Problem 1

Solve in either order:

(a). Construct, with justification, a subset A of $\mathbb{R}$ such that every point of A is isolated and $A' \neq \emptyset$.

(b). Rudin Chapter 2, problem 5, page 43: Construct a bounded set of real numbers with exactly three limit points.

(a). *Solution.* We provide the subset $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. First, if $a \in A$, denote this as $a = \frac{1}{n'}$. We show a is an isolated point: let $m = \min\left\{\left|\frac{1}{n'} - \frac{1}{n'+1}\right|, \left|\frac{1}{n'} - \frac{1}{n'-1}\right|\right\}$. Note that since for any $\overline{n} \geq n'$, we have that $\left|\frac{1}{n'+1} - \frac{1}{n'}\right| \leq \left|\frac{1}{\overline{n}} - \frac{1}{n'}\right|$, thus any ball around $\frac{1}{n'}$ that contains $\frac{1}{\overline{n}}$ must also contain $\frac{1}{n'+1}$. Now for any $\underline{n} \leq n'$, we have that $\left|\frac{1}{n'-1} - \frac{1}{n'}\right| \leq \left|\frac{1}{\underline{n}} - \frac{1}{n'}\right|$, thus any ball around $\frac{1}{n'}$ that contains $\frac{1}{\underline{n}}$ must also contain $\frac{1}{n'-1}$.

Consider the open set of R $U = \mathbb{B}[\frac{1}{n'}; \frac{1}{2}m)$. Note that $\frac{1}{n'-1} \notin U$ and $\frac{1}{n'+1} \notin U$. Thus by the contrapositive of the claims we just said, for any $n \in \mathbb{N}$ where $n \neq n'$, we have that $\frac{1}{n} \notin U$. Thus, $\frac{1}{n'}$ is isolated. Since this is true for arbitrary $n' \in \mathbb{N}$, every point in A is isolated.

Now, note that $0 \in A'$ so $A' \neq \emptyset$. If U is be an arbitrary open set in the neighbourhood of 0. Note that we will always have an element of A in U. Assume otherwise, that there exists an open set U such that $U \cap A = \emptyset$. Conside a ball in U, specifically $\mathbb{B}[0, r) \subseteq U$. Note by the Archimedean property of the reals, there exists $n \in \mathbb{N}$ such that $n \cdot 1 > r^{-1} > 0$, thus $0 < \frac{1}{n} < r$. But then $\frac{1}{n} \in \mathbb{B}[0, r)$, thus a contradiction. Thus, since $U \in \mathcal{N}(0)$ was arbitrary, we have that every open set in the neighbourhood of 0 has a non empty intersection with A, thus 0 is a limit piont of A. Thus, $A' \neq \emptyset$.

(b). *Solution.* We give the set $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{\frac{1}{n} + 10 : n \in \mathbb{N}\} \cup \{\frac{1}{n} + 20 : n \in \mathbb{N}\}$. From part (a) of this problem, we note that no element in A is a limit point of A, since they are all isolated (and thus cannot be limit points); the argument is the same, since the additional term just makes it so that we have three subsets of A that do not intersect. Furthermore, we can make identical arguments as from part (a) to show that 10 and 20 are in A', as well as 0. Thus, A has exactly three limit points.

Problem 2

(a). Give an example of two sets A and B in some HTS satisfying

$$\mathrm{int}(A \cup B) \neq \mathrm{int}(A) \cup \mathrm{int}(B)$$

(b). Give an example of two sets A and B in some HTS satisfying

$$\overline{A \cap B} \neq \overline{A} \cap \overline{B}$$

(c). Working $\mathbb{R}^k$ with the usual topology, express the open ball $\mathbb{B}[0;1)$ as a union of closed sets. Can $\mathbb{B}[0;1)$ be expressed as an intersection of closed sets?

(a). *Solution.* Let our HTS be $\mathbb{R}$, and let $A = [0,1]$ and $B = [1,2]$. We have $\mathrm{int}(A \cup B) = \mathrm{int}([0,2]) = (0,2)$ and $\mathrm{int}(A) \cup \mathrm{int}(B) = (0,1) \cup (1,2) = (0,2) \setminus \{1\}$. Hence we have shown $\mathrm{int}(A) \cup \mathrm{int}(B) = (0,2) \setminus \{1\} \neq (0,2) = \mathrm{int}(A \cup B)$, so we are done.

(b). *Solution.* Let our HTS be $\mathbb{R}$, and let $A = (0,1)$ and $B = (1,2)$. We have $\overline{A \cap B} = \overline{\emptyset} = (\mathbb{R}^o)^c$, but since A is open if and only if A^0 is open (from notes) and $\mathbb{R}$ must be open, we have $\overline{A \cap B} = \mathbb{R}^c = \emptyset$. Now, see that $\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$, where we have used the fact that in $\mathbb{R}$, $\overline{(a,b)} = (((a,b)^c)^o)^c = (((-\infty,a] \cup [b,\infty))^o)^c$ but taking the largest open subset we get $((-\infty,a) \cup (b,\infty))^c = [a,b]$. Hence, $\overline{A \cap B} = \emptyset \neq \{1\} = \overline{A} \cap \overline{B}$.

(c). *Solution.* We have

$$\mathbb{B}[0;1) = \bigcup_{n \in \mathbb{N}} \mathbb{B}\left[0; 1 - \frac{1}{n}\right]$$

To verify this equality, if $x \in \mathbb{B}[0;1)$ and $x \neq 0$ (the $x = 0$ case is trivial, x is definitely in the RHS) note that $x_k < 1$, so then by Archimedean, there exists some $n \in \mathbb{N}$ such that $0 < \frac{1-x_k}{<} n \cdot 1 \implies 0 < x_k < 1 - \frac{1}{n}$, thus $x \in [0; 1 - \frac{1}{n}]$, so x is in the RHS. Now if $x \in \bigcup_{n \in \mathbb{N}} \mathbb{B}[0; 1 - \frac{1}{n}]$, there exists some n such that $0 \leq x_k \leq 1 - \frac{1}{n}$. But then $0 \leq x_k < 1$ for all k, so $x \in \mathbb{B}[0;1)$, as desired.

We cannot write $\mathbb{B}[0;1)$ as an intersection of closed sets, since we know that for any HTS¡ the arbitrary intersection of closed sets is also closed, and $\mathbb{B}[0;1)$ is open by definition.

Problem 3

Define a family $\mathcal{T}$ of subsets of $\mathbb{R}$ as follows:

A set $G \subseteq \mathbb{R}$ belongs to $\mathcal{T}$ if and only if for every x in G, there exists $r > 0$ such that $[x, x + r) \subseteq G$.

(a). Prove that $(\mathbb{R}, \mathcal{T})$ is a HTS. (It is called the Sorgenfrey line.*)*

All our terminology – open set, closed set, boundary point, limit point, convergence – depends on what topology we use. Use the Sorgenfrey topology in parts (b)-(d):

(b) Show that the interval $[0, 1)$ is open.

(c) Find all boundary points of the interval $(0, 1)$.

(d) Let $s_n = -1/n$ and $t_n = 1/n$. Prove that one of these sequences converges to 0, and the other does not. Use the definition given in class, i.e. $x_n \to \hat{x}$ means that for every open set U containing $\hat{x}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$.

(a). *Solution.* First, note that $\mathbb{R}$ and $\emptyset$ are both open. If $x \in \mathbb{R}$, then $[x, x+1) \in \mathbb{R}$ as well, so $\mathbb{R} \in \mathcal{T}$. Additionally, $\emptyset$ vacuously satisfies our criteria, and so $\emptyset \in \mathcal{T}$ as well.

Now consider a collection $\mathcal{G}$ of open sets in $\mathcal{T}$. Let V denote the set $\bigcup \mathcal{G}$. If $x \in V$, then $x \in G$ for some $G \in \mathcal{G}$, and by assumption, since $G \in \mathcal{T}$, there exists some $r > 0$ such that $[x, x + r) \in G$, which means $[x, x + r) \in V$ as well (by definition of union). Thus, $V \in \mathcal{T}$ as well.

Now let $U_1, U_2, \cdots, U_N \in \mathcal{T}$ (for some $N \in \mathbb{N}$), and let V denote the set $U_1 \cap U_2 \cap \cdots \cap U_N$. If $x \in V$, then we must have that $x \in U_i$ for all $i \in \{1, \ldots, N\}$. And since $U_i \in \mathcal{T}$, there is some $r_i > 0$ such that $[x, x+r_i) \in U_i$. Let $r = \min_i\{r_i\} > 0$. Then surely $[x, x + r) \subseteq [x, x + r_i)$ for all $i \in \{1, \ldots, N\}$, thus $[x, x + r) \in U_i$ for all i, so $[x, x + r) \in V$. Thus, since $r > 0$, $V \in \mathcal{T}$.

Let $x, y \in \mathbb{R}$ such that $x \neq y$. WLOG assume $x < y$. Clearly $x \in [x, y)$ and $y \in [y, y + 1)$. These are clearly distinct subsets of $\mathbb{R}$. We now show that both are open. If $z \in [x, y)$, let $y - z = \delta > 0$. We have that $[z, z + \delta) = [z, y) \in [x, z)$. Thus, $[x, y)$ is an open set. If $w \in [y, y + 1)$, let $y + 1 - w = \delta > 0$. We have that $[w, w + \delta) = [w, y + 1) \in [y, y + 1)$. Thus, $[y, y + 1)$ is an open set. Thus, we have shown $(\mathbb{R}, \mathcal{T})$ is a HTS, as desired.

(b). *Solution.* Let $x \in [0, 1)$. Then let $1 - x = \delta > 0$. We have that $[x, x + \delta) = [x, 1) \in [0, 1)$. Thus, $[0, 1) \in \mathcal{T}$ and so is an open set.

(c). *Solution.* We claim the only boundary point of $(0, 1)$ is 0. If U is an open set containing 0, it is of the form $[a, b) \in \mathcal{T}$ such that $0 \in [a, b)$ ($b = a + r$, $r > 0 \implies b > a$). In order to have 0 fall in it, we must have $a \leq 0$ and $b > 0$. Since $0 \in U$ and $0 \notin (0, 1)$, then $U \cap (0, 1)^c \neq \emptyset$. Now, either $b \geq 1$ or $b < 1$. If it is the former, then we have that $(0, 1) \subset [a, b)$ and so $U \cap (0, 1) \neq \emptyset$. If it is the latter, then we have $b/2 \in [a, b)$ and $b/2 \in (0, 1)$, thus $U \cap (0, 1) \neq \emptyset$ again. This is sufficient to show that 0 is a boundary point.

We now show that there are no more boundary points of the set. Let $x \in \mathbb{R}$ and $x \neq 0$. If $x < 0$, the open set $[x, 0)$ contains x but does not intersect $(0, 1)$, so cannot be a boundary point for it. If $x \in (0, 1)$, then $[x, 1)$ contains x but does not intersect $(0, 1)^c$, so cannot be a boundary point for $(0, 1)$. And if $x \geq 1$, then $[x, x + 1)$ contains x but does not intersect $(0, 1)$ so cannot be a boundary point for it. Thus, since x was arbitrary, $x \in \mathbb{R}^*$ cannot be a boundary point for $(0, 1)$.

(d). *Solution.* We claim that s_n does not converge to 0. If $U = [0, 1)$, then we have that U is open from part (b) of this problem and it contains 0, however, for all $n \in \mathbb{N}$, we have that $-1/n < 0$ and so for no $n \in \mathbb{N}$ do we have $s_n \in U$. Thus, $s_n \notin U$ for all n, thus s_n does not converge to 0.

We claim that t_n does converge to 0. Consider an open set U that contains 0. That is, U is of the form $[a, b)$ where $b > a$ (since $b = a + r$ and $r > 0$ thus $b > a$) such that $0 \in U$. In order for this to happen, it is necessary that $a \leq 0$ and $b > 0$. Thus, it is sufficient to show that $0 < t_n < b$ for large enough n to show $t_n \in U$. Clearly, $t_n > 0$ for all $n \in \mathbb{N}$. Now, by Archimedean, there exists $N \in \mathbb{N}$ such that $N \cdot 1 > b^{-1} > 0$, thus $0 < \frac{1}{N} = t_N < b$. Furthermore, $\frac{1}{n} \leq \frac{1}{N}$ for $n \geq N$ (this is obvious, but if really desired, this can be shown with induction), thus for $n \geq N$, we have $0 < t_n < b$, thus $t_n \in U$. Since U was an arbitrary open set that contained 0, this is sufficient to show convergence of $t_n \to 0$.

Problem 4

Let A be a subset of a HTS $(X, \mathcal{T})$. The ***boundary of*** *A is a set denoted ∂A: we say $z \in \partial A$ if and only if every open U containing z satisfies both $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. Prove:*

(a). $\partial A = \overline{A} \cap \overline{A^c}$.

(b). A is closed if and only if $\partial A \subseteq A$.

(c). A is open if and only if $A \cap \partial A = \emptyset$.

(a). *Solution.* Let $x \in \overline{A} \cap \overline{A^c}$. Let $G \in \mathcal{N}(x)$. We first prove that $A \cap G \neq \emptyset$. If $x \in A$, we are done, since $\{x\} \in A \cap G$. Now, assume that $x \notin A$. For the sake of contradiction, assume that $A \cap G = \emptyset$. This means that $G \subseteq A^c$ Thus, A is closed (by the lemma from class). Then $A = \overline{A}$. But recall that $x \in \overline{A} = A$, thus a contradiction. We now prove that $A^c \cap G \neq \emptyset$. If $x \in A^c$, we are done, since $\{x\} \in A^c \cap G$. Now, assume that $x \notin A^c$. For the sake of contradiction, assume that $A^c \cap G = \emptyset$. This means that $G \subseteq A$ Thus, A^c is closed (by the lemma from class). Then $A^c = \overline{A^c}$. But recall that $x \in \overline{A^c} = A^c$, thus a contradiction. Thus, since G was arbitrary, we have both $A \cap G \neq \emptyset$ and $A^c \cap G \neq 0$ for all $G \in \mathcal{N}(x)$, which by definition, means that $x \in \partial A$. Thus, $\overline{A} \cap \overline{A^c} \subseteq \partial A$.

Now let $x \in \partial A$. We now prove that $x \in \overline{A}$. If $x \in A$, we are done, since $A \subset \overline{A}$. If $x \notin A$, since $\overline{A}$ is closed, we have that there exists some neighbourhood $U \in \mathcal{N}(x)$ such that $U \subseteq A^c$, and since $A^c \cap A = \emptyset$, this implies that $U \cap A = \emptyset$. But recall that since $x \in \partial A$, we have that $A \cap U \neq \emptyset$, thus, a contradiction. Hence, $x \in \overline{A}$. Now we prove that $x \in \overline{A^c}$. If $x \in A^c$, we are done, since $A^c \subset \overline{A^c}$. If $x \notin A^c$, since $\overline{A^c}$ is closed, we have that there exists some neighbourhood $U \in \mathcal{N}(x)$ such that $U \subseteq (A^c)^c = A$, and since $A^c \cap A = \emptyset$, this implies that $U \cap A^c = \emptyset$. But recall that since $x \in \partial A$, we have that $A^c \cap U \neq \emptyset$, thus, a contradiction. Hence, $x \in \overline{A^c}$. Thus, $x \in \overline{A} \cap \overline{A^c}$, so $\overline{A} \cap \overline{A^c} \subseteq \partial A$.

We have shown set inclusion in both directions, so $\partial A = \overline{A} \cap \overline{A^c}$.

(b). *Solution.* Assume that A is closed. Let $x \in \partial A$. From part (a) of this problem, we have $\partial A = \overline{A} \cap \overline{A^c}$, thus $z \in \overline{A} \cap \overline{A^c} \implies z \in \overline{A}$. But since A is closed, $A = \overline{A}$, thus $z \in A$. Since z was arbitrary, this implies that $\partial A \subseteq A$.

Now assume that $\partial A \subseteq A$. From part (a) of this problem, we have $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (((A^c)^c)^o)^c = \overline{A} \cap (A^o)^c = \overline{A} \setminus A^o$, thus $\partial A \cup A^o = \overline{A}$. Thus, since $\partial A \subseteq A$, by assumption, and $A^o \subseteq A$, we have $\overline{A} \subseteq A$. But $A \subseteq \overline{A}$ by definition, so $A = \overline{A}$. But this is true only A is closed.

(c). *Solution.* Assume that A is open. Then A^c is closed, and thus $A^c = \overline{A^c}$. Invoking part (a) of this problem, we have

$$
\begin{aligned}
A \cap \partial A &= A \cap (\overline{A} \cap \overline{A^c}) \\
&= (A \cap A^c) \cap \overline{A} \\
&= \emptyset \cap \overline{A} \\
&= \emptyset
\end{aligned}
$$

Now assume that $A \cap \partial A = \emptyset$. But then $A \cap \overline{A} \cap \overline{A^c} = \emptyset$. Since $A \subseteq \overline{A}$, we have $A \cap \overline{A} = A$, thus we have $A \cap \overline{A^c} = \emptyset$. Writing out the full definition for the closure of A^c, we have $\overline{A^c} = (((A^c)^c)^o)^c = (A^o)^c$, thus we have $A \cap (A^o)^c = \emptyset$. For the sake of contradiction, assume that A is not open. Then, there exists $x \in A$ such that $x \notin A^o$ (since $A^o \subset A$ and $A^o \neq A$). Then, $x \in (A^o)^c$. But then $x \in A \cap (A^o)^c) \implies A \cap (A^o)^c \neq \emptyset$, thus a contradiction. Hence A must be open.

Problem 5

Prove: For every set A in a HTS $(X, \mathcal{T})$, A' is closed.

Solution. It is sufficient to show that $\partial A' \subseteq A'$, by problem 4(b). Let $x \in \partial A'$. By definition, this means that for all open sets $U \in \mathcal{N}(x)$, we have $U \cap A' \neq \emptyset$. Thus, let $y \in U \cap A'$. Since U is an open set and $y \in U$, there exists $U' \in \mathcal{N}(y)$ such that $U' \subseteq U$. Now define $V = U' \setminus x$. Note that $y \in V$ still, and V is open, since $V = U' \setminus x = U' \cap \{x\}^c$ which is a finite intersection of open sets ($\{x\}^c$ is open since $\{x\}$ is closed), so is also open, thus $V \in \mathcal{N}(y)$ still. Now, since $y \in A'$, then for all open $W \in \mathcal{N}(y)$, we have $W \setminus \{y\} \cap A \neq \emptyset$, so $V \setminus \{y\} \cap A \neq \emptyset$. Thus, there exists $z \in V \setminus \{y\} \cap A \implies z \in U$ since $V \setminus \{y\} \subseteq U$. Thus, we se that there exists $z \in U \cap A$ where $z \neq x$, or $U \setminus \{x\} \cap A \neq \emptyset \implies x \in A'$. Hence, we have that $\partial A' \subseteq A'$, which shows that A' is closed.

Problem 6

Recall the sequence space ℓ^2 from HW07 Q3. Given a specific $M = (M_1, M_2, \dots)$ in ℓ^2, let

$$S = \{x \in \ell^2 \colon \forall n \in \mathbb{N}, |x_n| \le M_n\}.$$

Prove: every sequence $\left(x^{(n)}\right)$ in S has a convergent subsequence, whose limit lies in S.

Solution. Write out the sequence of the first components of our sequence, namely

$$\left(x_1^{(n)}\right) = x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \dots$$

Each term of $\left(x_1^{(n)}\right)$ is bounded by M_1, thus, by Bolzano-Weierstrass (which we can use since this is just a real-valued sequence), there exists a subsequence (n_{k_1}) such that $\left(x_1^{(n_{k_1})}\right)$ converges. This is a convergent sequence, so denote $\lim_{k_1} x_1^{(n_{k_1})} = s_1$; since $|x_1^{(i)}| \le M_1$ for all $i \in n_k$, the limit is at most M_1, thus $|s_1| \le M_1$ as well. We will now use the notation $k_1(i)$ to be the ith integer of n_{k_1}. Thus our new subsequence of elements in S is $\left(x^{(n_{k_1})}\right) = x^{(k_1(1))}, x^{(k_1(2))}, x^{(k_1(3))}, \dots$.

Now consider the second components of our subsequence $\left(x^{(n_{k_1})}\right)$:

$$\left(x_2^{(n_{k_1})}\right) = x_2^{(k_1(1))}, x_2^{(k_1(2))}, x_2^{(k_1(3))}, \dots$$

Again, by Bolzano Weierstrass from boundedness by M_2, there exists a subsequence of n_{k_1}, call it n_{k_2} such that $\left(x_2^{(n_{k_2})}\right)$ converges. Again, $k_2(i)$ is the ith integer of n_{k_2}.

Now, for any j, we can iteratively acquire a subsequence n_{k_j}. We write it them all out below:

$$\begin{aligned}
\left(x^{(n_{k_1})}\right) &= x^{(k_1(1))}, x^{(k_1(2))}, x^{(k_1(3))}, \dots \\
\left(x^{(n_{k_2})}\right) &= x^{(k_2(1))}, x^{(k_2(2))}, x^{(k_2(3))}, \dots \\
&\vdots \\
\left(x^{(n_{k_j})}\right) &= x^{(k_j(1))}, x^{(k_j(2))}, x^{(k_j(3))}, \dots
\end{aligned}$$

Some remarks to note:

(a). $\left(x^{(n_{k_j})}\right)$ is a subsequence of $\left(x^{(n_{k_{j-1}})}\right)$ for all $j = 2, 3, 4, \dots$.

(b). $\left(x_j^{(n_{k_j})}\right)$ converges as $k_j \to \infty$ (since each step, Bolzano Weierstrass lets us pick a convergent subsequence), whose value we denote s_j and $|s_j| \le M_j$ (by the argument we provided when $j = 1$).

(c). By the definition of a subsequence, if $x^{(m)}$ comes before $x^{(m')}$ in $\left(x^{(n_{k_j})}\right)$, we must have that $x^{(m)}$ comes before $x^{(m')}$ in $\left(x^{(n_{k_i})}\right)$ for all $1 \le i \le j$ by the definition of a subsequence.

We now pick entries from our diagonal to form a new subsequence of $\left(x^{(n)}\right)$, which we will call $\left(x^{(n_k)}\right)$:

$$\left(x^{(n_k)}\right) = x^{(k_1(1))}, x^{(k_2(2))}, x^{(k_3(3))}, \dots$$

Define $\hat{s} = s_{k_1(1)}, s_{k_2(2)}, s_{k_3(3)}, \dots$. Recall that $0 < |s_n| \le M_n \implies 0 < |s_n|^2 \le M_n^2$. Note since $\sum_n M_n^2$ converges by definition of being in ℓ^2, we also have that $\sum_n |s_n|^2$ converges, thus $\hat{s} \in \ell^2$ as well. Hence, $\hat{s} \in S$. Therefore, it is sufficient to show now that $\left(x^{(n_k)}\right) \to \hat{s}$ as $k \to \infty$.

From our previous homework, our notion of distance is

$$d(x, y) = \|x - y\| = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$$

Thus to say that $\left(x^{(n_k)}\right) \to \hat{s}$ as $k \to \infty$, we would need to show that for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$, such that for all $k > K$, we have $\sqrt{\sum_{i=1}^{\infty}(x_i^{(n_k)} - \hat{s}_i)^2} < \varepsilon$.

Thus let $\varepsilon > 0$ be arbitrary. Since $(M_n) \in \ell^2$, $\sum_n M_n^2$ converges; denote the value it converges to by L. Note that $(\sum_n M_{n_k})$ is a subsequence, and subsequences converge to the same limit. Then there exists $K - 1 \in \mathbb{N}$ such that for all $k \geq K - 1$, we have

$$\left| L - \sum_{i=1}^{k} M_i^2 \right| < \frac{\varepsilon^2}{4}$$

This applies when $k = K - 1$, thus

$$\left| L - \sum_{i=1}^{K-1} M_i^2 \right| < \frac{\varepsilon^2}{4} \implies \sum_{i=K}^{\infty} M_i^2 < \frac{\varepsilon^2}{4}$$

where the implication is because $L = \sum_{i=1}^{K-1} M_i^2 + \sum_{i=K}^{\infty} M_i^2$, and we remove the absolute value, because $M_i^2 > 0$ always, so the sum is > 0 always.

Now, note that for any k, $0 < |x_i^{(n_k)} - \hat{s}_i| \leq |x_i^{(n_k)}| + |\hat{s}_i| \leq 2M_i$ for all i. Since $0 < |x_i^{(n_k)} - \hat{s}_i|^2 \leq 4M_i^2$, and we know that $\sum_n 4M_i^2$ converges, we know that $\sum_n |x_i^{(n_k)} - \hat{s}_i|^2$ converges too. Thus, writing an infinite series means something, and so for any $p \in P$, we have $\sum_{i=p}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2 \leq \sum_{i=p}^{\infty} M_i^2$. Thus, if we let $P_K = \sum_{i=1}^{K-1} |x_i^{(n_k)} - \hat{s}_i|^2$, we have

$$\sum_{i=1}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2 = P_K + \sum_{i=K}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2 \leq P_K + 4\sum_{i=K}^{\infty} M_i^2$$

Taking the lim sup of both sides, we get

$$\limsup_{k\to\infty} \sum_{i=1}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2 \leq \limsup_{k\to\infty} \left(P_K + 4\sum_{i=K}^{\infty} M_i^2 \right) \leq \limsup_{k\to\infty P_K} + 4 \limsup_{k\to\infty} \sum_{i=K}^{\infty} M_i^2$$

On the right hand side, note that $\limsup P_K = 0$, since $x_i^{(n_k)} \to \hat{s}_i$ as $k \to \infty$ by construction (note (b) from above); additionally $\sum_{i=K}^{\infty} M_i^2$ does not vary with k. Thus, we have

$$\limsup_{k\to\infty} \sum_{i=1}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2 \leq 4\sum_{i=K}^{\infty} M_i^2 < \varepsilon^2$$

By the definition of lim sup, this gives us

$$\sum_{i=1}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2 < \varepsilon^2$$

hence

$$\sqrt{\sum_{i=1}^{\infty} |x_i^{(n_k)} - \hat{s}_i|^2} < \varepsilon$$

for all $k \geq K$, as desired.