## Problem 1

*Consider a real-valued sequence* $(x_n)$ *and a real number* $\hat{x}$*. Prove that the following are equivalent:*

*(a).* $x_n \to \hat{x}$,

*(b).* $\forall \varepsilon > 0, \exists N \in \mathbb{N} \colon \forall n \geq 23N, |x_n - \hat{x}| < 20\varepsilon$.

*Solution.* Assume (a). For any $\varepsilon > 0$, since $x_n$ converges to $\hat{x}$ (and $20\varepsilon > 0$), we have that there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - \hat{x}| < 20\varepsilon$. But since if $n \geq 23N$, it is certainly true that $n > N$, thus $|x_n - \hat{x}| < 20\varepsilon$ for all $n \geq 23N$, which is (b).

Now assume (b). For any $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{20}$. We know that there exists $N' \in \mathbb{N}$ such that for all $n \geq 23N'$, we have that $|x_n - \hat{x}| < 20\varepsilon' = \varepsilon$. But since $24N' \in \mathbb{N}$, let us fix $N = 24N'$, then for all $n \geq N > 23N'$, we have $|x_n - \hat{x}| < \varepsilon$ again, which, by definition, means $x_n \to \hat{x}$.

## Problem 2

*Extend our collection of equivalent formulations of the completeness property for $\mathbb{R}$ by proving that the following are equivalent (TFAE). Proceed directly, without relying on the completeness property in one of its other forms. (So, for example, do not assume existence of suprema and infima.)*

*(a). For any sequence of nonempty closed real intervals $I_1[a_1, b_1], I_2 = [a_2, b_2], \dots,$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ (such intervals are called "nested"), one has*

$$\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$$

*(b). Every bounded monotonic sequence in $\mathbb{R}$ converges. (Recall Rudin's Definition 3.13.)*

*(*Note*: The inverval notation $[a, b] = \{t \in \mathbb{R} \colon a \le t \le b\}$ is reserved for the case where both $a$ and $b$ are real numbers. To encode $\{t \in \mathbb{R} \colon t \ge 0\}$, for example, we would write $[0, +\infty)$, not $[0, +\infty]$.)*

*Solution.* We first prove (a) $\implies$ (b). Let be a bounded monotonic sequence in $\mathbb{R}$ be given, $(x_n)$. WLOG, let this be a monotonic increasing sequence (a similar argument will exist for decreasing, just flipping some inequalities). Let the upper bound be $M$. Define the interval $I_1 = [x_1, M]$. We can bisect $I_1$ into two subintervals. If there exists $N$ such that $x_N \ge (M - x_1)/2$ let $I_2$ be $[(M - x_1)/2, M]$, otherwise $I_2 = [x_1, (M - x_1)/2]$, (thus $(M - x_1)/2$ is an upper bound for all $x_n$ where $n \ge N$). In either case, $I_2$ contains the infinitely many points in the sequence of the form $x_n$ where $n \ge N$ (since $x_n$ is monotonically increasing and so is bounded below in the interval, and it is also bounded above in the interval). Define an arbitrary interval in this sequence $I_k$ by bisecting $I_{k-1}$ and choosing the interval $I_k$ as the lower interval if the $x_n$ are bounded by the midpoint, otherwise take the upper interval. Note that by construction, $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$. Thus, by (a), there exists some $\hat{x}$ that is in every interval $I_k$.

We now show $x_n$ converges to $\hat{x}$. Let $\varepsilon > 0$. Note the length of the interval $I_k$ is $(M - x_1)2^{-(k-1)}$. If $M - x_1 = 0$, then we are done, since for any $n \ge 1$, $x_n = \hat{x}$ (since $x_n \ge x_{n-1}$ but $x_n \le M$ and $M = x_1$, thus $x_n = x_1 = \hat{x}$ because $\hat{x} \in [x_1, x_1] \implies \hat{x} = x_1$) and $|x_n - \hat{x}| = 0 < \varepsilon$. Now assume $M - x_1 \neq 0$. By the Archimedean property, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{(M - x_1)}$. Note that $2^n > n$ for all $n \in \mathbb{N}$ (induction: when $n = 1$, $2^n = 2 > 1 = n$. Further, assume that $2^n > n$, then $2^{n+1} > 2^n 2 > 2n \ge n + 1$). Thus, $0 < \frac{1}{2^n} < \frac{1}{n} < \frac{\varepsilon}{M - x_1}$ which implies there is some $n \in \mathbb{N}$ such that $0 < \frac{M - x_1}{2^n} < \varepsilon$. But note that the in the interval $I_{N+1}$, the largest distance between two points is the length of the interval, that is $(M - x_1)2^{-N}$. Thus there exists $N \in \mathbb{N}$ such that for all $n \ge N + 1$,

$$|x_n - \hat{x}| \le (M - x_1)2^{-N} < \varepsilon$$

Thus $(x_n)$ converges.

We now prove that (b) $\implies$ (a). Consider the sequence of nonempty closed real intervals $I_n$ from the problem statement (ie. $I_n = [a_n, b_n]$, $a_n \le b_n$). Note that $a_n$ and $b_n$ form monotonic sequences. Since in order for $I_n \supseteq I_{n+1}$ to be true, we must have that $a_n \le a_{n+1}$ and $b_n \ge b_{n+1}$. Thus, $a_n$ is monotonically increasing and $b_n$ is monotonically decreasing. Note that the sequence $(a_n)$ must be bounded. We know that any $a_n \ge a_1$ since its monotone increasing, so its bounded below, and $a_n \le b_1$, otherwise $b_1 < a_n \le b_n$, which contradicts our assumption that $b_n$ monotonically decreases, so $(a_n)$ is bounded above.

By (b), we then know that $a_n$ converges to some value, call it $\hat{a}$. Note that $\hat{a} \ge a_n$ for all $n \in \mathbb{N}$, since $(a_n)$ is monotonically increasing: if there existed some $N$ where $\hat{a} < a_N$, then for all $n \ge N$, $\hat{a} < a_N \le a_n$ since the sequence is monotonically decreasing, which violates convergence to $\hat{a}$ (if $\varepsilon = |\hat{a} - a_N|$, then $\varepsilon < |\hat{a} - a_n|$ for all $n \ge N$). Furthermore, $\hat{a} \le b_n$. Otherwise, if $b_N > \hat{a}$, then $a_N \ge b_N$ and so $a_n > \hat{a}$ for all $n \ge N$, which then fails the assumption that $a_n$ converges to $\hat{a}$ in the same way as before.

Thus for any interval $I_n$, we have $a_n \le \hat{a} \le b_n$, thus $\hat{a} \in I_n$. But then $\hat{a}$ is in the intersections of all the intervals, thus $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$.

*Note: Questions 3-6 contribute to the major project of constructing $\mathbb{R}$ from $\mathbb{Q}$. Therefore they must be completed entirely in the context of the ratoinal numbers. Present solutions that make no reference at all to the completeness property of $\mathbb{R}$, in any of its equivalent forms.*

## Problem 3

*Introduce the following notation:*
$CS(\mathbb{Q})$: *the set of all Cauchy sequences with rational elements.*
$x, y, z$: *typical symbols for elements of $CS(\mathbb{Q})$. Thus, e.g., $x = (x_1, x_2, \dots)$.*
$R[x]$: *the subset of $CS(\mathbb{Q})$ associated with a given $x \in CS(\mathbb{Q})$ as follows:*

$$R[x] = \left\{ x' \in CS(\mathbb{Q})\colon \lim_{n\to\infty} |x'_n - x_n| = 0 \right\}.$$

$\Phi$: *the function that takes each rational number $q$ into the subset of $CS(\mathbb{Q})$ containing the corresponding constant sequence, i.e.,*

$$\Phi(q) = R[(q, q, \dots)] \quad \forall q \in \mathbb{Q}$$

*(a). Prove: $R[x] \neq \emptyset$ for every $x \in CS(\mathbb{Q})$.*

*(b). Prove: For any $x, y \in CS(\mathbb{Q})$, $R[x] = R[y] \iff R[x] \cap R[y] \neq \emptyset$*

(a). *Solution.* Define $x'$ where $x'_n = x_n$. Then for any $n \in \mathbb{N}$, $|x'_n - x_n| = 0$, so $\lim_{n\to\infty} |x'_n - x_n| = 0$, thus $x' \in R[x]$. And so $R[x] \neq \emptyset$.

(b). *Solution.* Assume $\mathcal{R}[x] = \mathcal{R}[y]$. By part (a) of this problem, these are nonempty sets. Thus, there exists an element in $\mathcal{R}[x]$ which must also be in $\mathcal{R}[y]$ by equality. But then $\mathcal{R}[x] \cap \mathcal{R}[y] \neq \emptyset$.

Now assume $\mathcal{R}[x] \cap \mathcal{R}[y] \neq \emptyset$. Thus, there exists $z \in CS(\mathbb{Q})$ such that $z \in \mathcal{R}[x]$ and $z \in \mathcal{R}[y]$. Since both $\mathcal{R}[x]$ and $\mathcal{R}[y]$ are nonempty by part (a), let $x' \in \mathcal{R}[x]$ and $y' \in \mathcal{R}[y]$ be arbitrary elements in their respective sets. We first prove that $\mathcal{R}[x] \subseteq \mathcal{R}[y]$ by showing $x' \in \mathcal{R}[y]$ since it is arbitrary. We will show first that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|z_n - x'_n| < \varepsilon$. Let $\varepsilon > 0$ be arbitrary. By definition of $z, x' \in \mathcal{R}[x]$, we have for $\varepsilon_1 = \frac{\varepsilon}{2}$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x'_n - x_n| < \varepsilon_1$, and for $\varepsilon_2 = \frac{\varepsilon}{2}$, there exists $N_2$ such that for all $n \geq N_2$, $|z_n - x_n| < \varepsilon_2$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$. Then we know for all $n \geq N$

$$|x'_n - z_n| \leq |x'_n - x_n| + |x_n - z_n| \leq \varepsilon_1 + \varepsilon_2 = \varepsilon$$

(where we have used the triangle inequality). Now, we will show first that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x'_n - y_n| < \varepsilon$, thus $x'_n \in \mathcal{R}[y] \implies \mathcal{R}[x] \subseteq \mathcal{R}[y]$ (since $x'_n$ was an arbitrary element of $\mathcal{R}[x]$). Let $\varepsilon > 0$ be arbitrary. By definition of $z \in \mathcal{R}[y]$, we have for $\varepsilon_1 = \frac{\varepsilon}{2}$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|z_n - y_n| < \varepsilon_1$, and using the fact we just proved, for $\varepsilon_2 = \frac{\varepsilon}{2}$, there exists $N_2$ such that for all $n \geq N_2$, $|z_n - x'_n| < \varepsilon_2$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$. Then we know for all $n \geq N$

$$|x'_n - y_n| \leq |x'_n - z_n| + |z_n - y_n| \leq \varepsilon_1 + \varepsilon_2 = \varepsilon$$

Thus $\mathcal{R}[x] \subseteq \mathcal{R}[y]$.

Finally, note that the argument to show that $y'_n \in \mathcal{R}[x]$ is identical, and we one would just need to swap the $x$'s and $y$'s, thus $\mathcal{R}[y] \subseteq \mathcal{R}[x]$ as well (since $y'_n$ was an arbitrary element of $\mathcal{R}[y]$). Therefore, $\mathcal{R}[x] = \mathcal{R}[y]$. This shows both directions of the implication.

## Problem 4

*Continue with the notation from Question 3. We would like to define a relatoin denoted "$<$" on $\mathbb{Q}^*$ as follows:*

$$R[x] < R[y] \iff \exists r > 0 (r \in \mathbb{Q}), \exists N \in \mathbb{N} \colon \forall n > N, y_n - x_n > r$$

*This relation looks like one that is familiar for rational numbers, but here it compares two* sets. *Each of the properties we take for granted when manipulating inequalities relating numbers requires careful thinking in this new context. Prove the following.*

(a). *Whenever $R[x'] = R[x]$ and $R[y'] = R[y]$ for some given $x, x', y, y' \in CS(\mathbb{Q})$, the definition proposed above gaurantees that*

$$R[x'] < R[y'] \iff R[x] < R[y].$$

*(that is, the proposed definition is unambiguous. Or, more conventionally, "the relation $<$ is well-defined".)*

(b). *If $x, y, z \in CS(\mathbb{Q})$ obey $R[x] < R[y]$ and $R[y] < R[z]$, then $R[x] < R[z]$.*

(c). *The inequality $R[x] < R[x]$ never happens, for any $x \in CS(\mathbb{Q})$.*

(d). *For any $p, q \in \mathbb{Q}$, we have $p < q \iff \Phi(p) < \Phi(q)$.*

(a). *Solution.* Assume $\mathcal{R}[x'] < \mathcal{R}[y']$. Let $r$ be given such there exists $N_1 \in \mathbb{N}$ where for all $n \geq N_1$, $y'_n - x'_n > r$. Since $\mathcal{R}[x'] = \mathcal{R}[x]$, if we let $\varepsilon = \frac{r}{2}$, we know there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $-\frac{r}{2} < x'_n - x_n < \frac{r}{2} \implies x_n - \frac{r}{2} < x'_n$. Thus, if $N' = \max\{N_1, N_2\} \in \mathbb{N}$, for all $n \geq N'$,

$$y'_n - x_n + \frac{r}{2} > y'_n - x'_n > r \implies y'_n - x_n > \frac{r}{2}$$

. Since $\mathcal{R}[y'] = \mathcal{R}[y]$, if we let $\varepsilon = \frac{r}{4}$, we know there exists $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$, $-\frac{r}{4} < y'_n - y_n < \frac{r}{4} \implies y'_n < y_n + \frac{r}{4}$. Thus, if $N = \max\{N_3, N'\} \in \mathbb{N}$, for all $n \geq N$,

$$y_n + \frac{r}{4} - x_n > y'_n - x_n > \frac{r}{2} \implies y_n - x_n > \frac{r}{4}$$

But if $0 < r \in \mathbb{Q}$, certainly $0 < \frac{r}{4} \in \mathbb{Q}$, thus $\mathcal{R}[x] < \mathcal{R}[y]$.

Now assume $\mathcal{R}[x] < \mathcal{R}[y]$. Note that we can do an identical proof as before, just swapping the $x$'s and $x'$'s and the $y$'s with $y'$'s, to show that this implies $\mathcal{R}[x'] < \mathcal{R}[y']$. Thus, we have shown both directions of the implication.

(b). *Solution.* Since $\mathcal{R}[x] < \mathcal{R}[y]$, there exists $0 < r_1 \in \mathbb{Q}$ where there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $y_n - x_n > r_1$. Secondly, since $\mathcal{R}[y] < \mathcal{R}[z]$, there exists $0 < r_2 \in \mathbb{Q}$ where there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $z_n - y_n > r_2 \implies z_n - r_2 > y_n$. Now let $N = \max\{N_1, N_2\}$, then for all $n \geq N$,

$$r_1 < y_n - x_n < z_n - r_2 - x_n \implies r_1 + r_2 < z_n - x_n$$

But if $0 < r_1, r_2 \in \mathbb{Q}$, certainly $0 < r_1 + r_2 \in \mathbb{Q}$, thus $\mathcal{R}[x] < \mathcal{R}[z]$.

(c). *Solution.* Note that for all $N \in \mathbb{N}$, we have that if $n \geq N$, $x_n - x_n = 0$. Thus, for any $r > 0$, $x_n - x_n < r$. Thus, we cannot have $\mathcal{R}[x] < \mathcal{R}[x]$.

(d). *Solution.* Define $p_n = p$ and $q_n = q$ for all $n \in \mathbb{N}$. Thus $\Phi(p) = \mathcal{R}[(p_n)]$, $\Phi(q) = \mathcal{R}[(q_n)]$ with this notation. Let $p < q$. This is true if and only if $0 < q - p = 2r$ (and note $2r \in \mathbb{Q}$ since the rationals are closed under addition). And this is true if and only if $q_n - p_n = 2r > 0$ for all $n \in \mathbb{N}$ (when the sequences are as defined above). Since $0 < 2r \in \mathbb{Q} \implies 0 < r \in \mathbb{Q}$, our inequality is true if and only if $q_n - p_n > r$ for all $n \in \mathbb{N}$. But then, this is our definition of $\mathcal{R}[(p_n)] < \mathcal{R}[(q_n)]$ (where $N = 1$), or equivalently, $\Phi(p) < \Phi(q)$. All of our statements imply both directions, so we have proven both directions of the implication.

## Problem 5

*Continue with the notation from Questions 3 and 4. Prove the following:*

*(a). For any $x \in CS(\mathbb{Q})$, exactly one of the following holds:*

$$R[x] < \Phi(0), \qquad R[x] = \Phi(0), \qquad \Phi(0) < R[x]$$

*(b). For each $x$ in $CS(\mathbb{Q})$, there exist $q, r \in \mathbb{Q}$ such that $\Phi(q) < R[x] < \Phi(r)$.*

*(c). For any $x, y \in CS(\mathbb{Q})$ with $R[x] < R[y]$, there exists $q \in \mathbb{Q}$ such that $R[x] < \Phi(q) < R[y]$.*

(a). *Solution.* We first show that at most, only one of the three statements can be true. First, if $\mathcal{R}[x] = \Phi(0)$, then by part (a) and part (c) of Problem 4, we have that $\mathcal{R}[x] < \Phi(0)$ and $\Phi(0) < \mathcal{R}[x]$ do not occur. Thus, we can only have $\mathcal{R}[x] = \Phi(0)$. Now let $\mathcal{R}[x] < \Phi(0)$. This means there exists some $r_1 > 0$ such that for some $N_1 \in \mathbb{N}$, $-x_n > r_1$ for all $n \geq N_1$. For the sake of contradiction, assume we have $\mathcal{R}[x] > \Phi(0)$ as well. Then there exists some $r_2 > 0$ such that for some $N_2 \in \mathbb{N}$, $x_n > r_2$ for all $n \geq N_2$. Taking $r = \min\{r_1, r_2\}$ and $N = \max\{N_1, N_2\}$, we have $-x_n > r$ and $x_n > r$ for all $n \geq N$, but these imply $x_n > r > -r > x_n$ which is a contradiction, by the trichotomy of order for rational numbers. Thus at most one of these is true.

Now we show that they cannot all be false, i.e. at least one is true. First, assume $\mathcal{R}[x] \not< \Phi(0)$ and $\mathcal{R}[x] \not> \Phi(0)$. Let $\varepsilon > 0$ be arbitrary. Then, for all $r > 0 (r \in \mathbb{Q})$ and for all $N \in \mathbb{N}$, there exists $N_1 \geq N$ such that $x_{N_1} \leq r + \frac{\varepsilon}{3}$, and $N_2 \geq N$ such that $-x_{N_2} \leq r + \frac{\varepsilon}{3}$. Let $r = \frac{\varepsilon}{3}$ and $N = 1$ then. Recall that $x$ is Cauchy, so for all $m, n \geq N_1$, $|x_m - x_n| < r$. So $x_m < r + x_{N_1} \leq 2r + \frac{\varepsilon}{3} = \varepsilon$. Furthermore, for all $m, n \geq N_2$, $|x_n - x_m| < r \implies -x_m < r - x_n \leq 2r + \frac{\varepsilon}{3} = \varepsilon \implies x_m > -\varepsilon$. Thus if $M = \max\{N_1, N_2\}$, then for all $m \geq M$, $-\varepsilon < x_m < \varepsilon \implies |x_m| < \varepsilon$, which is sufficent to show that $\lim_{m\to\infty} |x_m - 0| = 0$, thus $\mathcal{R}[x] = \Phi(0)$.

Therefore, we have proven that at least one of our statements must be true, and at most one is true, thus exactly one of them hold.

(b). *Solution.* Recall from the trichotomy that $\mathcal{R}[x] = \Phi(0)$ or $\mathcal{R}[x] > \Phi(0)$ or $\mathcal{R}[x] < \Phi(0)$. If it is the first, since $\Phi(0) < \Phi(1)$ (by part (d) of question 4), and $\Phi(0) > \Phi(-1)$, we must have $\Phi(-1) < \mathcal{R}[x] < \Phi(1)$ (by part (a) of question 4).

Now assume that $\mathcal{R}[x] > \Phi(0)$ (the case when $\mathcal{R}[x] < \Phi(0)$ won't be explicitly shown here, since the logic is identical and one must only change some signs and inequalities). We need now only to find $q \in \mathbb{Q}$ such that $\mathcal{R}[x] < \Phi(q)$. We need that there exists some $0 < r \in \mathbb{Q}$ such that there is some $N \in \mathbb{N}$ where for all $n \geq N$, $q - x_n > r$ or that $q - r > x_n$. But since $x_n$ is Cauchy, we have shown before it is bounded, thus we can find some $q$ and $r$ such that $q - r > x_n$, so we are done.

(c). *Solution.* From the definition of $\mathcal{R}[x] < \mathcal{R}[y]$, there exists $0 < r \in \mathbb{Q}$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $y_n - x_n > r$. But then $y_n > r + x_n > x_n$. Since each of these elements are rational, we have $\Phi(r + x_N)$ will work as our $q \in \mathbb{Q}$.

## Problem 6

*Continue with the notation from Questions 3 and 4. Prove the following:*

*If $x \in CS(\mathbb{Q})$ has $R[x] \neq \Phi(0)$, then there exists $z \in CS(\mathbb{Q})$ for which $R[x \cdot z] = \Phi(1)$.*

*Here $x \cdot z$ denotes the sequence whose $n$th term is $x_n z_n$. (Recall from Assignment 4, Question 6, that $x \cdot z \in CS(\mathbb{Q})$ whenever $x, z \in CS(\mathbb{Q})$.)*

*Solution.* If $\mathcal{R}[x] \neq \Phi(0)$, then part (b) of Problem 3 says that $\mathcal{R}[x] \cap \Phi(0) = \emptyset$. Then $(0, 0, \dots) \in \Phi(0)$ so $(0, 0, \dots) \notin \mathcal{R}[x]$. Then there exists some $\varepsilon' > 0$ such that for all $N_1 \in \mathbb{N}$, if $n \geq N_1$ then $|x_n| > \varepsilon'$. That is to say that, for all $n \geq N_1$, $x_n \neq 0$. Now define $z_n = x_n^{-1}$ for $n \geq N_1$ and $z_n = 2023$ when $1 \leq n < N_1$.

We claim that $z$ is Cauchy. Let $\varepsilon > 0$. Since $x$ is Cauchy, we know that there exists $N_2$ such that for all $m, n \geq N_2$, $|x_m - x_n| \leq \varepsilon\varepsilon'^2$. Let $N = \max\{N_1, N_2\}$. Then for all $m, n > N$:

$$\begin{aligned}
|z_m - z_n| &= \left| \frac{1}{x_m} - \frac{1}{x_n} \right| \\
&= \left| \frac{x_n - x_m}{x_n x_m} \right| \\
&= \frac{|x_n - x_m|}{|x_n||x_m|} \\
&< \frac{\varepsilon\varepsilon'^2}{|x_n||x_m|} \\
&< \frac{\varepsilon\varepsilon'^2}{\varepsilon'^2} \\
&= \varepsilon
\end{aligned}$$

thus $z$ is Cauchy as well. Therefore, we have that $x \cdot z$ is Cauchy from Assignment 4, Question 6 (b).

We now claim that $(1, 1, \dots) \in \mathcal{R}[x \cdot z]$. Let $0 < \varepsilon \in \mathbb{Q}$ be given. Then if $n \geq N$, we have that

$$|1 - x_n \cdot z_n| = |1 - x_n \cdot x_n^{-1}| = |1 - 1| = 0 < \varepsilon$$

which satisifies the definition that $(1, 1, \dots) \in \mathcal{R}[x \cdot z]$. But since $(1, 1, \dots) \in \Phi(1)$, $\Phi(1) \cap \mathcal{R}[x \cdot z] \neq \emptyset$, and from part (b) of Problem 3, we have that $\mathcal{R}[x \cdot z] = \Phi(1)$.

## Problem 7

*[Rudin problem 3.5] For any two real sequences $(a_n)$ and $(b_n)$, prove that the inequality*

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

*holds whenever the right side is not of the form $(+\infty)+(-\infty)$. Give a specific example to show that inequality may hold.*

*Solution.* First consider when $\limsup_{n\to\infty} a_n = +\infty$. Since we need not consider when $\limsup_{n\to\infty} b_n = -\infty$, and we know that $+\infty$ plus any other value is $+\infty$, our right hand side is $+\infty$, which bounds everything above in the extended reals, thus

$$\limsup_{n\to\infty}(a_n+b_n) \leq +\infty = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

A similar argument will also hold true if $a_n = -\infty$.

Now consider when $\limsup_{n\to\infty} a_n \neq +\infty, -\infty$, i.e. $a_n$ is bounded above and below. I'm going to use the Rudin definition of lim sup (which we proved was equivalent to ours in class), which says that

$$\limsup_{n\to\infty} a_n = \sup\{\hat{a} \colon \hat{a} = \lim_{k\to\infty} a_{n_k}\}$$

By Theorem 3.7 in Rudin, which states that the subsequential limits in $\mathbb{R}$ form a closed set, we know that $\sup\{\widehat{a+b} \colon \widehat{a+b} = \lim_{k\to\infty} a_{n_k} + b_{n_k}\} \in \{\widehat{a+b} \colon \widehat{a+b} = \lim_{k\to\infty} a_{n_k} + b_{n_k}\}$. Thus there exists a subsequence $\{n_k\}$ such that $\limsup_{n\to\infty} a_n + b_n = \lim_{k\to\infty} a_{n_k} + b_{n_k}$. We can choose a subsequence $\{n_{k_j}\}$ of $\{n_k\}$, such that $\lim_{j\to\infty} a_{n_{k_j}}$ converges, since $a_{n_k}$ is bounded so there is a convergent subsequence. Since any subsequence of a convergent sequence also converges to the same value,

$$\limsup_{n\to\infty}(a_n+b_n) = \lim_{k\to\infty}(a_{n_k}+b_{n_k}) = \lim_{j\to\infty}(a_{n_{k_j}}+b_{n_{k_j}})$$

Since $a_n$ is bounded above by assumption, so $\lim_{j\to\infty} a_{n_{k_j}}$ is not $\infty$ (and we're not subtracting by $\infty$), we must have that $b_{n_{k_j}}$ converges to

$$\lim_{j\to\infty} b_{n_{k_j}} = \lim_{j\to\infty}(a_{n_{k_j}}+b_{n_{k_j}}) - \lim_{j\to\infty} a_{n_{k_j}}$$

But note that, by the definition of lim sup being the supremum of subsequences, $\limsup_{n\to\infty} a_n \geq \lim_{j\to\infty} a_{n_{k_j}}$ and $\limsup_{n\to\infty} b_n \geq \lim_{j\to\infty} b_{n_{k_j}}$, thus

$$\limsup_{n\to\infty} \leq \limsup n\to\infty a_n + \limsup_{n\to\infty}$$

as desired.

Example for inequality: $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Note that $\limsup_{n\to\infty}(a_n+b_n) = \limsup_{n\to\infty} 0 = 0$. On the other hand, $\limsup_{n\to\infty} a_n = 1$ and $\limsup_{n\to\infty} b_n = 1$ (just consider the subsequence $n$ is even for $a_n$, and odd for $b_n$). Thus we get a strict inequality.

## Problem 8

*Let $X$ denote the collection of all functions $f\colon [0,1] \to \mathbb{R}$ for which the set of real numbers $f([0,1]) = \{f(x)\colon x \in [0,1]\}$ is bounded. For each $f \in X$, define*

$$\|f\| = \sup\{|f(x)|\colon x \in [0,1]\}$$

*Prove that for all real $c$ and all $f, g, h \in X$,*

*(a). $\|cf\| = |c|\|f\|$,*

*(b). $\|f+g\| \leq \|f\| + \|g\|$,*

*(c). $\|f-h\| - \|g-h\| \leq \|f-g\|$.*

*Give an example where (b) holds as a strict inequality ("<").*

(a). *Solution.* We first deal with the case when $c = 0$. Then

$$\begin{aligned}\|cf\| &= \sup\{|cf(x)|\colon x \in [0,1]\}\\ &= \sup\{0\colon x \in [0,1]\}\\ &= 0\end{aligned}$$

And note that since $f([0,1])$ is bounded, there is an upper bound, so $f([0,1])$ has a supremum $\beta \in \mathbb{R}$. But since $0 \cdot \beta = 0$, we have shown $\|cf\| = |c|\|f\|$ when $c = 0$.

Now let $c \neq 0$. We have

$$\begin{aligned}\|cf\| &= \sup\{|cf(x)|\colon x \in [0,1]\}\\ &= \sup\{|c||f(x)|\colon x \in [0,1]\}\end{aligned}$$

We now prove that $|c|\sup\{|f(x)|\colon x \in [0,1]\}$ is also a supremum for $|c||f([0,1])|$. Note that since $|f(x')| \leq \sup\{|f(x)|\colon x \in [0,1]\}$ for all $x' \in [0,1]$ (by definition of a supremum), we have $|c||f(x')| \leq \sup\{|f(x)|\colon x \in [0,1]\}$ for all $x' \in [0,1]$ (since $|c| > 0$). Thus, $|c|\sup\{|f(x)|\colon x \in [0,1]\}$ is an upper bound for $|c||f([0,1])|$. Let $\varepsilon > 0$. By the definition of supremum, we know that there exists $x' \in [0,1]$ such that $\sup\{|f(x)|\colon x \in [0,1]\} - \frac{\varepsilon}{|c|} < |f(x')|$ since $|c| > 0$. But then $|c|\sup\{|f(x)|\colon x \in [0,1]\} - \varepsilon < |c||f(x')|$. Thus, $|c|\sup\{|f(x)|\colon x \in [0,1]\}$ satisifies both our conditions to be the supremum of $|c||f([0,1])|$. But note that a supremum is unique (both are upper bounds and the only way for them to be simultaneously less than or equal to each other is if they are equal). Thus

$$\begin{aligned}|c|\sup\{|f(x)|\colon x \in [0,1]\} &= \sup\{|c||f(x)|\colon x \in [0,1]\}\\ \implies |c|\|f\| &= \|cf\|\end{aligned}$$

Thus we have shown this is true for all real $c$.

(b). *Solution.* Recall that since $|f(x') + g(x')| \leq |f(x')| + |g(x')|$ for every $x' \in [0,1]$ (triangle inequality), we have $|f(x') + g(x')| \leq \sup_{x\in[0,1]}\{|f(x)|\} + |g(x')| \leq \sup_{x\in[0,1]}\{|f(x)|\} + \sup_{x\in[0,1]}\{|g(x)|\}$, thus

$$\begin{aligned}\sup_{x\in[0,1]}\{|f(x)+g(x)|\} &\leq \sup_{x\in[0,1]}\{|f(x)|\} + \sup_{x\in[0,1]}\{|g(x)|\}\\ \implies \|f+g\| &\leq \|f\| + \|g\|\end{aligned}$$

We now give an example when this is a strict inequality: Let $f = 1$ and $g = -1$ for all $x \in [0,1]$. Then

$$\|f+g\| = \|0\| = 0 < 2 = 1 + 1 = \|f\| + \|g\|$$

(c). *Solution.* We let $f - g$ be "$f$" and $g - h$ be "$g$" from part (b). Then,

$$\|f - g + g - h\| \leq \|f-g\| + \|g-h\|$$

This implies

$$\|f-h\| - \|g-h\| \leq \|f-g\|$$