Math 321 Homework 2

(Completed with collaboration with Matthew Bull-Weizel)

Problem 1

Let $\mathcal{C} \subset [0,1]$ be the middle-third Cantor set. Let $\alpha\colon [0,1] \to [0,1]$ be the Cantor-Lebesgue function. In particular, α is (weakly) monotone increasing with $\alpha(0) = 0$, $\alpha(1) = 1$, and α is constant on every open interval $I \subset [0,1]$ with $I \cap \mathcal{C} = \emptyset$. Let

$$f(x) = \begin{cases} 1 & x \in \mathcal{C} \\ 0 & x \in [0,1] \setminus \mathcal{C} \end{cases}$$

Prove that

$$\overline{\int_0^1} f d\alpha = 1, \quad \text{and} \quad \underline{\int_0^1} f d\alpha = 0$$

Solution. We first prove the upper integral case. Let P be an arbitrary partition, $\{x_0, \ldots, x_n\}$. Recall that $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$ where $\Delta\alpha_i = (\alpha(x_i) - \alpha(x_{i-1}))$ and $M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$. Note that this is a finite sum, so we may rearrange our summation. We split $U(P, f, \alpha)$ up into a sum of the terms for the intervals that intersect $\mathcal{C}$, and the intervals that do not intersect $\mathcal{C}$:

$$U(P, f, \alpha) = \sum_{\mathcal{C} \cap [x_{i-1}, x_i] = \emptyset} M_i \Delta\alpha_i + \sum_{\mathcal{C} \cap [x_{i-1}, x_i] \neq \emptyset} M_i \Delta\alpha_i$$

For the sum on the left, α is constant on these intervals, so $\Delta\alpha_i = 0$, giving us

$$U(P, f, \alpha) + \sum_{\mathcal{C} \cap [x_{i-1}, x_i] \neq \emptyset} M_i \Delta\alpha_i$$

For the remaining sum, note that on these intervals, since they intersect $\mathcal{C}$, there is some $c \in \mathcal{C} \cap [x_{i-1}, x_i]$, so $f(c) = 1$, thus $M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\} = 1$ (the max value of this function is 1, so the supremum cannot be greater than 1). This is true for all the intervals in this sum, thus

$$\begin{aligned} U(P, f, \alpha) &= \sum_{\mathcal{C} \cap [x_{i-1}, x_i] \neq \emptyset} (\alpha(x_i) - \alpha(x_{i-1})) \\ &= \sum_{\mathcal{C} \cap [x_{i-1}, x_i] \neq \emptyset} (\alpha(x_i) - \alpha(x_{i-1})) + \sum_{\mathcal{C} \cap [x_{i-1}, x_i] = \emptyset} (\alpha(x_i) - \alpha(x_{i-1})) \\ &= \sum_{i=1}^n (\alpha(x_i) - \alpha(x_{i-1})) \\ &= \alpha(x_n) - \alpha(x_0) \\ &= 1 \end{aligned}$$

where we add the extra summation on the second line since, as we mentioned before, all of these terms are 0 (and so we are just adding 0 multiple times), and we can cancel all the terms in the fourth line because this is a telescoping series.

Hence, for any partition, $U(P, f, \alpha) = 1$. It is then clear that

$$\overline{\int_0^1} f d\alpha = \inf_P U(P, f, \alpha) = 1$$

We now turn to the lower integral case. We make the remark that for any interval $[a, b] \subset [0, 1]$, where $b > a$, we have $\mathcal{C}^c \cap [a, b] \neq \emptyset$. To prove, this we assume, for the sake of contradiction, that it is false, i.e. there is some interval $[a, b] \subset [0, 1]$ such that $\mathcal{C}^c \cap [a, b] = \emptyset$, which is equivalent to saying that $[a, b] \subset \mathcal{C}$ (and now we proceed with a standard proof). Recall the definition of the Cantor set: $C_0 = [0, 1]$, $C_{n+1} = C_n \setminus$

{the middle third interval of all intervals of C_n}, and $\mathcal{C} = \bigcap_{i=1}^n C_n$. Recall that in the ith iteration of the Cantor set's construction, we are removing the middle third of all current intervals; hence, our largest interval is $\dfrac{1}{3^i}$. And so, by the Archimedean property, we can find $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$ so $3^n > n > \frac{1}{b-a} > 0$, so $\frac{1}{3^n} < b - a$. And so $[a,b] \notin C_n \implies [a,b] \notin \mathcal{C}$, a contradiction. Hence, $\mathcal{C}^c \cap [a,b] \neq \emptyset$ for any interval $[a,b] \subset [0,1]$ such that $b > a$.

Now, let P be an arbitrary partition, $\{x_0, \dots, x_n\}$. Then for any $[x_{i-1}, x_i]$, we have $\mathcal{C}^c \cap [x_{i-1}, x_i] \neq \emptyset$. Let $a = \mathcal{C}^c \cap [x_{i-1}, x_i]$. By definition, $f(a) = 0$. Hence,

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i = \sum_{i=1}^n 0 = 0$$

since $m_i = \inf\{f(x) \colon [x_{i-1}, x_i]\} = 0$. Hence, for any partition, $L(P, f, \alpha) = 0$. It is then clear that

$$\underline{\int_0^1} f d\alpha = \sup_P L(P, f, \alpha) = 0$$

Problem 2

Given a rational number $r = \frac{p}{q} \in \mathbb{Q}$, we say that r has lowest form $\frac{p}{q}$ *if $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\gcd(p,q) = 1$ (.e., any possible cancellation has been performed). Let*

$$f(x) = \begin{cases} 0 & (x \notin \mathbb{Q}) \\ \frac{1}{q} & (x \in \mathbb{Q} \text{ has lowest form } \frac{p}{q}) \end{cases}$$

Prove that $f \in \mathcal{R}[0,1]$.

Solution. Let P be an arbitrary partition of $[0,1]$, $\{x_0, \dots, x_n\}$. Note that since the irrationals are dense in the reals, we have that every interval contains an irrational, so $m_i = \inf\{f(x) \colon x \in [x_{i-1}, x_i]\} = 0$ for all $1 \leq i \leq n$. Hence, $L(P, f) = \sum_{i=1}^n 0 = 0$ for any P.

Let $\varepsilon > 0$ be arbitrary. We will show that there exists some partition P_ε such that $U(P_\varepsilon, f) < \varepsilon$. By the Archimidean property, there is some $N \in \mathbb{N}$ such that $N > 3\varepsilon > 0 \implies \frac{3}{N} < \varepsilon$. Now consider the partition of P_ε that splits $[0,1]$ into N^3 equally sized intervals, each of length $\frac{1}{N^3}$.

Note that within $(0,1]$, $f(x) = \frac{1}{q}$ precisely when $x = \frac{p}{q}$ where $0 < p < q$, and $\gcd(p,q) = 1$. Technically, this is $\phi(q)$, however, we can get a good enough upper bound on it, that is $\phi(q) < q$, as there are only $q - 1$ possible values for p. Hence, if A_q is the number of $x \in (0,1)$ such that $f(x) = \frac{1}{q}$, then $A_q < q$.

Let us now consider the number of values of $x \in [0,1]$ such that $f(x) > \frac{1}{N}$. Denote this value B_N. Our left endpoint satisfy this, since $f(0) = 1$. Any $x \in (0,1]$ such that $f(x) = \frac{1}{q}$ where $q \leq N - 1$ also satisfies this, so we get to use our A_q from before. See $B_N = 1 + \sum_{q=1}^{N-1} A_q \leq \sum_{q=1}^{N-1} q = 1 + \frac{N(N-1)}{2}$.

Let $C_N < N^3$ be the number of partitions that contain all of the points $x \in [0,1]$ such that $f(x) > \frac{1}{N}$. Note that $C_N \leq B_N$, as at most, we have that all possible x are in distinct intervals. Now consider $U(P_\varepsilon, f) = \sum_{i=1}^n M_i \Delta x_i$ where $M_i = \sup\{f(x) \colon x \in [x_{i-1}, x_i]\}$ and $\Delta x_i = x_i - x_{i-1} = \frac{1}{N^3}$. Since this is a finite sum, we can rearrange and split up our sum into the intervals that contain some x such that $f(x) > \frac{1}{N}$, and those that don't. Assuming that our M_i take the upper bound for its possible value, i.e. 1 for the intervals that contain x such that $f(x) > \frac{1}{N}$ and

$\frac{1}{N}$ for the intervals that don't, we get the upper bound:

$$\begin{aligned}
U(P_\varepsilon, f) &\leq C_N(1)\frac{1}{N^3} + (N^3 - C_N)\left(\frac{1}{N}\right)\frac{1}{N^3} \\
&\leq \frac{1}{N^3}\left(B_N + (N^3 - C_N)\left(\frac{1}{N}\right)\right) \\
&\leq \frac{1}{N^3}\left(B_N + \frac{N^3}{N}\right) \\
&\leq \frac{1}{N^3}\left(1 + \frac{N^2 - N}{2} + N^2\right) \\
&\leq \frac{1}{N^3}\left(N^2 + N^2 + N^2\right) \\
&= \frac{3}{N} \\
&< \varepsilon
\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists some partition, namely P_ε, such that

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) = U(P_\varepsilon, f) < \varepsilon$$

and so by Rudin Theorem 6.6, $f \in \mathcal{R}[0,1]$.

Problem 3

Let $f\colon [0,1] \to \mathbb{R}$, and suppose f is discontinuous at each $c \in [0,1]$. In this problem, we will prove that $f \notin \mathcal{R}[0,1]$.

(a). For each $c \in [0,1]$, define

$$\omega_f(c) = \lim \delta \searrow 0\left(\sup\{f(x)\colon x \in (c-\delta, c+\delta) \cap [0,1]\} - \inf\{f(x)\colon x \in (c-\delta, c+\delta) \cap [0,1]\}\right)$$

(note that in the above, $\lim_{\delta\searrow 0}$ could be replaced by $\liminf$ or $\inf_{\delta>0}$, since the limit is monotone decreasing as $\delta \searrow 0$.) Prove that $\omega_f(c) > 0$ for each $c \in [0,1]$.

(b). For $t > 0$, define

$$\Omega_t = \{c \in [0,1]\colon \omega_f(c) \geq t\}$$

Prove that Ω_t is closed.

(c). Let $\{I_i\}_{i=1}^\infty$ be a set of open intervals (in $\mathbb{R}$), each of finite length. Syppose that $[0,1] \subset \bigcup_{i=1}^\infty I_i$. Prove that $\sum_{i=1}^\infty \ell(I_i) \geq 1$, where $\ell(I_i)$ is the length of the interval I_i (i.e. if $I = (a,b)$ then $\ell(I) = b - a$).

(d). Let $\{F_i\}_{i=1}^\infty$ be closed subsets of $[0,1]$, and suppose $[0,1] = \bigcup_{i=1}^\infty F_i$. Prove that there exists an index $i \in \mathbb{N}$ and a number $s > 0$ with the following property: If $I_1, \ldots, I_n$ are open intervals in $\mathbb{R}$ and $F_i \subset \bigcup_{j=1}^n I_j$, then $\sum_{j=1}^n \ell(I_j) \geq s$.

(e). Let f be as above. Prove that there is a number $\varepsilon > 0$ so that for every partition P of $[0,1]$ we have

$$U(P,f) - L(P,f) \geq \varepsilon$$

and hence f is not *Riemann integrable.* Hint*: ε might be a product of two numbers, which are related to questions b) and d).*

(a). *Solution.* By the definition of supremum and infinum, we have $\sup\{f(x)\colon x \in (c-\delta, c+\delta) \cap [0,1]\} - \inf\{f(x)\colon x \in (c-\delta, c+\delta) \cap [0,1]\} \geq 0$, and limits preserve non-strict inequalities, so $w_f(c) \geq 0$.

Assume for the sake of contradiction that there is some $c \in [0,1]$ such that $\omega_f(c) = 0$. Then for all ε, there is some $\Delta > 0$ such that $0 < \delta < \Delta$ implies $\sup\{f(x)\colon x \in (c-\delta, c+\delta) \cap [0,1]\} - \inf\{f(x)\colon x \in$

$(c-\delta, c+\delta) \cap [0,1]\} < \varepsilon$. Remark that, by the definition of supremum and infinum, for all $x \in (c-\delta, c+\delta)$, $\inf\{f(x) \colon x \in (c-\delta, c+\delta)\} \le f(x) \le \sup\{f(x) \colon x \in (c-\delta, c+\delta)\}$. This is also true for $x = c$: $\inf\{f(x) \colon x \in (c-\delta, c+\delta)\} \le f(c) \le \sup\{f(x) \colon x \in (c-\delta, c+\delta)\}$. Thus, $|f(x) - f(c)| \le |\sup\{f(x) \colon x \in (c-\delta, c+\delta)\} - \inf\{f(x) \colon x \in (c-\delta, c+\delta)\}| < \varepsilon$.

Therefore, for any $0 < \delta < \Delta$, $0 < |x - c| < \delta < \Delta$ (i.e. $x \in (c-\delta, c+\delta)$) implies $|f(x) - f(c)| < \varepsilon$. $\varepsilon > 0$ can be arbitrary, and so we get that f is continuous at c, which is a contradiction since we assume that f was discontinuous for all $c \in [0,1]$.

Hence, $\omega_f(c) > 0$ for all $c \in [0,1]$.

(b). *Solution.* We show that $\partial\Omega_t \subset \Omega_t$, which is equivalent to Ω_t being closed. If $\partial\Omega_t = \emptyset$, the set inclusion is trivial, and so we are done. So assume $\partial\Omega_t \ne \emptyset$. Then let $x \in \partial\Omega_t$. Hence, for all $\delta > 0$, we have

$$(x-\delta, x+\delta) \cap \Omega_t \ne \emptyset$$

Let $c \in (x-\delta, x+\delta) \cap \Omega_t$. By the definition of an open set, there is some open neighbourhood of c such that the neighbourhood is contained inside of $(x-\delta, x+\delta)$. Let us denote this $(c-\Delta, c+\Delta)$. Hence, $(c-\Delta, c+\Delta) \subset (x-\delta, x+\delta)$, so

$$\sup_{(x-\delta,x+\delta)} f(x) \ge \sup_{(c-\Delta,c+\Delta)} f(x) \ge \inf_{(c-\Delta,c+2\Delta)} f(x) \ge \inf_{(x-\delta,x+\delta)} f(x)$$

where the middle inequality is by the definition of sup, inf, and the outer two is because the supremum on the super set is always larger than the supremum on the sub set, and similarly for the infinum. So we have

$$\sup_{(x-\delta,x+\delta)} f(x) - \inf_{(x-\delta,c+\delta)} f(x) \ge \sup_{(c-\Delta,c+\Delta)} f(x) - \inf_{(c-\Delta,c+\Delta)} f(x) \ge t$$

where the last inequality is due to the fact that $c \in \Omega_t$. Hence, $x \in \Omega_t$. Therefore, $\partial\Omega_t \subset \Omega_t$, as desired.

(c). *Solution.* Since $[0,1]$ is compact by Heine-Borel, and $[0,1] \subset \bigcup_{i=1}^\infty I_i$ so $\{I_i\}_{i=1}^\infty$ is an open cover of $[0,1]$, we can extract a finite subcover $I_1, \dots, I_n$ such that $[0,1] \subset \bigcup_{i=1}^n I_i$.

We now prove that $\sum_{i=1}^n \ell(l_i) \ge 1$. Note that if two open intervals intersect, that is $J_1 = (a,b)$ and $J_2 = (c,d)$ and $c < b$ (but still $c > a$), then the sum of their lengths is less than the length of their union: $\ell(J_1) + \ell(J_2) = (b-a) + (d-c) > (c-a) + (d-c) = d - a = \ell(J_1 \cup J_2)$. If J_1 and J_2 are disjoint, we get that $\ell(J_1) + \ell(J_2) = \ell(J_1 \cup J_2)$, hence in general, $\ell(J_1) + \ell(J_2) \ge \ell(J_1 \cup J_2)$. We can repeat for J_3 comparing to $J_1 \cup J_2$: $\ell(J_1 \cup J_2 \cup J_3) \le \ell(J_1 \cup J_2) + \ell(J_3) \le \ell(J_1) + \ell(J_2) + \ell(J_3)$. It is clear now that we can do this n times to get that for n open intervals, we have $\sum_{i=1}^n \ell(J_i) \ge \ell(\bigcup_{i=1}^n J_i)$.

Note that if $J_1 = (a,b)$, $J_2 = (c,d)$ such that $J_1 \subset J_2$ then we have $c \le a$ and $d \ge b$, so $\ell(J_2) \ge \ell(J_1)$. So using this fact and $(0,1) \subset [0,1] \subset \bigcup_{i=1}^n I_i$, and what we showed before, we have

$$\sum_{i=1}^n \ell(I_i) \ge \ell\left(\bigcup_{i=1}^n I_i\right) \ge \ell((0,1)) = 1$$

Now, since $\ell(I_i) \ge 0$, adding more terms to our sum will only increase it, so $\sum_{i=1}^\infty \ell(I_i) \ge \sum_{i=1}^n \ell(I_i) \ge 1$.

(d). *Solution.* Let $\{F_i\}_{i=1}^\infty$ be closed subsets of $[0,1]$ such that $[0,1] = \bigcup_{i=1}^\infty F_i$. Assume, for the sake of contradiction, that for all $i \in \mathbb{N}$ and all $s > 0$, if $I_1^{(i)}, \dots, I_{n_i}^{(i)}$ are open intervals in $\mathbb{R}$ such that $F_i \subset \bigcup_{j=1}^{n_i} I_j^{(i)}$, then $\sum_{j=1}^n \ell(I_j^{(i)}) < s$.

Now let $1 > t > 0$. If $s = t/2^i$, we have $\sum_{j=1}^{n_i} \ell(I_j^{(i)}) < \frac{t}{2^i}$. We can sum over all $i \in \mathbb{N}$ to get

$$\sum_{i=1}^\infty \sum_{j=1}^{n_i} \ell(I_j^{(i)}) < \sum_{i=1}^\infty \frac{t}{2^i} = t < 1$$

where we used the geometric sum formula.

Now, see that $[0,1] = \bigcup_{i=1}^\infty F_i \subset \bigcup_{i=1}^\infty \bigcup_{j=1}^{n_i} I_j^{(i)}$, and so part (c) from above implies that $\sum_{i=1}^\infty \sum_{j=1}^{n_i} \ell(I_j^{(i)}) \ge 1$. This contradicts what we have just above, hence there does exist some $i \in \mathbb{N}$ and $s > 0$ such that if $I_1, \dots, I_n$ are open intervals in $\mathbb{R}$ and $F_i \subset \bigcup_{j=1}^n I_j$, we have $\sum_{j=1}^n \ell(I_j) \ge s$.

(e). *Solution.* Define $F_i = \Omega_{1/i}$ for $1 \leq i \leq \infty$. From part (b), our F_i are closed. Note that $\bigcup_{i=1}^{\infty} F_i = [0,1]$: firstly, $\bigcup_{i=1}^{\infty} F_i \subset [0,1]$ since each $F_i \subset [0,1]$ by construction of $\Omega_{1/i}$. Now let $c \in [0,1]$. By part (a), $\omega_f(c) > 0$, so there is some $t > 0$ such that $\omega_f(c) = t$. Then, by Archimedean principle, there is some $n \geq t > 0 \implies \frac{1}{n} \leq t$, so $w_f(c) = t \geq \frac{1}{n} \implies c \in \Omega_{1/n} = F_n$, thus, $c \in \bigcup_{i=1}^{n} F_i$. Therefore $[0,1] \subset \bigcup_{i=1}^{n} F_i$, hence $\bigcup_{i=1}^{\infty} F_i = [0,1]$. Hence, by part (d) of the problem, there exists some $t \in \mathbb{N}$ and $s > 0$ such that if $F_t \subset \bigcup_{j=1}^{n} I_j$, then $\sum_{j=1}^{n} \ell(I_j) \geq s$.

Fix our partition P of $[0,1]$, $\{x_0, \dots, x_n\}$. Let our I_i be the open sets (x_{i-1}, x_i) and $(x_i - \frac{s}{4(n+1)}, x_i + \frac{s}{4(n+1)})$. Note then that $[0,1] \subset \bigcup_{i=1}^{2n+1} I_i$. Thus, $F_a \subset \bigcup_{i=1}^{2n+1} I_i$ as well. We can then compute:

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta_i \geq \frac{1}{a}\left(\sum_{i=1}^{2n+1} \ell(I_i) - \sum_{i=1}^{n+1} \frac{s}{2(n+1)}\right) \geq \frac{1}{a}(s - \frac{s}{2}) = \frac{s}{2a}$$

where we got $M_i - m_i \geq \frac{1}{a}$ by the definition of $\Omega_{1/a}$, and we ignore points off of this set as this is a lower bound. Hence, by the contrapositive of Rudin Theorem 6.6, this is not Riemann integrable.