Problem 1

Use $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ *and a splitting argument to evaluate* $S = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$.

Solution. We have

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = S + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{n^4} = S + \sum_{n=1}^{\infty} \frac{1}{(2n)^4} = S + \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} = S + \frac{\pi^4}{16 \cdot 90}$$

Thus $S = \frac{\pi^4}{90} - \frac{\pi^4}{16 \cdot 90} = \frac{\pi^4}{96}$.

Problem 2

Test the following series for convergence. Treat all real values of the constant parameter p.

(a). $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$

(b). $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

(c). $\sum_{n=2}^{\infty} \frac{1}{n^p(\log n)}$

(d). $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

(a). *Solution.* Note that $\log n$ is monotonically increasing and $\log n > 0$ when $n \geq 2$. Let $p \leq 0$. Then $(\log n)^p$ is monotonically decreasing, so $a_n = \frac{1}{(\log n)^p}$ is monotonically increasing. Note that $a_2 > 0$ for all p, and $a_n \geq a_2$ for all n, thus (a_n) does not converge to zero because it is bounded below away from zero (one can use $\varepsilon = a_2$ to show failiure to converge). Thus, by the crude divergence, $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ does not converge.

Let $p > 0$. Since $\log n$ is monotonically increasing, $(\log n)^p$ is also monotonically increasing. Furthermore, for $n \geq 2$, $(\log n)^p > 0$. Thus, (a_n), where $a_n = \frac{1}{(\log n)^p}$, is a monotonically decreasing series.

Also note that

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} 2^k \frac{1}{(\log 2^k)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{2^k}{k^p}$$

Note that $\left(\frac{2^k}{k^p}\right)^{-1} = \frac{k^p}{2^k}$. Thus by Theorem 3.20 (d) in Rudin (where $\alpha = p$ and $p = 1$) we know that $\lim_{n\to\infty} \frac{k^p}{2^k} = 0$. But by question 6 (b) from homework 6, which states that if $x_n \to 0$ then $1/x_n$ cannot converge, we have that $\frac{2^k}{k^p}$ diverges. Thus, $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges as well (crude divergence test). Then, by the Cauchy Condensation Test, $\sum_{n=2}^{\infty} a_n$ must diverge as well (since (a_n) is monotonically decreasing and bounded below by 0). Hence, regardless of p, the series fails to converge.

(b). *Solution.* Let $a_n = \frac{1}{(\log n)^n}$ and

$$b_n = \begin{cases} \frac{1}{(\log 2)^2} & n = 2 \\ \frac{1}{(\log 3)^n} & n \geq 3 \end{cases}$$

Consider the series $\sum_{n=2}^{\infty} b_n = \frac{1}{(\log 2)^n} + \sum_{n=3}^{\infty} \frac{1}{(\log 3)^n}$. Note that our series is a geometric series, specifically $\log 3 > 1$ so $0 < \frac{1}{\log 3} < 1$, which is the common ratio r, and so we know that the series $\sum_{n=2}^{\infty} b_n$ converges.

Now, since $0 < \log 3 \leq \log n$ for $n \geq 3$, we have $0 < \frac{1}{\log n} < \frac{1}{\log 3} \implies 0 < \frac{1}{(\log n)^n} < \frac{1}{(\log 3)^n}$, thus $b_n \geq a_n = |a_n| \geq 0$ for all n, and thus by the comparison test, $\sum_{n=2}^{\infty} a_n$ must converge as well. (This is true regardless of p; it was not used.)

(c). *Solution.* Let $a_n = \frac{1}{n^p(\log n)}$. Let $p \leq 0$. Then $n > 1 \implies 0 < \frac{1}{n} < 1 \implies \frac{1}{n^p} \geq 1$. Furthermore, $0 < \log n < n \implies \frac{1}{\log n} > \frac{1}{n} > 0$. Thus $\frac{1}{n^p}\frac{1}{\log n} > \frac{1}{n} = \left|\frac{1}{n}\right| > 0$. Recall that $\sum_n \frac{1}{n} = +\infty$, so by the comparison test, $\sum_{n=2}^{\infty} a_n = +\infty$ as well, i.e. it fails to converge.

Now let $0 < p \leq 1$. Note that $(n+1)^p > n^p > 0$ and $\log n + 1 > \log n > 0$ for all n, thus $(n+1)^p \log n + 1 > n^p \log n > 0$, and taking the reciprocal, we get $a_n > a_{n+1} > 0$. Now see

$$\sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p(\log 2^k)} = \frac{1}{\log 2} \sum_{k=1}^{\infty} \frac{(2^k)^{1-p}}{k}$$

First, if $p = 1$, then our sum becomes $\frac{1}{\log 2} \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges, so by the Cauchy condensation test, $\sum_n a_n$ must diverge as well. Now let $p \neq 1$. Note that $\left(\frac{(2^k)^{1-p}}{k}\right)^{-1} = \frac{k}{(2^{1-p})^k}$. By Theorem 3.20 (d) in Rudin (where $\alpha = 1$ and $p = 2^{1-p} - 1 > 0$), we know that $\lim_{k\to\infty} \frac{k}{(2^{1-p})^k} = 0$. But then by question 6 (b) from homework 6, we have that $\frac{(2^k)^{1-p}}{k}$ must diverge as well. Thus $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges as well (crude divergence test). Then, by the Cauchy condensation test, $\sum_{n=2}^{\infty} a_n$ must diverge as well. Thus if $0 < p \leq 1$, the series fails to converge.

Finally, let $p > 1$. Note that for all $n \geq 3$, $\log n > 1$, so $0 < n^p < n^p \log n$, thus $0 < \frac{1}{n^p \log n} = \left|\frac{1}{n^p \log n}\right| < \frac{1}{n^p}$. Since $p > 1$, we know that $\sum_{n=3}^{\infty} \frac{1}{n^p}$ converges, thus by the comparison test, $\sum_{n=3}^{\infty} \frac{1}{n^p \log n}$ converges as well. Therefore, when $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n^p(\log n)}$ converges.

(d). *Solution.* Let $a_n = \frac{1}{n(\log n)^p}$. Let $p \leq 0$. Note that since $0 < \frac{1}{\log n} < 1$ for $n \geq 3$, we have $1 \leq \frac{1}{(\log n)^p}$. Thus, $\frac{1}{n(\log n)^p} \geq \frac{1}{n} = \left|\frac{1}{n}\right| > 0$. Furthermore, $\sum_n \frac{1}{n}$ diverges, thus by the comparison test, $\sum_{n=3}^{\infty} a_n$ diverges as well. Thus, when $p \leq 0$, $\sum_{n=2}^{\infty} a_n$ does not converge.

Now let $p > 0$. Note that $(n+1) > n > 0$ and $\log n + 1 > \log n > 0 \implies (\log n + 1)^p > (\log n)^p > 0$ for all $n \geq 2$. thus $(n+1)^p \log n + 1 > n^p \log n > 0$, and taking the reciprocal, we get $a_n > a_{n+1} > 0$. Consider

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

But the sum is just the p-series. Thus if $p \leq 1$, this sum diverges, and if $p > 1$, the sum converges. Thus, by the Cauchy condensation test, when $0 < p \leq 1$, $\sum_n a_n$ fails to converge; when $p > 1$, $\sum_n a_n$ converges.

Problem 3

Consider the set ℓ^2 consisting of all real sequences $x = (x_1, x_2, \dots)$ enjoying the special property that $\sum_n |x_n|^2$ converges. Define an inner product on ℓ^2 as follows:

$$\forall x, y \in \ell^2,\ \langle x, y\rangle := \sum_{n=1}^{\infty} x_n y_n$$

(a). Prove that the series in this definition converges.

Informally, this is the natural generalization of Euclidean k-space to the case $k = \aleph_0$; the inner product $\langle x, y\rangle$ in ℓ^2 is analogous to the dot product $x \bullet y$ in $\mathbb{R}^k$. It's only a small stretch to call the elements of ℓ^2 "vectors". Add further credibility to this interpretation by defining $\|x\| = \sqrt{\langle x, x\rangle}$ for each $x \in \ell^2$, and then proving

(b) $|\langle x, y\rangle| \le \|x\|\|y\|$ for all $x, y \in \ell^2$.

(c) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in \ell^2$.

This generalization has some limitations, however. In $\mathbb{R}^k$, any sequence of vectors $x^{(1)}, x^{(2)}, x^{(3)}, \dots,$ whose compoenent sequences converge must be a convergent sequence of vectors, and its limit can be identified by taking the limit in each component separetly. Show that this fails in ℓ^2, as follows:

(d) Construct a sequence $x^{(1)}, x^{(2)}, \dots,$ of vectors in ℓ^2 such that $\|x^{(n)}\| = 1$ for all n, and yet every $p \in \mathbb{N}$ the 'p-th component sequence' $\langle \mathbf{e}_p, x^{(n)}\rangle$ converges to 0 as $n \to \infty$. Here, just as in $\mathbb{R}^k$, $\mathbf{e}_p$ denotes the "standard unit vector" with exactly one nonzero entry, which is a 1 in position p.

(a). *Solution.* For all n, we have $(x_n + y_n)^2 \ge 0$, so $x_n^2 + y_n^2 \ge -2x_n y_n$, and $(x_n - y_n)^2 \ge 0$, so $x_n^2 + y_n^2 \ge 2x_n y_n$, hence $x_n^2 + y_n^2 \ge 2|x_n y_n| \ge |x_n y_n|$.

Let $b_n = x_n^2 + y_n^2 = |x_n|^2 + |y_n|^2$. We now show that $\sum_n b_n$ converges. Let $X_n = \sum_{k=0}^{n} |x_k|^2$ and $Y_n = \sum_{k=0}^{n} |y_k|^2$. Thus $X_n + Y_n = \sum_{k=0}^{n}(|x_k|^2 + |y_k|^2) = \sum_{k=0}^{n} b_k$. Since $\sum_n |x_n|^2$ and $\sum_n |y_n|^2$ converge, denote their limits as $X = \sum_n |x_n|^2 = \lim_{n\to\infty} X_n$ and $Y = \sum_n |y_n|^2 = \lim_{n\to\infty} Y_n$. So by the addition limit law, we have $\sum_n b_n = \lim_{n\to\infty} \sum_{k=0}^{n} b_k = \lim_n \to \infty(X_n + Y_n) = X + Y$, thus $\sum_n b_n$ converges.

Finally, by the comparison test, since $\sum_n (x_n^2 + b_n^2)$, we have that $\sum_n (x_n y_n)$ converges as well.

(b). *Solution.* Note that if we consider the first k terms of x_n and y_n as entries of a k-tuple, we have shown in class the Cauchy Schwartz inequality:

$$L_k = \left|\sum_{n=1}^{k} x_n y_n\right| \le \sqrt{\sum_{n=1}^{k} x_n^2}\sqrt{\sum_{n=1}^{k} y_n^2} = \sqrt{\sum_{n=1}^{k} x_n^2 \sum_{n=1}^{k} y_n^2} = R_k$$

Note that L_k and R_k are sequences that satisfy $L_k \le R_k$ for all k. Thus by the lemma from October 11 from the course notes, we also have

$$\liminf_{k\to\infty} L_k \le \liminf_{k\to\infty} R_k \quad \text{and} \quad \limsup_{k\to\infty} L_n \le \limsup_{k\to\infty} R_k$$

Furthermore, both $\sum_n x_n^2$ and $\sum_n y_n^2$ converge, so our limit laws tell us that their product must also converge, and R_k is just the square of a convergent sequence, and so R_k must also converge. Additionally, we proved in part (a) that L_k converges. Thus, both of their lim sups and lim infs must equal each other, so we get

$$\lim_{k\to\infty} L_k \le \lim_{k\to\infty} R_k$$

But extracting the definitions of L_k and R_k, this is just

$$\lim_{k\to\infty} \left|\sum_{n=1}^{k} x_n y_n\right| \le \lim_{k\to\infty} \sqrt{\sum_{n=1}^{k} x_n^2}\sqrt{\sum_{n=1}^{k} y_n^2}$$

$$\implies |\langle x, y\rangle| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \le \sqrt{\sum_{n=1}^{\infty} x_n^2}\sqrt{\sum_{n=1}^{\infty} y_n^2} = \|x\|\|y\|$$

(c). *Solution.* ff idk something to do with previous one. Note that if we consider the first k terms of x_n and y_n as entries of a k-tuple, we have shown in class the triangle inequality:

$$L_k = \sqrt{\sum_{n=1}^{k} x_n + y_n} \leq \sqrt{\sum_{n=1}^{k} x_n^2}\sqrt{\sum_{n=1}^{k} y_n^2} = \sqrt{\sum_{n=1}^{k} x_n^2 \sum_{n=1}^{k} y_n^2} = R_k$$

Note that L_k and R_k are sequences that satisfy $L_k \leq R_k$ for all k. Thus by the lemma from October 11 from the course notes, we also have

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Furthermore, both $\sum_n x_n^2$ and $\sum_n y_n^2$ converge, so our limit laws tell us that their product must also converge, and R_k is just the square of a convergent sequence, and so R_k must also converge. Additionally, we proved in part (a) that L_k converges. Thus, both of their lim sups and lim infs must equal each other, so we get

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But extracting the definitions of L_k and R_k, this is just

$$\lim_{k\to\infty}\left|\sum_{n=1}^{k} x_n y_n\right| \leq \lim_{k\to\infty}\sqrt{\sum_{n=1}^{k} x_n^2}\sqrt{\sum_{n=1}^{k} y_n^2}$$

$$\implies |\langle x, y\rangle| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \leq \sqrt{\sum_{n=1}^{\infty} x_n^2}\sqrt{\sum_{n=1}^{\infty} y_n^2} = \|x\|\|y\|$$

(d). *Solution.* ff

Problem 4

Given that the sequence $(s_n + 2s_{n+1})$ converges, prove that the sequence (s_n) converges.

Solution. Cauchy-ness: Since $(s_n + 2s_{n+1})$ converges, then for any ε', there exists some $N' \in \mathbb{N}$ such that for for all $n \geq N$ and for all $p \in \mathbb{N}$, we have $|2s_{n+p+1} + s_{n+p} - 2s_{n+1} - s_n| < \varepsilon'$.

We want to show that if $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $p \in \mathbb{N}$, we have $|s_{n+p} - s_n| < \varepsilon$. To do this then, we want some bound on $|2s_{n+p+1} - 2s_{n+1}|$. This kinda makes me feel like induction. If we let $p = 1$, then we have $|2s_{n+2} - 2s_{n+1}|$.

Maybe something about how

Let $\varepsilon > 0$ be arbitrary. Let $N = N'$. We will show that (s_n) is Cauchy. Let $N = N'$ (where N' is so that other series bounded by ε too). Let $n \geq N$. We now induct on p. Base case ($p = 1$): we have $|s_{n+1} - s_n|$. But note that $\varepsilon > |2s_{n+2} + s_{n+1} - 2s_{n+1} - s_n| = |2s_{n+2} - s_{n+1} - s_n|$.

Hmm, what if showing s_n gets arbitrarily close to $s_n + 2s_{n+1}$. ff

Let $a_n = s_n + 2s_{n+1}$. Then $s_n = a_n - 2a_{n+1} + 4a_{n+2} - + \cdots = \sum_{i=0}^{\infty}(-2)^i a_{n+i}$.

Problem 5

Prove that if $\sum_{n=1}^{\infty} a_n^2$ *converges, then* $\sum_{n=1}^{\infty} \frac{a_n}{n^q}$ *converges for any constant* $q > \frac{1}{2}$.

Solution. Apparently 3(a)... presumebly $a_n = x_n$ and $y_n = \frac{1}{n^q}$ (p-series so converges).

Problem 6

In parts (a)-(c) below, suppose $a_n > 0$ and $b_n > 0$ for all n, and define

$$A = \sum_{n=1}^{\infty} a_n, \quad B = \sum_{n=1}^{\infty} b_n$$

(a). *Prove the Limit Comparison Test: If b_n/a_n converges to a real number $L > 0$, then series A converges if and only if series B converges.*

(b). *Prove the Ratio Comparison Test: If $a_{n+1}/a_n \leq b_{n+1}/b_n$, convergence of series B implies convergence of series A. What if $a_{n+1}/a_n \leq b_n/b_{n-1}$ instead? [Clue: Start by finding upper and lower bounds for the sequence $r_n = a_n/b_n$.]*

(c). *Use (b) with $\zeta(p)$ to prove Raabe's Test: if $p > 1$ and $a_{n+1}/a_n \leq 1 - p/n$ for all n sufficiently large, then series A converges. [Clue: First show that $1 - px < (1-x)^p$ for all $x \leq 1$. Just use calculus.]*

(d). *Test $\sum_n a_n$ for convergence, where $a_n = \frac{1 \cdot 4 \cdots (3n+1)}{n^2 3^n n!}$.*

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

(d). *Solution.* ff

Problem 7

Prove: If each $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges. Does the converse hold?

Solution. ff Contrapositive Contrapositive

Problem 8

(a). *Prove: Given any $D \in \mathbb{R}$ and $\delta > 0$, there is a finite collection of numbers $a_1, a_2, \dots, a_N$ such that $D = a_1 + a_2 + \cdots + a_N$ and*

$$\delta > |a_1| > |a_2| > \cdots > |a_N| > 0$$

(b). *Let $(\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Explain how to construct a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying, simultaneously*

(i) $|x_n| > |x_{n+1}|$ for all n, and $x_n \to 0$ as $n \to \infty$, and

(ii) the sequence $(s_N)_{N\in\mathbb{N}}$ defined by $s_N = \sum_{n=1}^{N} x_n$ has $(\sigma_n)_n$ as a subsequence.

Discussion*: This show badly the converse of the Crude Divergence Test can fail: the series $\sigma_n x_n$ has terms tending to 0, yet its sequence of partial sums can be wild enough to hit all elements of the preassigned $(\sigma_n)_n$.*

(a). *Solution.* ff

(b). *Solution.* ff