Math 321 Assignment 1
Due Friday, January 19 at 9 am

Instructions

(i) Homework should be submitted using Canvas. Include your name and SID.

(ii) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(iii) Theorems stated in the text or proved in lecture do not need to be reproved. Any other statement should be justified rigorously.

In this homework, we will need several definitions. Let $I = [a, b]$ be an interval and $k \geq 0$ be an integer. If $f \colon I \to \mathbb{R}$ is a function that is k-times differentiable on I, then we define

$$\|f\|_{C^k(I)} = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x)|.$$

This quantity is called the "C^k norm of f." We define $C^k(I)$ to be the set of functions $f \colon I \to \mathbb{R}$ that satisfy the following two properties. (i): f is k-times differentiable on I, and (ii): $f^{(k)}$ is continuous on I. We define a metric on $C^k(I)$ as follows: $d(f, g) = \|f - g\|_{C^k(I)}$, i.e.

$$d(f, g) = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x) - g^{(j)}(x)|. \tag{1}$$

It is straightforward to verify that this is indeed a metric, but you do not have to do so for this homework.

1. Let $f(t) = e^t$; recall that f is monotone increasing, $f'(t) = f(t)$, and $f(0) = 1$. Let $P_n(t)$ be the n-th order Taylor polynomial of f at the point $x_0 = 0$, as discussed in lecture. Let $I = [-1, 1]$ and let $k \geq 1$ be an integer.
Using Taylor's theorem, prove that the sequence $\{P_n\}$ converges to f in the metric space $C^k(I)$.

Hints (i) Compute the Taylor polynomial $P_n(t)$. (ii) What is the derivative of P_n? (iii) What are the higher derivatives of P_n? (iv) How can you estimate each term in (1)?

2. Let $f(t) = e^t$. Let $P_n(t)$ be the n-th order Taylor polynomial of f at the point $x_0 = 0$.

a) Let $n \geq 1$. Prove that $n!P_n(1)$ is an integer.

b) Using part a) and Taylor's theorem, prove that Euler's number e is irrational. You may use the fact that e^t is strictly monotone increasing, and $0 < e < 3$.
Hint: if e were rational, then we could write $e = m/n$....

The next problem concerns monotone increasing functions, and will help prepare us for the Riemann–Stieltjes integral. Let $\alpha\colon [0,1] \to \mathbb{R}$ be increasing. Recall from last term that for every $c \in [0,1]$, $\lim_{x \searrow c} \alpha(x)$ and $\lim_{x \nearrow c} \alpha(x)$ always exist. Thus α is continuous at c if and only if $\lim_{x \searrow c} \alpha(x) = \lim_{x \nearrow c} \alpha(x)$. If α is not continuous at c, then $\lim_{x \nearrow c} \alpha(x) < \lim_{x \searrow c} \alpha(x)$, and we say α has a *jump discontinuity* at c.

3. Let $\alpha\colon [0,1] \to \mathbb{R}$ be montone increasing. Prove that the set of points $c \in [0,1]$ where α is not continuous is either finite (possibly empty), or countably infinite.