Problem 1

Who are your group members?

Solution. Nicholas Rees

Problem 2

(a). Show that the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

has the unique solution

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 4/2 \\ -1/2 \end{bmatrix}$$

(b). Say that $f\colon \mathbb{R} \to \mathbb{R}$ has $f'''(x)$ existing for all x. Say that $x_0, h \in \mathbb{R}$, and that $|f'''(\xi)| \leq M_3$ for all ξ between x_0 and $x_0 + 2h$. Use the fact that

$$\begin{aligned} f(x_0) &= f(x_0) \\ f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)M_3 \\ f(x_0+2h) &= f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2} f''(x_0) + O(h^3)M_3 \end{aligned}$$

to find a value of c_0, c_1, c_2 such that

$$c_0 f(x_0) + c_1 f(x_0+h) + c_2 f(x_0+2h) = hf'(x_0) + O(h^3)M_3$$

(c). To which formula on page 411 (Section 14.1) of [A&G] are parts (a) and (b) related? Explain.

(d). What is the significance of the solution of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

to approximating $f'(x_0)$? [You don't have to solve this system, just state what you can do with the solution c_0, c_1, c_2, c_3.]

(a). *Solution.* Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$. Note that $\det(A) = 1(4-2) - 0 + 0 = 2 \neq 0$ so A^{-1} exists. One can compute A^{-1} by finding the adjoint matrix and dividing by $\det(A)$, which gives $A^{-1} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix}$ (and one can check $AA^{-1} = A^{-1}A = I$). Thus, multiplying by A^{-1} on both sides, we get

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 2 \\ -1/2 \end{bmatrix}$$

hence c_0, c_1, c_2 must be $-3/2, 4/2, -1/2$, respectively, thus uniquely determining our solution to be the desired one.

(b). *Solution.* We seek to find a linear combination of $f(x_0), f(x_0+h), f(x_0+2h)$ that adds up to only being $hf'(x_0)+O(h^3)M_3$, using the given equalities. But this actually corresponds exactly to our linear transformation A from part (a), where we are changing bases from $f(x_0), f(x_0+h), f(x_0+2h)$ to $f(x_0), hf'(x_0), \frac{h^2}{2}f''(x_0)+O(h^3)M_3$, and our equations that dictate this transformation is represented in A (i.e. the coefficients of f, hf', and $\frac{h^2}{2}f''$ are the entries of A). Hence, our coefficients are the solution to part (a), namely, $c_0 = -3/2$, $c_1 = 4/2$, $c_2 = -1/2$. This gives us

$$-\frac{3}{2}f(x_0) + \frac{4}{2}f(x_0+h) - \frac{1}{2}f(x_0+2h) = hf'(x_0) + O(h^3)M_3$$

(c). *Solution.* This corresponds to the three-point, second order, one-sided formula for $f'(x_0)$, which was in section 2(b) of 14.1 in [A&G], written there as

$$f'(x_0) = \frac{1}{2h}\left(-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right) + \frac{h^2}{3}f'''(\xi)$$

where $\xi \in [x_0, x_0+2h]$. To get the formula we derived from parts (a) and (b), we bring in the $\frac{1}{2}$, multiply everything by h, and bring the third order term to the other side (M_3 let's ignore the specific ξ, and $O(h^3)$ let's ignore the specific coefficient of this term).

(d). *Solution.* This gives us a third order approximation of $f'(x_0)$. Recall by Taylor's theorem that if $f\colon \mathbb{R} \to \mathbb{R}$ has $f^{(4)}(x)$ existing for all x, and for $x_0, h \in \mathbb{R}$ that $|f^{(4)}(\xi)| \le M_4$ for all ξ between x_0 and x_0+3h, we have

$$\begin{aligned}
f(x_0) &= f(x_0)\\
f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0)\frac{h^3}{6}f'''(x_0) + O(h^4)M_4\\
f(x_0+2h) &= f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2}f''(x_0)\frac{(2h)^3}{6}f'''(x_0) + O(h^4)M_4\\
f(x_0+3h) &= f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2}f''(x_0)\frac{(3h)^3}{6}f'''(x_0) + O(h^4)M_4
\end{aligned}$$

The coefficients of these equations correspond to the ones in our matrix, thus, solving the system for c_0, c_1, c_2, c_3 would give us the coefficients for the equation

$$c_0 f(x_0) + c_1 f(x_0+h) + c_2 f(x_0+2h) + c_3 f(x_0+3h) = hf'(x_0) + O(h^4)M_4$$

Since the error term is higher order, this should decrease our error for small h, making this a more accurate approximation of $f'(x_0)$ (dividing everything by h shows it is third order).

Problem 3

Consider an ODE $y' = f(t,y)$, where as in [A&G], $y = y(t)$, and y' refers to dy/dt. Say that $f(t,y)$ is of the special form $f(t,y) = h(t)g(y)$, where g is a differentiable function and h is continuous. Then the ODE

$$y' = dy/dt = h(t)g(y)$$

is called a *separable differential equation*, and it can be solved by writing

$$\frac{dy}{g(y)} = h(t)dt$$

and taking indefinite integrals of both sides. See Section 2.4 (Separable ODE's) of UBC's Calculus 2 Textbook for details, including Example 2.4.2 there, where they solve the equation $y' = y^2$ (in this textbook, y' refers to dy/dx, as is common in math books).

(a). Solve the ODE $y' = y^3$ (here $y = y(t)$ and y' refers to dy/dt) in the same manner as $y' = y^2$ is solved in general form.

(b). Solve $y' = y^3$ for the initial condition $y(1) = 1$.

(c). Solve $y' = y^4$ for the initial condition $y(1) = 1$.

(d). Let $y(t)$ be as in part (b); for $t \geq 1$, when does $y(t)$ fail to exist, i.e., for what $T > 1$ does $y(t) \to \infty$ as $t \to T$?

(e). Same question for part (c).

(a). *Solution.* When $y \neq 0$

$$\frac{dy}{dt} = y^3 \implies \frac{dy}{y^3} = dt \underbrace{\implies}_{\int} \frac{y^{-2}}{-2} = t + C_1 \implies y^2 = \frac{1}{C - 2t}$$

Presumebly, we are restricting ourselves to real-valued solutions, so when $C - 2t > 0 \implies t < C/2$ we have $y(t) = \sqrt{(C-2t)^{-1}}$ or $y(t) = -\sqrt{(C-2t)^{-1}}$.

Now when $y = 0$, we have $y' = 0$ and $y^3 = 0$, hence the zero function also satisfies the ODE, giving our solutions

$$y = 0, (C - 2t)^{-1/2}, -(C - 2t)^{-1/2}$$

where the last two are only defined for $t < C/2$.

(b). *Solution.* Only one of our general forms has positive values, namely $y(t) = (C - 2t)^{-1/2}$. Solving, we get

$$1 = (C - 2(1))^{-1/2} \implies 1 = C - 2 \implies 3 = C$$

So our solution to the IVP is $y(t) = (3 - 2t)^{-1/2}$.

(c). *Solution.* We may assume that y is not the zero function, by the initial values. Then solving the general form gives

$$\frac{dy}{dt} = y^4 \implies \frac{dy}{y^4} = dt \underbrace{\implies}_{\int} \frac{y^{-3}}{-3} = t + C_1 \implies y^3 = \frac{1}{C - 3t} \implies y = (C - 3t)^{-1/3}$$

Plugging in the initial values, we get

$$1 = (C - 3(1))^{-1/3} \implies 1 = C - 3 \implies 4 = C$$

So our solution to the IVP is $y(t) = (4 - 3t)^{-1/3}$.

(d). *Solution.* I have already made some coments on this in part (a), but I claim that $y(t)$ will fail to exist when $t \geq C/2 = 3/2$. So let $T = 3/2$. When $t > T$, we have the square root of a negative, which is not defined in the reals. Now consider $y(t)$ as $t \to T$ (from the left). The denominator is going to 0 and the numerator is a fixed constant. Since both the denominator and numerator are positive, this gives that $y(t) \to \infty$ as $t \to T$. Hence, $y(t)$ from (b) fails to exist for $t \geq 3/2$.

(e). *Solution.* I claim that $y(t)$ will fail to exist when $4 - 3t = 0 \implies t = 4/3$. Consider $y(t)$ as $t \to 4/3$. From either direction, we have a denominator going to 0 and a numerator that is a fixed constant. From the left, the denominator is positive so it blows up to ∞, and from the right, the denominator is negative so it blows up to $-\infty$. Either way, the value does not exist. Other than that point, the function is well-defined, so $y(t)$ from (c) only fails to exist at $t = 4/3$.

Problem 4

Let $y(t) = (3 - 2t)^{-1/2}, z(t) = (4 - 3t)^{-1/3}$.

(a). Examine a plot of $y(t)$ and $z(t)$ for $1 \leq t < 4/3$. Is one of these functions larger than the other in the entire interval (as far as the plot shows)? (Here a simple answer will do. You might type `plot (3-2t)^(-1/2) and (4-3t)^(-1/3)` into Google, or something like that.)

(b). Show that $y' = y^3$ and $z' = z^4$ and that $y(1) = z(1) = 1$.

(c). Show that $y'(1) = z'(1) = 1$.

(d). Show that $y''(1) = 3$ and $z''(1) = 4$. [Hint: differentiate both sides of $y' = y^3$; similarly for z.]

(e). Show that for $h > 0$ and h sufficiently small we have $y(1+h) < z(1+h)$. [Hint: let $u(t) = z(t) - y(t)$; what are the values of $u(1), u'(1), u''(1)$?]

(a). *Solution.* Yes, $z(t) \geq y(t)$ on $1 \leq t < 4/3$ with equality only at $t = 1$.

(b). *Solution.* We can compute

$$\frac{dy}{dt} = \left(-\frac{1}{2}\right)(3-2t)^{-3/2}(-2) = (3-2t)^{-3/2} = y^3$$

and also

$$\frac{dz}{dt} = \left(-\frac{1}{3}\right)(4-3t)^{-4/3}(-3) = (4-3t)^{-4/3} = z^4$$

Furthermore, one can calculate $y(1) = (3-2)^{-1/2} = 1$ and $z(1) = (4-3)^{-1/3} = 1$.

(c). *Solution.* Using the results shown in part (b), we can compute

$$y'(1) = y(1)^3 = 1^3 = 1$$

$$z'(1) = z(1)^4 = 1^4 = 1$$

(d). *Solution.* We first compute y''. We start by differentiating $y' = y^3$ to get

$$y'' = 3y^2y' = 3y^2y^3 = 3y^5$$

Then

$$y''(1) = 3y(1)^5 = 3 \cdot 1^5 = 3$$

Similarly for z, we differentiate $z' = z^4$ to get

$$z'' = 4z^3z' = 4z^3z^4 = 4z^7$$

Then

$$z''(1) = 4z(1)^7 = 4 \cdot 1^7 = 4$$

(e). *Solution.* Let $u(t) = z(t) - y(t)$. Note $u'(t) = z'(t) - y'(t)$ and $u''(t) = z''(t) - y''(t)$. Then

$$\begin{aligned} u(1) &= z(1) - y(1) = 1 - 1 = 0 \\ u'(1) &= z'(1) - y'(1) = 1 - 1 = 0 \\ u''(1) &= z''(1) - y''(1) = 4 - 3 = 1 \end{aligned}$$

By Taylor's theorem, if $h \in \mathbb{R}$, since $u'''(1+h) = z'''(1+h) - y'''(1+h)$ exists for all h sufficiently small, there is some ξ between 1 and $1+h$ such that

$$u(1+h) = u(1) + hu'(1) + \frac{h^2}{2}u''(1) + \frac{h^3}{6}u'''(\xi) = \frac{h^2}{2} + \frac{h^3}{6}u'''(\xi)$$

Hence $z(1+h) - y(1+h) = \frac{h^2}{2} + \frac{h^3}{6}u'''(\xi)$ or $z(1+h) = y(1+h) + \frac{h^2}{2} + \frac{h^3}{6}u'''(\xi)$. When h is sufficiently small, our trailing term is negligible compared to our h^2 term, thus $z(1+h) \approx y(1+h) + \frac{h^2}{2}$. Note that $h^2/2 > 0$ for all h, and so we have that $z(1+h) > y(1+h)$ for h sufficiently small.