Problem 1

In a metric space (Y, d), suppose $S \subseteq Y$ is dense, i.e., $\overline{S} = Y$. Suppose that every cauchy sequence (x_n) for which $x_n \in S$ converges in Y. Prove that Y is complete.

Solution. Didn't do :)

Problem 2

(Nearest Points) Let (X, d) be a metric space, and let K be a nonempty subset of X. Define the functoin $d_K : X \to \mathbb{R}$ representing "the distance from K by

$$d_K(p) = \inf\{d(p, x) : x \in K, \quad p \in X$$

(a). Prove that $d_K(p) = d_{\overline{K}}(p)$ for all p in X.

(b). Prove that $|d_K(p) - d_K(q)| \leq d(p, q)$ for any $p, q \in X$.

(c). Suppose K is compact. Prove: $\forall p \in X, \exists \hat{x} \in K : d_K(p) = d(p, \hat{x})$.

(d). Prove that in the special metric space $\mathbb{R}^k$, the result in part (c) remains valid for every closed set K. (This is interesting because some closed sets are not compact.)

(a). *Solution.* Didn't do :)

(b). *Solution.* Didn't do :)

(c). *Solution.* Didn't do :)

(d). *Solution.* Didn't do :)

Problem 3

*Equip $\mathbb{R}$ with its usual topology. Recall that a set $A \subseteq \mathbb{R}$ is called **dense** iff $\overline{A} = \mathbb{R}$.*

(a). Prove: A set $A \subseteq \mathbb{R}$ is dense if and only if for every nonempty open interval (a, b), $A \cap (a, b) \neq \emptyset$.

(b). A sequence of sets $G_1, G_2, G_3, \ldots$ in $\mathbb{R}$ is given. Each G_k is open *and* dense. *Prove that $S = \bigcap_{k\in\mathbb{N}} G_k$ is dense. [Suggestion: Use the characterization from part (a). Construct a suitable Cauchy sequence.]*

(c). Prove that the subset $\mathbb{Q}$ of $\mathbb{R}$ cannot be expressed as a countable intersection of open sets. [Hint: Such a representation is incompatible with the result in (b).]

(a). *Solution.* First, assume that $A \subseteq \mathbb{R}$ is dense. That is to say, $\overline{A} = \mathbb{R}$, so $\mathbb{R} = A' \cup A$. Let (a, b) be a nonempty open interval of $\mathbb{R}$. Define $\alpha = (b - a)/2 \in \mathbb{R}$. Either $\alpha \in A'$ or $\alpha \in A$. If $\alpha \in A$, we are done. So let $\alpha \in A'$. Then by definition of a limit point, $(a, b) \setminus \{\alpha\}) \cap A \neq \emptyset$. So $(a, b) \cap A \neq \emptyset$, as desired.

Now assume that for every nonempty open interval (a, b), $A \cap (a, b) \neq \emptyset$. Let $x \in \mathbb{R}$. Either $x \in A$ or $x \notin A$. If $x \notin A$, then every open set around x excluding x, i.e. $(x-r, x+r) \setminus \{x\}$ must have a nonempty intersection with A, since $(x-r, x+r) \cap A \neq \emptyset$ and we already assumed that $x \notin A$. Hence, $x \in A'$. Thus, for any $x \in \mathbb{R}$, we have $x \in A \cup A'$. Thus, $\mathbb{R} \subseteq A \cup A'$. Now, by definition, $A \cup A' \subseteq \mathbb{R}$, so $\mathbb{R} = A \cup A' \implies \overline{A} = \mathbb{R}$.

(b). *Solution.* Didn't do :)

(c). *Solution.* Didn't do :)

Problem 4

Prove: If $E \subseteq \mathbb{R}$ is uncountable, then $E' \cap E \neq \emptyset$. [Hint: Try the contrapositive.]

Solution. We prove using the contrapositive, that is, assume that $E' \cap E = \emptyset$. That means that every point in E is an isolated point, hence for every point $x \in E$, there exists some $r > 0$ where $(x - r, x + r) \cap E = \{x\}$. Since the rationals are dense in the reals, there exists some $q_1, q_2 \in \mathbb{Q}$ such that $(q_1, q_2) \subseteq (x - r, x + r)$. Furthermore, we can make such pairs unique for each x, since we can just find rationals that are even closer (i.e. some q between q_1 and x, etc.). Let our set of such (q_i, q_j) pairs be denoted Q. We have that $|Q| \geq |E|$ by the obvious injective map from x to q_1. Furthermore, $\mathbb{Q}$ is countable, and $Q \subseteq \mathbb{Q} \times \mathbb{Q}$ is countable, hence $|Q|$ is countable. Hence, E is countable, as desired.

Problem 5

Let $(X, \mathcal{T})$ be a HTS and suppose K_1 and K_2 are nonempty compact sets in X with $K_1 \cap K_2 = \emptyset$. Prove that there are open sets U_1 and U_2 in X such that

$$K_1 \subseteq U_1, \quad K_2 \subseteq U_2, \quad U_1 \cap U_2 = \emptyset$$

Solution. Didn't finish :)

Problem 6

Given an enumeration of $\mathbb{Q}$ *as* $(q_1, q_2, q_3, \dots)$, *define* $f\colon \mathbb{R} \to (0,1)$ *by*

$$f(x) := \sum \left\{ \frac{1}{2^k} : q_k < x \right\}$$

Prove that f *is "lower semicontinuous", i.e. that the following set is open for every* $p \in \mathbb{R}$*:*

$$f^{-1}((p,+\infty)) = \{x \in \mathbb{R} \colon f(x) > p\}.$$

Solution. Didn't finish :)