Problem 1

Use $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ *and a splitting argument to evaluate* $S = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$.

Solution. We have

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = S + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{n^4} = S + \sum_{n=1}^{\infty} \frac{1}{(2n)^4} = S + \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} = S + \frac{\pi^4}{16 \cdot 90}$$

Thus $S = \frac{\pi^4}{90} - \frac{\pi^4}{16 \cdot 90} = \frac{\pi^4}{96}$.

Problem 2

Test the following series for convergence. Treat all real values of the constant parameter p.

(a). $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$

(b). $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

(c). $\sum_{n=2}^{\infty} \frac{1}{n^p(\log n)}$

(d). $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

(a). *Solution.* Note that $\log n$ is monotonically increasing and $\log n > 0$ when $n \geq 2$. Let $p \leq 0$. Then $(\log n)^p$ is monotonically decreasing, so $a_n = \frac{1}{(\log n)^p}$ is monotonically increasing. Note that $a_2 > 0$ for all p, and $a_n \geq a_2$ for all n, thus (a_n) does not converge to zero because it is bounded below away from zero (one can use $\varepsilon = a_2$ to show failiure to converge). Thus, by the crude divergence, $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ does not converge.

Let $p > 0$. Since $\log n$ is monotonically increasing, $(\log n)^p$ is also monotonically increasing. Furthermore, for $n \geq 2$, $(\log n)^p > 0$. Thus, (a_n), where $a_n = \frac{1}{(\log n)^p}$, is a monotonically decreasing series.

Also note that

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} 2^k \frac{1}{(\log 2^k)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{2^k}{k^p}$$

Note that $\left(\frac{2^k}{k^p}\right)^{-1} = \frac{k^p}{2^k}$. Thus by Theorem 3.20 (d) in Rudin (where $\alpha = p$ and $p = 1$) we know that $\lim_{n\to\infty} \frac{k^p}{2^k} = 0$. But by question 6 (b) from homework 6, which states that if $x_n \to 0$ then $1/x_n$ cannot converge, we have that $\frac{2^k}{k^p}$ diverges. Thus, $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges as well (crude divergence test). Then, by the Cauchy Condensation Test, $\sum_{n=2}^{\infty} a_n$ must diverge as well (since (a_n) is monotonically decreasing and bounded below by 0). Hence, regardless of p, the series fails to converge.

(b). *Solution.* Let $a_n = \frac{1}{(\log n)^n}$ and

$$b_n = \begin{cases} \frac{1}{(\log 2)^2} & n = 2 \\ \frac{1}{(\log 3)^n} & n \geq 3 \end{cases}$$

Consider the series $\sum_{n=2}^{\infty} b_n = \frac{1}{(\log 2)^n} + \sum_{n=3}^{\infty} \frac{1}{(\log 3)^n}$. Note that our series is a geometric series, specifically $\log 3 > 1$ so $0 < \frac{1}{\log 3} < 1$, which is the common ratio r, and so we know that the series $\sum_{n=2}^{\infty} b_n$ converges.

Now, since $0 < \log 3 \leq \log n$ for $n \geq 3$, we have $0 < \frac{1}{\log n} < \frac{1}{\log 3} \implies 0 < \frac{1}{(\log n)^n} < \frac{1}{(\log 3)^n}$, thus $b_n \geq a_n = |a_n| \geq 0$ for all n, and thus by the comparison test, $\sum_{n=2}^{\infty} a_n$ must converge as well. (This is true regardless of p; it was not used.)

(c). *Solution.* Let $a_n = \frac{1}{n^p(\log n)}$. Let $p \leq 0$. Then $n > 1 \implies 0 < \frac{1}{n} < 1 \implies \frac{1}{n^p} \geq 1$. Furthermore, $0 < \log n < n \implies \frac{1}{\log n} > \frac{1}{n} > 0$. Thus $\frac{1}{n^p}\frac{1}{\log n} > \frac{1}{n} = \left|\frac{1}{n}\right| > 0$. Recall that $\sum_n \frac{1}{n} = +\infty$, so by the comparison test, $\sum_{n=2}^{\infty} a_n = +\infty$ as well, i.e. it fails to converge.

Now let $0 < p \leq 1$. Note that $(n+1)^p > n^p > 0$ and $\log n + 1 > \log n > 0$ for all n, thus $(n+1)^p \log n + 1 > n^p \log n > 0$, and taking the reciprocal, we get $a_n > a_{n+1} > 0$. Now see

$$\sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p(\log 2^k)} = \frac{1}{\log 2} \sum_{k=1}^{\infty} \frac{(2^k)^{1-p}}{k}$$

First, if $p = 1$, then our sum becomes $\frac{1}{\log 2} \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges, so by the Cauchy condensation test, $\sum_n a_n$ must diverge as well. Now let $p \neq 1$. Note that $\left(\frac{(2^k)^{1-p}}{k}\right)^{-1} = \frac{k}{(2^{1-p})^k}$. By Theorem 3.20 (d) in Rudin (where $\alpha = 1$ and $p = 2^{1-p} - 1 > 0$), we know that $\lim_{k\to\infty} \frac{k}{(2^{1-p})^k} = 0$. But then by question 6 (b) from homework 6, we have that $\frac{(2^k)^{1-p}}{k}$ must diverge as well. Thus $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges as well (crude divergence test). Then, by the Cauchy condensation test, $\sum_{n=2}^{\infty} a_n$ must diverge as well. Thus if $0 < p \leq 1$, the series fails to converge.

Finally, let $p > 1$. Note that for all $n \geq 3$, $\log n > 1$, so $0 < n^p < n^p \log n$, thus $0 < \frac{1}{n^p \log n} = \left|\frac{1}{n^p \log n}\right| < \frac{1}{n^p}$. Since $p > 1$, we know that $\sum_{n=3}^{\infty} \frac{1}{n^p}$ converges, thus by the comparison test, $\sum_{n=3}^{\infty} \frac{1}{n^p \log n}$ converges as well. Therefore, when $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n^p(\log n)}$ converges.

(d). *Solution.* Let $a_n = \frac{1}{n(\log n)^p}$. Let $p \leq 0$. Note that since $0 < \frac{1}{\log n} < 1$ for $n \geq 3$, we have $1 \leq \frac{1}{(\log n)^p}$. Thus, $\frac{1}{n(\log n)^p} \geq \frac{1}{n} = \left|\frac{1}{n}\right| > 0$. Furthermore, $\sum_n \frac{1}{n}$ diverges, thus by the comparison test, $\sum_{n=3}^{\infty} a_n$ diverges as well. Thus, when $p \leq 0$, $\sum_{n=2}^{\infty} a_n$ does not converge.

Now let $p > 0$. Note that $(n+1) > n > 0$ and $\log n + 1 > \log n > 0 \implies (\log n + 1)^p > (\log n)^p > 0$ for all $n \geq 2$. thus $(n+1)^p \log n + 1 > n^p \log n > 0$, and taking the reciprocal, we get $a_n > a_{n+1} > 0$. Consider

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

But the sum is just the p-series. Thus if $p \leq 1$, this sum diverges, and if $p > 1$, the sum converges. Thus, by the Cauchy condensation test, when $0 < p \leq 1$, $\sum_n a_n$ fails to converge; when $p > 1$, $\sum_n a_n$ converges.

Problem 3

Consider the set ℓ^2 consisting of all real sequences $x = (x_1, x_2, \dots)$ enjoying the special property that $\sum_n |x_n|^2$ converges. Define an inner product on ℓ^2 as follows:

$$\forall x, y \in \ell^2,\ \langle x, y\rangle := \sum_{n=1}^{\infty} x_n y_n$$

(a). Prove that the series in this definition converges.

Informally, this is the natural generalization of Euclidean k-space to the case $k = \aleph_0$; the inner product $\langle x, y\rangle$ in ℓ^2 is analogous to the dot product $x \bullet y$ in $\mathbb{R}^k$. It's only a small stretch to call the elements of ℓ^2 "vectors". Add further credibility to this interpretation by defining $\|x\| = \sqrt{\langle x, x\rangle}$ for each $x \in \ell^2$, and then proving

(b) $|\langle x, y\rangle| \le \|x\|\|y\|$ for all $x, y \in \ell^2$.

(c) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in \ell^2$.

This generalization has some limitations, however. In $\mathbb{R}^k$, any sequence of vectors $x^{(1)}, x^{(2)}, x^{(3)}, \dots,$ whose compoenent sequences converge must be a convergent sequence of vectors, and its limit can be identified by taking the limit in each component separetly. Show that this fails in ℓ^2, as follows:

(d) Construct a sequence $x^{(1)}, x^{(2)}, \dots,$ of vectors in ℓ^2 such that $\|x^{(n)}\| = 1$ for all n, and yet every $p \in \mathbb{N}$ the 'p-th component sequence' $\langle \mathbf{e}_p, x^{(n)}\rangle$ converges to 0 as $n \to \infty$. Here, just as in $\mathbb{R}^k$, $\mathbf{e}_p$ denotes the "standard unit vector" with exactly one nonzero entry, which is a 1 in position p.

(a). *Solution.* For all n, we have $(x_n + y_n)^2 \ge 0$, so $x_n^2 + y_n^2 \ge -2x_n y_n$, and $(x_n - y_n)^2 \ge 0$, so $x_n^2 + y_n^2 \ge 2x_n y_n$, hence $x_n^2 + y_n^2 \ge 2|x_n y_n| \ge |x_n y_n|$.

Let $b_n = x_n^2 + y_n^2 = |x_n|^2 + |y_n|^2$. We now show that $\sum_n b_n$ converges. Let $X_n = \sum_{k=0}^{n} |x_k|^2$ and $Y_n = \sum_{k=0}^{n} |y_k|^2$. Thus $X_n + Y_n = \sum_{k=0}^{n}(|x_k|^2 + |y_k|^2) = \sum_{k=0}^{n} b_k$. Since $\sum_n |x_n|^2$ and $\sum_n |y_n|^2$ converge, denote their limits as $X = \sum_n |x_n|^2 = \lim_{n\to\infty} X_n$ and $Y = \sum_n |y_n|^2 = \lim_{n\to\infty} Y_n$. So by the addition limit law, we have $\sum_n b_n = \lim_{n\to\infty} \sum_{k=0}^{n} b_k = \lim_n \to \infty(X_n + Y_n) = X + Y$, thus $\sum_n b_n$ converges.

Finally, by the comparison test, since $\sum_n (x_n^2 + b_n^2)$, we have that $\sum_n (x_n y_n)$ converges as well.

(b). *Solution.* Note that if we consider the first k terms of x_n and y_n as entries of a k-tuple, we have shown in class the Cauchy Schwartz inequality:

$$L_k = \left|\sum_{n=1}^{k} x_n y_n\right| \le \sqrt{\sum_{n=1}^{k} x_n^2}\sqrt{\sum_{n=1}^{k} y_n^2} = \sqrt{\sum_{n=1}^{k} x_n^2 \sum_{n=1}^{k} y_n^2} = R_k$$

Note that L_k and R_k are sequences that satisfy $L_k \le R_k$ for all k. Thus by the lemma from October 11 from the course notes, we also have

$$\liminf_{k\to\infty} L_k \le \liminf_{k\to\infty} R_k \quad \text{and} \quad \limsup_{k\to\infty} L_n \le \limsup_{k\to\infty} R_k$$

Furthermore, both $\sum_n x_n^2$ and $\sum_n y_n^2$ converge, so our limit laws tell us that their product must also converge, and R_k is just the square of a convergent sequence, and so R_k must also converge. Additionally, we proved in part (a) that L_k converges. Thus, both of their lim sups and lim infs must equal each other, so we get

$$\lim_{k\to\infty} L_k \le \lim_{k\to\infty} R_k$$

But extracting the definitions of L_k and R_k, this is just

$$\lim_{k\to\infty}\left|\sum_{n=1}^{k} x_n y_n\right| \le \lim_{k\to\infty} \sqrt{\sum_{n=1}^{k} x_n^2}\sqrt{\sum_{n=1}^{k} y_n^2}$$

$$\implies |\langle x, y\rangle| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \le \sqrt{\sum_{n=1}^{\infty} x_n^2}\sqrt{\sum_{n=1}^{\infty} y_n^2} = \|x\|\|y\|$$

(c). *Solution.* ff idk something to do with previous one. Note that if we consider the first k terms of x_n and y_n as entries of a k-tuple, we have shown in class the triangle inequality:

$$L_k = \sqrt{\sum_{n=1}^{k} x_n + y_n} \leq \sqrt{\sum_{n=1}^{k} x_n^2} + \sqrt{\sum_{n=1}^{k} y_n^2} = R_k$$

Note that L_k and R_k are sequences that satisfy $L_k \leq R_k$ for all k. Thus by the lemma from October 11 from the course notes, we also have

$$\liminf_{k\to\infty} L_k \leq \liminf_{k\to\infty} R_k \quad \text{and} \quad \limsup_{k\to\infty} L_n \leq \limsup_{k\to\infty} R_k$$

Furthermore, both $\sum_n x_n^2$ and $\sum_n y_n^2$ converge, so our limit laws tell us that their sum must also converge, and R_k is just the square of a convergent sequence, and so R_k must also converge. Additionally, in order for $x, y \in \ell^2$, we must have that L_k converges. Thus, both of their lim sups and lim infs must equal each other, so we get

$$\lim_{k\to\infty} L_k \leq \lim_{k\to\infty} R_k$$

But extracting the definitions of L_k and R_k, this is just

$$\lim_{k\to\infty} \sqrt{\sum_{n=1}^{k} x_n + y_n} \leq \lim_{k\to\infty} \sqrt{\sum_{n=1}^{k} x_n^2} + \lim_{k\to\infty} \sqrt{\sum_{n=1}^{k} y_n^2}$$

$$\implies \|x+y\| = \sqrt{\sum_{n=1}^{\infty} x_n + y_n} \leq \sqrt{\sum_{n=1}^{\infty} x_n^2} + \sqrt{\sum_{n=1}^{\infty} y_n^2} = \|x\| + \|y\|$$

(d). *Solution.* Consider the sequence of vectors:

$$\begin{aligned}
x^{(1)} &= 1, 0, 0 \dots \\
x^{(2)} &= \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \dots \\
&\vdots \\
x^{(n)} &= \underbrace{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n \text{ times}} 0, 0, \dots
\end{aligned}$$

All of these vectors lie in ℓ^2, since they are just the sum of a finite sequence of terms. Note that $\|x^{(n)}\| = \sum_{k=0}^{n} \frac{1}{n} = 1$.

However, $\langle \mathbf{e}_p, x^{(n)} \rangle = \frac{1}{\sqrt{n}}$ if $p \leq n$ or $= 0$ if $p > n$. Regardless, as $n \to \infty$, $\frac{1}{\sqrt{n}} \to 0$ as well, thus $\langle \mathbf{e}_p, x^{(n)} \rangle \to 0$ as $n \to \infty$.

Problem 4

Given that the sequence $(s_n + 2s_{n+1})$ converges, prove that the sequence (s_n) converges.

Solution. Since $(s_n + 2s_{n+1})$ converges, then for any ε', there exists some $N' \in \mathbb{N}$ such that for for all $n \geq N$ and for all $p \in \mathbb{N}$, we have $|2s_{n+p+1} + s_{n+p} - 2s_{n+1} - s_n| < \varepsilon'$.

We want to show that if $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $p \in \mathbb{N}$, we have $|s_{n+p} - s_n| < \varepsilon$. To do this then, we want some bound on $|2s_{n+p+1} - 2s_{n+1}|$. This kinda makes me feel like induction. If we let $p = 1$, then we have $|2s_{n+2} - 2s_{n+1}|$.

Maybe something about how

Let $\varepsilon > 0$ be arbitrary. Let $N = N'$. We will show that (s_n) is Cauchy. Let $N = N'$ (where N' is so that other series bounded by ε too). Let $n \geq N$. We now induct on p. Base case ($p = 1$): we have $|s_{n+1} - s_n|$. But note that $\varepsilon > |2s_{n+2} + s_{n+1} - 2s_{n+1} - s_n| = |2s_{n+2} - s_{n+1} - s_n|$.

Hmm, what if showing s_n gets arbitrarily close to $s_n + 2s_{n+1}$.

If the above is still gibberish, it's because I had to submit in a rush and didn't end up solving/fixing this question.

Problem 5

Prove that if $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n^q}$ converges for any constant $q > \frac{1}{2}$.

Solution. First, note that $\sum_n |\frac{1}{n^q}|^2 = \sum_n \frac{1}{n^{2q}}$. Since $2q > 1$, this is a p-series that converges. Thus, by problem 3(a), we have that $\sum_{n=1}^{\infty} \frac{a_n}{n^q}$ will converge as well (where we substitute $x_n = a_n$ and $y_n = \frac{1}{n^q}$).

Problem 6

In parts (a)-(c) below, suppose $a_n > 0$ and $b_n > 0$ for all n, and define

$$A = \sum_{n=1}^{\infty} a_n, \quad B = \sum_{n=1}^{\infty} b_n$$

(a). *Prove the Limit Comparison Test: If b_n/a_n converges to a real number $L > 0$, then series A converges if and only if series B converges.*

(b). *Prove the Ratio Comparison Test: If $a_{n+1}/a_n \leq b_{n+1}/b_n$, convergence of series B implies convergence of series A. What if $a_{n+1}/a_n \leq b_n/b_{n-1}$ instead? [Clue: Start by finding upper and lower bounds for the sequence $r_n = a_n/b_n$.]*

(c). *Use (b) with $\zeta(p)$ to prove Raabe's Test: if $p > 1$ and $a_{n+1}/a_n \leq 1 - p/n$ for all n sufficiently large, then series A converges. [Clue: First show that $1 - px < (1-x)^p$ for all $x \leq 1$. Just use calculus.]*

(d). *Test $\sum_n a_n$ for convergence, where $a_n = \frac{1 \cdot 4 \cdots (3n+1)}{n^2 3^n n!}$.*

(a). *Solution.* Let $b_n/a_n \to L > 0$ as $n \to \infty$. Then for any $\varepsilon = L$, there exists some $N \in \mathbb{N}$ such that for $n \geq N$, we have $|b_n/a_n - L| < L$. Furthermore, since $a_n, b_n, L > 0$, we have

$$\frac{b_n}{a_n} - L = \left|\frac{b_n}{a_n}\right| - |L| \leq \left|\left|\frac{b_n}{a_n}\right| - |L|\right| \leq \left|\frac{b_n}{a_n} - L\right| < L$$

Thus $0 < |b_n| < 2La_n$. Assume that series A converges. Then $\lim_{n\to\infty} \sum_{k=N}^{n} (2La_k) = 2L \lim_{n\to\infty} \sum_{k=N}^{n} a_k$ converges as well. Thus by the comparison test, $\sum_{k=N}^{\infty} b_k$ converges, thus $\sum_{k=1}^{N-1} b_k + \sum_{k=N}^{\infty} b_k = B$ converges as well.

Now, note that if b_n/a_n converges to some positive value, a_n/b_n must converge to some positive value as well, specifically, L^{-1} (by Rudin Theorem 3.3 (d)), since $b_n/a_n \neq 0$ for all n and $L \neq 0$. Thus, we can repeat an identical argument to the one above to show that B converging implies A converging (where we just swap the a_n's and b_n's around).

(b). *Solution.* Let $a_{n+1}/a_n \leq b_{n+1}/b_n$. Then since $a_n, b_n > 0$, we have

$$\frac{a_n}{b_n} \geq \frac{a_{n+1}}{b_{n+1}}$$

This clearly has an upper bound, namely a_1/b_1, and since both $a_n, b_n > 0$ for all n, we must have $a_n/b_n > 0$ for all n, hence 0 is a lower bound for a_n/b_n. Thus, by the Monotone Convergence property, we have that a_n/b_n converges, which we'll denote to be a value, $L \geq 0$. If $L > 0$, then convergence of B implies convergence of A by the Limit Comparison Test above.

It remains to show this is still true when $L = 0$. If $a_n/b_n \to 0$ as $n \to \infty$, then if $\varepsilon = 1$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|\frac{a_n}{b_n}\right| < 1$. But $\frac{a_n}{b_n} = \left|\frac{a_n}{b_n}\right|$, thus $0 < a_n = |a_n| < b_n$. Assume that series B converges. Then $\lim_{n\to\infty} \sum_{k=N}^{n} b_n$ converges as well (all subsequences of a convergent sequence converge). Thus by the comparison test, $\sum_{k=N}^{\infty} a_k$ converges, thus $\sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} a_k = A$ converges as well.

Note that $a_{n+1}/a_n \leq b_n/b_{n-1}$ is also a valid statement too, since we can just reindex b_n by bumping everything along by 1, (say set $b'_n = b_{n-1}$ and $b'_1 = 1$). This doesn't alter the convergence of B (ie. $\sum_n b'_n$ converges if and only if $\sum_n b_n = B$ converges), since we are just adding a finite term, thus, the Ratio Comparison test still holds.

(c). *Solution.* We first invoke calculus to show that $1 - px \leq (1-x)^p$ when $0 \leq x \leq 1$ and $p > 1$. Note that

$$\frac{d}{dx}(1-x)^p|_{x=0} = -p(1-x)^{p-1}|_{x=0} = -p$$

Now, at $x_0 = 0$, $(1 - x_0)^p = 1 = y_0$. Thus the tangent line at x_0 is $y_0 - p(x - x_0) = 1 - px$. Realize that $(1-x)^p$ is convex on $x \in [0, 1]$ since $p > 1$ (one can confirm this with the second derivative, $p(p-1)(1-x)^{p-1}$). Thus, our tangent line lies below the curve on the interval, thus $1 - px \leq (1 - x)^p$.

Thus, we have for all n sufficiently large,

$$a_{n+1}/a_n \leq 1 - p/n \leq (1 - \frac{1}{n})^p = \left(\frac{n-1}{n}\right)^p$$

(which is valid, since $\frac{1}{n} \in [0, 1]$ for all $n \in \mathbb{N}$). Thus, if $b_n = \frac{1}{n^p}$, then by what we just proved in part (b) for the ratio comparison test, since $a_{n+1}/a_n \leq b_n/b_{n-1}$, and $B = \sum_n b_n$ converges since b_n is a p-series with $p > 1$, we can conclude that $\sum_n a_n = A$ converges as well.

(d). Solution. Note that for all n.

$$\begin{aligned} a_{n+1}/a_n &= \frac{(3(n+1)+1)n^2}{(n+1)^2 3(n+1)} \\ &= \frac{(3n+4)n^2}{3(n+1)^3} \\ &\leq \frac{(3(n+1)n^2}{3(n+1)^3} \\ &= \left(\frac{n}{n+1}\right)^2 \end{aligned}$$

Then $a_{n+1}/a_n \leq \left(\frac{n}{n+1}\right)^2$. Note that when $n \geq 2$, we have that $1 - \frac{1.1}{n} \geq \frac{2}{n} \geq a_n$. Thus for sufficiently large n, there exists $p > 1$ such that $a_{n+1}/a_n \leq 1 - p/n$, so by Raabe's test, $\sum_n a_n$ converges.

Problem 7

Prove: If each $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges. Does the converse hold?

Solution. We prove the contrapositive. Let each $a_n \geq 0$. Let $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converge. By the Monotone Convergence criterion, we must have that the sequence of partial sums of $\frac{a_n}{1+a_n}$ is bounded, say $0 \leq \frac{a_n}{1+a_n} < M$ and $M > 1$. ff wait this should be a partial sum say $0 \leq \sum_{n=1}^{k} \frac{a_n}{1+a_n} < M$. Then $a_n < M + Ma_n \implies -M < (M-1)a_n \implies -\frac{M}{M-1} < a_n$. The thing is... I doubt that M can turn into a bound for $\sum_n a_n$. ff

The converse of this is if $\sum_n a_n$ converges, then $\sum_n \frac{a_n}{1+a_n}$ converges as well. This is not true: we provide the counter-example $a_n =$

Problem 8

(a). Prove: Given any $D \in \mathbb{R}$ and $\delta > 0$, there is a finite collection of numbers $a_1, a_2, \dots, a_N$ such that $D = a_1 + a_2 + \cdots + a_N$ and

$$\delta > |a_1| > |a_2| > \cdots > |a_N| > 0$$

(b). Let $(\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Explain how to construct a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying, simultaneously

(i) $|x_n| > |x_{n+1}|$ for all n, and $x_n \to 0$ as $n \to \infty$, and

(ii) the sequence $(s_N)_{N\in\mathbb{N}}$ defined by $s_N = \sum_{n=1}^{N} x_n$ has $(\sigma_n)_n$ as a subsequence.

Discussion*: This show badly the converse of the Crude Divergence Test can fail: the series $\sigma_n x_n$ has terms tending to 0, yet its sequence of partial sums can be wild enough to hit all elements of the preassigned $(\sigma_n)_n$.*

(a). *Solution.* By the Archimedean property, there exists an integer q such that $q\delta > |D|$. integer we can take). Then let $l = |D|/q < \delta$. Then assign $|a_1| = l\frac{2q+1-1}{2q+1}$, $|a_2| = l\frac{2q+1-2}{2q+1}$, etc. $|a_j| = l\frac{2q+1-j}{2q+1}$ (we will assign signs later). Note that each of these are less than l, since the fraction is always less than 1, but never 0. Then letting j vary from 1 to $2q$, we have

$$\sum_{j=1}^{2q} |a_j| = \sum_{j=1}^{2q} l\frac{2q+1-j}{2q+1} = l\sum_{j=1}^{q}\left(\frac{2q+1-j}{2q+1} + \frac{2q+1+j}{2q+1}\right) = lq = |D|$$

Now, assign each $|a_n|$ the same sign as D, and one now sees that, $\delta > l > |a_1| > |a_2| > \cdots > |a_{2q}| > 0$; and $D = a_1 + a_2 + \cdots + a_{2q}$. Since q was an integer, this is a finite collection.

(b). *Solution.* We construct our sequence (x_n) inductively such that $(\sigma_n)_n$ is a subsequence of s_N. Let $x_1 = \sigma_1 \implies s_1 = \sigma_1$. Now, for any $k \in \mathbb{N}$, assume that x_j, for all $j \in \{1, \dots, N\}$, has been defined such that $s_N = \sigma_k$ and $|x_n| > |x_{n+1}|$ for all n. By part (a), if $\delta = \min\{|x_N|, \frac{1}{N}\}$ and $D = \sigma_{k+1} - \sigma_k$, there is some finite collection of M terms such that $\min\{|x_N|, \frac{1}{N}\} > |x_{N+1}| > |x_{N+2}| > \cdots > |x_{N+M}| > 0$ and $\sigma_{k+1} - \sigma_k = x_{N+1} + x_{N+2} + \cdots + x_{N+M}$. Thus, $s_{N+M} = s_N + x_{N+1} + x_{N+2} + \cdots + x_{N+M} = \sigma_k + \sigma_{k+1} - \sigma_k = \sigma_{k+1}$. Thus, our constructed sequence satisifes (ii).

To show it satisifies (i), note that we have $|x_n| > |x_{n+1}|$ for all n, by construction. Finally, to show that $x_n \to 0$ as $n \to \infty$, let $\varepsilon > 0$ be arbitrary. Then by Archimedean property, there exists some $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon} > 0 \implies 0 < \frac{1}{k} < \varepsilon$. But then we know that for all $n > k$, we have $|x_n| < \frac{1}{k} < \varepsilon$ by the construction, thus $x_n \to 0$ as $n \to \infty$, giving us (i) as well.