Problem 2 (Ch. 1.12)

Determine representatives of the conjugancy classes in S_5 and the number of elements in each class. Use this information to prove that the only normal subgroups of S_5 are 1, A_5, S_5.

Solution. Recall that the conjugancy classes of S_n are just the possible partitions of n elements. ff just silly calculations

Problem 4 (Ch. 1.12)

*Show that if a finite group G has a subgroup H of index n then H contains a normal subgroup of G of index a divisor of $n!$. (*Hint*: Consider the action of G on G/H by left translations.*

Solution. Now assume $n > 1$. So $G = x_1H \sqcup x_2H \sqcup \cdots \sqcup x_nH$ where $x_1 = 1_G$. Consider the action of G on G/H. g acts on the coset xH by gxH. The kernel of this map is the set of all G such that $gxH = xH$ for all $x \in H$. But this is just g' such that $x^{-1}g'x \in H$ for all $x \in H$. we claim that $g' \in H$?

Note that $n \mid |G|$ so $n \leq |G|$, but then a map representating the action of G on G/H is not injective. Thus, there is a nontrivial kernel. ff something something subset $N \in H$ has $g^{-1}Ng \in N$ for all $g \in G$.

Theorem 1.10 says action of G on G/H is equivalent to action of G on S (transtively) and $H = \text{Stab}\, x$, $x \in S$. ff

Wait, do we not have trivial group? $1 \in H$, 1 is a normal subgroup... ahh index! not order divides $n!$

Problem 5 (Ch. 1.12)

Let p be the smallest prime dividing the order of a finite group. Show that any subgroup of H of G of index p is normal.

Solution. Apply Problem 4 from 1.12 (above): if H is a subgroup of G of index p, we have that H contains a normal subgroup of G of index a divisor of $p!$. Note that if p is the smallest prime dividing the order of G, then there is no smaller value (other than 1) that divides the order of G, since if there were, then it must be composite (since p is assumed to be the smallest), but composite numbers are products of primes less than it, but then those primes are less than p, which we can't have. ff

Problem 6 (Ch. 1.12)

Show that evry group of order p^2, p is a prime, is abelian. Show that up to isomorphism there are only two such groups.

Solution. ff

Problem 8 (Ch. 1.12)

Let G act on S, H act on T, and assume $S \cap T = \emptyset$. Let $U = S \cup T$ and define for $g \in G$,$h \in H$, $s \in S$ $t \in T$; $(g, h)s = gs$, $(g, h)t = ht$. Show that this defines an action of $G \times H$ on U.

Solution. ff

Problem 9 (Ch. 1.9)

A group H is said to act on a group K by automorphisms *if we have an action of H on K and for every $h \in H$ the map $k \to hk$ of K is an automorphism. Suppose this is the case and let H be the product set $K \times H$. Define a binary composition in $K \times H$ by*

$$(k_1, h_1)(k_2, h_2) = (k_1(h_1k_2), h_1h_2)$$

and define $1 = (1, 1)$ – the units of K and H respectively. Verify that this defines a group such that $h \to (1, h)$ is a monomorphism of H into $K \times H$ and $k \to (k, 1)$ is a monomorphism of K into $K \times H$ whose image is a normal subgroup. G is called a semi-direct product of K and H. *Note that if H and K are finite than $|K \times H| = |K||H|$.*

Solution. ff