Problem 1

Who are your group members?

Solution. Nicholas Rees

Problem 2

(a). Show that the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

has the unique solution

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 4/2 \\ -1/2 \end{bmatrix}$$

(b). Say that $f\colon \mathbb{R} \to \mathbb{R}$ has $f'''(x)$ existing for all x. Say that $x_0, h \in \mathbb{R}$, and that $|f'''(\xi)| \leq M_3$ for all ξ between x_0 and $x_0 + 2h$. Use the fact that

$$\begin{aligned} f(x_0) &= f(x_0) \\ f(x_0 + h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)M_3 \\ f(x_0 + 2h) &= f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2} f''(x_0) + O(h^3)M_3 \end{aligned}$$

to find a value of c_0, c_1, c_2 such that

$$c_0 f(x_0) + c_1 f(x_0 + h) + c_2 f(x_0 + 2h) = hf'(x_0) + O(h^3)M_3$$

(c). To which formula on page 411 (Section 14.1) of [A&G] are parts (a) and (b) related? Explain.

(d). What is the significance of the solution of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

to approximating $f'(x_0)$? [You don't have to solve this system, just state what you can do with the solution c_0, c_1, c_2, c_3.]

(a). *Solution.* Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$. Note that $\det(A) = 1(4-2) - 0 + 0 = 2 \neq 0$ so A^{-1} exists. One can compute A^{-1} by finding the adjoint matrix and dividing by $\det(A)$, which gives $A^{-1} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix}$ (and one can check $AA^{-1} = A^{-1}A = I$). Thus, multiplying by A^{-1} on both sides, we get

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 2 \\ -1/2 \end{bmatrix}$$

hence c_0, c_1, c_2 must be $-3/2, 4/2, -1/2$, respectively, thus uniquely determining our solution to be the desired one.

(b). *Solution.* We seek to find a linear combination of $f(x_0), f(x_0+h), f(x_0+2h)$ that adds up to only being $hf'(x_0) + O(h^3)M_3$. But this actually corresponds exactly to our linear transformation A from part (a), where we are changing bases from $f(x_0), f(x_0+h), f(x_0+2h)$ to $f(x_0), hf'(x_0), \frac{h^2}{2}f''(x_0) + O(h^3)M_3$, and our equations that dictate this equivalence is represented in A. Hence, our coefficients are the solution to part (a), namely, $c_0 = -3/2$, $c_1 = 4/2$, $c_2 = -1/2$.

(c). *Solution.* Nicholas Rees

(d). *Solution.* Nicholas Rees

Problem 3

Consider an ODE $y' = f(t,y)$, where as in [A&G], $y = y(t)$, and y' refers to dy/dt. Say that $f(t,y)$ is of the special form $f(t,y) = h(t)g(y)$, where g is a differentiable function and h is continuous. Then the ODE

$$y' = dy/dt = h(t)g(y)$$

is called a *separable differential equation*, and it can be solved by writing

$$\frac{dy}{g(y)} = h(t)dt$$

and taking indefinite integrals of both sides. See Section 2.4 (Separable ODE's) of UBC's Calculus 2 Textbook for details, including Example 2.4.2 there, where they solve the equation $y' = y^2$ (in this textbook, y' refers to dy/dx, as is common in math books).

(a). Solve the ODE $y' = y^3$ (here $y = y(t)$ and y' refers to dy/dt) in the same manner as $y' = y^2$ is solved in general form.

(b). Solve $y' = y^3$ for the initial condition $y(1) = 1$.

(c). Solve $y' - y^4$ for the initial condition $y(1) = 1$.

(d). Let $y(t)$ be as in part (b); for $t \geq 1$, when does $y(t)$ fail to exist, i.e., for what $T > 1$ does $y(t) \to \infty$ as $t \to T$?

(e). Same question for part (c).

(a). *Solution.* Nicholas Rees

(b). *Solution.* Nicholas Rees

(c). *Solution.* Nicholas Rees

(d). *Solution.* Nicholas Rees

(e). *Solution.* Nicholas Rees

Problem 4

Let $y(t) = (3-2t)^{-1/2}, z(t) = (4-3t)^{-1/3}$.

(a). Examine a plot of $y(t)$ and $z(t)$ for $1 \leq t < 4/3$. Is one of these functions larger than the other in the entire interval 9as far as the plot shows)? (Here a simple answer will do. You might type `plot (3-2t)^(-1/2) and (4-3t)^(-1/3)` into Google, or something like that.)

(b). Show that $y' = y^3$ and $z' = z^4$ and that $y(1) = z(1) = 1$.

(c). Show that $y'(1) = z'(1) = 1$.

(d). Show that $y''(1) = 3$ and $z''(1) = 4$. [Hint: differentiate both sides of $y' = y^3$; similarly for z.]

(e). Show that for $h > 0$ and h sufficiently small we have $y(1+h) < z(1+h)$. [Hint: let $u(t) = z(t) - y(t)$; what are the values of $u(1), u'(1), u''(1)$?]