1 September 20

1.1 Sequences and Limits

A sequence in a given set X is simply a function $x\colon \mathbb{N} \to X$. It is often writen x_n instead of $x(n)$, and list just the values

$$x = (x_1, x_2, x_3, \dots)$$

Order matters: the sequence is different from the set $\{x_1, x_2, \dots\}$, e.g. $x_n = (-1)^{n+1}$ for $n \in \mathbb{N}$, but $\{(-1)^{n+1}\colon n \in \mathbb{N}\} = \{-1, 1\}$.

Definition 1 (Converges to $\hat{x}$). Given a sequence $(x_n)_{n \in \mathbb{N}}$ and a point $\hat{x}$, all in $X = \mathbb{R}$, saying the sequence (x_n) converges to $\hat{x}$ means

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}\colon \forall n > N, |x_n - \hat{x}| < \varepsilon$$

> You should very familiar with this definition. "Not familiar like yeah I know them, but familiar like your mom."
> Notation: When this happens, write

$$\hat{x} = \lim_{n \to \infty} x_n$$

or $x_n \to \hat{x}$ as $n \to \infty$.

What is this definition saying? Somebody walks in the door with some epislon, some error tolerance beyond our control, and we need to be ready to give an N for anything, typically depending on ε. N depicts a point on the number line where stuff starts to go right, all the other points are ε close to $\hat{x}$. Close, because absolute value is how we measure distance.

$$|x_n - \hat{x}| < \varepsilon \iff -\varepsilon < x_n - \hat{x} < \varepsilon \iff x_n \in (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$$

"When I go to the tailor, I think of this. For any ribbon width, you have to cover all but finite terms ($N(\varepsilon)$) of x_n with your fabric."

In the world of computer science, you might care about efficiency of picking N, but here, we don't, just need to find an N. So technically, we can always find a prime number where after everything goes right.

Definition 2 (Converges/Diverges). A real-valued sequence $(x_n)_{n \in \mathbb{N}}$ *converges* when

$$\exists \hat{x} \in \mathbb{R}\colon \lim_{n \to \infty} x_n = \hat{x}$$

It *diverges* when it does not converge.

(x_n) diverges $\iff \forall \hat{x} \in \mathbb{R}, \exists \varepsilon > 0, \forall N \in \mathbb{N}\colon |x_n - \hat{x}| \geq \varepsilon$. In other words, every candidate for the limit $\hat{x}$, comes with some tolerance so that no matter how far to the right we start out ("political joke"), a point further right will escape tolerance band.

If we want to reimagine this as story of closeness for vectors, complex numbers, or something else with a distance, the idea of a limit is the same (just have to reinterpret distance). If you were wondering why we jumped over metric spaces in chapter 2, this gives us some examples to motivate the metrics.

Some simple examples

(a). $x_n = \frac{1}{n}$ conberges to $\hat{x} = 0$.

> *Proof.* Given $\varepsilon > 0$, note $\frac{1}{\varepsilon} \in \mathbb{R}$ and Archimedes says $\exists N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Use that one: any $n > N$ will have $|x_n - \hat{x}| = |\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon$. $\square$

(b). $x_n = 1$ converges to $\hat{x} = 1$.

> *Proof.* Given $\varepsilon > 0$, pick $N = 320$. Now every $n > N$ makes $|x_n - \hat{x}| = 0 < \varepsilon$. This works... but many smaller choices for N would be okay too. $\square$

Less simple examples:

(a). $x_n = \frac{\sin n}{1+n+n^2+n^3+n^4+n^5}$ converges to $\hat{x} = 0$. Preview: for every $n \in \mathbb{N}$,

$$|x_n - \hat{x}| = \frac{|\sin n|}{1+n+n^2+n^3+n^4+n^5} < \frac{1}{0+n+0+0+0+0}$$

and $\frac{1}{n} < \varepsilon$ whenever $n > \frac{1}{\varepsilon}$. So pick any integer $N \geq \frac{1}{\varepsilon}$ and every $n > N$ will have $\frac{1}{n} < \varepsilon$ and make $|x_n - \hat{x}| < \varepsilon$.

Efficiency note: I chose the worst possible thing to outgrow $\sin n$ fast, but it does not matter.

2   September 22

Here is an example of a convergence proof, as it would be written in homework / from the book: the sequence $x = \frac{n^2 - 320n^{3/2}}{2n^2 - 80}$ converges to $\hat{x} = \frac{1}{2}$. (guessing the value for rationals, just look for dominating power).

Proof. Given $\varepsilon > 0$, choose integer $N \geq \max\{30, \left(\frac{750}{\varepsilon}\right)^2\}$. This works. Indeed, every $n > N$ obeys $n > 30$ so $n^2 > 900$, giving

$$2n^2 - 801 = n^2 + (n^2 - 801) > n^2 \tag{1}$$

$$\sqrt{n} > \frac{750}{\varepsilon} \tag{2}$$

Thus, if $n > N$,

$$\begin{aligned}
|x_n - \hat{x}| &= \left|\frac{n^2 - 320n^{3/2}}{2n^2 - 801} - \frac{1}{2}\right| \\
&= \left|\frac{2(n^2 - 320n^{3/2}) - (2n^2 - 801)}{2(2n^2 - 801)}\right| \\
&\leq \frac{640n^{3/2} + 801}{2(2n^2 - 801)} \\
&\leq \frac{640n^{3/2} + 801n^{3/2}}{2n^2} \\
&< \frac{1500n^{3/2}}{2n^2} \\
&= \frac{750}{\sqrt{n}} < \frac{750}{750/\varepsilon} = \varepsilon
\end{aligned}$$

$\square$

How do we come up with this? We start with coming up with the limit. Then, we try to make $|x_n - \hat{x}|$ small, or a simpler version small (and then hopefully simpler thing is slightly larger).

Another example: $\lim_{n \to \infty} n^{1/n} = 1$ (from the book itself).

Proof. Define $x_n = n^{1/n} - 1$; each $x_n > 0$. Recall $(1 + a) = 1 + na + \frac{n(n-1)}{2}a^2 + \cdots + na^{n-1} + a^n$ (fact of algebra). So $(1+a)^n \geq \frac{n(n-1)}{2}a^2$ for all $a > 0$. Thus $n = (1 + x_n)^2 \geq \frac{n(n-1)}{2}x_n^2$. Insist $n \geq 2$ to get $x_n > 0$. Reorganize $0 < x_n^2 \leq \frac{2}{n(n-1)}n \iff x_n \leq \sqrt{\frac{2}{n-1}}$. Solve $\sqrt{2}n - 1 < \varepsilon$ for $\frac{2}{n-1} < \varepsilon^2 \iff \frac{2}{\varepsilon^2} < n - 1$. So choosing $N \geq \max\{2, 1 + \frac{2}{\varepsilon^2}\}$ does the job. $\square$

This kind of is more accurate what he might expect on homework, folding the explanation job and the proof.

Let's look at a divergent example now: $x_n = (-1)^n$.

Proof. Fix any $\hat{x} \in \mathbb{R}$. Pick $\varepsilon = 1$. Fix $N \in \mathbb{N}$. Now consider n_e even, n_o odd, with $n_e > N, n_o > N$. Then $x_{n_e} = (-1)^{n_e} = 1$ and $x_{n_e} = (-1)^{n_e}$. So $2 = |x_{n_2} - x_{n_o}| = |(x_{n_e} - \hat{x}) + (\hat{x} - x_{n_o})| \leq |x_{n_e} - \hat{x}| + |x_{n_o} - \hat{x}|$ (Typical $\pm$ trick to "uncancel"). One of the terms on the RHS is ≥ 1. One of $n = n_e$ or $n = n_o$ completes the logical statement above. $\square$

2.1   Theory / Properties

(More in the notes)

Theorem 1 (Squeeze Theorem)**.** *Let* $(a_n), (x_n), (b_n$ *be real-valued sequences, and* $L \in \mathbb{R}$*. Assume*

(a). $a_n \to L$

(b). $b_n \to L$

(c). $a_n \leq x_n \leq b_n, \exists N \in \mathbb{N}, \forall n > N$

Then $x_n \to L$ *as* $n \to \infty$*.*

Proof. Given $\varepsilon > 0$, use (a) to get $N_a \in N$ such that

$$|a_n - L| < \varepsilon, \forall n > N_a$$

So $L - \varepsilon < a_n$. Use (b) to get $N_b \in \mathbb{N}$ such that

$$|b_n - L| < \varepsilon, \forall n > N_b$$

So $b_n < L + \varepsilon$. Use (c) to get $N_c \in \mathbb{N}$ such that

$$a_n \leq x_n \leq b_n, \forall n > N_c$$

Now if $N = \max\{N_a, N_b, N_c\}$, every $n > N$ does all three jobs:

$$L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon$$

Thus $|x_n - L| < \varepsilon$. $\square$

3   September 27

3.1   Completeness

Completeness is the property that makes $\mathbb{R}$ better than $\mathbb{Q}$. There are three ways to look at it (three properties $\mathbb{R}$ has $\mathbb{Q}$ doesn't, that kind of capture the same thing):

3.1.1   Cauchy Sequences

Definition 3. A sequence (x_n) is call *Cauchy* when

(a). $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $|x_m - x_n| \leq \varepsilon$ (last bit is what makes different than limit), OR EQUIVALENT DEFINITION:

(b). $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, \forall p \in \mathbb{N}, |x_{n+p} - x_n| < \varepsilon$

Proposition 1. *Every convergent sequence is Cauchy.*

Proof. Pick one: let (x_n) converge to $\hat{x}$. Estimate:

$$|x_n - x_m| = |(x_n - \hat{x}) + (\hat{x} - x_m)| \leq |x_n - \hat{x}| + |x_m - \hat{x}|$$

To get definition (a), let $\varepsilon > 0$ be given and use definition of $x_n \to \hat{x}$ with $\varepsilon' = \frac{\varepsilon}{2}$ to get $N \in \mathbb{N}$ such that $|x_k - \hat{x}| < \varepsilon'$ whenever $k > N$. This N works in (a), since $m, n \geq \mathbb{N} \implies |x_m - x_n| < \varepsilon' + \varepsilon' = \varepsilon$. $\square$

Corollary 1. *Any sequence that is not Cauchy must diverge.*

This is the contrapositive of the statement above. This corollary is useful for proving a sequence diverges.

Theorem 2 (Metric Completeness)**.** *Every Cauchy sequence converges (to a real limit) in* $\mathbb{R}$*.*

3.1.2   Bounded Sets

Theorem 3 (Order Completeness). *Given any nonempty $S \subseteq \mathbb{R}$, let*

$$A = \{a \in \mathbb{R} \colon \forall x \in S, a \leq x\}$$
$$B = \{b \in \mathbb{R} \colon \forall x \in S, b \geq x\}$$

Then

(a). Either $A = \emptyset$ or $A = (-\infty, \alpha]$ for some $\alpha \in \mathbb{R}$

(b). Either $B = \emptyset$ or $B = [\beta, +\infty)$ for some $\beta \in \mathbb{R}$.

Terminology: Say S is *bounded above* when $B \neq \emptyset$ and call each $b \in B$ an *upper bound for S*. S is *bounded below* when $A \neq \emptyset$; each $a \in A$ is a *lower bound for S*. *Bounded* means bounded above and bounded below.
You can see this property breaking with the rationals with $S = \{p \in \mathbb{Q} \colon p^2 < 2\}$.
When $B \neq \emptyset$, call β the *supremum* of S: $\beta = \sup(S)$. Useful characterization:

(a). $\forall x \in S$, $x \leq \beta$ (β is an upper bound for S)

(b). $\forall \gamma < \beta$, $\exists x \in S \colon \gamma < x$ (nothing less than β is an upper bound).

(c). Some would say β is the Least Upper Bound for S.

When $A \neq \emptyset$, $\alpha = \inf(S)$ is the *infininum* or the *greatest lower bound* for S.

3.1.3   Monotonic Sequences

Theorem 4. *Given any sequence (x_n) with*

$$x_1 \leq x_2 \leq x_3 \leq \cdots$$

either $x_n \to \infty$ or x_n converges to a real limit.

When we are saying goes to ∞, similar to $\varepsilon - N$ (definition in Rudin).
We will see all of these theorem's are logically equivalent, and then will prove one.

3.1.4   Linkages

(He's going fast to prove one of these theorems; wants to give Zahl informed students in 321, but also Zahl is teaching next class because Louwen is out of town)
These 3 viewpoints on completeness contain equivalent information. Each one implies the others. We will show Metric Completeness implies Order Completeness. Zahl can do what he wants, but Louwen wants him to show the other two linkages.

Theorem 5. *Metric completeness (Cauchy sequences must converge) implies order completeness (if $S \neq \emptyset$ bounded above, $\sup(S)$ exists).*

Proof. Let $S \subseteq \mathbb{R}$ be nonempty; define $B = \{b \in \mathbb{R} \colon \forall s \in S, s \leq b\}$. Assume $B \neq \emptyset$ and define sequence

$$b_n = \min\left(B \cap \left\{\frac{k}{2^n} \colon k \in \mathbb{Z}\right\}\right)$$

We will see this will be a Cauchy sequence; $\beta = \lim_{n \to \infty} b_n$ will have the properties defining $\sup(S)$. To show this, for each fixed n,

(a). $b_n - \frac{1}{2^n} \notin B \implies \exists s_n \in S \colon s_n > b_n - \frac{1}{2^n}$

(b). $b_{n+1} \leq b_n$ (min over a larger set of points)

(c). Using $b_{n+1} \in B$, $b_{n+1} \geq s_n$ for s_n above. Using (b), $b_{n+1} \geq s_n > b_n - \frac{1}{2^n} \iff 0 \leq b_n - b_{n+1} < \frac{1}{2^n}$

Now estimate

$$
\begin{aligned}
|b_{n+p} - b_n| &= b_n - b_{n+p} \\
&= (b_n - b_{n+1} + (b_{n+1} - b_{n+2}) + \cdots + (b_{n+p-1} - b_{n+p}) \\
&< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}} < \frac{2}{2^n}
\end{aligned}
$$

This is the key to showing (b_n) is Cauchy, and then we use previous stuff to imply order. Can't finish within class, so check lecture notes. □

4 September 29 (Zahl)

4.1 Subsequences

We will look at subsequences today, since will use it in the heart of a proof today.

Recall the definition of a sequence of real numbers (but also works for general sequences). A seqeunce of real numbers is a function $f\colon \mathbb{N} \to \mathbb{R}$ (or also notation $x\colon \mathbb{N} \to \mathbb{R}$), so the sequence is $f(1), f(2), \ldots$ (or $x(1), x(2), \ldots$). But we more often write this as $x_1, x_2, x_3, \ldots, (x_n)$.

A subsequence will be created by jumping over elements in a sequence. The important thing is that the indices do need to strictly increase.

Definition 4 (Subsequence). If $x\colon \mathbb{N} \to \mathbb{R}$ is a sequence, then a *subsequence* of x is a sequence of the form

$$
x \circ g \colon \mathbb{N} \to \mathbb{R}
$$

where $g\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing.

But no one uses this notation in practice. We more often write $x_{n_1}, x_{n_2}, \ldots, (x_{n_k})$, where $n_1 < n_2 < n_3 < \cdots$. We can continually take subsequences, and since naturals are infinte, our new subsequences can be infinite in size.

4.2 Completeness

Recall the three slightly related notions of completeness. These properties help distinguish the reals from the rationals, or higher dimensional Euclidean space, where some of these properties do not hold.

(a). Metric completeness property: every Cauchy sequence of real numbers converges.

- Recall that we showed that every convergent is Cauchy, and we are showing that the converse is true, but there exists other settings where this does not hold

(b). Order completeness (Least Upper Bound property): If $S \subset \mathbb{R}$ is non-empty, we define the set B to be the upper bounds of S: $B = \{b \in \mathbb{R} : \forall s \in S, s \leq b\}$. Then either $B = \emptyset$ or $B = [\beta, \infty)$ for some $\beta \in \mathbb{R}$.

- We making use of the canonical ordering of $\mathbb{R}$, we don't have such an obvious ordering for other spaces
- If S is any set that is unbounded above, then B is empty.
- Switch to Greek letters (β) for historic reasons (same with Dedekind cuts)
- We know order is transitive, but this is entirely consistent with $B = (\beta, \infty)$. It is not hard to imagine a world where we don't get β: the set of positive numbers whose square is less than 2. Can describe this entirely with the rationals, but there is no rational β for this set, will always be able to find a smaller rational.

(c). Monotone converge property: If (x_n) is monotone increasing, either $x_n \to \infty$, or (x_n) converges. (Could also have monotone decreasing and $-\infty$).

Theorem 6. *Properties (a),(b),(c) are equivalent.*

(Whenever setting up a result of this type, make it $(a) \implies (b)$, $(b) \implies (c)$, $(c) \implies (a)$ (reorder) otherwise people will be confused). We will eventually see that this holds for the reals, but not today.

We will prove that $(b) \implies (c)$ today.

Proof. Let (x_n) be a monotone increasing sequence. (Note on notation, we mean $x_1 \leq x_2 \leq \cdots$; some people will call this a weakly monotone increasing sequence, and with strict inequalities is strictly increasing). Let $S = \{x_n : n \in \mathbb{N}\}$ (the range of values taken by x). If S is not bounded above, then we claim that $x_n \to \infty$. (At the level of this course, expect us to fill in the details; a little fill-in: we know monotone increasing and attains the full range above, $B = \emptyset$) Otherwise, the set of upper bounds B is of the form $B = [\beta, \infty)$ (B is not empty because S is bounded above, the set of upper bounds).

Want to show x_n converges (the way to show convergence is just to come up with clever guess, and then show it converges to it). Lets show $x_n \to \beta$. We know $x_n \leq \beta$ for every n. Let $\varepsilon > 0$. Then $\exists x_n \in S$ such that $x_n > \beta - \varepsilon$. (If this were not true, $\beta - \varepsilon$ is an upper bound less than β.) So for all $n \geq N$, $x_n \geq x_N > \beta - \varepsilon$. i.e. $|x_n - \beta| < \varepsilon$. But then x_n converges to β. $\qquad\square$

We will now show that (c) $\implies$ (a). This is complicated enough to break into three pieces; each of these three pieces are useful for other places in the future (so will call a Lemma).

Lemma 1. *Every sequence of real numbers (doesn't have to be real?) has a (weakly) monotone subsequence.*

Won't try to prove, but let's understand. Note that monotone here can both be for increasing or decreasing. See that for $x_n = (-1)^n n$, we can find uncountably many monotone subsequences; think the power set of the natural numbers. But we just need the existence of a single monotone subsequence.

Lemma 2. *Every Cauchy sequence is bounded*

Complete confidence can prove in a line or two. Lemma 1 required some cleverness to prove, this is just definitions.

Lemma 3. *If (x_n) is Cauchy, (x_{n_k}) is a subsequence, and $x_{n_k} \to \hat{x}$ ($\forall \varepsilon > 0, \exists K$ such that $\forall k \geq K$, $|x_n - \hat{x} < \varepsilon$). Then $x_n \to \hat{x}$.*

We do need Cauchy: take the sequence $x_n = (-1)^n$.

Because every Cauchy sequence is bounded, and we have a monotone increasing subsequence, and we know it converges by monotone convergence property, we have that the Cauchy sequence converges.

5   October 4

5.1   Wrapping up Completeness Structure

We will prove one of the theorem's later, and will use the properties freely for now.

We will prove the third lemma from last class: If sequence (x_n) is Cauchy, and some subsequence (x_{n_k}) converges to $L \in \mathbb{R}$, then the full sequence obeys $\lim_{n \to \infty} x_n = L$.

Proof. Key:
$$|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L|$$
Let $\varepsilon > 0$ be given. Invent $\varepsilon' = \frac{\varepsilon}{2}$ and use Cauchy:
$$\exists N : \forall m, n \geq N, |x_m - x_n| < \varepsilon'$$
Because the subsequence convergence gives $K \in \mathbb{N}$ so big that $\forall k > K$ makes
$$|x_{n_k} - L| < \varepsilon'$$
Pick any $k^* > K$ (with n_{k^*}) and $n > N$ and use the first inequality from above:
$$|x_n - L| < |x_n - x_{n_{k^*}}| + |x_{n_{k^*}} - L| < \varepsilon' + \varepsilon' = \varepsilon$$

$\qquad\square$

The key thing is that we are going far enough down where our second term is under control, and our first is under control by Cauchy.

Hunting License (called because each property says there exists a real number that does stuff...): it is ok to use completeness in any of these three forms to solve HW and exams.

Corollary 2 (Bolzano-Weierstrass Theorem). *Every bounded sequence of real numbers is gauranteed to have a convergent subsequence.*

Some students showed him Tao, and Tao uses the word ε-steady to replace a sequence being Cauchy... just a name that is useful to remember though, like Bolzano-Weierstrass.

"Completeness will let you down if you're a rational person, but will ff if you're a real person."

5.2   Supremum and Infinum

Recall that the supremum is the least upper bound, and the infinum is the greatest lower bound. Giiven any set $S \subseteq \mathbb{R}$, we let
$$B = \{b \in \mathbb{R} \colon \forall x \in S, x \leq b\}$$
What can happen?

- $B = \emptyset$, if S has no upper bound (e.g. $S = \mathbb{Z}$)

- $B = [\beta, +\infty)$, if S has an upper bound (hence $\beta = \sup(S)$)

- $B = (-\infty, \infty)$, if $S = \emptyset$

We can extend the definition of "sup" to cover all 3 cases: $\sup(S) = +\infty$ if S has no upper bound and $\sup(S) = -\infty$ if $S = \emptyset$.

There is also a symmetric extension for inf: $\inf(S) = -\infty$ if S has no lower bound and $\inf(S) = +\infty$ if $S = \emptyset$.

This also allows sup and inf to operate on sets that includes $+\infty, -\infty$.

5.2.1   Upper and Lower Limits

Definition 5 (Limit Superior). Given a real sequence (x_n), define

$$\limsup_{n \to \infty} x_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} x_k \right)$$

Definition 6 (Limit Inferior). Given a real sequence (x_n), define

$$\liminf_{n \to \infty} x_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} x_k \right)$$

Halloween joke: "limb soup" is pronounciation of the first. "I'm not playing the age card, but I got dad jokes plus!"

- $x_n = \frac{1}{n}$, then
$$\limsup_{n \to \infty} \frac{1}{n} = \inf_{n \in \mathbb{N}} \left(\sup\{\frac{1}{k} \colon k \geq n\} \right) = \inf_{n \in \mathbb{N}} \{\frac{1}{n}\} = 0$$

If you have a convergent sequence to start with, limsup is going to match your idea what the limit is.

- $x_n = (-1)^n + \frac{1}{n}$, then

$$\limsup_{k \to \infty}(x_n) = \inf_n \sup\{(-1)^k + \frac{1}{k} \colon k \geq n\}$$

$$= \inf_n \{1 + 2j \text{ if } k = 2j \text{ even, OR} - 1 + \frac{1}{2j} \text{ if } k = 2j - 1 \text{ odd}\}$$

$$= \inf_n \{1 + \frac{1}{\lceil n/2 \rceil}\}$$

$$= 1$$

or something like that... fix at home.

$$\liminf_{n\to\infty}(x_n) = \sup_n[\inf_{k\geq n}(-1)^k + \frac{1}{k}]$$
$$= \sup_n\{-1\}$$
$$= -1$$

This thing is bouncing around 1 and -1 with a negligiblle pertrubation. We can also tell that it has two subsequences that do converge (n even will converge to 1 and n odd will converge to -1). This is what is being captured with $\limsup$ and $\liminf$.

6 October 6

6.1 Upper and Lower Limits (continued)

Recall, for any real sequeunce, $x = (x_n)$,

$$\overline{\lim_{n\to\infty}} = \limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n}\right)$$

$$\underline{\lim_{n\to\infty}} = \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}}\left(\inf_{k\geq n}\right)$$

gives values in $\mathbb{R}\cup\{\pm\infty\}$ ff missing proposition.

Proof. For each $n\in\mathbb{N}$, consider "tail",

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$$

and let $i_n = \inf(T_n), s_n = \sup(T_n)$. Clearly $i_n \leq s_n$, and $i_n \leq i_{n+1}$, $s_n \geq s_{n+1}$. For any $m, n \in \mathbb{N}$, pick any $N > \max\{\}$. ehh... I showed up late, now I'm lost　　　□

"Don't look at a Cauchy sequence after 9pm... I lost sleep."

Proposition 2. *Let $x = (x_n)$ be a real sequence. Define $\mu = \liminf_{n\to\infty} x_n$, $M = \limsup_{n\to\infty} x_n$.*

(a). If $l = \lim_{k\to\infty} x_{n_k}$ for some subsequence (x_{n_k}) of x, then $\mu \leq l \leq M$.

(b). There exist subsequences (x_{n_j}) and (x_{n_k}) of x along which $\mu = \lim_{j\to\infty} x_{n_j}$ and $M = \lim_{k\to\infty} x_{n_k}$.

This is actually how Rudin defines the $\limsup$ and $\liminf$. But we will show the two defintions are equivalent. Read notes for proof.
Next time: we are doing half the job of constructing the reals. The other half is on HW5.