

Problem 1

Prove: If $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges. Is this statement still true if the word “absolutely” is removed?

Solution. It is sufficient to show that $\sum_n a_n b_n$ is absolutely convergent. Consider the series $\sum_n |a_n b_n| = \sum_n |a_n| |b_n|$. Since $\lim_{n \rightarrow \infty} a_n = 0$ (by the contrapositive of the “crude” divergence test since $\sum_n a_n$ converges), we have that a_n is bounded, and so $|a_n|$ is bounded as well (the upper bound is just the max of the lower and upper bound of a_n , and it is bounded below by 0). Let $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then $|a_n| |b_n| < M |b_n|$. We have that $\sum_n M |b_n|$ converges, since if $s_N = \sum_{n=1}^N |b_n|$, then

$$\sum_n M |b_n| = \lim_{n \rightarrow \infty} M |b_0| + M |b_1| + \cdots + M |b_n| = \lim_{n \rightarrow \infty} M (|b_0| + |b_1| + \cdots + |b_n|) = \lim_{n \rightarrow \infty} M s_n$$

and since (s_n) converges (by the absolute convergence of b_n), by our constant multiplication limit law, $(M s_n) = \sum_n M |b_n|$ converges as well.

Now since $0 \leq |a_n b_n| \leq M |b_n|$, by the comparison test, $\sum_n |a_n b_n|$ converges, thus $\sum_n a_n b_n$ is absolutely convergent, which implies that $\sum_n a_n b_n$ converges.

Problem 2

For each series below, find the set of $x \in \mathbb{R}$ where the series converges.

(a). $\sum_{n=1}^{\infty} c^{n^2} (x-1)^n$ ($c > 0$ const.)

(b). $\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(c). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[\frac{x+1}{2x+1} \right]^n$

(d). $\sum_{n=1}^{\infty} \left[\frac{(2n)!}{n(n!)^2} \right] (x-e)^n$

(a). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = c^{n^2}(x-1)^n$ and $\alpha = \limsup_n |a_n|^{1/n}$. We can compute

$$\alpha = \limsup_n \left| c^{n^2}(x-1)^n \right|^{1/n} = \limsup_n |c^n(x-1)| = |x-1| \limsup_n c^n$$

where we've brought the exponent n out in the first step, since $|a^n b^n| = |ab|^n$.

If $x = 1$, then $\alpha = 0$, so the series converges by the root test regardless of c . Now let $x \in \mathbb{R} \setminus \{1\}$. We know that $\lim_{n \rightarrow \infty} c^n \rightarrow +\infty$ if $c > 1$, so $\limsup_n c^n = +\infty$, thus the series diverges for all x . Additionally, $\lim_{n \rightarrow \infty} c^n = 0$ if $0 < c < 1$, so $\limsup_n c^n = 0$, thus the series converges for all x . Finally, if $c = 1$, $a_n = (x-1)^n$ which is a geometric series: it will converge when $|x-1| < 1 \implies 0 < x < 2$ and will diverge otherwise.

In summary:

- If $c > 1$, $x \in \{1\}$ makes the series converge
- If $c = 1$, $x \in (0, 2)$ makes the series converge
- If $0 < c < 1$, $x \in \mathbb{R}$ makes the series converge

(b). *Solution.* Let $a_n = \frac{x^n(1-x^n)}{n}$. Let $x \in \{0, 1\}$. Then $a_n = 0$ for all n , thus the series converges. Let $x = -1$. Then our series is $\sum_n a_n = \sum_{n \text{ odd}} \frac{2}{n}$. We can rewrite our sum to be $\sum_n \frac{1}{2\lfloor (n-1)/2 \rfloor + 1}$ (since if $n = 2k$, $2\lfloor (n-1)/2 \rfloor + 1 = n-1$: the odd number directly below it, and if $n = 2k+1$, $2\lfloor (n-1)/2 \rfloor + 1 = n$: itself) and then since $0 < 2\lfloor (n-1)/2 \rfloor + 1 < n$ so $0 < \left| \frac{1}{n} \right| \leq \frac{1}{2\lfloor (n-1)/2 \rfloor + 1}$, comparison test says this series diverges.

Now consider when $|x| > 1$. Then we claim there exists an $N \in \mathbb{N}$ such that $x^n(1-x^n) < -1$ for all $n \geq N$. We prove this by considering when n is positive and negative. Let $x > 1$. Note that there exists an N such that $x^n > 2$ for all $n \geq N$: using the inequality from Problem 4(a) of Homework 6 since $x > 1$, we have that $x^n > x^n - 1 \geq n(x-1)$ and then invoke Archimedean property to find N such that $N(x-1) > 2$, it's trivial to see that $n \geq N$ also implies $x^n > 2$. Now if $n \geq N$, we have $1-x^n < -1$ and since $x^n > 1$, we have $x^n(1-x^n) < 1-x^n < -1$. Now let $x < -1$. If n is even, $x^n(1-x^n) = |x|^n(1-|x|^n)$, and we have the same N from when $x > 1$ to have $x^n(1-x^n) < -1$. If n is odd, $x^n(1-x^n) = (-1)|x|^n(1-(-1)|x|^n) = -|x|^n(|x|^n+1)$, and using the N from before, we have $-|x|^n(|x|^n+1) < -2(x^n+1) < -2 < -1$. This proves our claim. But then for all $n \geq N$, we have that

$$\frac{x^n(1-x^n)}{n} < \frac{-1}{n} < 0 \implies 0 < \frac{1}{n} = \left| \frac{1}{n} \right| < -\frac{x^n(1-x^n)}{n}$$

And so by the comparison test, $\sum_n -a_n$ diverges. But this is true only if $\sum_n a_n$ diverges, since if $s_N = \sum_{n=1}^N a_n$ and $s'_N = \sum_{n=1}^N -a_n$, we have that $s'_N - s_N = -s_N$, and if s_N converged as $N \rightarrow \infty$, constant multiplication limit law would tell us that s'_N would converge as well. Thus, if $|x| > 1$, we have that the series diverges.

Now consider when $0 < x < 1$. Consider $\sum_k 2^k \frac{x^{2^k}(1-x^{2^k})}{2^k} = \sum_k x^{2^k}(1-x^{2^k})$. We have $0 < x^{2^k}(1-x^{2^k}) < x^{2^k} < x^k$ (where the inequality is due to the fact that x^a is monotonically decreasing when $0 < x < 1$, and $2^k > k$), and $\sum_k x^k$ converges since it is geometric series with ratio $x < 1$. Thus, by the comparison test,

we have that $\sum_k 2^k \frac{x^{2^k}(1-x^{2^k})}{2^k}$ converges as well. Finally, $\frac{x^n(1-x^n)}{n}$ is monotonically decreasing and bounded below by 0: all the terms are positive, so $a_n > 0$ for all n ; now see

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}(1-x^{n+1})n}{x^n(1-x^n)(n+1)} < x \frac{1-x^{n+1}}{1-x^n}$$

and $x \frac{1-x^{n+1}}{1-x^n} \rightarrow x$ as $n \rightarrow \infty$ (limit laws), thus for sufficiently large n , we have that $\frac{a_{n+1}}{a_n} < x + \varepsilon$ where setting $\varepsilon = 1 - x > 0$ gives $\frac{a_{n+1}}{a_n} < 1$, hence the series is monotonically decreasing past that point. Thus, by Cauchy condensation, the series converges when $0 < x < 1$ (technically, Cauchy Condensation only tells us that the series converges starting from our n where the series begins to be monotonically decreasing, but then we have the sum of a convergent series and a finite sum, which itself converges).

It remains to consider the case when $-1 < x < 0$. Now if n is odd, we have $a_n = \frac{(-1)|x|^n(1+|x|^n)}{n} < \frac{-2|x|^n}{n}$. Since $\lim_{n \rightarrow \infty} \frac{2n}{(1/|x|)^n} = 0$ by Rudin Theorem 3.20 (d), we have some N such that $\left(\frac{1}{|x|}\right)^n > 2n > 0$ for all $n \geq N$, thus $0 < 2|x|^n < \frac{1}{n}$, thus $|a_n| < \frac{2|x|^n}{n} < \frac{1}{n^2}$. Furthermore, when n is even, we have $a_n = \frac{|x|^n(1-|x|^n)}{n} < \frac{|x|^n(1+|x|^n)}{n} < \frac{1}{n^2}$ as well. Thus $|a_n| < \frac{1}{n^2}$ for all $n \geq N$. And since $\sum_n \frac{1}{n^2}$ is a convergent p -series ($p > 1$), the comparison test tells us that $\sum_n a_n$ converges as well.

In summary: the series converges when $x \in (-1, 1]$, and diverges otherwise.

(c). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = \frac{1}{\sqrt{n}} \left[\frac{x+1}{2x+1} \right]^n$ and $\alpha = \limsup_n |a_n|^{1/n}$. We can compute

$$\alpha = \limsup_n \left| \frac{1}{\sqrt{n}} \left[\frac{x+1}{2x+1} \right]^n \right|^{1/n} = \limsup_n (n^{1/(2n)})^{-1} \left| \frac{x+1}{2x+1} \right| = \left| \frac{x+1}{2x+1} \right| \limsup_n (n^{1/(2n)})^{-1} = \left| \frac{x+1}{2x+1} \right|$$

where our final equality is due to $\lim_{n \rightarrow \infty} n^{1/(2n)} = 1$, and so $\liminf_n n^{1/2n} = 1$ (lim inf agrees with convergent limits), and by Problem 8(c) from homework 4, $\limsup_n (n^{1/(2n)})^{-1} = 1^{-1} = 1$ (we've also used the fact $|a^n| = |a|^n$ for our first equality). Now, ratio test gives convergence when $|x+1| < |2x+1|$.

(d). *Solution.* ff

Problem 3

Discuss the series whose n th terms are shown below:

$$a_n = (-1)^n \frac{n^n}{(n+1)^{n+1}},$$

$$c_n = (-1)^n \frac{(n+1)^n}{n^n},$$

$$b_n = \frac{n^n}{(n+1)^{n+1}},$$

$$d_n = \frac{(n+1)^n}{n^n}.$$

Solution. ff

Problem 4

Suppose $x_1 \geq x_2 \geq x_3 \geq \cdots$ and $\lim_{n \rightarrow \infty} x_n = 0$. Show that the following series converges:

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \pm \cdots .$$

Solution. Note that the sequence is bounded below by 0. We have a sum of geometric series.

Problem 5

(a). *Prove: if $a_n \geq a_{n+1} \geq 0$ for all n , and $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} na_n = 0$.*

(b). *Prove: If $\sum (b_n^2/n)$ converges, $\frac{1}{N} \sum_{j=1}^N b_j \rightarrow 0$ as $N \rightarrow \infty$.*

[Hint: In part (a), it's enough to prove that $\frac{1}{2}na_n \rightarrow 0$.]

(a). *Solution.* It would be sufficient to show that $\sum_n \frac{1}{2}na_n$ converges. This converges if and only if $\sum_n 2^n a_n$ by the Cauchy condensation... this is not Cauchy condensation because of n ... but what about $n < 2^k$.

Since $a_n \geq a_{n+1} \geq 0$ for all n and $\sum a_n$ converges, the Cauchy Condensation Test gives $\sum_k 2^k a_{2^k}$ converges as well. Since our sequence monotonically decreases and is always positive, we have $2^k a_n \leq 2^k a_{2^k}$ for $n \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$. Note that for any k , $2^k \leq n < 2^{k+1}$ implies $\frac{n}{2} < 2^k$. Thus, $0 \leq |\frac{n}{2}a_n| = \frac{n}{2}a_n < 2^k a_n \leq 2^k a_{2^k}$ for $2^k \leq n < 2^{k+1}$. ff

(b). *Solution.* In Problem 8(a) from homework 3, we proved that if $a_n \rightarrow 0$ as $n \rightarrow \infty$, then $(a_1 + a_2 + \cdots + a_n)/n \rightarrow 0$ as well. Thus, it is sufficient to show that $b_n \rightarrow 0$ as $n \rightarrow \infty$.

Since $\sum (b_n^2/n)$ converges, by the crude divergence test, we have that $b_n^2/n \rightarrow 0$ as $n \rightarrow \infty$. ff

Problem 6

Define $f(\theta) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)\theta)$. Determine the domain of f , namely, the set of all real θ where the series converges, by completing the steps below.

(a). Obtain the following identities, valid for each $n \in \mathbb{N}$ at all points where $\sin \theta \neq 0$:

$$C_n(\theta) = \cos(\theta) + \cos(3\theta) + \cos(5\theta) + \cdots + \cos((2n-1)\theta) = \frac{\sin(2n\theta)}{2 \sin \theta},$$

$$S_n(\theta) = \sin(\theta) + \sin(3\theta) + \sin(5\theta) + \cdots + \sin((2n-1)\theta) = \frac{1 - \cos(2n\theta)}{2 \sin \theta},$$

[Suggestion: Use geometric sums of complex numbers, with $e^{it} = \cos(t) + i \sin(t)$.]

(b). Prove that the domain of f is the interval $(-\infty, +\infty)$.

(c). Find a sequence (θ_n) such that $\theta_n \rightarrow 0$ and $S_n(\theta_n) \rightarrow +\infty$ as $n \rightarrow \infty$. Explain why your solution in part (b) is correct in spite of the evident unboundedness of the sequence $(S_n(\theta_n))$.

(a). Solution. ff

(b). Solution. ff

(c). Solution. ff