Problem 1

Who are your group members?

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Problem 2

Using the expansion $e^{Ah} = I + hA + (1/2)(hA)^2 + O(h^3)$ for $|h|$ small (i.e., near 0) (where the constant in $O(h^3)$ depends on A), show that for any fixed square matrices A, B of the same dimensions,

$$e^{(A+B)h} - e^{Ah}e^{Bh} = (1/2)(BA - AB)h^2 + O(h^3)$$

for $|h|$ small.

Solution. We just algebraically verify:

$$\begin{aligned} e^{Ah}e^{Bh} &= (I + hA + (1/2)(hA)^2 + O(h^3))(I + hB + (1/2)(hB)^2 + O(h^3)) \\ &= I + hB + (h^2/2)B^2 + hA + h^2AB + (h^3/2)AB^2 + (h^2/2)A^2 + (h^3/2)A^2B + (h^4/4)A^2B^2 + O(h^3) \\ &= I + h(A+B) + (h^2/2)B^2 + h^2AB + (h^2/2)A^2 + O(h^3) \\ &= I + h(A+B) + (h^2/2)(B^2 + 2AB + A^2) + O(h^3) \end{aligned}$$

Then

$$\begin{aligned} &e^{(A+B)h} - e^{Ah}e^{Bh} \\ &\quad = I + h(A+B) + (1/2)(h(A+B))^2 + O(h^3) - \big(I + h(A+B) + (h^2/2)(B^2 + 2AB + A^2) + O(h^3)\big) \\ &\quad = \frac{1}{2}(A^2 + AB + BA + B^2 - B^2 - 2AB - A^2)h^2 + O(h^3) \\ &\quad = (1/2)(BA - AB)h^2 + O(h^3) \end{aligned}$$

as desired.

Problem 3

Recall that Euler's method for solving $y' = f(y)$ subject to $y(t_0) = y_0$ involves (page 485 of [A&G], where a, c there are t_0, y_0 here) (1) choosing an $h > 0$, (2) setting $t_i = t_0 + ih$, and (3) approximating $y(t_i)$ as y_i using the formula

$$y_{i+1} = y_i + hf(y_i)$$

Now consider the ODE where $f(y) = |y|^{1/2}$.

(a). Say that for some $i \geq 1$, $y_{i+1} = 0$. Show that y_i is either 0 or $-h^2$.

(b). Say that for some $i \geq 1$, $y_{i+1} < 0$. Show that:

- (i). $y_i < 0$; and
- (ii). using part (i), setting $x = \sqrt{-y_i}$, i.e., x is the positive square root of $-y_i$, show that

$$x = \frac{h + \sqrt{h^2 - 4y_{i+1}}}{2},$$

(where $\sqrt{h^2 - 4y_{i+1}}$ refrs to the positive square root) and hence

$$y_i = -\left(\frac{h + \sqrt{h^2 - 4y_{i+1}}}{2}\right)^2$$

(c). Show that if for some $i \geq 1$, $y_{i+1} < 0$ and $y_{i+1} = -uh^2$ for a real $u > 0$, then

$$y_i = -h^2 g(u) \quad \text{where} \quad g(u) = \frac{1 + 2u + \sqrt{1+4u}}{2}$$

(a). *Solution.* Plugging in our values for variables, we have $0 = y_i + h|y_i|^{1/2}$, and rearranging gives $-y_i = h|y_i|^{1/2}$. First, note that $y_i = 0$ satisfies our equation. Now, assume that $y_i \neq 0$. Squaring both sides gives $y_i^2 = h^2|y_i| \implies |y_i| = h^2$, where we can divide by $|y_i|$ because it is nonzero. If $y_i = h^2$, we have $0 = h^2 + h|h^2|^{1/2} = h^2 + h^2$. But squares are nonnegative, and the sum of two nonnegative numbers is also nonnegative; but also we don't have equality becuase $y_i \neq 0 \implies h^2 \neq 0$, hence $0 = h^2 + h^2 > 0$, so we cannot have $y_i = h^2$. Now if $y_i = -h^2$, we have $0 = -h^2 + h|-h^2|^{1/2} = -h^2 + h^2 = 0$, so $y_i = -h^2$ works.

Hence, we must either have $y_i = 0$ or $y_i = -h^2$.

(b). *Solution.* (i). We prove the contrapositive. That is, for some $i \geq 1$, assume that $y_i \geq 0$. Note $h|y_i|^{1/2} \geq 0$ since $h > 0$ and $|x|^{1/2} \geq 0$ for all $x \in \mathbb{R}$. Hence, $y_i + h|y_i|^{1/2} = y_{i+1} \geq 0$. Therefore, we have shown $y_{i+1} < 0 \implies y_i < 0$.

(ii). Rearranging the Euler's method formula, we get

$$y_{i+1} = y_i + h|y_i|^{1/2}$$

Plugging in $x = \sqrt{-y_i}$, note that $y_i = -x^2$ and since $-y_i = |y_i|$ since $y_i < 0$ by part (i), we have $|y_i|^{1/2} = \sqrt{-y_i} = x$. Then subbing back in, we get

$$y_{i+1} = -x^2 + hx \implies x^2 - hx + y_{i+1} = 0$$

The quadratic equation gives us

$$x = \frac{h \pm \sqrt{h^2 - 4y_{i+1}}}{2}$$

But $y_{i+1} < 0 \implies -4y_{i+1} > 0 \implies h^2 - 4y_{i+1} > h^2 \implies h = \sqrt{h^2} < \sqrt{h^2 - 4y_{i+1}}$ hence $\frac{h - \sqrt{h^2 - 4y_{i+1}}}{2} < 0$, but we said x was positive. Thus

$$x = \frac{h + \sqrt{h^2 - 4y_{i+1}}}{2}$$

(which is obviously positive, since we are adding two positive values in the numerator).
Now, we substitute $x = \sqrt{-y_i}$ to recover

$$\sqrt{-y_i} = \frac{h + \sqrt{h^2 - 4y_{i+1}}}{2} \implies y_i = -\left(\frac{h + \sqrt{h^2 - 4y_{i+1}}}{2}\right)^2$$

as desired.

(c). *Solution.* Using the formula from the previous part, we plug in $y_{i+1} = -uh^2$ to get

$$\begin{aligned} y_i &= -\left(\frac{h+\sqrt{h^2+4uh^2}}{2}\right)^2 \\ &= -\left(\frac{h+h\sqrt{1+4u}}{2}\right)^2 \\ &= -\left(\frac{h(1+\sqrt{1+4u})}{2}\right)^2 \\ &= -h^2\left(\frac{1+\sqrt{1+4u}}{2}\right)^2 \\ &= -h^2\left(\frac{1+2\sqrt{1+4u}+1+4u}{4}\right) \\ &= -h^2\left(\frac{1+2u+\sqrt{1+4u}}{4}\right) \\ &= -h^2 g(u) \end{aligned}$$

as desired.

Problem 4

Consider the ODE $y' = |y|^{1/2}$, with subject to $y(0) = y_0$ and arbitrary $h > 0$.

(a). Using the results of the previous exercise, find a value of $y_0 < 0$ (as a function of h) such that (in exact arithmetic) $y_1 = 0$, and therefore $y_2 = 0$, $y_3 = 0$, etc.

(b). Using the results of the previous exercise, find a value of $y_0 < 0$ (as a function of h) such that (in exact arithmetic) $y_1 < 0$ and $y_2 = y_3 = \cdots = 0$.

(c). Use the code from Homework 2 called `chaotic_sqrt.m`, or implement your own Euler's method solver for $y' = |y|^{1/2}$. What does MATLAB report for $y(2)$ on the initial condition $y(0) = y_0$ where $h = 2/N$ for the values: $N = 2, 4, 16, 64, 65, 63, 99, 100, 101$ and $y_0 = -(2/N)^2 = -h^2$? Make a table of computed values of $y(2)$. Which values of N above yields something likely due to roundoff or truncation error in finite precision? (Therefore you might want to type something like `chaotic_sqrt(0,2,63,-4/(63^2));` into MATLAB for $N = 63$.)

(d). Use the code from Homework 2 called `chaotic_sqrt.m`, or implement your own Euler's method solver for $y' = |y|^{1/2}$. What does MATLAB report for $y(2)$ on the initial condition $y(0) = y_0$ where $h = 2/N$ for the values: $N = 7, 8, 9, 10, 100, 101, 10^6, 10^6 + 1$ and for

$$y_0 = -h^2\left(\frac{3+\sqrt{5}}{2}\right)?$$

Type this same expression into MATLAB as written above (e.g., don't write down an approximation for $\sqrt{5}$). (Therefore you might want to type something like `chaotic_sqrt(0,2,100,-(4/(100^2))*(3+sqrt(5))/2);` into MATLAB for $N = 100$.) Which values of N above yields something likely due to roundoff or truncation error in finite precision?

(a). *Solution.* Since $y_1 = 0$, and our ODE is $y' = |y|^{1/2}$, we can invoke question 3(a) to get $y_0 = 0, -h^2$. Since we ask for $y_0 \neq 0$, we have $y_0 = -h^2$, which is less than 0 since $h^2 > 0$.

(b). *Solution.* As with the previous part, we can use our results from question 3. Now, since $y_2 = 0$, we can invoke question 3(a) to get $y_1 = 0, -h^2$. But since we say $y_1 \neq 0$, this gives us $y_1 = -h^2$. Now, from question 3(c) (since $y_1 = y_{0+1} < 0$), we have $y_{0+1} = -uh^2$ where $u = 1$, and so we get $y_0 = -h^2 g(1)$. But $g(1) = \frac{1+2(1)+\sqrt{1+4(1)}}{2} = \frac{3+\sqrt{5}}{2}$. Hence $y_0 = -h^2\frac{3+\sqrt{5}}{2}$, which is less than 0 since $h^2 > 0$.

(c). *Solution.* Here is my table of computed values:

N	$y(2)$
2	0
4	0
16	0
64	0
65	0
63	0.8042
99	0
100	0
101	0.8703

By part (a) of this problem, since $y_0 = -h^2$ regardless of N, $y_1 = y_2 = \cdots = 0$. Hence, theory says $y(2) = 0$. Out of the above values, $N = 63, 101$ are instances of something due to roundoff or truncation error in finite precision, since they are not equal to 0.

(d). *Solution.* Here is my table of computed values:

N	$y(2)$
7	0
8	0
9	0
10	0
100	0.8507
101	0
10^6	0
$10^6 + 1$	1.000

By part (b) of this problem, since $y_0 = -h^2 \left(\frac{3+\sqrt{5}}{2}\right)$ regardless of N, $y_2 = y_3 = \cdots = 0$. Hence, theory says $y(2) = 0$ when $N \geq 2$. Out of the above values, $N = 100, 10^6 + 1$ are instances of something due to roundoff or truncation error in finite precision, since they are not equal to 0.