1 January 9

1.1 Complex Numbers

Starts with $\mathbb{N} = \{1, 2, 3, \dots\}$. We can solve $x + 2 = 5$ ($x = 3$), but we cannot solve $x + 5 = 2$. So we introduce $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Now $x + a = b$ is always solvable in $\mathbb{Z}$ ($a, b \in \mathbb{Z}$), namely $x = b - a \in \mathbb{Z}$. So consider $2x = 8$. This has the solution $x = 4 \in \mathbb{Z}$. But it's easy to come up with equations like this that aren't solvable in $\mathbb{Z}$, namely $8x = 2$. So we enlarge our system of numbers to $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$. Now we can solve $ax = b$ for $a, b \in \mathbb{Q}$ as long as $a \neq 0$.

Remark 1. If we tried to add another number ∞ to $\mathbb{Q}$ so that ∞ is a solution to $0x = 1$, this would lead to a breakdown of the rules of arithmetic because $0 \cdot a = 0$ for all a by distributive law ($0 \cdot a + 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a = 0 + 0 \cdot a \implies 0 \cdot a = 0$).

We can now do linear algebra: in $\mathbb{Q}$, we can solve all linear equations and systems of linear equations.

From $\mathbb{Q}$ to $\mathbb{R}$: we want to do calculus. Put in all limits of monotone increasing bounded sequences, e.g.

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e \notin \mathbb{Q}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{pi^2}{6} \notin \mathbb{Q}$$

Actually, calculating the above limit is a highlight of this course.

As a consequence, we get the intermediate value theorem: $f\colon [a, b] \to \mathbb{R}$ continuous, $f(a) < 0$, $f(b) > 0$, then $\exists x \in (a, b)$ such that $f(x) = 0$. Also the extremal value theorem: $f\colon [a, b] \to \mathbb{R}$ continuous, then $\exists x \in [a, b]$ such that $\forall y \in [a, b]$, $f(x) \geq f(y)$. In particuluar, say $a > 0$, then $f(x) = x^2 - a$ on the interval $[0, 1 + a]$, $f(0) = -a < 0$ and $f(1+a) = (1+a)^2 - a = 1 + a + a^2 > 2$, so by the IVT: $\exists x \in \mathbb{R}$ such that $f(x) = 0$. So $x^2 - a = 0$ has a solution in $\mathbb{R}$. So we have a solution to this quadratic equation in $\mathbb{R}$. The notation we use is $\sqrt{a}$. Positive real numbers have square roots in $\mathbb{R}$. So we can sole all quadratic equations $x^2 + bx + c = 0$ if $b^2 - 4c \geq 0$, namely $x = -\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$.

Now we go from $\mathbb{R}$ to $\mathbb{C}$: if $b^2 - 4c < 0$, we cannot solve $x^2 + bx + c$ in $\mathbb{R}$.

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{-1}\sqrt{4c - b^2}$$

where $\sqrt{4c - b^2} \in \mathbb{R}$. So we need to make sense of $\sqrt{-1}$, and then we can solve all quadratic equations $x^2 + bx + c = 0$ where $b, c \in \mathbb{R}$. We simply add the symbol $i := \sqrt{-1}$ to $\mathbb{R}$. We then get the solutions $x = \alpha \pm i\beta$ where $\alpha = -\frac{b}{2}$ and $\beta = \frac{1}{2}\sqrt{4c - b^2}$ where $\alpha, \beta \in \mathbb{R}$. We call i the "imaginary unit" and write numbers as $\alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$. We do our calculations the usual way using the extra rule $i^2 = -1$.

Miracle: this leads to a coherent system of numbers $\mathbb{C}$, the complex numbers, where all quadratic equations can be solved, and we can do calculus (the contents of this course).

Some definitions of the operations:

$$+\colon (a + ib) + (c + id) = (a + c) + i(b + d)$$
$$\times\colon (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Now, this was somewhat informal. So formally, we define $\mathbb{C} = \mathbb{R}^2$ (assuming $\mathbb{R}$ is given). Addition is the same as vector addition. The multiplication is $(a, b)(c, d) = (ac - bd, ad + bc)$. One can check that this multiplication is commutative, associative, satisifies the distributive law, there is a multiplicative unit $(1, 0)$, and every nonzero complex number has a multiplicative inverse: $(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$. Hence, we can freely divide (multiplying by the multiplcative inverse) by nonzero complex numbers. So $\mathbb{C}$ is a field (see [BMPS]).

We can map $\mathbb{R}$ to $\mathbb{C}$ by $a \mapsto (a, 0)$. So geometrically, $\mathbb{C}$ is the plane and $\mathbb{R}$ is the x-axis. This is a "field morphism", i.e. it respects addition and multiplication and sends the multiplication unit to the multiplication unit (so $(a \cdot b, 0) = (a, 0) \cdot (b, 0)$ and $(a + b, 0) = (a, 0) + (b, 0)$). We have $\alpha \in \mathbb{R}$, $(a, b) \in \mathbb{C}$, scalar multiplication: $\alpha(a, b) = (\alpha a, \alpha b)$ and complex multiplication: $(\alpha, 0) \cdot (a, b) = (\alpha a, \alpha\beta)$. So we identify $\mathbb{R}$ with its image in $\mathbb{C}$. Standard basis of $\mathbb{C} = \mathbb{R}^2$: $(1, 0), (0, 1)$. We can abbreviat $1 = (1, 0)$ and $i = (0, 1)$. Write $(a, b) = a(1, 0) + b(0, 1) = a1 + bi = a + ib$. We can check that $i^2 = -1$: $(0, 1) \cdot (0, 1) = (-1, 0) = -1$.

We write $z \in \mathbb{C}$ as $z = a + ib$, $a, b \in \mathbb{R}$. We call a the real part and b the imaginary part, and write $a = \mathrm{Re}(z), b = \mathrm{Im}(z)$. $|a + ib| = \sqrt{a^2 = b^2}$ as the norm / absolute valuue / modulus of $z = a + ib$.

1.1.1 Polar form

It is often convenient to write complex numbers in a different form. Imagining z as a point on the Cartesian plane, we let r be the distance from the origin and θ the angle z sweeps out. We can compute $a = r\cos\theta$ and $b = r\sin\theta$. So $a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. r is the modulus of $a + ib$ and we call θ the argument. The argument is ambiguous, but we can restrict $\theta \in (-\pi, \pi]$, which is called the principal value of the argument. $r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$ if and only if $r = s$ and $\phi - \theta \in 2\pi\mathbb{Z}$.

With this, we can get a geometric meaning of multiplication. Fix $z = r(\cos\theta+i\sin\theta)$. Consider the "multiplying by z" map $\mathbb{C} \to \mathbb{C}$ where $w \mapsto zw$. Write (x, y) as $\binom{x}{y}$.

$$\begin{aligned}
w = \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto r(\cos\theta + i\sin\theta) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\
&= r(\cos\theta + i\sin\theta) \cdot (x + iy) \\
&= rx\cos\theta - ry\sin\theta + i(yr\cos\theta + xr\sin\theta) \\
&= \begin{pmatrix} rx\cos\theta - ry\sin\theta \\ ry\cos\theta + rx\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

where we have the rotation matrix. So we are stretching w by the modulus and rotating it by the argument.