Math 321 Homework 5

The purpose of this homework is to prove the monotone convergence theorem for Riemann integrable functions, i.e. if $\{f_n\}$ is a monotone sequence of Riemann integrable functions that converge pointwise to an integrable function f, then $\int_a^b f_n \to \int_a^b f dx$. Equivalently, if $\{f_n\}$ is a monotone sequence of integrable functions that converge pointwise to 0, then $\int_a^b f dx \to 0$. First we will need a few definitions.

Definition 1. A *step function* is a function $f\colon \mathbb{R} \to \mathbb{R}$ or $[a,b] \to \mathbb{R}$ that can be written in the form $\sum_{i=1}^n b_i \chi_{A_i}$ (here χ_{A_i} denotes the indicator function of the set A_i), where

- $b_i \in \mathbb{R}$ for each i.
- The sets $\{A_1, \ldots, A_n\}$ are disjoint, and $\bigcup_{i=1}^n A_i = \mathbb{R}$ (or $\bigcup A_i = [a,b]$, if the domain of f is $[a,b]$).
- Each set A_i is either a point, a (possibly infinite) open interval, a closed interval, or a (possibly infinite) half-open interval.

Note that step functions on $[a,b]$ are always Riemann integrable, since they are bounded and have finitely many points of discontinuity.

Definition 2. We say a step function is *equally spaced* if f is of the form $\sum_{i=1}^n b_i \chi_{A_i}$, where each A_i is an interval (open, closed, or half-open), and each interval has the same length.

Problem 1

Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Let $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$. Let $\varepsilon > 0$. Prove that there are equally spaced step functions $g, h\colon [a,b] \to \mathbb{R}$ so that $m \le g(x) \le f(x) \le h(x) \le M$ for all $x \in [a,b]$, and

$$\int_a^b h(x)dx - \varepsilon \le \int_a^b f(x)dx \int_a^b g(x)dx + \varepsilon$$

Solution. ff

Problem 2

Let $f\colon [a,b] \to \mathbb{R}$ be an equally-spaced step function. Let $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$. Let $\varepsilon > 0$. Prove that there are continuous functions $g, h\colon [a,b] \to \mathbb{R}$ so that $m \le g(x) \le f(x) \le h(x) \le M$ for all $x \in [a,b]$, and

$$\int_a^b h(x)dx - \varepsilon \le \int_a^b f(x)dx \le \int_a^b g(x)dx + \varepsilon$$

Remark 1. The above statement is true for arbitrary step functions (i.e. not necessarily equally spaced), but takes more effort to prove.

Solution. ff

Problem 3

Let $\{f_n\}$ be a sequence of functions from $[a,b] \to [0,\infty)$. Suppose that each f_n is Riemann integrable on $[a,b]$, and $f_n \to 0$ pointwise. For each n, let $g_n\colon [a,b] \to \mathbb{R}$ be a continuous function with $0 \le g_n(x) \le f_n(x)$ (the existence of such a g_n is gauranteed by problems 1 & 2). For each $k \ge 1$, let $h_k(x) = \min\{g_1(x), \ldots, g_k(x)\}$. Prove that $h_k \to 0$ uniformly.

Solution. ff

Problem 4

Let $x_1, \ldots, x_n$ be non-negative real numbers, and let $y = \min\{x_1, \ldots, x_n\}$. Prove that

$$x_n \leq y + \sum_{i=1}^{n-1} (\max\{x_i, \ldots, x_n\} - x_i)$$

Hint: *Note that $y = x_j$ for some index j, and $x_n = x_j + (x_n - x_j)$.*

Solution. ff

Problem 5

Let $\{f_n\}$ be a sequence of Riemann integrable functions from $[a,b] \to [0,\infty)$ and suppose that $\{f_n(x)\}$ is monotone decreasing for each $x \in [a,b]$. For each index n, let $g_n\colon [a,b] \to \mathbb{R}$ with $0 \leq g_n(x) \leq f_n(x)$. Prove that for each $i \leq n$ we have

$$\int_a^b (\max\{g_i(x), \ldots, g_n(x)\} - g_i(x))dx \leq \int_a^b f_i(x)dx - \int_a^b g_i(x)dx$$

Solution. ff

Problem 6

Let $\{f_n\}$ be a sequence of Riemann integrable functions from $[a,b] \to [0,\infty)$ and suppose that $\{f_n(x)\}$ is monotone decreasing for each $x \in [a,b]$. Let $\varepsilon > 0$. For each index n, let $g_n\colon [a,b] \to \mathbb{R}$ be a continuous function with $0 \leq g_n(x) \leq f_n(x)$ and $\int_a^b f_n(x) \leq \int_a^b g_n(x) + \varepsilon/2^n$. Define $h_k(x) = \min\{g_1(x), \ldots, g_k(x)\}$. Prove that for each n,

$$\int_a^b g_n(x)dx \leq \int_a^b h_n(x)dx + \varepsilon$$

Solution. ff

Problem 7

Let $\{f_n\}$ be a sequence of Riemann integrable functions from $[a,b] \to [0,\infty)$ andsuppose that $\{f_n(x)\}$ is monotone decreasing for each $x \in [a,b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$$

Solution. ff