Problem 1 (Ch. 1.13)

Show that if P is a Sylow subgruop, then $N(N(P)) = N(P)$

Solution. I'm pretty sure that $N(P)$ is just the Sylow p-subgroups where $p^m = |P|$. Recall that $N(H) = \{g \in G \mid gHg^{-1} = H\}$,

Let G be our ambient group of order $p^n m$ where p is a prime and $p \nmid m$. Then let P be a Sylow p-subgroup, that is, of order p^n. Recall that if P' is another subgroup of G with order p^n, then there exists $a \in G$ such that $P' = aPa^{-1}$ by Sylow's Theorem. ff

We have $N(P) = \{g \in G \mid gPg^{-1} = P\}$, and so $N(N(P)) = \{g_2 \in G \mid g_2 N(P) g_2^{-1} = N(P)\}$. Set inclusion.

Let $a \in N(P)$ (so $gPg^{-1} = P$).

Consider the action of the G on the set of Sylow p-subgroups by conjugation. Recall from equation (41) from Jacobson p. 76 that $\operatorname{Stab} aPa^{-1} = a(\operatorname{Stab} P)a^{-1}$ where $a \in G$. Note that $N(P)$ is the stablizer of P under the conjugancy action, thus $N(aPa^{-1}) = aN(P)a^{-1}$ for all $a \in G$. ff something about how G acting on Sylow p-subgropus by conjugation is transitive.

Problem 2 (Ch. 1.13)

Show that there are no simple groups of order 148 or of order 56.

Solution. ff lol Evan Chen problem.

Problem 3 (Ch. 1.13)

Show that there is no simple group of order pq, p, and q primes (cf. exercise 5, p. 77).

Solution. ff

Problem 4 (Ch. 1.13)

Show that every non-abelian group of order 6 is isomorphic to S_3.

Solution. ff

Problem 5 (Ch. 1.13)

Determine the number of non-isomorphic groups of order 15.

Solution. ff

An element of order 2 in a group is called an *involution.* An important insight into the structure of a finite group is obtained by studying its involutions and their centralizers. The next five excercises give a program for characterizing S_5 in this way. In all of these exercises, as well as in the rest of this set, G is a finite group.

Problem 6 (Ch. 1.13)

Let u and v be distinct involutions in G. Show that $\langle u, v\rangle$ is (isomorphic to) a dihedral group.

Solution. ff

Problem 7 (Ch. 1.13)

Let u and v be involutions in G. Show that if uv is of odd order than u and v are conjugate in G ($v = gug^{-1}$).

Solution. ff

Problem 8 (Ch. 1.13)

Let u and v be involutions in G such that uv has even order $2n$, so $w = (uv)^n$ is an involution. Show that $u, v \in C(w)$.

Solution. ff

Problem 9 (Ch. 1.13)

*Suppose G contains exactly two conjugacy classes of involutions. Let u_1 and u_2 be non-conjugate involutions in G. Let $c_i = |C(u_i)|, i = 1, 2$. Let $S_i, i = 1, 2$, be the set of ordered pairs (x, y) with x conjugate to u_1, y conjugate to u_2, and $(xy)^n = u_i$ for some n. Let $s_i = |S_i|$. Prove that $|G| = c_1 s_2 + c_2 s_1$. (*Hint*: Count the number of ordered pairs (x, y) with x conjugate to u_1 and y conjugate to u_2 in two ways. First, this number is $(|G|/c_1)(|G|/c_2)$. Since x is no conjugate to y, exercises 7 and 8 imply that for $n = o(xy)/2$, $(xy)^n$ is conjugate to either u_1 or u_2. This implies that $(|G|/c_1)(|G|/c_2) = (|G|/c_1)s_1 + (|G|/c_2)s_2$.)*

Solution. ff