1 Problem 1

Find all integers $n > 1$ with the property that for each positive divisor d of n, we also have that

$$(d + 2) \mid (n + 2)$$

Solution. ff $\square$

2 Problem 2

Find all positive integers m and n such that

$$2^m - 3^n = 7$$

Solution. We can rearrange our equation to get

$$2^m = 7 + 3^n \tag{1}$$

Obviously, any m that satisfies the above equation will also satisfy $2^m = 2 \cdot 2^{m-1} \equiv 7 \pmod 3$ (since $3 \mid 3^n$ for any n). That is to say, the set of solutions M_1 to equation (1) (elements in M_1 are of the form 2^m) is a subset of the set of solutions M_2 to our subsequent relation, $M_1 \subset M_2$. If X is the set of solutions $x \in \mathbb{Z}$ to $2x \equiv 7 \pmod 3$, then clearly $M_2 \subset X$.

Recall proposition 7.2 (B) from the course notes: if $a, b, m \in \mathbb{Z}$ with $m \neq 0$ and $d = gcd(a, m)$, then if $d \mid b$, the congruence equation $ax \equiv b \pmod m$ has exactly d solutions. Since 2 and 3 are coprime, we have that $d = 1$, so $d \mid 7$, thus $2x \equiv 7 \pmod 3$ has exactly one solution. Thus, X has exactly one element. Therefore, since $M_1 \subset X$, equation (1) has at most one solution.

We can verify that there does exist such a solution, namely when $m = 4$ and $n = 2$, then we have $2^4 - 3^2 = 16 - 9 = 7$. □

3 Problem 3

Let $k \in \mathbb{N}$. Show that there exists k consecutive positive integers with the property that no integer from this set is of the form $a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

Solution. Note if $k = a^2 + b^2$, then $0 = a^2 - k + b^2$, which only

If $k = a^2 + b^2$, then $k \equiv a^2 + b^2 \pmod{a} \equiv b^2 \pmod{a}$ and $k \equiv a^2 + b^2 \pmod{b} = a^2 \pmod{b}$. ff □

4 Problem 4

As always, for each positive integer m, we have that $d(m)$ is the number of the positive divisors of m; also, we let $\phi(m)$ be the corresponding value of the Euler ϕ-function. Then compute the following limits:

$$\lim_{n\to\infty} \frac{n!}{d(n!)\phi(n!)}$$

$$\lim_{n\to\infty} \frac{n!}{2^{d(n!)}}$$

Solution.
Limit 1: $d(mn) = d(m)d(n), \phi(mn) = \phi(m)\phi(n)$ when m, n coprime. We can see that $d(n!) = d(p_1^{e_1})d(\prod_{i=2}^k p_i^{e_i}) = \prod(e_i+1)$. And can also check $\phi(n!) = \prod(p_j^{e_j} - p_{j-1}^{e_{j-1}}$ or something from $\phi(p_1 \cdots p_k) = \prod \phi(p_i) = \prod(p_i - 1)$. Simpler case: $d(p_1 \cdots p_k) = \prod_i^k d(p_i) = \prod_i^k 2 = 2^k$ and $\phi(p_1 \cdots p_k) = ff$.

Limit 2: ff □