Problem 10 (Ch. 1.8)

Let G be a finite group, A and B non-vacuous subsets of G. Show that $G = AB$ if $|A| + |B| > |G|$.

Solution. We prove the contrapostive. That is, assume that there exists some $g \in G$ such that $g \neq ab$ for any $a \in A, b \in B$. $A \neq gB$

If $|A| + |B| > |G|$, we must have $A \cap B \neq \emptyset$, since A and B are nonempty subgroups of G (and so their respective sets are subsets of G's). Thus, there exists some $g \in A$ and $g \in B$. Also, we must have some $g' \notin A$ and $g' \notin B$.

Note that A must have finite index, since G is a finite group and A has positive order, so let $[G : A] = r$. Lagrange's theorem gives specifically $|G|/|A| = r$. We have $G = A \sqcup Ag_1 \sqcup Ag_2 \sqcup \cdots \sqcup Ag_r$. Similarly, let $l = |G|/|B|$ be the index of B in G. We have $2|A| + 2|B| > |A|r + |B|l \implies (2 - r)|A| + (2 - l)|B| > 0$.

Prove that $|AB| = |G|$?

Problem 11 (Ch. 1.8)

*Let G be a group of order $2k$ where k is odd. Show that G contains a subgroup of index 2. (*Hint: *Consider the permutation group G_L of left translations and use exercise 13, p.36.*

Solution. Since G is of even order, by exercise 13, there exists some $a \in G$ such that $a \neq 1$ and $a^2 = 1$. ff

Problem 2 (Ch. 1.9)

ff

Solution. ff

Problem 4 (Ch. 1.9)

ff

Solution. ff

Problem 5 (Ch. 1.9)

ff

Solution. ff

Problem 8 (Ch. 1.9)

ff

Solution. ff