Math 321 Assignment 6
Due Friday, March 1 at 9 am

Instructions

(i) Homework should be submitted using Canvas. Include your name and SID.

(ii) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(iii) Theorems stated in the text (Chapters 1-7) or proved in lecture do not need to be reproved. Any other statement should be justified rigorously.

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1. Let $\{f_n\}$ and f be functions from $[0,1] \to \mathbb{R}$. Suppose that f_n, f have bounded variation on $[0,1]$. Define $g_n(x) = TV[f_n|_{[0,x]}]$ and $g(x) = TV[f|_{[0,x]}]$ (recall Homework 3 for relevant definitions).

a) (2 pts) Suppose that $f_n \to f$ pointwise. Is it true that $g_n \to g$ pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

b) (3 pts) Suppose that $f_n \to f$ uniformly. Is it true that $g_n \to g$ uniformly? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

c) (3 pts) Suppose that $g_n \to g$ pointwise. Is it true that $f_n \to f$ pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

2 (6 pts)**.** For $n \in \mathbb{N}$, let $f_n : [-1,1] \to [0,\infty)$ be: (i) continuous, (ii) obey $\int_{-1}^{1} f_n(x)dx = 1$, and (iii) be such that f_n converges to 0 uniformly on $[-1,-c] \cup [c,1]$ for every $c \in (0,1)$. Suppose $g : [-1,1] \to \mathbb{R}$ is bounded, Riemann integrable, and continuous at 0. Prove that $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)g(x)dx = g(0)$.
Hint: $g(0) = \int_{-1}^{1} f_n(x)g(0)dx$.

3. Let $c \in \mathbb{R}$. For each $n \in \mathbb{N}$ and $x \in [0,1]$, define $f_n(x) = n^c x^3 (1-x^4)^n$.

a) (1 pt) Prove that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$ and determine the limit (you should justify any steps in your computation).

b) (3 pts) Determine the values of c for which the convergence in part (a) is uniform. Prove that your answer is correct.

c) (2 pts) For which values of c do we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx?$$

Prove that your answer is correct.