Solutions to CS511 Homework 07

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Exercise 1. Open Lecture Slides 29, I, Analytical Tableaux for Classical First-Order Logic. Do Exercise 1 on page 14.

Hint: To show that $\{\phi 1, \phi 2, \phi 3\} \models \phi$ is equivalent to showing $\{\phi 1, \phi 2, \phi 3\} \vdash \phi$ (by completeness), which is equivalent to showing $\vdash (\phi 1 \land \phi 2 \land \phi 3) \to \phi$. These equivalences hold for formal proofs carried out according to the rules of natural deduction, and they hold again when analytic tableaux are used as a formal-proof system.

Show that $\Gamma \models \phi$

where:

$\Gamma = \{\forall x \forall y \forall z (P(x,y) \land P(y,z) \to P(x,z)), \forall x \forall y (P(x,y) \to P(y,x))\}$
$\phi = \forall x \forall y \forall z (P(x,y) \land P(z,y) \to P(x,z))$

Solutions to First-Order Ground Tableaux Proof:

We negate the formula we want to prove and show it leads to a contradiction:

$$
\begin{array}{c}
1.\ \neg(\forall x \forall y \forall z (P(x,y) \land P(z,y) \to P(x,z))) \\
| \\
2.\ \exists x \exists y \exists z (P(x,y) \land P(z,y) \land \neg P(x,z)) \\
| \\
3.\ P(x_1,y_1) \land P(z_1,y_1) \land \neg P(x_1,z_1) \\
| \\
4.\ P(x_1,y_1) \\
| \\
5.\ P(z_1,y_1) \\
| \\
6.\ \neg P(x_1,z_1) \\
| \\
7.\ P(y_1,z_1) \quad \text{(from 2, 5)} \\
| \\
8.\ P(x_1,z_1) \quad \text{(from 1, 4, 7)} \\
| \\
\text{X}
\end{array}
$$

Explanation:

1. Line 1: Negation of the conclusion
2. Line 2: Converting universal quantifier to existential
3. Line 3: Instantiation with fresh variables x_1, y_1, z_1
4. Lines 4-6: Conjunction elimination
5. Line 7: From $P(z_1, y_1)$ and symmetry axiom
6. Line 8: From $P(x_1, y_1)$, $P(y_1, z_1)$ and transitivity axiom
7. X: Contradiction between lines 6 and 8

Since we reached a contradiction, the original formula is valid.

Exercise 2. Open Lecture Slides 29, I, Analytical Tableaux for Classical First-Order Logic. Do Exercise 2 on page 14.

Hint: Review the hint in the preceding exercise

Show that $\Gamma \models \phi$

where:

$\Gamma = \{\forall x Q(a, x, x), \forall x \forall y \forall z(Q(x, y, z) \rightarrow Q(x, s(y), s(z))), \forall x \forall y \forall z(Q(x, y, z) \rightarrow Q(y, x, z))\}$
$\phi = \exists x Q(s^{(2)}(a), s^{(3)}(a), x)$

where Q is a ternary predicate symbol, s is a unary function symbol, and a is a constant symbol.

Solution using ground tableaux method: First-Order Ground Tableaux Proof

1. $\neg(\exists x Q(s^{(2)}(a), s^{(3)}(a), x))$
2. $\forall x Q(a, x, x)$
3. $Q(a, a, a)$ [from 2]
4. $Q(a, s(a), s(a))$ [from 2,3]
5. $Q(s(a), a, s(a))$ [from 3,4]
6. $Q(s(a), s^{(2)}(a), s^{(2)}(a))$ [from 4,5]
7. $Q(s^{(2)}(a), s(a), s^{(2)}(a))$ [from 5,6]
8. $Q(s^{(2)}(a), s^{(2)}(a), s^{(3)}(a))$ [from 6,7]
9. $Q(s^{(2)}(a), s^{(3)}(a), s^{(3)}(a))$ [from 7,8]
10. $\forall x \neg Q(s^{(2)}(a), s^{(3)}(a), x)$ [from 1]
11. $\neg Q(s^{(2)}(a), s^{(3)}(a), s^{(3)}(a))$ [from 10]

X

The tableau closes because we derived a contradiction between lines 9 and 11, proving that $\Gamma \models \phi$.

PROBLEM 1 Open EML.Chapter 6.pdf. Do part Exercise 113 on page 69

Hint: This is a continuation of the discussion in lecture yesterday (Thursday, October 17). As sug- gested in lecture, read carefully and understand Example 112 on pages 68-69 before embarking on Exercise 113.

Queens Problem II

Task:

We need to construct a first-order sentence ψ that characterizes solutions of the Queens Problem using a unary function q instead of a binary relation. The structure $M \stackrel{\text{def}}{=} (\mathbb{N}, =, +, <, 0, q)$ interprets q as a function where $q(i) = 0$ means no queen in row i, and $q(i) = j$ means a queen is placed at position (i, j).

Solution:

The first-order sentence ψ can be defined as:

$$\psi \stackrel{\text{def}}{=} \psi_{\text{base}} \wedge (\psi_{\text{fin}} \vee \psi_{\text{inf}}) \wedge \psi_{\text{valid}} \wedge \psi_{\text{noattack}}$$

where:

$$\psi_{\text{base}} \stackrel{\text{def}}{=} q(0) \approx 0$$

$$\begin{aligned}\psi_{\text{fin}} \stackrel{\text{def}}{=} {} & \exists n > 0.(\forall i.(i > n \to q(i) \approx 0) \wedge \\ & \forall i.(0 < i \leq n \to 0 < q(i) \leq n))\end{aligned}$$

$$\psi_{\text{inf}} \stackrel{\text{def}}{=} \forall i > 0.\exists j > 0.(q(i) \approx j)$$

$$\begin{aligned}\psi_{\text{valid}} \stackrel{\text{def}}{=} {} & \forall i > 0.\forall j > 0.(q(i) \approx j \to \\ & \forall k > 0.(k \not\approx i \to q(k) \not\approx j))\end{aligned}$$

$$\begin{aligned}\psi_{\text{noattack}} \stackrel{\text{def}}{=} {} & \forall i_1 > 0.\forall i_2 > 0.(i_1 \not\approx i_2 \wedge q(i_1) \not\approx 0 \wedge q(i_2) \not\approx 0 \to \\ & |i_1 - i_2| \not\approx |q(i_1) - q(i_2)|)\end{aligned}$$

where:

- ψ_{base} ensures $q(0) = 0$
- ψ_{fin} handles finite solutions

- ψ_{inf} handles infinite solutions
- ψ_{valid} ensures no two queens share a column
- ψ_{noattack} ensures no two queens are on the same diagonal

The absolute value notation $|x - y|$ can be expressed in first-order logic as:

$$|x - y| \approx z \stackrel{\text{def}}{=} ((x - y \approx z) \vee (y - x \approx z)) \wedge z \geq 0$$

ON LEAN-4

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

Exercise 3. From Macbeth's book

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

Exercise 4. From Macbeth's book

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean

PROBLEM 2. From Macbeth's book

Solution

https://github.com/nich-ikech/CS511-hw-macbeth/blob/main/cs511HwSolutions/hw07/hw07_nicholas_ikechukwu.lean