CS 511, Fall 2024, Lecture Slides 19

First-Order Logic: Semantics

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first-order models (structures)

- given a vocabulary, aka signature or similarity type, $(\mathcal{F}, \mathcal{P})$:
  - a set $\mathcal{F}$ of **function** symbols
    (including **constant** symbols as zero-ary function symbols)
  - a set $\mathcal{P}$ of **predicate** symbols
- a **model** $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:
  - a non-empty set A, the **universe** or **domain** of concrete values
  - for every 0-ary $c \in \mathcal{F}$, a concrete element $c^{\mathcal{M}}$
  - for every n-ary $f \in \mathcal{F}$, with $n \geqslant 1$, a concrete function $f^{\mathcal{M}} : A^n \to A$
  - for every n-ary $P \in \mathcal{P}$, with $n \geqslant 1$, a concrete predicate $P^{\mathcal{M}} \subseteq A^n$

interpretation of open WFF's requires an environment

- We need an environment to interpret WFF's with **free variables**.

- an **environment** or **look-up table** for model $\mathcal{M} \triangleq (A, \mathcal{P}^{\mathcal{M}}, \mathcal{F}^{\mathcal{M}})$:

$$\ell : \{\text{all variables}\} \to A$$

- $\ell[x \mapsto a]$ denotes an adjustment of ℓ at variable x:

$$\ell[x \mapsto a](y) \triangleq \begin{cases} a & \text{if } x \text{ and } y \text{ are the same variable} \\ \ell(y) & \text{otherwise} \end{cases}$$

satisfaction of WFF's w.r.t. model $\mathcal{M}$ and look-up table ℓ

- interpretation of terms:

$$t^{\mathcal{M},\ell} \triangleq \begin{cases} \ell(x) & \text{if } t = x \\ c^{\mathcal{M}} & \text{if } t = c \text{ where } c \text{ is constant symbol} \\ f^{\mathcal{M}}(t_1^{\mathcal{M},\ell}, \ldots, t_n^{\mathcal{M},\ell}) & \text{if } t = f(t_1, \ldots, t_n) \text{ where } f \text{ is } n\text{-ary with } n \geqslant 1 \end{cases}$$

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- interpretation of WFF's:
  - $\mathcal{M}, \ell \models (t_1 \approx t_2)$ iff $t_1^{\mathcal{M},\ell} = t_2^{\mathcal{M},\ell}$
  - $\mathcal{M}, \ell \models P(t_1, \ldots, t_n)$ iff $\langle t_1^{\mathcal{M},\ell}, \ldots, t_n^{\mathcal{M},\ell} \rangle \in P^{\mathcal{M}}$
  - $\mathcal{M}, \ell \models \varphi \vee \psi$ iff $\mathcal{M}, \ell \models \varphi$ **or** $\mathcal{M}, \ell \models \psi$
  - $\mathcal{M}, \ell \models \varphi \wedge \psi$ iff $\mathcal{M}, \ell \models \varphi$ **and** $\mathcal{M}, \ell \models \psi$
  - $\mathcal{M}, \ell \models \varphi \to \psi$ iff $\mathcal{M}, \ell \models \psi$ **whenever** $\mathcal{M}, \ell \models \varphi$
  - $\mathcal{M}, \ell \models \neg\varphi$ iff it is **not** the case that $\mathcal{M}, \ell \models \varphi$
  - $\mathcal{M}, \ell \models \forall x\, \varphi$ iff $\mathcal{M}, \ell[x \mapsto a] \models \varphi$ for **every** $a \in A$
  - $\mathcal{M}, \ell \models \exists x\, \varphi$ iff $\mathcal{M}, \ell[x \mapsto a] \models \varphi$ for **some** $a \in A$

semantic entailment, semantic validity, satisfiability

- WFF φ is **satisfiable** iff
  there is some $\mathcal{M}$ and some ℓ such that $\mathcal{M}, \ell \models \varphi$

- WFF φ is **semantically valid** iff
  for every $\mathcal{M}$ and every ℓ it is the case that $\mathcal{M}, \ell \models \varphi$

let Γ be a set of WFF's:

- Γ is **satisfiable** iff
  there is some $\mathcal{M}$ and some ℓ such that $\mathcal{M}, \ell \models \Gamma$,
  i.e., $\mathcal{M}, \ell \models \varphi$ for every $\varphi \in \Gamma$

- **semantic entailment**: $\Gamma \models \psi$ iff
  for every $\mathcal{M}$ and every ℓ, it holds that $\mathcal{M}, \ell \models \Gamma$ implies $\mathcal{M}, \ell \models \psi$

tautologies and (semantical) validities

- **validities** (or **semantical validities**) are WFF's that are satisfied by (or true in) every model $\mathcal{M}$ and environment ℓ

- **tautologies** are a proper subset of the first-order **validities**

- in **propositional logic**, the two notions coincide

- in **first-order logic**, a **tautology** is a WFF that can be obtained by taking a tautology of propositional logic and uniformly replacing each propositional atom (or variable) by a first-order formula (one formula per propositional atom)

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- **example of a first-order validity which is not a tautology**:
$(\forall x\, \varphi) \;\to\; (\neg\exists x\, \neg\varphi)$

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