1 Problem 1

Let $\{a_n\}_{n\geq 0}$ *be a sequence defined as follows:*

$$a_0 = 0; a_1 = 1; a_2 = 2 \text{ and}$$

$$a_{n+3} = 5^n \cdot a_{n+2} + n^2 \cdot a_{n+1} + 11a_n \text{ for } n \geq 0$$

Prove that there exist infinitely many $n \in \mathbb{N}$ *such that* $2023 \mid a_n$.

Solution. ff

2 Problem 2

Let $n \in \mathbb{N}$. Find the number of solutions for the congruence equation:

$$x^3 \equiv 1 \pmod{n}$$

Solution. ff

3 Problem 3

As always, $\phi(\cdot)$ is the Euler-ϕ function.
Let α be any real number in the interval $[0, 1]$. Prove that there exists an infinite sequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{\phi(n_k)}{n_k} = \alpha$$

Solution. If $n = \prod_{i=1}^{\pi(n)} p_i^{\alpha_i}$, then

$$\phi(n) = \prod_{i=1}^{\pi(n)} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}) = n \prod_{\substack{p\mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

Thus

$$\frac{\phi(n)}{n} = \prod_{\substack{p\mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

To show that $\{\frac{\phi(n)}{n}\}$ is dense in $(0, 1)$, let $x, y \in (0, 1)$ where $x < y$, then we claim there exists n such that $x < \frac{\phi(n)}{n} < y$. It is sufficient to consider when $x, y \in \mathbb{Q} \cap (0, 1)$. Let $x = p_1'/q_1$ and $y = p_2'/q_2$ where $p_i' < q_i$ and $\gcd(p_i', q_i) = 1$. We can rewrite them to have the same denomonator: $x = p_1/q$ and $y = p_2/q$, where $p_1 < p_2 < q$. Note $1 - \frac{1}{p} = \frac{p-1}{p}$. Choose the p such that $p \mid q$ (I want something more, like the product of all the primes is q). So let's just assume that q's prime decomposition only has exponents 1 (and then could show this is dense) so then we want $p_1 < \prod(p - 1) < p_2$. Perhaps if it is too big, then we find more p later and multiply it down. If it is too small, ff

This is the problem: we can't write every rational number as a product of $(p-1)/p$, or even every rational with denominator whose prime number decomposition do not have extra exponents. But somehow we achieve density.

Note that $\prod(1 - \frac{1}{p})$ converges iff $\sum -\frac{1}{p}$ converges (stack exchange link in source code). But I don't think we really care.

Maybe useful: the rationals in simplified form do not contain any shared primes between the numerator and denominator.

Maybe we do a completely different approach to density. What if $n = q$ or something.