CS 511, Fall 2024, Lecture Slides 06 – Appendix

Examples of Strengthening the Induction Hypothesis

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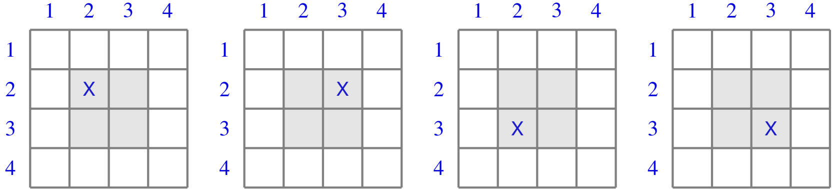
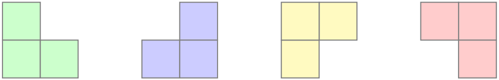
When We May Have to Strengthen the Induction Hypothesis

Below is a 4×4 grid, which is to be entirely covered with *L-shaped trominoes*, except for exactly one *X-designated cell*, which is always centrally located:



Four possible *L-shaped trominoes*:



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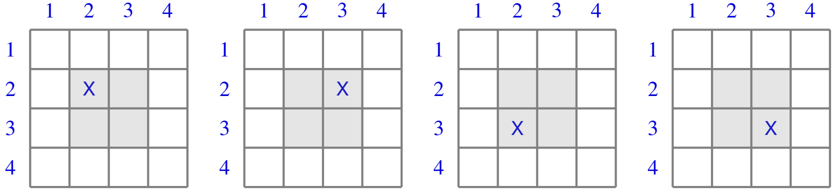

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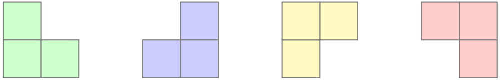
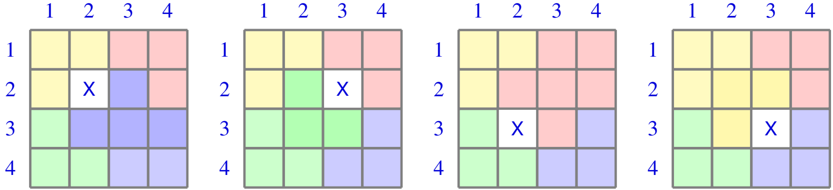


Four possible solutions, one for each of the four possible positions of the X-*designated cell*:



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- **Theorem**:

  > For every $n \geqslant 2$ and for every grid of size $2^n \times 2^n$, we can cover the entire grid with L-shaped trominoes, except for a X-*designated cell* that is centrally located.

- *Proof*: Proceed by simple induction on $n \geqslant 2$.

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  Hint 2: Prove the following stronger theorem, which implies the theorem above:

  For every $n \geqslant 2$ and for every grid of size $2^n \times 2^n$, we can cover the entire grid with L-shaped trominoes, except for one X-*designated cell*, **which we can place anywhere on the grid.**

  For example, for the grid of dimension $2^3 \times 2^3 = 8 \times 8$, we may have the following input configuration, where the X-*designated cell* is not one of the four centrally located cells:

	1	2	3	4	5	6	7	8
1								
2						X		
3								
4								
5								
6								
7								
8								

Example 2: Why & How To Strengthen *Induction Hypothesis*

- **Theorem**:

  The sum of the squares of the first $n \geqslant 1$ inverses of natural numbers is < 2:

  $$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2$$

- *Proof*: Proceed by simple induction on $n \geqslant 1$.

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  The sum of the squares of the first $n \geqslant 1$ inverses of natural numbers is $\leqslant 2 - \dfrac{1}{n}$:

  $$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leqslant 2 - \frac{1}{n}$$

  Hint 3: Together with *Hint 2*, you will need to show that for all $n \geqslant 1$, it holds that:

  $$\frac{1}{n} - \frac{1}{(n+1)^2} > \frac{1}{n+1}$$

  using very simple and easy arithmetical calculation.

Example 3: Why & How To Strengthen *Induction Hypothesis*

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  Every planar graph G can have its edges directed such that the in-degree of each vertex is $\leqslant 3$.

- *Proof*: Proceed by simple induction on the number $n \geqslant 3$ of vertices in G.

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  Hint 2: Prove the following stronger theorem, which implies the theorem above:

  Every plane graph can have its edges directed such that the in-degree of every vertex is $\leqslant 3$, and the in-degree of every vertex on the boundary is $\leqslant 2$.

  Proof sketch: Let G be a plane graph and let the boundary vertices be denoted $v_1, v_2, v_3, \ldots, v_n$ in cyclic order, where $n \geqslant 3$. Define the subgraph H of G by deleting v_2 and all incident edges. Then use the induction hypothesis to direct the edges of H, and extend this to G by making v_2 have two in-coming edges (from v_1 and v_3) and all other edges be out-going. The base case is trivial, so the result follows by induction on $n \geqslant 3$.

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