Problem 1

Prove the following theorem (terminology is given below):

Suppose X is compact and $f : X \to \mathbb{R}$ is lower semicontinuous. Then f is bounded below on X, and there exists a point $z \in X$ satisfying $f(z) \leq f(x)$ for all $x \in X$.

Recall that in a HTS $(X, \mathcal{T})$, a function $f : X \to \mathbb{R}$ is called lower semicontinuous *if the following set is closed for every $p \in \mathbb{R}$:*

$$f^{-1}((-\infty, p]) = \{x \in X : f(x) \leq p\}.$$

(One approach uses the family of closed sets $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$.)

Solution. We first prove that f is bounded below on X, that is, $\inf f(x) > -\infty$. For the sake of contradiction, assume the opposite, that $\inf f(x) = -\infty$. Consider $f^{-1}((-\infty, p])$ for some $p \in \mathbb{R}$. ff

Consider the family of closed sets of $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$, call it $\mathcal{F}$. First, remark that each element in $\mathcal{F}$ is nonempty, otherwise $f^{-1}(-\infty, p])$ is nonempty, thus there is no $x_0 \in X$ where $f(x_0) \in (-\infty, p]$ and so $p \leq \inf f(x)$, which we assumed not true. Secondly, by the assumption that f is lower semicontinuous, each element in $\mathcal{F}$ is also closed. Finally, note that $\mathcal{F}$ has the finite intersection property: let $N \in \mathbb{N}$ and $F_1, \ldots, F_N$ are sets in $\mathcal{F}$, which we can write explicitly as $F_i = f^{-1}((-\infty, p_i])$ where $p_i > \inf f(x)$; denote $p_0 = \min_i\{p_i\}$. Then $F_0 = f^{-1}((-\infty, p_0]) \subseteq F_i$ for all $1 \leq i \leq N$, and since we're just minimizing over a finite number of sets, $F_0 \in \{F_1, \ldots, F_n\} \subseteq \mathcal{F}$, thus

$$\bigcap_{i=1}^{N} F_i = f^{-1}((-\infty, p_o]) = F_0 \neq \emptyset$$

so we have the finite intersection property.

Now, since we're in a a HTS and X is compact, any collection of elements of $\mathcal{F}$ has nonempty intersection, by the theorem proved in class (every element is a subset of X and are closed, and any finite collection has the finite intersection property). Notably, $\bigcap \mathcal{F} \neq \emptyset$. This means that there exists some $z \in X$ where $z \in \bigcap \mathcal{F}$. Then, for all $p > \inf f(x)$, we have $z \in f^{-1}((-\infty, p])$. ff

Problem 2

Let (X, d) be a metric space, with $K \subseteq X$ a compact set. Prove that whenever $\mathcal{G}$ is an open cover for K, there exists $r < 0$ with this property: for every pair of points $x, y \in K$ obeying $d(x, y) < r$, some open set $G \in \mathcal{G}$ contains both x and y.

Solution. Let $G_1, G_2, \ldots, G_N$ be the finite subcover of K such that $G_i \in \mathcal{G}$ for all $i \in 1, 2, \ldots, N$ and $K \subseteq \bigcup_{1 \le i \le N} G_i$, which we know exists from the compactness of K. Define $I_{i,j} := G_i \cap G_j$ where $1 \le i < j \le N$. Note then that $I_{i,j}$ is open as well. For each $I_{i,j}$, pick some $x_{i,j} \in I_{i,j}$ if $I_{i,j} \ne \emptyset$. Then, since X is a metric space and $I_{i,j}$ is an open set, we must have that there exists some $r_{i,j} > 0$ such that $\mathbb{B}[x_{i,j}; r_{i,j}) \subseteq I_{i,j}$ (if $I_{i,j} = \emptyset$, just let $r_{i,j} = 1$). Let $r = \min_{1 \le i < j \le N}\{r_{i,j}\}$. Since there are only finitely many $r_{i,j}$, all of them greater than 0, we must have $r > 0$ as well.

Now consider any $x, y \in K$ such that $d(x, y) < r$. Consider $\mathbb{B}[x; r)$. By our distance condition, we have $y \in [x; r)$. It is sufficient to show now that there exists some $G \in \mathcal{G}$ such that $\mathbb{B}[x; r) \subseteq G$. Since $\{G_i\}_{1 \le i \le N}$ is a covering of all of K, there exists some $G_x \in \{G_i\}_{1 \le i \le N}$ such that $x \in G_x$. If $\mathbb{B}[x; r) \subseteq G_x$, we are done. For the sake of contradiction, assume then that $\mathbb{B}[x; r) \not\subseteq G_x$. hmm it's not this, it's either in G_x, or in G_x'... consider the G_i for which x is in?

I don't even have that $\mathbb{B}[x; r) \subseteq K$. Hmm, so we don't want to lose y, because we'll always have $x \in \partial K$ (since K is closed) violate our condition with the ball. ff

So let $x \in G_x$ and $y \in G_y$. We claim that either $x \in G_y$ or $y \in G_x$ (this is trivial if $G_x = G_y$, so we now assume that $G_x \ne G_y$). For the sake of contradiction, assume this is false: $x, y \notin G_x \cap G_y$. Then by construction, for some $x_{x,y} \in G_x \cap G_y$, $x, y \notin \mathbb{B}[x_{x,y}, r)$. Then $d(x_{x,y}, x) > r$ and $d(x_{x,y}, y) > r$. We hav $d(x, y) \le d(x_{x,y}, x) + d(x_{x,y}, y)$. Hmm... the thing on the right is bounded below by $2r$, and the thing on the left is bounded above by r. We want to use this inequality in a different way I think, like $d(x_{x,y}, x) < d(x, y)$ and $d(x_{x,y}, y) < d(x, y)$ somehow. Maybe something to do with them being in the same G_i?

So we have $d(x_{x,y}, x) < d(x, y) + d(x_{x,y}, y)$. But $d(x_{x,y}, y) < d(x, y) + d(x_{x,y}, x)$, thus $d(x_{x,y}, x) < 2d(x, y) + d(x_{x,y}, x) \implies 0 < 2d(x, y)$ trivially. ff So the claim I want to make is that the intersection between two open sets in a metric space is closer to each other than they are to each other.

Hmm, we might need to define r in terms of points that are not in the same G_i, but are close to each other. Think of a square donut that his a tiny slit that doesn't touch, and an open cover of two sets that intersect on the non-slit side.

So it seems the problem is when I fix the slit, I get a problem with the intersection. When I fix the intersection, I get a problem with the slit. I will think about this later.

So Tighe said to do a proof by contradiction.

Problem 3

Define the set-valued "projection" mapping $p_1 : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R})$ *by*

$$p_1(S) = \{x_1 \in \mathbb{R} : (x_1, x_2) \in S \text{ for some } x_2\}, \qquad S \subseteq \mathbb{R}^2$$

(a). If S *is bounded, must* $p_1(S)$ *be bounded? (Why or why not?)*

(b). If S *is closed, must* $p_1(S)$ *be closed? (Why or why not?)*

(c). If S *is compact, must* $p_1(S)$ *be compact? (Why or why not?)*

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 4

Recall the set ℓ^2 from HW07 Q3, and the standard "unit vectors" $\hat{\ }_p = (0, 0, \dots, 0, 1, 0, \dots)$, where the only nonzero entry in $\hat{\ }_p$ occurs in component p. For any x in ℓ^2 and subset $V \subseteq \ell^2$, write

$$\Omega(x; V) = \{y \in \ell^2 \colon -1 < (v, y - x) < 1, \forall v \in V\}.$$

Then define a collection $\mathcal{T}$ of subsets of ℓ^2 by saying $G \in \mathcal{T}$ if and only if every point $x \in G$ has the property that $x \in \Omega(x; V) \subseteq G$ for some finite set $V \subseteq \ell^2$.

(a). Prove that $\Omega(x; V) \in \mathcal{T}$ of every finite set $V \subseteq \ell^2$ and point $x \in \ell^2$.

(b). Prove that $(\ell^2, \mathcal{T})$ is a Hausdorff Topological Space.

(c). Let $S = \{\hat{\ }_p \colon p \in \mathbb{N}\}$. Prove that $0 \in S'$. (Here 0 denotes $(0, 0, \dots)$, the "origin in ℓ^2.) Note*: This fact proves that $\mathcal{T}$ is different from the metric topology on ℓ^2.*

(d). Prove that every G in $\mathcal{T}$ has the property: for every x in G, there exists $r > 0$ such that

$$G \supseteq \mathbb{B}[x; r) = \{y \in \ell^2 \colon \|y - x\| < r\}.$$

This fact proves that every set considered "open" in $\mathcal{T}$ is also open in the metric topology on ℓ^2. This explains why $\mathcal{T}$ gets called "the weak topology" and the metric topology is also called "the strong topology."

(e). Prove that the following set is closed in the weak topology of ℓ^2: $\mathbb{B}[0; 1] = \{y \in \ell^2 \colon \|y\| \le 1\}$.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

(d). *Solution.* ff

(e). *Solution.* ff