Math 321 Homework 9

For the next problems, recall that if (X, d) is a metric space then $\mathcal{C}_{\mathbb{R}}(X)$ is the algebra of bounded continuous functions $f: X \to \mathbb{R}$.

Problem 1

Let K be a compact metric space and let $x_0 \in K$. Let $\mathcal{A} \subset \mathcal{C}_{\mathbb{R}}(K)$ be an algebra that separates points, and vanishes only at the point x_0, i.e. $f(x_0) = 0$ for all $f \in \mathcal{A}$, and for each $x \in K$ with $x \neq x_0$, there exists $g \in \mathcal{A}$ so that $g(x) \neq 0$.

(a). Let $\mathcal{A}'$ be the set of functions of the form $f + c$, where $f \in \mathcal{A}$ and $c \in \mathbb{R}$. Prove that $\mathcal{A}'$ is an algebra, $\mathcal{A}'$ separates points, and $\mathcal{A}'$ vanishes at no point of K.

(b). Prove that

$$\overline{\mathcal{A}} = \{f \in \mathcal{C}_{\mathbb{R}}(K) : f(x_0) = 0\}$$

(a). *Solution.* We show that $\mathcal{A}'$ is an algebra. Let $f, g \in \mathcal{A}$ and $a, b \in \mathbb{R}$. Note $f + a, g + b$ are arbitrary elements of $\mathcal{A}'$.

(a) $(f + a) + (g + b) = (f + g) + (a + b)$. Note that $f + g \in \mathcal{A}$ and $a + b \in \mathbb{R}$, hence the sum is also in $\mathcal{A}'$.
(b) $(f + a)(g + b) = fg + ag + bf + ab$. Note that $fg, ag, bf \in \mathcal{A}$, hence $fg + ag + bf \in \mathcal{A}$, and $ab \in \mathbb{R}$, hence the product is also in $\mathcal{A}'$.
(c) $c(f + a) = cf + ca$. Note that $cf \in \mathcal{A}$ and $ca \in \mathbb{R}$, hence the scalar product is also in $\mathcal{A}'$. This confirms that $\mathcal{A}'$ is an algebra.

To see that $\mathcal{A}'$ separates points, let $x, y \in X$ such that $x \neq y$. Then there exists $f \in \mathcal{A}$ where $f(x) \neq f(y)$. Furthermore, $f = f + 0 \in \mathcal{A}'$, so $\mathcal{A}'$ separates the points x, y as well, and these were arbitrary, so $\mathcal{A}'$ separates all points.

Finally, we show that $\mathcal{A}'$ vanishes at no point. If $x \neq x_0$, then there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. Hence $f = f + 0 \in \mathcal{A}'$, and so $\mathcal{A}'$ does not vanish at x as well. Now, if $x = x_0$, then for any $f \in \mathcal{A}$, we have $f + 1 \in \mathcal{A}'$ and $(f + 1)(x_0) = f(x_0) + 1 = 1 \neq 0$, hence there is a function that doesn't vanish at x_0. This covers all possibilities for $x \in K$, hence $\mathcal{A}'$ vanishes at no point of K. □

(b). *Solution.* ff I will come back to this when I am less hungry, but basically we are just copying the proof. □

Problem 2

Let K be a compact metric space. Let $\mathcal{A} \subset \mathcal{C}_{\mathbb{R}}(K)$ be an algebra that separates points. Prove that the closure $\overline{\mathcal{A}}$ consists of either: (i) $\mathcal{C}_{\mathbb{R}}(K)$, or (ii) all continuous functions f on K such that $f(x_0) = 0$ for some fixed $x_0 \in K$.

Solution. We will show that if (i) is not true, then (ii) must be true, assuming that $\mathcal{A} \subset \mathcal{C}_{\mathbb{R}}(K)$ is an algebra that separates points. This follows from the contrapositive of Stone-Weierstrass. Assume that $\overline{\mathcal{A}} \neq \mathcal{C}_{\mathbb{R}}(K)$. Hence, $\mathcal{A}$ either fails to separate points or it vanishes at a point. We assume that it separates points, hence it vanishes at some point. This means that there is some $x_0 \in K$ where $f(x_0) = 0$ for all $f \in \mathcal{A}$. However, since it separates points, we cannot have some other $x_1 \in K$ where $f(x_1) = 0$ for all $f \in \mathcal{A}$, otherwise it does not separat x_0 and x_1. This satisfies the hypothesis for Problem 1 above, and so by 1b, we have $\overline{\mathcal{A}} = \{f \in \mathcal{C}_{\mathbb{R}}(K) : f(x_0) = 0\}$, as desired. □

Problem 3

Let $f: [0, 1]^2 \to \mathbb{R}$ be continuous, let $\varepsilon > 0$. Prove that there exists $n \in \mathbb{N}$ and continuous functions $g_1, \ldots, g_n, h_1, \ldots, h_n : [0, 1] \to \mathbb{R}$ so that

$$\sup_{(x,y)\in[0,1]^2} \left| f(x, y) - \sum_{i=1}^{n} g_i(x)h_i(y) \right| < \varepsilon \tag{1}$$

Solution. We claim that the set defined by

$$\mathcal{A} = \{f(x,y) \in \mathcal{C}_{\mathbb{R}}([0,1]^2) \colon f(x,y) = \sum_{i=1}^{n} g_i(x)h_i(y), n \in \mathbb{N}, g_i, h_i \in \mathcal{C}([0,1])\} \subset \mathcal{C}_{\mathbb{R}}([0,1]^2)$$

is an algebra. See if $g_i, h_i, g'_j, h'_j \in \mathcal{C}_{\mathbb{R}}([0,1])$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ for some $n, m \in N$,

(a). To see closure under addition, we have

$$\sum_{i=1}^{n} g_i(x)h_i(y) + \sum_{j=1}^{m} g'_j(x)h'_j(y) = \sum_{k=1}^{n+m} g''_k(x)h''_k(y)$$

where $g''_k = g_k$ when $1 \leq k \leq n$ and $g''_k = g'_{k-n}$ when $n+1 \leq k \leq m$, and similarly for h''_k. Clearly, since $g''_k, h''_k \in \mathcal{C}_{\mathbb{R}}([0,1])$ as well, we have that the sum is in $\mathcal{A}$ as well.

(b). To see closure under multiplication, we have

$$\left(\sum_{i=1}^{n} g_i(x)h_i(y)\right)\left(\sum_{j=1}^{m} g'_j(x)h'_j(y)\right) = \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} g_i(x)g'_j(x)h_i(y)h'_j(y)$$

But note that the product of continuous functions is also continuous, and so $g_i(x)g'_j(x), h_i(y)h'_j(y) \in \mathcal{C}_{\mathbb{R}}([0,1])$ still, and we get a finite sum of nm these elements. Hence, the product is in $\mathcal{A}$.

(c). Finally, we have for $c \in \mathbb{R}$, $c\sum_{i=1}^{n} g_i(x)h_i(y) = \sum_{i=1}^{n} cg_i(x)h_i(y)$ and $cg_i(x) \in \mathcal{C}_{\mathbb{R}}([0,1])$ since a continuous function multiplied by a scalar is still continuous, hence any sclar multiple is in $\mathcal{A}$.

Hence, we have that $\mathcal{A}$ is an algebra.

To see that $\mathcal{A}$ separates points, consider $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ such that $(x_1,y_1) \neq (x_2,y_2)$. ff

So Stone-Weierstrass gives us that there an algebra converges uniformly. This requirement is converging uniformly: $f_n = \sum_{i=1}^{n} g_i(x)h_i(y)$. We have to show that this is an algebra. I want to prove that functions of the form $\sum_{i=1}^{n} g_i(x)h_i(y)$ where g, h are continuous, form an algebra. Then we get a uniformly convergent sequence, eventually works. ff □

Problem 4

Let α and β be monotone non-decreasing continuous real-valued functions on $[0,1]$, with $\alpha(0) = \beta(0) = 0$. Suppose that, for all $n = 0,1,2,3,\ldots$,

$$\int_0^1 e^{-nx}d\alpha(x) = \int_0^1 e^{-nx}d\beta(x)$$

(a). Prove that if $f\colon [0,1] \to \mathbb{R}$ is continuous then $\int_0^1 f(x)d\alpha(x) = \int_0^1 f(x)d\beta(x)$

(b). Does it follow that $\alpha(x) = \beta(x)$ for all $x \in [0,1]$? Prove it or give a counterexample.

(a). *Solution.* ff Something about how the finite sum is an algebra, and then theorem 6.12 and 7.16 for the win. □

(b). *Solution.* Something that equals on the continuous, but not on the not continuous (implication doesn't go this way, but it is a necessary condition). Hmm... we require that α, β continuous. ff □