Lecture-15

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Quotes of the day: Dr. Joshua Zahl 02/09/2024

No quotes today :(

Theorem: Dini's uniform convergence theorem (Baby Rudin 7.13)

Let $(\mathcal{M}, d)$ be a compact metric space (i.e., $[a, b]$), $\{f_n\}$ a sequence of functions, $f_n : \mathcal{M} \to \mathbb{R}$. Suppose that

(a) Each f_n is continuous.

(b) f_n converges *point-wise* to some continuous $f : \mathcal{M} \to \mathbb{R}$.

(c) $f_{n+1}(x) \geq f_n(x)$ for each $x \in \mathcal{M}$, $n \in \mathbb{N}$.

Then, $f_n \to f$ *uniformly* on $\mathcal{M}$.

Proof. Let $g_n = f - f_n$. Then, (a) g_n is continuous, (b) $g_n \to 0$ point-wise, (c) $g_n(x) \geq g_{n+1}(x) \geq 0$ for all $n \in \mathbb{N}$.

Goal: Prove $g_n \to 0$ uniformly, i.e.,

$$\text{For all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that for all } n > N, x \in \mathcal{M}, 0 \leq g_n(x) < \varepsilon.$$

Since g_n is monotonically decreasing, it is sufficient to show for all $x \in \mathcal{M}$, $g_n(x) < \varepsilon$.

Let $\mathcal{K}_n = g_n^{-1}([\varepsilon, \infty))$, $\mathcal{K}_n$ is closed, hence compact ($\mathcal{M}$ compact). Since $\{g_n\}$ is decreasing, $\mathcal{K}_n$ are nested, i.e., $\mathcal{K}_{n+1} \subseteq \mathcal{K}$. Since $g_n \to 0$ point-wise, for each $x \in \mathcal{M}$, there exists n such that $g_n(x) < \varepsilon \Rightarrow x \notin \mathcal{K}_n$. Since x was arbitrary, $\bigcap_{n=1}^{\infty} \mathcal{K}_n = \emptyset$. By theorem 2.36, there exists $N \in \mathbb{N}$ such that $\mathcal{K}_N = \emptyset$, i.e.,

$$\begin{aligned} &g_N(x) < \varepsilon \quad \text{for all } x \in \mathcal{M} \\ \Rightarrow &g_n(x) < \varepsilon \quad \text{for all } x \in \mathcal{M}, n \geq N \\ \Rightarrow &|g_n(x)| < \varepsilon \quad \text{for all } x \in \mathcal{M}, n \geq N. \end{aligned}$$

□

Definition: Supremum norm

Let $(\mathcal{X}, d)$ be a non-empty metric space. Define

$$\mathscr{C}(\mathcal{X}) = \{f : \mathcal{X} \to \mathbb{C} : f \text{ is bounded and continuous}\}.$$

For each $f \in \mathscr{C}(\mathcal{X})$, define the "supremum norm"

$$\|f\| = \sup_{x \in \mathcal{X}} |f(x)|, \text{ for } f \in \mathscr{C}(\mathcal{X}), \|f\| < \infty.$$

Note. If $\mathcal{X}$ is compact in the above definition, f being bounded is superfluous.

Notation 1 (Alternate notation)**.** Some other notation for the supremum norm is: $\|f\|_{\mathscr{C}(\mathcal{X})}$, $\|f\|_{\mathscr{C}^0(\mathcal{X})}$, $\|f\|_\infty$, where the first one is probably the best one.

Note that $\mathscr{C}(\mathcal{X})$ is a vector space over $\mathbb{C}$, with $\|\cdot\|$ as the norm. For this, we have

1. $\|f\| \geq 0$, $\|f\| = 0$ iff $f(x) = 0$ for all $x \in \mathcal{X}$, i.e., $f = 0$.
2. For $\lambda \in \mathbb{C}$, $\|\lambda f\| = \lambda \|f\|$.
3. $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\| \Rightarrow \|f+g\| \leq \|f\| + \|g\|$.

Thus, if we define $\varrho(f, g) = \|f - g\|$, then ϱ is a metric, and $(\mathscr{C}(\mathcal{X}), \varrho)$ is a metric space. Therefore,

$$\begin{aligned} f_n \to f \text{ uniformly } &\iff \text{for all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that for all } x \in \mathcal{X}, \text{ for all } n > N, |f_n(x) - f(x)| < \varepsilon \\ &\iff \text{for all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that for all } n > N, \|f - f_n\| < \varepsilon \\ &\iff f_n \to f \text{ in the metric space } \mathscr{C}(\mathcal{X}). \end{aligned}$$

Theorem: Baby Rudin 7.15

$\mathscr{C}(\mathcal{X})$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence (in $\mathscr{C}(\mathcal{X})$). By theorem 7.8 (Cauchy's criteria), $f_n \to f$ uniformly for some $f : \mathcal{X} \to \mathbb{C}$. by corollary 7.12, f is continuous, since it is the uniform limit of a continuous function. Finally, f is bounded, and $f_n \to f$ uniformly, so there exists $N \in \mathbb{N}$ such that $|f(x) - f_N(x)| < 1$ for all $x \in \mathcal{X}$, so

$$\begin{aligned} &|f(x)| < |f_N(x)| + 1 \leq \|f_N\| + 1 \\ \Rightarrow &\|f\| < \|f_N\| + 1 < \infty, \end{aligned}$$

so f is bounded, and hence $f \in \mathscr{C}(\mathcal{X})$. $\square$