## Problem 1

*Let* $0 < a_1 < b_1$ *and define* $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{a_n+b_n}{2}$, $n \in \mathbb{N}$.

(a). *Prove that the sequences* $(a_n)$ *and* $(b_n)$ *both converge. (Suggestion: Use induction to prove* $0 < a_n < a_{n+1} < b_{n+1} < b_n$*.)*

(b). *Prove that the sequences* $(a_n)$ *and* $(b_n)$ *have the same limit.*

(a). *Solution.* First, we use induction to prove $0 < a_n < a_{n+1} < b_{n+1} < b_n$. We do this in steps: first we prove that $0 < a_n$ and $0 < b_n$ for all $n \in \mathbb{N}$. When $n = 1$, by assumption, $0 < a_1$ and $0 < b_1$. Now let $n = j$, and $0 < a_j, b_j$. Then $0 < \sqrt{a_j b_j} = a_{j+1}$ and $0 < \frac{a_j+b_j}{2} = b_{j+1}$ as desired. Thus $0 < a_n, b_n, \forall n \in \mathbb{N}$.

Now we prove that $a_n < b_n$ for all $n \in \mathbb{N}$. When $n = 1$, by assumption, $a_1 < b_1$. Now let $n = j$, and $a_j < b_j$. Then $a_{j+1} = \sqrt{a_j b_j}$ and $b_{j+1} = \frac{a_j+b_j}{2}$. We have

$$\begin{aligned} b_{j+1}^2 &= (a_j^2 + 2a_j b_j + b_j^2)/4 \\ &> (2a_j b_j + 2a_j b_j)/4 \\ &= a_j b_j = a_{j+1}^2 \end{aligned}$$

where the second line is because $a_j - b_j > 0 \implies (a_j - b_j)^2 > 0$ so $a_j^2 - 2a_j b_j + b_j^2 > 0 \implies a_j^2 + b_j^2 > 2a_j b_j$. But since we know that both $a_{j+1}, b_{j+1} > 0$, then $b_{j+1}^2 > a_{j+1}^2 \implies |b_{j+1}| > |a_{j+1}| \implies b_{j+1} > a_{j+1}$, which closes the induction.

Now we prove that $a_n < a_{n+1}$ and $b_n > \frac{a_n+b_n}{2}$. We have $a_{n+1} = \sqrt{a_n b_n} > \sqrt{a_n^2} = a_n$ (since $x^2 > y^2 \implies x > y$ when $x, y > 0$). Furthermore, $b_{n+1} = \frac{a_n+b_n}{2} > \frac{2b_n}{2} = b_n$. Both of these are true for all $n \in \mathbb{N}$, so we have fully shown

$$0 < a_n < a_{n+1} < b_{n+1} < b_n$$

Note then that $(a_n)$ is monotonically increasing, and bounded by $a_1$ and $b_1$, and $(b_n)$ is monotonically decreasing, and bounded by $a_1$ and $b_1$ as well. Thus by the Monotone Convergence Theorem, and the fact $\mathbb{R}$ is complete, we have that $(a_n)$ and $(b_n)$ both converge.

(b). *Solution.* We claim that $b_n - a_n < (b_1 - a_1)/2^{n-1}$ for all $n \geq 2$. This is true in the base case ($n = 2$): $b_2 - a_2 = (b_1 + a_1)/2 - \sqrt{a_1 b_1} < (2b_1)/2 + (a_1 - b_1)/2 + \sqrt{a_1^2} = b_1 - a_1 - (b_1 - a - 1)/2 = (b_1 - a_1)/2$. Now assume that $b_n - a_n < (b_1 - a_1)/2^{n-1}$. Then

$$\begin{aligned} b_{n+1} - a_{n+1} &= \frac{b_n + a_n}{2} - \sqrt{a_n b_n} \\ &< \frac{2b_n}{2} + \frac{a_n - b_n}{2} - \sqrt{a_n^2} \\ &= b_n - a_n - \frac{b_n - a_n}{2} \\ &= \frac{b_n - a_n}{2} \\ &< \frac{b_1 - a_1}{2^n} \end{aligned}$$

Thus we have closed the induction.

Now let $\alpha \in \mathbb{R}$ be the value that $(a_n)$ converges to (as proven will exist with part (a)). Let $\varepsilon > 0$. Let $\varepsilon' = \varepsilon/2$. We know when there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N$, we have $|\alpha - a_n| < \varepsilon'$. Additionally, by Archimedes, there exists some $N_2 \in \mathbb{N}$ such that $0 < \frac{b_1 - a_1}{\varepsilon'} < N_2$, and $N_2 \leq 2^{N_2}$, thus $\varepsilon' > (b_1 - a_1)/2^{N_2 - 1}$. Since $(b_1 - a_1)/2^{n-1}$ is a monotonically decreasing function, this $\varepsilon' > (b_1 - a_1)/2^{n-1}$ for all $n \geq N_2$. So $0 < b_n - a_n < \varepsilon'$ for all $n \geq N_2$. Thus $|b_n - \alpha| < |\varepsilon' + a_n - \alpha| \leq |\varepsilon'| + |a_n - \alpha| < \varepsilon' + \varepsilon' = \varepsilon$, thus $b_n \to \alpha$ as well (where the first substitution is valid, since $b_n, a_n, \varepsilon' > 0$).

## Problem 2

*(a). Suppose $(z_n)_{n\in\mathbb{N}}$ is a bounded sequence with integer values.*

*(a) Prove that both numbers below are integers:*

$$\lambda = \liminf_{n\to\infty} z_n \qquad \mu = \limsup_{n\to\infty} z_n$$

*(b) Prove that there are infinitely many integers $n$ for which $z_n = \lambda$.*

*(b). Let $d_n = p_{n+1} - p_n$ $(n \in \mathbb{N})$ denote the sequence of prime differences, built from the sequence of primes*

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$$

*(a) Prove that $\limsup_{d\to\infty} = +\infty$.*

*(b) Some mathematicians believe that*

$$\delta := \liminf_{n\to\infty} d_n = 2 \tag{1}$$

*However, the best estimate of $\delta$ known to date is $2 \leq \delta \leq 246$. Identify by name a famous unsolved problem in mathematics that is equivalent to proving or disproving line (1). (After giving the name, clearly explain the required relationship.)*

(a). (a) *Solution.* Let $E$ be the set of number $x$ such that there exists a convergent subsequence $\{n_k\}$ where $z_{n_k}$ converges to $x$ (which we know is nonempty by Bolzano-Weierstrass, because $(z_n)$ is bounded). We claim that each $x \in E$ is an integer. Since every convergent sequence is Cauchy, there must exist some $N$ where for all $k, k' \geq N$, $|z_{n_k} - z_{n'_k}| < \frac{1}{2}$, but since $z_{n_k}, z_{n'_k}$ are integers, this is only true when $z_{n_k} = z_{n'_k}$. Thus, for all $k \geq N$ $z_{n_k}$ is the same integer, call it $j$. $x$ must equal $j$; otherwise say $x = j + \delta$ for some fixed $\delta \in \mathbb{R}$, then if $\varepsilon = \delta$, for all $k \geq N$, $|z_{n_k} - x| = |j - j \pm \delta| = |\delta| \not< \varepsilon$, which contradicts that $s_{n_k}$ converges to $x$. Thus $x = j$, which is an integer. Therefore, every element in $E$ is an integer.

Now, recall definition 3.16 in Rudin: $\lambda = \liminf_{n\to\infty} z_n = \inf E$ and $\mu = \limsup_{n\to\infty} z_n = \sup E$. Furthermore, theorem 3.17 in Rudin tells us that $\lambda \in E$ and $\mu \in E$. But every element in $E$ is an integer, thus $\lambda, \mu \in \mathbb{Z}$.

(b) *Solution.* Recall that from the previous part of the question that $\lambda \in E$ where $E$ are all such $x$ where there is a subsequence $z_{n_k}$ that converges to $x$. So there is a subsequence $z_{n_l}$ that converges to $\lambda$. In order for $z_{n_l} \to \lambda$, we must have that for all $\varepsilon$, there is some $N \in \mathbb{N}$ such that for all $l \geq N$, we have $|z_{n_l} - \lambda| < \varepsilon$. If $\varepsilon = \frac{1}{2}$, since $z_{n_l}, \lambda$ are integers, we must have that $z_{n_l} = \lambda$. This is true for all $l \geq N$, which there are infinitely many of in, so there are infinitely many $z_{n_l} = \lambda$, thus there are infitly many $n$ where $z_{n_l} = \lambda$ $(n = n_l$ when $l \geq N)$.

(b). (a) *Solution.* Let $k$ be given. We want to show that

(b) *Solution.* ff

## Problem 3

(a). *Prove: For any sequences $(a_n)$ and $(b_n)$ of nonnegative real numbers,*

$$\limsup_{n\to\infty}(a_nb_n) \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right),$$

*provided the right side does not involve the product of* $0$ *and* $\infty$*.*

(b). *Give an example in which the result of part (a) holds with a strict inequality.*

(a). *Solution.* Let $\alpha = \limsup_{n\to\infty} a_n$ and $\beta = \limsup_{n\to\infty} b_n$.

First let us consider when $\alpha = +\infty$. Regardless of $\beta$ (since we are assuming $\beta > 0$), our right hand side becomes $+\infty$, and regardless of the left-hand side, in the extended reals, our inequality holds. WLOG, this is also true when $\beta = +\infty$.

Now we can assume $\alpha > 0$, $\beta > 0$. Let $A > \alpha$. Then by definition, $A > \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} a_k\right)$, thus there exists some $N_1 \in \mathbb{N}$ such that $A > \sup_{k\geq N_1} a_k$, so $A > a_k$ when $k \geq N_1$. Similarly, if $B > \beta$, there exists some $N_2 \in \mathbb{N}$ such that $B > b_k$ when $k \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $k \geq N$, $a_k < A$ and $b_k < B$, thus $a_kb_k < AB$ since both sides are nonnegative, and $A > \alpha \geq 0$ and $B > \beta \geq 0$. Thus $\sup_{k\geq N}(a_kb_k) \leq AB$. Any lower bound for a set of values indexed by $n$ must be less than or equal to each of them (e.g. one where $n = N$). Thus the previous inequality implies

$$\limsup_{n\to\infty}(a_nb_n) = \inf_{n\in\mathbb{N}}\sup_{k\geq n}(a_nb_n) \leq AB$$

Since this holds for arbitrary $A > \alpha$ and $B > \beta$, the real number on the left cannot exceed $\alpha\beta$. Thus, we must have

$$\limsup_{n\to\infty}(a_nb_n) \leq \alpha\beta = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(b). *Solution.* We give the sequences $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $a_nb_n = -1$, so $\limsup_{n\to\infty}(a_nb_n) = -1$. Additionally, $\limsup_{n\to\infty} a_n = 1$, since for any $n \in \mathbb{N}$, there always exists some $a_k = 1$ where $k \geq n$, and so $\limsup_{n\to\infty} = \inf_{n\in\mathbb{N}}\sup_{k\geq n} a_n = \inf_{n\in\mathbb{N}} 1 = 1$); and for the same reason, $\limsup_{n\to\infty} b_n = 1$. Thus

$$\limsup_{n\to\infty}(a_nb_n) = -1 < 1 = \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right)$$

## Problem 4

(a). *Show that for any $r \geq 1$, one has*

$$n(r-1) \leq r^n - 1 \leq nr^{n-1}(r-1) \quad \forall n \in \mathbb{N}$$

(b). *Prove that for each real $a \geq 1$, the following sequences converges:*

$$x_n = n\left(a^{1/n} - 1\right), \quad n \in \mathbb{N}$$

(c). *Prove that the sequence in (b) also converges for each real $a \in (0,1)$.*

(d). *Let $L(x) = \lim_{n\to\infty} n\left(x^{1/n} - 1\right)$ for $x > 0$. Prove that $L(ab) = L(a) + L(b)$ for all $a > 0, b > 0$.*

Note: *Present solutions that use only methods discussed in MATH 320. No calculus, please!*

(a). *Solution.* Recall the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ for any $n \in \mathbb{N}$. Since $r \geq 1$, applying the identity when $b = r$ and $a = 1$, we get

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + 1) \leq (r-1)nr^{n-1}$$

since each $r^{n-j} \leq r^{n-1}$ $(1 \leq j \leq n)$, and there are $n$ many $r^{n-j}$ in the factor.

To prove the second inequality, we use the identity again, however

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + 1) \geq n(r-1)$$

since each $r^{n-j} \geq 1$ $(1 \leq j \leq n)$, and there are $n$ many $r^{n-j}$ in the factor. Thus we have shown $n(r-1) \leq r^n - 1 \leq nr^{n-1}(r-1)$.

(b). *Solution.* We have that for all $n \in \mathbb{N}$, by part (a) of this problem, $x_n = n\left(a^{1/n} - 1\right) \leq a - 1$, thus $x_n$ is bounded above. Additionally, $a \geq 1 = 1^n \implies a^{1/n} \geq 1$, thus $n(a^{1/n} - 1) \geq 0$, and so $x_n$ is bounded below. Furthermore, we claim that $(x_n)$ is monotonically decreasing: $x_{n+1} = (n+1)(a^{1/(n+1)} - 1) = n(a^{1/(n+1)} - 1) + a^{1/(n+1)} - 1$. That is $x_n - x_{n+1} \geq 0$. We see

$$\begin{aligned}
x_n - x_{n+1} &= n\left(a^{1/n} - 1\right) - (n+1)\left(a^{1/(n+1)} - 1\right) \\
&\geq n^2\left(a^{1/n^2} - 1\right) - (n+1)\left(a^{1/(n+1)} - 1\right) \\
&=
\end{aligned}$$

I can't figure this one out, but let's just assume it's monotonically decreasing. Since it is bounded and monotonically decreasing, it converges.

(c). *Solution.* The sequence is bounded above by 0, since $0 < a < 1 = 1^n \implies a^{1/n} < 1$, thus $n(a^{1/n} - 1) < 0$. Furthermore, it is bounded below, and monotonically decreasing (same as above). Thus the series converges.

(d). *Solution.* We have

$$\begin{aligned}
L(ab) &= \lim_{n\to\infty} n((ab)^{1/n} - 1) \\
&= \lim_{n\to\infty} n(a^{1/n}b^{1/n} - 1) \\
&= \lim_{n\to\infty} n/2(-(a^{1/n} - b^{1/n})^2 - 2 + a^{2/n} + b^{2/n}) \\
&=
\end{aligned}$$

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