Problem 1

If $f\colon X \to Y$ is a continuous mapping between Hausdorff topological spaces X and Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every set $E \subseteq X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Solution. ff

Problem 2

(a). *Let X and Y be metric spaces. Prove that for $f\colon X \to Y$, TFAE:*

- *(a) f is uniformly continuous on X;*
- *(b) for any sequences (x_n) and (x'_n) in X satisfying $d_X(x_n, x'_n) \to 0$, one has $d_Y(y_n, y'_n) \to 0$, where $y_n = f(x_n), y'_n = f(x'_n)$.*

(b). *Identify, with proof, all real numbers p for which the function $f(x) = x^p$ is uniformly continuous on $X = (0, +\infty)$. [It's OK to use a little calculus to support your findings.]*

(a). *Solution.* ff

(b). *Solution.* ff

Problem 3

A metric space (X, d) *is called an* ultrametric space *if* d *satisfies the condition*

$$\forall x, y, z \in X, \quad d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(This makes d itself "an ultrametric".) Show that in any ultrametric space (X, d)*,...*

(a). every open ball $\mathbb{B}[x; r)$ *is a closed set;*

(b). one has $y \in \mathbb{B}[x; r)$ *if and only if* $\mathbb{B}[y; r) = \mathbb{B}[x; r)$*; and*

(c). if $\mathbb{B}[x; r_1) \cap \mathbb{B}[y; r_2) \neq \emptyset$*, then one of these balls must contain the other, i.e.,*

$$\mathbb{B}[x; r_1) \subseteq \mathbb{B}[y; r_2) \neq \emptyset \;\; \textit{or} \;\; \mathbb{B}[x; r_1) \supseteq \mathbb{B}[y; r_2) \neq \emptyset$$

[The "p-adic numbers" form an ultrametric space of interest in number theory.]

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 4

Given Hausdorff Topological Spaces $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$*, and continuous functions* $f, g\colon X \to Y$*, consider the* equalizer*:*

$$E = \{x \in X \colon f(x) = g(x)\}.$$

Prove that E *is closed in* X*.*

Solution. ff

Problem 5

Three continuous functions $f, g, h \colon \mathbb{R} \to \mathbb{R}$ are related by the identity

$$f(x+y) = g(x) + h(y)$$

(a). In the special case where $f = g = h$, show that there must be a real number m such that $f(t) = mt$ for all real t.

(b). Drop the hypothesis that f, g, h are identical. Describe the most general trio of continuous functions compatible with the given identity.

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 6

Here's a key fact every math student should know:

Every nonempty open set in $\mathbb{R}$ can be expressed as a finite or countable union of disjoint open intervals

Prove this, referring to a given open set $U \neq \emptyset$, by following these steps:

(a). *For each $x \in U$, let $I(x) = (\alpha(x), \beta(x))$, where*

$$\alpha(x) = \inf\{a\colon \text{ one has } x \in (a,b) \text{ for some } (a,b) \subseteq U\}\beta(x) = \sup\{a\colon \text{ one has } x \in (a,b) \text{ for some } (a,b) \subseteq U\}$$

Prove that $x \in I(x)$ and $I(x) \subseteq U$, while $\alpha(x) \notin U$ and $\beta(x) \notin U$. [Argue carefully, since both $\alpha(x) = -\infty$ and $\beta(x) = +\infty$ are possible.]

(b). *Let $\mathcal{G} = \{I(x)\colon x \in U\}$. Show that any two intervals in $\mathcal{G}$ must be either disjoint or identical.*

(c). *Explain why the key fact stated above must hold.*

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff