Math 321 Homework 3

Problem 1

Let $f \in \mathcal{R}[a,b]$ and $0 < p < \infty$. Define

$$\|f\|_p = \left(\int_a^b |f|^p dx \right)^{1/p}$$

(a). Prove that for $0 < p < \infty$, $|f|^p \in \mathcal{R}[a,b]$ (and hence the above definition makes sense).

(b). If f is continuous, prove that

$$\lim_{p\to\infty} \|f\|_p = \sup\{|f(x)| \colon x \in [a,b]\}.$$

(c). For f fixed, define $\phi(p) = \|f\|_p^p$. Using Rudin Problem 6.10, prove that $p \mapsto \log \phi(p)$ is convex on $(0,\infty)$ (recall Rudin problem 4.23 for the definition of convexity, and its consequences). Do not submit the proof of Rudin Problem 6.10 (but I encourage you to do it; it is a good exercise).

Remark 1. Since convex functions are continuous (see Rudin Problem 4.23), you have just shown that ϕ and hence $p \mapsto \|f\|_p$ are continuous.

(a). *Solution.* ff follows from Rudin 6.12

(b). *Solution.* Note that since f is continuous, so is $|f|$ (ff). Furthermore, the function is defined on the closed and bounded set $[a,b]$, which in $\mathbb{R}$ is compact, so $|f|$ attains its maximum value on $[a,b]$ (ff). 6.21(d) gives us an upper bound.

Hmm maybe like IVT. We have that sup is an upper bound. Prove monotonically increasing. Then assume some lower value is an upper bound, but $|f|$ attains this by IVT (or attains halfway between it and the sup), and somehow argue that the integral is larger.

ff

(c). *Solution.* ff

Problem 2

Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions from $\mathbb{R} \to \mathbb{R}$ that converge pointwise. Must it be true that $\{f_n \circ g_n\}$ converges pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

Solution. It can be false. We provide the counter-example: define $f_n(x)$ on $0 < x \le \frac{1}{n}$ as $f_n(x) = x$, and we periodically extend this function off of $(0, \frac{1}{n}]$ so that $f_n(x+k) = f_n(x)$ for any $k \in \mathbb{Z}$; now let

$$g_n(x) = \begin{cases} n, & 0 < x \le \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

as well.

Clearly, both of these are functions from $\mathbb{R} \to \mathbb{R}$. Furthermore, we get that both of these sequences of functions converge pointwise to the 0 function. To see this for f, note that f_n attans its maximum value at $x = \frac{1}{n}$, since on $x \in (0, \frac{1}{n}]$, f_n is monotone increasing and this is the right most value, and since f_n is periodic, the largest value this function attains is the same as the largest value it attains on this interval. Furthermore, $f_n(\frac{1}{n}) = \frac{1}{n}$. Let $\varepsilon > 0$ and $x \in \mathbb{R}$ be fixed, then Archimedean gives us some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and so for any $n \ge N$, we have $|f_n(x) - 0| = f_n(x) \le \frac{1}{n} \le \frac{1}{N} < \varepsilon$, which means that f_n is pointwise convergent to 0 for all $x \in \mathbb{R}$.

To see this for g, fix some $\varepsilon > 0$ and some $x \in \mathbb{R}$. Archimedean gives us some $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. By the definition of g_n, when $n \ge N$, we have that $|g_n(x) - 0| = g_n(x) = 0 < \varepsilon$ (since $x > \frac{1}{N} \ge \frac{1}{n}$). Hence, g_n is pointwise convergent to 0 for all $x \in \mathbb{R}$.

Now let us consider the composition, $\{f_n \circ g_n\}$. ff considering the values pointwise seem like a pain. Make some argument that $f_n \circ g_n = n$, which obviously converges pointwise nowhere.

Problem 3

Let E be a set and let $(M_1, d_1), (M_2, d_2)$ be metric spaces with the discrete metric (i.e. $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ if $x \neq y$). Let $\{g_n\}$ be a sequence of functions from $E \to M_1$, and let $\{f_n\}$ be a sequence of functions from $M_1 \to M_2$. Suppose that $\{f_n\}$ and $\{g_n\}$ converge pointwise. Must it be true that $\{f_n \circ g_n\}$ converges pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

Solution. We claim that this is true, specifically if $g\colon E \to M_1$ and $f\colon M_1 \to M_2$ are functions such that $g_n \to g$ and $f_n \to f$ pointwise, $f_n \circ g_n \to f \circ g$ pointwise as well.

Let $\varepsilon > 0$ and $x \in E$. Since $g_n \to g$, for all $\varepsilon' > 0$, there exists some N_1 such that $d(g_n(x), g(x)) < \varepsilon'$ for all $n > N_1$. If we let $\varepsilon' = \frac{1}{2}$, since this is the discrete metric so $d(g_n(x), g(x))$ can only either be 1 or 0, this tells us that for all $n \geq N_1$, $d(g_n(x), g(x)) = 0$, i.e. $g_n(x) = g(x)$.

Note $g(x) \in M_1$. Since $f_n \to f$, for all $\varepsilon' > 0$, there exists some N_2 such that $d(f_n(g(x)), f(g(x))) < \varepsilon'$ for all $n > N_2$. If we let $\varepsilon' = \frac{1}{2}$, since this is the discrete metric so $d(f_n(g(x)), f(g(x))$ can only either be 1 or 0, this tells us that for all $n \geq N_2$, $d(f_n(g(x)), f(g(x))) = 0$, i.e. $f_n(g(x)) = f(g(x))$.

We now consider $\{f_n \circ g_n\}$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, $f_n(g_n(x)) = f_n(g(x)) = f(g(x))$. Thus, $d(f_n(g_n(x)), f(g(x))) = d(f(g(x)), f(g(x))) = 0 < \varepsilon$.

ε, x were arbitrary, hence $\{f_n \circ g_n\}$ converges pointwise.

Problem 4

Let $\{f_n\}$ be a sequence of functions in $\mathcal{R}[a, b]$, let $f \in \mathcal{R}[a, b]$, let $f_n \to f$ pointwise, and suppose that $\{f_n(x)\}$ is monotone increasing for each $x \in [a, b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Solution. We want $\inf_{P'} U(P', f) - \inf_P U(P, f_n) < \varepsilon$. We know that for any $x \in [a, b]$, there exists some N such that for all $n \geq N$, $f(x) - f_n(x) < \varepsilon$; also $f_n(x) \leq f_{n+1}(x)$.

The integrals are a monotone sequence of functions, their supremum is $\int_a^b f(x)dx$.

I think $f_n \to f$ uniformly because monotone.

Once we have a uniform bound, then we get that for any $\int f_n dx > \int_a^b f - \varepsilon dx = \int_a^b f dx - \varepsilon(b - a)$.

Probably prove that $f \geq f_n$ for all n.

We first prove that f_n converge uniformly. We do this by verifying the Cauchy criterion for uniform convergence (Rudin Theorem 7.8), which we can apply, since our functions are maps into $\mathbb{R}$ (presumebly, since we have not defined the Riemann integral otherwise), which is a complete metric space. So let $\varepsilon > 0$ be aribtrary. ff

Note that it must converge to f specifically, since ff (theorem doesn't specify what it converges to).

For the sake of contradiction, assume that f_n does not converge to f uniformly. Then there exists some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there is some $n \geq N$ and some $x \in [a, b]$ such that $f(x) - f_n(x) \geq \varepsilon$. But then $f_n(x) \notin$

Let $\varepsilon > 0$ be arbitrary. Let B be the lower bound on f_1, i.e. $B \leq f_1 \leq f_n \leq f$ for all n (by monotonicity). Let $\delta_0 = \sup_{x\in[a,b]}\{f(x) - B\}$ Note $\delta_0 > 0$ for all $x \in [a, b]$, since $f(x) - B \geq f_1(x) - B \geq 0$ (and we can ignore equality in the first inequality, because if that was the case, $f(x) = f_1(x) \implies f(x) = f_n(x)$ and so uniform convergence is trivial... maybe make another paragraph for this ff), and must exist in \$$R$, since f must be bounded above, say by M, and $0 < f(x) - B \leq M - B \in \mathbb{R}$.

Now, define $\delta_n = \delta_0/2^n$. Note that since $f_n \to f$ pointwise, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $f_n > f - \varepsilon$, otherwise Since $f_n \in \mathcal{R}[a, b]$, f_n is bounded on $[a, b]$.

Consider $\varepsilon_1 = \sup_{x\in[a,b]}\{f(x) - f_n(x)\}$. This satisfies uniformity for ε_1.

The Tighe Cook: Let $\varepsilon > 0$, and define $\varepsilon' = \frac{\varepsilon}{2(b-a)}$. Consider the set of subintervals of $[a, b]$, denote it X. Define a set of tags for each subinterval, namely $S = \{s\colon s \in I, I \in X\}$. Conversely, let I_s be the subinterval associated with $s \in S$, and δ_s is the length of I_s. For each $k \in \mathbb{N}$, let $L_k = \{s\colon g_k(s) < \varepsilon', s \in S\}$. By pointwise convergence, there exists some k_s such that $s \in L_{k_s}$. Note that $\bigcup_{s\in S} I_s$ covers $[a, b]$ (sus with the boundrary), and since $[a, b]$

is compact, we can extract a finite subcover, so we have $G_1, G_2, \ldots, G_N$ that covers $[a, b]$. Then there is a finite k such that all the $s \in L_k$ for all the s centered in the G_i.

Define $g_n = f - f_n$. Note that for any I_s where $s \in L_k$, where we have

$$\left| \int_{I_s} g_n - g_n(s)\delta_s \right| < \varepsilon' \delta_s$$

(For any g_n, $\exists \delta_n$ such that P finr than δ_n, so we have the inequality above).

And then we're done:

$$\int_a^b g_n = \sum_{I_s \in P} \int_{I_s} g_n \leq g_n \sum_s (g_n(s) + \varepsilon')\delta)s \leq 2\varepsilon'(b-a) = \varepsilon$$