Problem 1

Solve in either order:

(a). Construct, with justification, a subset A of $\mathbb{R}$ such that every point of A is isolated and $A' \neq \emptyset$.

(b). Rudin Chapter 2, problem 5, page 43: Construct a bounded set of real numbers with exactly three limit points.

(a). *Solution.* We provide the subset $A = \{\frac{1}{n} \colon n \in \mathbb{N}\}$. First, if $a \in A$, denote this as $a = \frac{1}{n'}$. We show a is an isolated point: let $m = \min\left\{\left|\frac{1}{n'} - \frac{1}{n'+1}\right|, \left|\frac{1}{n'} - \frac{1}{n'-1}\right|\right\}$. Note that since for any $\overline{n} \geq n'$, we have that $\left|\frac{1}{n'+1} - \frac{1}{n'}\right| \leq \left|\frac{1}{\overline{n}} - \frac{1}{n'}\right|$, thus any ball around $\frac{1}{n'}$ that contains $\frac{1}{\overline{n}}$ must also contain $\frac{1}{n'+1}$. Now for any $\underline{n} \leq n'$, we have that $\left|\frac{1}{n'-1} - \frac{1}{n'}\right| \leq \left|\frac{1}{\underline{n}} - \frac{1}{n'}\right|$, thus any ball around $\frac{1}{n'}$ that contains $\frac{1}{\underline{n}}$ must also contain $\frac{1}{n'-1}$.

Consider the open set of R $U = \mathbb{B}[\frac{1}{n'}; \frac{1}{2}m)$. Note that $\frac{1}{n'-1} \notin U$ and $\frac{1}{n'+1} \notin U$. Thus by the contrapositive of the claims we just said, for any $n \in \mathbb{N}$ where $n \neq n'$, we have that $\frac{1}{n} \notin U$. Thus, $\frac{1}{n'}$ is isolated. Since this is true for arbitrary $n' \in \mathbb{N}$, every point in A is isolated.

Now, note that $0 \in A'$ so $A' \neq \emptyset$. If U is be an arbitrary open set in the neighbourhood of 0. Note that we will always have an element of A in U. Assume otherwise, that there exists an open set U such that $U \cap A = \emptyset$. Conside a ball in U, specifically $\mathbb{B}[0, r) \subseteq U$. Note by the Archimedean property of the reals, there exists $n \in \mathbb{N}$ such that $n \cdot 1 > r^{-1} > 0$, thus $0 < \frac{1}{n} < r$. But then $\frac{1}{n} \in \mathbb{B}[0, r)$, thus a contradiction. Thus, since $U \in \mathcal{N}(0)$ was arbitrary, we have that every open set in the neighbourhood of 0 has a non empty intersection with A, thus 0 is a limit piont of A. Thus, $A' \neq \emptyset$.

(b). *Solution.* We give the set $A = \{\frac{1}{n} \colon n \in \mathbb{N}\} \cup \{\frac{1}{n} + 10 \colon n \in \mathbb{N}\} \cup \{\frac{1}{n} + 20 \colon n \in \mathbb{N}\}$. From part (a) of this problem, we note that no element in A is a limit point of A, since they are all isolated (and thus cannot be limit points); the argument is the same, since the additional term just makes it so that we have three subsets of A that do not intersect. Furthermore, we can make identical arguments as from part (a) to show that 10 and 20 are in A', as well as 0. Thus, A has exactly three limit points.

Problem 2

(a). Give an example of two sets A and B in some HTS satisfying

$$\mathrm{int}(A \cup B) \neq \mathrm{int}(A) \cup \mathrm{int}(B)$$

(b). Give an example of two sets A and B in some HTS satisfying

$$\overline{A \cap B} \neq \overline{A} \cap \overline{B}$$

(c). Working $\mathbb{R}^k$ with the usual topology, express the open ball $\mathbb{B}[0;1)$ as a union of closed sets. Can $\mathbb{B}[0;1)$ be expressed as an intersection of closed sets?

(a). *Solution.* Let our HTS be $\mathbb{R}$, and let $A = [0,1]$ and $B = [1,2]$. We have $\mathrm{int}(A \cup B) = \mathrm{int}([0,2]) = (0,2)$ and $\mathrm{int}(A) \cup \mathrm{int}(B) = (0,1) \cup (1,2) = (0,2) \setminus \{1\}$. Hence we have shown $\mathrm{int}(A) \cup \mathrm{int}(B) = (0,2) \setminus \{1\} \neq (0,2) = \mathrm{int}(A \cup B)$, so we are done.

(b). *Solution.* Let our HTS be $\mathbb{R}$, and let $A = (0,1)$ and $B = (1,2)$. We have $\overline{A \cap B} = \overline{\emptyset} = (\mathbb{R}^o)^c$, but since A is open if and only if A^0 is open (from notes) and $\mathbb{R}$ must be open, we have $\overline{A \cap B} = \mathbb{R}^c = \emptyset$. Now, see that $\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$, where we have used the fact that in $\mathbb{R}$, $\overline{(a,b)} = (((a,b)^c)^o)^c = (((-\infty, a] \cup [b, \infty))^o)^c$ but taking the largest open subset we get $((-\infty, a) \cup (b, \infty))^c = [a,b]$. Hence, $\overline{A \cap B} = \emptyset \neq \{1\} = \overline{A} \cap \overline{B}$.

(c). *Solution.* ff

Problem 3

Define a family $\mathcal{T}$ of subsets of $\mathbb{R}$ as follows:

A set $G \subseteq \mathbb{R}$ belongs to $\mathcal{T}$ if and only if for every x in G, there exists $r > 0$ such that $[x, x+r) \subseteq G$.

(a). Prove that $(\mathbb{R}, \mathcal{T})$ is a HTS. (It is called the Sorgenfrey line.*)*

All our terminology – open set, closed set, boundary point, limit point, convergence – depends on what topology we use. Use the Sorgenfrey topology in parts (b)-(d):

(b) Show that the interval $[0, 1)$ is open.

(c) Find all boundary points of the interval $(0, 1)$.

(d) Let $s_n = -1/n$ and $t_n = 1/n$. Prove that one of these sequences converges to 0, and the other does not. Use the definition given in class, i.e. $x_n \to \hat{x}$ means that for every open set U containing $\hat{x}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$.

(a). *Solution.* Hmm... basically the open sets are just those are open on one end.

(b). *Solution.* Let $x \in [0, 1)$. Then let $1 - x = \delta > 0$. We have that $[x, x + \delta) = [x, 1) \in [0, 1)$. Thus, $[0, 1) \in \mathcal{T}$ and so is an open set.

(c). *Solution.* ff

(d). *Solution.* ff

Problem 4

*Let A be a subset of a HTS $(X, \mathcal{T})$. The **boundary of** A is a set denoted ∂A: we say $z \in \partial A$ if and only if every open U containing z satisfies both $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. Prove:*

(a). $\partial A = \overline{A} \cap \overline{A^c}$.

(b). A is closed if and only if $\partial A \subseteq A$.

(c). A is open if and only if $A \cap \partial A = \emptyset$.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 5

Prove: For every set A in a HTS $(X, \mathcal{T})$, A' is closed.

Solution. ff

Problem 6

Recall the sequence space ℓ^2 from HW07 Q3. Given a specific $M = (M_1, M_2, \dots)$ in ℓ^2, let

$$S = \{x \in \ell^2 \colon \forall n \in \mathbb{N}, |x_n| \leq M_n\}.$$

Prove: every sequence $\left(x^{(n)}\right)$ in S has a convergent subsequence, whose limit lies in S.

Solution. Write out the sequence of the first components of our sequence, namely

$$(x_1)^{(n)} = x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \dots$$

Each term of $(x_1)^{(n)}$ is bounded by M_1, thus, by Bolzano-Weierstrass (which we can use since this is just a real-valued sequence), there exists a subsequence (n_{k_1}) such that $(x_1)^{(n_{k_1})}$ converges. Denote this value that it converges to as s_1.

Now consider the second components of our subsequence (n_{k_1}), namely

$$(x_2)^{(n_{k_1})} = x_2^{(n_1)}, x_2^{(n_2)}, x_2^{(n_3)}, \dots$$

Again, by Bolzano Weierstrass from boundedness by M_2, there exists a subsequence of n_{k_1}, call it n_{k_2} such that $(x_2)^{(n_{k_2})}$ converges. Denote this value that it converges to as s_2. Note that since subsequences of convergent sequences converge to the same value, we still have that $(x_1)^{(n_{k_2})} = s_1$.

Now, for any j, we can iteratively acquire a subsequence n_{k_j} such that $(x_j)^{(n_{k_j})}$ converges, call that value s_j, and for all $i < j$, we have $(x_i)^{(n_{k_j})} = s_i$.

Thus, we have that $\lim_{j\to\infty} \left(x^{(n_{k_j})}\right) = s_1, s_2, s_3, \dots = \hat{s}$, and one can confirm that $\hat{s} \in S$.