Problem 1

Prove: If $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges. Is this statement still true if the word "absolutely" is removed?

Solution. It is sufficient to show that $\sum_n a_n b_n$ is absolutely convergent. Consider the series $\sum_n |a_n b_n| = \sum_n |a_n||b_n|$. Since $\lim_{n\to\infty} a_n = 0$ (by the contrapositive of the "crude" divergence test since $\sum_n a_n$ converges), we have that a_n is bounded, and so $|a_n|$ is bounded as well (the upper bound is just the max of the lower and upper bound of a_n, and it is bounded below by 0). Let $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then $|a_n||b_n| < M|b_n|$. We have that $\sum_n M|b_n|$ converges, since if $s_N = \sum_{n=1}^{N} |b_n|$, then

$$\sum_n M|b_n| = \lim_{n\to\infty} M|b_0| + M|b_1| + \cdots + M|b_n| = \lim_{n\to\infty} M(|b_0| + |b_1| + \cdots + |b_n|) = \lim_{n\to\infty} Ms_n$$

and since (s_n) converges (by the absolute convergence of b_n), by our constant multiplication limit law, $(Ms_n) = \sum_n M|b_n|$ converges as well.

Now since $0 \leq |a_n b_n| \leq M|b_n|$, by the comparison test, $\sum_n |a_n b_n|$ converges, thus $\sum_n a_n b_n$ is absolutely convergent, which implies that $\sum_n a_n b_n$ converges.

Problem 2

For each series below, find the set of $x \in \mathbb{R}$ where the series converges.

(a). $\sum_{n=1}^{\infty} c^{n^2}(x-1)^n$ *($c > 0$ const.)*

(b). $\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(c). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[\frac{x+1}{2x+1}\right]^n$

(d). $\sum_{n=1}^{\infty} \left[\frac{(2n)!}{n(n!)^2}\right](x-e)^n$

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

(d). *Solution.* ff

Problem 3

Discuss the series whose nth terms are shown below:

$$a_n = (-1)^n \frac{n^n}{(n+1)^{n+1}}, \qquad b_n = \frac{n^n}{(n+1)^{n+1}},$$
$$c_n = (-1)^n \frac{(n+1)^n}{n^n}, \qquad d_n = \frac{(n+1)^n}{n^n}.$$

Solution. ff

Problem 4

Suppose $x_1 \geq x_2 \geq x_3 \geq \cdots$ *and* $\lim_{n\to\infty} x_n = 0$. *Show that the following series converges:*

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \pm \cdots .$$

Solution. Note that the sequence is bounded below by 0. We have a sum of geometric series.

Problem 5

(a). Prove: if $a_n \geq a_{n+1} \geq 0$ for all n, and $\sum a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.

(b). Prove: If $\sum(b_n^2/n)$ converges, $\frac{1}{N}\sum_{j=1}^{N} b_j \to 0$ as $N \to \infty$.

[Hint: In part (a), it's enough to prove that $\frac{1}{2}na_n \to 0$.]

(a). *Solution.* It would be sufficient to show that $\sum_n \frac{1}{2}na_n$ converges. This converges if and only if $\sum_n 2^n a_n$ by the Cauchy condensation... this is not Cauchy condensation because of n... but what about $n < 2^k$.

Since $a_n \geq a_{n+1} \geq 0$ for all n and $\sum a_n$ converges, the Cauchy Condensation Test gives $\sum_k 2^k a_{2^k}$ converges as well. Since our sequence monotonically decreases and is always positive, we have $2^k a_n \leq 2^k a_{2^k}$ for $n \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$. Note that for any k, $2^k \leq n < 2^{k+1}$ implies $\frac{n}{2} < 2^k$. Thus, $0 \leq |\frac{n}{2}a_n| = \frac{n}{2}a_n < 2^k a_n \leq 2^k a_{2^k}$ for $2^k \leq n < 2^{k+1}$. ff

(b). *Solution.* In Problem 8(a) from homework 3, we proved that if $a_n \to 0$ as $n \to \infty$, then $(a_1 + a_2 + \cdots a_n)/n \to 0$ as well. Thus, it is sufficent to show that $b_n \to 0$ as $n \to \infty$.

Since $\sum(b_n^2/n)$ converges, by the crude divergence test, we have that $b_n^2/n \to 0$ as $n \to \infty$. ff

Problem 6

Define $f(\theta) = \sum_{k=1}^{\infty} \frac{1}{2k-1}\sin((2k-1)\theta)$. *Determine the domain of* f, *namely, the set of all real* θ *where the series converges, by completing the steps below.*

(a). Obtain the following identities, valid for each $n \in \mathbb{N}$ *at all points where* $\sin\theta \neq 0$:

$$C_n(\theta) = \cos(\theta) + \cos(3\theta) + \cos(5\theta) + \cdots + \cos((2n-1)\theta) = \frac{\sin(2n\theta)}{2\sin\theta},$$

$$S_n(\theta) = \sin(\theta) + \sin(3\theta) + \sin(5\theta) + \cdots + \sin((2n-1)\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta},$$

[Suggestion: Use geometric sums of complex numbers, with $e^{it} = \cos(t) + i\sin(t)$.*]*

(b). Prove that the domain of f *is the interval* $(-\infty, +\infty)$.

(c). Find a sequence (θ_n) *such that* $\theta_n \to 0$ *and* $S_n(\theta_n) \to +\infty$ *as* $n \to \infty$. *Explain why your solution in part (b) is correct in spite of the evident unboundedness of the sequence* $(S_n(\theta_n))$.

(a). Solution. ff

(b). Solution. ff

(c). Solution. ff