Math 321 Homework 8

For the next problem, we will need the following definition. Let $N \geq 1$ be an integer and let s, t be integers. We define the half-open square

$$S_{s,t;N} = \left[\frac{s}{N}, \frac{s+1}{N}\right) \times \left[\frac{t}{n}, \frac{t+1}{N}\right)$$

Observe that for each fixed N, $\mathbb{R}^2$ is a disjoint union of the half-open squares $\{S_{s,t;N} : s, t \in \mathbb{Z}\}$. We say a function $f : \mathbb{R}^2 \to \mathbb{C}$ or $\mathbb{R}^2 \to \mathbb{R}$ is "constant on half-open squares at resolution N" if f is constant on each $S_{s,t;N}$, i.e. f can be written as

$$f(x, y) = \sum_{s,t \in \mathbb{Z}} a_{s,t} \chi_{s,t;N}(x, y)$$

Problem 1

(a). Let $N \geq 1$ be an integer and let $f : \mathbb{R}^2 \to \mathbb{R}$ have compact support and be constant on half-open squares at resolution N. For each $x, y \in \mathbb{R}$, define

$$g(x) = \int_{-\infty}^{\infty} f(x, y)dy, \quad h(y) = \int_{-\infty}^{\infty} f(x, y)dx,$$

Prove that g and h are integrable on $\mathbb{R}$, and that

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} h(y)dy$$

(and that both integrals converge).

(b). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and have compact support. For each $x, y \in \mathbb{R}$, define

$$g(x) = \int_{-\infty}^{\infty} f(x, y)dy, \quad h(y) = \int_{-\infty}^{\infty} f(x, y)dx,$$

Prove that g and h are integrable on $\mathbb{R}$, and that

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} h(y)dy$$

Remark 1. You have just proved that for $f : \mathbb{R}^2 \to \mathbb{R}$ continuous with compact support, we have

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x, y)dx\right) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x, y)dy\right) dx$$

(a). *Solution.* Since f has compact support, there exists some s', t' such that $a_{s,t} = 0$ when $|s| > s'$ or $|t| > t'$. Alternatively, $\text{supp}(f) \subset [-M, M] \times [-M, M]$ for some $M \in \mathbb{R}$. Thus, using Rudin Theorem 6.12(a) (our sum is finite, so there are no problems with using the theorem),

$$\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-M}^{M} \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} a_{s,t}\chi_{s,t;N}(x,y)dy \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \int_{t/N}^{(t+1)/N} a_{s,t}\chi_{s,t;N}(x,y)dy \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} a_{s,t} \int_{t/N}^{(t+1)/N} \chi_{s,t;N}(x,y)dy \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \frac{a_{s,t}}{N}\chi_{s,t;N}(x)
\end{aligned}$$

And the integral converges, via 6.12(a) as well. Similarly, we can do the same for $h(y)$ to get

$$h(y) = \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \frac{a_{s,t}}{N}\chi_{s,t;N}(y)$$

Since our s', t' are the same as before, we still have $\operatorname{supp}(g) \subset [-M, M]$ and $\operatorname{supp}(h) \subset [-M, M]$. Hence, using Rudin Theorem 6.12(a) again, we see

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &= \int_{-M}^{M} \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \frac{a_{s,t}}{N}\chi_{s,t;N}(x)dx \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \int_{s/N}^{(s+1)/N} \frac{a_{s,t}}{N}\chi_{s,t;N}(x)dx \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \frac{a_{s,t}}{N} \int_{s/N}^{(s+1)/N} \chi_{s,t;N}(x)dx \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \frac{a_{s,t}}{N^2}
\end{aligned}$$

And again, the integral converges 6.12(a). Similarly,

$$\int_{-\infty}^{\infty} h(y)dy = \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\le s',|t|\le t'}} \frac{a_{s,t}}{N^2}$$

These are equal, hence we have shown

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} h(y)dy$$

(b). *Solution.* We first show that $g(x)$ is integrable. Since $f(x,y)$ has compact support, there is some $M \in \mathbb{N}$ such that $\text{supp}(f) \subset [-M,M],[-M,M]$. Since $f(x,y)$ continuous in a compact set, it is uniformly continuous on its support. Let $\varepsilon > 0$. Then for $\varepsilon' = \varepsilon/(4M)$, we have that there exists $\delta > 0$ such that for all $(x_1,y),(x_2,y) \in [-M,M] \times [-M,M]$ where $d((x_1,y),(x_2,y)) < \delta$ (the typical metric d), we have $|f(x_1,y) - f(x_2,y)| < \varepsilon'$. Thus by Theorem 6.13

$$|g(x_1) - g(x_2)| = \left|\int_{-M}^{M} f(x_1,y) - f(x_2,y)dy\right| \leq \int_{-M}^{M} |f(x_1,y) - f(x_2,y)|dy \leq \varepsilon' \int_{-M}^{M} dy = 2M\varepsilon' = \varepsilon/2 < \varepsilon$$

Therefore, $g(x)$ is continuous. Furthermore, when $x \notin [-M,M]$, we have $\int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} 0dy = 0$ and so $\text{supp}(g) \subset [-M,M]$, so $g(x)$ has compact support as well. Thus, $\int_{-\infty}^{\infty} g(x)dx = \int_{-M}^{M} g(x)dx$ exists. In an identical manner, one can show that $h(y)$ is integrable by showing that it is continuous with compact support.

Let $a_{s,t;N}$ be the value of $f(x,y)$ at the center of $S_{s,t;N}$. Define

$$f_N(x,y) = \sum_{s,t\in\mathbb{Z}} a_{s,t}\chi_{s,t;N}(x,y)$$

$f(x,y)$ is uniformly continuous, so for any $\varepsilon > 0$, we get a $\delta > 0$ such that $|f(x_1,y_1) - f(x_2,y_2)| < \varepsilon$ for all $(x_1,y_1),(x_2,y_2) \in [-M,M] \times [-M,M]$ where $d((x_1,y_1),(x_2,y_2)) < \delta$. Note that we can choose N large enough so that all points in $S_{s,t;N}$ are within δ of the center (the furthest points at the corners are $\sqrt{2}/(2N)$ from the center, which we can obviously make arbitrarily small). Let all N written below be this N, or larger.

Using the definitions we provided in part (a) (for s',t'), we have (with appropriate triangle inequality and Rudin 6.13):

$$\begin{aligned}
\left|g(x) - \int_{-M}^{M} f_N(x,y)dy\right| &= \left|\int_{-M}^{M} f(x,y)dy - \int_{-M}^{M} f_N(x,y)dy\right| \\
&= \left|\int_{-M}^{M} f(x,y) - f_N(x,y)dy\right| \\
&= \left|\sum_{\substack{s,t\in\mathbb{Z} \\ |s|\leq s',|t|\leq t'}} \int_{t/N}^{(t+1)/N} f(x,y) - a_{s,t;N}dy\right| \\
&= \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\leq s',|t|\leq t'}} \int_{t/N}^{(t+1)/N} |f(x,y) - a_{s,t;N}|dy \\
&\leq \sum_{\substack{s,t\in\mathbb{Z} \\ |s|\leq s',|t|\leq t'}} \varepsilon \int_{t/N}^{(t+1)/N} dy \\
&< \sum_{\substack{t\in\mathbb{Z} \\ |t|\leq t'}} \frac{\varepsilon}{N} = \frac{T}{N}\varepsilon
\end{aligned}$$

where T is the number of $T = 2MN$, the number of squares along the y direction that is inside $[-M,M]$. T is also the number of squares along the x direction inside $[-M,M]$, by symmetry.

In the same method, we can get

$$\left|h(y) - \int_{-\infty}^{\infty} f_N(x,y)dx\right| < \frac{T}{N}\varepsilon$$

Since $f_N(x,y)$ is defined from part (a), we can interchange the order of integration to get equal integrals, which gives us

$$\begin{aligned}\left|\int_{-\infty}^{\infty} g(x)dx - \int_{-\infty}^{\infty} h(y)dx\right| &= \left|\int_{-M}^{M} g(x)dx - \int_{-M}^{M} h(y)dx\right| \\ &\le \left|\int_{-M}^{M} g(x)dx - \int_{-M}^{M}\int_{-M}^{M} f_N(x,y)dydx\right| \\ &+ \left|\int_{-M}^{M} h(x)dx - \int_{-M}^{M}\int_{-M}^{M} f_N(x,y)dxdy\right| \\ &\le \left|\int_{-M}^{M} \left(g(x)dx - \int_{-M}^{M} f_N(x,y)dy\right)dx\right| \\ &+ \left|\int_{-M}^{M} \left(h(x)dx - \int_{-M}^{M} f_N(x,y)dx\right)dy\right| \\ &< \left|\int_{-M}^{M} \frac{T}{N}\varepsilon dx\right| + \left|\int_{-M}^{M} \frac{T}{N}\varepsilon dy\right| \\ &= 4M\frac{2MN}{N}\varepsilon\end{aligned}$$

and since $\varepsilon > 0$ was abitrary, and the rest are constants determined before, we can make this bound arbitrarily small. Thus, the integrals are equal, i.e.

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} h(y)dy$$

Problem 2

Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous and have compact support. Prove that

$$\int_{-\infty}^{\infty} |f * g(x)|dx \le \left(\int_{-\infty}^{\infty} |f(x)|dx\right)\left(\int_{-\infty}^{\infty} |g(x)|dx\right)$$

Hint*: Problem* 1 *might be useful.*

Solution. (Disclaimer: sorry for the f', g' notation below, I originally proved it for f, g, but realized I needed it for $|f|, |g|$, and so I wanted to minimize the amount of extra typing I'd need.)

Let $|f| = f'$ and $|g| = g'$. Since f, g are continuous with compact support, f', g' both also have compact support and are continuous. We now show that $f' * g'(x)$ is continuous and has compact support as well. Note that $f' * g'$ has compact support. We have compact sets K_f, K_g such that $\text{supp}(f') \subset K_f$ and $\text{supp}(g') \subset K_g$. Heine-Borel says that both of these sets are bounded, and so there exists $M \in \mathbb{R}$ such that $K_f, K_g \subset [-M, M]$. See $f' * g'(x)$ is defined for all $x \in \mathbb{R}$, since $f' * g'(x) = \int_{-\infty}^{\infty} f'(t)g'(x-t)dt = \int_{-M}^{M} f'(t)g'(x-t)dt$ (since $f'(t) = 0$ when $t \notin [-M, M]$), and $f'(t)g'(x-t)$ is the product of two continuous functions, and so is integrable and exists on $[-M, M]$ for any $x \in \mathbb{R}$. Furthermore, when $x \notin [-2M, 2M]$, we have that $g'(x-t) = 0$ since $t \in [-M, M]$, and so $\int_{-M}^{M} f'(t)g'(x-t)dt = 0$ as well (since $f'(t)$ is bounded since it is continuous). Thus, $f' * g'(x)$ has support contained in $[-2M, 2M]$, and so has compact support.

We also note that $f' * g'(x)$ is continuous. Since g' is continuous on a compact set $[-2M, 2M]$, it is uniformly continuous on it. Let $\varepsilon > 0$. Let $\varepsilon' = \varepsilon\left(2\int_{-2M}^{2M} f'(t)dt\right)^{-1}$. Then there exists $\delta > 0$ such that $|g'(x) - g'(y)| < \varepsilon'$

when $|x-y|<\delta$. So if $|x-y|<\delta$ (and so $|x-t-(y-t)|<\delta$, using Rudin Theorem 6.12 and 6.13 gives us

$$\begin{aligned}
|f'*g'(x)-f'*g'(y)| &\leq \left|\int_{-2M}^{2M} f'(t)g'(x-t)dt - \int_{-2M}^{2M} f'(t)g'(y-t)dt\right| \\
&= \left|\int_{-2M}^{2M} f'(t)(g'(x-t)-g'(y-t))dt\right| \\
&\leq \int_{-2M}^{2M} |f'(t)||g'(x-t)-g'(y-t)|dt \\
&\leq \varepsilon' \int_{-2M}^{2M} |f'(t)|dt \\
&= \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

Hence, $f'*g'(x)$ is continuous.

Note that since $f*g(x)$, is continuous, it is Riemann integrable, particularly since $f*g(x)$ is compactly supported, we have $f*g(x)\in\mathcal{R}[-2M,2M]$. Furthermore, clearly this means $\text{supp}(|f*g(x)|)=\text{supp}(f*g(x))\subset[-2M,2M]$ as well. Hence, Rudin Theorem 6.13 gives us

$$|f*g(x)| = \left|\int_{-\infty}^{\infty} f(t)g(x-t)dt\right| = \left|\int_{-2M}^{2M} f(t)g(x-t)dt\right| \leq \int_{-2M}^{2M} |f(t)g(x-t)|dt = \int_{-\infty}^{\infty} |f(t)||g(x-t)|dt = f'*g'(x)$$

Now, we can invoke Rudin Theorem 6.12(b) with the inequality from above, and then use the result proven in Problem 1(b), since $f'g'(x)$ is continuous and has compact support:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f*g(x)|dx &\leq \int_{-\infty}^{\infty} f'*g'(x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)||g(x-t)|dxdt \\
&= \int_{-\infty}^{\infty} |f(t)| \int_{-\infty}^{\infty} |g(x-t)|dxdt \\
&= \int_{-\infty}^{\infty} |f(t)| \int_{-\infty}^{\infty} |g(x)|dxdt \\
&= \int_{-\infty}^{\infty} |g(x)dx \int_{-\infty}^{\infty} |f(t)|dt
\end{aligned}$$

as desired. Note that above, we have used the fact that for, fixed t, we have $\int_{-\infty}^{\infty}|g(x-t)|dx=\int_{-\infty}^{\infty}|g(x)|dx$: since $\text{supp}(|g(x)|)\subset[-M,M]$, we have $\int_{-\infty}^{\infty}|g(x-t)|dx=\int_{-M+t}^{M+t}|g(x-t)|dx=\int_{-M}^{M}|g(x)|dx=\int_{-\infty}^{\infty}|g(x)|dx$ (this technically uses Rudin Theorem 6.19, where $\phi(y)=y+t$, but this is also trivial to see as redefining $x=x-t$, and the integrator function $x-t$ acts the same as x). $\square$

Problem 3

Let $f\colon\mathbb{R}\to\mathbb{R}$, and suppose f can be uniformly approximated by polynomials. Prove that f must be a polynomial.
Hint*: If $P_n\to f$ uniformly, consider the sequence $P_{n+1}-P_n$.*

Solution. By the Cauchy Criterion (Rudin Theorem 7.8), we have that for any $\varepsilon>0$, there exists some N such that for all $m,n\geq N$ and all $x\in\mathbb{R}$, $|P_m(x)-P_n(x)|<\varepsilon$. Note that P_m-P_n must be a constant. To see this, note that the difference of polynomials is also a polynomial, and if the difference were not a constant, then our polynomial has some degree $l\geq 1$, and so there is some $a\in\mathbb{R}^*$ such that $P_m(x)-P_n(x)=ax^l+q(x)$ where $q(x)$ is the rest of the polynomial, and has degree less than l; then as $x\to\infty$, we know that $ax^l+q(x)\to\pm\infty$ (the sign is the same as a). (This is a standard result, if you really want a proof: let A be the max of the coefficients of $q(x)$, then for

$x > 1$, $ax^l + q(x) \geq ax^l - lAx^{l-1} = (ax^l - lA)x^{l-1} \geq ax^l - nA$, which very clearly diverges to $\pm\infty$ as $x \to \infty$, so the original polynomial does as well.) But this contradicts that $|P_m(x) - P_n(x)| = |ax^l + q(x)| < \varepsilon$ for all $x \in \mathbb{R}$. Thus, $P_m(x) - P_n(x)$ is a constant.

Let $c_n = P_n(x) - P_N(x)$ where $n > N$ from before. We then have for $m > N$, and all $x \in \mathbb{R}$ that

$$|c_m - c_n| = |P_m - P_N - P_n + P_N| = |P_m - P_n| < \varepsilon$$

thus $\{c_n\}$ is a real valued sequence that is Cauchy. Hence, since $\mathbb{R}$ is complete, we get that $\{c_n\}$ converges, say to some $c \in \mathbb{R}$.

Now, for all $\varepsilon > 0$, there is some $N' \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|c_n - c| < \varepsilon/2$. Also, since $P_n \to f$ uniformly, we have that there exists some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|f(x) - P_n| < \varepsilon/2$ for all $x \in \mathbb{R}$. So when $n \geq \max\{N_1, N_2\}$, we have that for all $x \in \mathbb{R}$,

$$|f(x) - P_N(x) - c| \leq |f(x) - P_n(x)| + |P_n(x) - P_N(x) - c| < \frac{\varepsilon}{2} + |c_n - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, we get that $f(x) - P_N(x) - c = 0$ for all $x \in \mathbb{R}$, thus $f(x) = P_N(x) + c$, which is a polynomial, so $f(x)$ is a polynomial. $\square$

For the next problem, we need the following definition. Let (X, d) be a metric space and let $\alpha > 0$. We say that a function $f \in \mathcal{C}(X)$ is "Hölder's continuous of exponent α" if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

is finite.

Problem 4

(a). Prove that if X is compact then $\{f \in \mathcal{C}(X) \colon \|f\| \leq 1$ and $N_\alpha(f) \leq 1\}$ is a compact subset of $\mathcal{C}(X)$.

(b). Prove that $\{f \in C([0,1]) \colon \|f\| \leq 1\}$ is not *a compact subset of $\mathcal{C}([0,1])$.*

(a). *Solution.* Recall from Problem 3 of Homework 7 that it is sufficient to show that $\mathcal{F} = \{f \in \mathcal{C}(X) \colon |f| \leq 1$ and $N_\alpha(f) \leq 1\}$ is closed, bounded, and equicontinuous.

We have that $\mathcal{F}$ is bounded, since $0 \in \mathcal{F}$ and for all $f \in \mathcal{F}$, we have $d(0, f) \leq 1 < 2$, since $\|f\| \leq 1$.

We show now that $\mathcal{F}$ is equicontinuous. Note that for $f \in \mathcal{F}$, since $N_\alpha(f) \leq 1$,

$$|f(x) - f(y)| \leq N_\alpha(f) d(x,y)^\alpha \leq d(x,y)^\alpha$$

Let $\varepsilon > 0$. Then we can choose $\delta = \varepsilon^{1/\alpha}$ to get that for any $f \in \mathcal{F}$ that when $d(x,y) < \delta$, we have

$$|f(x) - f(y)| < \delta^\alpha = \varepsilon$$

hence $\mathcal{F}$ is equicontinuous.

We now show that $\mathcal{F}$ is closed. That is, every limit point of $\mathcal{F}$ is also in $\mathcal{F}$. That is, for any f such that there is some sequence $f_n \in \mathcal{F}$ where $f_n \to f$ uniformly (the metric of $C(\mathcal{X})$), then $f \in \mathcal{F}$ as well. Since $\mathcal{C}(X)$ with this metric is a complete metric space (Rudin Theorem 7.15), we have $f \in \mathcal{C}(X)$. Also, for all $\varepsilon > 0$, there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\|f - f_n\| < \varepsilon$. Thus

$$\|f\| \leq \|f - f_n\| + \|f_n\| \leq 1 + \varepsilon$$

But since ε can be made arbitrarily small, we get the nonstrict inequality $\|f\| \leq 1$ as well. Finally, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $\|f - f_n\| < \varepsilon/2$, and for any $x, y \in X$ where $x \neq y$, we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq 2\|f - f_n\| + d(x,y)^\alpha \leq \varepsilon + d(x,y)^\alpha$$

Since ε can be made arbitrarily small, we get the nonstrict inequality $|f(x) - f(y)| \leq d(x,y)^\alpha$. $d(x,y) \neq 0$, so we get $N_\alpha(f) = \frac{|f(x)-f(y)|}{d(x,y)^\alpha} \leq 1$. Thus, this verifies all the conditions on $\mathcal{F}$, thus $f \in \mathcal{F}$, and so $\mathcal{F}$ is closed.

Finally, this shows that $\mathcal{F}$ is a compact subset of $\mathcal{C}(X)$.

(b). *Solution.* Define for $a \in \mathbb{R}^+$.

$$f_a = \begin{cases} ax & x \in [0, 1/a] \\ 1 & x \in (1/a, 1] \end{cases}$$

Note that for any a, $f_a \in \mathcal{C}(X)$. When $x \in [0, 1/a)$ and $x \in (1/a, 1]$, f_a are functions which are known to be continuous. If $x = 1/a$, then $\lim_{x \nearrow \frac{1}{a}} f_a(x) = 1 = \lim_{x \searrow \frac{1}{a}} f_a(x)$, so it is continuous at $\frac{1}{a}$, and so continuous everywhere. Also, clearly $\|f_a\| = 1$, hence $f_a \in \{f \in \mathcal{C}([0, 1]) \colon \|f\| \leq 1\}$ for all $a \in \mathbb{R}^+$.

We now show that $\{f \in \mathcal{C}([0, 1]) \colon \|f\| \leq 1\}$ is not compact. We do this by showing that the set is not equicontinuous, and so by Problem 3 of Homework 7, the set is not compact in $\mathcal{C}([0, 1])$. Let $\varepsilon = 1$. Then for all $\delta > 0$, let $x = \delta/2, y = 0$ so $d(x, y) < \delta$, and $f = f_{2/\delta}$, so we get

$$|f(x) - f(y)| = |f_{2/\delta}(\delta/2)| = 1 \geq \varepsilon$$

This shows that $\{f \in \mathcal{C}([0, 1]) \colon \|f\| \leq 1\}$ is not equicontinuous, and hence not compact.