Problem 1

Prove the following theorem (terminology is given below):

Suppose X is compact and $f\colon X \to \mathbb{R}$ is lower semicontinuous. Then f is bounded below on X, and there exists a point $z \in X$ satisfying $f(z) \leq f(x)$ for all $x \in X$.

Recall that in a HTS $(X, \mathcal{T})$, a function $f\colon X \to \mathbb{R}$ is called lower semicontinuous *if the following set is closed for every $p \in \mathbb{R}$:*

$$f^{-1}((-\infty, p]) = \{x \in X \colon f(x) \leq p\}.$$

(One approach uses the family of closed sets $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$.)

Solution. We first prove that f is bounded below on X, that is, $\inf f(x) > -\infty$. For the sake of contradiction, assume the opposite, that $\inf f(x) = -\infty$. Consider $f^{-1}((-\infty, p])$ for some $p \in \mathbb{R}$. ff

Supposedly, we just have $f^{-1}((-\infty, p])^c = f^{-1}((-\infty, p]^c) = f^{-1}((p, \infty))$ which is open. And maybe a union? And open covers over all of X, and can get a finite subcover.ff

Consider the family of closed sets of $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$, call it $\mathcal{F}$. First, remark that each element in $\mathcal{F}$ is nonempty, otherwise $f^{-1}(-\infty, p])$ is nonempty, thus there is no $x_0 \in X$ where $f(x_0) \in (-\infty, p]$ and so $p \leq \inf f(x)$, which we assumed not true. Secondly, by the assumption that f is lower semicontinuous, each element in $\mathcal{F}$ is also closed. Finally, note that $\mathcal{F}$ has the finite intersection property: let $N \in \mathbb{N}$ and $F_1, \ldots, F_N$ are sets in $\mathcal{F}$, which we can write explicitly as $F_i = f^{-1}((-\infty, p_i])$ where $p_i > \inf f(x)$; denote $p_0 = \min_i\{p_i\}$. Then $F_0 = f^{-1}((-\infty, p_0]) \subseteq F_i$ for all $1 \leq i \leq N$, and since we're just minimizing over a finite number of sets, $F_0 \in \{F_1, \ldots, F_n\} \subseteq \mathcal{F}$, thus

$$\bigcap_{i=1}^{N} F_i = f^{-1}((-\infty, p_o]) = F_0 \neq \emptyset$$

so we have the finite intersection property.

Now, since we're in a a HTS and X is compact, any collection of elements of $\mathcal{F}$ has nonempty intersection, by the theorem proved in class (every element is a subset of X and are closed, and any finite collection has the finite intersection property). Notably, $\bigcap \mathcal{F} \neq \emptyset$. This means that there exists some $z \in X$ where $z \in \bigcap \mathcal{F}$. Then, for all $p > \inf f(x)$, we have $z \in f^{-1}((-\infty, p])$. ff

Problem 2

Let (X, d) be a metric space, with $K \subseteq X$ a compact set. Prove that whenever $\mathcal{G}$ is an open cover for K, there exists $r < 0$ with this property: for every pair of points $x, y \in K$ obeying $d(x, y) < r$, some open set $G \in \mathcal{G}$ contains both x and y.

Solution. Let $G_1, G_2, \ldots, G_N$ be the finite subcover of K such that $G_i \in \mathcal{G}$ for all $i \in 1, 2, \ldots, N$ and $K \subseteq \bigcup_{1 \leq i \leq N} G_i$, which we know exists from the compactness of K. Define $I_{i,j} := G_i \cap G_j$ where $1 \leq i < j \leq N$. Note then that $I_{i,j}$ is open as well. For each $I_{i,j}$, pick some $x_{i,j} \in I_{i,j}$ if $I_{i,j} \neq \emptyset$. Then, since X is a metric space and $I_{i,j}$ is an open set, we must have that there exists some $r_{i,j} > 0$ such that $\mathbb{B}[x_{i,j}; r_{i,j}) \subseteq I_{i,j}$ (if $I_{i,j} = \emptyset$, just let $r_{i,j} = 1$). Let $r = \min_{1 \leq i < j \leq N}\{r_{i,j}\}$. Since there are only finitely many $r_{i,j}$, all of them greater than 0, we must have $r > 0$ as well.

Now consider any $x, y \in K$ such that $d(x, y) < r$. Consider $\mathbb{B}[x; r)$. By our distance condition, we have $y \in [x; r)$. It is sufficient to show now that there exists some $G \in \mathcal{G}$ such that $\mathbb{B}[x; r) \subseteq G$. Since $\{G_i\}_{1 \leq i \leq N}$ is a covering of all of K, there exists some $G_x \in \{G_i\}_{1 \leq i \leq N}$ such that $x \in G_x$. If $\mathbb{B}[x; r) \subseteq G_x$, we are done. For the sake of contradiction, assume then that $\mathbb{B}[x; r) \not\subseteq G_x$. hmm it's not this, it's either in G_x, or in G_x'... consider the G_i for which x is in?

I don't even have that $\mathbb{B}[x; r) \subseteq K$. Hmm, so we don't want to lose y, because we'll always have $x \in \partial K$ (since K is closed) violate our condition with the ball. ff

So let $x \in G_x$ and $y \in G_y$. We claim that either $x \in G_y$ or $y \in G_x$ (this is trivial if $G_x = G_y$, so we now assume that $G_x \neq G_y$). For the sake of contradiction, assume this is false: $x, y \notin G_x \cap G_y$. Then by construction, for some $x_{x,y} \in G_x \cap G_y$, $x, y \notin \mathbb{B}[x_{x,y}, r)$. Then $d(x_{x,y}, x) > r$ and $d(x_{x,y}, y) > r$. We hav $d(x, y) \leq d(x_{x,y}, x) + d(x_{x,y}, y)$. Hmm... the thing on the right is bounded below by $2r$, and the thing on the left is bounded above by r. We want to use this inequality in a different way I think, like $d(x_{x,y}, x) < d(x, y)$ and $d(x_{x,y}, y) < d(x, y)$ somehow. Maybe something to do with them being in the same G_i?

So we have $d(x_{x,y}, x) < d(x, y) + d(x_{x,y}, y)$. But $d(x_{x,y}, y) < d(x, y) + d(x_{x,y}, x)$, thus $d(x_{x,y}, x) < 2d(x, y) + d(x_{x,y}, x) \implies 0 < 2d(x, y)$ trivially. ff So the claim I want to make is that the intersection between two open sets in a metric space is closer to each other than they are to each other.

Hmm, we might need to define r in terms of points that are not in the same G_i, but are close to each other. Think of a square donut that his a tiny slit that doesn't touch, and an open cover of two sets that intersect on the non-slit side.

So it seems the problem is when I fix the slit, I get a problem with the intersection. When I fix the intersection, I get a problem with the slit. I will think about this later.

So Tighe said to do a proof by contradiction.

Problem 3

Define the set-valued "projection" mapping $p_1\colon \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R})$ by

$$p_1(S) = \{x_1 \in \mathbb{R}\colon (x_1, x_2) \in S \text{ for some } x_2\}, \qquad S \subseteq \mathbb{R}^2$$

(a). If S is bounded, must $p_1(S)$ be bounded? (Why or why not?)

(b). If S is closed, must $p_1(S)$ be closed? (Why or why not?)

(c). If S is compact, must $p_1(S)$ be compact? (Why or why not?)

(a). *Solution.* It must. If S is bounded, then by definition, there exists $x \in S$ and $R > 0$ such that $S \subseteq \mathbb{B}[x; R)$. Using the the standard metric on $\mathbb{R}^2$ (namely $d(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$), this means for any $y \in S$, we have $d(x, y) < r$, or $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < R$. Consider $x_1 = p_1(x)$. Then for any $y_1 \in p_1(S)$ (using the standard metric on $\mathbb{R}$, $d(x, y) = |y - x|$), we have

$$d(x_1, y_1) = |y_1 - x_1| = \sqrt{(y_1 - x_1)^2} \leq \sqrt{(y_1 - x_1)^2 + (y' - x_2)^2} < R$$

where $y' \in p^{-1}(y_1)$, and so the last inequality follows from the boundedness of S. Thus, $p_1(S) \subseteq \mathbb{B}[x_1; R)$, so $p_1(S)$ is bounded.

(b). *Solution.* This is not true. We provide the counter-example $S = \{(2^{-n}, 2^n) \in \mathbb{R}^2 \colon n \in \mathbb{N}\}$.

We first prove that S is closed. Note that $S' = \emptyset$. To see this, for the sake of contradiction, let $s \in S'$. Then for some sequence s_n of distinct elements of S, we have $\lim_{n\to\infty} s_n = s$ (by the proposition proven in class). Unraveling the definition of the limit, this means that for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ where $\forall n \geq N$, we have $d(s, s_n) < \varepsilon$. For the sake of contradiction, assume that this is true; then let $\varepsilon = \frac{1}{2}$, which gives us some N where $d(s, s_n) < \frac{1}{2}$ when $n \geq N$. But note that for any $s_n, s_{n+1} \in S$, since $s_n \neq s_{n+1}$, we have that $d(s_n, s_{n+1}) > 2$ (by construction, since $2 \leq 2^{n+1} - 2^n = y_{s_{n+1}} - y_{s_{n+1}} = \sqrt{(y_{s_{n+1}} - y_{s_{n+1}})^2} \leq \sqrt{(y_{s_{n+1}} - y_{s_{n+1}})^2 + (x_{s_{n+1}} - x_{s_{n+1}})^2} = d(s_n, s_n + 1)$). Thus $2 \leq d(s_{n+1}, s_n) \leq d(s, s_n) + d(s, s_{n+1}) \leq \frac{1}{2} + d(s, s_{n+1}) \implies \frac{3}{2} < d(s, s_{n+1})$. But this contradicts our assumption, since $n + 1 > n \geq N$, but $d(s, s_{n+1} > \frac{3}{2} > \frac{1}{2} = \varepsilon$. Thus, there are no limit points of S, so $S' = \emptyset$.

Now recall the theorem proven in class, $\overline{S} = S \cup S'$. Since $S' = \emptyset$, this leaves us $\overline{S} = S$. But recall that this is true only if S is closed.

We now prove that $p_1(S)$ is not closed. Note that $p_1(S) = \{2^{-n} \colon n \in \mathbb{N}\}$. See that $0 \in p_1(S)'$ but $0 \notin p_1(S)$. The second of these is obvious, $0 < 2^{-n}$ for all $n \in \mathbb{N}$. To see that 0 is a limit point, we have $\lim_{n\to\infty} 2^{-n} = 0$ (obviously, we are in $\mathbb{R}$), and $2^{-n} \in p_1(S)$ are distinct points, thus $0 \in p_1(S)'$ (by our proposition in metric spaces). Thus, $p_1(S) \neq p_1(S) \cup p_1(S)' = \overline{p_1(S)}$. But this is true only if $p_1(S)$ is not closed. Hence, S is closed but $p_1(S)$ is not closed.

(c). *Solution.* It must. Assume that S is compact. Consider an open cover $\mathcal{G}$ of $p_1(S)$. ff

Problem 4

Recall the set ℓ^2 from HW07 Q3, and the standard "unit vectors" $\hat{\mathbf{e}}_p = (0, 0, \dots, 0, 1, 0, \dots)$, where the only nonzero entry in $\hat{\mathbf{e}}_p$ occurs in component p. For any x in ℓ^2 and subset $V \subseteq \ell^2$, write

$$\Omega(x; V) = \{y \in \ell^2 \colon -1 < \langle v, y - x\rangle < 1, \forall v \in V\}.$$

Then define a collection $\mathcal{T}$ of subsets of ℓ^2 by saying $G \in \mathcal{T}$ if and only if every point $x \in G$ has the property that $x \in \Omega(x; V) \subseteq G$ for some finite set *$V \subseteq \ell^2$.*

(a). *Prove that $\Omega(x; V) \in \mathcal{T}$ for every finite set $V \subseteq \ell^2$ and point $x \in \ell^2$.*

(b). *Prove that $(\ell^2, \mathcal{T})$ is a Hausdorff Topological Space.*

(c). *Let $S = \{\hat{\mathbf{e}}_p \colon p \in \mathbb{N}\}$. Prove that $0 \in S'$. (Here 0 denotes $(0, 0, \dots)$, the "origin in ℓ^2.)* Note*: This fact proves that $\mathcal{T}$ is different from the metric topology on ℓ^2.*

(d). *Prove that every G in $\mathcal{T}$ has the property: for every x in G, there exists $r > 0$ such that*

$$G \supseteq \mathbb{B}[x; r) = \{y \in \ell^2 \colon \|y - x\| < r\}.$$

This fact proves that every set considered "open" in $\mathcal{T}$ is also open in the metric topology on ℓ^2. This explains why $\mathcal{T}$ gets called "the weak topology" and the metric topology is also called "the strong topology."

(e). *Prove that the following set is closed in the weak topology of ℓ^2: $\mathbb{B}[0; 1] = \{y \in \ell^2 \colon \|y\| \leq 1\}$.*

(a). *Solution.* Let $x' \in \Omega(x; V)$. Want to show there exists a finite set $V' \subseteq \ell^2$ such that $\Omega(x'; V') \subseteq \Omega(x, V)$. Then $-1 < \langle v, x' - x\rangle < 1$ for all $v \in V$. This is equivalent to

$$-1 < \sum_{n=1}^{\infty} v_n(x'_n - x_n) < 1$$

for all $v \in V$. ff

(b). *Solution.* We have that $\emptyset \in \mathcal{T}$, since there does not exist $x \in \emptyset$ so it satisfies our condition to be in $\mathcal{T}$ vacuously. We also have $\ell^2 \in \mathcal{T}$, since $\Omega(x; V)$ is composed of elements of ℓ^2, and so for any $x \in \ell^2$, $\Omega(x; V) \subseteq \ell^2$.

Now consider $\mathcal{G} \subseteq \mathcal{T}$. Consider an arbitrary element $x \in \bigcup \mathcal{G}$. Then for some $G \in \mathcal{G}$, we have $x \in G$. Then $\Omega(x; V) \subseteq G \subseteq \bigcup \mathcal{G}$ for some finite set $V \subseteq \ell^2$ since $G \in \mathcal{T}$, so $\bigcup \mathcal{G} \in \mathcal{T}$ as well.

Now consider $U_1, \dots, U_N \in \mathcal{T}$ where $N \in \mathbb{N}$. Consider an arbitrary element $x \in \bigcap_i^N U_i$. Then for all $1 \leq i \leq N$, $x \in U_i$. Then by definition of each U_i being in $\mathcal{T}$, we have that there exists a finite set $V_i \subseteq \ell^2$ such that $\Omega(x; V_i) \subseteq U_i$. ff

Finally, let $x, y \in \ell^2$ such that $x \neq y$. ff this one doesn't look bad, we use part (a)

(c). *Solution.* Let $U \in \mathcal{N}(0)$ be an arbitrary open set, ie. $U \in \mathcal{T}$ such that $0 \in U$. We want to show that $(U \setminus \{0\}) \cap S \neq \emptyset$; since $0 \notin S$ anyway, we just need to show $U \cap S \neq \emptyset$.

Since $0 \in U$, there exists a finite set $V \subseteq \ell^2$ such that $\Omega(0; V) \subseteq U$. If $V = \emptyset$, then $\Omega(0; V) = \ell^3$ since $-1 < \langle v, y - x\rangle < 1$ is now vacuously true for all $y \in \ell^2$; then $\Omega(0; V) \cap S$ since $\hat{\mathbf{e}}_1 \in \ell^2 \cap S$, and since $\Omega(0; V) \subseteq U$, $U \cap S \neq \emptyset$. So now assume V is not empty. Denote the elements of V as v^i where $1 \leq i \leq k$. Then since $v^i \in \ell^2$, we must have that $\lim_n (v^i_n)^2 = 0$ (crude divergence test). Then there exists some N_i where $(v^i_{N_i})^2 < 1$ by the definition of convergence. Let $N = \min_i\{N_i\}$. Then $-1 < v^i_N < 1$ as well. See

$$\langle v^i, \hat{\mathbf{e}}_N\rangle = \sum_{n=1}^{\infty} v^i_n (\hat{\mathbf{e}}_N)_n = v^i_N$$

Thus $\hat{\mathbf{e}}_N \in \Omega(0, V)$ since $-1 < \langle v, \hat{\mathbf{e}}_N - 0\rangle = v_N < 1$ for all $v \in V$. Thus, $\hat{\mathbf{e}}_N \in \Omega(0, V) \subseteq U$. Since $\hat{\mathbf{e}}_N \in S$, thus shows that $S \cap U \neq \emptyset$, so we are done since U was arbitrary (this works for any open $U \in \mathcal{N}(0)$).

(d). *Solution.* ff

(e). *Solution.* ff

Recap of what still needs to be done:

- Wrapping up Q1: bounded case and end of attaining minimum
- All of Q2 (with contradiction)
- 3(b) (closed implies closed)
- 4(a), 4(b) with the second two conditions for HTS, 4(d), 4(e)