

Problem 1

Prove the following theorem (terminology is given below):

Suppose X is compact and $f: X \rightarrow \mathbb{R}$ is lower semicontinuous. Then f is bounded below on X , and there exists a point $z \in X$ satisfying $f(z) \leq f(x)$ for all $x \in X$.

Recall that in a HTS (X, \mathcal{T}) , a function $f: X \rightarrow \mathbb{R}$ is called lower semicontinuous if the following set is closed for every $p \in \mathbb{R}$:

$$f^{-1}((-\infty, p]) = \{x \in X: f(x) \leq p\}.$$

(One approach uses the family of closed sets $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$.)

Solution. We first prove that f is bounded below on X , that is, $\inf f(x) > -\infty$. For the sake of contradiction, assume the opposite, that $\inf f(x) = -\infty$. Consider $f^{-1}((-\infty, p])$ for some $p \in \mathbb{R}$. ff

Supposedly, we just have $f^{-1}((-\infty, p])^c = f^{-1}((-\infty, p]^c) = f^{-1}((p, \infty))$ which is open. And maybe a union? And open covers over all of X , and can get a finite subcover. ff

Consider the family of closed sets of $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$, call it \mathcal{F} . First, remark that each element in \mathcal{F} is nonempty, otherwise $f^{-1}((-\infty, p])$ is nonempty, thus there is no $x_0 \in X$ where $f(x_0) \in (-\infty, p]$ and so $p \leq \inf f(x)$, which we assumed not true. Secondly, by the assumption that f is lower semicontinuous, each element in \mathcal{F} is also closed. Finally, note that \mathcal{F} has the finite intersection property: let $N \in \mathbb{N}$ and F_1, \dots, F_N are sets in \mathcal{F} , which we can write explicitly as $F_i = f^{-1}((-\infty, p_i])$ where $p_i > \inf f(x)$; denote $p_0 = \min_i \{p_i\}$. Then $F_0 = f^{-1}((-\infty, p_0]) \subseteq F_i$ for all $1 \leq i \leq N$, and since we're just minimizing over a finite number of sets, $F_0 \in \{F_1, \dots, F_N\} \subseteq \mathcal{F}$, thus

$$\bigcap_{i=1}^N F_i = f^{-1}((-\infty, p_0]) = F_0 \neq \emptyset$$

so we have the finite intersection property.

Now, since we're in a HTS and X is compact, any collection of elements of \mathcal{F} has nonempty intersection, by the theorem proved in class (every element is a subset of X and are closed, and any finite collection has the finite intersection property). Notably, $\bigcap \mathcal{F} \neq \emptyset$. This means that there exists some $z \in X$ where $z \in \bigcap \mathcal{F}$. Then, for all $p > \inf f(x)$, we have $z \in f^{-1}((-\infty, p])$. ff

Problem 2

Let (X, d) be a metric space, with $K \subseteq X$ a compact set. Prove that whenever \mathcal{G} is an open cover for K , there exists $r > 0$ with this property: for every pair of points $x, y \in K$ obeying $d(x, y) < r$, some open set $G \in \mathcal{G}$ contains both x and y .

Solution. For the sake of contradiction, assume that for all $r > 0$, there are some $x_r, y_r \in K$ such that $d(x_r, y_r) < r$ but for any $G \in \mathcal{G}$, x_r, y_r are not both in G . Note that Since \mathcal{G} covers K , we have that both x_r, y_r are in some $G \in \mathcal{G}$, so have $x \in G_x$ and $y \in G_y$ but $G_x \neq G_y$.

Hmm, have a two sequences go towards each other.

TFAE: K is compact and every sequence (x_n) in K has a convergent subsequence whose limit lies in K .

TFAE: A is an open set and for every $x \in A$ and every sequence (x_n) obeying $x_n \rightarrow x$, one has $x_n \in A$ for all n sufficiently large.

Evan has a solution ff

Problem 3

Define the set-valued “projection” mapping $p_1: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathcal{P}(\mathbb{R})$ by

$$p_1(S) = \{x_1 \in \mathbb{R}: (x_1, x_2) \in S \text{ for some } x_2\}, \quad S \subseteq \mathbb{R}^2$$

- (a). If S is bounded, must $p_1(S)$ be bounded? (Why or why not?)
- (b). If S is closed, must $p_1(S)$ be closed? (Why or why not?)
- (c). If S is compact, must $p_1(S)$ be compact? (Why or why not?)
- (a). *Solution.* It must. If S is bounded, then by definition, there exists $x \in S$ and $R > 0$ such that $S \subseteq \mathbb{B}[x; R]$. Using the standard metric on \mathbb{R}^2 (namely $d(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$), this means for any $y \in S$, we have $d(x, y) < R$, or $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < R$. Consider $x_1 = p_1(x)$. Then for any $y_1 \in p_1(S)$ (using the standard metric on \mathbb{R} , $d(x, y) = |y - x|$), we have

$$d(x_1, y_1) = |y_1 - x_1| = \sqrt{(y_1 - x_1)^2} \leq \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < R$$

where $y' \in p^{-1}(y_1)$, and so the last inequality follows from the boundedness of S . Thus, $p_1(S) \subseteq \mathbb{B}[x_1; R]$, so $p_1(S)$ is bounded.

- (b). *Solution.* This is not true. We provide the counter-example $S = \{(2^{-n}, 2^n) \in \mathbb{R}^2: n \in \mathbb{N}\}$.

We first prove that S is closed. Note that $S' = \emptyset$. To see this, for the sake of contradiction, let $s \in S'$. Then for some sequence s_n of distinct elements of S , we have $\lim_{n \rightarrow \infty} s_n = s$ (by the proposition proven in class). Unraveling the definition of the limit, this means that for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ where $\forall n \geq N$, we have $d(s, s_n) < \varepsilon$. For the sake of contradiction, assume that this is true; then let $\varepsilon = \frac{1}{2}$, which gives us some N where $d(s, s_n) < \frac{1}{2}$ when $n \geq N$. But note that for any $s_n, s_{n+1} \in S$, since $s_n \neq s_{n+1}$, we have that $d(s_n, s_{n+1}) > 2$ (by construction, since $2 \leq 2^{n+1} - 2^n = y_{s_{n+1}} - y_{s_n} = \sqrt{(y_{s_{n+1}} - y_{s_n})^2} \leq \sqrt{(y_{s_{n+1}} - y_{s_n})^2 + (x_{s_{n+1}} - x_{s_n})^2} = d(s_n, s_{n+1})$). Thus $2 \leq d(s_{n+1}, s_n) \leq d(s, s_n) + d(s, s_{n+1}) \leq \frac{1}{2} + d(s, s_{n+1}) \implies \frac{3}{2} < d(s, s_{n+1})$. But this contradicts our assumption, since $n + 1 > n \geq N$, but $d(s, s_{n+1}) > \frac{3}{2} > \frac{1}{2} = \varepsilon$. Thus, there are no limit points of S , so $S' = \emptyset$.

Now recall the theorem proven in class, $\bar{S} = S \cup S'$. Since $S' = \emptyset$, this leaves us $\bar{S} = S$. But recall that this is true only if S is closed.

We now prove that $p_1(S)$ is not closed. Note that $p_1(S) = \{2^{-n}: n \in \mathbb{N}\}$. See that $0 \in p_1(S)'$ but $0 \notin p_1(S)$. The second of these is obvious, $0 < 2^{-n}$ for all $n \in \mathbb{N}$. To see that 0 is a limit point, we have $\lim_{n \rightarrow \infty} 2^{-n} = 0$ (obviously, we are in \mathbb{R}), and $2^{-n} \in p_1(S)$ are distinct points, thus $0 \in p_1(S)'$ (by our proposition in metric spaces). Thus, $p_1(S) \neq p_1(S) \cup p_1(S)' = \bar{p_1(S)}$. But this is true only if $p_1(S)$ is not closed. Hence, S is closed but $p_1(S)$ is not closed.

- (c). *Solution.* Now, assume that S is compact. Consider an open cover \mathcal{G} of $p_1(S)$. If $G \in \mathcal{G}$, extract the “open column” corresponding to G , namely $G^2 = \{(x, y) \in \mathbb{R}^2: x \in G, y \in \mathbb{R}\}$. Notice two things: each G^2 is open, and \mathcal{G}^2 , the collection of all G^2 such that $G \in \mathcal{G}$, covers S .

To see the first, let $g \in G^2$ and denote $g = (x_0, y_0)$. Since $x_0 \in G$, we have that there exists r such that $\mathbb{B}_1[x_0, r] = \{x \in \mathbb{R}: |x_0 - x| < r\} \subseteq G$ by the fact that G is open. We claim that $\mathbb{B}_2[g, r) \subset G^2$ as well; for the sake of contradiction, assume there is $(x_1, y_1) \in \mathbb{B}_2[g, r)$ but $(x_1, y_1) \notin G^2$. Then $x_1 \notin G$ so $x_1 \notin \mathbb{B}_1[x_0, r)$, which give us $|x_0 - x_1| \geq r$. But then $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \geq \sqrt{r^2 + (y_1 - y_0)^2} \geq \sqrt{r^2} = r$, which contradicts the fact that $(x_1, y_1) \in \mathbb{B}_2[g, r)$. Thus, $\mathbb{B}_2[g, r) \subset G^2$, and since $g \in G$ was arbitrary, this proves that G^2 is open.

To see the second claim, that \mathcal{G}^2 covers S , let $s = (x_0, y_0) \in S$. Then since \mathcal{G} covers $p_1(S)$, we have $p_1(s) = x_0 \in G$ for some $G \in \mathcal{G}$. Then $(x_0, y_0) \in G^2$ by definition, and $G^2 \in \mathcal{G}^2 \subseteq \bigcup_{G^2 \in \mathcal{G}^2} G^2$. Thus $s \in \bigcup_{G^2 \in \mathcal{G}^2} G^2$, so \mathcal{G}^2 covers S since $s \in S$ was arbitrary.

Hence, \mathcal{G}^2 is an open cover of S . By the compactness of S , there exists a finite subcover of \mathcal{G}^2 , which we can denote as $G_1^2, G_2^2, \dots, G_N^2$. So $S \subseteq \bigcup_i G_i^2$. But recall that each $G_i^2 \in \mathcal{G}^2$ had some corresponding $G \in \mathcal{G}$ by

definition (recall that the x -values of each G^2 were determined by some G), so we have a finite collection of open sets $G_1, G_2, \dots, G_N \in \mathcal{G}$. We prove that this is a finite subcover of $p_1(S)$. Let $x_0 \in p_1(S)$. Then $\exists y_0$ such that $(x_0, y_0) \in S$. Then $(x_0, y_0) \in G_j^2$ for some $1 \leq j \leq N$. But then by definition of G_j^2 , $x_0 \in G_j$. Therefore, since $x_0 \in p_1(S)$ was arbitrary, $p_1(S) \subseteq \bigcup_i G_i$, hence $G_1, \dots, G_N \in \mathcal{G}$ is a finite subcover, thus $p_1(S)$ is compact.

Problem 4

Recall the set ℓ^2 from HW07 Q3, and the standard “unit vectors” $\hat{e}_p = (0, 0, \dots, 0, 1, 0, \dots)$, where the only nonzero entry in \hat{e}_p occurs in component p . For any x in ℓ^2 and subset $V \subseteq \ell^2$, write

$$\Omega(x; V) = \{y \in \ell^2 : -1 < \langle v, y - x \rangle < 1, \forall v \in V\}.$$

Then define a collection \mathcal{T} of subsets of ℓ^2 by saying $G \in \mathcal{T}$ if and only if every point $x \in G$ has the property that $x \in \Omega(x; V) \subseteq G$ for some finite set $V \subseteq \ell^2$.

- Prove that $\Omega(x; V) \in \mathcal{T}$ for every finite set $V \subseteq \ell^2$ and point $x \in \ell^2$.
- Prove that (ℓ^2, \mathcal{T}) is a Hausdorff Topological Space.
- Let $S = \{\hat{e}_p : p \in \mathbb{N}\}$. Prove that $0 \in S'$. (Here 0 denotes $(0, 0, \dots)$, the “origin in ℓ^2 .”) Note: This fact proves that \mathcal{T} is different from the metric topology on ℓ^2 .
- Prove that every G in \mathcal{T} has the property: for every x in G , there exists $r > 0$ such that

$$G \supseteq \mathbb{B}[x; r) = \{y \in \ell^2 : \|y - x\| < r\}.$$

This fact proves that every set considered “open” in \mathcal{T} is also open in the metric topology on ℓ^2 . This explains why \mathcal{T} gets called “the weak topology” and the metric topology is also called “the strong topology.”

- Prove that the following set is closed in the weak topology of ℓ^2 : $\mathbb{B}[0; 1] = \{y \in \ell^2 : \|y\| \leq 1\}$.
- Solution.* Let $x' \in \Omega(x; V)$. Want to show there exists a finite set $V' \subseteq \ell^2$ such that $\Omega(x'; V') \subseteq \Omega(x; V)$. Then $-1 < \langle v, x' - x \rangle < 1$ for all $v \in V$. This is equivalent to

$$-1 < \sum_{n=1}^{\infty} v_n(x'_n - x_n) < 1$$

for all $v \in V$. Let v' be the sequence $v'_n(y_n - x_n) = v'_n(y_n - x'_n)$, ff

- Solution.* We have that $\emptyset \in \mathcal{T}$, since there does not exist $x \in \emptyset$ so it satisfies our condition to be in \mathcal{T} vacuously. We also have $\ell^2 \in \mathcal{T}$, since $\Omega(x; V)$ is composed of elements of ℓ^2 , and so for any $x \in \ell^2$, $\Omega(x; V) \subseteq \ell^2$.

Now consider $\mathcal{G} \subseteq \mathcal{T}$. Consider an arbitrary element $x \in \bigcup \mathcal{G}$. Then for some $G \in \mathcal{G}$, we have $x \in G$. Then $\Omega(x; V) \subseteq G \subseteq \bigcup \mathcal{G}$ for some finite set $V \subseteq \ell^2$ since $G \in \mathcal{T}$, so $\bigcup \mathcal{G} \in \mathcal{T}$ as well.

Now consider $U_1, \dots, U_N \in \mathcal{T}$ where $N \in \mathbb{N}$. Consider an arbitrary element $x \in \bigcap_{i=1}^N U_i$. Then for all $1 \leq i \leq N$, $x \in U_i$. Then by definition of each U_i being in \mathcal{T} , we have that there exists a finite set $V_i \subseteq \ell^2$ such that $\Omega(x; V_i) \subseteq U_i$. ff

Finally, let $x, y \in \ell^2$ such that $x \neq y$. We have that $y_N \neq x_N$ for some $N \in \mathbb{N}$. Let $v \in \ell^2$ be defined by $v_N = 2(y_N - x_N)^{-1}$ and $v_n = 0$ for all other n . Define $V = \{v\}$. We claim that $x \in \Omega(x; V)$ and $y \in \Omega(y; V)$ are disjoint (and they are open sets by part (a) of this problem). Let $x' \in \Omega(x; V)$. It is sufficient to show $x' \notin \Omega(y; V)$. We have $-1 < \langle v, x' - x \rangle = \sum_n v_n(x'_n - x_n) = 2(x'_N - x_N)/(y_N - x_N) < 1$, or $0 \leq |2(x'_N - x_N)/(y_N - x_N)| < 1$, so $0 \leq |x'_N - x_N| < \frac{1}{2}|y_N - x_N|$. Note $|y_N - x_N| \leq |x'_N - y_N| + |x'_N - x_N| < |x'_N - y_N| + \frac{1}{2}|y_N - x_N|$ thus $\frac{1}{2}|y_N - x_N| < |x'_N - y_N|$. But then $1 < |2(x'_N - y_N)/(y_N - x_N)|$ and $\sum_n v_n(x'_n - y_n) = 2(x'_N - y_N)/(y_N - x_N)$. Thus $\sum_n v_n(x'_n - y_n) < -1$ or $\sum_n v_n(x'_n - y_n) > 1$, in either case, $x' \notin \Omega(y; V)$.

This satisfies all the conditions for a HTS, thus (ℓ^2, \mathcal{T}) is a HTS.

- Solution.* Let $U \in \mathcal{N}(0)$ be an arbitrary open set, ie. $U \in \mathcal{T}$ such that $0 \in U$. We want to show that $(U \setminus \{0\}) \cap S \neq \emptyset$; since $0 \notin S$ anyway, we just need to show $U \cap S \neq \emptyset$.

Since $0 \in U$, there exists a finite set $V \subseteq \ell^2$ such that $\Omega(0; V) \subseteq U$. If $V = \emptyset$, then $\Omega(0; V) = \ell^3$ since $-1 < \langle v, y - x \rangle < 1$ is now vacuously true for all $y \in \ell^2$; then $\Omega(0; V) \cap S$ since $\hat{e}_1 \in \ell^2 \cap S$, and since $\Omega(0; V) \subseteq U$, $U \cap S \neq \emptyset$. So now assume V is not empty. Denote the elements of V as v^i where $1 \leq i \leq k$.

Then since $v^i \in \ell^2$, we must have that $\lim_n (v_n^i)^2 = 0$ (crude divergence test). Then there exists some N_i where $(v_{N_i}^i)^2 < 1$ by the definition of convergence. Let $N = \min_i \{N_i\}$. Then $-1 < v_N^i < 1$ as well. See

$$\langle v^i, \hat{e}_N \rangle = \sum_{n=1}^{\infty} v_n^i (\hat{e}_N)_n = v_N^i$$

Thus $\hat{e}_N \in \Omega(0, V)$ since $-1 < \langle v, \hat{e}_N - 0 \rangle = v_N < 1$ for all $v \in V$. Thus, $\hat{e}_N \in \Omega(0, V) \subseteq U$. Since $\hat{e}_N \in S$, thus shows that $S \cap U \neq \emptyset$, so we are done since U was arbitrary (this works for any open $U \in \mathcal{N}(0)$).

(d). *Solution.* ff

(e). *Solution.* ff Minkowski sum?

Recap of what still needs to be done:

- Wrapping up Q1: bounded case and end of attaining minimum
- All of Q2 (with contradiction)
- 3(b) (closed implies closed)
- 4(a), 4(b) with the second two conditions for HTS, 4(d), 4(e)