Problem 1

Prove the following theorem (terminology is given below):

Suppose X is compact and $f\colon X \to \mathbb{R}$ is lower semicontinuous. Then f is bounded below on X, and there exists a point $z \in X$ satisfying $f(z) \leq f(x)$ for all $x \in X$.

Recall that in a HTS $(X, \mathcal{T})$, a function $f\colon X \to \mathbb{R}$ is called lower semicontinuous *if the following set is closed for every $p \in \mathbb{R}$:*

$$f^{-1}((-\infty, p]) = \{x \in X\colon f(x) \leq p\}.$$

(One approach uses the family of closed sets $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$.)

Solution. Consider the family of closed sets of $f^{-1}((-\infty, p])$ satisfying $p > \inf f(x)$, call it $\mathcal{F}$. First, remark that each element in $\mathcal{F}$ is nonempty, otherwise $f^{-1}((-\infty, p])$ is empty, thus there is no $x_0 \in X$ where $f(x_0) \in (-\infty, p]$ and so $p \leq \inf f(x)$, which we assumed not true. Secondly, by the assumption that f is lower semicontinuous, each element in $\mathcal{F}$ is also closed. Finally, note that $\mathcal{F}$ has the finite intersection property: let $N \in \mathbb{N}$ and $F_1, \ldots, F_N$ are sets in $\mathcal{F}$, which we can write explicitly as $F_i = f^{-1}((-\infty, p_i])$ where $p_i > \inf f(x)$; denote $p_0 = \min_i\{p_i\}$. Then $F_0 = f^{-1}((-\infty, p_0]) \subseteq F_i$ for all $1 \leq i \leq N$, and since we're just minimizing over a finite number of sets, $F_0 \in \{F_1, \ldots, F_n\} \subseteq \mathcal{F}$, thus

$$\bigcap_{i=1}^{N} F_i = f^{-1}((-\infty, p_o]) = F_0 \neq \emptyset$$

so we have the finite intersection property.

Now, since we're in a a HTS and X is compact, any collection of elements of $\mathcal{F}$ has nonempty intersection, by the theorem proven in class (every element is a subset of X and are closed, and any finite collection has the finite intersection property). Notably, $\bigcap \mathcal{F} \neq \emptyset$. This means that there exists some $z \in X$ where $z \in \bigcap \mathcal{F}$. Then, for all $p > \inf f(x)$, we have $z \in f^{-1}((-\infty, p])$. If $x \in X$, then $z \in f^{-1}((-\infty, f(x)])$, thus $f(z) \leq f(x)$. Therefore, f is bounded below on X, specifically by $f(z)$ where $z \in X$, since $f(z) \leq f(x)$ for all $x \in X$.

Problem 2

Let (X, d) be a metric space, with $K \subseteq X$ a compact set. Prove that whenever $\mathcal{G}$ is an open cover for K, there exists $r < 0$ with this property: for every pair of points $x, y \in K$ obeying $d(x, y) < r$, some open set $G \in \mathcal{G}$ contains both x and y.

Solution. For the sake of contradiction, assume that for all $r > 0$, there are some $x_r, y_r \in K$ such that $d(x_r, y_r) < r$ but for any $G \in \mathcal{G}$, x_r, y_r are not both in G. Note that Since $\mathcal{G}$ covers K, we have that both x_r, y_r are in some $G \in \mathcal{G}$, so have $x \in G_x$ and $y \in G_y$ but $G_x \neq G_y$.

Hmm, have a two sequences go towards each other.

TFAE: K is compact and every sequence (x_n) in K has a convergent subsequence whose limit lies in K.

TFAE: A is an open set and for every $x \in A$ and every sequence (x_n) obeying $x_n \to x$, one has $x_n \in A$ for all n sufficiently large.

Evan has a solution ff

Problem 3

Define the set-valued "projection" mapping $p_1\colon \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R})$ *by*

$$p_1(S) = \{x_1 \in \mathbb{R}\colon (x_1, x_2) \in S \text{ for some } x_2\}, \qquad S \subseteq \mathbb{R}^2$$

(a). If S is bounded, must $p_1(S)$ be bounded? (Why or why not?)

(b). If S is closed, must $p_1(S)$ be closed? (Why or why not?)

(c). If S is compact, must $p_1(S)$ be compact? (Why or why not?)

(a). *Solution.* It must. If S is bounded, then by definition, there exists $x \in S$ and $R > 0$ such that $S \subseteq \mathbb{B}[x; R)$. Using the the standard metric on $\mathbb{R}^2$ (namely $d(x,y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$), this means for any $y \in S$, we have $d(x,y) < r$, or $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < R$. Consider $x_1 = p_1(x)$. Then for any $y_1 \in p_1(S)$ (using the standard metric on $\mathbb{R}$, $d(x,y) = |y - x|$), we have

$$d(x_1, y_1) = |y_1 - x_1| = \sqrt{(y_1 - x_1)^2} \leq \sqrt{(y_1 - x_1)^2 + (y' - x_2)^2} < R$$

where $y' \in p^{-1}(y_1)$, and so the last inequality follows from the boundedness of S. Thus, $p_1(S) \subseteq \mathbb{B}[x_1; R)$, so $p_1(S)$ is bounded.

(b). *Solution.* This is not true. We provide the counter-example $S = \{(2^{-n}, 2^n) \in \mathbb{R}^2 \colon n \in \mathbb{N}\}$.

We first prove that S is closed. Note that $S' = \emptyset$. To see this, for the sake of contradiction, let $s \in S'$. Then for some sequence s_n of distinct elements of S, we have $\lim_{n\to\infty} s_n = s$ (by the proposition proven in class). Unraveling the definition of the limit, this means that for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ where $\forall n \geq N$, we have $d(s, s_n) < \varepsilon$. For the sake of contradiction, assume that this is true; then let $\varepsilon = \frac{1}{2}$, which gives us some N where $d(s, s_n) < \frac{1}{2}$ when $n \geq N$. But note that for any $s_n, s_{n+1} \in S$, since $s_n \neq s_{n+1}$, we have that $d(s_n, s_{n+1}) > 2$ (by construction, since $2 \leq 2^{n+1} - 2^n = y_{s_{n+1}} - y_{s_{n+1}} = \sqrt{(y_{s_{n+1}} - y_{s_{n+1}})^2} \leq \sqrt{(y_{s_{n+1}} - y_{s_{n+1}})^2 + (x_{s_{n+1}} - x_{s_{n+1}})^2} = d(s_n, s_n + 1)$). Thus $2 \leq d(s_{n+1}, s_n) \leq d(s, s_n) + d(s, s_{n+1}) \leq \frac{1}{2} + d(s, s_{n+1}) \implies \frac{3}{2} < d(s, s_{n+1})$. But this contradicts our assumption, since $n + 1 > n \geq N$, but $d(s, s_{n+1} > \frac{3}{2} > \frac{1}{2} = \varepsilon$. Thus, there are no limit points of S, so $S' = \emptyset$.

Now recall the theorem proven in class, $\overline{S} = S \cup S'$. Since $S' = \emptyset$, this leaves us $\overline{S} = S$. But recall that this is true only if S is closed.

We now prove that $p_1(S)$ is not closed. Note that $p_1(S) = \{2^{-n} \colon n \in \mathbb{N}\}$. See that $0 \in p_1(S)'$ but $0 \notin p_1(S)$. The second of these is obvious, $0 < 2^{-n}$ for all $n \in \mathbb{N}$. To see that 0 is a limit point, we have $\lim_{n\to\infty} 2^{-n} = 0$ (obviously, we are in $\mathbb{R}$), and $2^{-n} \in p_1(S)$ are distinct points, thus $0 \in p_1(S)'$ (by our proposition in metric spaces). Thus, $p_1(S) \neq p_1(S) \cup p_1(S)' = \overline{p_1(S)}$. But this is true only if $p_1(S)$ is not closed. Hence, S is closed but $p_1(S)$ is not closed.

(c). *Solution.* Now, assume that S is compact. Consider an open cover $\mathcal{G}$ of $p_1(S)$. If $G \in \mathcal{G}$, extract the "open column" corresponding to G, namely $G^2 = \{(x,y) \in \mathbb{R}^2 \colon x \in G, y \in \mathbb{R}\}$. Notice two things: each G^2 is open, and $\mathcal{G}^2$, the collection of all G^2 such that $G \in \mathcal{G}$, covers S.

To see the first, let $g \in G^2$ and denote $g = (x_0, y_0)$. Since $x_0 \in G$, we have that there exists r such that $\mathbb{B}_1[x_0, r) = \{x \in \mathbb{R} \colon |x_0 - x| < r\} \subseteq G$ by the fact that G is open. We claim that $\mathbb{B}_2[g, r) \subset G^2$ as well; for the sake of contradiction, assume there is $(x_1, y_1) \in \mathbb{B}_2[g, r)$ but $(x_1, y_1) \notin G^2$. Then $x_1 \notin G$ so $x_1 \notin \mathbb{B}_1[x_0, r)$, which give us $|x_0 - x_1| \geq r$. But then $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \geq \sqrt{r^2 (y_1 - y_0)^2} \geq \sqrt{r^2} = r$, which contradicts the fact that $(x_1, y_1) \in \mathbb{B}_2[g, r)$. Thus, $\mathbb{B}_2[g, r) \subset G^2$, and since $g \in G$ was arbitrary, this proves that G^2 is open.

To see the second claim, that $\mathcal{G}^2$ covers S, let $s = (x_0, y_0) \in S$. Then since $\mathcal{G}$ covers $p_1(S)$, we have $p_1(s) = x_0 \in G$ for some $G \in \mathcal{G}$. Then $(x_0, y_0) \in G^2$ by definition, and $G^2 \in \mathcal{G}^2 \subseteq \bigcup_{G^2 \in \mathcal{G}^2} G^2$. Thus $s \in \bigcup_{G^2 \in \mathcal{G}^2} G^2$, so $\mathcal{G}^2$ covers S since $s \in S$ was arbitrary.

Hence, $\mathcal{G}^2$ is an open cover of S. By the compactness of S, there exists a finite subcover of $\mathcal{G}^2$, which we can denote as $G_1^2, G_2^2, \ldots, G_N^2$. So $S \subseteq \bigcup_i G_i^2$. But recall that each $G_i^2 \in \mathcal{G}^2$ had some corresponding $G \in \mathcal{G}$ by definition (recall that the x-values of each G^2 were determined by some G), so we have a finite collection of open sets $G_1, G_2, \ldots, G_N \in \mathcal{G}$. We prove that this is a finite subcover of $p_1(S)$. Let $x_0 \in p_1(s)$. Then $\exists y_0$ such that $(x_0, y_0) \in S$. Then $(x_0, y_0) \in G_j^2$ for some $1 \leq j \leq N$. But then by definition of G_j^2, $x_0 \in G_j$. Therefore, since $x_0 \in p_1(S)$ was arbitrary, $p_1(S) \subseteq \bigcup_i G_i$, hence $G_1, \ldots, G_N \in \mathcal{G}$ is a finite subcover, thus $p_1(S)$ is compact.

Problem 4

Recall the set ℓ^2 from HW07 Q3, and the standard "unit vectors" $\hat{\mathbf{e}}_p = (0, 0, \dots, 0, 1, 0, \dots)$, where the only nonzero entry in $\hat{\mathbf{e}}_p$ occurs in component p. For any x in ℓ^2 and subset $V \subseteq \ell^2$, write

$$\Omega(x; V) = \{y \in \ell^2 \colon -1 < \langle v, y - x\rangle < 1, \forall v \in V\}.$$

Then define a collection $\mathcal{T}$ of subsets of ℓ^2 by saying $G \in \mathcal{T}$ if and only if every point $x \in G$ has the property that $x \in \Omega(x; V) \subseteq G$ for some finite set $V \subseteq \ell^2$.

(a). *Prove that $\Omega(x; V) \in \mathcal{T}$ for every finite set $V \subseteq \ell^2$ and point $x \in \ell^2$.*

(b). *Prove that $(\ell^2, \mathcal{T})$ is a Hausdorff Topological Space.*

(c). *Let $S = \{\hat{\mathbf{e}}_p \colon p \in \mathbb{N}\}$. Prove that $0 \in S'$. (Here 0 denotes $(0, 0, \dots)$, the "origin in ℓ^2.)* Note: *This fact proves that $\mathcal{T}$ is different from the metric topology on ℓ^2.*

(d). *Prove that every G in $\mathcal{T}$ has the property: for every x in G, there exists $r > 0$ such that*

$$G \supseteq \mathbb{B}[x; r) = \{y \in \ell^2 \colon \|y - x\| < r\}.$$

This fact proves that every set considered "open" in $\mathcal{T}$ is also open in the metric topology on ℓ^2. This explains why $\mathcal{T}$ gets called "the weak topology" and the metric topology is also called "the strong topology."

(e). *Prove that the following set is closed in the weak topology of ℓ^2: $\mathbb{B}[0; 1] = \{y \in \ell^2 \colon \|y\| \leq 1\}$.*

(a). *Solution.* Let $x' \in \Omega(x; V)$. Want to show there exists a finite set $V' \subseteq \ell^2$ such that $\Omega(x'; V') \subseteq \Omega(x, V)$. Then $-1 < \langle v, x' - x\rangle < 1$ for all $v \in V$. This is equivalent to

$$-1 < \sum_{n=1}^{\infty} v_n(x'_n - x_n) < 1$$

for all $v \in V$. Let v' be the sequence $v_n(y_n - x_n) = v'_n(y_n - x'_n)$, $v_n y_n - v_n x_n = v'_n y_n - v'_n x'_n$ if $v'_n = v_n x_n / x'_n$ we get $v_n y_n = v_n y_n x_n / x'_n$.

Let v' be the sequence defined by $v'_n = v_n x_n / x'_n$.

Let $y \in \Omega(x'; V')$. Then

$$-1 < \sum_{n=1}^{\infty} v'_n(y_n - x'_n) < 1$$

for all $v' \in V'$. Then

$$-1 < \sum_{n=1}^{\infty} v_n(y_n x_n / x'_n - x_n) < 1$$

ff

Actually, let $v'_n = v_n(x'_n - x_n)$. It is clear that $v' \in \ell^2$, since ff

Or can use Cauchy-Schwartz: $|\langle v, y - x\rangle| \leq \|v\| \|y - x\|$ ff

(b). *Solution.* We have that $\emptyset \in \mathcal{T}$, since there does not exist $x \in \emptyset$ so it satisfies our condition to be in $\mathcal{T}$ vacuously. We also have $\ell^2 \in \mathcal{T}$, since $\Omega(x; V)$ is composed of elements of ℓ^2, and so for any $x \in \ell^2$, $\Omega(x; V) \subseteq \ell^2$.

Now consider $\mathcal{G} \subseteq \mathcal{T}$. Consider an arbitrary element $x \in \bigcup \mathcal{G}$. Then for some $G \in \mathcal{G}$, we have $x \in G$. Then $\Omega(x; V) \subseteq G \subseteq \bigcup \mathcal{G}$ for some finite set $V \subseteq \ell^2$ since $G \in \mathcal{T}$, so $\bigcup \mathcal{G} \in \mathcal{T}$ as well.

Now consider $U_1, \dots, U_N \in \mathcal{T}$ where $N \in \mathbb{N}$. Consider an arbitrary element $x \in \bigcap_i^N U_i$. Then for all $1 \leq i \leq N$, $x \in U_i$. Then by definition of each U_i being in $\mathcal{T}$, we have that there exists a finite set $V_i \subseteq \ell^2$ such that $\Omega(x; V_i) \subseteq U_i$. Note that by definition, if $y \in \Omega(x; V_i \cup V_j)$, then $y \in \Omega(x; V_i)$ and $y \in \Omega(x; V_j)$. So let $V = \cup_i V_i$. This is still a finite set, since we are just unioning a finite number of finite sets. Then if

$y \in \Omega(x;V)$, we have that $y \in \Omega(x;V_i)$ for all $1 \le i \le N$, thus $y \in U_i$. Since $y \in \Omega(x;V)$ was arbitrary, we have that $\Omega(x;V) \subseteq U_i$ for all $1 \le i \le N$, thus $\Omega(x;V) \subseteq \bigcap_i^N U_i$, hence $\bigcap_i^N U_i \in \mathcal{T}$ as well.

Finally, let $x, y \in \ell^2$ such that $x \neq y$. We have that $y_N \neq x_N$ for some $N \in \mathbb{N}$. Let $v \in \ell^2$ be defined by $v_N = 2(y_N - x_N)^{-1}$ and $v_n = 0$ for all other n. Define $V = \{v\}$. We claim that $x \in \Omega(x;V)$ and $y \in \Omega(y;V)$ are disjoint (and they are open sets by part (a) of this problem). Let $x' \in \Omega(x;V)$. It is sufficient to show $x' \notin \Omega(y;V)$. We have $-1 < \langle v, x'-x\rangle = \sum_n v_n(x'_n - x_n) = 2(x'_N - x_N)/(y_N - x_N) < 1$, or $0 \le |2(x'_N - x_N)/(y_N - x_N)| < 1$, so $0 \le |x'_N - x_N| < \frac{1}{2}|y_N - x_N|$. Note $|y_N - x_N| \le |x'_N - y_N| + |x'_N - x_N| < |x'_N - y_N| + \frac{1}{2}|y_N - x_N|$ thus $\frac{1}{2}|y_N - x_N| < |x'_N - y_N|$. But then $1 < |2(x'_N - y_N)/(y_N - x_N)|$ and $\sum_n v_n(x'_n - y_n) = 2(x'_N - y_N)/(y_N - x_N)$. Thus $\sum_n v_n(x'_n - y_n) < -1$ or $\sum_n v_n(x'_n - y_n) > 1$, in either case, $x' \notin \Omega(y;V)$.

This satisfies all the conditions for a HTS, thus $(\ell^2, \mathcal{T})$ is a HTS.

(c). *Solution.* Let $U \in \mathcal{N}(0)$ be an arbitrary open set, ie. $U \in \mathcal{T}$ such that $0 \in U$. We want to show that $(U \setminus \{0\}) \cap S \neq \emptyset$; since $0 \notin S$ anyway, we just need to show $U \cap S \neq \emptyset$.

Since $0 \in U$, there exists a finite set $V \subseteq \ell^2$ such that $\Omega(0;V) \subseteq U$. If $V = \emptyset$, then $\Omega(0;V) = \ell^3$ since $-1 < \langle v, y-x\rangle < 1$ is now vacuously true for all $y \in \ell^2$; then $\Omega(0;V) \cap S$ since $\hat{\mathbf{e}}_1 \in \ell^2 \cap S$, and since $\Omega(0;V) \subseteq U$, $U \cap S \neq \emptyset$. So now assume V is not empty. Denote the elements of V as v^i where $1 \le i \le k$. Then since $v^i \in \ell^2$, we must have that $\lim_n (v^i_n)^2 = 0$ (crude divergence test). Then there exists some N_i where $(v^i_{N_i})^2 < 1$ by the definition of convergence. Let $N = \min_i\{N_i\}$. Then $-1 < v^i_N < 1$ as well. See

$$\langle v^i, \hat{\mathbf{e}}_N\rangle = \sum_{n=1}^{\infty} v^i_n (\hat{\mathbf{e}}_N)_n = v^i_N$$

Thus $\hat{\mathbf{e}}_N \in \Omega(0,V)$ since $-1 < \langle v, \hat{\mathbf{e}}_N - 0\rangle = v_N < 1$ for all $v \in V$. Thus, $\hat{\mathbf{e}}_N \in \Omega(0,V) \subseteq U$. Since $\hat{\mathbf{e}}_N \in S$, thus shows that $S \cap U \neq \emptyset$, so we are done since U was arbitrary (this works for any open $U \in \mathcal{N}(0)$).

(d). *Solution.* If $x \in G$, there exists a finite set $V \subseteq \ell^2$ where $\Omega(x;V) \subseteq G$. If there exists some $v_0 \in V$ such that $\|v_0\| = 0$, then v_0 must be the zero sequence (otherwise $\|v_0\| = \sum_n v_0^2 > 0$), but then regardless of $y \in \ell^2$, $\langle v_0, y-x\rangle = \sum_n 0(y_n - x_n) = 1$, so $G = \ell^2$, and so $\mathbb{B}[x;1) \subseteq \ell^2 = G$ and we are done (note we just made $r = 1 > 0$ here).

Now consider the remaining case when $0 < \|v_i\|$ where $v_i \in \{v_1, \ldots, v_N\} = V$. Let $r = \min_i\{\|v_i\|^{-1}\}$. Note that since is just the minimum of a finite number of values, all greater than zero, we have $r > 0$ as well. Let $y \in \mathbb{B}[x;r)$. Then $\|y - x\| < r \le \|v\|^{-1}$ for all $v \in V$. Thus $\|v\|\|y-x\| < 1$. Now using Cauchy-Schwartz (which we proved for this norm in homework 7), we have $|\langle v, y-x\rangle \le \|v\|\|y-x\| < 1$, but this is equivalent to $-1 < \langle v, y-x\rangle < 1$, so $y \in \Omega(x;V)$ (since this was true for any $v \in V$), thus $y \in G$. Since $y \in \mathbb{B}[x;r)$ was arbitrary, this means $\mathbb{B}[x;r) \subseteq G$, as desired.

(e). *Solution.* It is sufficient to show that $\mathbb{B}[0;1]^c = \{y \in \ell^2 \colon \|y\| > 1\}$ is open. Let $x \in \mathbb{B}[0;1]^c$. Consider $y \in \Omega(x;V)$ (ff don't forget to choose V). Then $|\langle v, y-x\rangle| < 1$ for all $v \in V$. ff

we want to show that $\|y\| > 1$. It would be sufficient to show that $|\langle v, y-x\rangle|/\|v\| > 1$. So we want

$$\left|\sum_{n=1}^{\infty} v_n(y_n - x_n)\right| > \sqrt{\sum_{n=1}^{\infty} v_n^2} > 0$$

$$\left|\sum_{n=1}^{\infty} v_n(y_n - x_n)\right|^2 > \sum_{n=1}^{\infty} v_n^2 > 0$$

Problem 8 of last homework (it was a bonus problem, but solutions were still posted): if A, B are closed and one of them are bounded, then $A + B$ is closed. $\mathbb{B}[0,1] = \{y \in \ell^2 \colon \|y\| \le 1\}$. oh but this was for $A, B \subseteq \mathbb{R}$.

Boundary point?? We claim $\partial\mathbb{B}[0;1]$ are the $y \in \ell^2$ such that $\|y\| = 1$. This means $\sum_n y_n^2 = 1$. Let $x \in \Omega$... neighbourhoods? Show that $\Omega(y,V)$ has some sequences z $\|z\| > 1$ and $\|z\| \le 1$ (but this is trivial from y). ff

Recap of what still needs to be done:

- Evan Chen Q2
- 4(a), 4(e)