1 January 9

1.1 Complex Numbers

Starts with $\mathbb{N} = \{1, 2, 3, \dots\}$. We can solve $x + 2 = 5$ ($x = 3$), but we cannot solve $x + 5 = 2$. So we introduce $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Now $x + a = b$ is always solvable in $\mathbb{Z}$ ($a, b \in \mathbb{Z}$), namely $x = b - a \in \mathbb{Z}$. So consider $2x = 8$. This has the solution $x = 4 \in \mathbb{Z}$. But it's easy to come up with equations like this that aren't solvable in $\mathbb{Z}$, namely $8x = 2$. So we enlarge our system of numbers to $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$. Now we can solve $ax = b$ for $a, b \in \mathbb{Q}$ as long as $a \neq 0$.

Remark 1. If we tried to add another number ∞ to $\mathbb{Q}$ so that ∞ is a solution to $0x = 1$, this would lead to a breakdown of the rules of arithmetic because $0 \cdot a = 0$ for all a by distributive law ($0 \cdot a + 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a = 0 + 0 \cdot a \implies 0 \cdot a = 0$).

We can now do linear algebra: in $\mathbb{Q}$, we can solve all linear equations and systems of linear equations.

From $\mathbb{Q}$ to $\mathbb{R}$: we want to do calculus. Put in all limits of monotone increasing bounded sequences, e.g.

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e \notin \mathbb{Q}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{pi^2}{6} \notin \mathbb{Q}$$

Actually, calculating the above limit is a highlight of this course.

As a consequence, we get the intermediate value theorem: $f\colon [a, b] \to \mathbb{R}$ continuous, $f(a) < 0$, $f(b) > 0$, then $\exists x \in (a, b)$ such that $f(x) = 0$. Also the extremal value theorem: $f\colon [a, b] \to \mathbb{R}$ continuous, then $\exists x \in [a, b]$ such that $\forall y \in [a, b]$, $f(x) \geq f(y)$. In particuluar, say $a > 0$, then $f(x) = x^2 - a$ on the interval $[0, 1 + a]$, $f(0) = -a < 0$ and $f(1+a) = (1+a)^2 - a = 1 + a + a^2 > 2$, so by the IVT: $\exists x \in \mathbb{R}$ such that $f(x) = 0$. So $x^2 - a = 0$ has a solution in $\mathbb{R}$. So we have a solution to this quadratic equation in $\mathbb{R}$. The notation we use is $\sqrt{a}$. Positive real numbers have square roots in $\mathbb{R}$. So we can sole all quadratic equations $x^2 + bx + c = 0$ if $b^2 - 4c \geq 0$, namely $x = -\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$.

Now we go from $\mathbb{R}$ to $\mathbb{C}$: if $b^2 - 4c < 0$, we cannot solve $x^2 + bx + c$ in $\mathbb{R}$.

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{-1}\sqrt{4c - b^2}$$

where $\sqrt{4c - b^2} \in \mathbb{R}$. So we need to make sense of $\sqrt{-1}$, and then we can solve all quadratic equations $x^2 + bx + c = 0$ where $b, c \in \mathbb{R}$. We simply add the symbol $i := \sqrt{-1}$ to $\mathbb{R}$. We then get the solutions $x = \alpha \pm i\beta$ where $\alpha = -\frac{b}{2}$ and $\beta = \frac{1}{2}\sqrt{4c - b^2}$ where $\alpha, \beta \in \mathbb{R}$. We call i the "imaginary unit" and write numbers as $\alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$. We do our calculations the usual way using the extra rule $i^2 = -1$.

Miracle: this leads to a coherent system of numbers $\mathbb{C}$, the complex numbers, where all quadratic equations can be solved, and we can do calculus (the contents of this course).

Some definitions of the operations:

$$+\colon (a + ib) + (c + id) = (a + c) + i(b + d)$$
$$\times\colon (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Now, this was somewhat informal. So formally, we define $\mathbb{C} = \mathbb{R}^2$ (assuming $\mathbb{R}$ is given). Addition is the same as vector addition. The multiplication is $(a, b)(c, d) = (ac - bd, ad + bc)$. One can check that this multiplication is commutative, associative, satisifies the distributive law, there is a multiplicative unit $(1, 0)$, and every nonzero complex number has a multiplicative inverse: $(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$. Hence, we can freely divide (multiplying by the multiplcative inverse) by nonzero complex numbers. So $\mathbb{C}$ is a field (see [BMPS]).

We can map $\mathbb{R}$ to $\mathbb{C}$ by $a \mapsto (a, 0)$. So geometrically, $\mathbb{C}$ is the plane and $\mathbb{R}$ is the x-axis. This is a "field morphism", i.e. it respects addition and multiplication and sends the multiplication unit to the multiplication unit (so $(a \cdot b, 0) = (a, 0) \cdot (b, 0)$ and $(a + b, 0) = (a, 0) + (b, 0)$). We have $\alpha \in \mathbb{R}$, $(a, b) \in \mathbb{C}$, scalar multiplication: $\alpha(a, b) = (\alpha a, \alpha b)$ and complex multiplication: $(\alpha, 0) \cdot (a, b) = (\alpha a, \alpha\beta)$. So we identify $\mathbb{R}$ with its image in $\mathbb{C}$. Standard basis of $\mathbb{C} = \mathbb{R}^2$: $(1, 0), (0, 1)$. We can abbreviat $1 = (1, 0)$ and $i = (0, 1)$. Write $(a, b) = a(1, 0) + b(0, 1) = a1 + bi = a + ib$. We can check that $i^2 = -1$: $(0, 1) \cdot (0, 1) = (-1, 0) = -1$.

We write $z \in \mathbb{C}$ as $z = a + ib$, $a, b \in \mathbb{R}$. We call a the real part and b the imaginary part, and write $a = \mathrm{Re}(z), b = \mathrm{Im}(z)$. $|a + ib| = \sqrt{a^2 = b^2}$ as the norm / absolute valuue / modulus of $z = a + ib$.

1.1.1 Polar form

It is often convenient to write complex numbers in a different form. Imagining z as a point on the Cartesian plane, we let r be the distance from the origin and θ the angle z sweeps out. We can compute $a = r\cos\theta$ and $b = r\sin\theta$. So $a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. r is the modulus of $a + ib$ and we call θ the argument. The argument is ambiguous, but we can restrict $\theta \in (-\pi, \pi]$, which is called the principal value of the argument. $r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$ if and only if $r = s$ and $\phi - \theta \in 2\pi\mathbb{Z}$.

With this, we can get a geometric meaning of multiplication. Fix $z = r(\cos\theta + i\sin\theta)$. Consider the "multiplying by z" map $\mathbb{C} \to \mathbb{C}$ where $w \mapsto zw$. Write (x, y) as $\binom{x}{y}$.

$$\begin{aligned}
w = \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto r(\cos\theta + i\sin\theta) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\
&= r(\cos\theta + i\sin\theta) \cdot (x + iy) \\
&= rx\cos\theta - ry\sin\theta + i(yr\cos\theta + xr\sin\theta) \\
&= \begin{pmatrix} rx\cos\theta - ry\sin\theta \\ ry\cos\theta + rx\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

where we have the rotation matrix. So we are scaling w by the modulus r and rotating it by the argument θ.

2 January 11

More about the argument: let $z = 1(\cos\pi/3 + i\sin\pi/3)$. The possible values of $\arg(z)$ are $\ldots, \pi/3 - 2\pi, \pi/3, \pi/3 + 2\pi, \pi/3 + 4\pi, \cdots = \pi/3 + 2\pi\mathbb{Z}$ where $2\pi\mathbb{Z} = 2\pi\{\ldots, -1, 0, 1, 2, \ldots\} = \{\ldots, -2\pi, 0, 2\pi, 4\pi, \ldots\}$. Then $\mathrm{Arg}(z) = \pi/3 + 2\pi\mathbb{Z} = \{\ldots, \pi/3 - 2\pi, \pi/3, \pi/3 + 2\pi, \pi/3 + 4\pi, \ldots\}$. Hence, $\mathrm{Arg}(z) :=$ the multivalued argument of z. It is an example of a multifunction which associates to each $z \in \mathbb{C}$ a *set* of complex numbers. So $\mathrm{Arg}(\frac{1}{2} + \frac{1}{2}i\sqrt{3}) = \frac{\pi}{3} + 2\pi\mathbb{Z}$. The *principal argument* of z is the unique $\pi \in \mathrm{Arg}(z)$ such that $-\pi < \theta \leq \pi$. Notation: $\arg(z)$ is the principal argument. (Warning: other sources use different notation.) E.g. $\arg(-1) = \pi$ but $\mathrm{Arg}(-1) = \pi + 2\pi\mathbb{Z}$.

2.1 Complex Conjugation

Definition 1 (Complex Conjugate). $\overline{a + ib} := a - ib$ (reflection across the real axis).

Some properties of the conjugate:

- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} + \overline{wz}$
- $|z|^2 = z\overline{z}$

See that if $z = a + ib$, then

$$z\overline{z} = (a + ib)(a - ib) = a^2 - i^2b + aib - aib = a^2 + b^2 = |a + ib|^2$$

2.1.1 The standard way to divide complex numbers

We have previously defined the multiplicative inverse of a complex number, but the standard way to divide is actually using the conjugate. We have

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}$$

2.2 nth roots of complex numbers

Proposition 1 (De Moivre's formula)**.**

$$z = r(\cos\theta + i\sin\theta)$$
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

To find a third root of $z = r(\cos\theta + i\sin\theta)$, divide θ by 3 and extract a third root of r (can always find a real nth root because $r \geq 0$):

$$w_1 = \sqrt[3]{r}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$$

then $w_1^3 = z$. But there are in fact 3 3rd roots of z. The others are found by dividing the circle with radius $\sqrt[3]{r}$ equally:

$$w_2 = \sqrt[3]{r}\left(\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\right)$$
$$w_3 = \sqrt[3]{r}\left(\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)\right)$$

Another example $(1+i)^8 = 1 + 8i + \binom{8}{2}i^2 + \binom{8}{3}i^4 + \cdots$ which isn't something we want to work with. Since $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, we can actually write (and compute) it much easier:

$$(1+i)^8 = \sqrt{2}^8\left(\cos\left(8\frac{\pi}{4}\right) + i\sin\left(8\frac{\pi}{4}\right)\right) = 16\left(\cos(2\pi) + i\sin(2\pi)\right) = 16$$

The fact that this is a real number is because $1 + i$ is a vertex of the octagon that has a vertex along the x-axis.

2.3 Phase

The *phase* of $z \neq 0$ is $\frac{z}{|z|}$. The phase of z is the complex number of modulus 1 with the same argument. The phase keeps track of the "angle" without the ambiguity in the argument.

A phase portraint is used for visualization. We associate colours to the phases. Red is associated with the positive real numbers, green with $\theta = 2\pi/3$ and blue with $-2\pi/3$. Then yellow is $\pi/3$, magenta is $-\pi/3$, and cyan is the negative real numbers.

2.4 The complex exponential

Recall for $x \in \mathbb{R}$,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

This also works for complex numbers. If $z \in \mathbb{C}$:

$$e^z = 1 + z + \frac{1}{2}z^2 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}z^n$$

For $z = i\theta$ where $\theta \in \mathbb{R}$ (purely imaginary) we get

$$\begin{aligned} e^z = e^{i\theta} &= 1 + i\theta - \frac{1}{2}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 + - \cdots \\ &= 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

So if all these infinte sums behave proerply, we deduce from this Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

From now on, z in polar coordinates will be written $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where r is the modulus and θ is the argument from before. We get the famous identity with this formula:

$$e^{2\pi i} = 1$$

In fact, $e^{2\pi i\mathbb{Z}} = \cos(2\pi\mathbb{Z}) + i\sin(2\pi\mathbb{Z}) = 1$. The complex exponential function ha speriod $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1 = e^z$$