Solutions to CS511 Homework 01

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Exercise 1 [LCS, page 79]: Exercise 1.2.1. parts (h), (i), and (j)

Prove the validity of the following sequents:

Sequent Proofs

(h) $p \vdash (p \to q) \to q$

Proof: We will use natural deduction to prove this sequent.

1. p (Premise)

2.	$p \to q$	(Assumption)
3.	q	(Modus Ponens 1, 2)

4. $(p \to q) \to q$ ($\to$-Introduction 2-3)

This proof is valid because:

- We start with the premise p.
- We assume $p \to q$ (for $\to$-Introduction).
- We use Modus Ponens with lines 1 and 2 to derive q.
- We conclude $(p \to q) \to q$ by $\to$-Introduction, discharging the assumption in line 2.

□

(i) $(p \to r) \wedge (q \to r) \vdash p \wedge q \to r$

Proof: We will use natural deduction to prove this sequent.

1. $(p \to r) \wedge (q \to r)$ (Premise)
2. $p \to r$ ($\wedge$-Elimination 1)
3. $q \to r$ ($\wedge$-Elimination 1)

4.	$p \wedge q$	(Assumption)
5.	p	($\wedge$-Elimination 4)
6.	q	($\wedge$-Elimination 4)
7.	r	(Modus Ponens 2, 5)

8. $p \wedge q \to r$ ($\to$-Introduction 4-7)

This proof is valid because:

- We start with the premise $(p \to r) \wedge (q \to r)$.
- We use $\wedge$-Elimination to derive $p \to r$ and $q \to r$.
- We assume $p \wedge q$ (for $\to$-Introduction).
- We use $\wedge$-Elimination to derive p and q separately.
- We use Modus Ponens with $p \to r$ and p to derive r.
- We conclude $p \wedge q \to r$ by $\to$-Introduction, discharging the assumption in line 4.

$\square$

(j) $q \to r \vdash (p \to q) \to (p \to r)$

Proof: We will use natural deduction to prove this sequent.

1.	$q \to r$	(Premise)
2.	$p \to q$	(Assumption)
3.	p	(Assumption)
4.	q	(Modus Ponens 2, 3)
5.	r	(Modus Ponens 1, 4)
6.	$p \to r$	($\to$-Introduction 3-5)
7.	$(p \to q) \to (p \to r)$	($\to$-Introduction 2-6)

This proof is valid because:

- We start with the premise $q \to r$.
- We assume $p \to q$ (for outer $\to$-Introduction).
- We assume p (for inner $\to$-Introduction).
- We use Modus Ponens twice to derive r.
- We conclude $p \to r$ by $\to$-Introduction, discharging the assumption in line 3.
- We conclude $(p \to q) \to (p \to r)$ by $\to$-Introduction, discharging the assumption in line 2.

□

Exercise 2 [LCS, page 84]: Exercise 1.4.2. parts (g), (h), and (i).

Compute the complete truth table of the formulae:

Complete Truth Tables

(g) $((p \to q) \to p) \to p$

p	q	$p \to q$	$(p \to q) \to p$	$((p \to q) \to p) \to p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Table 1: Truth table for $((p \to q) \to p) \to p$

(h) $((p \lor q) \to r) \to ((p \to r) \lor (q \to r))$

p	q	r	$p \lor q$	$(p \lor q) \to r$	$p \to r$	$q \to r$	$(p \to r) \lor (q \to r)$	$((p \lor q) \to r) \to ((p \to r) \lor (q \to r))$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Table 2: Truth table for $((p \lor q) \to r) \to ((p \to r) \lor (q \to r))$

(i) $(p \to q) \to (\neg p \to \neg q)$

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$(p \to q) \to (\neg p \to \neg q)$
T	T	T	F	F	T	T
T	F	F	F	T	T	T
F	T	T	T	F	F	F
F	F	T	T	T	T	T

Table 3: Truth table for $(p \to q) \to (\neg p \to \neg q)$

PROBLEM 1 [LCS, page 87]: Exercise 1.5.3. parts (b) and (c).

3. Adequate Set of Connectives

(b) Showing that if $C \subseteq \{\neg, \wedge, \vee, \rightarrow, \bot\}$ is adequate, then $\neg \in C$ or $\bot \in C$

Proof by contradiction:

Assume that C is adequate and that neither $\neg \in C$ nor $\bot \in C$.

Let v be a valuation that assigns T to all atomic propositions. Consider any formula ϕ constructed using only connectives from C. We will prove by structural induction that $v(\phi) = \mathrm{T}$ for all such ϕ.

Base case:

- If ϕ is an atomic proposition, then $v(\phi) = \mathrm{T}$ by definition of v.

Inductive step:

- If $\phi = \psi \wedge \chi$, then $v(\phi) = v(\psi) \wedge v(\chi) = \mathrm{T} \wedge \mathrm{T} = \mathrm{T}$
- If $\phi = \psi \vee \chi$, then $v(\phi) = v(\psi) \vee v(\chi) = \mathrm{T} \vee \mathrm{T} = \mathrm{T}$
- If $\phi = \psi \rightarrow \chi$, then $v(\phi) = v(\psi) \rightarrow v(\chi) = \mathrm{T} \rightarrow \mathrm{T} = \mathrm{T}$

Therefore, any formula constructed using only connectives from C will always evaluate to T under valuation v.

However, the formula $\bot$ (false) should always evaluate to F under any valuation. Since C is assumed to be adequate, it must be able to express $\bot$, which is impossible given our proof.

This contradiction shows that our initial assumption must be false. Therefore, if C is adequate, then $\neg \in C$ or $\bot \in C$.

(c) Is $\{\leftrightarrow, \neg\}$ adequate?

Claim: The set $\{\leftrightarrow, \neg\}$ is adequate for propositional logic.

Proof:

To prove adequacy, we need to show that we can express all other connectives using only $\leftrightarrow$ and $\neg$. We'll do this by providing equivalent formulas for $\wedge$, $\vee$, and $\rightarrow$.

1. Expressing $\wedge$: $p \wedge q \equiv \neg(p \leftrightarrow \neg q)$

Proof of equivalence:

- When p and q are both T, $\neg q$ is F, so $p \leftrightarrow \neg q$ is F, and $\neg(p \leftrightarrow \neg q)$ is T.
- In all other cases, either p is F or q is F (or both), so $p \leftrightarrow \neg q$ is T, and $\neg(p \leftrightarrow \neg q)$ is F.

2. Expressing $\vee$: $p \vee q \equiv (p \leftrightarrow q) \leftrightarrow (p \leftrightarrow p)$

Proof of equivalence:

- When either p or q (or both) are T, $p \leftrightarrow q$ is not equivalent to $p \leftrightarrow p$ (which is always T), so the overall expression is T.
- When both p and q are F, $p \leftrightarrow q$ is T, which is equivalent to $p \leftrightarrow p$, so the overall expression is F.

3. Expressing $\rightarrow$: $p \rightarrow q \equiv \neg p \leftrightarrow (p \leftrightarrow q)$

Proof of equivalence:

- When p is T and q is F, $\neg p$ is F, $p \leftrightarrow q$ is F, so $\neg p \leftrightarrow (p \leftrightarrow q)$ is T.
- In all other cases, $p \rightarrow q$ is T, and our expression also evaluates to T.

Since we can express $\wedge$, $\vee$, and $\rightarrow$ using only $\leftrightarrow$ and $\neg$, and we already have $\neg$, the set $\{\leftrightarrow, \neg\}$ is adequate for propositional logic.

With Lean 4

Exercise 3 Write the script of the Lean 4 proof for Example 1.3.4 in Macbeth's book. The book gives the "proof by hand" in full, but does not give its mechanized version in Lean 4.

Solutions in one file at: https://github.com/nich-ikech/CS511-hw-macbeth/blob/d7a44938a14b51e12a670a41a9c27f0f70de6e46/cs511HwSolutions/hw01/hw01.lean

Script link for Exercise 3

https://github.com/nich-ikech/CS511-hw-macbeth/blob/d7a44938a14b51e12a670a41a9c27f0f70de6e46/cs511HwSolutions/hw01/hw01_3.lean

Exercise 4 Write the script of the Lean 4 proof for Example 1.3.9 in Macbeth's book. Again here, the book gives the "proof by hand", but does not give its mechanized version in Lean 4.

Script link for Exercise 4

https://github.com/nich-ikech/CS511-hw-macbeth/blob/d7a44938a14b51e12a670a41a9c27f0f70de6e46/cs511HwSolutions/hw01/hw01_4.lean

Problem 2: Do Exercise 7, in Section 1.3.11, in Macbeth's book.

Script link for Problem 2 - Exercise 7

https://github.com/nich-ikech/CS511-hw-macbeth/blob/d7a44938a14b51e12a670a41a9c27f0f70de6e46/cs511HwSolutions/hw01/hw01_p2.lean