Problem 1 (Ch. 1.6)

Write $(456)(567)(671)(123)(234)(345)$ *as a product of disjoint cycles.*

Solution. We can explicitly write down where each element in the domain of the product of the 3-cycles, specifically $1, 2, \ldots, 7$, go according to the $(467)(567)(671)(123)(234)(345)$:

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 7 \\ 3 &\mapsto 3 \\ 4 &\mapsto 4 \\ 5 &\mapsto 5 \\ 6 &\mapsto 6 \\ 7 &\mapsto 1 \end{aligned}$$

By inspection, we can find that the disjoint cycles in this is (1247), (56) and (3), thus

$$(467)(567)(671)(123)(234)(345) = (127)$$

Problem 2 (Ch. 1.6)

Show that if $n \geq 3$ *then* A_n *is generated by the* 3*-cycles* (abc).

Solution. Let $E_n = \{(abc) : a, b, c \in \mathbb{N}, a, b, c \leq n\}$ be the set of 3-cycles in S_n. We seek to prove that $\langle E_n \rangle = A_n$. Let $\eta \in \langle E_n \rangle$. Since $(abc) = (ac)(ab)$, each 3-cycle (abc) in η can be decomposed into two transpositions, so if η was the product of k 3-cycles, η is equal to the product of $2k$ transpositions, which is even, thus $\eta \in A_n$. Thus $E_n \subseteq A_n$.

Now let $\alpha \in A_n$. Note that by the first part of question 5 of this homework, we can decompose α into a product of transpositions of the form $(1\ m)$ where $1 \leq m \leq n$ (it is fine to cite this solution; my solution to 5 does not rely on this problem, and if I really wanted to, I could write the proof of 5 above this). Note that this will always be an even number of transpositions, by the definition of α being even. Now, consider pairing up neighbouring elements $(1\ m_1)(1\ m_2)$ from the decomposition of α (and each will have a pair by evenness of number of transpositions). Then we can rewrite this as $(1\ m_1)(1\ m_2) = (1\ m_2\ m_1)$ (one can easily verify this is true). But then α can just be written as a product of 3-cycles, so $\alpha \in E_n$. Thus $A_n \subseteq E_n$ since α was arbitrary. This shows that $A_n = E_n$.

Problem 3 (Ch. 1.6)

Determine the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

Solution. Call our permutation α. Consider when n is even. Then we can rewrite $n = 2k$. Then one can easily see that we can decompose this permutation as the product of disjoint cycles below

$$(1\ n)(2\ [n-1]) \cdots (k\ \ [k+1])$$

Then, there are $k = \frac{n}{2}$ transpositions, thus the sign of α is the parity of k, or $\operatorname{sgn}(\alpha) = (-1)^k$.

If n is odd, we can rewrite as $n = 2k + 1$. We then decompose the permutation as the following product of disjoint cycles:

$$(1\ n)(2\ [n-1]) \cdots (k\ [k+1])$$

Then, there are $k = (n-1)/2$ transpositions, thus the sign of the permutation is the parity of k, or $\operatorname{sgn}(\alpha) = (-1)^k$.

We can summarize these two statements by noticing that this result is dependent on $n \pmod 4$. If n is even, and if $n/2$ is even, then our permutation is even; if $n/2$ is odd, then our permutation is odd. So $n \equiv 0 \pmod 4 \implies \operatorname{sgn}(\alpha) = 1$ and $n \equiv 2 \pmod 4 \implies \operatorname{sgn}(\alpha) = -1$. If n is odd, and if $(n-1)/2$ is even, then our permutation is even; if $(n-1)/2$ is odd, then our permutation is odd. So $n \equiv 1 \pmod 4 \implies \text{sgn}(\alpha) = 1$ and $n \equiv 3 \pmod 4 \implies \text{sgn}(\alpha) = -1$. Thus

$$\text{sgn}(\alpha) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 4 \\ -1 & \text{if } n \equiv 2, 3 \pmod 4 \end{cases}$$

Problem 4 (Ch. 1.6)

Show that if α is any permutation then

$$\alpha(i_1 i_2 \cdots i_r)\alpha^{-1} = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r))$$

Solution. First note that $\alpha(i_1 i_2 \cdots i_r)\alpha^{-1} = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r))$ if and only if

$$\alpha(i_1 i_2 \cdots i_r) = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r))\alpha$$

It is sufficient to show that the permutations on either side map each element of the ambient set to the same element.

Consider $j \notin \{i_1, \dots, i_r\}$. Then our left hand side maps j to $\alpha(j)$ (since a cycle will just map any element not included in the cycle to itself). Furthermore, $\alpha(j) \notin \{\alpha(i_1)\cdots\alpha(i_r)\}$ since permutations are injective, thus the right hand side maps j to $\alpha(j)$ as well.

Now let $m \in \mathbb{N}, 1 \le m \le r$ be arbitrary. Then

$$\big(\alpha \circ (i_1 i_2 \cdots i_r)\big)(i_m) = \alpha(i_{m+1}) = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_m)\alpha(i_{m+1})\cdots\alpha(i_r))(\alpha(i_m)) = \big((\alpha(i_1)\cdots\alpha(i_r a)) \circ \alpha\big)(i_m)$$

But since this is true for all elements in $\{i_1, \dots, i_r\}$, and we already proved the case for all the elements not in this set, we have shown that our two permutations map every element in the ambient set to the same element, and so permutations on the left and the right are equal, thus our original two permutations are equal as well.

Problem 5 (Ch. 1.6)

Show that S_n is generated by the $n-1$ transpositions $(12), (13), \cdots, (1n)$ and also by the $n-1$ transpositions $(12), (23), \cdots, (n-1n)$.

Solution. First note that $\langle (12), \dots, (1n) \rangle \subseteq S_n$. This follows from the definition of S_n.

Now, let $\alpha \in S_n$ be an arbitrary permutation. It was shown in Jacobson that we can decompose any α into a product of transpositions. But for any transposition $(a\ b)$, $a, b \in \{1, \dots, n\}$, we can easily see that $(a\ b) = (1\ a)(1\ b)(1\ a)$ (one can easily verify this is true). But this applies to every transposition in the decomposition of α, thus α can be decomposed in terms of a product of elements of the form $(1\ a)(1\ b)(1\ a)$. But then $\alpha \in \langle (12), \dots, (1n) \rangle$, so $S_n \subseteq \langle (12), \dots, (1n) \rangle$. Therefore, we have shown $S_n = \langle (12), \dots, (1n) \rangle$.

We now seek to prove $S_n = \langle (12), (23), \dots, (n-1n) \rangle$. Note that $\langle (12), (23), \dots, (n-1n) \rangle \subseteq S_n$, by definition of S_n. So it sufficies to prove that $S_n \subseteq \langle (12), (23), \dots, (n-1n) \rangle$. But since we have just shown $S_n = \langle (12), \dots, (1n) \rangle$, it suffices to prove that $\langle (12), \dots, (1n) \rangle \subseteq \langle (12), (23), \dots, (n-1n) \rangle$. Consider $(1m)$ where $1 \le m \le n$. Then

$$(1m) = (m-1\ m)(m-2\ m-1)\cdots(23)(12)(23)\cdots(m-2\ m-1)(m-1\ m)$$

One can check this by manually checking that both sides of the equality map each element $i \in \{1, 2, \dots, n\}$ to the same element (since for any two distinct elements in S_n, there exists $i \in \{1, 2, \dots, n\}$ where they map i to a different element in $\{1, 2, \dots, n\}$, thus they are the same if there is no such i). First, note that if $i \neq 1, m$, then i is mapped to itself with the permutation on both sides of the equality. If $i = 1, m$, the permutation on the left transposes them. The permutation on the right maps m to 1, since m gets mapped to the integer directly below it with each transposition until (12), and then 1 is never permuted again, so m stays at 1; and it maps 1 to m, since it is not transposed before (12), and then after, each transposition maps it to the integer directly above it until it reaches m. Thus the permutations on either side of the equality are equal. But since m was arbitrary, any element in $\langle (12), \dots, (1n) \rangle$ can be written as a product of transpositions from $\{(12), (23), \dots, (n-1\ n)\}$, thus $\langle (12), \dots, (1n) \rangle \subseteq \langle (12), (23), \dots, (n-1\ n) \rangle$. And so we have $S_n \subseteq \langle (12), (23), \dots, (n-1\ n) \rangle$, therefore $S_n = \langle (12), (23), \dots, (n-1\ n) \rangle$.