Problem 1

Prove or disprove, giving plenty of detail:

(a). *If (x_n) is a real sequence obeying $x_n \to +\infty$, then $x_n \leq x_{n+1}$ for all n sufficiently large.*

(b). *If $(x_n)_{n=1}^{\infty}$ is a real sequence obeying $x_n \to +\infty$, then (x_n) has a subsequence $(x_{n_k})_{k=1}^{\infty}$ satisfying $x_{n_k} \leq x_{n_{k+1}}$ for all k.*

(a). *Solution.* This is not true, we provide the counterexample

$$(x_n) = \begin{cases} 2n, & \text{if } n \text{ even} \\ n, & \text{if } n \text{ odd} \end{cases}$$

Note that $x_n \to +\infty$ still, since if $M \in \mathbb{R}$ is arbitrary, then let $N = \max\{M+1, 1\}$, and then for any $n \geq N$, $(x_n) > M$. But note that if n is even, then $x_n = 2n > n + 1 = x_{n+1}$. Thus the hypothesis is true, but the conclusion is false, disproving the claim.

(b). *Solution.* This is true. We will prove this through construction of the subsequence. We do so with induction on k. Let let $n_1 = 1$, so x_1 is the first element in the subsequence. Now let an arbitrary term in the subsequence $x_{n_{k'}}$ be given, where $k' \geq 1$. Since $x_n \to +\infty$, if $M = x_{n_{k'}}$, we know that there exists an N such that for all $n > N$, $x_n \geq x_{n_{k'}}$. Certainly $x_{N+1} \geq x_{n_{k'}}$, thus let $n_{k'+1} = N + 1$. Thus $x_{n_{k'+1}} \geq x_{n_{k'}}$ Since this is true for any $k' \geq 1$, by induction, we have shown that there exists a subsequence such that $x_{n_k} \leq x_{n_{k+1}}$ for all k.

Problem 2

Decide whether these sequences converge or diverge. Then present detailed ε, N proofs confirming your decisions.

$$\text{(a)} a_n = n\left(\sqrt{1+\frac{1}{n}} - 1\right) \qquad \text{(b)} b_n = \frac{(-1)^n n}{n+1}$$

(a). *Solution.* This sequence will converge, specifically, to $\frac{1}{2}$. Let $\varepsilon > 0$. Then let $N =??$.

$$\begin{aligned}
\left|n\left(\sqrt{1+\frac{1}{n}}-1\right)-\frac{1}{2}\right|^2 &= n^2\left(1+\frac{1}{n}-2\sqrt{1+\frac{1}{n}}+1\right) - n\left(\sqrt{1+\frac{1}{n}}-1\right)+\frac{1}{4} \\
&= 2n^2+n-2n^2\sqrt{1+\frac{1}{n}} - n\sqrt{1+\frac{1}{n}}+n+\frac{1}{4} \\
&= 2n^2\left(1-\sqrt{1+\frac{1}{n}}\right)+n\left(2-\sqrt{1+\frac{1}{n}}\right)+\frac{1}{4} \\
&< 2n^2\left(1-\sqrt{1+\frac{1}{n}}\right)+2n+\frac{1}{4} \\
&< 2n^2+2n+\frac{1}{4} = \varepsilon^2 \\
&\implies (2n+\frac{1}{2})^2 = \varepsilon^2
\end{aligned}$$

(b). *Solution.* This sequence will diverge. (Basically equal to 1 in the limit, and then jumping back and forth... set $\varepsilon = 1$.)

Problem 3

Let us extend our familiar idea of addition by defining a generalized sum, $\sum$, that assigns a value in $\mathbb{R} \cup \{+\infty\}$ to every subset of the real interval $[0, +\infty)$. the first step is easy: let $\sum(\emptyset) = 0$, and for any nonempty set $F = \{a_1, a_2, \ldots, a_n\}$ in $[0, +\infty)$, let

$$\sum(F) = a_1 + a_2 + \cdots + a_n$$

Now suppose A is any nonempty subset of $[0, +\infty)$: define

$$\sum(A) = \sup\{\sum(F) \colon F \text{ is a finite subset of } A\}$$

(This is clearly consistent with the previous setup when A is finite.) Prove:

(a). If $\sum(A)$ is defined and finite, then A is finite or countable.

(b). If $A = \{a_1, a_2, \ldots\}$ with all $a_n > 0$, then $\sum(A) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$. (Work in $\mathbb{R} \cup \{+\infty\}$.)

(a). *Solution.* Let $\sum(A)$ be defined and finite. Then there exists some finite least upper bound on

$$\{\sum(F) \colon F \text{ is a finite subset of } A\}$$

Maybe that if $A = \mathbb{N}$, $\sum(A)$ is not finite (we just keep taking larger subsets; or see part (b)), and so then if A is uncountable, this sum is even larger? And then contrapositive. Might be hard to work with it when A is a finite interval.

ff

(b). *Solution.* ff

Problem 4

Nonempty sets X and Y and a function $f\colon X \times Y \to \mathbb{R}$ are given. Assume $f(X \times Y)$ is bounded. Define $M_1\colon X \to \mathbb{R}$ and $W_2\colon Y \to \mathbb{R}$ as follows:

$$M_1(x) = \sup\{f(x,y)\colon y \in Y\}, \quad W_2(y) = \inf\{f(x,y)\colon x \in X\}$$

(a). Prove that $\sup_Y W_2 \le \inf_X M_1$. Note: This is shorthand for

$$\sup\{W_2(y)\colon y \in Y\} \le \inf\{M_1(x)\colon x \in X\}$$

A nice restatement of the result in original notation is worth rembering:

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) \le \inf_{x \in X} \sup_{y \in Y} f(x,y)$$

(b). Show by example that strict inequality is posssible in (a).

(a). *Solution.* ff

(b). *Solution.* Let $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows

$$f(x,y) = \begin{cases} 0, & \text{if } |x| \le 1 \\ 1, & \text{otherwise} \end{cases}$$

Then for all y, $\inf_X f(x,y) = 0$ (we just choose $x = 0$), and so $\sup_Y \inf_X f(x,y) = 0$. Now, for all x, $\sup_Y$ shoot I messed up, can be 0 sometimes.

Problem 5

Prove: For every nonempty set S of positive real numbers,

$$\textit{either}\quad \bigcap_{s\in S}[0,s)=[0\inf(S)) \quad \textit{or} \quad \bigcap_{s\in S}[0,s)=[0\inf(S)]$$

Include, with proof, a simple test involving the number $\inf(S)$ *and the set S that predicts exactly which outcome will occur.*

Solution. ff

Problem 6

All of the sequences in this problem have rational elements. Give direct proofs of the following:

(a). If (x_n) and (y_n) are Cauchy sequences, then $s_n = x_n + y_n$ defines a Cauchy sequence.

(b). If (x_n) and (y_n) are Cauchy sequences, then $p_n = x_n y_n$ defines a Cauchy sequence.

(c). If (x_n) is a Cauchy sequence and (y_n) is a sequence satisfyin $(y_n - x_n) \to 0$ as $n \to \infty$, then (y_n) is a Cauchy sequence.

(Work entirely in $\mathbb{Q}$: do not mention $\mathbb{R}$ or use any of its distinctive properties.)

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 7

Given some $\lambda \in (0,1)$ and $a_0, a_1 \in \mathbb{R}$, define a sequence $(a_n)_{n\geq 0}$ recursively as follows:

$$a_n = (1-\lambda)a_{n-1} + \lambda a_{n-2}, \quad n = 2, 3, 4, \ldots$$

(a). Prove that the sequence (a_n) must converge. (Try for a method that does not rely on part (b).)

(b). Express $\alpha = \lim_{n\to\infty} a_n$ in terms of λ, a_0, a_1.

Note: This question is inspired by its special case $\lambda = \frac{1}{2}$, which appeared (with no hint) on the final exam for MATH 120 in December 2009.

(a). *Solution.* Since this sequence is in $\mathbb{R}$, we are sufficient to show that (a_n) is Cauchy, by the metric completeness of $\mathbb{R}$.

Hmm... for any a_n, furthest point away is a_{n+1} I think. But need to prove that.

We show that for all n, $a_n \in (a_{n-1}, a_{n-2})$ when $a_{n-1} < a_{n_2}$, and $a_n = (a_{n-2}, a_{n-1})$ when $a_{n-1} > a_{n_2}$ (need to mention when $a_{n-1} = a_{n-2}$ only occurs when $a_0 = a_1$, which is trivial). Note $a_n = (1-\lambda)a_{n-1} + \lambda a_{n-2} = a_{n-1} + \lambda(a_{n-2} - a_{n-1})$. We first prove the case when $a_{n-1} < a_{n-2}$ We have then

$$\begin{aligned} a_n &= a_{n-1} + \lambda(a_{n-2} - a_{n-1}) \\ &> a_{n-1} \end{aligned}$$

where we get the last line because $\lambda(a_{n-2} - a_{n-1}) > 0$ since $a_{n-2} > a_{n-1}$. Furthermore,

$$\begin{aligned} a_n &= (1-\lambda)a_{n-1} + \lambda a_{n-2} \\ &< (1-\lambda)a_{n-2} + \lambda a_{n-2} \\ &= a_{n_2} \end{aligned}$$

(where the second line is because $\lambda < 1$ and $a_{n-2} > a_{n-1}$). Thus, $a_n \in (a_{n-1}, a_{n-2})$. Likewise, when $a_{n-1} > a_{n-2}$, we have

$$\begin{aligned} a_n &= (1-\lambda)a_{n-1} + \lambda a_{n-2} \\ &< (1-\lambda)a_{n-1} + \lambda a_{n-2} \\ &= a_{n-1} \end{aligned}$$

(where the second line is because $\lambda > 0$ and $a_{n-2} < a_{n-1}$). Furthermore,

$$\begin{aligned} a_n &= a_{n-1} + \lambda(a_{n-2} - a_{n-1}) \\ &> a_{n-1} \end{aligned}$$

(where the last line is because $\lambda > 0$ and $a_{n-2} < a_{n-1}$, so $\lambda(a_{n-2} - a_{n-1} < 0)$. Thus, $a_n \in (a_{n-2}, a_{n-1})$. For the remainder of this discussion, we assume without loss of generality that $a_{n-1} < a_{n-2}$. Since n was arbitrary, this holds for any $n \in \{2, 3, \ldots\}$.

Note then that for all $p \in \mathbb{N}$, $|a_{n+p} - a_n| < |a_{n-1} - a_n|$. But

$$|a_{n-1} - a_n| = |a_{n-1} - (1-\lambda)a_{n-1} - \lambda a_{n-2}| = |\lambda a_{n-1} - \lambda a_{n-2}| = \lambda |a_{n-1} - a_{n-2}|$$

So our problem now is about finding the bound between subsequent terms instead of the difference between arbitrary terms. Hmm... this gives a nice recursive formula, where

$$|a_n - a_{n-1}| = \lambda^{n-1}|a_1 - a_0|$$

Thus let $\varepsilon > 0$. We let $N = \left\lceil \log_\lambda \left(\frac{\varepsilon}{|a_1 - a_0|} \right) \right\rceil + 1$. Then for any $n \geq N$ and $p \in \mathbb{N}$, we have

$$\begin{aligned}
|x_n - x_{n+p}| &< \lambda^{n-1}|a_1 - a_0| \\
&< \lambda^{N-1}|a_1 - a_0| && \text{since } 0 < \lambda < 1 \\
&= \lambda^{\left\lceil \log_\lambda \left(\frac{\varepsilon}{|a_1 - a_0|} \right) \right\rceil}|a_1 - a_0| \\
&< \lambda^{\log_y \left(\frac{\varepsilon}{|a_1 - a_0|} \right)}|a_1 - a_0| && \text{whatever real exponentiation means} \\
&= \varepsilon
\end{aligned}$$

Thus our sequence is Cauchy, thus it converges.

$a_n - a_{n-1} = -\lambda(a_{n-1} - a_{n-2})$.

(b). *Solution.* I think I can take the recursive formula thing to show that

$$\alpha = \lim_{n\to\infty} a_1 + |a_1 - a_0| \sum_{n=1}^{\infty} (-1)^{n+1} \lambda^n$$

(I think exact sign of the term depends on whether $a_1 > a_0$ or not. This is just a geometric series $\frac{\lambda}{1+\lambda}$. But then

$$\alpha = a_1 + |a_1 - a_0| \frac{\lambda}{1 + \lambda}$$

Problem 8

For any nonempty set S of positive *real numbers, define $S^{-1} = \{x^{-1} : x \in S\}$. Prove:*

(a). $\inf(S) = 0 \iff \sup(S^{-1}) = +\infty$

(b). $0 < \inf(S) < +\infty \iff \sup(S^{-1}) < +\infty$, *and when these are true one has* $\sup(S^{-1}) = [\inf(S)]^{-1}$.

Taken together, items (a) and (b) provide some rationale for the symbolic equations "$1/0^+ = +\infty$" and "$1/(+\infty) = 0^+$". (These are "symbolic" because the usual rules of algebra are not available: we cannot infer a value for $(0^+)(+\infty)$). Using these symbolic equations, prove

(c). If $x_n > 0$ for each n, then $\limsup_{n\to\infty}(x_n^{-1}) = (\liminf_{n\to\infty} x_n)^{-1}$

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff