Problem 1

Prove the following theorem (terminology is given below):

Suppose X is compact and $f\colon X \to \mathbb{R}$ is lower semicontinuous. Then F is bounded below on X, and there exists a point $z \in X$ satisfying $f(z) \leq f(x)$ for all $x \in X$.

Recall that in a HTS $(X, \mathcal{T})$, a function $f\colon X \to \mathbb{R}$ is called lower semicontinuous *if the following set is closed for every $p \in \mathbb{R}$:*

$$f^{-1}((-\infty, p]) = \{x \in X \colon f(x) \leq p\}.$$

(One approach uses the family of closed sets $f^{-1}((-\infty, p])$ satisfyin $p > \inf f(x)$.)

Solution. ff

Problem 2

Let (X, d) be a metric space, with $K \subseteq X$ a compact set. Prove that whenever $\mathcal{G}$ is an open cover for K, there exists $r < 0$ with this property: for every pair of points $x, y \in K$ obeying $d(x, y) > r$, some open set $G \in \mathcal{G}$ contains both x and y.

Solution. Let $G_1, G_2, \ldots, G_N$ be the finite subcover of K such that $G_i \in \mathcal{G}$ for all $i \in 1, 2, \ldots, N$ and $K \subseteq \bigcup_{1 \leq i \leq N} G_i$, which we know exists from the compactness of K. Since G_i is open, there exists some r_i such that $\mathbb{B}[x, r_i) \subseteq G_i$ for all $x \in G$. Then, let $r := \min_i \{r_i\}$. ff

Problem 3

Define the set-valued "projection" mapping $p_1 \colon \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R})$ *by*

$$p_1(S) = \{x_1 \in \mathbb{R} \colon (x_1, x_2) \in S \text{ for some } x_2\}, \qquad S \subseteq \mathbb{R}^2$$

(a). If S is bounded, must $p_1(S)$ be bounded? (Why or why not?)

(b). If S is closed, must $p_1(S)$ be closed? (Why or why not?)

(c). If S is compact, must $p_1(S)$ be compact? (Why or why not?)

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 4

Recall the set ℓ^2 from HW07 Q3, and the standard "unit vectors" $\hat{\ }_p = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the only nonzero entry in $\hat{\ }_p$ occurs in component p. For any x in ℓ^2 and subset $V \subseteq \ell^2$, write

$$\Omega(x; V) = \{y \in \ell^2 \colon -1 < (v, y - x) < 1, \forall v \in V\}.$$

Then define a collection $\mathcal{T}$ of subsets of ℓ^2 by saying $G \in \mathcal{T}$ if and only if every point $x \in G$ has the property that $x \in \Omega(x; V) \subseteq G$ for some finite set *$V \subseteq \ell^2$.*

(a). Prove that $\Omega(x; V) \in \mathcal{T}$ of every finite set $V \subseteq \ell^2$ and point $x \in \ell^2$.

(b). Prove that $(\ell^2, \mathcal{T})$ is a Hausdorff Topological Space.

(c). Let $S = \{\hat{\ }_p \colon p \in \mathbb{N}\}$. Prove that $0 \in S'$. (Here 0 denotes $(0, 0, \ldots)$, the "origin in ℓ^2.) Note*: This fact proves that $\mathcal{T}$ is different from the metric topology on ℓ^2.*

(d). Prove that every G in $\mathcal{T}$ has the property: for every x in G, there exists $r > 0$ such that

$$G \supseteq \mathbb{B}[x; r) = \{y \in \ell^2 \colon \|y - x\| < r\}.$$

This fact proves that every set considered "open" in $\mathcal{T}$ is also open in the metric topology on ℓ^2. This explains why $\mathcal{T}$ gets called "the weak topology" and the metric topology is also called "the strong topology."

(e). Prove that the following set is closed in the weak topology of ℓ^2: $\mathbb{B}[0; 1] = \{y \in \ell^2 \colon \|y\| \leq 1\}$.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

(d). *Solution.* ff

(e). *Solution.* ff