Math 321 Homework 1

In this homework, we will need several definitions. Let $I = [a, b]$ be an interval and $k \geq 0$ be an integer. If $f\colon I \to \mathbb{R}$ is a function that is k-times differentiable on I, then we define

$$\|f\|_{C^k(I)} = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x)|.$$

This quantity is called the "C^k norm of f." We define $C^k(I)$ to be the set of functions $f\colon I \to \mathbb{R}$ that satisfy the following two properties. (i): f is k-times differentiable on I, and (ii): $f^{(k)}$ is continuous on I. We define a metric on $C^k(I)$ as follows: $d(f, g) = \|f - g\|_{C^k(I)}$, i.e.

$$d(f, g) = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x) - g^{(j)}(x)|. \tag{1}$$

It is straightforward to verify that this is indeed a metric, but you do not have to do so for this homework.

Problem 1

Let $f(t) = e^t$; recall that f is monotone increasing, $f'(t) = f(t)$, and $f(0) = 1$. Let $P_n(t)$ be the n-th order Taylor polynomial of f at the point $x_0 = 0$, as discussed in lecture. Let $I = [-1, 1]$ and let $k \geq 1$ be an integer.

Using Taylor's theorem, prove that the sequence $\{P_n\}$ converges to f in the metric space $C^k(I)$.

Hints *(i) Compute the Taylor polynomial $P_n(t)$. (ii) What is the derivative of P_n? (iii) What are the higher derivatives of P_n? (iv) How can you estimate each term in* (1)*?*

Solution. Recall that for $f(t)$, the n-th ordered Taylor polynomial at $x_0 = 0$ is

$$P_n(t) = \sum_{i=0}^{n} \frac{x^i}{i!}$$

Furthermore, note that the j-th derivative of $P_n(t)$ is 0 if $j > n$ and

$$\frac{d^j}{dx^j} P_n(t) = \frac{d^j}{dx^j} \sum_{i=0}^{j-1} \frac{x^i}{i!} + \sum_{i=j}^{n} \frac{d^j}{dx^j} \frac{x^i}{i!} = 0 + \sum_{i=j}^{n} \frac{1}{i!} \frac{i!}{(i-j)!} x^{i-j} = \sum_{i=j}^{n} \frac{x^{i-j}}{(i-j)!} = \sum_{i=0}^{n-j} \frac{x^i}{i!} = P_{n-j}(t) \tag{2}$$

when $j \leq n$.

Recall from Taylor's theorem that, since e^t is continuous and it's $(n+1)$ derivative always exists (simple induction, since $f'(t) = f(t)$), there exists c_n between t and 0 such that

$$e^t = P_n(t) + \frac{f^{(n+1)}(c_n)}{(n+1)!} t^{n+1} = P_n(t) + \frac{e^{c_n}}{(n+1)!} t^{n+1} \tag{3}$$

We remark that since e^t is k-times differentiable for any k, $e^t \in C^k(I)$. So we now only need to show $d(e^t, P_n) < \varepsilon$

for arbitrary $\varepsilon > 0$. Consider $d(e^t, P_n)$ in $C^k(I)$ when we fix $n \geq k$.

$$\begin{aligned}
d(e^t, P_n) &= \sum_{j=0}^{k} \sup_{t \in I} |e^t - P_n^{(j)}(t)| \\
&= \sum_{j=0}^{k} \sup_{t \in I} |e^t - P_{n-j}(t)| && \text{applying (2)} \\
&= \sum_{j=0}^{k} \sup_{t \in I} \left| P_{n-j}(t) + \frac{e^{c_{n-j}}}{(n-j+1)!} t^{n-j+1} - P_{n-j}(t) \right| && \text{applying (3)} \\
&= \sum_{j=0}^{k} \sup_{t \in I} \left| \frac{e^{c_{n-j}}}{(n-j+1)!} t^{n-j+1} \right| \\
&= \sum_{j=0}^{k} \frac{1}{(n-j+1)!} \sup_{t \in I} \left| e^{c_{n-j}} t^{n-j+1} \right| \\
&\leq \sum_{j=0}^{k} \frac{e}{(n-j+1)!} \\
&\leq \frac{ke}{(n-k+1)!} \\
&\leq \frac{ke}{n-k}
\end{aligned}$$

where the 6th line is done by the following reasoning: since $c_{n-j} \leq 1$ always, and since e^t is monotonically increasing, $e^{c_{n-j}} \leq e^1$ and so $|e^{c_{n-j}} t^{n-j+1}| \leq |et^{n-j+1}|$ and taking the supremum of both sides preserves weak inequalities, giving

$$\sup_{t \in I} |e^{c_{n-j}} t^{n-j+1}| \leq \sup_{t \in I} |et^{n-j+1}|$$

Now, since the maximum value $|t^x|$ can obtain when $t \in I$ (and $x > 0$, since $n - j + 1 \geq n - k + 1 \geq 1$) is 1, we have

$$|et^{n-j+1}| = |e||t^{n-j+1}| < e \implies \sup_{t \in I} |et^{n-j+1}| \leq e$$

giving us the desired inequality.

Let $\varepsilon > 0$. Let $N = \max\{k, \lceil 2ke/\varepsilon + k \rceil\}$. Then for all $n \geq N$, we have

$$d(e^t, P_n) \leq \frac{ke}{n-k} \leq \frac{ke}{N-k} \leq \frac{ke}{2ke/\varepsilon + k - k} = \frac{\varepsilon}{2} < \varepsilon$$

Hence, $\{P_n\}$ converges to $f = e^t$ in $C^k(I)$.

Problem 2

Let $f(t) = e^t$. Let $P_n(t)$ be the n-th order Taylor polynomial of f at the point $x_0 = 0$.

(a). Let $n \geq 1$. Prove that $n!P_n(1)$ is an integer.

(b). Using part (a) and Taylor's theorem, prove that Euler's number e is irrational. You may use the fact that e^t is strictly monotone increasing, and $0 < e < 3$.

Hint*: if e were rational, then we could write $e = m/n$....*

(a). *Solution.* See

$$n!P_n(t) = n! \sum_{i=0}^{n} \frac{t^i}{i!} = \sum_{i=0}^{n} \left(n(n-1) \cdots (i+1) t^i \right)$$

Now when $t = 1$, each term is an integer since the integers are closed under multiplication. The sum will also be an integer since integers are closed under addition, so $n!P_n(1)$ is an integer as well.

(b). *Solution.* Assume, for the sake of contradiction, that $e \in \mathbb{Q}$, that is to say, $e = m/n$, for some $m \in \mathbb{Z}, n \in \mathbb{N}$. Note then $n!e = m(n-1)! \in \mathbb{Z}$. Let $n' = \max\{2, n\}$. We also have $n'!e \in \mathbb{Z}$ (if $n' \neq n$ then $2 > n$, which means $n = 1 \implies e = m$, and $2e = 2m \in \mathbb{Z}$). Also recall by Taylor's theorem that

$$e = P_{n'}(1) + \frac{f^{(n'+1)}(x)}{(n'+1)!} = P_{n'}(1) + \frac{e^x}{(n'+1)!}$$

for some $x \in (0, 1)$. Then

$$n'!e = n'!P_{n'}(t) + \frac{e^x}{n'+1}$$

Since $0 < e < 3$ and $0 < x < 1$, we have $0 < e^x < 3$, and so since $n' \geq 2$, $n' + 1 > e^x \implies \frac{e^x}{n'+1} \notin \mathbb{Z}$. However, by part (a), we know that $n'!P_{n'}(t) \in \mathbb{Z}$ and so $\frac{e^x}{n'+1} = n'!e - n'!P_{n'}(t) \in \mathbb{Z}$ (since this is the difference of two integers), which is a contradiction.

Problem 3

The next problem concerns monotone increasing functions, and will help prepare us for the Riemann–Stieltjes integral. Let $\alpha\colon [0,1] \to \mathbb{R}$ be increasing. Recall from last term that for every $c \in [0,1]$, $\lim_{x \searrow c} \alpha(x)$ and $\lim_{x \nearrow c} \alpha(x)$ always exist. Thus α is continuous at c if and only if $\lim_{x \searrow c} \alpha(x) = \lim_{x \nearrow c} \alpha(x)$. If α is not continuous at c, then $\lim_{x \nearrow c} \alpha(x) < \lim_{x \searrow c} \alpha(x)$, and we say α has a *jump discontinuity* at c.

Let $\alpha\colon [0,1] \to \mathbb{R}$ be montone increasing. Prove that the set of points $c \in [0,1]$ where α is not continuous is either finite (possibly empty), or countably infinite.

Solution. Consider the set $D \subset [0,1]$ where $D = \{c \mid \alpha \text{ has a jump discontinuity at } c\}$. We seek to show there exists a injective map from D to $\mathbb{Q}$, and so the cardinality of D is at most the cardinality of $\mathbb{Q}$, hence D is at most countable. By the density of the rationals, since $\lim_{x \nearrow c} \alpha(x) < \lim_{x \searrow c} \alpha(x)$ when $c \in D$, there is some rational q such that $\lim_{x \nearrow c} \alpha(x) < q < \lim_{x \searrow c} \alpha(x)$. Let $\phi\colon D \to \mathbb{Q}$ be defined by $c \mapsto q_c$ where q_c is one such rational such that $\lim_{x \nearrow c} \alpha(x) < q_c < \lim_{x \searrow c} \alpha(x)$.

We now prove injectivity of ϕ. Let $c_1, c_2 \in D$ such that $c_1 \neq c_2$. WLOG let $c_1 < c_2$. Let $m = \frac{c_1 + c_2}{2}$. Since α is monotonically increasing, we have $\alpha(c_1) \leq \alpha(m) \leq \alpha(c_2)$. Clearly $\inf_{c_1 < x < m} \alpha(x) \leq \alpha(m)$ and $\sup_{m < x < c_2} \alpha(x) \geq \alpha(m)$, and $\lim_{x \searrow c_1} \alpha(x) = \inf_{c_1 < x < m} \alpha(x)$ and $\lim_{x \nearrow c_2} \alpha(x) = \sup_{m < x < c_2} \alpha(x)$ by Rudin Theorem 4.29, so

$$q_{c_1} < \lim_{x \searrow c_1} \alpha(x) \leq \alpha(m) \leq \lim_{x \nearrow c_2} \alpha(x) < q_{c_2}$$

Hence, $q_{c_1} \neq q_{c_2}$ which implies $\phi(c_1) \neq \phi(c_2)$, which shows the injectivity of the map. Hence, D is at most countable (either finite or countably infinite).