Math 321 Homework 6

Problem 1

Let $\{f_n\}$ and f be functions from $[0,1] \to \mathbb{R}$. Suppose that f_n, f have bounded variation on $[0,1]$. Define $g_n(x) = TV[f_n|_{[0,x]}]$ and $g(x) = TV[f|_{[0,x]}]$ (recall Homework 3 for the relevant defintions).

(a). Suppose that $f_n \to f$ pointwise. Is it true that $g_n \to g$ pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

(b). Suppose that $f_n \to f$ uniformly. Is it true that $g_n \to g$ uniformly? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

(c). Suppose that $g_n \to g$ pointwise. Is it true that $f_n \to f$ pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

(a). *Solution.* Consider the functions

$$f_n(x) = \begin{cases} n & x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

We claim that $f_n \to 0$ pointwise. Let $x \in [0,1]$, and $\varepsilon > 0$. If $x = 0$, we are done. If $x \neq 0$, note that Archimedes gives us an $N \in \mathbb{N}$ such that $Nx > 1 \implies x > \frac{1}{N} \geq \frac{1}{n}$ for any $n \geq N$. Hence, for all f_n where $n \geq N$, $|f_n(x)| = 0 < \varepsilon$. Hence, $f_n \to 0$.

We now show that f_n and $f = 0$ both have bounded variation on $[0,1]$. Clearly, f has bounded variation, and fruthermore, $g(x) = 0$ (this follows directly from the definition: $\sum_i \Delta f_i = \sum 0 = 0$). Now consider f_n. If $P = \{x_1, \ldots, x_l\}$ is a partition of $[0,1]$, then if P contains no point in $(0, \frac{1}{n})$, then all $f(x_i) = 0$ and so $V(f_n, P) = \sum_i |f(x_i) - f(x_{i-1})| = \sum_i 0 = 0$. If P does contain points in $(0, \frac{1}{n})$, then we must have $x_1 \in (0, \frac{1}{n})$, and some $s \in \mathbb{N}_{\leq l}$ such that $x_s \geq \frac{1}{n}$ but $x_{s-1} < \frac{1}{n}$. Then $V(f, P) = \sum_i |f(x_i) - f(x_{i-1})| = |f(x_1) - f(x_0)| + \sum_{i=2}^{s-1} |f(x_i) - f(x_{i-1})| + |f(x_s) - f(x_{s-1})| + \sum_{i=s+1}^{l} |f(x_i) - f(x_{i-1})| = n + 0 + n + 0 = 2n$. Then $TV[f_n] = 2n$, and so it has bounded variation. Note that if we restrict our domain, our analysis is the same as above to conlude that $TV[f_n|_{[0,x]}] = 2n$ when $0 < x \leq 1$, and $V[f_n|_{[0,0]}] = 0$ by definition. Hence, $g_n(x) = \begin{cases} 0 & x = 0 \\ 2n & 0 < x \leq 1 \end{cases}$. This does not converge pointwise to $g(x)$ at $x = 1$: for all $n \in \mathbb{N}$, $|g_n(1) - g(1)| \geq 1$, and so they never get within $\varepsilon = \frac{1}{2}$. Hence, we have $f_n \to f$ pointwise and both of bounded variation, but g_n does not converge pointwise to g.

(b). *Solution.* Consider the functions

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in \{\frac{j}{n+1} : j \in \mathbb{N}_{<n}\} \\ 0 & \text{otherwise} \end{cases}$$

We claim that $f_n \to 0$ uniformly. Let $\varepsilon > 0$. Archimedes gives us some $N \in \mathbb{N}$ such that $N\varepsilon > 1 \implies \varepsilon > \frac{1}{N} \geq \frac{1}{n}$ for all $n \geq N$. Then for all $n \geq N, x \in [0,1]$, $|f_n(x) - 0| \leq \frac{1}{n} < \varepsilon$. So $f_n \to 0$ uniformly.

We know that $f = 0$ has bounded variation, so we now show that f_n has bounded variation. Consider some partition $P = \{x_1, \ldots, x_l\}$. Clearly, $V(f, P)$ is maximized when P contains all of $\{\frac{j}{n+1} : j \in \mathbb{N}_{<n}\}$ and there exists some x_i between every $\frac{j}{n}, \frac{j+1}{n}$ (and the end points), since the only positive terms in the sum $\sum_i |f(x_i) - f(x_{i-1})|$ are all equal (with a value of $\frac{1}{n}$) and are precisely those where one of x_i or x_{i-1} are in $\{\frac{j}{n+1} : j \in \mathbb{N}_{\leq n}\}$, and the other is not. There is at most $2n$ of such terms, so $\sup_P V(f, P) = ((2n)\frac{1}{n} = 2$. Hence, $TV[f_n] = 2$ for all n.

To show that g_n doesn't converge to g uniformly, it is sufficient to show that g_n doesn't even converge to g pointwise at some point, say $x = 1$. Clearly, $g(x) = 0$, see part (a) of this problem, and $g_n(1) = TV[f_n] = 2$. Hence, $|g_n(1) - g(1)| = 2$ for all $n \in \mathbb{N}$, and so they never get within $\varepsilon = 1$, meaning $g_(1) \not\to g(1)$. Thus, we have $f_n \to f$ and both of bounded variation, but g_n does not converge uniformly to g.

(c). *Solution.* Consider $f_n(x) = 1$ and $f(x) = 0$. This gives $g_n(x) = 0$ and $g(x) = 0$, identical to our method for $f(x) = 0$ from part (a) of this problem. Hence, we have $g_n = g$ for all $n \in \mathbb{N}$, and so $|g_n(x) - g(x)| = 0 < \varepsilon$ for any $\varepsilon > 0$ and $x \in [0, 1]$, thus $g_n \to g$. However, clearly f_n does not converge to f, as $|f_n - f| = 1$ for all $n \in \mathbb{N}$ and so they never get within $\varepsilon = \frac{1}{2}$. Thus, $g_n \to g$ pointwise, but f_n does not converge pointwise to f.

Problem 2

For $n \in \mathbb{N}$, let $f_n \colon [-1, 1] \to [0, \infty)$ be: (i) continuous, (ii) obey $\int_{-1}^{1} f_n(x)dx = 1$, and (iii) be such that f_n converges to 0 uniformly on $[-1, -c] \cup [c, 1]$ for every $c \in (0, 1)$. Suppose $g \colon [-1, 1] \to \mathbb{R}$ is bounded, Riemann integrable, and continuous at 0. Prove that $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)g(x)dx = g(0)$.
Hint: $g(0) = \int_{-1}^{1} f_n(x)g(0)dx$.

Solution. Given the hint, we have the following to be equivalent if the limit exists:

$$\begin{aligned}\lim_{n\to\infty} \int_{-1}^{1} f_n(x)gxdx = g(0) &\iff \lim_{n\to\infty} \int_{-1}^{1} f_n(x)gxdx - \int_{-1}^{1} f_n(x)g(0)dx = 0 \\ &\iff \lim_{n\to\infty} \int_{-1}^{1} f_n(x)(g(x) - g(0))dx = 0\end{aligned}$$

where we can bring the integral inside the limit because it is just a constant (even with a f_n term, the value is a constant value), and we can bring the second term inside the integral by Rudin 6.12(a).

Let $\varepsilon > 0$ be given. Since g is continuous, there is some $0 < \delta < 1$ such that $|x - 0| \le \delta$ implies that $|g(x) - g(0)| < \frac{\varepsilon}{3}$. Since g is bounded, we have some $M \ge 0$ such that $|g(x)| \le M$ for all $x \in [-1, 1]$. Since $f_n \to f$ uniformly converges on $[-1, -\delta]$, Rudin 7.16 gives us an $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, we have

$$\left| \int_{-1}^{-\delta} f_n(x)dx - 0 \right| < \frac{\varepsilon}{6M}$$

Similarly, since $f_n \to f$ uniformly on $[\delta, 1]$, 7.16 gives an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, we have

$$\left| \int_{\delta}^{1} f_n(x)dx - 0 \right| < \frac{\varepsilon}{6M}$$

Let $N = \max\{N_1, N_2\}$.

Rudin 6.13 tells us that since $g(x) - g(0) \in \mathcal{R}[-1, 1]$ (since $g(x)\ in \mathcal{R}[-1, 1]$), then so is $|g(x) - g(0)|$ and

$$\left| \int_{-1}^{1} f_n(x)(g(x) - g(0))dx \right| \le \int_{-1}^{1} |f_n(x)||g(x) - g(0)|dx = \int_{-1}^{1} f_n(x)|g(x) - g(0)|dx$$

Finally, Rudin 6.12(c) lets us split up the integral, to see that when $n \ge N$, we have

$$\begin{aligned}\left| \int_{-1}^{1} f_n(x)(g(x) - g(0))dx \right| &\le \int_{-1}^{1} f_n(x)|g(x) - g(0)|dx \\ &= \int_{-1}^{-\delta} f_n(x)|g(x) - g(0)|dx + \int_{-\delta}^{\delta} f_n(x)|g(x) - g(0)|dx + \int_{\delta}^{1} f_n(x)|g(x) - g(0)|dx \\ &\le \int_{-1}^{-\delta} f_n(x)2Mdx + \int_{-\delta}^{\delta} f_n(x)\frac{\varepsilon}{3}dx + \int_{\delta}^{1} f_n(x)2Mdx \\ &\le 2M\frac{\varepsilon}{6M} + \frac{\varepsilon}{3}\int_{-1}^{1} f_n(x)dx + 2M\frac{\varepsilon}{6M} \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

(where expanding the domain of our integral was an upper bound, since f is always nonnegative, and 6.12(b) and (c) gives us the bound we want by comparing f and $f' = f$ on $[-\delta, \delta]$ and 0 everywhere else). Therefore, we have shown

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)(g(x) - g(0))dx = 0$$

as desired.

Problem 3

Let $c \in \mathbb{R}$. For each $n \in \mathbb{N}$ and $x \in [0,1]$, define $f_n(x) = n^c x^3 (1-x^4)^n$.

(a). Prove that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$ and determine the limit (you should justify any steps in your computation).

(b). Determine the values of c for which the convergence in part (a) is uniform. Prove that your answer is correct.

(c). For what values of c do we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx?$$

Prove that your answer is correct.

(a). *Solution.* Let $x \in [0,1]$ be fixed. If $x = 0, 1$, then $f_n(x) = 0$ for all n, and so $\lim_{n\to\infty} f_n(x)$ exists and is 0. Now assume that $x \neq 0, 1$. Note that since $0 < (1-x^4) < 1$, we have $1 < (1-x^4)^{-1}$, and so there is some $p > 0$ (specifically $p = (1-x^4)^{-1} - 1$) such that $(1-x^4)^{-1} = 1 + p$ and so

$$(1-x^4)^n = ((1-x^4)^{-1})^{-n} = \frac{1}{(1+p)^n}$$

Thus, we can compute the limit

$$\lim_{n\to\infty} n^c x^3 (1-x^4)^n = x^3 \lim_{n\to\infty} \frac{n^c}{(1+p)^n} = x^3(0) = 0$$

where we apply Rudin 3.3(b) and Rudin 3.20(d) (where $c = \alpha$). Hence, we have shown $f(x) = \lim_{n\to\infty} f_n(x)$ exists and is equal to 0 for all $x \in [0,1]$.

(b). *Solution.* It is sufficient to show that $M_n = \sup_x |f_n(x)| \to 0$ as $n \to \infty$ by Rudin 7.6.

Note that since $f_n(x)$ is continuous (product of continuous functions) on a compact domain, $f_n(x)$ attains its extrema. Math 100 gives us the first derivative test: consider

$$\begin{aligned} f_n'(x) &= n^c(3x^2(1-x^4)^n - x^3 n(1-x^4)^{n-1} 4x^3) \\ &= n^c x^2 (1-x^4)^{n-1}(3(1-x^4) - 4nx^4) \\ &= n^c x^2 (1-x^4)^{n-1}(3 - (3+4n)x^4) \end{aligned}$$

The extrema of $f_n'(x)$ hence occur precisely when $f_n'(x) = 0$, or when $x = 0, 1, \left(\frac{3}{3+4n}\right)^{1/4}$. But recall that $f_n(0) = f(1) = 0$, but $f_n(0.1) = n^c(0.1)^3(1-0.1^4)^n > 0$ since it is three positive factors multiplied to each other. Hence, $M_n = f\left(\left(\frac{3}{3+4n}\right)^{1/4}\right)$. So it is sufficient for us to consider the condition on c for when $\lim_{n\to\infty} f\left(\left(\frac{3}{3+4n}\right)^{1/4}\right) = 0$.

We can plug in our value:

$$\begin{aligned} f_n\left(\left(\frac{3}{3+4n}\right)^{1/4}\right) &= n^c\left(\frac{3}{3+4n}\right)^{3/4}\left(1-\frac{3}{3+4n}\right)^n \\ &= n^c\left(\frac{3}{3+4n}\right)^{3/4}\left(\frac{3+4n-3}{3+4n}\right)^n \\ &= n^{c-3/4}\left(\frac{3n}{3+4n}\right)^{3/4}\left(\frac{1}{3/4n+1}\right)^n \end{aligned}$$

Using the appropriate theorems from Rudin Chapter 3 (3.3,3.31), we can conlude

$$\begin{aligned} \lim_{n\to\infty} M_n &= \left(\lim_{n\to\infty} n^{c-3/4}\right)\left(\lim_{n\to\infty}\left(\frac{3}{4+3/n}\right)^{3/4}\right)\left(\lim_{n\to\infty}\left(\frac{1}{3/4n+1}\right)^n\right) \\ &= \left(\lim_{n\to\infty} n^{c-3/4}\right)\left(\frac{3}{4}\right)^{3/4} e^{-3/4} \end{aligned}$$

Hence, our function uniformly converges if and only if $\lim_{n\to\infty} n^{c-3/4} = 0$. This only occurs when $c - \frac{3}{4} < 0$ (Rudin 3.20), therefore our requirement is that $c \in (-\infty, 3/4)$.

(c). *Solution.* Let $F_n(x) = -\frac{n^c}{4}\left(\frac{(1-x^4)^{n+1}}{n+1}\right)$. Since $F'(x)$ is differentiable (polynomials are differentiable) and $F_n'(x) = f_n(x)$, $f_n(x) \in \mathcal{R}[0,1]$ since it is continuous becuase it is the product of continuous functions, the fundamental theorem of calculus gives us

$$\int_0^1 f_n(x)dx = F_n(1) - F_n(0) = \frac{n^c}{4(n+1)}$$

We need this value to go to 0 as $n \to \infty$, since $\int_0^1 f(x)dx = \int_0^1 0dx = 0$.

We have $0 < \frac{n^c}{4(n+1)} < \frac{n^{c-1}}{4}$. When $c < 1$, the fraction goes to 0. Otherwise, it does not. Hence, we require $c \in (-\infty, 1)$.