Problem 1

If $f\colon X \to Y$ is a continuous mapping between Hausdorff topological spaces X and Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every set $E \subseteq X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Solution. Let $y \in f(\overline{E})$, then there is some $x \in \overline{E}$ such that $f(x) = y$. If $x \in E$, we have that $y \in f(E)$ hence $y \in \overline{f(E)}$. If $x \notin E$, we must have that $x \in E'$ (since $\overline{E} = E \cup E'$).

So now assume that $x \in E'$. If $y \in f(E)$, we have $y \in \overline{f(E)}$ as well. So now also assume that $y \notin f(E)$. Consider some arbitrary open set W containing y. Since f is continuous, $U = f^{-1}(W)$ is open, and furthermore, $x \in U$. Since $x \in E'$, we have that $U \setminus \{x\} \cap E \neq \emptyset$. So there is some $x_1 \in U \setminus \{x\}$ and $x_1 \in E$. So $f(x_1) \in f(E)$ and $f(x_1) \in f(U \setminus \{x\}) \subseteq f(U) = W$. Hence, we have that $f(x_1) \in f(E) \cap W \neq \emptyset$, and since $y \notin f(E)$, we have $f(x_1) \in f(E) \cap (W \setminus \{y\})$. Hence, $f(E) \cap (W \setminus \{y\}) \neq \emptyset$. But since W was an arbitrary open set containing y, this gives us that $y \in f(E)'$. Thus, $y \in \overline{f(E)}$.

In each case, whether $x \in E$, $x \notin E$ and $y \in f(E)$, or $x \notin E$ and $y \notin f(E)$, we have shown that $y \in \overline{f(E)}$. Since y was an arbitrary point in $f(\overline{E})$, this shows that $f(\overline{E}) \subseteq \overline{f(E)}$ as deired.

To show that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$, we give the following example: let X be $\mathbb{R}$ with the discrete topology, and Y be $\mathbb{R}$ with the usual topology (from the metric). As we have mentioned in class, these are both Hausdorff. Recall that, by definition, every set in X is open, which implies that for any $U \subseteq X$, U^c is open as well, so every set in X is closed as well. Hence, if we let $E \subseteq X$ be the interval (a,b), since E is closed, we have $E = \overline{E} = (a,b)$. Now let f be the identity map. This is obviously continuous, since the pre-image of every open set in Y is a set in X, which is open. So we have $f(\overline{E}) = f((a,b)) = (a,b)$. Note though that $\overline{f(E)} = \overline{(a,b)} = [a,b]$. So $f(\overline{E}) = (a,b) \subsetneq [a,b] = \overline{f(E)}$. Hence, we have some continuous f that maps between two Hausdorff spaces, and there is some $E \subseteq X$ such that $f(\overline{E}) \subsetneq \overline{f(E)}$.

Problem 2

(a). Let X and Y be metric spaces. Prove that for $f\colon X \to Y$, TFAE:

- *(a) f is uniformly continuous on X;*
- *(b) for any sequences (x_n) and (x'_n) in X satisfying $d_X(x_n, x'_n) \to 0$, one has $d_Y(y_n, y'_n) \to 0$, where $y_n = f(x_n), y'_n = f(x'_n)$.*

(b). Identify, with proof, all real numbers p for which the function $f(x) = x^p$ is uniformly continuous on $X = (0, +\infty)$. [It's OK to use a little calculus to support your findings.]

(a). *Solution.* We first prove $(a) \implies (b)$. So assume that f is uniformly continuous on X. Let (x_n), (x'_n) be arbitrary sequences in X such that $d_X(x_n, x'_n) \to 0$. Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous on X, we get some $\delta > 0$ such that $\forall t \in \mathbb{B}_X[x_n; \delta)$, $d_Y(f(x_n), f(t)) < \varepsilon$. Now using the definition of $d_X(x_n, x'_n) \to 0$, we have that for some N, $d(x_n, x'_n) < \delta$ for all $n \geq N$. Hence, $x'_n \in \mathbb{B}_X[x_n; \delta)$, thus $d_Y(f(x_n), f(x'_n)) < \varepsilon$ for all $n \geq N$. But then if $y_n = f(x_n)$, $y'_n = f(x'_n)$, since ε was arbitrary, this is the definition for $d_Y(y_n, y'_n) \to 0$.

Now, to prove $(b) \implies (a)$, we use contraposition, so assume that f is not uniformly continuous on X. Then there is some $\varepsilon' > 0$ such that for any $\delta > 0$, there is some $s \in X$ where for some $t \in \mathbb{B}_X[s; \delta)$, we have $d(f(s), f(t)) \geq \varepsilon'$. Consider $\delta_n = \frac{1}{n}$. For each δ_n, from above, we can get s, t where $t \in \mathbb{B}_X[s; \delta_n)$, but $d(f(s), f(t)) \geq \varepsilon'$; so for each δ_n, we let $x_n = s$ and $x'_n = t$. See that $d_X(x_n, x'_n) \to 0$, since for any $\varepsilon > 0$, using the Archimedean property, we can find $N > \frac{1}{\varepsilon} > 0$ so $\frac{1}{N} = \delta_N < \varepsilon$ and $\forall n \geq N$, $\delta_n \leq \delta_N < \varepsilon$, so for all $n \geq N$, $x'_n \in \mathbb{B}_X[x_n; \delta_n) \implies d_X(x_n, x'_n) < \delta_n < \varepsilon$. However, notice that if $y_n = f(x_n)$ and $y'_n = f(x'_n)$, we have $d_Y(y_n, y'_n) \not\to 0$, since regardless of n, we have $d_Y(y_n, y'_n) \geq \varepsilon'$, hence $d_Y(y_n, y'_n)$ cannot converge to 0. This shows that $(a) \iff (b)$, as desired.

(b). *Solution.* We note that $|x|^p$ is not uniformly continuous on X when $p < 0$ and $p > 1$: via calculus, the slopes are increasing at some point, and so for any ε, we can always find an two points x_1, x_2 such that $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| \geq \varepsilon$.

In the case when $p = 0$, this is uniformly continuous trivially: we just pick $\delta = \varepsilon/2$.

In the $p \in (0, 1]$ case, we have uniform continiuity. I ran out of time, but the essential idea is that since the slope doesn't increase, we can find δ small enough that points are going to be close enough together that $|f(x_1) - f(x_2)| < \varepsilon$... might use a little MVT.

Problem 3

A metric space (X, d) *is called an* ultrametric space *if* d *satisfies the condition*

$$\forall x, y, z \in X, \quad d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(This makes d itself "an ultrametric".) Show that in any ultrametric space (X, d)*,...*

(a). every open ball $\mathbb{B}[x; r)$ *is a closed set;*

(b). one has $y \in \mathbb{B}[x; r)$ *if and only if* $\mathbb{B}[y; r) = \mathbb{B}[x; r)$*; and*

(c). if $\mathbb{B}[x; r_1) \cap \mathbb{B}[y; r_2) \neq \emptyset$*, then one of these balls must contain the other, i.e.,*

$$\mathbb{B}[x; r_1) \subseteq \mathbb{B}[y; r_2) \neq \emptyset \;\; \textit{or} \;\; \mathbb{B}[x; r_1) \supseteq \mathbb{B}[y; r_2) \neq \emptyset$$

[The "p-adic numbers" form an ultrametric space of interest in number theory.]

(a). *Solution.* So normally, triangle inequality gives us $d(x, z) \leq d(x, y) + d(y, z)$.

By a corollary from class, it is sufficient to show that $\mathbb{B}[x; r)' \subseteq \mathbb{B}[x; r)$. If $\mathbb{B}[x; r)' = \emptyset$, we are done. So assume that there exists some $y \in \mathbb{B}[x; r)'$. So for any $r \geq r_1 > 0$, we have $\mathbb{B}[x; r) \cap (\mathbb{B}[y; r_1) \setminus \{y\}) \neq \emptyset$. So there is some $z \in \mathbb{B}[x; r)$ and $z \in \mathbb{B}[y; r_1) \setminus \{y\}$. Hence $d(x, z) < r$ and $0 < d(y, z) < r_1$. Using the inequality, we then get that $d(x, y) < \max\{r, r_1\}$. Since $r \geq r_1$, we have $d(x, y) < r \implies y \in \mathbb{B}[x, r)$. Since $y \in \mathbb{B}[x; r)'$ was arbitrary, this shows that $\mathbb{B}[x; r)' \subseteq \mathbb{B}[x; r)$, so $\mathbb{B}[x; r)$ is closed.

(b). *Solution.* Assume that $y \in \mathbb{B}[x; r)$. So $d(x, y) < r$. Let $z \in \mathbb{B}[y; r)$, so $d(y, z) < r$. Thus, $d(x, z) \leq \max\{d(x, y), d(y, z)\} < r$. Hence, $z \in \mathbb{B}[x; r)$, and since z was arbitrary, we get $\mathbb{B}[y; r) \subseteq \mathbb{B}[x; r)$. Now let $z \in \mathbb{B}[x; r)$, so $d(x, z) < r$. Thus, $d(y, z) \leq \max\{d(x, y), d(x, z)\} < r$. Hence, $z \in \mathbb{B}[y; r)$, and since z was arbitrary, we get $\mathbb{B}[y; r) \subseteq \mathbb{B}[x; r)$. Therefore, we get that $\mathbb{B}[x; r) = \mathbb{B}[y; r)$.

Now assume that $\mathbb{B}[y; r) = \mathbb{B}[x; r)$. Since $y \in \mathbb{B}[y; r)$, we must then have that $y \in \mathbb{B}[x; r)$ by equality. Thus, we have $y \in \mathbb{B}[x; r)$ if and only if $\mathbb{B}[x; r) = \mathbb{B}[y; r)$.

(c). *Solution.* If $\mathbb{B}[x; r_1) \cap \mathbb{B}[y; r_2) \neq \emptyset$, then $\mathbb{B}[x; r_1) \neq \emptyset \neq \mathbb{B}[y; r_2)$, and there exists some $z \in \mathbb{B}[x; r_1)$ and $z \in \mathbb{B}[y; r_2)$. So $d(x, z) < r_1$ and $d(y, z) < r_2$. Assume that $\mathbb{B}[x; r_1) \not\supseteq \mathbb{B}[y; r_2)$. Then there is some point $y_1 \in \mathbb{B}[y; r_2)$ such that $y_1 \notin \mathbb{B}[x; r_1)$. So $d(y, y_1) < r_2$ but $d(x, y_1) \geq r_1$.

We now show that this gives us $r_2 \geq r_1$. Note that $d(y_1, z) \leq \max\{d(y_1, y), d(z, y)\} \leq r_2$. Now we have that $\max\{d(y_1, z), d(x, z)\} \geq d(y_1, x) \geq r_1$. However, $d(x, z) < r_1$, so to have our left hand side $\geq$ than the right, we must have that $d(y_1, z) \geq r_1 > d(x, z)$. Thus, $r_1 \leq d(y_1, z) \leq r_2$ so by transitivity of order in the reals, we get that $r_1 \leq r_2$ as well.

Now take any $x_1 \in \mathbb{B}[x; r_1) \neq \emptyset$, We know that $d(x_1, z) \leq \max\{d(x_1, x), d(x, z)\} < r_1$. Furthermore, we know that $d(y, z) < r_2$, so $d(x_1, y) \leq \max\{d(x_1, z), d(z, y)\} < \max\{r_1, r_2\} = r_2$. So $x_1 \in \mathbb{B}[y; r_2)$. But since $x_1 \in \mathbb{B}[x_1; r_1)$ was arbitrary, this gives us that $\mathbb{B}[x; r_1) \subseteq \mathbb{B}[y; r_2)$. Therefore we must have $\mathbb{B}[y; r_2) \subseteq \mathbb{B}[x; r_1)$, or if this is not true, we have proven that we must then have $\mathbb{B}[x; r_1) \subseteq \mathbb{B}[y; r_2)$. Furthermore, we stated our balls are not $\emptyset$ at the start, and so we have proven what is desired.

Problem 4

Given Hausdorff Topological Spaces $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$*, and continuous functions* $f, g\colon X \to Y$*, consider the* equalizer*:*

$$E = \{x \in X \colon f(x) = g(x)\}.$$

Prove that E *is closed in* X*.*

Solution. For the sake of contradiction, assume that E is not closed in X. Then $E \subsetneq \overline{E}$. So there is some $x_1 \in \overline{E}$ but $x_1 \notin E$. Recall the following theorem from class: for continuous function f_1, f_2 and $Q \subseteq X$, if $f_1(q) = f_2(q)$ for all $q \in Q$, then $f_1(x) = f_2(x)$ for all $x \in \overline{Q}$. Using the theorem with the definition of E then, we must have that $f(x) = g(x)$ for all $x \in \overline{E}$, which includes x_1, so $f(x_1) = g(x_1)$. But then $x_1 \in E$, a contradiction, since we assumed that $x_1 \notin E$. Hence, we must have that $E = \overline{E}$ and so E is closed.

Problem 5

Three continuous functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ are related by the identity

$$f(x+y) = g(x) + h(y)$$

(a). In the special case where $f = g = h$, show that there must be a real number m such that $f(t) = mt$ for all real t.

(b). Drop the hypothesis that f, g, h are identical. Describe the most general trio of continuous functions compatible with the given identity.

(a). *Solution.* Since $f(x+y) = f(x) + f(y)$, we get that $f(2y) = 2f(y)$. Now, we can prove that $f(ky) = kf(y)$ by induction, since if $f(ky) = kf(y)$, we have $f((k+1)y) = f(ky) + f(y) = (k+1)f(y)$, and our base case is done above. Hence $f(ky) = kf(y)$ for all $k \in \mathbb{N}$ and $y \in \mathbb{R}$.

Now, when $x = y = 0$, we have that $f(0) = 2f(0)$, so $f(0) = 0$ is the only value that satisfies this. So $0 = f(0) = f(ky + (-ky)) = f(ky) + f(-ky)$ so $f(-ky) = -kf(y) \implies f(ky) = kf(y)$ is true for any $k \in \mathbb{Z}$ as well.

Now since $f(y) = qf(y/q)$, we get that $\frac{1}{q}f(y) = f(y/q)$ when $q \in \mathbb{Z} \setminus \{0\}$. Soo $f(\frac{p}{q}y) = \frac{p}{q}f(y)$ for any $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. So $f(ky) = kf(y)$ when $k \in \mathbb{Q}$ as well.

Now let $y \in \mathbb{R}$ be fixed. We have that two functions, $f(ky)$ and $kf(y)$ are fixed for all $k \in \mathbb{Q}$. Since f is continuous, and $\mathbb{Q}$ is dense in $\mathbb{R}$, by the theorem in class, this gives us that $f(ky) = kf(y)$ when $k \in \mathbb{R}$ as well. Since $y \in \mathbb{R}$ is arbitrary, this means that $f(ky) = kf(y)$ for all $k, y \in \mathbb{R}$.

It is clear then that for $t \in \mathbb{R}$, we have $f(t) = tf(1)$. So if $f(1) = m \in \mathbb{R}$, we recover $f(t) = mt$, as desired.

(b). *Solution.* Sorry ran out of time...

Problem 6

Here's a key fact every math student should know:

Every nonempty open set in $\mathbb{R}$ can be expressed as a finite or countable union of disjoint open intervals

Prove this, referring to a given open set $U \neq \emptyset$, by following these steps:

(a). For each $x \in U$, let $I(x) = (\alpha(x), \beta(x))$, where

$$\alpha(x) = \inf\{a\colon \text{ one has } x \in (a,b) \text{ for some } (a,b) \subseteq U\}\beta(x) = \sup\{a\colon \text{ one has } x \in (a,b) \text{ for some } (a,b) \subseteq U\}$$

Prove that $x \in I(x)$ and $I(x) \subseteq U$, while $\alpha(x) \notin U$ and $\beta(x) \notin U$. [Argue carefully, since both $\alpha(x) = -\infty$ and $\beta(x) = +\infty$ are possible.]

(b). Let $\mathcal{G} = \{I(x)\colon x \in U\}$. Show that any two intervals in $\mathcal{G}$ must be either disjoint or identical.

(c). Explain why the key fact stated above must hold.

(a). *Solution.* Sorry ran out of time...

(b). *Solution.* Sorry ran out of time...

(c). *Solution.* The basic argument would go somethng like, our $I(x)$ will cover any open set in $\mathbb{R}$ by (a), and there are at most countable many of them by, and so by (b), they are disjoint.