Math 321 Homework 3

The goal of the next few problems is to understand which functions can be written as a difference of two increasing functions. We begin with several definitions.

Let $f\colon [a,b] \to \mathbb{R}$ and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. Define

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

This is called the *variation of f with respect to the partition* $[a,b]$. Define

$$TV|f| = \sup_P V(f,P),$$

where the supremum is taken over all partitions of $[a,b]$. this is called the *total variation of f on* $[a,b]$. We say that f has *bounded variatoin on* $[a,b]$ if $TV|f| < \infty$. For $c \in (a,b]$, define $TF|f_{[a,c]}|$ to be the total variation of $f\colon [a,c] \to \mathbb{R}$ (i.e. f is restricted to the interval $[a,c] \subset [a,b]$). We define $TV|f_{[a,a]}| = 0$.

Problem 1

Let $\alpha, \beta\colon [a,b] \to \mathbb{R}$ be (weakly) monotone increasing. Prove that $f(x) = \alpha(x) - \beta(x)$ has bounded variation on $[a,b]$.

Solution. Basically, f is bounded by α and β (watch for negatives) and monotone functions have bounded variation.

Problem 2

Let $f\colon [a,b] \to \mathbb{R}$ have bounded variation on $[a,b]$.

(a). For $x \in [a,b]$, define

$$g(x) = TV|f_{[a,x]}|.$$

Prove that g is (weakly) monotone increasing.

(b). For $x \in [a,b]$, define

$$h(x) = f(x) + TV|f_{[a,x]}|.$$

Prove that h is (weakly) monotone increasing.

(c). Prove that f can be written as $f(x) = \alpha(x) - \beta(x)$, where $\alpha, \beta\colon [a,b] \to \mathbb{R}$ are (weakly) monotone increasing.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 3

Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2\colon [a,b] \to \mathbb{R}$ are (weakly) monotone increasing, and $\alpha_1(x) - \beta_1(x) = \alpha_2(x) - \beta_2(x)$ for all $x \in [a,b]$. Prove that for every continuous $f\colon [a,b] \to \mathbb{R}$, we have

$$\int_a^b f d\alpha_1 - \int_a^b f d\beta_1 = \int_a^b f d\alpha_2 - \int_a^b f d\beta_2$$

Remark. *You have just proven that if $\gamma\colon [a,b] \to \mathbb{R}$ has bounded variation and $f\colon [a,b] \to \mathbb{R}$ is continuous, then we can define*

$$\int_a^b f d\gamma = \int_a^b f d\alpha - \int_a^b f d\beta,$$

where $\gamma = \alpha - \beta$ with α, β monotone increasing; the RHS of this expression does not depend on the specific decomposition $\gamma = \alpha - \beta$ that is chosen.

Solution. ff

Problem 4

(a). Let $f \in \mathcal{R}[a,b]$. For $n \geq 1$, let P_n be the partition with $n+1$ equally spaced points in $[a,b]$, i.e. if $b-a=d$, then

$$P_n = \left\{a, a+\frac{d}{n}, a+\frac{2d}{n}, \ldots, a+\frac{nd}{n} = b\right\} \tag{1}$$

Prove that

$$\lim_{n\to\infty} \left(U(P_n, f) - L(P_n, f)\right) = 0.$$

(b). Let $\alpha\colon [a,b] \to \mathbb{R}$ be monotone increasing, let $f \in \mathcal{R}_\alpha[a,b]$, and let P_n be as defined in (1). Must it be true that

$$\lim_{n\to\infty} \left(U(P_n, f, \alpha) - L(P_n, f, \alpha)\right) = 0? \tag{2}$$

If so, prove it. If not, give a counter-example (i.e. a choice of interval $[a,b]$, a choice of monotone increasing α, and a choice of $f \in \mathcal{R}_\alpha[a,b]$ for which (2 fails) and show that your counter-example is correct.

(a). *Solution.* ff

(b). *Solution.* ff