Math 321 Homework 3

(Including work made in collaboration with Arsam Najafian, Oscar Poitras, and Sushrut Tadwalkar.)

The goal of the next few problems is to understand which functions can be written as a difference of two increasing functions. We begin with several definitions.

Let $f\colon [a,b] \to \mathbb{R}$ and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. Define

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

This is called the *variation of f with respect to the partition* $[a,b]$. Define

$$TV|f| = \sup_P V(f,P),$$

where the supremum is taken over all partitions of $[a,b]$. this is called the *total variation of f on* $[a,b]$. We say that f has *bounded variation on* $[a,b]$ if $TV|f| < \infty$. For $c \in (a,b]$, define $TF|f_{[a,c]}|$ to be the total variation of $f\colon [a,c] \to \mathbb{R}$ (i.e. f is restricted to the interval $[a,c] \subset [a,b]$). We define $TV|f_{[a,a]}| = 0$.

Problem 1

Let $\alpha, \beta\colon [a,b] \to \mathbb{R}$ be (weakly) monotone increasing. Prove that $f(x) = \alpha(x) - \beta(x)$ has bounded variation on $[a,b]$.

Solution. Let P be some partition, $P = \{x_0, \dots, x_n\}$. Note that for any monotone increasing function γ, we have $\gamma(x_i) - \gamma(x_{i-1}) > 0 \implies |\gamma(x_i) - \gamma(x_{i-1})| = \gamma(x_i) - \gamma(x_{i-1})$. We have

$$\begin{aligned}
V(f,P) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{n} |\alpha(x_i) - \beta(x_i) - \alpha(x_{i-1} + \beta(x_{i-1}))| \\
&\le \sum_{i=1}^{n} \left(|\alpha(x_i) - \alpha(x_{i-1})| + |\beta(x_{i-1}) - \beta(x_i)|\right) \\
&= \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| + \sum_{i=1}^{n} |\beta(x_i) - \beta(x_{i-1})| \\
&= \sum_{i=1}^{n} (\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i=1}^{n} (\beta(x_i) - \beta(x_{i-1})) \\
&= \alpha(b) - \alpha(a) + \beta(b) - \beta(a) =: M
\end{aligned}$$

where we take advantage that both are telescoping series. This is true for any partition of $[a,b]$, and so $TV|f| = \sup_P V(f,P) = M < \infty$. Hence, f has bounded variation on $[a,b]$.

Problem 2

Let $f\colon [a,b] \to \mathbb{R}$ have bounded variation on $[a,b]$.

(a). For $x \in [a,b]$, define

$$g(x) = TV|f_{[a,x]}|.$$

Prove that g is (weakly) monotone increasing.

(b). For $x \in [a,b]$, define

$$h(x) = f(x) + TV|f_{[a,x]}|.$$

Prove that h is (weakly) monotone increasing.

(c). Prove that f can be written as $f(x) = \alpha(x) - \beta(x)$, where $\alpha, \beta\colon [a,b] \to \mathbb{R}$ are (weakly) monotone increasing.

(a). *Solution.* Let $b \geq x_2 > x_1 \geq a$. If $x_1 = a$, then $TV|f_{[a,x_1]}| = 0$. Since $x_2 > a$, there is some partition of $[a, x_2]$ so $V(f, P)$ is defined, and this is the sum of nonzero elements, so $TV|f_{[a,x_2]}| \geq 0$ as well. Then $TV|f_{[a,x_2]}| \geq TV|f_{[a,x_1]}|$.

Now assume that $x_1 \neq a$. So let $P = \{a = y_0, y_1, \ldots, y_n = x_1\}$ be any partition of $[a, x_1]$. Note that $P^+ = P \cup \{x_2\}$ is a partition of $[a, x_2]$. Then $V(f, P) = \sum_{i=1}^n |f(y_i) - f(y_{i-1})| \leq \sum_{i=1}^n |f(y_i) - f(y_{i-1})| + |f(x_2) - f(x_1)| = V(f, P^+) \leq TV|f_{[a,x_2]}|$. This is true for any partition P of $[a, x_1]$, so $TV|f_{[a,x_2]}|$ is an upper bound for all $V(f, P)$, and so must be greater than the supremum, namely $TV|f_{[a,x_2]}| \geq TV|f_{[a,x_1]}|$.

In either case, we have shown that $x_2 > x_1 \implies g(x_2) \geq g(x_2)$, which means that g is (weakly) monotone increasing.

(b). *Solution.* We want to show $|f(x_2) - f(x_1)| \leq TV|f_{[a,x_2]}| - TV|f_{[a,x_1]}|$ for all $a \leq x_1 < x_2 \leq b$. We deal first with when $a = x_1$. Then $TV|f_{[a,x_1]}| = 0$. So if we hav a partition $P = \{a, x_2\}$, we have $|f(x_2) - f(x_1)| = V(P, f) \leq TV|f_{[a,x_2]}| = TV|f_{[a,x_2]}| - TV|f_{[a,x_1]}|$ as desired.

It now remains to show the inequality when $a < x_1 < x_2 \leq b$. For the sake of contradiction, assume that there exists c, d such that $a < c < d \leq b$ but $|f(d) - f(c)| > TV|f_{[a,d]}| - TV|f_{[a,c]}|$. Then $|f(d) - f(c)| + TV|f_{[a,c]}| - TV|f_{[a,d]}| > 0$ is some fixed positive value, which we'll denote with $\eta > 0$. So $TV|f_{[a,d]}| + \eta = |f(d) - f(c)| + TV|f_{[a,c]}|$.

Now, since $TV|f_{[a,c]}| = \sup_P V(P, f)$ (which is well-defined, since $c > a$), there exists some partition P such that $TV|f_{[a,c]}| - \eta/2 \leq V(P, f)$. Note that $P^+ = P \cup \{d\}$ is a partition of $[a, d]$. We see now that

$$\begin{aligned}
|f(d) - f(c)| + TV|f_{[a,c]}| &\leq |f(d) - f(c)| + V(P, f) + \eta/2 \\
&= V(P^+, f) + \eta/2 \\
&\leq TV|f_{[a,d]}| + \eta/2 \\
&< TV|f_{[a,d]}| + \eta \\
&= |f(d) - f(c)| + TV|f_{[a,c]}|
\end{aligned}$$

But there is a strict inequality between values that are equal, thus a contradiction.

So if $x_2 > x_1$, we have $|f(x_2) - f(x_1)| \leq TV|f_{[a,x_2]}| - TV|f_{[a,x_1]}|$. We have $f(x_1) - f(x_2) \leq |f(x_1) - f(x_2)| = |f(x_1) - f(x_2)|$, so $f(x_1) + TV|f_{[a,x_1]}| \leq f(x_2) + TV|f_{[a,x_2]}|$. Hence, $x_2 > x_1 \implies h(x_2) \geq h(x_1)$, which means that h is (weakly) monotone increasing.

(c). *Solution.* Note that $f(x) = h(x) - g(x)$ from above, and we have just shown that $g, h\colon [a, b] \to \mathbb{R}$ are (weakly) monotone increasing.

Problem 3

Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2\colon [a, b] \to \mathbb{R}$ are (weakly) monotone increasing, and $\alpha_1(x) - \beta_1(x) = \alpha_2(x) - \beta_2(x)$ for all $x \in [a, b]$. Prove that for every continuous $f\colon [a, b] \to \mathbb{R}$, we have

$$\int_a^b f d\alpha_1 - \int_a^b f d\beta_1 = \int_a^b f d\alpha_2 - \int_a^b f d\beta_2$$

Remark. *You have just proven that if $\gamma\colon [a, b] \to \mathbb{R}$ has bounded variation and $f\colon [a, b] \to \mathbb{R}$ is continuous, then we can define*

$$\int_a^b f d\gamma = \int_a^b f d\alpha - \int_a^b f d\beta,$$

where $\gamma = \alpha - \beta$ with α, β monotone increasing; the RHS of this expression does not depend on the specific decomposition $\gamma = \alpha - \beta$ that is chosen.

Solution. Since f is continuous on $[a, b]$, we have that $f \in \mathcal{R}_{\alpha_1}[a, b]$, $f \in \mathcal{R}_{\alpha_2}[a, b]$, $f \in \mathcal{R}_{\beta_1}[a, b]$, and $f \in \mathcal{R}_{\beta_2}[a, b]$, by Rudin Theorem 6.8. Hence, $f \in \mathcal{R}_{\alpha_1+\beta_2}[a, b], \mathcal{R}_{\alpha_2+\beta_1}[a, b]$, and by Rudin Theorem 6.12 (e), we get

$$\int_a^b f d\alpha_1 + \int_a^b f d\beta_2 = \int_a^b f d(\alpha_1 + \beta_2) = \int_a^b f d(\alpha_2 + \beta_1) = \int_a^b f d\alpha_2 + \int_a^b f d\beta_1$$

where the equality in the middle is because $\alpha_1(x) + \beta_2(x) = \alpha_2(x) + \beta_1(x)$, and so they are the same integral. We then get

$$\int_a^b f d\alpha_1 - \int_a^b f d\beta_1 = \int_a^b f d\alpha_2 - \int_a^b f d\beta_2$$

as desired.

Problem 4

(a). Let $f \in \mathcal{R}[a,b]$. For $n \geq 1$, let P_n be the partition with $n+1$ equally spaced points in $[a,b]$, i.e. if $b-a=d$, then

$$P_n = \left\{a, a+\frac{d}{n}, a+\frac{2d}{n}, \dots, a+\frac{nd}{n} = b\right\} \tag{1}$$

Prove that

$$\lim_{n\to\infty} (U(P_n,f) - L(P_n,f)) = 0.$$

(b). Let $\alpha\colon [a,b] \to \mathbb{R}$ be monotone increasing, let $f \in \mathcal{R}_\alpha[a,b]$, and let P_n be as defined in (1). Must it be true that

$$\lim_{n\to\infty} (U(P_n,f,\alpha) - L(P_n,f,\alpha)) = 0? \tag{2}$$

If so, prove it. If not, give a counter-example (i.e. a choice of interval $[a,b]$, a choice of monotone increasing α, and a choice of $f \in \mathcal{R}_\alpha[a,b]$ for which (2) fails and show that your counter-example is correct.

(a). *Solution.* Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$, there exists some partition P such that $U(P,f) - L(P,f) < \varepsilon/2$. Let K be the number of points in P. Let $A = \min_i\{x_i - x_{i-1}\}$, the smallest distance between points in P. By the Archimedean property, there exists some N_1 such that we have $\frac{1}{N_1} < \frac{A}{4}$, and so when $n \geq N_1$, we have $\frac{1}{n} \leq \frac{1}{N_1} < \frac{A}{4}$. Now let $n \geq N_1$. Consider $P_n = \{a, a+\frac{d}{n}, \cdots, a+d = b\} = \{y_0, y_1, \dots, y_r\}$. Then every element in P is contained in either one or two intervals of the form $[y_{j-1}, y_j]$ (the two interval case is when x_i falls on some y_j). Furthermore, since $y_j - y_{j-2} = \frac{2}{N_1} < \frac{A}{2} < A \leq x_i - x_{i-1}$ for any j, i, we have that any contiguous pairs of intervals $[y_{j-2}, y_{j-1}], [y_{j-1}, y_j]$ that contain some x_i do not contain any other $x_{i'}$ (otherwise we would violate the inequality).

Let $B = \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$ (which is a real number, since $f \in \mathcal{R}[a,b]$ only when f is bounded, by definition). By the Archimedean property, there exists some N_2 such that $\frac{1}{N_2} < \frac{\varepsilon}{4}\frac{1}{KdB}$ and so when $n \geq N_2$, we have $\frac{1}{n} \leq \frac{1}{N_2} < \frac{\varepsilon}{4}\frac{1}{KdB} \implies \frac{2KdB}{n} < \frac{\varepsilon}{2}$.

Now let $N = \max\{N_1, N_2\}$. For all $n \geq N$, fix our P_n. Let $S = \{j\colon j \in \mathbb{N}_{\leq r}, [y_{j-1}, y_j] \cap P \neq \emptyset\}$. We have $\#S \leq 2K$, by what we mentioned earlier about how each element of P is in at most 2 intervals $[y_{j-1}, y_j]$. Now, note that $P^* = \{y_j\colon j \notin S\} \cup P$ is a refinement of P, and denote its elements $\{z_1, z_2, \dots\}$. So

$$\sum_{j\notin S} (M_j - m_j)\Delta y_j \leq \sum_{k=1}^{\#P^*-1} (M_k - m_k)\Delta z_k = U(P^*,f) - L(P^*,f) \leq U(P,f) - L(P,f) < \frac{\varepsilon}{2}$$

where the first inequality is true, because we are only adding terms (all of which are positive), and the second inequality is true by Rudin Theorem 6.4 since P^* is a refinement of P.

We now make our final push:

$$
\begin{aligned}
U(P_n, f) - L(P_n, f) &= \sum_j (M_j - m_j)\Delta y_j \\
&\leq \sum_{j \notin S} (M_j - m_j)\Delta y_j + \sum_{j \in S} (M_j - m_j)\Delta y_j \\
&\leq \sum_{j \notin S} (M_j - m_j)\Delta y_j + \sum_{j \in S} B\Delta y_j \\
&< \frac{\varepsilon}{2} + \sum_{j \in S} B\frac{d}{n} \\
&\leq \frac{\varepsilon}{2} + 2K\frac{dB}{n} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

as desired.

(b). *Solution.* We claim that this is not true. Let $[a, b] = [1, 2]$, $\alpha \colon [1, 2] \to \mathbb{R}$ be a variant of the Heaviside step function:

$$
\alpha(x) = \begin{cases} 0 & x \in [1, \sqrt{2}] \\ 1 & x \in (\sqrt{2}, 2] \end{cases}
$$

And we define $f \colon [1, 2] \to \mathbb{R}$ below:

$$
f(x) = \begin{cases} 0 & x \in [1, \sqrt{2}) \\ 1 & x \in [\sqrt{2}, 2] \end{cases}
$$

Let $\varepsilon > 0$. We give the partition $P = \{1, \sqrt{2}, 2\}$. Then $U(P, f, \alpha) = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 = 1 \cdot 0 + 1 \cdot 1 = 1$ and $L(P, f, \alpha) = m_1\Delta\alpha_1 + m_2\Delta\alpha_2 = 0 \cdot 0 + 1 \cdot 1 = 1$. Hence, $U(P, f, \alpha) - L(P, f, \alpha) = 1 - 1 = 0 < \varepsilon$. Therefore, $f \in \mathcal{R}_\alpha[a, b]$.

However, consider P_n for any n (as defined in (1)). Denote its elements $1 = x_0 < x_1 < \cdots < x_n = 2$. Note that $\sqrt{2} \notin P_n$ since all elements of P_n are rational and $\sqrt{2} \notin \mathbb{Q}$. Thus, there exists some k such that $x_{k-1} < \sqrt{2} < x_k$. Note that for $[a, x_{k-1}]$, α is constant, and for $[x_k, b]$, α is constant, so $\Delta\alpha_i = 0$ whenever $i \neq k$. We can then compute

$$
U(P_n, f, \alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) = M_k(\alpha(x_k) - \alpha(x_{k-1})) = 1 \cdot 1 = 1
$$

$$
L(P_n, f, \alpha) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})) = m_k(\alpha(x_k) - \alpha(x_{k-1})) = 0 \cdot 1 = 0
$$

Since this is true for all P_n, no matter how large n is, we have $\lim_{n\to\infty}(U(P_n, f, \alpha) - L(P_n, f, \alpha)) = 1$, which obviously shows (2) fails.