Complex differentiation. (follow Howie)

Defn. Let $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ be a complex function.

f is (complex) differentiable at $c \in D$ if

$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$ exists. This limit is called the <u>derivative</u> of f at c, notation $f'(c)$

$$\boxed{f'(c) = \lim_{z \to c} \frac{f(z) - f(c)}{z - c}}$$

Standard results from calculus apply (sums, products, quotient, chain)
(same proof)

Obviously, $z \mapsto z$, $f(z) = z$ is diff'ble at all $c \in \mathbb{C}$:

$$\lim_{z \to c} \frac{z - c}{z - c} = 1 \quad \text{so} \quad f'(c) = 1 \text{ for all } c \in \mathbb{C}.$$

So polynomials are diff'ble: $f(z) = 2z^2 + 3i\,z + 1$

$$f'(z) = 4z + 3i.$$

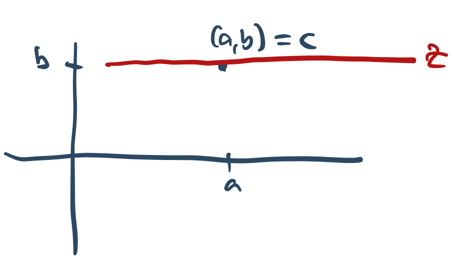
$g(z) = \frac{z^2 + i}{z - i}$ is diff'ble for all $c \neq i$.

Work out what this means for real / imaginary parts.

$z = x + iy$ $\qquad f(z) = f(x+iy) = u(x+iy) + i\,v(x+iy)$

$$= u(x,y) + i\,v(x,y) \qquad u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

$c = a + ib \quad a, b \in \mathbb{R}.$



fix z to be on the horizontal line through c.

$z = x + iy$ $\quad$ <u>$y = b$</u> fixed, x variable.

$$\lim_{x\to a}\frac{f(x+iy)-f(a+ib)}{x+iy-(a+ib)} = \lim_{x\to a}\frac{u(x,y)+iv(x,y)-u(a,b)-iv(a,b)}{x+iy-a-ib}$$

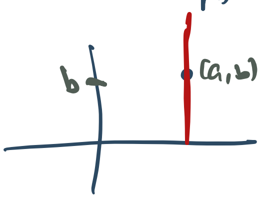
$(y=b)$

$$= \lim_{x\to a}\frac{u(x,b)+iv(x,b)-u(a,b)-iv(a,b)}{x+\cancel{ib}-a-\cancel{ib}}$$

$$= \lim_{x\to a}\frac{u(x,b)-u(a,b)}{x-a} + i\lim_{x\to a}\frac{v(x,b)-v(a,b)}{x-a}$$

$$\boxed{f'(a+ib) = \left.\frac{\partial u}{\partial x}\right|_{(a,b)} + i\left.\frac{\partial v}{\partial x}\right|_{(a,b)}}$$

Now we fix $x=a$ and let y vary:



$$f'(a+ib) = \lim_{y\to b}\frac{u(a,y)+iv(a,y)-\left(u(a,b)+iv(a,b)\right)}{(\cancel{a}+iy)-(\cancel{a}+ib)}$$

$$= \lim_{y\to b}\frac{u(a,y)-u(a,b)+i\left(v(a,y)-v(a,b)\right)}{i(y-b)}$$

$$= \lim_{y\to b}\frac{v(a,y)-v(a,b)}{y-b} - i\lim_{y\to b}\frac{u(a,y)-u(a,b)}{y-b}$$

$$\boxed{f'(a+ib) = \left.\frac{\partial v}{\partial y}\right|_{(a,b)} - i\left.\frac{\partial u}{\partial y}\right|_{(a,b)}}$$

So if f is complex diff'ble at $c=(a+ib)=(a,b)$ then

$$\left.\frac{\partial u}{\partial x}\right|_{(a,b)} = \left.\frac{\partial v}{\partial y}\right|_{(a,b)} \qquad \left.\frac{\partial v}{\partial x}\right|_{(a,b)} = -\left.\frac{\partial u}{\partial y}\right|_{(a,b)}$$

Cauchy – Riemann equations

Feb 1

<u>Example.</u> $f(z) = \bar{z}$

$$f(x+iy) = \overline{x+iy} = x - iy = u(x,y) + iv(x,y)$$

So $u(x,y) = x$ and $v(x,y) = -y$.

Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = -1$ (at all pts of $\mathbb{C}$)

CRI: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$: $1 = -1$ holds nowhere in $\mathbb{C}$.

CRII: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$: $0 = 0$.

So $f(z) = \bar{z}$ is nowhere cplx diff'ble.

<u>Example.</u> $f(z) = |z|^2 = z\bar{z}$.

$$f(x+iy) = (x+iy)(x-iy) = x^2 - i\cancel{xy} + i\cancel{yx} + y^2 = x^2+y^2$$

$u(x,y) = x^2+y^2$ $\quad v(x,y) = 0$.

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial u}{\partial y} = 2y$ $\quad \frac{\partial v}{\partial x} = 0$ $\quad \frac{\partial v}{\partial y} = 0$ (at all pts of $\mathbb{C}$)

CRI: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$: $2x = 0$ only solution $x = 0$.

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$: $2y = 0$ only solution $y = 0$.

only solution to both equations is $(x,y) = (0,0)$ or $z = 0$.

So $f(z) = |z|^2$ is not (cplx) diff'ble anywhere, except for maybe at the origin.

Let's consider the origin: $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{|z|^2 - 0}{z}$

$$= \lim_{z\to 0} \frac{z\bar{z}}{z} = \lim_{z\to 0} \bar{z} = \overline{\lim_{z\to 0} z} = \bar{0} = 0 .$$

Since this limit exists, $f(z) = |z|^2$ is (cplx) diffble at $z=0$

and $f'(0) = 0$.

Warning: just because the C.R. eqtns hold at some point $z=c$, does not mean that $f(z)$ is (cplx) diffble at $z=c$:

<u>Example</u>. $f(x+iy) = \sqrt{|xy|}$ at the origin. $c=0$.

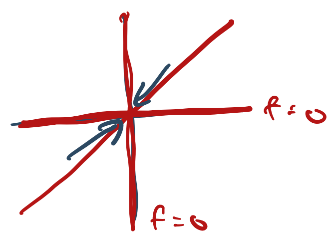
$$u(x,y) = \sqrt{|xy|} \qquad v(x,y) = 0.$$

on the real axis $(y=0)$ and on the imaginary axis $(x=0)$ we have $u(x,y)=0$.

So $\frac{\partial u}{\partial x}\big|_0$ and $\frac{\partial u}{\partial y}\big|_0$ exist and are 0. (also $\frac{\partial v}{\partial x}\big|_0 = 0$, $\frac{\partial v}{\partial y}\big|_0 = 0$)

So the CR. eqtns hold at $c=0$.

<u>But</u>: f is not (cplx) diffble at $c=0$:



consider f restricted to the line $x=y$.

$$f(x+ix) = \sqrt{|x^2|} = |x|$$

$$\lim_{x\to 0^+} \frac{|x|-|0|}{x-0} = 1 \qquad \lim_{x\to 0^-} \frac{|x|-|0|}{x-0} = -1$$

so $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist. f not cplx diffble at $c=0$.

<u>Theorem</u> let $D = D(c,r)$ disc centred at $c \in \mathbb{C}$, radius $r \in \mathbb{R}$, $r>0$.

$$D(c,r) = \{ z \in \mathbb{C} \mid |z-c| < r \}$$

Suppose that $D \subset$ domain of the complex function $f = u+iv$

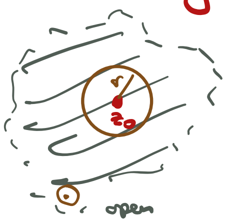
Suppose that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist <u>and</u> are continuous at all pts of D

Then if the CR eqtns hold at c, then f is cplx diffble at c.

(If the CR eqtns hold at all pts of D, then f is holomorphic ~~cplx diffble~~ at all pts of D.)

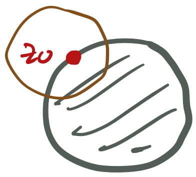
Defn. A subset $U \subset \mathbb{C}$ is open if

For every $z_0 \in U$ there exists an $r > 0$ s.t. $D(z_0, r) \subset U$.



e.g. $\{z \mid |z - z_0| < r\}$ is open

$\{z \mid |z - z_0| \leq r\}$ is not open



not open

Defn If $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ a function,

If f is cplx diff'ble at every pt $c \in U$, then f is called

holomorphic or analytic. If $U = \mathbb{C}$ then f is entire.

So $f(z) = |z|^2$ is cplx diff'ble at 0 but not holomorphic at 0.

(holomorphic requires diff'ble on some open set.)