Problem 1

Let $0 < a_1 < b_1$ *and define* $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{a_n + b_n}{2}$, $n \in \mathbb{N}$.

(a). *Prove that the sequences* (a_n) *and* (b_n) *both converge. (Suggestion: Use induction to prove* $0 < a_n < a_{n+1} < b_{n+1} < b_n$*.)*

(b). *Prove that the sequences* (a_n) *and* (b_n) *have the same limit.*

(a). *Solution.* First, we use induction to prove $0 < a_n < a_{n+1} < b_{n+1} < b_n$. We do this in steps: first we prove that $0 < a_n$ and $0 < b_n$ for all $n \in \mathbb{N}$. When $n = 1$, by assumption, $0 < a_1$ and $0 < b_1$. Now let $n = j$, and $0 < a_j, b_j$. Then $0 < \sqrt{a_j b_j} = a_{j+1}$ and $0 < \frac{a_j + b_j}{2} = b_{j+1}$ as desired. Thus $0 < a_n, b_n, \forall n \in \mathbb{N}$.

Now we prove that $a_n < b_n$ for all $n \in \mathbb{N}$. When $n = 1$, by assumption, $a_1 < b_1$. Now let $n = j$, and $a_j < b_j$. Then $a_{j+1} = \sqrt{a_j b_j}$ and $b_{j+1} = \frac{a_j + b_j}{2}$. We have

$$\begin{aligned} b_{j+1}^2 &= (a_j^2 + 2a_j b_j + b_j^2)/4 \\ &> (2a_j b_j + 2a_j b_j)/4 \\ &= a_j b_j = a_{j+1}^2 \end{aligned}$$

where the second line is because $a_j - b_j > 0 \implies (a_j - b_j)^2 > 0$ so $a_j^2 - 2a_j b_j + b_j^2 > 0 \implies a_j^2 + b_j^2 > 2a_j b_j$. But since we know that both $a_{j+1}, b_{j+1} > 0$, then $b_{j+1}^2 > a_{j+1}^2 \implies |b_{j+1}| > |a_{j+1}| \implies b_{j+1} > a_{j+1}$, which closes the induction.

Now we prove that ff

(b). *Solution.* ff

Problem 2

(a). Suppose $(z_n)_{n\in\mathbb{N}}$ is a bounded sequence with integer values.

(a) Prove that both numbers below are integers:

$$\lambda = \liminf_{n\to\infty} z_n \qquad \mu = \limsup_{n\to\infty} z_n$$

(b) Prove that there are infinitely many integers n for which $z_n = \lambda$.

(b). Let $d_n = p_{n+1} - p_n$ ($n \in \mathbb{N}$) denote the sequence of prime differences, built from the sequence of primes

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$$

(a) Prove that $\limsup_{d\to\infty} = +\infty$.

(b) Some mathematicians believe that

$$\delta := \liminf_{n\to\infty} d_n = 2 \tag{1}$$

However, the best estimate of δ known to date is $2 \leq \delta \leq 246$. Identify by name a famous unsolved problem in mathematics that is equivalent to proving or disproving line (1). (After giving the name, clearly explain the required relationship.)

(a). (a) *Solution.* ff

(b) *Solution.* ff

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(b) *Solution.* ff

Problem 3

(a). Prove: For any sequences (a_n) *and* (b_n) *of nonnegative real numbers,*

$$\limsup_{n\to\infty}(a_n b_n) \le \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right),$$

provided the right side does not involve the product of 0 *and* ∞*.*

(b). Give an example in which the result of part (a) holds with a strict inequality.

(a). *Solution.* ff

(b). *Solution.* ff

Problem 4

(a). *Show that for any $r \geq 1$, one has*

$$n(r-1) \leq r^n - 1 \leq nr^{n-1}(r-1) \quad \forall n \in \mathbb{N}$$

(b). *Prove that for each real $a \geq 1$, the following sequences converges:*

$$x_n = n\left(a^{1/n} - 1\right), \quad n \in \mathbb{N}$$

(c). *Prove that the sequence in (b) also converges for each real $a \in (0,1)$.*

(d). *Let $L(x) = \lim_{n\to\infty} n\left(x^{1/n} - 1\right)$ for $x > 0$. Prove that $L(ab) = L(a) + L(b)$ for all $a > 0, b > 0$.*

Note: *Present solutions that use only methods discussed in MATH 320. No calculus, please!*

(a). *Solution.* Recall the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ for any $n \in \mathbb{N}$. Since $r \geq 1$, applying the identity when $b = r$ and $a = 1$, we get

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + 1) \leq (r-1)nr^{n-1}$$

since each $r^{n-j} \leq r^{n-1}$ $(1 \leq j \leq n)$, and there are n many r^{n-j} in the factor.

To prove the second inequality, we use the identity again, however

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + 1) \geq n(r-1)$$

since each $r^{n-j} \geq 1$ $(1 \leq j \leq n)$, and there are n many r^{n-j} in the factor. Thus we have shown $n(r-1) \leq r^n - 1 \leq nr^{n-1}(r-1)$.

(b). *Solution.* We have that for all $n \in \mathbb{N}$, by part (a) of this problem, $x_n = n\left(a^{1/n} - 1\right) \leq a - 1$, thus x_n is bounded above. Furthermore, since $a \geq 1$, we must have $a^{1/n} = b \geq 0$ for all $n \in \mathbb{N}$, otherwise if $a^{1/n'} < 0$ for some n', then $a < 1$, a contradiction (ff). weird inequality stuff.

(c). *Solution.* ff

(d). *Solution.* ff