## Problem 1

*Prove: If $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges. Is this statement still true if the word "absolutely" is removed?*

*Solution.* It is sufficient to show that $\sum_n a_n b_n$ is absolutely convergent. Consider the series $\sum_n |a_n b_n| = \sum_n |a_n||b_n|$. Since $\lim_{n\to\infty} a_n = 0$ (by the contrapositive of the "crude" divergence test since $\sum_n a_n$ converges), we have that $a_n$ is bounded, and so $|a_n|$ is bounded as well (the upper bound is just the max of the lower and upper bound of $a_n$, and it is bounded below by 0). Let $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then $|a_n||b_n| < M|b_n|$. We have that $\sum_n M|b_n|$ converges, since if $s_N = \sum_{n=1}^{N} |b_n|$, then

$$\sum_n M|b_n| = \lim_{n\to\infty} M|b_0| + M|b_1| + \cdots + M|b_n| = \lim_{n\to\infty} M(|b_0| + |b_1| + \cdots + |b_n|) = \lim_{n\to\infty} Ms_n$$

and since $(s_n)$ converges (by the absolute convergence of $b_n$), by our constant multiplication limit law, $(Ms_n) = \sum_n M|b_n|$ converges as well.

Now since $0 \leq |a_n b_n| \leq M|b_n|$, by the comparison test, $\sum_n |a_n b_n|$ converges, thus $\sum_n a_n b_n$ is absolutely convergent, which implies that $\sum_n a_n b_n$ converges.

## Problem 2

*For each series, find the set of $x \in \mathbb{R}$ where the series converges.*

(a). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = c^{n^2}(x-1)^n$ and $\alpha = \limsup_n |a_n|^{1/n}$. We can compute

$$\alpha = \limsup_n \left| c^{n^2}(x-1)^n \right|^{1/n} = \limsup_n |c^n(x-1)| = |x-1| \limsup_n c^n$$

where we've brought the exponent $n$ out in the first step, since $|a^n b^n| = |ab|^n$.

If $x = 1$, then $\alpha = 0$, so the series converges by the root test regardless of $c$. Now let $x \in \mathbb{R} \setminus \{1\}$. We know that $\lim_{n\to\infty} c^n \to +\infty$ if $c > 1$, so $\limsup_n c^n = +\infty$, thus the series diverges for all $x$. Additionally, $\lim_{n\to\infty} c^n = 0$ if $1 > c > 0$, so $\limsup_n c^n = 0$, thus the series converges for all $x$. Finally, if $c = 1$, $a_n = (x-1)^n$ which is a geometric series: it will converge when $|x-1| < 1 \implies 0 < x < 2$ and will diverge otherwise.

In summary:

- If $c > 1$, $x \in \{1\}$ makes the series converge
- If $c = 1$, $x \in (0, 2)$ makes the series converge
- If $0 < c < 1$, $x \in \mathbb{R}$ makes the series converge

(b). *Solution.* Let $a_n = \frac{x^n(1-x^n)}{n}$. Let $x \in \{0, 1\}$. Then $a_n = 0$ for all $n$, thus the series converges. Let $x = -1$. Then our series is $\sum_n a_n = \sum_{n \text{ odd}} \frac{2}{n}$. We can rewrite our sum to be $\sum_n \frac{1}{2\lfloor (n-1)/2 \rfloor + 1}$ (since if $n = 2k$, $2\lfloor (n-1)/2 \rfloor + 1 = n-1$: the odd number directly below it, and if $n = 2k+1$, $2\lfloor (n-1)/2 \rfloor + 1 = n$: itself) and then since $0 < 2\lfloor (n-1)/2 \rfloor + 1 < n$ so $0 < \left|\frac{1}{n}\right| \le \frac{1}{2\lfloor (n-1)/2 \rfloor + 1}$, comparison test says this series diverges (since the harmonic series diverges to infinity).

Now consider when $|x| > 1$. Then we claim there exists an $N \in \mathbb{N}$ such that $x^n(1-x^n) < -1$ for all $n \ge N$. We prove this by considering when $n$ is positive and negative. Let $x > 1$. Note that there exists an $N$ such that $x^n > 2$ for all $n \ge N$: using the inequality from Problem 4(a) of Homework 6 since $x > 1$, we have that $x^n > x^n - 1 \ge n(x-1)$ and then invoke Archimedean property to find $N$ such that $N(x-1) > 2$, it's trivial to see that $n \ge N$ also implies $x^n > 2$. Now if $n \ge N$, we have $1 - x^n < -1$ and since $x^n > 1$, we have $x^n(1-x^n) < 1 - x^n < -1$. Now let $x < -1$. If $n$ is even, $x^n(1-x^n) = |x|^n(1-|x|^n)$, and we have the same $N$ from when $x > 1$ to have $x^n(1-x^n) < -1$. If $n$ is odd, $x^n(1-x^n) = (-1)|x|^n(1-(-1)|x|^n) = -|x|^n(|x|^n+1)$, and using the $N$ from before, we have $-|x|^n(|x|^n+1) < -2(x^n+1) < -2 < -1$. This proves our claim. But then for all $n \ge N$, we have that

$$\frac{x^n(1-x^n)}{n} < \frac{-1}{n} < 0 \implies 0 < \frac{1}{n} = \left|\frac{1}{n}\right| < -\frac{x^n(1-x^n)}{n}$$

And so by the comparison test, $\sum_n -a_n$ diverges. But this is true only if $\sum_n a_n$ diverges, since if $s_N = \sum_{n=1}^N a_n$ and $s'_N = \sum_{n=1}^N -a_n$, we have that $s'_N = -\sum_{n=1}^N a_n = -s_N$, and if $s_N$ converged as $N \to \infty$, constant multiplication limit law would tell us that $s'_N$ would converge as well. Thus, if $|x| > 1$, we have that the series diverges.

Now consider when $0 < x < 1$. Consider $\sum_k 2^k \frac{x^{2^k}(1-x^{2^k})}{2^k} = \sum_k x^{2^k}(1-x^{2^k})$. We have $0 < x^{2^k}(1-x^{2^k}) < x^{2^k} < x^k$ (where the inequality is due to the fact that $x^a$ is monotonically decreasing when $0 < x < 1$, and $2^k > k$), and $\sum_k x^k$ converges since it is geometric series with ratio $x < 1$. Thus, by the comparison test, we have that $\sum_k 2^k \frac{x^{2^k}(1-x^{2^k})}{2^k}$ converges as well. Finally, $\frac{x^n(1-x^n)}{n}$ is monotonically decreasing and bounded below by 0: all the terms are positive, so $a_n > 0$ for all $n$; now see

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}(1-x^{n+1})n}{x^n(1-x^n)(n+1)} < x\frac{1-x^{n+1}}{1-x^n}$$

and $x\frac{1-x^{n+1}}{1-x^n} \to x$ as $n \to \infty$ (limit laws), thus for sufficiently large $n$, we have that $\frac{a_{n+1}}{a_n} < x + \varepsilon$ where setting $\varepsilon = 1 - x > 0$ gives $\frac{a_{n+1}}{a_n} < 1$, hence the series is monotonically decreasing past that point. Thus, by Cauchy condensation, the series converges when $0 < x < 1$ (technically, Cauchy Condensation only tells us that the series converges starting from our $n$ where the series begins to be monotonically decreasing, but then we have the sum of a convergent series and a finite sum, which itself converges).

It remains to consider the case when $-1 < x < 0$. Now if $n$ is odd, we have $a_n = \frac{(-1)|x|^n(1+|x|^n)}{n^n} < \frac{-2|x|^n}{n}$. Since $\lim_{n\to\infty} \frac{2n}{(1/|x|)^n} = 0$ by Rudin Theorem 3.20 (d), we have some $N$ such that $\left(\frac{1}{|x|}\right)^n > 2n > 0$ for all $n \geq N$, thus $0 < 2|x|^n < \frac{1}{n}$, thus $|a_n| < \frac{2|x|^n}{n} < \frac{1}{n^2}$. Furthermore, when $n$ is even, we have $a_n = \frac{|x|^n(1-|x|^n)}{n} < \frac{|x|^n(1+|x|^n)}{n} < \frac{1}{n^2}$ as well. Thus $|a_n| < \frac{1}{n^2}$ for all $n \geq N$. And since $\sum_n \frac{1}{n^2}$ is a convergent $p$-series ($p > 1$), the comparison test tells us that $\sum_n a_n$ converges as well.

In summary: the series converges when $x \in (-1, 1]$, and diverges otherwise.

(c). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = \frac{1}{\sqrt{n}}\left[\frac{x+1}{2x+1}\right]^n$ and $\alpha = \limsup_n |a_n|^{1/n}$. Note that if $x = -\frac{1}{2}$, none of our terms exist, so we ignore that value. We can compute

$$\alpha = \limsup_n \left|\frac{1}{\sqrt{n}}\left[\frac{x+1}{2x+1}\right]^n\right|^{1/n} = \limsup_n (n^{1/(2n)})^{-1}\left|\frac{x+1}{2x+1}\right| = \left|\frac{x+1}{2x+1}\right| \limsup_n (n^{1/(2n)})^{-1} = \left|\frac{x+1}{2x+1}\right|$$

where our final equality is due to $\lim_{n\to\infty} n^{1/(2n)} = 1$, and so $\liminf_n n^{1/2n} = 1$ (lim inf agrees with convergent limits), and by Problem 8(c) from homework 4, $\limsup_n (n^{1/(2n)})^{-1} = 1^{-1} = 1$ (we've also used the fact $|a^n| = |a|^n$ for our first equality). When $x > 0$, we have $|2x+1| = 2x+1 > x+1 = |x+1|$. When $-\frac{2}{3} < x < 0$, $|2x+1| = -2x-1 < x+1 = |x+1|$. When $x < -\frac{2}{3}$, $|2x+1| > |x+1|$. Now, ratio test gives convergence when $|x+1| < |2x+1|$. Thus, when $x \in (-\infty, -\frac{2}{3}) \cup (0, \infty)$, $\alpha < 1$, thus ratio test says the series converges. When $x \in (-\frac{2}{3}, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$, $\alpha > 1$, thus the ratio test says the series diverges.

If $x = -\frac{2}{3}$, we have $a_n = (-1)^n \frac{1}{\sqrt{n}}$. Note that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, thus the series is monotonically decreasing. Thus, the alternating series test says the series converges.

Finally, if $x = 0$, $a_n = \frac{1}{\sqrt{n}}$, thus $\sum_n a_n$ diverges since this is a $p$-series where $p = \frac{1}{2} < 1$.

In summary: the series converges when $x \in (-\infty, -\frac{2}{3}] \cup (0, \infty)$.

(d). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = \left[\frac{(2n)!}{n(n!)^2}\right](x-e)^n$ and define $\overline{\alpha} = \limsup_n \left|\frac{a_{n+1}}{a_n}\right|$ and $\underline{\alpha} = \liminf_n \left|\frac{a_{n+1}}{a_n}\right|$. $a_n = 0$ only when $x = e$. In this case, $a_n = 0$, thus $\sum_n a_n$ converges. Now assume that $x \neq e$. We can compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2n+2)(2n+1)n}{(n+1)^2(n+1)}(x-e)\right| = |x-e|\frac{4n^2+2n}{n^2+2n+1}$$

We see $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-e|\lim_{n\to\infty}\frac{4+2/n}{1+2/n+1/n^2} = 4|x-e|$ (applying limit laws for multiplication and division). Thus, $\overline{\alpha} = \underline{\alpha} = 4|x-e|$ since the limit exists. Thus, the series converges when $4|x-e| < 1 \implies |x-e| < \frac{1}{4} \implies e-\frac{1}{4} < x < e+\frac{1}{4}$ and diverges when $4|x-e| > 1 \implies |x-e| > \frac{1}{4} \implies x-e > \frac{1}{4} \implies x > e+\frac{1}{4}$ and $x - e < -\frac{1}{4} \implies x < e - \frac{1}{4}$ by the ratio test.

If $x = e + \frac{1}{4}$, we have $a_n = \left[\frac{(2n)!}{n(n!)^2}\right]\left(\frac{1}{4}\right)^n$. Then, $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{1}{|x-e|}\frac{1+2/n+1/n^2}{4+2/n} - 1\right) = \lim_{n\to\infty}\frac{n+2+1/n-n-1/2}{1+1/(2n)} = 2 > 1$, thus by Raabe's test, the series converges.

If $x = e - \frac{1}{4}$, we have $a_n = \left[\frac{(2n)!}{n(n!)^2}\right]\left(-\frac{1}{4}\right)^n$. But we have already proven $\sum_n |a_n|$ converges (above), and so the series converges absolutely, thus this series converges.

In summary: the series converges when $x \in [e - \frac{1}{4}, e + \frac{1}{4}]$.

## Problem 3

*Discuss the series whose nth terms are shown below:*

$$a_n = (-1)^n \frac{n^n}{(n+1)^{n+1}}, \qquad b_n = \frac{n^n}{(n+1)^{n+1}},$$
$$c_n = (-1)^n \frac{(n+1)^n}{n^n}, \qquad d_n = \frac{(n+1)^n}{n^{n+1}}.$$

*Solution.* $a_n$ will converge. We have that

$$\frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} < \frac{1}{n+1}$$

Then, since $\lim_{n\to\infty} \frac{1}{n+1}$, by Squeeze test, we have that $\lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = 0$ as well (since it is bounded below by 0). Furthermore, note that $|a_{n+1}| = \frac{(n+1)^{2(n+1)}}{n^n(n+2)^{n+2}} |a_n|$, and since $\frac{(n+1)^{2(n+1)}}{n^n(n+2)^{n+2}} < 1$ for all $n$, $|a_n|$ is monotonically decreasing. Thus, applying the alternating series test, we get that $\sum_n a_n$ converges.

Note that $b_n$ fails to converge. Let $s_n = \frac{1}{nb_n} = \frac{(n+1)^{n+1}}{n^{n+1}} = \left(\frac{n+1}{n}\right)^{n+1}$. By the definition of $e$ in Rudin, we have that $s_n \to e$ as $n \to \infty$. Thus, for all $\varepsilon > 0$, we have that there exists $N$ such that for all $n \geq N$ such that $|s_n - e| < \varepsilon$. Let $\varepsilon = 1$. Thus, $0 < e - 1 < s_n < 1 + e$. Rearranging, we get that $0 < \frac{1}{1+e}\frac{1}{n} < b_n$. Thus, by the comparison test, since $\sum_n \frac{1}{n}$ diverges to infinity, we must have that $b_n$ diverges as well. This means that although $a_n$ is convergent, it is not absolutely convergent.

Note that $c_n$ fails to converge. Since $\frac{n+1}{n} > 1$ for all $n \in \mathbb{N}$, we have $\left(\frac{n+1}{n}\right)^n > 1$. Thus $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \geq 1 > 0$ if the limit exists. Regardless, this implies $\lim_{n\to\infty} c_n \neq 0$, and so the crude divergence test tells us $\sum_n c_n$ fails to converge (and so fails to absolutely converge).

Note that $d_n$ fails to converge. We have $d_n = \frac{(n+1)^n}{n^{n+1}} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n}$. Since $\left(\frac{n+1}{n}\right)^n > 1$ for all $n$, we have that $d_n > \frac{1}{n} = \left|\frac{1}{n}\right| > 0$. Thus, by the comparison test, $\sum_n d_n$ diverges to infinity, since the harmonic series also diverges to infinity (and so $\sum_n d_n$ also fails to absolutely converge).

## Problem 4

*Suppose* $x_1 \geq x_2 \geq x_3 \geq \cdots$ *and* $\lim_{n\to\infty} x_n = 0$. *Show that the following series converges:*

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \pm \cdots .$$

*Solution.* Let $s_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Note that since $\lim_{n\to\infty} x_n = 0$, we have that $\lim_{n\to\infty} s_n = 0$ as well, by Problem 8(a) from homework 3. Furthermore, note that $s_n$ is monotonically decreasing: for the sake of contradiction, assume the opposite, that is, there exists a $s_k$ such that $s_k > s_{k-1}$ (maybe assume that this is the first such $k$ff). Then

$$s_k = \frac{x_1}{k} + \frac{x_2}{k} + \cdots + \frac{x_{k-1}}{k} + \frac{x_k}{k} > \frac{x_1}{k-1} + \frac{x_2}{k-1} + \cdots + \frac{x_{k-1}}{k-1} = s_{k-1}$$

Then we have

$$\frac{x_k}{k} > \frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1} - \frac{x_1 + x_2 + \cdots + x_{k-1}}{k} = \frac{x_1 + x_2 + \cdots + x_{k-1}}{k(k-1)}$$

Or

$$x_k > \frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1} \geq \frac{(k-1)x_{k-1}}{k-1} = x_{k-1}$$

but this contradicts the assumption that $x_n$ are monotonically decreasing, thus $s_k \leq s_{k-1}$.

Now let $b_n = (-1)^{n+1}$. Note that $b_1 + b_2 + \cdots + b_N$ is bounded (ie. it always either 0 or 1). Thus, we can apply Dirichlet's Theorem to the series $\sum_n s_n b_n$, which says that

$$\sum_n s_n b_n = x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - + \cdots$$

converges.

## Problem 5

*(a). Prove: if $a_n \geq a_{n+1} \geq 0$ for all $n$, and $\sum a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.*

*(b). Prove: If $\sum(b_n^2/n)$ converges, $\frac{1}{N}\sum_{j=1}^N b_j \to 0$ as $N \to \infty$.*

*[Hint: In part (a), it's enough to prove that $\frac{1}{2}na_n \to 0$.]*

(a). *Solution.* Since $a_n \geq a_{n+1} \geq 0$ for all $n$ and $\sum a_n$ converges, the Cauchy Condensation Test gives $\sum_k 2^k a_{2^k}$ converges as well. Since $a_n$ monotonically decreases and is always positive, we have $2^k a_n \leq 2^k a_{2^k}$ for $n \in \mathbb{N}$ such that $2^k \leq n$. Note that for any $k$, $n < 2^{k+1}$ implies $\frac{n}{2} < 2^k$. Thus, $0 \leq \frac{n}{2}a_n < 2^k a_n \leq 2^k a_{2^k}$ for $2^k \leq n < 2^{k+1}$. Not that $2^k a_{2^k} \to 0$ as $k \to \infty$. Thus, givn $\varepsilon > 0$, there exists $K$ such that $0 < 2^k a_{2^k} < \varepsilon$ for all $k \geq K$. Then if $n \geq 2^K$, there exists $k \geq K$ such that $2^k \leq n < 2^{k+1}$, and so $\frac{1}{2}na_n < 2^k a_{2^k} < \varepsilon$. Thus $\frac{1}{2}na_n \to 0$ as $n \to \infty$. But then $na_n \to 0$ as well; this is easy to see: if $na_n$ diverged, then $\frac{1}{2}na_n$ would diverge as well (one can simply consider $\varepsilon/2$ and get breaking of convergence), and if $na_n$ converges to a value, the constant multiplication limit law tells us that $\frac{1}{2}na_n$ would converge to half that value, and so $\lim_{n\to\infty} na_n = 0$.

(b). *Solution.* We first prove that if $a_n > 0$ and $\sum_n a_n$ converges, then $\frac{1}{N}\sum_{n=1}^N na_n \to 0$ as $N \to \infty$. Thus, let $\sum_{n=1}^\infty a_n = L$. We see that

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^N na_n &= \frac{1}{N}\sum_{n=1}^N \sum_{j=1}^n a_n \\
&\leq \frac{1}{N}\sum_{j=1}^N \sum_{n=j}^N a_n
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^N na_n \leq \frac{1}{N}\sum_{n=1}^N \sum_{n=j}^\infty a_n \\
\implies &\frac{1}{N}\sum_{n=1}^N na_n \leq \frac{1}{N}\sum_{n=1}^N \left(L - \sum_{n=1}^{j-1} a_n\right) \\
\implies &\frac{1}{N}\sum_{n=1}^N na_n \leq L - \frac{1}{N}\sum_{n=1}^N \sum_{n=1}^{j-1} a_n
\end{aligned}$$

If $s_j = \sum_{n=1}^{j-1} a_n$, by Problem 8(a) from homework 3, we have that $c_j \to L$ as $j \to \infty$, thus $\frac{1}{N}\sum_{n=1}^N c_n \to L$. Thus, by Squeeze theorem, we have that $\frac{1}{N}\sum_{n=1}^N na_n \to 0$ as $N \to \infty$.

Now, applying this, we get that $\frac{1}{N}\sum n = 1^N b_n^2 \to 0$ as $N \to \infty$. Now, note that $\frac{1}{N}\sqrt{\sum_{n=1}^N \sum_{n=1}^N b_n^2}$ (this is done by Cauchy-Schwartz). Thus, using triangle inequality, we arrive at

$$0 \leq \left|\frac{1}{N}\sum_{n=1}^N b_n\right| \leq \frac{1}{N}\sum_{n=1}^N |b_n| \leq \sqrt{\frac{1}{N}\sum_{n=1}^N b_n^2}$$

and so by Squeeze theorem, since $\frac{1}{N}\sum_{n=1}^N b_n^2 \to 0$ (and so it's square root certainly does), we have that $\frac{1}{N}\sum_{n=1}^N b_n \to 0$ as $N \to \infty$, as desired.

## Problem 6

*Define* $f(\theta) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)\theta)$*. Determine the domain of* $f$*, namely, the set of all real* $\theta$ *where the series converges, by completing the steps below.*

*(a). Obtain the following identities, valid for each* $n \in \mathbb{N}$ *at all points where* $\sin\theta \neq 0$*:*

$$C_n(\theta) = \cos(\theta) + \cos(3\theta) + \cos(5\theta) + \cdots + \cos((2n-1)\theta) = \frac{\sin(2n\theta)}{2\sin\theta},$$

$$S_n(\theta) = \sin(\theta) + \sin(3\theta) + \sin(5\theta) + \cdots + \sin((2n-1)\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta},$$

*[Suggestion: Use geometric sums of complex numbers, with* $e^{it} = \cos(t) + i\sin(t)$*.]*

*(b). Prove that the domain of* $f$ *is the interval* $(-\infty, +\infty)$*.*

*(c). Find a sequence* $(\theta_n)$ *such that* $\theta_n \to 0$ *and* $S_n(\theta_n) \to +\infty$ *as* $n \to \infty$*. Explain why your solution in part (b) is correct in spite of the evident unboundedness of the sequence* $(S_n(\theta_n))$*.*

(a). *Solution.* We have $\sum_{k=1}^{n} e^{i(2k-1)\theta} = C_n(\theta) + iS_n(\theta)$. We can rewrite our sum as $\sum_{k=0}^{n-1} e^{i(2k+1)\theta}$. But this is a geometric series with common ratio $e^{2i\theta}$ and initial value $e^{i\theta}$, thus

$$C_n(\theta) + iS_n(\theta) = e^{i\theta}\frac{1-(e^{2i\theta})^n}{1-e^{2i\theta}} = \frac{1-\cos(2n\theta) - i\sin(2n\theta)}{e^{-i\theta} - e^{i\theta}}$$

But note that

$$\frac{1}{e^{-i\theta} - e^{i\theta}} = \frac{1}{\cos(-\theta) + i\sin(-\theta) - \cos(\theta) - i\sin\theta} = \frac{1}{-2i\sin(\theta)} = \frac{i}{2\sin\theta}$$

Thus

$$C_n(\theta) + iS_n(\theta) = \frac{\sin(2n\theta)}{2\sin\theta} + i\frac{1-\cos(2n\theta)}{2\sin\theta}$$

For equality, the real components must equal the real components, and the imaginary components must equal the imaginary components, thus since $C_n(\theta)$ and $S_n(\theta)$ are strictly real-valued functions, we have

$$C_n(\theta) = \frac{\sin(2n\theta)}{2\sin\theta}, \quad S_n(\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta}$$

as desired.

(b). *Solution.* We seek to show that the series converges for all $\theta \in \mathbb{R}$. Recall that $|\sin(x)| \geq \frac{2x}{\pi}$ (Piazze @331), and so $\frac{1}{2k-1}\sin((2k-1)\theta)$

Fix some $\theta \in \mathbb{R}$. Then $S_n(\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta} \leq$, thus $|\sum_k \sin((2k-1)\theta)| = |S_n(\theta)| \leq \frac{1}{2\sin\theta}$. Thus $S_n(\theta)$ is a bounded sequence. Additionally, $\frac{1}{2k-1} \to 0$ as $n \to \infty$, and is monotonically decreasing. Thus, we can apply Dirichlet's theorem to get that $\sum_k \frac{\sin((2k-1)\theta)}{2k-1}$ converges. Since $\theta \in \mathbb{R}$ was arbitrary, we have that $f(\theta)$ has domain $(-\infty, \infty)$.

(c). *Solution.* Consider $(\theta_n) = \frac{1}{n}$. Then $S_n(\theta_n) = \frac{1-\cos 2}{2\sin\frac{1}{n}}$. We have that $\sin(\frac{1}{n}) \to 0$ as $n \to \infty$ (since $\sin(x) \to 0$ as $x \to \infty$). Thus $S_n(\theta_n) \to \infty$ as $n \to \infty$, since we are just multiplying a divergent sequence by a constant.

This doesn't change the convergence of part (b): notably, $f(\theta) = S_n(\theta)/(2k-1)$: this reciprocal term is "controlling" the growth of $S_n(\theta)$. So divergence here does not give divergence for $f(\theta)$.