Problem 1

Consider a real-valued sequence (x_n) and a real number $\hat{x}$. Prove that the following are equivalent:

(a). $x_n \to \hat{x}$,

(b). $\forall \varepsilon > 0, \exists N \in \mathbb{N}\colon \forall n \geq 23N, |x_n - \hat{x}| < 20\varepsilon$.

Solution. Assume (a). For any $\varepsilon > 0$, since x_n converges to $\hat{x}$ (and $20\varepsilon > 0$), we have that there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - \hat{x}| < 20\varepsilon$. But since if $n \geq 23N$, it is certainly true that $n > N$, thus $|x_n - \hat{x}| < 20\varepsilon$ for all $n \geq 23N$, which is (b).

Now assume (b). For any $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{20}$. We know that there exists $N' \in \mathbb{N}$ such that for all $n \geq 23N'$, we have that $|x_n - \hat{x}| < 20\varepsilon' = \varepsilon$. But since $24N' \in \mathbb{N}$, let us fix $N = 24N'$, then for all $n \geq N > 23N'$, we have $|x_n - \hat{x}| < \varepsilon$ again, which, by definition, means $x_n \to \hat{x}$.

Problem 2

Extend our collection of equivalent formulations of the completeness property for $\mathbb{R}$ by proving that the following are equivalent (TFAE). Proceed directly, without relying on the completeness property in one of its other forms. (So, for example, do not assume existence of suprema and infima.)

(a). For any sequence of nonempty closed real intervals $I_1[a_1, b_1], I_2 = [a_2, b_2], \ldots,$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ (such intervals are called "nested"), one has

$$\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$$

(b). Every bounded monotonic sequence in $\mathbb{R}$ converges. (Recall Rudin's Definition 3.13.)

*(*Note*: The inverval notation $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$ is reserved for the case where both a and b are real numbers. To encode $\{t \in \mathbb{R} : t \geq 0\}$, for example, we would write $[0, +\infty)$, not $[0, +\infty]$.)*

Solution. Backwards direction: a_n and b_n are bounded monotone sequences (a_n and b_n are reals, so bounded above/below, and bounded below/above by each other). Either a_n, b_n converge to the same number, then nonempty, or some distance apart, so also nonempty.

Now assume (b). Consider the sequence of nonempty closed real intervals I_n from the problem statement. Note that a_n and b_n form monotonic sequences. Since in order for $I_n \supseteq I_{n+1}$ to be true, we must have that $a_n \geq a_{n+1}$ and $b_n \leq b_{n+1}$. Thus, a_n is monotonically decreasing and b_n is monotonically decreasing. Note that the sequence (a_n) must be bounded. We know that any $a_n \leq a_1$ since its monotone decreasing, so its bounded above, and $a_n \geq b_1$, otherwise $b_1 > a_n \geq b_n$, which contradicts our assumption that b_n monotonically increases, so (a_n) is bounded below. Likewise, (b_n) is bounded above and below with the same argument, swapping the a's and b's. ff

Note: Questions 3-6 contribute to the major project of constructing $\mathbb{R}$ from $\mathbb{Q}$. Therefore they must be completed entirely in the context of the ratoinal numbers. Present solutions that make no reference at all to the completeness property of $\mathbb{R}$, in any of its equivalent forms.

Problem 3

Introduce the following notation:
$CS(\mathbb{Q})$: *the set of all Cauchy sequences with rational elements.*
x, y, z: *typical symbols for elements of $CS(\mathbb{Q})$. Thus, e.g., $x = (x_1, x_2, \dots)$.*
$R[x]$: *the subset of $CS(\mathbb{Q})$ associated with a given $x \in CS(\mathbb{Q})$ as follows:*

$$R[x] = \left\{ x' \in CS(\mathbb{Q}) \colon \lim_{n\to\infty} |x'_n - x_n| = 0 \right\}.$$

Φ: *the function that takes each rational number q into the subset of $CS(\mathbb{Q})$ containing the corresponding constant sequence, i.e.,*

$$\Phi(q) = R[(q, q, \dots)] \quad \forall q \in \mathbb{Q}$$

(a). *Prove: $R[x] \neq \emptyset$ for every $x \in CS(\mathbb{Q})$.*

(b). *Prove: For any $x, y \in CS(\mathbb{Q})$, $R[x] = R[y] \iff R[x] \cap R[y] \neq \emptyset$*

(a). *Solution.* Define x' where $x'_n = \begin{cases} x_n + 1 & \text{if } n = 1 \\ x_n & \text{if } n > 1 \end{cases}$. Then for any $n > 1$, $|x'_n - x_n| = 0$, so $\lim_{n\to\infty} |x'_n - x_n| = 0$, thus $x' \in R[x]$. And so $R[x] \neq \emptyset$.

(b). *Solution.* Something about converging to the same value.ff

Problem 4

Continue with the notation from Question 3. We would like to define a relatoin denoted "<" on $\mathbb{Q}^*$ as follows:

$$R[x] < R[y] \iff \exists r > 0 (r \in \mathbb{Q}), \exists N \in \mathbb{N} \colon \forall n > N, y_n - x_n > r$$

This relation looks like one that is familiar for rational numbers, but here it compares two sets. *Each of the properties we take for granted when manipulating inequalities relating numbeers requires careful thinking in this new context. Prove the following.*

(a). Whenever $R[x'] = R[x]$ and $R[y'] = R[y]$ for some given $x, x', y, y' \in CS(\mathbb{Q})$, the definition proposed above gaurantees that

$$R[x'] < R[y'] \iff R[x] < R[y].$$

(that is, the proposed definition is unambiguous. Or, more conventionally, "the relation $<$ is well-defined".)

(b). If $x, y, z \in CS(\mathbb{Q})$ obey $R[x] < R[y]$ and $R[y] < R[z]$, then $R[x] < R[z]$.

(c). The inequality $R[x] < R[x]$ never happens, for any $x \in CS(\mathbb{Q})$.

(d). For any $p, q \in \mathbb{Q}$, we have $p < q \iff \Phi(p) < \Phi(q)$.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

(d). *Solution.* ff

Problem 5

Continue with the notation from Questions 3 and 4. Prove the following:

(a). For any $x \in CS(\mathbb{Q})$, exactly one of the following holds:

$$R[x] < \Phi(0), \qquad R[x] = \Phi(0), \qquad \Phi(0) < R[x]$$

(b). For each x in $CS(\mathbb{Q})$, there exist $q, r \in \mathbb{Q}$ such that $\Phi(q) < R[x] < \Phi(r)$.

(c). For any $x, y \in CS(\mathbb{Q})$ with $R[x] < R[y]$, there exists $q \in \mathbb{Q}$ such that $R[x] < \Phi(q) < R[y]$.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 6

Continue with the notation from Questions 3 and 4. Prove the following:

$$\textit{If } x \in CS(\mathbb{Q}) \textit{ has } R[x] \neq \Phi(0), \textit{ then there exists } z \in CS(\mathbb{Q}) \textit{ for which } R[x \cdot z] = \Phi(1).$$

Here $x \cdot z$ *denotes the sequence whose* n*th term is* $x_n z_n$*. (Recall from Assignment 4, Question 6, that* $x \cdot z \in CS(\mathbb{Q})$ *whenever* $x, z \in CS(\mathbb{Q})$*.)*

Solution. ff

Problem 7

[Rudin problem 3.5] For any two real sequences (a_n) *and* (b_n)*, prove that the inequality*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

holds whenever the right side is not of the form $(+\infty) + (+\infty)$*. Give a specific example to show that inequality may hold.*

Solution. ff

Problem 8

Let X denote the collection of all functions $f\colon [0,1] \to \mathbb{R}$ for which the set of real numbers $f([0,1]) = \{f(x)\colon x \in [0,1]\}$ is bounded. For each $f \in X$, define

$$\|f\| = \sup\{|f(x)|\colon x \in [0,1]\}$$

Prove that for all rael c and all $f, g, h \in X$,

(a). $\|cf\| = |c|\|f\|$,

(b). $\|f + g\| \leq \|f\| + \|g\|$,

(c). $\|f - h\| - \|g - h\| \leq \|f - g\|$.

Give an example where (b) holds as a strict inequality ("<").

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff