)$ such that $\mathcal{A} \models \varphi$. Relative to this Σ-structure $\mathcal{A}$, we next define a Herbrand Σ-structure $\mathcal{H}$ such that $\mathcal{H} \models \varphi$. By definition, a Herbrand Σ-structure $\mathcal{H}$ is of the form:

$$\mathcal{H} \ \stackrel{\text{def}}{=} \ \big(\mathsf{Terms}(\Sigma, \varnothing),\, \mathcal{R}^{\mathcal{H}},\, \mathcal{F},\, \mathcal{C}\big)$$

where only the interpretation $\mathcal{R}^{\mathcal{H}}$ needs to be specified, which we do as follows. For every relation symbol $R \in \mathcal{R}$ of some arity $n \geqslant 0$ and all ground terms $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$, we set the truth value of the atom $R(t_1, \ldots, t_n) \in \mathsf{Atoms}(\Sigma, \varnothing)$ as follows:

$$R^{\mathcal{H}}(t_1, \ldots, t_n) \ \stackrel{\text{def}}{=} \ R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}).$$

So far, we have not used any information about the given φ, except that it has a model $\mathcal{A}$. To reach the desired conclusion, we prove a stronger result, namely: *For all $\psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$, it holds that $\mathcal{A} \models \psi \ \Leftrightarrow \mathcal{H} \models \psi$.* Note that this assertion holds for all $\psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$, and not only for the given φ. We prove this assertion by induction on the number of connectives in $\{\neg, \wedge, \vee, \rightarrow\}$ occurring in ψ. Remaining details of Part 1, and all of Part 2, are left as an exercise. $\square$

Exercise 59. Complete the proof of Theorem 58: (1) Write the details of the induction to complete the proof of the implication "$\Rightarrow$" in Part 1 of the theorem. (2) Explain how the implication "$\Rightarrow$" in Part 1 implies the implication "$\Rightarrow$" in Part 2 of the theorem. $\square$

We extend our analysis to the case when the equality symbol "$\approx$" occurs in the syntax of wff's. By definition, the interpretation of a ground term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ in a Herbrand Σ-structure is the term t itself. This poses a problem: Whereas two syntactically distinct ground terms t_1 and

[22] If you are familiar with notions of universal algebra, you will recognize that $\big(\mathsf{Terms}(\Sigma, \varnothing),\, =,\, \mathcal{F},\, \mathcal{C}\big)$ is what is called a *term algebra* or also an *absolutely free algebra*. In a term algebra, the signature is limited to $\mathcal{F} \cup \mathcal{C}$ where $\mathcal{C} \neq \varnothing$; there are no underlying relations other than equality "$=$"; and "$=$" always denotes syntactic equality between uninterpreted ground terms.

t_2 may be interpreted to the same element in the universe A of a Σ-structure $\mathcal{A}$, *i.e.*, so that $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$, they cannot be equated in the corresponding Herbrand Σ-structure.

So, how shall we construct a Herbrand Σ-structure $\mathcal{H}$ where satisfiability matches satisfiability in a given Σ-structure $\mathcal{A}$ with equality? The following suggests itself as a natural solution: In the $\mathcal{H}$ to be constructed, we add a new binary relation, denoted $\mathsf{eq}^{\mathcal{H}}$, which is a *congruence relation* on the universe $\mathsf{Terms}(\Sigma, \varnothing)$ whose congruence classes are each the set of all ground terms that are equated (*i.e.*, interpreted to the same element) in the given Σ-structure $\mathcal{A}$. To be precise, the augmented Herbrand structure is specified as:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing), \mathsf{eq}^{\mathcal{H}}, \mathcal{R}^{\mathcal{H}}, \mathcal{F}, \mathcal{C}\big)$$

and satisfies two conditions, the first of which is reproduced from the proof of Theorem 58:

1. For all $R \in \mathcal{R}$ of arity $n \geqslant 0$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$, $R^{\mathcal{H}}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$;
2. For all $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$, $\quad \mathsf{eq}^{\mathcal{H}}(t_1, t_2) \;\Leftrightarrow\; t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$.

Note that the signature of the constructed $\mathcal{H}$ is $\Sigma \cup \{\mathsf{eq}\}$, whereas the signature of the given $\mathcal{A}$ with equality is still Σ (or $\Sigma \cup \{\approx\}$, if we ignore our standing assumption that "$\approx$" is not part of a signature). Though our notation does not indicate it, purposely in order to avoid the clutter, the interpretation of eq and the interpretation of every $R \in \mathcal{R}$ in $\mathcal{H}$ depends on $\mathcal{A}$.[23]

To develop some intuition of how the congruence classes of $\mathsf{eq}^{\mathcal{H}}$ may behave, do Exercise 76 before proceeding further. Consider now a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure, separate from any Σ-structure $\mathcal{A}$:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing), \mathsf{eq}^{\mathcal{H}}, \mathcal{R}^{\mathcal{H}}, \mathcal{F}, \mathcal{C}\big)$$

We want the relation $\mathsf{eq}^{\mathcal{H}}$ to behave in a particular way, not as an arbitrary binary relation, but as a *congruence relation* on the universe $\mathsf{Terms}(\Sigma, \varnothing)$. Hence, for a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure $\mathcal{H}$ to be *well-behaved*, we require that $\mathsf{eq}^{\mathcal{H}}$ satisfies the conditions of a congruence, namely:

1. *reflexivity*:
   $\mathsf{eq}^{\mathcal{H}}(t, t)$ for all $t \in \mathsf{Terms}(\Sigma, \varnothing)$,
2. *symmetry*:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, t_2)$ then $\mathsf{eq}^{\mathcal{H}}(t_2, t_1)$ for all $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$,
3. *transitivity*:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, t_2)$ and $\mathsf{eq}^{\mathcal{H}}(t_2, t_3)$ then $\mathsf{eq}^{\mathcal{H}}(t_1, t_3)$ for all $t_1, t_2, t_3 \in \mathsf{Terms}(\Sigma, \varnothing)$,
4. *compatible with* $\mathcal{F}$:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, u_1), \ldots, \mathsf{eq}^{\mathcal{H}}(t_n, u_n)$ and $f \in \mathcal{F}$ has arity $n \geqslant 1$,
   then $\mathsf{eq}^{\mathcal{H}}\big(f(t_1, \ldots, t_n), f(u_1, \ldots, u_n)\big)$ for all $t_1, u_1, \ldots, t_n, u_n \in \mathsf{Terms}(\Sigma, \varnothing)$,
5. *compatible with* $\mathcal{R}$:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, u_1), \ldots, \mathsf{eq}^{\mathcal{H}}(t_n, u_n)$ and $R \in \mathcal{R}$ has arity $n \geqslant 0$,
   then $R^{\mathcal{H}}(t_1, \ldots, t_n) \;\Leftrightarrow\; R^{\mathcal{H}}(u_1, \ldots, u_n)$ for all $t_1, u_1, \ldots, t_n, u_n \in \mathsf{Terms}(\Sigma, \varnothing)$.

Exercise 60. Let $\mathcal{A}$ be a Σ-structure and $\mathcal{H}$ a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure. Show that if $\mathcal{H}$ is induced by $\mathcal{A}$, then $\mathcal{H}$ is well-behaved; *i.e.*, the relation $\mathsf{eq}^{\mathcal{H}}$ satisfies the preceding 5 conditions. □

Definition 61 (*Enforcing* $\mathsf{eq}^{\mathcal{H}}$ *as a Congruence Relation*)**.** Instead of qualifying a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure as being *well-behaved*, we can omit the qualifier and require instead that it satisfies the following set of ground atomic wff's $\Delta_{\mathsf{eq}} \subseteq \mathsf{Atoms}\big(\Sigma \cup \{\mathsf{eq}\}, \varnothing\big)$, which is the least set such that:

$$\Delta_{\mathsf{eq}} \supseteq \Big\{ \mathsf{eq}(t, t) \;\Big|\; t \in \mathsf{Terms}(\Sigma, \varnothing) \Big\} \cup$$

[23] A more precise notation would be therefore to write "$\mathcal{H}(\mathcal{A})$" instead of just "$\mathcal{H}$", making it clear that two distinct Σ-structures $\mathcal{A}_1$ and $\mathcal{A}_2$ induce two distinct Herbrand structures $\mathcal{H}(\mathcal{A}_1)$ and $\mathcal{H}(\mathcal{A}_2)$.

$$\Big\{\, \mathsf{eq}(t_1,t_2) \;\Big|\; \mathsf{eq}(t_2,t_1) \text{ and } t_i \in \mathsf{Terms}(\Sigma,\varnothing) \,\Big\} \cup$$
$$\Big\{\, \mathsf{eq}(t_1,t_3) \;\Big|\; \mathsf{eq}(t_1,t_2),\ \mathsf{eq}(t_2,t_3), \text{ and } t_i \in \mathsf{Terms}(\Sigma,\varnothing) \,\Big\} \cup$$
$$\Big\{\, \mathsf{eq}\big(f(t_1,\dots,t_n), f(u_1,\dots,u_n)\big) \;\Big|\; \mathsf{eq}(t_1,u_1),\dots,\mathsf{eq}(t_n,u_n), \text{ and } t_i,u_i \in \mathsf{Terms}(\Sigma,\varnothing) \,\Big\} \cup$$
$$\Big\{\, R(t_1,\dots,t_n) \;\Big|\; \mathsf{eq}(t_1,u_1),\dots,\mathsf{eq}(t_n,u_n),\ R(u_1,\dots,u_n), \text{ and } t_i,u_i \in \mathsf{Terms}(\Sigma,\varnothing) \,\Big\}.$$

The five parts in the definition of Δ_{eq} correspond to the five conditions preceding Exercise 60. Because Δ_{eq} is the least such subset of ground atoms, if $\mathcal{H}$ satisfies no ground atoms other than Δ_{eq}, then $\mathsf{eq}^{\mathcal{H}}$ coincides with equality on the set of ground terms and each $\mathsf{eq}^{\mathcal{H}}$-congruence class will be a singleton set. In general, however, $\mathcal{H}$ satisfies additional ground atoms and some $\mathsf{eq}^{\mathcal{H}}$-congruence classes are not singleton sets. □

We define a translation $\boxed{\approx\ \mapsto \mathsf{eq}}$ from $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\},\varnothing)$ to $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\},\varnothing)$ as follows:[24]

$$\boxed{\approx\ \mapsto \mathsf{eq}}(\varphi) \stackrel{\text{def}}{=} \begin{cases} \mathsf{eq}(t_1,t_2) & \text{if } \varphi = (t_1 \approx t_2),\\ \varphi & \text{if } \varphi \in \mathsf{Atoms}(\Sigma,\varnothing),\\ \neg\left(\boxed{\approx\ \mapsto \mathsf{eq}}(\psi)\right) & \text{if } \varphi = \neg\psi,\\ \left(\boxed{\approx\ \mapsto \mathsf{eq}}(\varphi_1)\right) \diamond \left(\boxed{\approx\ \mapsto \mathsf{eq}}(\varphi_2)\right) & \text{if } \varphi = \varphi_1 \diamond \varphi_2 \text{ and } \diamond \in \{\wedge,\vee,\to\}\ . \end{cases}$$

In words, all that $\boxed{\approx\ \mapsto \mathsf{eq}}$ does is to replace every atom $(t_1 \approx t_2)$ by $\mathsf{eq}(t_1,t_2)$.

We write $\boxed{\mathsf{eq} \mapsto\ \approx}$ for the inverse translation from $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\},\varnothing)$ to $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\},\varnothing)$, which simply replaces every $\mathsf{eq}(t_1,t_2)$ by $(t_1 \approx t_2)$. We omit a formal definition of $\boxed{\mathsf{eq} \mapsto\ \approx}$. Both translations, $\boxed{\approx\ \mapsto \mathsf{eq}}$ and $\boxed{\mathsf{eq} \mapsto\ \approx}$ are needed to write the next theorem and its proof.

Theorem 62 (Intermediate Herbrand Theorem)**.** *Let φ be an arbitrary wff and Γ an arbitrary set of wff's in $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\},\varnothing)$. Let $\psi \stackrel{\text{def}}{=} \boxed{\approx\ \mapsto \mathsf{eq}}(\varphi)$ and $\Delta \stackrel{\text{def}}{=} \boxed{\approx\ \mapsto \mathsf{eq}}(\Gamma)$. It then holds that:*

1. *φ is satisfiable $\Leftrightarrow$ ψ has a well-behaved Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*
2. *Γ is satisfiable $\Leftrightarrow$ Δ has a well-behaved Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*

Equivalently, using Δ_{eq} from Definition 61, it holds that:

1. *φ is satisfiable $\Leftrightarrow$ $\{\psi\} \cup \Delta_{\mathsf{eq}}$ has a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*
2. *Γ is satisfiable $\Leftrightarrow$ $\Delta \cup \Delta_{\mathsf{eq}}$ has a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*

Proof. For the "$\Rightarrow$" implication in both parts of the theorem, the steps here are very similar to those in the proof of Theorem 58, which we leave as an exercise.

In contrast to the proof of Theorem 58, the "$\Leftarrow$" implication here requires extra care. Consider the "$\Leftarrow$" implication in Part 1, the same issues applies to Part 2. Suppose that ψ has a well-behaved Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model $\mathcal{H}$. To show φ is satisfiable, we want a model for it.

We recover φ from ψ by applying $\boxed{\mathsf{eq} \mapsto\ \approx}$ to ψ, *i.e.*, $\varphi = \boxed{\mathsf{eq} \mapsto\ \approx}(\psi)$, but for which $\mathcal{H}$ cannot be a model because the signature of φ is not $\mathcal{H}$'s signature: φ uses "$\approx$", which is not in $\mathcal{H}$'s signature. One case is simple, however, which occurs when every congruence class defined by the congruence $\mathsf{eq}^{\mathcal{H}}$ consists of a single element, in which case it suffices to replace $\mathsf{eq}^{\mathcal{H}}$ by equality "$=$" to make $\mathcal{H}$ a model of φ.

[24] See footnote 16 for our convention of naming transformations of syntax.

The general case is when some of the congruence classes of $\mathsf{eq}^{\mathcal{H}}$ are not singleton sets. In this case, we first define a homomorphic image (or quotient) of $\mathcal{H}$ modulo $\mathsf{eq}^{\mathcal{H}}$, which produces a $(\Sigma \cup \{\mathsf{eq}\})$-structure $\mathcal{H}'$ where every congruence class is a singleton set. It is then easy to argue that $\mathcal{H}'$ is a model of ψ. Finally, by replacing $\mathsf{eq}^{\mathcal{H}'}$ by equality "=" in $\mathcal{H}'$, we obtain a model $\mathcal{H}''$ of φ. □

Exercise 63. As much as you can, write the details of the proof of Theorem 62. For the the implication "⇒", use the proof of Theorem 58 and Exercise 59 as a guide for what you need to do. For the converse implication "⇐", you will need to brush up your knowledge of what a *homomorphic image*, or *quotient structure*, modulo a congruence relation is. □

4.2 Compactness and Completeness in ZOL

A close examination of the Herbrand theory shows that the ground atoms in $\mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ are all that matters in the specification of a Herbrand model $\mathcal{H}$, more precisely, in the interpretation of the relations in $\{\mathsf{eq}\} \cup \mathcal{R}$. The interpretations of the symbols in $\mathcal{F} \cup \mathcal{C}$ are themselves and the same in all Herbrand models, and thus play no role in differentiating Herbrand models.

And since there are only propositional connectives, but no variables and no quantifiers in ground atoms, each ground atom plays the role of a propositional variable. This suggests the introduction of fresh variables, one for every ground atom; more precisely, we introduce a fresh set $\mathcal{Y}$ of propositional variables by:

$$\mathcal{Y} \;=\; \Big\{ Y_\alpha \;\Big|\; \alpha \in \mathsf{Atoms}\big(\Sigma \cup \{\mathsf{eq}\}, \varnothing\big) \Big\}.$$

Each member of $\mathcal{Y}$ is named by the upper-case letter "Y" subscripted with a ground atom α. (We use upper-case "Y" to keep these new variables separate from other variables in these notes.) For the rest of this section we consider only $\mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$, which includes $\mathsf{Atoms}(\Sigma, \varnothing)$ as a proper subset.

We define a translation from ZOL to PL, named "$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}$" suggestively:

$$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}} \;:\; \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing) \;\to\; \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Y})$$

such that for all $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$:

$$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi) \;\stackrel{\text{def}}{=}\; \begin{cases} Y_\alpha & \text{if } \varphi = \alpha \in \mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing), \\ \neg\,\big(\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\psi)\big) & \text{if } \varphi = \neg\psi, \\ \big(\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi_1)\big) \diamond \big(\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi_2)\big) & \text{if } \varphi = \varphi_1 \diamond \varphi_2 \text{ and } \diamond \in \{\wedge, \vee, \to\}\ . \end{cases}$$

Informally in words, all that $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}$ does is to replace every atomic wff $\alpha \in \mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ by the propositional variable Y_α.

We write $\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}$ for the inverse transformation from $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Y})$ to $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ which simply replaces every variable Y_α by the corresponding ground atom $\alpha \in \mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$. We omit a formal definition of $\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}$.

As usual, we extend the definitions of $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}$ and $\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}$ to sets of wff's, in the obvious way:

$$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Gamma) \;\stackrel{\text{def}}{=}\; \Big\{ \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi) \;\Big|\; \varphi \in \Gamma \Big\},$$

$$\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}(\Delta) \;\stackrel{\text{def}}{=}\; \Big\{ \boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}(\psi) \;\Big|\; \psi \in \Delta \Big\}.$$

We are now ready to state a *transfer principle* from ZOL to PL.

Lemma 64 (Transfer Principle)**.** *Let Γ be an arbitrary set, possibly finite, of wff's in* $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, and let* $\Delta \stackrel{\text{def}}{=} \boxed{\approx\ \mapsto \mathsf{eq}}(\Gamma)$*. It then holds that:*

1. *Γ is satisfiable (in the sense of* zeroth-order logic*)* $\Leftrightarrow$
   $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$ *is satisfiable (in the sense of* propositional logic*).*
2. *Γ is finitely satisfiable (in the sense of* zeroth-order logic*)* $\Leftrightarrow$
   $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$ *is finitely satisfiable (in the sense of* propositional logic*).*

Δ_{eq} *is the set of ground atoms from Definition 61 which enforce that a Herbrand* $(\Sigma \cup \{\mathsf{eq}\})$*-structure is well-behaved.*

Proof. It suffices to prove Part 1 since Γ is possibly a finite set. For the proof of Part 1, it suffices to show, by Theorem 62:

> $\Delta \cup \Delta_{\mathsf{eq}}$ has a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model $\Leftrightarrow$
> $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$ is satisfiable (in the sense of *propositional logic*).

Let $\mathcal{Y}'$, a subset of $\mathcal{Y}$, be the set of propositional variables occurring in $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$. For the "$\Rightarrow$" implication, we assume there is a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model for $\Delta \cup \Delta_{\mathsf{eq}}$, and then derive from it a truth assignment $\sigma : \mathcal{Y} \to \{\mathit{false}, \mathit{true}\}$ such that $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$. We omit the easy details.

For the converse implication "$\Leftarrow$", starting from a truth assignment $\sigma : \mathcal{Y} \to \{\mathit{false}, \mathit{true}\}$ such that $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$, we define a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model $\mathcal{H}$ for $\Delta \cup \Delta_{\mathsf{eq}}$. All we need to specify is the interpretation of every symbol in $\mathcal{R} \cup \{\mathsf{eq}\}$ in $\mathcal{H}$. We specify the interpretation of $R \in \mathcal{R}$ by assigning a truth value to every member of the set:

$$\{\, R^{\mathcal{H}}(t_1, \ldots, t_n) \mid R \text{ has arity } n \text{ and } t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing) \,\}.$$

An expression of the form $R(t_1, \ldots, t_n)$ is an atom, call it α, in the set $\mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$, to which corresponds a variable $Y_\alpha \in \mathcal{Y}$. We now define:

$$R^{\mathcal{H}}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \begin{cases} \mathit{false} & \text{if } \sigma(Y_\alpha) = \mathit{false}, \\ \mathit{true} & \text{if } \sigma(Y_\alpha) = \mathit{true}. \end{cases}$$

Because $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta)$, we next use structural induction (easy details omitted) on an arbitrary $\varphi \in \Delta$, to conclude that $\mathcal{H} \models \Delta$.

So far, we have defined a Herbrand Σ-model $\mathcal{H}$ for Δ. We need to expand $\mathcal{H}$ to $(\Sigma \cup \{\mathsf{eq}\})$-model for Δ_{eq}. We proceed in the same way as for $R \in \mathcal{R}$, by assigning a truth value to every member of the set:

$$\{\, \mathsf{eq}^{\mathcal{H}}(t_1, t_2) \mid t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing) \,\},$$

guaranteeing the $\mathsf{eq}^{\mathcal{H}}$ is a congruence relation. We use the fact that $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta_{\mathsf{eq}})$, together with structural induction on wff's in Δ_{eq}, to conclude that $\mathcal{H} \models \Delta_{\mathsf{eq}}$. All remaining details omitted. $\square$

Theorem 65 (Compactness for Zeroth-Order Logic, Version I)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary set of wff's. We then have: Γ is satisfiable $\Leftrightarrow$ Γ is finitely satisfiable.*

Proof. This follows from Theorem 2 and Lemma 64. $\square$

Corollary 66 (Compactness for Zeroth-Order Logic, Version II)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary set of wff's, and* $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary wff. We then have:* $\Gamma \models \varphi$ $\Leftrightarrow$ *there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \models \varphi$*.*

Proof. This follows from Corollary 7, Lemma 64, and Theorem 65. Some of the details reproduce details in the proof of Corollary 7, all left as an exercise. □

Exercise 67. As much as you can, supply the details in the proof of Corollary 66. □

Lemma 68 is a weak form of the Completeness Theorem; it does not need Compactness for its proof.

Lemma 68. *Let $\varphi_1, \ldots, \varphi_n, \psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$. If $\varphi_1, \ldots, \varphi_n \models \psi$ then $\varphi_1, \ldots, \varphi_n \vdash \psi$.*

Proof. **(MORE TO COME)** □

Theorem 69 (Completeness for Zeroth-Order Logic)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$, an arbitrary set of wff's, and $\psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$, an arbitrary wff. It holds that if $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. A straightforward consequence of Corollary 66 and Lemma 68. The details here are almost identical to the details in the proof of Theorem 13, which is Completeness for PL. □

4.3 Applications and Exercises

We revisit some of the earlier applications, using the formalism and conventions of ZOL, and then add a few fresh ones. Our first Example 70 and Exercise 71 reproduce many of the ideas in our earlier analysis of *graph coloring* in Exercise 21, but now using the conventions and formalism of ZOL.

Example 70 (*Graph Coloring*)**.** We take a countably infinite simple graph G to be a structure of the form $G \overset{\text{def}}{=} (\mathbb{N}, E)$ where $E \subseteq \mathbb{N} \times \mathbb{N}$. The set $\mathbb{N}$ of natural numbers is the set of vertices, and $E(m, n)$ means that there is an edge from vertex m to vertex n.

A graph $G \overset{\text{def}}{=} (\mathbb{N}, E)$ is *simple* if there are no self-loops and no multiple edges between the same two vertices. The signature consists of a single binary relation, $\Sigma = \{R\}$, whose interpretation in G is E, *i.e.*, $R^G = E$. We have no quantifiers and no variables in ZOL; hence, in order to express properties involving vertices and edges, we expand the signature Σ to Σ', with the latter including a constant symbol a_n for every vertex $n \in \mathbb{N}$:

$$\Sigma' \overset{\text{def}}{=} \{R\} \cup \{\, a_n \mid n \in \mathbb{N}\,\}.$$

To require that all the constant symbols in $\{a_n \,|\, n \in \mathbb{N}\}$ are interpreted distinctly in a Σ'-structure, we define:

$$\Theta_1 \overset{\text{def}}{=} \{\, \neg(a_m \approx a_n) \mid m, n \in \mathbb{N} \text{ and } m \neq n\,\}.$$

In a Σ'-structure $\mathcal{M} \overset{\text{def}}{=} (M, \ldots)$ such that $\mathcal{M} \models \Theta_1$, we can take the universe M to be a superset of $\mathbb{N}$ and the interpretation of every a_n to be n, *i.e.*, $a_n^{\mathcal{M}} = n$. A particular Σ'-structure is an expansion G' of $G = (\mathbb{N}, E)$ where all the elements of the universe $\mathbb{N}$ are now distinguished constants:

$$G' \overset{\text{def}}{=} (\mathbb{N}, E, 0, 1, 2, \ldots)$$

in order to make the signature of G' match Σ'. We express the property that the graph is undirected by the following set Θ_2 of wff's over signature Σ':

$$\Theta_2 \overset{\text{def}}{=} \{\, R(a_m, a_n) \to R(a_n, a_m) \mid m, n \in \mathbb{N} \text{ and } m \neq n\,\},$$

i.e., we view an undirected edge between vertices m and n as two directed edges, one in each direction. The property that there are no self-loops is expressed by:

$$\Theta_3 \stackrel{\text{def}}{=} \{ \neg R(a_n, a_n) \mid n \in \mathbb{N} \}.$$

Suppose we are interested in the colorability of one particular simple graph $G = (\mathbb{N}, E)$. We assume therefore that the three preceding subsets $\Theta_1, \Theta_2, \Theta_3 \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma' \cup \{\approx\}, \varnothing)$ have been defined to make the following satisfaction by the expansion G' hold:

$$G' \models \Theta_1 \cup \Theta_2 \cup \Theta_3.$$

The set $\Theta_1 \cup \Theta_2 \cup \Theta_3$ is what is called the *diagram* of G, which uniquely specifies the structure of G, as well as that of its expansion G', up to isomorphism. The following is an easy fact:

Claim: *Let $\mathcal{M}$ be a Σ'-structure. The graph G' is isomorphically embedded in $\mathcal{M}$ iff $\mathcal{M} \models \Theta_1 \cup \Theta_2 \cup \Theta_3$.*

We next introduce k unary relation symbols $\{C_1, \ldots, C_k\}$ to express the k-colorability of G, where $k \geqslant 1$. We thus expand the signature once more to Σ'':

$$\Sigma'' \stackrel{\text{def}}{=} \Sigma' \cup \{ C_1, \ \ldots, C_k \}.$$

We express the property that *every vertex is assigned a color* by writing:

$$\Phi_1 \stackrel{\text{def}}{=} \Big\{ C_1(a_n) \vee \cdots \vee C_k(a_n) \ \Big| \ n \in \mathbb{N} \Big\}.$$

The property that *every vertex is assigned at most one color* is expressed by:

$$\Phi_2 \stackrel{\text{def}}{=} \Big\{ C_i(a_n) \to \bigwedge \{ \neg C_j(a_n) \mid 1 \leqslant j \leqslant k \text{ and } i \neq j \} \ \Big| \ n \in \mathbb{N} \Big\}.$$

The property that *no two adjacent vertices have the same color* is expressed by:

$$\Phi_3 \stackrel{\text{def}}{=} \Big\{ R(a_m, a_n) \to \neg\big(C_i(a_m) \wedge C_i(a_n)\big) \ \Big| \ m, n \in \mathbb{N} \text{ and } 1 \leqslant i \leqslant k \Big\}.$$

The set of wff's $\Phi_1 \cup \Phi_2 \cup \Phi_3$ over signature Σ'' define conditions for the k-colorability of a graph $G = (\mathbb{N}, E)$ or its expansion $G' = (\mathbb{N}, E, 0, 1, 2, \ldots)$, but it does not put conditions on elements outside $\mathbb{N}$ in a Σ''-structure $\mathcal{M}$ such that:

$$\mathcal{M} \models \Phi_1 \cup \Phi_2 \cup \Phi_3.$$

Let Γ be the set of wff's in $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma_2 \cup \{\approx\}, \varnothing)$ defined by:

$$\Gamma \stackrel{\text{def}}{=} \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Phi_1 \cup \Phi_2 \cup \Phi_3.$$

A model $\mathcal{M}$ of Γ is a Σ''-structure of the form:

$$\mathcal{M} \stackrel{\text{def}}{=} \Big(M, \ R^{\mathcal{M}}, C_1^{\mathcal{M}}, \ \ldots \ , C_k^{\mathcal{M}}, a_0^{\mathcal{M}}, a_1^{\mathcal{M}}, a_2^{\mathcal{M}}, \ \ldots \Big)$$

The reduct $\mathcal{M}_0 \stackrel{\text{def}}{=} (M, \ R^{\mathcal{M}})$ is obtained from $\mathcal{M}$ by omitted all the unary relations $\{ C_1^{\mathcal{M}}, \ \ldots \ , C_k^{\mathcal{M}} \}$ and all the distinguished constants $\{ a_n^{\mathcal{M}} \mid n \in \mathbb{N} \}$. By the **Claim** above, the particular graph $G = (\mathbb{N}, E)$ of our interest is isomorphically embedded in $\mathcal{M}_0$.

Exercise 71 (*Graph Coloring* (continued))**.** The background for this exercise is Example 70. We are interested in the k-colorability of a particular countably infinite simple graph $G = (\mathbb{N}, E)$ which is otherwise arbitrary.

1. Provide the details of the proof of the **Claim** in Example 70.
2. Prove that if every finite subgraph of G is k-colorable, then so is G k-colorable.
   Hint: Apply Compactness of ZOL to Γ.

3. Prove that if the given $G = (\mathbb{N}, E)$ is a planar graph, then G is 4-colorable.
   Hint: Every finite planar graph is 4-colorable. □

Exercise 72 (*Basic Herbrand Theorem* (specialized))**.** Let $\Sigma = \{R, f, g, c_0, c_1\}$ be a first-order signature, where R is a binary relation symbol, f is a unary function symbol, g is a binary function symbol, and c_0 and c_1 are constant symbols. In this exercise you are asked to revisit parts of the proof of Theorem 58 relative to a particular Σ-structure $\mathcal{A}$. More precisely, you have to show below that, given an arbitrary $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$:

$$\mathcal{A} \models \varphi \;\Leftrightarrow\; \mathcal{H} \models \varphi,$$

where $\mathcal{H}$ is the Herbrand Σ-structure induced by $\mathcal{A}$. We define $\mathcal{A} \stackrel{\text{def}}{=} \big(A,\, R^{\mathcal{A}},\, f^{\mathcal{A}},\, g^{\mathcal{A}},\, {c_0}^{\mathcal{A}},\, {c_1}^{\mathcal{A}}\big)$ where $A = \mathbb{N}$ and:

$$\begin{aligned}
R^{\mathcal{A}} &= \{\,(m,n) \mid m,n \in \mathbb{N} \text{ with } m < n\,\} && (R^{\mathcal{A}} \text{ is the binary less-than relation on } \mathbb{N})\\
f^{\mathcal{A}} &= \{\,(m,m+2) \mid m \in \mathbb{N}\,\} && (f^{\mathcal{A}} \text{ is the two-successor function on } \mathbb{N})\\
g^{\mathcal{A}} &= \{\,(m,n,m+n) \mid m,n \in \mathbb{N}\,\} && (g^{\mathcal{A}} \text{ is the addition function on } \mathbb{N})
\end{aligned}$$

and ${c_0}^{\mathcal{A}} = 0$ and ${c_1}^{\mathcal{A}} = 1$. More succintly, we can therefore write $\mathcal{A} \stackrel{\text{def}}{=} \big(\mathbb{N},\, <,\, f^{\mathcal{A}},\, +,\, 0,\, 1\big)$ where we use $<$ and $+$ in infix position. There are four parts:

1. We want you to write an inductive definition of $\mathsf{Terms}(\Sigma, \varnothing)$ in a particular way. The set $\mathsf{Terms}(\Sigma, \varnothing)$ is the smallest such that:

   $$\mathsf{Terms}(\Sigma, \varnothing) \supseteq \{\,c_0, c_1\,\} \;\cup\; \{\,f(t) \mid t \in \mathsf{Terms}(\Sigma, \varnothing)\} \;\cup\; \cdots \quad \cdots$$

   You have to complete the definition.

2. Define the Herbrand Σ-structure $\mathcal{H}$ induced by $\mathcal{A}$:

   $$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing),\, R^{\mathcal{H}},\, f,\, g,\, c_0,\, c_1\big)$$

   As in the proof of Theorem 58, you only need to define the relation $R^{\mathcal{H}}$.

3. Let $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$ be arbitrary terms. Define an algorithm that decides whether $\mathcal{H} \models R^{\mathcal{H}}(t_1, t_2)$ or $\mathcal{H} \not\models R^{\mathcal{H}}(t_1, t_2)$ depending on the syntactic structure of t_1 and t_2.

   Hint 1: An initial part of your algorithm transforms an arbitrary $t \in \mathsf{Terms}(\Sigma, \varnothing)$ into a term u of the form $f^{(p)}c_0$ or $f^{(p)}c_1$ such that $t^{\mathcal{A}} = u^{\mathcal{A}}$, for some $p \geqslant 0$, where we write $f^{(p)}$ to denote that f is applied p times.

   Hint 2: The rest of your algorithm decides whether $\mathcal{H} \models R^{\mathcal{H}}(u_1, u_2)$ or $\mathcal{H} \not\models R^{\mathcal{H}}(u_1, u_2)$ where each of u_1 and u_2 is of the form $f^{(p)}c_0$ or $f^{(p)}c_1$.

4. Prove, for every $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$, that $\mathcal{A} \models \varphi \;\Leftrightarrow\; \mathcal{H} \models \varphi$ by structural induction on the definition of φ. □

Exercise 73 (*Intermediate Herbrand Theorem* (specialized))**.** This is a continuation of Exercise 72. We now consider an expansion of $\mathcal{A}$ with equality, *i.e.*, $\mathcal{A} \stackrel{\text{def}}{=} \big(\mathbb{N},\, =,\, <,\, f^{\mathcal{A}},\, +,\, 0,\, 1\big)$. We call this expansion with the same name "$\mathcal{A}$" in order to minimize the notational clutter.

Consider an arbitrary $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$ and let $\psi \stackrel{\text{def}}{=} \boxed{\approx\, \mapsto \mathsf{eq}}(\varphi)$. Recall that ψ is just φ after every occurrence of "$\approx$" is replaced by "eq". Consider the Herbrand structure $\mathcal{H}$ induced by $\mathcal{A}$, as defined in the paragraph after Exercise 59:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing),\, \mathsf{eq}^{\mathcal{H}},\, R^{\mathcal{H}},\, f,\, g,\, c_0,\, c_1\big)$$

This is an expansion, with the binary relation $\mathsf{eq}^{\mathcal{H}}$, of the Herbrand structure $\mathcal{H}$ in Exercise 72. According to Exercise 60, $\mathcal{H}$ is a well-behaved Herbrand structure. Reviewing the proof of Theorem 62, it is not difficult to see that $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{H} \models \psi$ and thus deciding whether φ is satisfied in $\mathcal{A}$ can be transferred to the question of whether ψ is satisfied in the Herbrand $\mathcal{H}$.

In this exercise, we construct a different structure $\mathcal{H}'$, derived from $\mathcal{H}$, such that $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{H}' \models \psi$, for an alternative way of deciding satisfiability in $\mathcal{A}$.

1. For every term $t \in \mathsf{Terms}(\Sigma, \varnothing)$, we write $[\,\mathsf{eq}^{\mathcal{H}}(t,*)\,]$ to denote the congruence class of t relative to the congruence $\mathsf{eq}^{\mathcal{H}}$:

$$[\,\mathsf{eq}^{\mathcal{H}}(t,*)\,] \stackrel{\text{def}}{=} \Big\{ u \in \mathsf{Terms}(\Sigma, \varnothing) \;\Big|\; \mathsf{eq}^{\mathcal{H}}(t,u) \Big\} = \Big\{ u \in \mathsf{Terms}(\Sigma, \varnothing) \;\Big|\; t^{\mathcal{A}} = u^{\mathcal{A}} \Big\}.$$

   For each of the following questions, if your answer is positive, give a term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ as a specific example. If your answer is negative, justify it in no more than a couple of lines:

   (a) Is there a term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ such that its congruence class $[\,\mathsf{eq}^{\mathcal{H}}(t,*)\,]$ is infinite?

   (b) Is there a term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ such that its congruence class $[\,\mathsf{eq}^{\mathcal{H}}(t,*)\,]$ is finite?

   (c) For every bound $n \in \mathbb{N}$, is there a term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ such that for every term $u \in [\,\mathsf{eq}^{\mathcal{H}}(t,*)\,]$, it holds that u mentions at most n occurrences of function symbol f?

   (d) For every bound $n \in \mathbb{N}$, is there a term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ such that for every term $u \in [\,\mathsf{eq}^{\mathcal{H}}(t,*)\,]$, it holds that u mentions at most n occurrences of function symbol g?

2. We define the *set of terms modulo* $\mathsf{eq}_{\mathcal{H}}$ as follows:

$$(\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}}) \stackrel{\text{def}}{=} \Big\{ [\,\mathsf{eq}^{\mathcal{H}}(t,*)\,] \;\Big|\; t \in \mathsf{Terms}(\Sigma, \varnothing) \Big\}$$

   Note that $(\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}})$ is a set of disjoint subsets of terms which form a partition of $\mathsf{Terms}(\Sigma, \varnothing)$. The desired structure $\mathcal{H}'$ is of the form:

$$\mathcal{H}' \stackrel{\text{def}}{=} \Big((\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}}),\ \mathsf{eq}^{\mathcal{H}'},\ R^{\mathcal{H}'},\ f^{\mathcal{H}'},\ g^{\mathcal{H}'},\ C_0,\ C_1 \Big)$$

   where the relations and functions in the signature of $\mathcal{H}'$ are defined as follows:

$$\begin{aligned}
\mathsf{eq}^{\mathcal{H}'} &\stackrel{\text{def}}{=} \Big\{ (T_1,T_2) \in (\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}})^2 \;\Big|\; \mathsf{eq}^{\mathcal{H}}(t_1,t_2) \text{ for all } (t_1,t_2) \in T_1 \times T_2 \Big\} \\
R^{\mathcal{H}'} &\stackrel{\text{def}}{=} \Big\{ (T_1,T_2) \in (\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}})^2 \;\Big|\; R^{\mathcal{H}}(t_1,t_2) \text{ for all } (t_1,t_2) \in T_1 \times T_2 \Big\} \\
f^{\mathcal{H}'}(T) &\stackrel{\text{def}}{=} \Big\{ f(t) \;\Big|\; t \in T \Big\} \qquad \text{for all } T \in (\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}}) \\
g^{\mathcal{H}'}(T_1,T_2) &\stackrel{\text{def}}{=} \Big\{ g(t_1,t_2) \;\Big|\; (t_1,t_2) \in T_1 \times T_2 \Big\} \qquad \text{for all } T_1, T_2 \in (\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}}) \\
C_0 &\stackrel{\text{def}}{=} [\,\mathsf{eq}^{\mathcal{H}}(c_0,*)\,] \\
C_1 &\stackrel{\text{def}}{=} [\,\mathsf{eq}^{\mathcal{H}}(c_1,*)\,]
\end{aligned}$$

   where we write $(\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}})^2$ to denote $(\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}}) \times (\mathsf{Terms}(\Sigma, \varnothing)/\mathsf{eq}_{\mathcal{H}})$.

   Your task is to prove that $\mathcal{H}'$ is isomorphic to $\mathcal{A}$. This implies that $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{H}' \models \psi$. $\square$

Example 74 (*Arithmetic*)**.** A zeroth-order language for arithmetic over the set $\mathbb{N}$ of natural numbers may include the equality relation $=$, the order relation $<$, the binary operations $+$ and $\times$, the successor operation S, and the constants 0 and 1 (or all of them $0, 1, 2, 3, \ldots$ with 2 being a shorthand for $S(1)$, 3 a shorthand for $S(S(1))$, etc.). The structure under consideration is therefore $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, =, <, +, \times, S, 0, 1)$. An example of zeroth-order wff's are the wff's expressing the Pythagorean property $i \times i + j \times j = k \times k$, which are satisfied in $\mathcal{N}$ for some triples but not all triples (i, j, k) of natural numbers. We ignore here the difference between the equation and the uninterpreted wff corresponding to it.

If we want to express other facts of airthmetic, we may expand the signature with other relations and operations. We may for example add a unary relation *prime*( ) which is *true* or *false*

$\otimes$	e	1	2	3
e	e	1	2	3
1	1	e	3	2
2	2	3	e	1
3	3	2	1	e

$(\,)^{-1}$	
e	e
1	1
2	2
3	3

Figure 4.1: Group $\mathcal{K}$ in Example 75. I omit the superscript "$\mathcal{K}$" on e for clarity, $\mathcal{K}$'s group identity.

depending on whether its argument is a prime or not. We can now formally express that (i, j) is a *Twin-Prime Pair* by writing:

$$prime(i) \wedge prime(j) \wedge (i + 2 = j)$$

which says "there are two prime numbers whose difference is 2", or that (i, j) is a *Bertrand-Chebyshev Pair*:

$$prime(i) \wedge prime(j) \wedge (i + i > j)$$

which says "there are two prime numbers whose difference is less than the smaller of the two".[25]

Example 75 (*Groups*)**.** In algebra, a group $\mathcal{G}$ is said to be a 2-generated group if its universe can be generated by at most two of its elements, using the binary group operation. Such a group can be specified as a Σ-structure with signature $\Sigma \stackrel{\text{def}}{=} \{f, i, \mathsf{e}, a, b\}$:

$$\mathcal{G} \;\stackrel{\text{def}}{=}\; \big(G, \approx^{\mathcal{G}}, f^{\mathcal{G}}, i^{\mathcal{G}}, \mathsf{e}^{\mathcal{G}}, a^{\mathcal{G}}, b^{\mathcal{G}}\big),$$

where $f^{\mathcal{G}}$ is the binary group operation, $i^{\mathcal{G}}$ is the unary inverse operation, $\mathsf{e}^{\mathcal{G}}$ is the group identity, and $\{a^{\mathcal{G}}, b^{\mathcal{G}}\}$ are the two generators.

A particular 2-generated group is the Klein group $\mathcal{K}$ whose universe K has 4 elements, $K = \{\mathsf{e}, 1, 2, 3\}$. (For simplicity, we do not distinguish between the constant symbol e and its interpretation $\mathsf{e}^{\mathcal{K}}$ in $\mathcal{K}$.) $\mathcal{K}$ can be specified as $\big(K, =, \otimes, (\,)^{-1}, \mathsf{e}, 1, 2\big)$. Because K is finite, its two operations $\{\otimes, (\,)^{-1}\}$ can be conveniently specified by the tables in Figure 4.1. Examples of *zeroth-order* wff's that are satisfied by the Klein group are the following three atomic wff's:

$$f(a, a) \approx \mathsf{e}, \quad f(b, b) \approx \mathsf{e}, \quad f\big(f(a, b), f(a, b)\big) \approx \mathsf{e},$$

or as interpreted equations and writing $\otimes = f^{\mathcal{K}}$ in infix position:

$$1 \otimes 1 \;=\; \mathsf{e}, \qquad 2 \otimes 2 \;=\; \mathsf{e}, \qquad (1 \otimes 2) \otimes (1 \otimes 2) \;=\; \mathsf{e},$$

which turn out to fully specify the Klein group; more precisely, those three atomic wff's turn out to imply every other *zeroth-order* wff satisfied by $\mathcal{K}$ (not shown here).

Another particular and more common 2-generated group is the set $\mathbb{Z}$ of all integers under addition, which can be specified as $\mathcal{Z} \stackrel{\text{def}}{=} (\mathbb{Z}, =, +, -, 0, 1, -1)$, where binary addition $+$, unary negation $-$, and constants 0, 1, and -1, are the respective interpretations of the symbols f, i, e, a, and b.

Exercise 76 (*Groups*)**.** There are two parts in this exercise.

[25] There are infinitely many *Pythagorean Triples*, a relatively easy fact which can be proved by hand. The related problem of the *Boolean Pythagorean Triples*, which asks whether $\mathbb{N}$ can be divided into two parts such that neither part contains a triple, is notoriously difficult and has – so far – required the use of automated theorem provers (to prove that it is impossible to so divide $\mathbb{N}$). There are infinitely many *Bertrand-Chebyshev Pairs*, a fact for which a proof has been automated using different interactive proof assistants, notably in Coq and Isabelle. As for *Twin-Prime Pairs*, it is still not known whether there are infinitely many of them!

1. Consider the Klein group $\mathcal{K}$ in Example 75. Its universe K has 4 elements $\{\mathsf{e}, 1, 2, 3\}$ (or, more precisely, $\{\mathsf{e}^{\mathcal{K}}, 1, 2, 3\}$), whereas the universe $\mathsf{Terms}(\Sigma, \varnothing)$ of ground terms is infinite. The latter is generated from the constants in $\{\mathsf{e}, a, b\}$ by applying functions f and i repeatedly. The elements "1" and "2" in K are the interpretations of a and b. The induced congruence $\mathsf{eq}^{\mathcal{H}}$ on $\mathsf{Terms}(\Sigma, \varnothing)$ has therefore 4 congruence classes, one for each element in $\{\mathsf{e}, 1, 2, 3\}$. Note there is no constant symbol in the signature which corresponds to the element "3" in K. Compute a few ground terms (at least three, say) in each of the congruence classes.
2. Consider the additive group $\mathcal{Z}$ of all integers in Example 75. Its universe $\mathbb{Z}$ is infinite, and so is the universe of uninterpreted ground terms $\mathsf{Terms}(\Sigma, \varnothing)$ generated from constant symbols $\{\mathsf{e}, a, b\}$ by applying function symbols f and i. Relative to $\mathcal{Z}$, the induced congruence $\mathsf{eq}^{\mathcal{H}}$ on $\mathsf{Terms}(\Sigma, \varnothing)$ has infinitely many congruence classes, one class for each integer. Select a few integers (say, $\{0, 1, 2, 3\}$) and compute a few ground terms (at least three) in each of the congruence classes corresponding to the selected integers. $\square$