Math 321 Homework 1

In this homework, we will need several definitions. Let $I = [a, b]$ be an interval and $k \geq 0$ be an integer. If $f \colon I \to \mathbb{R}$ is a function that is k-times differentiable on I, then we define

$$\|f\|_{C^k(I)} = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x)|.$$

This quantity is called the "C^k norm of f." We define $C^k(I)$ to be the set of functions $f \colon I \to \mathbb{R}$ that satisfy the following two properties. (i): f is k-times differentiable on I, and (ii): $f^{(k)}$ is continuous on I. We define a metric on $C^k(I)$ as follows: $d(f, g) = \|f - g\|_{C^k(I)}$, i.e.

$$d(f, g) = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x) - g^{(j)}(x)|. \tag{1}$$

It is straightforward to verify that this is indeed a metric, but you do not have to do so for this homework.

Problem 1

Let $f(t) = e^t$; recall that f is monotone increasing, $f'(t) = f(t)$, and $f(0) = 1$. Let $P_n(t)$ be the n-th order Taylor polynomial of f at the point $x_0 = 0$, as discussed in lecture. Let $I = [-1, 1]$ and let $k \geq 1$ be an integer.

Using Taylor's theorem, prove that the sequence $\{P_n\}$ converges to f in the metric space $C^k(I)$.

Hints *(i) Compute the Taylor polynomial $P_n(t)$. (ii) What is the derivative of P_n? (iii) What are the higher derivatives of P_n? (iv) How can you estimate each term in (1)?*

Solution. Recall that for $f(t)$, the n-th ordered Taylor polynomial at $x_0 = 0$ is

$$P_n(t) = \sum_{i=0}^{n} \frac{x^i}{i!}$$

Furthermore, note that the j-th derivative of $P_n(t)$ is 0 if $j > n$ and

$$\frac{d^j}{dx^j} P_n(t) = \frac{d^j}{dx^j} \sum_{i=0}^{j-1} \frac{x^i}{i!} + \sum_{i=j}^{n} \frac{d^j}{dx^j} \frac{x^i}{i!} = 0 + \sum_{i=j}^{n} \frac{1}{i!} \frac{i!}{(i-j)!} x^{i-j} = \sum_{i=j}^{n} \frac{x^{i-j}}{(i-j)!} = \sum_{i=0}^{n-j} \frac{x^i}{i!} = P_{n-j}(t)$$

when $j \geq n$.

Recall from Taylor's theorem that there exists c between t and 0 such that

$$e^t = P_n(t) + \frac{f^{(n+1)}(c)}{(n+1)!} t^{n+1} = P_n(t) + \frac{e^c}{(n+1)!} t^{n+1}$$

Then we make some argument about how e^c is maximal at 1 in I, but this will decrease arbitrarily, so $\{P_n\} \to f$.
ff

Problem 2

Let $f(t) = e^t$. Let $P_n(t)$ be the n-th order Taylor polynomial of f at the point $x_0 = 0$.

(a). Let $n \geq 1$. Prove that $n! P_n(1)$ is an integer.

(b). Using part (a) and Taylor's theorem, prove that Euler's number e is irrational. You may use the fact that e^t is strictly monotone increasing, and $0 < e < 3$.

Hint*: if e were rational, then we could write $e = m/n$....*

(a). *Solution.* See

$$n!P_n(t) = n!\sum_{i=0}^{n} \frac{x^i}{i!} = \sum_{i=0} (n-i)!x^i$$

Now when $x = 1$, each term is an integer, and the integers are closed under addition, so $n!P_n(1)$ is an integer as well.

(b). *Solution.* Assume, for the sake of contradiction, that $e \in \mathbb{Q}$, that is to say, $e = m/n$, for some $m \in \mathbb{Z}, n \in \mathbb{N}$. Note $n!e = m(n-1)! \in \mathbb{Z}$. Also recall by Taylor's theorem that

$$e = P_n(1) + \frac{f^{(n+1)}(x)}{(n+1)!} = P_n(1) + \frac{e^x}{(n+1)!}$$

for some $x \in (0,1)$. Then

$$n!e = n!P_n(t) + \frac{e^x}{n+1}$$

Let $n \geq 2$. Since $0 < e < 3$ and $0 < x < 1$, we have $0 < e^x < 3$, and so when $n \geq 2$, $n+1 > e^x \implies \frac{e^x}{n+1} \notin \mathbb{Z}$. However, by part (a), we know that $n!P_n(t) \in \mathbb{Z}$ and so $\frac{e^x}{n+1} = n!e - n!P_n(t) \in \mathbb{Z}$, which is a contradiction.

Problem 3

The next problem concerns monotone increasing functions, and will help prepare us for the Riemann–Stieltjes integral. Let $\alpha\colon [0,1] \to \mathbb{R}$ be increasing. Recall from last term that for every $c \in [0,1]$, $\lim_{x\searrow c} \alpha(x)$ and $\lim_{x\nearrow c} \alpha(x)$ always exist. Thus α is continuous at c if and only if $\lim_{x\searrow c} \alpha(x) = \lim_{x\nearrow c} \alpha(x)$. If α is not continuous at c, then $\lim_{x\nearrow c} \alpha(x) < \lim_{x\searrow c} \alpha(x)$, and we say α has a *jump discontinuity* at c.

Let $\alpha\colon [0,1] \to \mathbb{R}$ be montone increasing. Prove that the set of points $c \in [0,1]$ where α is not continuous is either finite (possibly empty), or countably infinite.

Solution. For the sake of contradiction, assume that the set of discontinuous points is uncountably infinite. ff Recall that $\lim_{x\searrow c} \alpha(x) = a$ when for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x \in (c, c+\delta)$, $|\alpha(x) - a| < \varepsilon$. Sequences: if $\alpha(x_n) \to a$ as $n \to \infty$ for all sequences $\{x_n\}$ in $(c,1]$ such that $x_n \to c$.