Math 300 Homework 1

Problem 1

Find each of the following limits:

(a). $\lim\limits_{z\to 2i} \dfrac{z^2+9}{2z^2+8}$

(b). $\lim\limits_{z\to \infty} \dfrac{3z^2-2z}{z^2-iz+8}$

(c). $\lim\limits_{z\to 5} \dfrac{3z}{z^2-(5-i)z-5i}$

(d). $\lim\limits_{z\to \infty} (8z^3+5z+2)$

(e). $\lim\limits_{z\to \infty} e^z$

(a). *Solution.*

$$\begin{aligned}\lim_{z\to 2i} \frac{z^2+9}{2z^2+8} &= \frac{\lim_{z\to 2i}(z^2+9)}{\lim_{z\to 2i}(2z^2+8)} \\ &= \frac{-8+9}{-8+8} \\ &= \frac{1}{0} \\ &= \infty\end{aligned}$$

(b). *Solution.*

$$\begin{aligned}\lim_{z\to\infty} \frac{3z^2-2z}{z^2-iz+8} &= \lim_{z\to\infty} \frac{3-\frac{2}{z}}{1-\frac{i}{z}+\frac{8}{z^2}} \\ &= \frac{\lim_{z\to\infty}(3-\frac{2}{z})}{\lim_{z\to\infty}(1-\frac{i}{z}+\frac{8}{z^2})} \\ &= \frac{3-\frac{2}{\infty}}{1-\frac{i}{\infty}+\frac{8}{\infty}} \\ &= \frac{3}{1} \\ &= 3\end{aligned}$$

(c). *Solution.*

$$\begin{aligned}\lim_{z\to 5} \frac{3z}{z^2-(5-i)z-5i} &= \lim_{z\to 5} \frac{3}{z-(5-i)-\frac{5i}{z}} \\ &= \frac{\lim_{z\to 5} 3}{\lim_{z\to 5}(z-(5-i)-\frac{5i}{z})} \\ &= \frac{3}{\infty-(5-i)-\frac{5i}{\infty}} \\ &= \frac{3}{\infty-0} \\ &= 0\end{aligned}$$

(d). *Solution.*

$$\begin{aligned}
\lim_{z\to\infty}(8z^3+5z+2) &= \lim_{z\to\infty} z^3(8+\frac{5}{z^2}+\frac{2}{z^3}) \\
&= (\lim_{z\to\infty} z^3)(\lim_{z\to\infty}(8+\frac{5}{z^2}+\frac{2}{z^3})) \\
&= \infty\cdot(8+\frac{5}{\infty}+\frac{2}{\infty})) \\
&= \infty\cdot(8+0+0) \\
&= \infty
\end{aligned}$$

(e). *Solution.* We claim that $\lim_{z\to\infty} e^z$ does not exist. Let $z=a+bi$. Then $e^z=e^ae^{ib}=e^a(\cos b+i\sin b)$. We note that if $b=0$, if $z\to\infty$, we can either have $a\to\infty$ or $a\to-\infty$. It is a common result from first year calculus that $\lim_{a\to\infty} e^a=\infty$ and $\lim_{a\to-\infty} e^a=0$.

For the sake of contradiction, assume that $\lim_{z\to\infty} e^z=L$, $L\in\mathbb{C}$. Then for $\varepsilon>0$, there exists $M>0$ such that $|z|>M \implies |e^z-L|<\varepsilon$. We also know that there is some N such that if $a>N$ where $a\in\mathbb{R}$, then $e^a>L+\varepsilon$. Fix $\varepsilon>0$. This gives us an M that satisfies our limit inequality, and an N that satisfies the inequaliity above. So if $M_1=\max\{N,M\}$, then if $z\in\mathbb{R}$ such that $z>M_1\geq M$, then we have $e^z>L+\varepsilon \implies e^z-L>\varepsilon$. Now since $\varepsilon>0$, $e^z>L$, so $e^z-L>0 \implies e^z-L=|e^z-L|$ so $|e^z-L|>\varepsilon$, which is a contradiction.

Now for the sake of contradiction, assume that $\lim_{z\to\infty} e^z=\infty$. Then for any $N>0$, there exists $M>0$ such that $|z|>M \implies |e^z|>N$. Note that for any $M>0$, for any $a>M$ where $a\in\mathbb{R}$, we have $0<e^{-a}<e^0=1$. So if $N=2$, for any $M>0$, if $z\in(-\infty,0)$ such that $|z|>M$, we still have $e^z\leq 2$, which is a contradiction.