Problem 1

Who are your group members?

Solution. Nicholas Rees

Problem 2

Familarize yourself with basic MATLAB syntax, and make sure you understand what each line in the file `start_here.txt` is doing, and what the commands in `exponential_of_a_matrix.txt` are doing. Answer the following questions with MATLAB (but just write down the answer).

(a). Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

What is the largest interger n such that each entry of $e^A - \sum_{i=0}^{15} A^i/i!$ is of absolute value at most 10^{-n}?

(b). Same question for

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a). *Solution.* $n = 13$.

(b). *Solution.* $n = 13$.

Problem 3

Create the files `apple.m`, `apple_bad.m`, `apple_worse.m`, `apple_quiet.m` and see how they are implementing Euler's method to solve $y' = 2y$ subject to $y(1) = 3$, in order to find $y(2)$. (The files `apple_bad.m`, `apple_worse.m` will produce error messages; they are just there as a cautionary note.) You might also have a look at `chaotic_sqrt.m`.

(a). Use Euler's method to solve $y' = |y|^{1/2}$ subject to $y(t_0) = y_0$ to find the value of $y(t_{\text{end}})$, where $t_0 = -2$, $y_0 = -1$, and $t_{\text{end}} = 2$. Use step size $h = (t_{\text{end}} - t_0)/N$, where $N = 1000$ and $N = 100000$. What values do you get?

(b). Same question, but with $y_0 = 0$, $t_0 = 0$, and $t_{\text{end}} = 2$. Use $N = 10000$.

(c). Same question, but with $y_0 = 10^{-20}$.

(d). Same question, but with $y_0 = 10^{-40}$.

(e). How do you explain the difference between parts (c,d) and part (b)?

(a). *Solution.* When $N = 1000$, we get $y(t_{\text{end}}) = 1.0036$. When $N = 100000$, we get $y(t_{\text{end}}) = 1.0000$.

(b). *Solution.* We get $y(t_{\text{end}}) = 0$.

(c). *Solution.* We get $y(t_{\text{end}}) = 0.9985$.

(d). *Solution.* We get $y(t_{\text{end}}) = 0.9982$.

(e). *Solution.* From an algorithmic point of view, when $y_0 = 0$, Euler's formula, no matter how many iterations, will always give $y(t + ih) = 0$, since it computes $y_{i+1}(t_0 + (i+1)h) = y_i + h|y_i|^{1/2}$ so $y_i = 0 \implies y_{i+1}$, and $y_0 = 0$, so by induction, $y(t_{\text{end}}) = y(t_N) = 0$. If, instead, we had a nonzero value, $y(t_N)$ will at least sum to a positive value.

From a theory point of view, it is because $y = 0$ is a solution to the differential equation, and the unique solution to the initial value problem $y_0 = t_0 = 0$ is $y(t) = 0$. On the other hand, we get a nontrivial solution when $y_0 > 0$, so it approaches a similar value to what we had before in part (a).

Problem 4

If $y\colon \mathbb{R} \to \mathbb{R}$ is a function and $T \in \mathbb{R}$, then the *translation of y by T*, denoted $\mathrm{Trans}_T(y)$, refers to the function z given by

$$\forall t \in \mathbb{R}, \quad z(t) = y(t - T)$$

(or, equivalently, $z(t+T) = y(t)$) (hence $z(T) = y(0)$, $z(T+1) = y(1)$, etc.). Similarly, the *time reversal of y at time T*, denoted $\mathrm{Reverse}_T(y)$ refers to the function z given by

$$\forall t \in \mathbb{R}, \quad z(t) = y(2T - t)$$

(hence $z(T) = y(T)$, and $z(T+a) = y(T-a)$).

(a). If $T_1, T_2 \in \mathbb{R}$ and y is any function, what is

$$\mathrm{Trans}_{T_1}(\mathrm{Trans}_{T_2}(y))$$

in simpler terms?

(b). If $T \in \mathbb{R}$ and y is any function, what is

$$\mathrm{Reverse}_T(\mathrm{Reverse}_T(y))$$

in simpler terms?

(c). If $T_1, T_2 \in \mathbb{R}$ and y is any function, what is

$$\mathrm{Reverse}_{T_1}(\mathrm{Reverse}_{T_2}(y))$$

in simpler terms?

(d). Show that if for some function $f\colon \mathbb{R} \to \mathbb{R}$, y satisfies the ODE $y' = f(y)$ globally (meaning $y'(t) = f(y(t))$ for all $t \in \mathbb{R}$), then $z = \mathrm{Trans}_T(y)$ satisfies the same ODE, i.e., $z' = f(z)$ (globally).

(e). Show directly that if $y(t) = e^{At}$ for some $A \in \mathbb{R}$, and if z satisfies the ODE $z' = Az$ with $z(t) > 0$ for some $t \in \mathbb{R}$, then z is a translation of y, **PROVIDED THAT** $A \neq 0$.

(f). If similarly $y' = f(y)$ globally, then $z = \mathrm{Reverse}_T(y)$ satisfies the ODE $z' = -f(z)$.

(g). If similarly $y'' = f(y)$ globally, then $z = \mathrm{Reverse}_T(y)$ satisfies the ODE $z'' = f(z)$.

(a). *Solution.* We can compute:

$$\begin{aligned}
\mathrm{Trans}_{T_1}(\mathrm{Trans}_{T_2}(y)) &= \mathrm{Trans}_{T_1}(y(t - T_2)) \\
&= y((t - T_1) - T_2) \\
&= y(t - T_1 - T_2)
\end{aligned}$$

(b). *Solution.* We can compute:

$$\begin{aligned}
\mathrm{Reverse}_T(\mathrm{Reverse}_T(y)) &= \mathrm{Reverse}_T(y(2T - t)) \\
&= y(2T - (2T - t)) \\
&= y(t)
\end{aligned}$$

(c). *Solution.* We can compute:

$$\begin{aligned}
\mathrm{Reverse}_{T_1}(\mathrm{Reverse}_{T_2}(y)) &= \mathrm{Reverse}_{T_1}(y(2T_2 - t)) \\
&= y(2T_2 - (2T_1 - t)) \\
&= y(t + 2T_2 - 2T_1)
\end{aligned}$$

(d). *Solution.* Assume that $y'(t) = f(y(t))$ for all $t \in \mathbb{R}$. Note $z(t) = \text{Trans}_T(y) = y(t-T)$. Hence, $z'(t) = y'(t-T) \cdot 1 = f(y(t-T)) = f(z(t))$ for all $t \in \mathbb{R}$. So we have shown $z' = f(z)$ globally.

(e). *Solution.* Let $A \in \mathbb{R}$, and $z' = Az$. Since $A \neq 0$ and $z(t) > 0$ for some $t \in \mathbb{R}$ so z is not the zero function, we can divide by Az to get

$$\begin{aligned}\frac{dz}{dt} = Az &\implies \frac{1}{Az}dz = dt \\ &\implies \int \frac{1}{Az}dz = \int dt \\ &\implies \frac{1}{A}\ln z = t + C \\ &\implies z = e^{A(t+C)}\end{aligned}$$

Hence, $z = y(t+C)$ and so z is a translation of y by $-C$, i.e. $z = \text{Trans}_{-C}(y)$.

(f). *Solution.* Assume that $y'(t) = f(y(t))$ for all $t \in \mathbb{R}$. Note $z(t) = \text{Reverse}_T(y) = y(2T-t)$. Hence $z'(t) = y'(2T-t) \cdot (-1) = -y'(2T-t) = -f(y(2T-t) = -f(z(t))$ for all $t \in \mathbb{R}$. So we have shown $z' = -f(z)$.

(g). *Solution.* Assume that $y''(t) = f(y(t))$ for all $t \in \mathbb{R}$. Note $z(t) = \text{Reverse}_T(y) = y(2T-t)$. Hence $z'(t) = y'(2T-t) \cdot (-1) = -y'(2T-t)$ for all $t \in \mathbb{R}$. We take another derivative to get $z''(t) = -y''(2T-t) \cdot (-1) = y''(2T-t) = f(y(2T-t)) = f(z(t))$ for all $t \in \mathbb{R}$. So we have shown $z'' = f(z)$.