Problem 1

(a). *Solution.* This is not true, we provide the counterexample

$$(x_n) = \begin{cases} 2n, & \text{if } n \text{ even} \\ n, & \text{if } n \text{ odd} \end{cases}$$

Note that $x_n \to +\infty$ still, since if $M \in \mathbb{R}$ is arbitrary, then we can let $N = \max\{M + 1 < 1\}$, and then for any $n \geq N$, $(x_n) > M$. but note that if n is even, then $x_n = 2n > n + 1 = x_{n+1}$. Thus, the hypothesis is true, but the conclusion is false, disproving the claim.

(b). *Solution.* This is true. We will prove this through construction of the subsequence. We do so with induction on k. Let $n_1 = 1$, so x_1 is the first element in the subsequence. Now, let an arbitrary term in the subsequence x_{n_k}, be given, where $k' \geq 1$. Since $x_n \to +\infty$, if $M = x_{n_{k'}}$, we know that there exists an N such that for all $n > N$, $x_n \geq x_{n_{k'}}$. Certainly, $x_{N+1} \geq x_{n_{k'}}$, thus let $n_{k'+1} = N + 1$. Thus $x_{n_{k'+1}} \geq x_{n_{k'}}$. Since this is true for any $k' \geq 1$ by induction, we hvae shown that there exists a subsequence such that $x_{n_k} \leq x_{n_{k+1}}$ for all k.

Problem 2

(a). *Solution.* See photo

(b). *Solution.* We seek to prove that b_n is not Cauchy, which implies that it does not converge. Let $\varepsilon = 1$. Then for all $N \in \mathbb{N}$, there exist $m, n \geq N$, which we can explicitly set $n = N, m = N + 1$, where

$$\begin{aligned}
|b_m - b_n| &= \left| \frac{(-1)^{N+1}(N+1)}{N+2} - \frac{(-1)^N N}{N+1} \right| \\
&= \left| (-1)^{N+1} \left(\frac{N+1}{N+2} + \frac{N}{N+1} \right) \right| \\
&= \left| \frac{(N+1)^2 + N(N+2)}{(N+1)(N+2)} \right| \\
&> \left| 2N^2 + 4N + 1 \right| \\
&> N \geq 1 = \varepsilon
\end{aligned}$$

where the third step is true since $(N + 1), (N + 2) > 1$ since $N \geq 1$, and similarly with the final step. Thus, our sequence is not Cauchy, therefore it does not converge.

Problem 3

(a). *Solution.* Let $\beta = \sum(A)$ Recall that for $\sum(A)$ to be the supremum of $\{\sum(F) \colon F = \text{ finite subset of } A\}$, we have that for all $F \subset A$, we have that $\beta \geq \sum(F)$, and that for any $\varepsilon > 0$, there exists $F \subset A$ such that $\beta - \varepsilon < \sum(F)$.

Now, define F_j to be a finite subset in A such that

$$\beta - \frac{1}{j} < \sum(F_j) \leq \beta$$

Note that $\sum(F_{j_1} \cup F_{j_2}) \geq \max\{\sum(F_{j_1}), \sum(F_{j_2})\}$, since all elements in A are nonnegative, so adding more elements to a set will increase the sum (or keep it the same if adding 0 or an element already in the original set), and so adding all of F_{j_2} to F_{j_1} will increase the sum of F_{j_1} by all of the elements in F_{j_2} but not in F_{j_1}. This is true for any number of unions, since we are always just adding elements, and at a minimum, the sum of the unioned set will be as small as the largest sum of one of the finite set.

Consider the set $\bigcup_{j \in \mathbb{N}} F_j$ of all such finite sets as defined above. We claim that $A = \bigcup_j F_j$. Let $f \in \bigcup_j F_j$. Then for some j, $f \in F_j$. But by definition, this is a subset of A, thus $f \in A$. Thus, $\bigcup_j F_j \subseteq A$. Now let $a \in A$. If $a = 0$, ff For the sake of contradiction, let $a \notin \bigcup_j F_j$. We then have $\sum(\{a\} \cup \bigcup_j F_j) = a + \sum(\bigcup_j F_j)$, since if we take the sum of all elements, we can just add a to it, and since a is positive, it will increase the sum. Thus

$$\beta \geq \sum(\{a\} \cup \bigcup_j F_j) = a + \sum(\bigcup_j F_j)$$

But this implies

$$\beta - a \geq \sum\left(\bigcup_j F_j\right)$$

But recall that for any $j \in \mathbb{N}$, $\beta - \frac{1}{j} < \sum(F_j) \leq \sum(\bigcup_j F_j)$, and by the Archimedean property, we can always find j such that $0 < \frac{1}{j} < a$, thus we have $\beta - a < \beta - \frac{1}{j} < \sum(\bigcup_j F_j)$. Thus, we have a contradiction. So $a \in \bigcup_j F_j$ and $A \subseteq \bigcup_j F_j$. Therefore, $A = \bigcup_j F_j$. And since $\bigcup_j F_j$ is an at most countable union of finite sets, which we know is at most countable, A must be at most countable.

(b). *Solution.* ff

Problem 4

(a). *Solution.* Fix an arbitrary element y' in Y. By definition of the infinum, we know that

$$W_2(y') \leq f(x, y')$$

for all $x \in X$. Since at least one element in $\{f(x, y') \colon y' \in Y\}$ is an upper bound for $W_2(y)$, and $\sup_Y f(x, y)$ is an upper bound for $\{f(x, y') \colon y' \in Y\}$, we have

$$\sup_Y W_2(y) \leq \sup_Y f(x, y) = M_1(x)$$

The value on the left is a lower bound for all $x \in X$, and the greatest lower bound is greater than or equal to any lower bound, thus

$$\sup_Y W_2(y) \leq \inf_X M_1(x)$$

as desired.

(b). *Solution.* We let $X, Y = \{0, 1\}$, and define $f \colon X \times Y \to \mathbb{R}$ as follows

$$\begin{aligned} f(0, 1) &= 1 \\ f(1, 0) &= 2 \\ f(0, 0) &= 3 \\ f(1, 1) &= 4 \end{aligned}$$

Note that

Problem 5

Solution. ff

Problem 6

(a). *Solution.* :(

(b). *Solution.* :(

(c). *Solution.* :(

Problem 7

(a). *Solution.* See photo

(b). *Solution.* See photo

Problem 8

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff