Problem 1

Who are your group members?

Solution. Nicholas Rees

Problem 2

The point of this exercise is to define a type of ODE known as *central force problem*, and to show that such ODE's satisfy a *conservation of energy*. In thie exercise $\| \quad \|$ refers to the L^2 norm $\| \quad \|$. Let $\mathbf{X}\colon \mathbb{R} \to \mathbb{R}^2$, i.e. $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$, where x_1, x_2 are functions $\mathbb{R} \to \mathbb{R}$, and similarly for $\mathbf{z} = \mathbf{z}(t) = (z_1(t), z_2(t))$.

(a). With the usual dot product:

$$\mathbf{x} \bullet \mathbf{z} = x_1 z_1 = x_2 z_2$$

(hence all the above depend on t), show that

$$\frac{d}{dt}(\mathbf{x} \bullet \mathbf{z}) = \mathbf{x} \bullet \dot{\mathbf{z}} + \dot{\mathbf{x}} \bullet \mathbf{z}$$

where $\dot{}$ denote d/dt (as usual in celestial mechanics).

(b). Show that

$$\frac{d}{dt}(\|\mathbf{x}\|^2) = \frac{d}{dt}(\mathbf{x} \bullet \mathbf{x}) = 2\mathbf{x} \bullet \dot{\mathbf{x}}$$

(c). Show that if m is a constant, then

$$\frac{d}{dt}(m\|\dot{\mathbf{x}}\|^2) = 2m\dot{\mathbf{x}} \bullet \ddot{\mathbf{x}}$$

(d). Show that

$$\frac{d}{dt}\|\mathbf{x}\| = \frac{d}{dt}\sqrt{x_1^2 + x_2^2} = \frac{1}{\|\mathbf{x}\|}\mathbf{x} \bullet \dot{\mathbf{x}}$$

(e). Show that if $U\colon (0, \infty) \to \mathbb{R}$ is a differentiable function, whose derivative is u, then

$$\frac{d}{dt}U(\|\mathbf{z}\|) = u(\|\mathbf{z}\|)\frac{1}{\|\mathbf{z}\|}\mathbf{z} \bullet \dot{\mathbf{z}}$$

(f). We say that a function $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^2$ satisfies a *central force law* if for some real $m > 0$ and $u\colon (0, \infty) \to \mathbb{R}$ we have

$$m\ddot{\mathbf{x}} = -mu(\|\mathbf{x}\|)\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{1}$$

(at times u may extend to a function $[0, \infty) \to \mathbb{R}$, but for Newton's Law of Gravitation, $u(0) = +\infty$). Show that in this case

$$\text{Energy} \;=\; \text{Energy}(t) := \frac{1}{2}m\|\dot{\mathbf{x}}\|^2 + mU(\|\mathbf{x}\|) \tag{2}$$

is independent of t (where, as in part (e), $U' = u$). [Hint: show that d/dt applied to Energy(t) is zero.]

(a). *Solution.* We have

$$\begin{aligned}
\frac{d}{dt}(\mathbf{x} \bullet \mathbf{z}) &= \frac{d}{dt}(x_1 z_1 + x_2 z_2) \\
&= \frac{d}{dt}x_1 z_1 + \frac{d}{dt}x_2 z_2 \\
&= \dot{x_1} z_1 + x_1 \dot{z_1} + \dot{x_2} z_2 + x_2 \dot{z_2} \\
&= (\dot{x_1} z_1 + \dot{x_2} z_2) + (x_1 \dot{z_1} + x_2 \dot{z_2}) \\
&= \dot{\mathbf{x}} \bullet \mathbf{z} + \mathbf{x} \bullet \dot{\mathbf{z}}
\end{aligned}$$

(b). *Solution.* Recall $\|\mathbf{x}\|^2 = \mathbf{x} \bullet \mathbf{x}$. We have

$$\begin{aligned}\frac{d}{dt}(\|\mathbf{x}\|^2) &= \frac{d}{dt}(\mathbf{x} \bullet \mathbf{x}) \\ &= \frac{d}{dt}(x_1^2 + x_2^2) \\ &= 2x_1\dot{x_1} + 2x_2\dot{x_2} \\ &= 2\mathbf{x} \bullet \dot{\mathbf{x}}\end{aligned}$$

(c). *Solution.* We have

$$\begin{aligned}\frac{d}{dt}(m\|\dot{\mathbf{x}}\|^2) &= m\frac{d}{dt}(\dot{\mathbf{x}} \bullet \dot{\mathbf{x}}) \\ &= m\frac{d}{dt}(\dot{x_1}^2 + \dot{x_2}^2) \\ &= m(2\dot{x_1}\ddot{x_1} + 2\dot{x_2}\ddot{x_2}) \\ &= 2m\dot{\mathbf{x}} \bullet \ddot{\mathbf{x}}\end{aligned}$$

(d). *Solution.* We have

$$\begin{aligned}\frac{d}{dt}\|\mathbf{x}\| &= \frac{d}{dt}\sqrt{x_1^2 + x_2^2} \\ &= \frac{1}{2}(x_1^2 + x_2^2)^{-1/2}\frac{d}{dt}(x_1^2 + x_2^2) \\ &= \frac{1}{2}(x_1^2 + x_2^2)^{-1/2}(2x_1\dot{x_1} + 2x_2\dot{x_2}) \\ &= \frac{1}{\sqrt{x_1^2 + x_2^2}}(x_1\dot{x_1} + x_2\dot{x_2}) \\ &= \frac{1}{\|\mathbf{x}\|}\mathbf{x} \bullet \dot{\mathbf{x}}\end{aligned}$$

(e). *Solution.* We have

$$\begin{aligned}\frac{d}{dt}U(\|\mathbf{z}\|) &= u(\|\mathbf{z}\|)\frac{d}{dt}\|\mathbf{z}\| \\ &= u(\|\mathbf{z}\|)\frac{1}{\|\mathbf{z}\|}\mathbf{z} \bullet \dot{\mathbf{z}}\end{aligned}$$

where we get the last step from part (d).

(f). *Solution.* We take the derivative of the energy, see

$$\begin{aligned}\frac{d}{dt}\text{Energy}(t) &= \frac{d}{dt}\left(\frac{1}{2}m\|\dot{\mathbf{x}}\|^2 + mU(\|\mathbf{x}\|)\right) \\ &= \frac{1}{2}\frac{d}{dt}(m\|\dot{\mathbf{x}}\|^2) + m\frac{d}{dt}U(\|\mathbf{x}\|) \\ &= \frac{1}{2}(2m\dot{\mathbf{x}} \bullet \ddot{\mathbf{x}}) + mu(\|\mathbf{x}\|)\frac{\mathbf{x}}{\|\mathbf{x}\|} \bullet \dot{\mathbf{x}} \\ &= m\dot{\mathbf{x}} \bullet \ddot{\mathbf{x}} - m\ddot{\mathbf{x}} \bullet \dot{\mathbf{x}} \\ &= 0\end{aligned}$$

where we applied parts (c) and (e) in the third line, the fact that $\mathbf{x}$ satisifes the central force law in the fourth line, and the commutativity of the dot product in the last line (since $\mathbf{x}\bullet\mathbf{y} = x_1y_1 + x_2y_2 = y_1x_1 + y_2x_2 = \mathbf{y}\bullet\mathbf{x}$). Hence, energy does not change with time, and so energy is conserved since it is independent of t.

Problem 3

Consider the central force problem (1), where $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^2$, for fixed u, U as in Problem (2). We may write (1) as a 4-dimensional ODE by setting $\mathbf{y} = (y_1, y_2, y_3, y_4) = (\dot{x_1}, \dot{x_2}, x_1, x_2)$ and letting

$$r = \|\mathbf{x}\| = \sqrt{y_3^2 + y_4^2}$$

and hence

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{f}(\mathbf{y}), \quad \text{where} \quad \mathbf{f}(\mathbf{y}) = \begin{bmatrix} -u(r)y_3/(mr) \\ -u(r)y_4/(mr) \\ y_1 \\ y_2 \end{bmatrix}, \quad r = \sqrt{y_3^2 + y_4^2} \tag{3}$$

Notice that the energy of the system Energy(t) can therefore be written as

$$E(\mathbf{y}) = \frac{1}{2}m\|(y_1, y_2)\|^2 + mU(\|(y_3, y_4)\|)$$

and hence $E(\mathbf{y}(t))$ is independent of t. One way to test the accuracy of numerical (i.e., approximate) solutions to (3) is to see if the approximation to $E(\mathbf{y}(t))$ changes in time. Newton's Law of Gravitation (to predict how planets move around the sun) fixes a real constant $g > 0$, and takes $U(r) = -g/r$ and so $u(r) = g/r^2$.

(a). Consider the case where $m = g = 1$, and we solve (3) subject to

$$\mathbf{y}(0) = [0, 0.8, 1, 0] \tag{4}$$

Use MATLAB to generate points $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N$ using Euler's method

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}(\mathbf{y}_i), \quad i = 0, 1, \ldots, N-1$$

with the following values of h, N:

(i) First take $h = 0.1$ and $N = 600$. (Hence you are approximating $y(t)$ for $0 \le t \le iN = 60$.) What is $E(\mathbf{y}_0)$, and $E(\mathbf{y}_N)$? Does $E(\mathbf{y}_i)$ always increase, always decrease, or does it fluctuate in both directions? Does the approximate ellipse that $\mathbf{y}_i$ traces out (in its 3rd and 4th components, which approximates $\mathbf{x}(t)$) seem to get larger in time or get smaller (or is it roughly the same)?

(ii) Next take $h = 0.01$ and $N = 6000$. (Hence you are still approximating $y(t)$ for $0 \le t \le 60$, but presumebly the approximation is better.) Same questions.

(iii) Same questions with $h = 0.001$ and $N = 60000$.

(b). Same questions with $h = 0.1$ and $N = 600$, but this time use the (explicit) trapezoidal method.

[Hint: you are welcome to use the program `Euler_Central_Force.m` that I'm supplying; and can observe Euler's method in the above three cases by typing each of the three lines:

```
Euler_Central_Force(1,1,[0,.8,1,0],0.1,600,.05,1)
Euler_Central_Force(1,1,[0,.8,1,0],0.01,6000,.05,10)
Euler_Central_Force(1,1,[0,.8,1,0],0.0001,600000,.05,1000)
```

For the trapezoid rule, you'll probably want to modify `Euler_Central_Force.m` by replacing the line:

```
for i=1:N
    yvals(i+1,:) = yvals(i,:) + h * f( yvals(i,:) );
end
```

with something like:

```
for i=1:N
    Y = yvals(i,:) + h * f( yvals(i,:) );
    yvals(i+1,:) = yvals(i,:) + (h/2) * ( f( yvals(i,:) ) + f( Y ));
end
```

However, you may likely be able to do a better job by writing your own code. Also, you can likely answer these questions without plotting anything; plotting just makes the answers easier to see.]

(a). *Solution.* We first use Euler's method.

(i) $E(\mathbf{y}_0) = -0.68$ and $E(\mathbf{y}_N) = -0.0175$. The magnitude of the energy almost always decreases for the time period, but very marginally increases at the very end. The ellipse it traces out gets bigger.

(ii) $E(\mathbf{y}_0) = -0.68$ and $E(\mathbf{y}_N) = -0.2137$. The magnitude of the energy always decreased during the time period. The ellipse it traces out gets bigger, but slower than before.

(iii) $E(\mathbf{y}_0) = -0.68$ and $E(\mathbf{y}_N) = -0.4980$. The magnitude of the energy always decreased during the time period. The ellipse it traces stays roughly the same, but it does grow slightly in the time.

(b). *Solution.* We now use the trapezoidal method.

(i) $E(\mathbf{y}_0) = -0.68$ and $E(\mathbf{y}_N) = -0.4071$. The magnitude of the energy fluctuated, but did decrease overall during the time period. The ellipse it traces stays roughly the same, but it does grow slightly in the time.

(ii) $E(\mathbf{y}_0) = -0.68$ and $E(\mathbf{y}_N) = -0.6794$. The magnitude of the energy fluctuated, staying relatively the same. The ellipse it traces stays roughly the same.

(iii) $E(\mathbf{y}_0) = -0.68$ and $E(\mathbf{y}_N) = -0.6800$. The magnitude of the energy stayed relatively the same. The ellipse it traces stays roughly the same.

Problem 4

Consider the recurrence $x_{n+2} - x_n = 0$.

(a). Write the general solution as $x_n = c_1 r_1^n + c_2 r_2^n$ for some values of r_1, r_2.

(b). Given the initial conditions $x_0 = 5$ and $x_1 = 7$, solve for c_1, c_2, and use these values to get a formula for x_n for any n.

(c). Given the initial conditions $x_0 = 5$ and $x_1 = 7$, what are the values of $x_9, x_{10}, x_{11}, x_{12}$? Was the above method of solving for c_1, c_2 the quickest way to determine these values?

(d). Write the recurrence as $\mathbf{y}_{n+1} = A\mathbf{y}_n$, where

$$\mathbf{y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

what is A? What are the values of $A^9, A^{10}, A^{11}, A^{12}$?

(a). *Solution.* Let us assume there is an $r \in \mathbb{R}$ such that $x_n = r^n$, and assume $r \neq 0$ since we are looking for nontrivial solutions. Then our recurrence becomes $r^{n+2} - r^n = 0$. We can divide out by r^n to get $r^2 - 1 = 0$. By inspection, $r_1 = 1$, $r_2 = -1$ both solve this equation. Now, by the linearity of the recurrence relation, if $c_1, c_2 \in \mathbb{R}$, we get the general solution

$$x_n = c_1(1)^n + c_2(-1)^n$$

(b). *Solution.* If $x_0 = 5$ and $x_1 = 7$, then we get the system

$$\begin{cases} 5 = c_1(1)^0 + c_2(-1)^0 = c_1 + c_2 \\ 7 = c_1(1)^1 + c_2(-1)^1 = c_1 - c_2 \end{cases}$$

Adding the two equations together gives us $12 = 2c_1$ or $c_1 = 6$, and so $c_2 = -1$. Then, we get the formula

$$x_n = 6(1)^n - (-1)^n = 6 - (-1)^n$$

(c). *Solution.* We have

$$x_9 = 7$$
$$x_{10} = 5$$
$$x_{11} = 7$$
$$x_{12} = 5$$

The fastest way was not through solving for c_1, c_2. Notice that $x_{n+2} = x_n$, and so every even x_n will be the same value, and every odd x_n will be of the same value. Hence, x_0 determines x_{10} and x_{12}, and x_1 determines x_9 and x_{11}.

(d). *Solution.* We have

$$\mathbf{y}_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A\mathbf{y}_n$$

So $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. And so if $n \in \mathbb{N}$ is even, i.e. $n = 2k$ for some $k \in \mathbb{N}$, then

$$A^n = A^{2k} = (A^2)^k = I^k = I$$

and if n is odd, i.e. $n = 2k+1$, then

$$A^n = A^{2k+1} = A(A^2)^k = AI^k = A$$

Hence,

$$A^9 = A^{11} = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$A^{10} = A^{12} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 5

Consider the recurrence $x_{n+2} - 4x_{n+1} + 4x_n = 0$ for all $n \in \mathbb{Z}$, with x_0, x_1 given (but arbitrary).

(a). Since the general solution of this recurrence (seen in class) is $x_n = c_1 2^n + c_2 n 2^n$, solve for c_0, c_1 in terms of x_0, x_1. Use this to get a formula for x_n for given x_0, x_1.

(b). Write the recurrence as $\mathbf{y}_{n+1} = A\mathbf{y}_n$, where

$$\mathbf{y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

what is A? Using the fact that $2I - A = N$ where $N^2 = 0$ (the zero matrix), derive a formula for A^n for any $n = 0, 1, 2, \ldots$. Use this to derive a formula for x_n in terms of x_0, x_1.

(c). For a small but nonzero ε, consider the recurrence

$$(\sigma - 2)(\sigma - 2 - \varepsilon)(x_n) = 0$$

i.e.,

$$x_{n+2} - (4 + \varepsilon)x_{n+1} + (4 + 2\varepsilon)x_n = 0$$

We can write this as a recurrence $\mathbf{y}_{n+1} = A(\varepsilon)\mathbf{y}_n$, where

$$\mathbf{y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

and

$$A = \begin{bmatrix} 4 + \varepsilon & -4 - 2\varepsilon \\ 1 & 0 \end{bmatrix}$$

From the general theory of recurrences, we know that A has an eigenvalue 2 with eigenvector $[2;1]$, and an eigenvalue $2+\varepsilon$ with eigenvector $[2+\varepsilon;1]$, and hence

$$A(\varepsilon) = S(\varepsilon)\begin{bmatrix}2 & 0\\ 0 & 2+\varepsilon\end{bmatrix}(S(\varepsilon))^{-1}, \quad \text{where} \quad S(\varepsilon) = \begin{bmatrix}2 & 2+\varepsilon\\ 1 & 1\end{bmatrix} \tag{5}$$

For any n, find

$$\lim_{\varepsilon\to 0}(A(\varepsilon))^n$$

using (5). Show that it agrees with the matrix in part (b). [Hint: the formula

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

shows that

$$(S(\varepsilon))^{-1} = \frac{1}{-\varepsilon}\begin{bmatrix}1 & -2-\varepsilon\\ -1 & 2\end{bmatrix}$$

it suffices to write

$$S(\varepsilon) = M_1 + \varepsilon M_2, \quad \begin{bmatrix}2 & 0\\ 0 & 2+\varepsilon\end{bmatrix}^n = M_3 + \varepsilon M_4 + O(\varepsilon^2), \quad \begin{bmatrix}1 & -2-\varepsilon\\ -1 & 2\end{bmatrix} = M_5 + \varepsilon M_6$$

and to consider the constant and order ε terms of

$$(M_1+\varepsilon M_2)(M_3+\varepsilon M_4)(M_5+\varepsilon M_6)$$

]

(a). *Solution.* We have the system

$$\begin{cases} x_0 = c_1 2^0 + c_2(0)2^0\\ x_1 = c_1 2^1 + c_2(1)2^1\end{cases}$$

The first equation just gives us $c_1 = x_0$. Plugging this into the second equation, we have $x_1 = 2x_0 + 2c_2$, and so $c_2 = (x_1 - 2x_0)/2$. Plugging this into the formula for x_n, we have

$$x_n = x_0 2^n + \frac{x_1 - 2x_0}{2} n 2^n$$

(b). *Solution.* We have

$$\mathbf{y}_{n+1} = \begin{bmatrix}x_{n+2}\\ x_{n+1}\end{bmatrix} = \begin{bmatrix}4 & -4\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_{n+1}\\ x_n\end{bmatrix} = A\mathbf{y}_n$$

So $A = \begin{bmatrix}4 & -4\\ 1 & 0\end{bmatrix}$. Note that $2I - A = \begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix} =: N$. See that $N^2 = \begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0$.

We claim that $A^n = 2^n I - n\cdot 2^{n-1}N$. We prove this with induction. For the base case $n=0$, see that $A^0 = I$, and $2^0 I - 0\cdot 2^{0-1}N = I$ so they agree. Now, assume that $A^n = 2^n I - n\cdot 2^{n-1}N$. We prove that $A^{n+1} = 2^{n+1}I - (n+1)\cdot 2^n N$. See

$$\begin{aligned}
A^{n+1} &= A^n A\\
&= (2^n I - n\cdot 2^{n-1}N)(2I - N)\\
&= 2^{n+1}I^2 - 2n\cdot 2^{n-1}NI - 2^n IN + n\cdot 2^{n-1}N^2\\
&= 2^{n+1}I - n\cdot 2^n N - 2^n N + 0\\
&= 2^{n+1} - (n+1)\cdot 2^n N
\end{aligned}$$

Hence, we have proven the formula for all $n \in \mathbb{N}_0$, $A^n = 2^n I - n\cdot 2^{n-1}N$. We can write this explicitly:

$$A^n = 2^n I - n\cdot 2^{n-1}N = \begin{bmatrix}2^n + n2^n & -n2^{n+1}\\ n2^{n-1} & 2^n - n2^n\end{bmatrix} = \begin{bmatrix}(1+n)2^n & -n\cdot 2^{n+1}\\ n\cdot 2^{n-1} & (1-n)2^n\end{bmatrix}$$

We now derive a formula for x_n in terms of x_0, x_1. Note that $\mathbf{y}_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$. It is clear that $\mathbf{y}_n = A^n \mathbf{y}_0$ for all $n \in \mathbb{N}_0$: $\mathbf{y}_0 = A^0 \mathbf{y}_0 = I\mathbf{y}_0$, and if we assume $\mathbf{y}_n = A^n \mathbf{y}_0$, then $\mathbf{y}_{n+1} = A\mathbf{y}_n = A^{n+1}\mathbf{y}_0$ by the definition of A, closing the induction. Thus, we have

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \mathbf{y}_n = A^n \mathbf{y}_0 = \begin{bmatrix} (1+n)2^n & -n \cdot 2^{n+1} \\ n \cdot 2^{n-1} & (1-n)2^n \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

Looking only at the bottom row, we get the explicit formula

$$x_n = (n \cdot 2^{n-1})x_1 + ((1-n)2^n)x_0$$

One can actually check that this matches the formula from part (a), but this is not required of me.

(c). *Solution.* According to the hint, we can rewrite our matrices:

$$S(\varepsilon) = \begin{bmatrix} 2 & 2+\varepsilon \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(S(\varepsilon))^{-1} = \frac{1}{-\varepsilon}\left(\begin{bmatrix} 1 & -2-\varepsilon \\ -1 & 2 \end{bmatrix}\right) = \frac{1}{-\varepsilon}\left(\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\right)$$

Now recall that powers of diagonal matrices are just the diagonal matrix with the entries raised to the same power, or see one can simply iterate like below:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^n = \underbrace{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\cdots\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}}_{n \text{ times}} = \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix}\underbrace{\cdots\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}}_{n-2 \text{ times}} = \begin{bmatrix} a^n & 0 \\ 0 & d^n \end{bmatrix}$$

Hence

$$\begin{bmatrix} 2 & 0 \\ 0 & 2+\varepsilon \end{bmatrix}^n = \begin{bmatrix} 2^n & 0 \\ 0 & (2+\varepsilon)^n \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n + n2^{n-1}\varepsilon + O(\varepsilon^2) \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & n2^{n-1} \end{bmatrix} + O(\varepsilon^2)$$

where we have used the binomial theorem. Now we assign

$$M_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 0 \\ 0 & n2^{n-1} \end{bmatrix}, \quad M_5 = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

We note that

$$(M_1 + \varepsilon M_2)(M_3 + \varepsilon M_4)(M_5 + \varepsilon M_6) = M_1M_3M_5 + \varepsilon(M_1M_3M_6 + M_1M_4M_5 + M_2M_3M_5) + O(\varepsilon^2)$$

We can compute

$$M_1M_3M_5 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2^{n+1} & 2^{n+1} \\ 2^n & 2^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2^{n+1} - 2^{n+1} & -2^{n+2} + 2^{n+2} \\ 2^n - 2^n & -2^{n+1} + 2^{n+1} \end{bmatrix} = 0$$

$$\begin{aligned}
M_1M_3M_6 + M_1M_4M_5 + M_2M_3M_5 &= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & n2^{n-1} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2^n \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & n2^n \\ 0 & n2^{n-1} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2^n \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -2^{n+1} \\ 0 & -2^n \end{bmatrix} + \begin{bmatrix} -n2^n & n2^{n+1} \\ -n2^{n-1} & n2^n \end{bmatrix} + \begin{bmatrix} -2^n & 2^{n+1} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2^n - n2^n & n2^{n+1} \\ -n2^{n-1} & -2^n + n2^n \end{bmatrix}
\end{aligned}$$

Hence we can finally compute

$$\begin{aligned}
(A(\varepsilon))^n &= \left(S(\varepsilon)\begin{bmatrix}2 & 0\\ 0 & 2+\varepsilon\end{bmatrix}(S(\varepsilon))^{-1}\right)^n\\
&= S(\varepsilon)\begin{bmatrix}2 & 0\\ 0 & 2+\varepsilon\end{bmatrix}^n (S(\varepsilon))^{-1}\\
&= \frac{1}{-\varepsilon}\left((M_1+\varepsilon M_2)(M_3+\varepsilon M_4)(M_5+\varepsilon M_6)+O(\varepsilon^2)\right)\\
&= \frac{1}{-\varepsilon}\left(0+\varepsilon\begin{bmatrix}-2^n-n2^n & n2^{n+1}\\ -n2^{n-1} & -2^n+n2^n\end{bmatrix}+O(\varepsilon^2)\right)\\
&= \begin{bmatrix}2^n+n2^n & -n2^{n+1}\\ n2^{n-1} & 2^n-n2^n\end{bmatrix}+O(\varepsilon)
\end{aligned}$$

So if we take the limit, we get

$$\lim_{\varepsilon\to 0}(A(\varepsilon))^n = \begin{bmatrix}(1+n)2^n & -n\cdot 2^{n+1}\\ n\cdot 2^{n-1} & (1-n)2^n\end{bmatrix}$$

which is in fact the same matrix we got in part (b).