Math 321 Homework 3
(Including work made in collaboration with Tighe McAsey.)

Problem 1

Let $f \in \mathcal{R}[a,b]$ and $0 < p < \infty$. Define

$$\|f\|_p = \left(\int_a^b |f|^p dx\right)^{1/p}$$

(a). Prove that for $0 < p < \infty$, $|f|^p \in \mathcal{R}[a,b]$ (and hence the above definition makes sense).

(b). If f is continuous, prove that

$$\lim_{p\to\infty} \|f\|_p = \sup\{|f(x)| : x \in [a,b]\}.$$

(c). For f fixed, define $\phi(p) = \|f\|_p^p$. Using Rudin Problem 6.10, prove that $p \mapsto \log\phi(p)$ is convex on $(0,\infty)$ (recall Rudin problem 4.23 for the definition of convexity, and its consequences). Do not submit the proof of Rudin Problem 6.10 (but I encourage you to do it; it is a good exercise).

Remark 1. Since convex functions are continuous (see Rudin Problem 4.23), you have just shown that ϕ and hence $p \mapsto \|f\|_p$ are continuous.

(a). *Solution.* By Rudin Theorem 6.13(b), since $f \in \mathcal{R}[a,b]$, we have $|f| \in \mathcal{R}[a,b]$. Then, since p is finite, by applying Rudin Theorem 6.13(a) p times, we get that $|f|^p \in \mathcal{R}[a,b]$ as well.

(b). *Solution.* Note that since f is continuous, so is $|f|$ since composition of continuous functions is also continuous. Similarly, $|f|^p$ is also continuous. Furthermore, the function is defined on the closed and bounded set $[a,b]$, which in $\mathbb{R}$ is compact, so $|f|$ attains its maximum value M on $[a,b]$, say at point $e \in [a,b]$. So $|f| \le M$ on $[a,b]$. Since $f(x) = x^p$ is a monotonically (strictly) increasing function on $[0,\infty)$, and $0 \le |f|$, our inequality is preserved if we raise it to p, so $|f|^p \le M^p$ on $[a,b]$. We then have $\int_a^b |f|^p dx \le M^p(b-a)$ by Rudin Theorem 6.12(d) (and part (a), which says $|f|^p \in \mathcal{R}[a,b]$). Since $f(x) = x^{1/p}$ is a monotonically (strictly) increasing function on $[0,\infty)$, and $0 \le |f|^p \implies 0 \le \int_a^b |f|^p dx$, our inequality is preserved if we raise it to $1/p$, so

$$\int_a^b |f|^p dx \le M(b-a)^{1/p}$$

Now when we take the limit $p \to \infty$, since limits preserve non-strict inequalities, we get

$$\lim_{p\to\infty} \int_a^b |f|^p dx \le M \lim_{p\to\infty} (b-a)^{1/p} = M$$

where we use the fact $b-a > 0$ (otherwise integral is 0 and this fact actuall fails) and Rudin Theorem 3.20(b): $\lim_{n\to\infty} \sqrt[n]{p} = 1$ when $p > 0$. So $\lim_{p\to\infty}\|f\|_p$ has an upper bound, specifically, $M = \sup\{|f(x)| : x \in [a,b]\}$. So either $\lim_{p\to\infty}\|f\|_p = M$, or there exists some $L < M$ such that $\lim_{p\to\infty}\|f\|_p$. We proceed with showing such an L cannot exist.

We prove that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $p \ge N$, we have $\|f\|_p \ge M - \varepsilon$. Let $\varepsilon > 0$. We may assume that $M - \varepsilon > 0$, otherwise $\|f\|_p > 0 \ge M - \varepsilon$ for all $p \in \mathbb{N}$, and we are done. Let c, d be the points $a \le c < d \le b$ defined as follows:

- c: If $f(x) \ge M - \varepsilon/2$ for all $x \in [a,e]$, then let $c = a$. Otherwise, there is some $y \in [a,e]$ such that $f(y) < M - \varepsilon/2$, and so by intermediate value theorem, since f is continuous, there is some point $c' \in (y,e)$ such that $f(c') = M - \varepsilon/2$. We let c be the rightmost point, i.e., $f(x) \in (M - \varepsilon/2, M)$ when $x \in (c,e)$ (and we can pick such a c, otherwise, if $f(x) = M - \varepsilon/2$ arbitrarily close to e, we would break continuity since $f(e) = M$).

- d: Identical as above, i.e. $d = b$ when $f(x) \geq M - \varepsilon/2$ for all $x \in [e, b]$; otherwise, d is the leftmost point so that $f(d) = M - \varepsilon/2$.

Note that we get the strict inequality on account of $c < e < d$.

Since $|f|^p$ is positive, $|f|^p \geq |f_{[c,d]}|^p$ where $f_{[c,d]} = \begin{cases} f(x) & x \in [c,d] \\ 0 & \text{otherwise} \end{cases}$, and $\int_c^d |f|^p dx = \int_a^b |f_{[c,d]}|^p dx$, so we have

$$\int_a^b |f|^p dx \geq \int_a^b |f_{[c,d]}|^p = \int_c^d |f|^p dx \geq (M - \varepsilon)^p (d - c) > 0$$

Raising it to $1/p$ (which doesn't change inequalities) gives

$$\|f\|_p \geq (M - \varepsilon)(d - c)^{1/p}$$

If $(d - c) \geq 1$, then we have $\|f\|_p \geq M - \varepsilon$ for all $p \in \mathbb{N}$ (since $(d - c)^{1/p} \geq 1$ for all $p \in \mathbb{N}$), and we are done. Now, assume $(d - c) < 1$. Since $(d - c) > 0$, Rudin 3.20(b) gives the limit $\lim_{p\to\infty} \sqrt[p]{d - c} = 1$. Thus, there exists some $N \in \mathbb{N}$ such that for all $p \geq N$, we have $0 < 1 - (d - c)^{1/p} < \varepsilon/(2M)$. Rearranging gives $(d - c)^{1/p} > 1 - \varepsilon/(2M)$. Then for all $p \geq N$,

$$\|f\|_p > (M - \varepsilon/2)(1 - \varepsilon/(2M)) = M - M(\varepsilon/(2M)) - \varepsilon/2 + \varepsilon^2/(4M) \geq M - \varepsilon + \varepsilon^2/(4M) > M - \varepsilon$$

which is what we wanted to show.

Hence, if $\lim_{p\to\infty}\|f\|_p = L < M$, there exists $N \in \mathbb{N}$ such that for all $p \geq N$, $\|f\|_p > M - (M - L)/2 > M - (M - L) = L$, which is a contradiction. This only leaves $\lim_{p\to\infty}\|f\|_p = M = \sup\{|f(x)| : x \in [a, b]\}$.

(c). *Solution.* Let $\lambda \in (0, 1)$ and let $p, p' \in (0, \infty)$. Since $\lambda + (1 - \lambda) = 1$, and $|f|^{\lambda p}, |f|^{(1-\lambda)p'} \in \mathcal{R}[a, b]$ from part (a), we can use Hölder's inequality:

$$\begin{aligned}
\log \phi(\lambda p + (1 - \lambda)p') &= \log \left(\int_a^b |f|^{\lambda p + (1-\lambda)p'} dx \right) \\
&= \log \left(\int_a^b (|f|^p)^\lambda (|f|^{p'})^{1-\lambda} dx \right) \\
&\leq \log \left| \int_a^b (|f|^p)^\lambda (|f|^{p'})^{1-\lambda} dx \right| \\
&\leq \log \left(\left(\int_a^b \left|(|f|^p)^\lambda\right|^{1/\lambda} dx \right)^\lambda \left(\int_a^b \left|(|f|^{p'})^{1-\lambda}\right|^{1/(1-\lambda)} dx \right)^{1-\lambda} \right) \\
&= \log \left(\left(\int_a^b |f|^p dx \right)^\lambda \left(\int_a^b |f|^{p'} dx \right)^{1-\lambda} \right) \\
&= \lambda \log \left(\int_a^b |f|^p dx \right) + (1 - \lambda) \log \left(\int_a^b |f|^{p'} dx \right) \\
&= \lambda \log \phi(p) + (1 - \lambda) \log \phi(p')
\end{aligned}$$

where we are using the fact that since log is a monotone increasing function, it preserves inequalities (and also eliminating some $|\cdot|$ since $|f| > 0 \implies |f|^x > 0$).

Problem 2

Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions from $\mathbb{R} \to \mathbb{R}$ that converge pointwise. Must it be true that $\{f_n \circ g_n\}$ converges pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

Solution. It can be false. We provide the counter-example: define $g_n(x)$ on $0 < x \leq \frac{1}{n}$ as $g_n(x) = x$, and we periodically extend this function off of $(0, \frac{1}{n}]$ so that $g_n(x+k) = g_n(x)$ for any $k \in \mathbb{Z}$; now let

$$f_n(x) = \begin{cases} n, & 0 < x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

as well.

Clearly, both of these are functions from $\mathbb{R} \to \mathbb{R}$. Furthermore, we get that both of these sequences of functions converge pointwise to the 0 function. To see this for g, note that g_n attains its maximum value at $x = \frac{1}{n}$, since on $x \in (0, \frac{1}{n}]$, g_n is monotone increasing and this is the right most value, and since g_n is periodic, the largest value this function attains is the same as the largest value it attains on this interval. Furthermore, $g_n(\frac{1}{n}) = \frac{1}{n}$. Let $\varepsilon > 0$ and $x \in \mathbb{R}$ be fixed, then Archimedean gives us some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and so for any $n \geq N$, we have $|g_n(x) - 0| = g_n(x) \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$, which actually uniformly bounds g_n, and so we must have f_n is pointwise convergent to 0 for all $x \in \mathbb{R}$.

To see this for f, fix some $\varepsilon > 0$ and some $x \in \mathbb{R}$. Archimedean gives us some $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. By the definition of f_n, when $n \geq N$, we have that $|f_n(x) - 0| = f_n(x) = 0 < \varepsilon$ (since $x > \frac{1}{N} \geq \frac{1}{n}$). Hence, f_n is pointwise convergent to 0 for all $x \in \mathbb{R}$.

Now let us consider the composition, $f_n \circ g_n$. If $x \in \mathbb{R}$, then $g_n(x)$ maps x to some value in $(0, \frac{1}{n})$, which then $f_n(x)$ would map to n. Hence, $f_n \circ g_n = n$. As $n \to \infty$, it is clear that for every x, this function diverges to infinity, and so obviously does not converge pointwise.

Problem 3

Let E be a set and let $(M_1, d_1), (M_2, d_2)$ be metric spaces with the discrete metric (i.e. $d(x,y) = 0$ if $x = y$, and $d(x,y) = 1$ if $x \neq y$). Let $\{g_n\}$ be a sequence of functions from $E \to M_1$, and let $\{f_n\}$ be a sequence of functions from $M_1 \to M_2$. Suppose that $\{f_n\}$ and $\{g_n\}$ converge pointwise. Must it be true that $\{f_n \circ g_n\}$ converges pointwise? If so, prove it. If not, give a counter-example and prove that your counter-example is correct.

Solution. We claim that this is true, specifically if $g\colon E \to M_1$ and $f\colon M_1 \to M_2$ are functions such that $g_n \to g$ and $f_n \to f$ pointwise, $f_n \circ g_n \to f \circ g$ pointwise as well.

Let $\varepsilon > 0$ and $x \in E$. Since $g_n \to g$, for all $\varepsilon' > 0$, there exists some N_1 such that $d(g_n(x), g(x)) < \varepsilon'$ for all $n > N_1$. If we let $\varepsilon' = \frac{1}{2}$, since this is the discrete metric so $d(g_n(x), g(x))$ can only either be 1 or 0, this tells us that for all $n \geq N_1$, $d(g_n(x), g(x)) = 0$, i.e. $g_n(x) = g(x)$.

Note $g(x) \in M_1$. Since $f_n \to f$, for all $\varepsilon' > 0$, there exists some N_2 such that $d(f_n(g(x)), f(g(x))) < \varepsilon'$ for all $n > N_2$. If we let $\varepsilon' = \frac{1}{2}$, since this is the discrete metric so $d(f_n(g(x)), f(g(x))$ can only either be 1 or 0, this tells us that for all $n \geq N_2$, $d(f_n(g(x)), f(g(x))) = 0$, i.e. $f_n(g(x)) = f(g(x))$.

We now consider $\{f_n \circ g_n\}$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, $f_n(g_n(x)) = f_n(g(x)) = f(g(x))$. Thus, $d(f_n(g_n(x)), f(g(x))) = d(f(g(x)), f(g(x))) = 0 < \varepsilon$.

ε, x were arbitrary, hence $\{f_n \circ g_n\}$ converges pointwise.

Problem 4

Let $\{f_n\}$ be a sequence of functions in $\mathcal{R}[a,b]$, let $f \in \mathcal{R}[a,b]$, let $f_n \to f$ pointwise, and suppose that $\{f_n(x)\}$ is monotone increasing for each $x \in [a,b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Solution. First, some definitions. If $P = \{x_0, \ldots, x_n\}$ is a partition, define $D(P) = \max_i\{x_i - x_{i-1}\}$. Secondly, $|I|$ will denote the length of an interval (either open or closed or neither, e.g. $|(a,b]| = b - a$).

We now collect some useful facts.

Lemma 1. *If $f \in \mathcal{R}[a,b]$ and $\varepsilon > 0$, then there exists some $\delta > 0$ such that if P is some partition of $[a,b]$ where $D(P) < \delta$ then $U(P,f) - L(P,f) < \varepsilon$.*

Proof. Let $f \in \mathcal{R}[a,b]$, $\varepsilon > 0$ be given. There must exist a partition, call it P^* such that $U(P^*, f) - L(P^*, f) < \varepsilon/3$. We claim $\delta = \min_i\{x_i - x_{i-1}\}$ is the value we want. Let P be a partition of $[a,b]$ such that $D(P) < \delta$. Then an interval in P is in at most two intervals in P^*. For all of the intervals of P entirely contained within one interval of P^*, call it $[x_{i-1}, x_i]$, their upper Riemann sum is bounded above by $M_i(x_i - x_{i-1})$ (and likewise with their lower). For the intervals $[y_{i-1}, y_i]$ of P that are between two intervals of P^* (and there are at most $\#P^*$ of them, since there are only those many boundaries between intervals), say $[x_{i-2}, x_{i-1}], [x_{i-1}, x_i]$, then $\sup\{f(x) \colon x \in [y_{i-1}, y_i]\}(y_i - y_{i-1})$ is less than $M_{i-1}(x_{i-1} - x_{i-2}) + M_i(x_i - x_{i-1})$, since $\sup\{f(x) \colon x \in [y_{i-1}, y_i]\} \le \max\{M_{i-1}, M_i\}$, and $y_i - y_{i-1} < \delta \le (x_{i-1} - x_{i-2}), (x_i - x_{i-1})$ (and likewise with the lower sum and infinum). Since there are $\#P^*$ of these points, an upper bound on the upper Riemann sum of these points is twice $U(P^*, f)$. Hence,

$$U(P, f) - L(P, f) \le 3U(P^*, f) - 3L(P^*, f) = 3(U(P^*, f) - L(P^*, f)) < \varepsilon$$

Lemma 2. *Let $f \in \mathcal{R}[a,b]$ be nonnegative, $\varepsilon > 0$, and δ given by Lemma 1 from f and ε. For any $a \le u < v \le b$ where $0 < (v - u) < \delta$ and $s \in [u, v]$, we have*

$$\int_u^v f dx < \varepsilon + f(s)(v - u)$$

Proof. By Lemma 1, we have that $U(P, f) - L(P, f) < \varepsilon$. Let $P^* = P \cup \{u, v\}$, a refinement of P. Hence, $U(P^*, f) - L(P^*, f) < \varepsilon$ by Rudin Theorem 6.7. If u and v are entirely contained within the same interval of P, i.e. some i such that $x_{i-1} \le u < v \le x_i$, then we have $\int_u^v f dx$ is less than some value in the upper sum, namely $M_{[u,v]}(v - u)$, and likewise, $f(s)(v - u)$ is greater than some value in the lower sum, namely $m_{[u,v]}(v - u)$. Hence, since all the terms are nonnegative since f is nonnegative

$$\int_u^v f dx - f(s)(v - u) \le U(P^*, f) - L(P^*, f) < \varepsilon$$

The case where there is a a point from P in between is handled identically by just removing that point: since we still have $D(P^*) < \delta$, we still keep our ε bound.

Let $\varepsilon > 0$, and define $\varepsilon' = \frac{\varepsilon}{4(b-a)+1}$. Define $g_n = f - f_n$. Clearly, g_n is a nonnegative, monotone decreasing in n, and $g_n \to 0$ pointwise, by the properties of f_n. We seek to show that there exists some $K \in \mathbb{N}$ such that for all $n \ge K$, $\int_a^b g_n(x)dx < \varepsilon$.

Invoking Lemma 1, we define $\delta_n > 0$ such that for any partition P of $[a,b]$ where $D(P) < \delta_n$, we have $U(P, g_n) - L(P, g_n) < 2^{-n}\varepsilon'$. Define $k(s)$ be the least k such that $g_k(s) < \varepsilon'$ for any $s \in [a,b]$, which must exist since $g_n \to g$ pointwise. Now define $I_s = (s - \delta_{k(s)}/2, s + \delta_{k(s)}/2)$. For all $s \in [a,b]$, $s \in I_s$, so $\{I_s\}_{s \in [a,b]}$ is an open cover of $[a,b]$. Since this interval is closed and bounded, it is compact in $\mathbb{R}$, so we can extract a finite subcover $I_{s_1}, \dots, I_{s_N}$, particularly one that does not have any redudancy (i.e. $I_{s_i} \not\subset I_{s_j}$ when $i \ne j$), which we can do because removing a redundant open set doesn't change the union of the sets. Define $J_{s_i} = \overline{I_{s_i}} \cap [a,b]$, hence $\bigcup_{i=1}^N J_{s_i} = [a,b]$.

Finally, for all $n \ge K = \max\{k(s_1), \dots, k(s_N)\}$, we have

$$\begin{aligned}
\int_a^b g_n(x)dx &\le \sum_{i=1}^N \int_{J_{s_i}} g_n(x)dx && (1)\\
&\le \sum_{i=1}^N \int_{J_{s_i}} g_K(x)dx && (2)\\
&\le \sum_{i=1}^N \int_{J_{s_i}} g_{k(s_i)}(x)dx && (3)\\
&\le \sum_{i=1}^N \left(\varepsilon' 2^{-k(s_i)} + g_{k(s_i)}(s_i)|J_{s_i}|\right) && (4)\\
&\le \varepsilon' \sum_{i=1}^N \left(2^{-k(s_i)} + |J_{s_i}|\right) && (5)\\
&\le \varepsilon'(1 + 2(b-a)) = \varepsilon && (6)
\end{aligned}$$

Where (1) is done by splitting up $[a, b]$ centered at the s_i by Rudin Theorem 6.12(c), and then expanding the surrounding interval so that we are integrating over all of J_{s_i} only increases the value since g_n is nonnegative; (2) is because g_n is monotone decreasing; (3) is by definition of K and monotone decreasing; (4) is by Lemma 2; (5) is by definition of $k(s_i)$ so that $g_{k(s_i)}(s_i) < \varepsilon'$; and (6) since the at any point in $[a, b]$, at most two intervals J_{s_i} overlap, otherwise we would have a nested interval which we constructed to avoid, and so since the J_{s_i} cover $[a, b]$, at most each $[a, b]$ is inside two intervals, so $\sum_{i=1}^{N} |J_{s_i}| \leq 2(b - a)$; the dyadic terms are bounded above by 1 by just taking our k large enough, which we were allowed to do.

Therefore, we have that for all $n \geq K$, $\int_a^b g_n(x)dx = \int_a^b (f(x) - f_n(x))dx = \int_a^b f(x)dx - \int_a^b f_n dx < \varepsilon$, hence, $\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b f(x)dx$.