Problem 1

Find all positive integers n for which there exist $a, b \in \mathbb{N}$ such that

$$n^2 = 2^a + 2^b$$

Solution. Let n be a positive integer such that there are $a, b \in \mathbb{N}$ such that $n^2 = 2^a + 2^b$.

First, consider when $a = b$. Then $n^2 = 2^{a+1}$. Obviously, if a, b is odd, i.e. $a = 2k - 1$ for some $k \in \mathbb{N}$, then $2^{a+1} = 2^{2k} = (2^k)^2$ which is a perfect square, so $n = 2^k$. We claim that we cannot have a, b even. Otherwise, we would have $a = 2k$, and so $n^2 = 2^{a+1} = 2^{2k+1} = 2 \cdot (2^k)^2$, but note that if $x^2 = zy^2$, $x, y, z \in \mathbb{Z}$, then $x = \sqrt{z}y \implies \sqrt{z} \in \mathbb{Q}$, hence $n^2 = 2(2^k)^2 \implies \sqrt{2} \in \mathbb{Q}$, which we know is not true. Thus, we have our first possible form of n, that is $n = 2^k$ for any $k \in \mathbb{N}$.

Now we let $a \neq b$. WLOG let $a > b$. Then we can write

$$n^2 = 2^b(2^{a-b} + 1)$$

Note that b must be even, since $2^b, (2^{a-b} + 1)$ are coprime, so both factors must be perfect squares. Let $c = a - b$. Since $2^c + 1$ is a perfect square, there exists $m \in \mathbb{N}$ such that $m^2 = 2^c + 1 \implies 2^c = (m+1)(m-1)$. Since the only prime that divides 2^c is 2, only 2 divides the right hand side, so we must have that $m + 1$ and $m - 1$ are powers of 2 as well. So there are $i, j \in \mathbb{N}$ such that $m + 1 = 2^i$ and $m - 1 = 2^j$. Note $i > j$. But

$$2 = m + 1 - (m - 1) = 2^i - 2^j = 2^j(2^{i-j} - 1)$$

We must then have $j = 1$, otherwise if $j > 1$, we have $2^j > 2$ and $2^{i-j} - 1 > 2^1 - 1 = 1$, so the right hand side would be greater than 2. But if $j = 1$, we must then also get that $i = 2$. So $2^{a-b} + 1 = 2^2 \cdot 2^1 + 1 = 3$. So n must be of the form $3 \cdot 2^{2k}$ where $k \in \mathbb{N}$.

Therefore, our n is of the form $n = 2^k$ or $n = 3 \cdot 2^{2k}$ where $k \in \mathbb{N}$.

Problem 2

Find all integers x and y for which

$$x^3 - y^2 = 9$$

Solution. Since $x^3 - y^2$ is odd, if there is a solution, either x^3 is odd and y^2 is even, or x^3 is even and y^2 is odd. Since taking the nth power of a number does not change whether it is even or odd, our cases are equivalent to when x is odd and y is even, and when x is even and y is odd.

First, consider when x is even and y is odd. Then we can rewrite $x = 2n$ and $y = 2m + 1$ where $n, m \in \mathbb{Z}$. Then $(2n)^3 - (2m+1)^2 = 8n^3 - 4m^2 - 4m - 1 = 9 \implies 8n^3 - 4m^2 - 4m = 10$. But the left hand side is congruent to 0 modulo 4 and the right hand side is congruent to 2 modulo 4, hence our left sides does not equal our right, for any m, n. Thus, there does not exist even x and odd y that solves $x^3 - y^2 = 9$.

Now consider when x is odd and y is even. Note that $x \equiv 1 \pmod 4$, for if $x \not\equiv 1 \pmod 4 \implies x \equiv -1$ then $x^3 \equiv -1 \cdot -1 \cdot -1 \equiv -1 \pmod 4$ so $x^3 - 8 \equiv -1 \pmod 4$, however, since $\exists k \in \mathbb{Z}$ such that $y = 2k$ because its even, we have $(2k)^2 + 1 = 4k^2 + 1 \equiv 1 \pmod 4$, but then $x^3 - 8 \neq y^2 + 1$, so no solutions exist when $x \equiv -1 \pmod 4$. Note that we can't have $x = 1$ since then $x^3 - y^2 < 0 < 9$. Then, since $x \equiv 1 \pmod 4$, $x \geq 5 \implies x - 2 \geq 3$. Since $2 \nmid x - 2$, then there exists at least one odd prime that divides $x - 2$. Let p be any prime that divides $x - 2$. Then since $(x-2)(x^2 + 2x + 4) = x^3 - 8 = y^2 + 1$, we get that $p \mid y^2 + 1$, hence $y^2 \equiv -1 \pmod p$. Then by the definition of the Legendre symbol, $\left(\frac{-1}{p}\right) = 1$. Then by Proposition 19.3, since p is odd, $p \equiv 1 (\bmod r)$. Now, since this is true for every prime in the prime decomposition of $x - 2$, we have that $x - 2 \equiv 1 \cdot 1 \cdots 1 \equiv 1 \pmod 4$, thus $x \equiv -1 \pmod 4$. But this contradicts our assumption that $x \equiv 1 \pmod 4$.

This covers all of our possible cases for x, y, so we cannot have a solution $x, y \in \mathbb{Z}$.

Problem 3

Prove that for each positive integers x and y, if the fractional part $\{\sqrt[3]{y}\}$ equals the fractional part $\{\sqrt{x}\}$, then we must have that x is a perfect square, while y is a perfect cube.

Solution. We have that $\{\sqrt{x}\} - \{\sqrt[3]{y}\} = 0$, and since $\{z\} = z - \lfloor z \rfloor$, we have that $\sqrt{x} - \lfloor \sqrt{x} \rfloor - \sqrt[3]{y} + \lfloor \sqrt[3]{y} \rfloor = 0$, so $\sqrt{x} - \sqrt[3]{y} = \lfloor \sqrt{x} \rfloor - \lfloor \sqrt[3]{y} \rfloor =: n \in \mathbb{Z}$. Then $\sqrt[3]{y} = \sqrt{x} - n$. So $y = (\sqrt{x} - n)^3 = x\sqrt{x} - xn + \sqrt{x}n^2 - n^3$. Since $(y + xn + n^3)/(x + n^2) \in \mathbb{Q}$ (we can divide by x since it is positive), $\sqrt{x} \in \mathbb{Q}$. Then Proposition 24.1 implies that $\sqrt{x} \in \mathbb{N}$, so x is a perfect square. But then the fractional part $\{\sqrt{x}\}$ is zero, and by assumption, the fractional part of $\{\sqrt[3]{y}\}$ is zero, so y is a perfect cube.