Problem 1

Prove: If $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges. Is this statement still true if the word "absolutely" is removed?

Solution. It is sufficient to show that $\sum_n a_n b_n$ is absolutely convergent. Consider the series $\sum_n |a_n b_n| = \sum_n |a_n||b_n|$. Since $\lim_{n\to\infty} a_n = 0$ (by the contrapositive of the "crude" divergence test since $\sum_n a_n$ converges), we have that a_n is bounded, and so $|a_n|$ is bounded as well (the upper bound is just the max of the lower and upper bound of a_n, and it is bounded below by 0). Let $|a_n| \le M$ for all $n \in \mathbb{N}$. Then $|a_n||b_n| < M|b_n|$. We have that $\sum_n M|b_n|$ converges, since if $s_N = \sum_{n=1}^{N} |b_n|$, then

$$\sum_n M|b_n| = \lim_{n\to\infty} M|b_0| + M|b_1| + \cdots + M|b_n| = \lim_{n\to\infty} M(|b_0| + |b_1| + \cdots + |b_n|) = \lim_{n\to\infty} Ms_n$$

and since (s_n) converges (by the absolute convergence of b_n), by our constant multiplication limit law, $(Ms_n) = \sum_n M|b_n|$ converges as well.

Now since $0 \le |a_n b_n| \le M|b_n|$, by the comparison test, $\sum_n |a_n b_n|$ converges, thus $\sum_n a_n b_n$ is absolutely convergent, which implies that $\sum_n a_n b_n$ converges.

Problem 2

For each series below, find the set of $x \in \mathbb{R}$ where the series converges.

(a). $\sum_{n=1}^{\infty} c^{n^2}(x-1)^n$ *($c > 0$ const.)*

(b). $\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(c). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[\frac{x+1}{2x+1}\right]^n$

(d). $\sum_{n=1}^{\infty} \left[\frac{(2n)!}{n(n!)^2}\right](x-e)^n$

(a). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = c^{n^2}(x-1)^n$ and $\alpha = \limsup_n |a_n|^{1/n}$. We can compute

$$\alpha = \limsup_n \left|c^{n^2}(x-1)^n\right|^{1/n} = \limsup_n |c^n(x-1)| = |x-1| \limsup_n c^n$$

where we've brought the exponent n out in the first step, since $|a^n b^n| = |ab|^n$.

If $x = 1$, then $\alpha = 0$, so the series converges by the root test regardless of c. Now let $x \in \mathbb{R} \setminus \{1\}$. We know that $\lim_{n\to\infty} c^n \to +\infty$ if $c > 1$, so $\limsup_n c^n = +\infty$, thus the series diverges for all x. Additionally, $\lim_{n\to\infty} c^n = 0$ if $1 > c > 0$, so $\limsup_n c^n = 0$, thus the series converges for all x. Finally, if $c = 1$, $a_n = (x-1)^n$ which is a geometric series: it will converge when $|x-1| < 1 \implies 0 < x < 2$ and will diverge otherwise.

In summary:

- If $c > 1$, $x \in \{1\}$ makes the series converge
- If $c = 1$, $x \in (0, 2)$ makes the series converge
- If $0 < c < 1$, $x \in \mathbb{R}$ makes the series converge

(b). *Solution.* Let $a_n = \frac{x^n(1-x^n)}{n}$. Let $x \in \{0, 1\}$. Then $a_n = 0$ for all n, thus the series converges. Let $x = -1$. Then our series is $\sum_n a_n = \sum_{n \text{ odd}} \frac{2}{n}$. We can rewrite our sum to be $\sum_n \frac{1}{2\lfloor(n-1)/2\rfloor+1}$ (since if $n = 2k$, $2\lfloor(n-1)/2\rfloor + 1 = n-1$: the odd number directly below it, and if $n = 2k+1$, $2\lfloor(n-1)/2\rfloor + 1 = n$: itself) and then since $0 < 2\lfloor(n-1)/2\rfloor + 1 < n$ so $0 < \left|\frac{1}{n}\right| \le \frac{1}{2\lfloor(n-1)/2\rfloor+1}$, comparison test says this series diverges (since the harmonic series diverges to infinity).

Now consider when $|x| > 1$. Then we claim there exists an $N \in \mathbb{N}$ such that $x^n(1-x^n) < -1$ for all $n \ge N$. We prove this by considering when n is positive and negative. Let $x > 1$. Note that there exists an N such that $x^n > 2$ for all $n \ge N$: using the inequality from Problem 4(a) of Homework 6 since $x > 1$, we have that $x^n > x^n - 1 \ge n(x-1)$ and then invoke Archimedean property to find N such that $N(x-1) > 2$, it's trivial to see that $n \ge N$ also implies $x^n > 2$. Now if $n \ge N$, we have $1 - x^n < -1$ and since $x^n > 1$, we have $x^n(1-x^n) < 1 - x^n < -1$. Now let $x < -1$. If n is even, $x^n(1-x^n) = |x|^n(1-|x|^n)$, and we have the same N from when $x > 1$ to have $x^n(1-x^n) < -1$. If n is odd, $x^n(1-x^n) = (-1)|x|^n(1-(-1)|x|^n) = -|x|^n(|x|^n+1)$, and using the N from before, we have $-|x|^n(|x|^n+1) < -2(x^n+1) < -2 < -1$. This proves our claim. But then for all $n \ge N$, we have that

$$\frac{x^n(1-x^n)}{n} < \frac{-1}{n} < 0 \implies 0 < \frac{1}{n} = \left|\frac{1}{n}\right| < -\frac{x^n(1-x^n)}{n}$$

And so by the comparison test, $\sum_n -a_n$ diverges. But this is true only if $\sum_n a_n$ diverges, since if $s_N = \sum_{n=1}^N a_n$ and $s'_N = \sum_{n=1}^N -a_n$, we have that $s'_N - \sum_{n=1}^N a_n = -s_N$, and if s_N converged as $N \to \infty$, constant multiplication limit law would tell us that s'_N would converge as well. Thus, if $|x| > 1$, we have that the series diverges.

Now consider when $0 < x < 1$. Consider $\sum_k 2^k \frac{x^{2^k}(1-x^{2^k})}{2^k} = \sum_k x^{2^k}(1-x^{2^k})$. We have $0 < x^{2^k}(1-x^{2^k}) < x^{2^k} < x^k$ (where the inequality is due to the fact that x^a is monotonically decreasing when $0 < x < 1$, and

$2^k > k$), and $\sum_k x^k$ converges since it is geometric series with ratio $x < 1$. Thus, by the comparison test, we have that $\sum_k 2^k \frac{x^{2^k}(1-x^{2^k})}{2^k}$ converges as well. Finally, $\frac{x^n(1-x^n)}{n}$ is monotonically decreasing and bounded below by 0: all the terms are positive, so $a_n > 0$ for all n; now see

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}(1-x^{n+1})n}{x^n(1-x^n)(n+1)} < x\frac{1-x^{n+1}}{1-x^n}$$

and $x\frac{1-x^{n+1}}{1-x^n} \to x$ as $n \to \infty$ (limit laws), thus for sufficiently large n, we have that $\frac{a_{n+1}}{a_n} < x + \varepsilon$ where setting $\varepsilon = 1 - x > 0$ gives $\frac{a_{n+1}}{a_n} < 1$, hence the series is monotonically decreasing past that point. Thus, by Cauchy condensation, the series converges when $0 < x < 1$ (technically, Cauchy Condensation only tells us that the series converges starting from our n where the series begins to be monotonically decreasing, but then we have the sum of a convergent series and a finite sum, which itself converges).

It remains to consider the case when $-1 < x < 0$. Now if n is odd, we have $a_n = \frac{(-1)|x|^n(1+|x|^n)}{n} < \frac{-2|x|^n}{n}$. Since $\lim_{n\to\infty} \frac{2n}{(1/|x|)^n} = 0$ by Rudin Theorem 3.20 (d), we have some N such that $\left(\frac{1}{|x|}\right)^n > 2n > 0$ for all $n \geq N$, thus $0 < 2|x|^n < \frac{1}{n}$, thus $|a_n| < \frac{2|x|^n}{n} < \frac{1}{n^2}$. Furthermore, when n is even, we have $a_n = \frac{|x|^n(1-|x|^n)}{n} < \frac{|x|^n(1+|x|^n)}{n} < \frac{1}{n^2}$ as well. Thus $|a_n| < \frac{1}{n^2}$ for all $n \geq N$. And since $\sum_n \frac{1}{n^2}$ is a convergent p-series ($p > 1$), the comparison test tells us that $\sum_n a_n$ converges as well.

In summary: the series converges when $x \in (-1, 1]$, and diverges otherwise.

(c). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = \frac{1}{\sqrt{n}}\left[\frac{x+1}{2x+1}\right]^n$ and $\alpha = \limsup_n |a_n|^{1/n}$. Note that if $x = -\frac{1}{2}$, none of our terms exist, so we ignore that value. We can compute

$$\alpha = \limsup_n \left|\frac{1}{\sqrt{n}}\left[\frac{x+1}{2x+1}\right]^n\right|^{1/n} = \limsup_n (n^{1/(2n)})^{-1}\left|\frac{x+1}{2x+1}\right| = \left|\frac{x+1}{2x+1}\right| \limsup_n (n^{1/(2n)})^{-1} = \left|\frac{x+1}{2x+1}\right|$$

where our final equality is due to $\lim_{n\to\infty} n^{1/(2n)} = 1$, and so $\liminf_n n^{1/2n} = 1$ (lim inf agrees with convergent limits), and by Problem 8(c) from homework 4, $\limsup_n (n^{1/(2n)})^{-1} = 1^{-1} = 1$ (we've also used the fact $|a^n| = |a|^n$ for our first equality). When $x > 0$, we have $|2x+1| = 2x+1 > x+1 = |x+1|$. When $-\frac{2}{3} < x < 0$, $|2x+1| = -2x-1 < x+1 = |x+1|$. When $x < -\frac{2}{3}$, $|2x+1| > |x+1|$. Now, ratio test gives convergence when $|x+1| < |2x+1|$. Thus, when $x \in (-\infty, -\frac{2}{3}) \cup (0, \infty)$, $\alpha < 1$, thus ratio test says the series converges. When $x \in (-\frac{2}{3}, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$, $\alpha > 1$, thus the ratio test says the series diverges.

If $x = -\frac{2}{3}$, we have $a_n = (-1)^n\frac{1}{\sqrt{n}}$. Note that $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, thus the series is monotonically decreasing. Thus, the alternating series test says the series converges.

Finally, if $x = 0$, $a_n = \frac{1}{\sqrt{n}}$, thus $\sum_n a_n$ diverges since this is a p-series where $p = \frac{1}{2} < 1$.

In summary: the series converges when $x \in (-\infty, -\frac{2}{3}] \cup (0, \infty)$.

(d). *Solution.* Fix some arbitrary $x \in \mathbb{R}$. Let $a_n = \left[\frac{(2n)!}{n(n!)^2}\right](x-e)^n$ and define $\overline{\alpha} = \limsup_n \left|\frac{a_{n+1}}{a_n}\right|$ and $\underline{\alpha} = \liminf_n \left|\frac{a_{n+1}}{a_n}\right|$. $a_n = 0$ only when $x = e$. In this case, $a_n = 0$, thus $\sum_n a_n$ converges. Now assume that $x \neq e$. We can compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2n+2)(2n+1)n}{(n+1)^2(n+1)}(x-e)\right| = |x-e|\frac{4n^2+2n}{n^2+2n+1}$$

We see $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-e|\lim_{n\to\infty}\frac{4+2/n}{1+2/n+1/n^2} = 4|x-e|$ (applying limit laws for multiplication and division). Thus, $\overline{\alpha} = \underline{\alpha} = 4|x-e|$ since the limit exists. Thus, the series converges when $4|x-e| < 1 \implies |x-e| < \frac{1}{4} \implies e - \frac{1}{4} < x < e + \frac{1}{4}$ and diverges when $4|x-e| > 1 \implies |x-e| > \frac{1}{4} \implies x - e > \frac{1}{4} \implies x > e + \frac{1}{4}$ and $x - e < -\frac{1}{4} \implies x < e - \frac{1}{4}$ by the ratio test.

If $x = e + \frac{1}{4}$, we have $a_n = \left[\frac{(2n)!}{n(n!)^2}\right]\left(\frac{1}{4}\right)^n$ ff only need to prove this case: this is absolute value of other one, so absolute convergence.

Problem 3

Discuss the series whose nth terms are shown below:

$$a_n = (-1)^n \frac{n^n}{(n+1)^{n+1}}, \qquad b_n = \frac{n^n}{(n+1)^{n+1}},$$
$$c_n = (-1)^n \frac{(n+1)^n}{n^n}, \qquad d_n = \frac{(n+1)^n}{n^{n+1}}.$$

Solution. Let $\alpha = \limsup_n |a_n|^{1/n}$. Then

$$\alpha = \limsup_n \left(\frac{n^n}{(n+1)^{n+1}}\right)^{1/n} = \limsup_n \frac{n}{n+1}\frac{1}{(n+1)^{1/n}}$$

Hmm I think $\limsup = 1$, so this won't work.ff

Note that c_n fails to converge. Since $\frac{n+1}{n} > 1$ for all $n \in \mathbb{N}$, we have $\left(\frac{n+1}{n}\right)^n > 1$. Thus $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \geq 1 > 0$ if the limit exists. Regardless, this implies $\lim_{n\to\infty} c_n \neq 0$, and so the crude divergence test tells us $\sum_n c_n$ fails to converge (and so fails to absolutely converge).

Note that d_n fails to converge. We have $d_n = \frac{(n+1)^n}{n^{n+1}} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n}$. Since $\left(\frac{n+1}{n}\right)^n > 1$ for all n, we have that $d_n > \frac{1}{n} = \left|\frac{1}{n}\right| > 0$. Thus, by the comparison test, $\sum_n d_n$ diverges to infinity, since the harmonic series also diverges to infinity (and so $\sum_n d_n$ also fails to absolutely converge).

Problem 4

Suppose $x_1 \geq x_2 \geq x_3 \geq \cdots$ *and* $\lim_{n\to\infty} x_n = 0$. *Show that the following series converges:*

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \pm \cdots .$$

Solution. By the alternating series test, we know that $\sum_n (-1)^n x_n$ converges. Now note that $\frac{1}{n}(x_1 + x_2 + \cdots + x_n) > x_n$.

Want to get hmm maybe even somethng about how inside each parantheses looks like a series too?

Note that the sequence is bounded below by 0. We have a sum of geometric series.

Problem 5

(a). *Prove: if $a_n \geq a_{n+1} \geq 0$ for all n, and $\sum a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.*

(b). *Prove: If $\sum(b_n^2/n)$ converges, $\frac{1}{N}\sum_{j=1}^{N} b_j \to 0$ as $N \to \infty$.*

[Hint: In part (a), it's enough to prove that $\frac{1}{2}na_n \to 0$.]

(a). *Solution.* It would be sufficient to show that $\sum_n \frac{1}{2}na_n$ converges. This converges if and only if $\sum_n 2^n a_n$ by the Cauchy condensation... this is not Cauchy condensation because of n... but what about $n < 2^k$.

Since $a_n \geq a_{n+1} \geq 0$ for all n and $\sum a_n$ converges, the Cauchy Condensation Test gives $\sum_k 2^k a_{2^k}$ converges as well. Since our sequence monotonically decreases and is always positive, we have $2^k a_n \leq 2^k a_{2^k}$ for $n \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$. Note that for any k, $2^k \leq n < 2^{k+1}$ implies $\frac{n}{2} < 2^k$. Thus, $0 \leq |\frac{n}{2}a_n| = \frac{n}{2}a_n < 2^k a_n \leq 2^k a_{2^k}$ for $2^k \leq n < 2^{k+1}$. ff

(b). *Solution.* In Problem 8(a) from homework 3, we proved that if $a_n \to 0$ as $n \to \infty$, then $(a_1+a_2+\cdots a_n)/n \to 0$ as well. Thus, it is sufficent to show that $b_n \to 0$ as $n \to \infty$.

Since $\sum(b_n^2/n)$ converges, $\liminf_n \left|\frac{b_{n+1}^2/(n+1)}{b_n^2/n}\right| \leq 0$ (contrapositive of part (b) of the ratio test). ff

But need to prove that $b_n^2/n \geq b_{n+1}^2/(n+1) \geq 0$, then can invoke part (a) and we're done.

Problem 6

Define $f(\theta) = \sum_{k=1}^{\infty} \frac{1}{2k-1}\sin((2k-1)\theta)$*. Determine the domain of* f*, namely, the set of all real* θ *where the series converges, by completing the steps below.*

(a). Obtain the following identities, valid for each $n \in \mathbb{N}$ *at all points where* $\sin\theta \neq 0$*:*

$$C_n(\theta) = \cos(\theta) + \cos(3\theta) + \cos(5\theta) + \cdots + \cos((2n-1)\theta) = \frac{\sin(2n\theta)}{2\sin\theta},$$

$$S_n(\theta) = \sin(\theta) + \sin(3\theta) + \sin(5\theta) + \cdots + \sin((2n-1)\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta},$$

[Suggestion: Use geometric sums of complex numbers, with $e^{it} = \cos(t) + i\sin(t)$*.]*

(b). Prove that the domain of f *is the interval* $(-\infty, +\infty)$*.*

(c). Find a sequence (θ_n) *such that* $\theta_n \to 0$ *and* $S_n(\theta_n) \to +\infty$ *as* $n \to \infty$*. Explain why your solution in part (b) is correct in spite of the evident unboundedness of the sequence* $(S_n(\theta_n))$*.*

(a). Solution. We have $\sum_{k=1}^{n} e^{i(2k-1)\theta} = C_n(\theta) + iS_n(\theta)$. We can rewrite our sum as $\sum_{k=0}^{n-1} e^{i(2k+1)\theta}$. But this is a geometric series with common ratio $e^{2i\theta}$ and initial value $e^{i\theta}$, thus

$$C_n(\theta) + iS_n(\theta) = e^{i\theta}\frac{1-(e^{2i\theta})^n}{1-e^{2i\theta}} = \frac{1-\cos(2n\theta) - i\sin(2n\theta)}{e^{-i\theta} - e^{i\theta}}$$

But note that

$$\frac{1}{e^{-i\theta} - e^{i\theta}} = \frac{1}{\cos(-\theta) + i\sin(-\theta) - \cos(\theta) - i\sin\theta} = \frac{1}{-2i\sin(\theta)} = \frac{i}{2\sin\theta}$$

Thus

$$C_n(\theta) + iS_n(\theta) = \frac{\sin(2n\theta)}{2\sin\theta} + i\frac{1-\cos(2n\theta)}{2\sin\theta}$$

For equality, the real components must equal the real components, and the imaginary components must equal the imaginary components, thus since $C_n(\theta)$ and $S_n(\theta)$ are strictly real-valued functions, we have

$$C_n(\theta) = \frac{\sin(2n\theta)}{2\sin\theta}, \quad S_n(\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta}$$

as desired.

(b). Solution. We seek to show that the series converges for all $\theta \in \mathbb{R}$. Recall that $|\sin(x)| \geq \frac{2x}{\pi}$ (Piazze @331), and so $\frac{1}{2k-1}\sin((2k-1)\theta)$

It is suffient to show if

(c). Solution. Consider $(\theta_n) = \frac{1}{n}$. Then $S_n(\theta_n) = \frac{1-\cos 2}{2\sin\frac{1}{n}}$. We have that $\sin(\frac{1}{n}) \to 0$ as $n \to \infty$ (since $\sin(\frac{1}{n}) \to 0$ as $n \to \infty$...).