1 Problem 1

Find all integers $n > 1$ with the property that for each positive divisor d of n, we also have that

$$(d+2) \mid (n+2)$$

Solution. We first show no even integers satisfy our property. If n is even, there exists some $k \in \mathbb{N}$ such that $2k = n$, so $k \mid n$. Furthermore, $n+2$ is even as well, so $\frac{n+2}{2}$ is an integer, specifically $\frac{n+2}{2} = \frac{n}{2} + 1 = k + 1$. So $2(k+1) = n+2 \implies k+1 \mid n+2$. Furthermore, $k+1$ must be the greatest possible divisor of $n+2$ that is not equal to $n+2$, since if a divisor d was greater than $k+1$, there is some integer $1 < m < 2$ where $dm = n+2$, but no such integer m exists. But $k+2 > k+1$, and $k+2 = \frac{n}{2} + 2 \neq n+2$ when $n > 1$, so $k+2 \nmid n+2$. So there exists a divisor of n such that two more than it is not a divisor of $n+2$.

Now, we consider odd composite integers n. Then there exists an integers $1 < d, q < n$ (not necessarily distinct) such that $qd = n$ (so $d \mid n$). Without loss of generality, let $d \geq q$. Then $n+2 = qd+2$. Note that both q and d must be odd, otherwise n would be divisible by 2 and would be even, which would be against our assumption. We can write $n+2 = qd+2+2q-2q = q(d+2) - 2(q-1)$, so

$$(n+2)/(d+2) = q - 2(q-1)/(d+2)$$

But since $d \geq q$, $d+2 > q-1$ so $d+2 \nmid q-1$. Furthermore, $q-1$ is even so $2(q-1)$ is even, but $d+2$ is odd, so $d+2 \nmid 2(q-1)$, so $2(q-1)/(d+2)$ is not an integer, thus our term on the right is not an integer. But then $d+2 \nmid n+2$. Thus odd composite integers do not satisfy our property either.

Finally, consider the only remaining possibility, when n is an odd prime number. The only such divisors of this is n and 1. $n+2 \mid n+2$ trivially. If and only if $1+2 = 3 \nmid n+2$, we have our desired property then. So if n is prime and of the form $n = 3j-2$ for some $j \in \mathbb{N}$, then n must satisfiy our property. We have shown that no other such n can satisfy our property, thus this is all the possible solutions. □

2 Problem 2

Find all positive integers m and n such that

$$2^m - 3^n = 7$$

Solution. We can rearrange our equation to get

$$2^m = 7 + 3^n \tag{1}$$

Obviously, any m that satisfies the above equation will also satisfy $2^m = 2 \cdot 2^{m-1} \equiv 7 \pmod 3$ (since $3 \mid 3^n$ for any n). That is to say, the set of solutions M_1 to equation (1) (elements in M_1 are of the form 2^m) is a subset of the set of solutions M_2 to our subsequent relation, $M_1 \subset M_2$. If X is the set of solutions $x \in \mathbb{Z}$ to $2x \equiv 7 \pmod 3$, then clearly $M_2 \subset X$.

Recall proposition 7.2 (B) from the course notes: if $a, b, m \in \mathbb{Z}$ with $m \neq 0$ and $d = gcd(a, m)$, then if $d \mid b$, the congruence equation $ax \equiv b \pmod m$ has exactly d solutions. Since 2 and 3 are coprime, we have that $d = 1$, so $d \mid 7$, thus $2x \equiv 7 \pmod 3$ has exactly one solution. Thus, X has exactly one element. Therefore, since $M_1 \subset X$, equation (1) has at most one solution.

We can verify that there does exist such a solution, namely when $m = 4$ and $n = 2$, then we have $2^4 - 3^2 = 16 - 9 = 7$. □

3 Problem 3

Let $k \in \mathbb{N}$. Show that there exists k consecutive positive integers with the property that no integer from this set is of the form $a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

Solution. Let $k \in \mathbb{N}$ be arbitrary. Let m be the product of the squares of the first k primes q of the form $q = 4j + 3$ (for some $j \in \mathbb{N}$), i.e. $m = \prod_{i=1}^{k} q_i^2$.

Note that we can always k-many q_i. To prove this, for the sake of contradiction, assume there are only $r < k$ many such primes of this form, $q_1, q_2, \ldots, q_r$ (where $q_1 < q_2 < \cdots < q_r$). Note that $n = 4q_1 q_2 \cdots q_r - 1$ is of the form $4j + 3$, but $n > q_r$, so by assumption, n cannot be prime, so is composite. Further, note that none of the q_i and 2 divide n, thus all the primes in the prime factor decomposition of n is of the form $4j + 1$, thus $n \equiv 1 \pmod 4$ which is a contradiction. Thus n is a prime greater than q_r and $n = 4j - 1$. We can do this indefinitely to get k many primes of the form $4j + 3$.

Now, consider the system

$$\begin{aligned}
x &\equiv q_1 - 1 \pmod{q_1^2} \\
x &\equiv q_2 - 2 \pmod{q_2^2} \\
&\vdots \\
x &\equiv q_k - k \pmod{q_k^2}
\end{aligned}$$

By the Chinese Remainder Theorem, there exists a unique solution to the system modulo m, which we'll call x_0. Then, let $x_i = x_0 + i$ for $1 \leq i \leq k$. Note that $x_i = q_i + nq_i^2$, thus $q_i \mid x_i$ but $q_i^2 \nmid x_i$ (and so $q_i^s \nmid x_i$ for all $s \geq 2$). Thus, there are k consecutive integers $x_1, x_2, \ldots, x_k$ whose prime number decomposition that contain a prime of the form $4j+3$ with exponent 1. But by Theorem 13.4 from the notes, for all $a, b \in \mathbb{Z}$, $a^2 + b^2 \neq x_i$ for all $1 \leq i \leq k$, since the exponent of q_i is $\gamma_i = 1$, which is not even. $\square$

4 Problem 4

As always, for each positive integer m, we have that $d(m)$ is the number of the positive divisors of m; also, we let $\phi(m)$ be the corresponding value of the Euler ϕ-function. Then compute the following limits:

$$\lim_{n\to\infty} \frac{n!}{d(n!)\phi(n!)}$$

$$\lim_{n\to\infty} \frac{n!}{2^{d(n!)}}$$

Solution.
Limit 1: Note that the prime factor decomposition of $n!$ contains every prime that came before it, since if p is prime and $p < n$, $p \mid n!$ by definition of factorial. Thus, if we enumerate the primes in order (ie. $p_1 = 2$, $p_2 = 3$, etc.), let p_r be the greatest prime less than or equal to n. Then we can write $n! = \prod_{i=1}^{r} p_i^{\alpha_i}$ thus

$$\frac{n!}{d(n!)\phi(n!)} = \prod_{i=1}^{r} \frac{p_i^{\alpha_i}}{d(p_i^{\alpha_i})\phi(p_i^{\alpha_i})} = \prod_{i=1}^{r} \frac{p_i^{\alpha_i}}{(\alpha_i+1)p_i^{\alpha_i-1}(p-1)} = \prod_{i=1}^{r} \frac{p_i}{(\alpha_i+1)(p_i-1)}$$

Note that for any

Limit 2: We are proving $\frac{a_{n+1}}{a_n} = (n+1)2^{d(n!)-d((n+1)!)} < 1$. If $n+1$ is prime, then ff see photo.

Now if $n+1$ is not prime, we claim the inequality $n+1 < 2^{\prod_{\beta_i \neq 0}(\alpha_i+\beta_i+1)}$. If $n+1$ is a prime squared, something something 48^2 has 6 factors of 7... we show that it works. Somehow, this links to when $n+1$ contains any prime squared.

Now, our remaining case is when $n+1$ is a sequence of primes with exponent 1.

Current strategy: if $d((n+1)!) - d(n!) > n \implies d((n+1)!) > n + d(n!)$, we are done since $(n+1)2^{-n}$ is definitely less than 1. Asymptotoically, this looks like $d(n!) > n^2$. □