## Problem 1

Who are your group members?

*Solution.* Nicholas Rees

## Problem 2

Recall the (forward) Euler's method (see class notes or [A&G], page 485) and the (explicit) trapazoidal method (class notes or [A&G], page 494), which solves $y' = f(t, y)$ subject to $y(t_0) = y_0$ (1) choosing a small $h > 0$ and setting $t_i = t_0 + ih$ (as usual), and (2) takes $y_i$ (an approximation of $y(t_i)$) to produce $y_{i+1}$, for $i = 0, 1, 2, \ldots$ via the two formulas:

$$Y_i = y_i + hf(t_i, y_i) \quad y_{i+1} = y_i + h\frac{f(t_i, y_i) + f(t_{i+1}, Y_i)}{2}$$

(hence $Y_i$ is what Euler's method gives for $y_{i+1}$).

(a). Consider the trapezoidal method for $y' = y$ with $y(t_0) = y_0$. Write down a formula for $y_{i+1}$ of the trapezoidal method in terms of $y_i$.

(b). Say that we want to approximate $y(1)$ given $y(0) = 1$ (i.e. $t_0 = 0$, $y_0 = 1$). Choose a natural $N$ (which we think of as large), set $h = 1/N$; show that $y_N$ in the trapezoidal method approximation of $y(1)$ equals

$$y_{N,\text{trap}} = \left(1 + \frac{1}{N} + \frac{1}{2N^2}\right)^N.$$

(c). Show that Euler's method for the same ODE and intial condition gives $y_{N,\text{Eul}} = (1 + 1/N)^N$.

(d). Recall that actual solution to $y' = y$ with $y(0) = 1$ has $y(1) = e$. Show that $\ln(y_{N,\text{trap}})$ is closer to the actutal value of $\ln(y(1))$ (namely $\ln(e) = 1$) than $\ln(y_{N,\text{Eul}})$, as $N \to \infty$ (i.e., for all $N$ sufficiently large). [It may be helpful to recall that $\ln(1 + \varepsilon) = \varepsilon - \varepsilon^2/2 + O(\varepsilon^3)$ for $|\varepsilon|$ small.]

(a). *Solution.*

$$y_{i+1} = y_i + h\frac{y_i + Y_i}{2} = y_i + h\frac{y_i + y_i + hy_i}{2} = \frac{2y_i + 2hy_i + h^2y_i}{2} = y_i\left(1 + h + \frac{h^2}{2}\right)$$

(b). *Solution.* Fix some $N$ large and let our step size be $h = 1/N$. We prove by induction that $y_i = y_0(1+\frac{1}{N}+\frac{1}{2N^2})^i$ for all $i \in \mathbb{N}$. The base case when $i = 1$: from part (a), we have $y_1 = y_0(1 + h + \frac{h^2}{2}) = y_0(1 + \frac{1}{N} + \frac{1}{2N^2})^1$, as desired. Now assume that $y_i = y_0(1 + \frac{1}{N} + \frac{1}{2N^2})^i$, we prove the formula for $y_{i+1}$. From part (a), we have

$$y_{i+1} = y_i(1 + h + \frac{h^2}{2}) = y_i(1 + \frac{1}{N} + \frac{1}{2N^2}) = y_0\left(1 + \frac{1}{N} + \frac{1}{2N^2}\right)^{i+1}$$

as desired.

Hence, the trapezoidal method at the $N$th step, which is the approximation at $y(Nh) = y(1)$ gives $y_{N,\text{trap}} = (1 + \frac{1}{N} + \frac{1}{2N^2})^N$.

(c). *Solution.* Recall that $y_{i+1} = y_i(1 + h)$ for Euler's method. Denote this equation $(*)$. Fix some $N$ large and let our step size be $h = 1/N$. We prove by induction that $y_i = y_0(1 + \frac{1}{N})^i$ for all $i \in \mathbb{N}$. The base case when $i = 1$: from equation $(*)$, we have $y_1 = y_0(1 + h) = y_0(1 + \frac{1}{N})^1$, as desired. Now assume that $y_i = y_0(1 + \frac{1}{N})^i$, we prove the formula for $y_{i+1}$. From equation $(*)$, we have

$$y_{i+1} = y_i(1 + h) = y_i(1 + \frac{1}{N}) = y_0\left(1 + \frac{1}{N}\right)^{i+1}$$

as desired.

Hence, Euler's method at the $N$th step, which is the approximation at $y(Nh) = y(1)$ gives $y_{N,\text{Eul}} = (1+\frac{1}{N})^N$.

(d). *Solution.* We can compute

$$\begin{aligned}
\ln(y_{N,\text{trap}}) &= \ln\left(\left(1+\frac{1}{N}+\frac{1}{2N^2}\right)^N\right)\\
&= N\ln\left(1+\frac{1}{N}+\frac{1}{2N^2}\right)\\
&= N\left(\frac{1}{N}+\frac{1}{2N^2}-\frac{1}{2}\left(\frac{1}{N}+\frac{1}{2N^2}\right)^2+O\left(\left(\frac{1}{N}+\frac{1}{2N^2}\right)^3\right)\right)\\
&= 1+\frac{1}{2N}-\frac{1}{2N}-\frac{1}{2N^2}-\frac{1}{4N^2}+NO\left(\frac{1}{N^3}\right)\\
&= 1+O\left(\frac{1}{N^2}\right)
\end{aligned}$$

where we replaced $O\left(\left(\frac{1}{N}+\frac{1}{2N^2}\right)^3\right)$ with $O\left(\frac{1}{N^3}\right)$ since $(a+\frac{1}{2}a^2)^3 = O(a^3)$. We also see

$$\begin{aligned}
\ln(y_{N,\text{Eul}}) &= \ln\left(\left(1+\frac{1}{N}\right)^N\right)\\
&= N\ln\left(1+\frac{1}{N}\right)\\
&= N\left(\frac{1}{N}-\frac{1}{2N^2}+O\left(\frac{1}{N^3}\right)\right)\\
&= 1-\frac{1}{2N}+O\left(\frac{1}{N^2}\right)
\end{aligned}$$

As $N\to\infty$, $\frac{1}{N^2}$ gets small (much faster than $1/N$), and we can neglect it the most, so we can see that while $\ln(y_{N,\text{trap}})$ only differs from $\ln(y(1))$ on order of $\frac{1}{N^2}$, $\ln(y_{N,\text{Eul}})$ differs on order of $\frac{1}{N}$, which is a much larger error when $N$ large.

## Problem 3

In this exercise we will solve the ODE $y' - 2y = t^2$.

(a). Show that there are constants $a, b, c$ such that $y(t) = at^2 + bt + c$ is a solution to $y' - 2y = t^2$.

(b). Show that if $y' - 2y = t^2$ and $z' - 2z = 0$, then $(y+z)' - 2(y+z) = t^2$.

(c). Show that if $y' - 2y = t^2$ and $(y+z)' - 2(y+z) = t^2$, then $z' - 2z = 0$.

(d). Solve for all $z' - 2z = 0$ (which is called the "homogenous form of the equation $y' - 2y = t^2$), and use this to write down the general solution to $y' - 2y = t^2$.

(e). Show that for any $t_0, y_0$, your general solution in part (d) has a unique $y$ such that $y(t_0) = y_0$.

(a). *Solution.* If $a = -1/2$, $b = 1/2$, $c = -1/4$, then $y(t) = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{4}$ and $y'(t) = -t + \frac{1}{2}$. Plugging it into the differential equation, we have $y' - 2y = -t + \frac{1}{2} + t^2 + t - \frac{1}{2} = t^2$, hence $y(t)$ is a solution to $y' - 2y = t^2$.

(b). *Solution.* Assume that $y' - 2y = t^2$ and $z' - 2z = 0$. Then

$$(y+z)' - 2(y+z) = y' + z' - 2y - 2z = y' - 2y + z' - 2z = t^2 + 0 = t^2$$

where we have used the linearity of the derivative.

(c). *Solution.* Assume that $y' - 2y = t^2$ and $(y+z)' - 2(y+z) = t^2$. Then by the linearity of the derivative, we have $y' + z' - 2y - 2z = t^2 \implies y' - 2y + z' - 2z = t^2 \implies t^2 + z' - 2z = t^2$, and subtracting $t^2$ on both sides, we are left with $z' = 2z = 0$.

(d). *Solution.* We can rewrite the differential equation as $z' = 2z$, and this is now one of our classic solutions (derived on previous homeworks), $z(t) = Ae^{2t}$, $A \in \mathbb{R}$ constant; one can verify this by plugging it in (and unique from standard theorems).

As we saw from parts (b) and (c), any and all solutions to $y' - 2y = t^2$ is of the form $y_1 + z$ where $y_1$ is a particular solution to the ODE, and $z$ is the general solution to the homogenous form of the ODE. Hence, using part (a) and the earlier part of (d), the general solution is of the form

$$y(t) = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{4} + Ae^{2t}$$

(e). *Solution.* Let $t_0, y_0$ be fixed. Then the solution from (d) gives

$$y_0 = -\frac{1}{2}t_0^2 + \frac{1}{2}t_0 - \frac{1}{4} + Ae^{2t_0}$$

Which implies

$$A = \frac{y_0 + \frac{1}{2}t_0^2 - \frac{1}{2}t_0 + \frac{1}{4}}{e^{2t_0}}$$

(and we need not worry about dividing by zero, since $e^x \neq 0$ for all $x \in \mathbb{R}$). $A$ is determined uniquely be $y_0, t_0$, which gives us the unique solution

$$y(t) = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{4} + \left(\frac{y_0 + \frac{1}{2}t_0^2 - \frac{1}{2}t_0 + \frac{1}{4}}{e^{2t_0}}\right)e^{2t}$$

## Problem 4

In this exericise we will solve the recurrence relation $x_{n+1} - 2x_n = n^2$ for all $n \in \mathbb{Z}$.

(a). Show that there are constants $a, b, c$ such that $x_n = an^2 + bn + c$ is a solution to $x_{n+1} - 2x_n = n^2$.

(b). What is the general solution to $x_{n+1} - 2x_n = 0$?

(c). Use the above two parts to write a general solution to the recurrence equation $x_{n+1} - 2x_n = n^2$, and explain your reasoning in term of Problem (2) above and the appropriate notation of the "homogenous form of the recurrence $x_{n+1} - 2x_n = n^2$.

(a). *Solution.* If $a = -1$, $b = -2$, $c = -3$, then $x_n = -n^2 - 2n - 3$ and $x_{n+1} = -1(n+1)^2 + -2(n+1) - 3 = -n^2 - 4n - 6$. Plugging these together into the recurrence relation, we have

$$x_{n+1} - 2x_n = -n^2 - 4n - 6 + 2n^2 + 4n + 6 = n^2$$

Hence, $x_n$ is a solution to $x_{n+1} - 2x_n = n^2$.

(b). *Solution.* This is equivalent to $x_{n+1} = 2x_n$. As we discussed in class, this is a solution of the form $Ar^n$, where $r \in \mathbb{R}$ is a constant, and $A$ is a constant determined by $x_0$. For our specific case, $x_n = x_0 2^n$ works.

(c). *Solution.* We can use the same logic from the previous problem, and it would be simple to show as well, that any solution to the recurrence relation is the sum of a particular solution and the general solution to the homogenous form of the recurrence relation. Hence, using parts (a) and (b), we have the general solution

$$x_n = -n^2 - 2n - 3 + A2^n$$

## Problem 5

(a). At what value of $n$ does MATLAB declare $(1/2)^n$ to be 0? There are a number of ways of doing this, but one way is to examine the values of $x$ generated by:

```
clear
for n=1:1100, x{n}=(1/2)^n; end

x
```

(you may need the extra blank line above if you copy and paste).

(b). Find the general solution to the recurrence $x_{n+2} = (3/2)x_{n+1} - (1/2)x_n$.

(c). Find the special case of this recurrence subject to the initial conditions $x_1 = 1$ and $x_2 = 1/2$.

(d). Use MATLAB to compute the solution of this recurrence subject to $x_1 = 1$ and $x_2 = 1/2$ for $x_1, x_2, \ldots, x_{100}$ via:

```
clear
x{1} = 1;
x{2} = 1/2;
for n=1:98, x{n+2} = (3/2)*x{n+1}-(1/2)*x{n}; end

x
```

What does MATLAB report for the values of

```
x{100} * 2^99
x{100} * 2^99 - 1
```

(e). Now run the same code to generate $x_1, x_2, \ldots, x_{1200}$ using

```
clear
x{1} = 1;
x{2} = 1/2;
for n=1:1198, x{n+2} = (3/2)*x{n+1}-(1/2)*x{n}; end

x
```

Anwser the following:

(i) Does MATLAB report $x_n = 0$ for some value of $n \leq 1200$?

(ii) What is the smallest value of $n_0$ such that the values that MATLAB reports for $(1/2)^{n_0-1}$ and $x\{n_0\}$ are not equal?

(iii) For the value of $n_0$, by examining the values of $(3/2) * x\{n_0 - 1\}$ and $(1/2) * x\{n_0 - 2\}$, can you see where an error in precision occurs?

(iv) What repeating pattern do you see in the values of $x\{n\}$ for $n \geq n_0$? [Hint: It may be simpler to report this pattern in multiples of $(1/2)^{1074}$.] [Warning: if you type something like

```
for n=1070:1100, {n,x{n},x{n}*2^1074}, end
```

then the last cell value will be `Inf`, since $2^{1074}$ will yield `Inf`. However, if you type:

```
for n=1070:1100, {n,x{n},(x{n}*2^900)*2^174}, end
```

then you'll get the answers you want.

(a). *Solution.* For any $n \geq 1075$.

(b). *Solution.* Didn't do :)

(c). *Solution.* Didn't do :)

(d). *Solution.* It produces 1 and 0, respectively.

(e). *Solution.*

(i) didn't do :0

(ii) didn't do :0

(iii) Didn't do :)

(iv) Didn't do :)