

Problem 1

Prove the following theorem (terminology is given below):

Suppose X is compact and $f: X \rightarrow \mathbb{R}$ is lower semicontinuous. Then F is bounded below on X , and there exists a point $z \in X$ satisfying $f(z) \leq f(x)$ for all $x \in X$.

Recall that in a HTS (X, \mathcal{T}) , a function $f: X \rightarrow \mathbb{R}$ is called lower semicontinuous if the following set is closed for every $p \in \mathbb{R}$:

$$f^{-1}((-\infty, p]) = \{x \in X : f(x) \leq p\}.$$

(One approach uses the family of closed sets $f^{-1}((-\infty, p])$ satisfyin $p > \inf f(x)$.)

Solution. ff

Problem 2

Let (X, d) be a metric space, with $K \subseteq X$ a compact set. Prove that whenever \mathcal{G} is an open cover for K , there exists $r < 0$ with this property: for every pair of points $x, y \in K$ obeying $d(x, y) < r$, some open set $G \in \mathcal{G}$ contains both x and y .

Solution. Let G_1, G_2, \dots, G_N be the finite subcover of K such that $G_i \in \mathcal{G}$ for all $i \in 1, 2, \dots, N$ and $K \subseteq \bigcup_{1 \leq i \leq N} G_i$, which we know exists from the compactness of K . Define $I_{i,j} := G_i \cap G_j$ where $1 \leq i < j \leq N$. Note then that $I_{i,j}$ is open as well. For each $I_{i,j}$, pick some $x_{i,j} \in I_{i,j}$ if $I_{i,j} \neq \emptyset$. Then, since X is a metric space and $I_{i,j}$ is an open set, we must have that there exists some $r_{i,j} > 0$ such that $\mathbb{B}[x_{i,j}; r_{i,j}] \subseteq I_{i,j}$ (if $I_{i,j} = \emptyset$, just let $r_{i,j} = 1$). Let $r = \min_{1 \leq i < j \leq N} \{r_{i,j}\}$. Since there are only finitely many $r_{i,j}$, all of them greater than 0, we must have $r > 0$ as well.

Now consider any $x, y \in K$ such that $d(x, y) < r$. Consider $\mathbb{B}[x; r]$. By our distance condition, we have $y \in [x; r)$. It is sufficient to show now that there exists some $G \in \mathcal{G}$ such that $\mathbb{B}[x; r] \subseteq G$. Since $\{G_i\}_{1 \leq i \leq N}$ is a covering of all of K , there exists some $G_x \in \{G_i\}_{1 \leq i \leq N}$ such that $x \in G_x$. If $\mathbb{B}[x; r] \subseteq G_x$, we are done. For the sake of contradiction, assume then that $\mathbb{B}[x; r] \not\subseteq G_x$. Hmm it's not this, it's either in G_x , or in G'_x ... consider the G_i for which x is in?

I don't even have that $\mathbb{B}[x; r] \subseteq K$. Hmm, so we don't want to lose y , because we'll always have $x \in \partial K$ (since K is closed) violate our condition with the ball. ff

So let $x \in G_x$ and $y \in G_y$. We claim that either $x \in G_y$ or $y \in G_x$ (this is trivial if $G_x = G_y$, so we now assume that $G_x \neq G_y$). For the sake of contradiction, assume this is false: $x, y \notin G_x \cap G_y$. Then by construction, for some $x_{x,y} \in G_x \cap G_y$, $x, y \notin \mathbb{B}[x_{x,y}, r]$. Then $d(x_{x,y}, x) > r$ and $d(x_{x,y}, y) > r$. We have $d(x, y) \leq d(x_{x,y}, x) + d(x_{x,y}, y)$. Hmm... the thing on the right is bounded below by $2r$, and the thing on the left is bounded above by r . We want to use this inequality in a different way I think, like $d(x_{x,y}, x) < d(x, y)$ and $d(x_{x,y}, y) < d(x, y)$ somehow. Maybe something to do with them being in the same G_i ?

So we have $d(x_{x,y}, x) < d(x, y) + d(x_{x,y}, y)$. But $d(x_{x,y}, y) < d(x, y) + d(x_{x,y}, x)$, thus $d(x_{x,y}, x) < 2d(x, y) + d(x_{x,y}, x) \implies 0 < 2d(x, y)$ trivially. ff So the claim I want to make is that the intersection between two open sets in a metric space is closer to each other than they are to each other.

Hmm, we might need to define r in terms of points that are not in the same G_i , but are close to each other. Think of a square donut that has a tiny slit that doesn't touch, and an open cover of two sets that intersect on the non-slit side.

Problem 3

Define the set-valued “projection” mapping $p_1: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathcal{P}(\mathbb{R})$ by

$$p_1(S) = \{x_1 \in \mathbb{R}: (x_1, x_2) \in S \text{ for some } x_2\}, \quad S \subseteq \mathbb{R}^2$$

(a). If S is bounded, must $p_1(S)$ be bounded? (Why or why not?)

(b). If S is closed, must $p_1(S)$ be closed? (Why or why not?)

(c). If S is compact, must $p_1(S)$ be compact? (Why or why not?)

(a). *Solution.* ff

(b). *Solution.* ff

(c). *Solution.* ff

Problem 4

Recall the set ℓ^2 from HW07 Q3, and the standard “unit vectors” $\hat{p} = (0, 0, \dots, 0, 1, 0, \dots)$, where the only nonzero entry in \hat{p} occurs in component p . For any x in ℓ^2 and subset $V \subseteq \ell^2$, write

$$\Omega(x; V) = \{y \in \ell^2 : -1 < (v, y - x) < 1, \forall v \in V\}.$$

Then define a collection \mathcal{T} of subsets of ℓ^2 by saying $G \in \mathcal{T}$ if and only if every point $x \in G$ has the property that $x \in \Omega(x; V) \subseteq G$ for some finite set $V \subseteq \ell^2$.

- (a). Prove that $\Omega(x; V) \in \mathcal{T}$ of every finite set $V \subseteq \ell^2$ and point $x \in \ell^2$.
- (b). Prove that (ℓ^2, \mathcal{T}) is a Hausdorff Topological Space.
- (c). Let $S = \{\hat{p} : p \in \mathbb{N}\}$. Prove that $0 \in S'$. (Here 0 denotes $(0, 0, \dots)$, the “origin in ℓ^2 .”) Note: This fact proves that \mathcal{T} is different from the metric topology on ℓ^2 .
- (d). Prove that every G in \mathcal{T} has the property: for every x in G , there exists $r > 0$ such that

$$G \supseteq \mathbb{B}[x; r) = \{y \in \ell^2 : \|y - x\| < r\}.$$

This fact proves that every set considered “open” in \mathcal{T} is also open in the metric topology on ℓ^2 . This explains why \mathcal{T} gets called “the weak topology” and the metric topology is also called “the strong topology.”

- (e). Prove that the following set is closed in the weak topology of ℓ^2 : $\mathbb{B}[0; 1] = \{y \in \ell^2 : \|y\| \leq 1\}$.
- (a). Solution. ff
- (b). Solution. ff
- (c). Solution. ff
- (d). Solution. ff
- (e). Solution. ff