Problem 1

Let $0 < a_1 < b_1$ *and define* $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{a_n+b_n}{2}$, $n \in \mathbb{N}$.

(a). *Prove that the sequences* (a_n) *and* (b_n) *both converge. (Suggestion: Use induction to prove* $0 < a_n < a_{n+1} < b_{n+1} < b_n$*.)*

(b). *Prove that the sequences* (a_n) *and* (b_n) *have the same limit.*

(a). *Solution.* First, we use induction to prove $0 < a_n < a_{n+1} < b_{n+1} < b_n$. We do this in steps: first we prove that $0 < a_n$ and $0 < b_n$ for all $n \in \mathbb{N}$. When $n = 1$, by assumption, $0 < a_1$ and $0 < b_1$. Now let $n = j$, and $0 < a_j, b_j$. Then $0 < \sqrt{a_j b_j} = a_{j+1}$ and $0 < \frac{a_j+b_j}{2} = b_{j+1}$ as desired. Thus $0 < a_n, b_n, \forall n \in \mathbb{N}$.

Now we prove that $a_n < b_n$ for all $n \in \mathbb{N}$. When $n = 1$, by assumption, $a_1 < b_1$. Now let $n = j$, and $a_j < b_j$. Then $a_{j+1} = \sqrt{a_j b_j}$ and $b_{j+1} = \frac{a_j+b_j}{2}$. We have

$$\begin{aligned} b_{j+1}^2 &= (a_j^2 + 2a_j b_j + b_j^2)/4 \\ &> (2a_j b_j + 2a_j b_j)/4 \\ &= a_j b_j = a_{j+1}^2 \end{aligned}$$

where the second line is because $a_j - b_j > 0 \implies (a_j - b_j)^2 > 0$ so $a_j^2 - 2a_j b_j + b_j^2 > 0 \implies a_j^2 + b_j^2 > 2a_j b_j$. But since we know that both $a_{j+1}, b_{j+1} > 0$, then $b_{j+1}^2 > a_{j+1}^2 \implies |b_{j+1}| > |a_{j+1}| \implies b_{j+1} > a_{j+1}$, which closes the induction.

Now we prove that $a_n < a_{n+1}$ and $b_n > \frac{a_n+b_n}{2}$. We have $a_{n+1} = \sqrt{a_n b_n} > \sqrt{a_n^2} = a_n$ (since $x^2 > y^2 \implies x > y$ when $x, y > 0$). Furthermore, $b_{n+1} = \frac{a_n+b_n}{2} > \frac{2b_n}{2} = b_n$. Both of these are true for all $n \in \mathbb{N}$, so we have fully shown

$$0 < a_n < a_{n+1} < b_{n+1} < b_n$$

Note then that (a_n) is monotonically increasing, and bounded by a_1 and b_1, and (b_n) is monotonically decreasing, and bounded by a_1 and b_1 as well. Thus by the Monotone Convergence Theorem, and the fact $\mathbb{R}$ is complete, we have that (a_n) and (b_n) both converge.

(b). *Solution.* We claim that $b_n - a_n < (b_1 - a_1)/2^{n-1}$ for all $n \geq 2$. This is true in the base case ($n = 2$): $b_2 - a_2 = (b_1 + a_1)/2 - \sqrt{a_1 b_1} < (2b_1)/2 + (a_1 - b_1)/2 + \sqrt{a_1^2} = b_1 - a_1 - (b_1 - a - 1)/2 = (b_1 - a_1)/2$. Now assume that $b_n - a_n < (b_1 - a_1)/2^{n-1}$. Then

$$\begin{aligned} b_{n+1} - a_{n+1} &= \frac{b_n + a_n}{2} - \sqrt{a_n b_n} \\ &< \frac{2b_n}{2} + \frac{a_n - b_n}{2} - \sqrt{a_n^2} \\ &= b_n - a_n - \frac{b_n - a_n}{2} \\ &= \frac{b_n - a_n}{2} \\ &< \frac{b_1 - a_1}{2^n} \end{aligned}$$

Thus we have closed the induction.

Now let $\alpha \in \mathbb{R}$ be the value that (a_n) converges to (as proven will exist with part (a)). Let $\varepsilon > 0$. Let $\varepsilon' = \varepsilon/2$. We know when there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N$, we have $|\alpha - a_n| < \varepsilon'$. Additionally, by Archimedes, there exists some $N_2 \in \mathbb{N}$ such that $0 < \frac{b_1 - a_1}{\varepsilon'} < N_2$, and $N_2 \leq 2^{N_2}$, thus $\varepsilon' > (b_1 - a_1)/2^{N_2 - 1}$. Since $(b_1 - a_1)/2^{n-1}$ is a monotonically decreasing function, this $\varepsilon' > (b_1 - a_1)/2^{n-1}$ for all $n \geq N_2$. So $0 < b_n - a_n < \varepsilon'$ for all $n \geq N_2$. Thus $|b_n - \alpha| < |\varepsilon' + a_n - \alpha| \leq |\varepsilon'| + |a_n - \alpha| < \varepsilon' + \varepsilon' = \varepsilon$, thus $b_n \to \alpha$ as well (where the first substitution is valid, since $b_n, a_n, \varepsilon' > 0$).

Problem 2

(a). Suppose $(z_n)_{n\in\mathbb{N}}$ is a bounded sequence with integer values.

(a) Prove that both numbers below are integers:

$$\lambda = \liminf_{n\to\infty} z_n \qquad \mu = \limsup_{n\to\infty} z_n$$

(b) Prove that there are infinitely many integers n for which $z_n = \lambda$.

(b). Let $d_n = p_{n+1} - p_n$ ($n \in \mathbb{N}$) denote the sequence of prime differences, built from the sequence of primes

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$$

(a) Prove that $\limsup_{d\to\infty} = +\infty$.

(b) Some mathematicians believe that

$$\delta := \liminf_{n\to\infty} d_n = 2 \tag{1}$$

However, the best estimate of δ known to date is $2 \le \delta \le 246$. Identify by name a famous unsolved problem in mathematics that is equivalent to proving or disproving line (1). (After giving the name, clearly explain the required relationship.)

(a). (a) *Solution.* Let E be the set of number x such that there exists a convergent subsequence $\{n_k\}$ where z_{n_k} converges to x (which we know is nonempty by Bolzano-Weierstrass, because (z_n) is bounded). We claim that each $x \in E$ is an integer. Since every convergent sequence is Cauchy, there must exist some N where for all $k, k' \ge N$, $|z_{n_k} - z_{n'_k}| < \frac{1}{2}$, but since $z_{n_k}, z_{n'_k}$ are integers, this is only true when $z_{n_k} = z_{n'_k}$. Thus, for all $k \ge N$ z_{n_k} is the same integer, call it j. x must equal j; otherwise say $x = j + \delta$ for some fixed $\delta \in \mathbb{R}$, then if $\varepsilon = \delta$, for all $k \ge N$, $|z_{n_k} - x| = |j - j \pm \delta| = |\delta| \not< \varepsilon$, which contradicts that s_{n_k} converges to x. Thus $x = j$, which is an integer. Therefore, every element in E is an integer.

Now, recall definition 3.16 in Rudin: $\lambda = \liminf_{n\to\infty} z_n = \inf E$ and $\mu = \limsup_{n\to\infty} z_n = \sup E$. Furthermore, theorem 3.17 in Rudin tells us that $\lambda \in E$ and $\mu \in E$. But every element in E is an integer, thus $\lambda, \mu \in \mathbb{Z}$.

(b) *Solution.* Recall that from the previous part of the question that $\lambda \in E$ where E are all such x where there is a subsequence z_{n_k} that converges to x. So there is a subsequence z_{n_l} that converges to λ. In order for $z_{n_l} \to \lambda$, we must have that for all ε, there is some $N \in \mathbb{N}$ such that for all $l \ge N$, we have $|z_{n_l} - \lambda| < \varepsilon$. If $\varepsilon = \frac{1}{2}$, since z_{n_l}, λ are integers, we must have that $z_{n_l} = \lambda$. This is true for all $l \ge N$, which there are infinitely many of in, so there are infinitely many $z_{n_l} = \lambda$, thus there are infitly many n where $z_{n_l} = \lambda$ ($n = n_l$ when $l \ge N$).

(b). (a) *Solution.* Let k be arbitrary. It is sufficient to show that there k many consecutive composite numbers, because then there is some n such that $d_n > k$, and since k was arbitrary, $\limsup_{n\to\infty} d_n = \inf_{n\in\mathbb{N}} \sup_{k\ge N} d_n = \inf_{n\in\mathbb{N}} +\infty = +\infty$ (this works regardless of n, since we have that are $k+n$ consecutive composite numbers, which surely k of them occur after n).

Thus, let $k \in \mathbb{N}$. Note that $(k+1)!+2, (k+1)!+3, \ldots, (k+1)!+k+1 = \{(k+1)!+j : 2 \le j \le k+1\}$ are k consecutive integers. Furthermore, $(k+1)!+j$ is divisible by $j \ne 1$, since $j \le k+1$, and so $j \mid (k+1)!$ (by definition of the factorial) and $j \mid j$. Thus, all of the consecutive integers are not prime. So we are done.

(b) *Solution.* The problem is the Twin Prime Conjecture. This states that there are infinitely many primes that differ by 2. This is equivalent to $\liminf_{n\to\infty} d_n = 2$, since $\liminf_{n\to\infty} d_n = \sup_{n\in\mathbb{N}} \inf_{k\ge n} d_n = 2$ only when regardless of how far we go down the real line (however big n is), we can find two primes that differ by only 2 ($\inf_{k\ge n} d_n = 2$).

Problem 3

(a). *Prove: For any sequences (a_n) and (b_n) of nonnegative real numbers,*

$$\limsup_{n\to\infty}(a_nb_n) \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right),$$

provided the right side does not involve the product of 0 *and* ∞.

(b). *Give an example in which the result of part (a) holds with a strict inequality.*

(a). *Solution.* Let $\alpha = \limsup_{n\to\infty} a_n$ and $\beta = \limsup_{n\to\infty} b_n$.

First let us consider when $\alpha = +\infty$. Regardless of β (since we are assuming $\beta > 0$), our right hand side becomes $+\infty$, and regardless of the left-hand side, in the extended reals, our inequality holds. WLOG, this is also true when $\beta = +\infty$.

Now we can assume $\alpha > 0$, $\beta > 0$. Let $A > \alpha$. Then by definition, $A > \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} a_k\right)$, thus there exists some $N_1 \in \mathbb{N}$ such that $A > \sup_{k\geq N_1} a_k$, so $A > a_k$ when $k \geq N_1$. Similarly, if $B > \beta$, there exists some $N_2 \in \mathbb{N}$ such that $B > b_k$ when $k \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $k \geq N$, $a_k < A$ and $b_k < B$, thus $a_kb_k < AB$ since both sides are nonnegative, and $A > \alpha \geq 0$ and $B > \beta \geq 0$. Thus $\sup_{k\geq N}(a_kb_k) \leq AB$. Any lower bound for a set of values indexed by n must be less than or equal to each of them (e.g. one where $n = N$). Thus the previous inequality implies

$$\limsup_{n\to\infty}(a_nb_n) = \inf_{n\in\mathbb{N}}\sup_{k\geq n}(a_nb_n) \leq AB$$

Since this holds for arbitrary $A > \alpha$ and $B > \beta$, the real number on the left cannot exceed $\alpha\beta$. Thus, we must have

$$\limsup_{n\to\infty}(a_nb_n) \leq \alpha\beta = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(b). *Solution.* We give the sequences $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $a_nb_n = -1$, so $\limsup_{n\to\infty}(a_nb_n) = -1$. Additionally, $\limsup_{n\to\infty} a_n = 1$, since for any $n \in \mathbb{N}$, there always exists some $a_k = 1$ where $k \geq n$, and so $\limsup_{n\to\infty} = \inf_{n\in\mathbb{N}}\sup_{k\geq n} a_n = \inf_{n\in\mathbb{N}} 1 = 1$); and for the same reason, $\limsup_{n\to\infty} b_n = 1$. Thus

$$\limsup_{n\to\infty}(a_nb_n) = -1 < 1 = \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right)$$

Problem 4

(a). Show that for any $r \geq 1$, one has

$$n(r-1) \leq r^n - 1 \leq nr^{n-1}(r-1) \quad \forall n \in \mathbb{N}$$

(b). Prove that for each real $a \geq 1$, the following sequences converges:

$$x_n = n\left(a^{1/n} - 1\right), \quad n \in \mathbb{N}$$

(c). Prove that the sequence in (b) also converges for each real $a \in (0,1)$.

(d). Let $L(x) = \lim_{n\to\infty} n\left(x^{1/n} - 1\right)$ for $x > 0$. Prove that $L(ab) = L(a) + L(b)$ for all $a > 0, b > 0$.

Note: *Present solutions that use only methods discussed in MATH 320. No calculus, please!*

(a). *Solution.* Recall the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ for any $n \in \mathbb{N}$. Since $r \geq 1$, applying the identity when $b = r$ and $a = 1$, we get

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + 1) \leq (r-1)nr^{n-1}$$

since each $r^{n-j} \leq r^{n-1}$ $(1 \leq j \leq n)$, and there are n many r^{n-j} in the factor.

To prove the second inequality, we use the identity again, however

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + 1) \geq n(r-1)$$

since each $r^{n-j} \geq 1$ $(1 \leq j \leq n)$, and there are n many r^{n-j} in the factor. Thus we have shown $n(r-1) \leq r^n - 1 \leq nr^{n-1}(r-1)$.

(b). *Solution.* We have that for all $n \in \mathbb{N}$, by part (a) of this problem, $x_n = n\left(a^{1/n} - 1\right) \leq a - 1$, thus x_n is bounded above. Additionally, $a \geq 1 = 1^n \implies a^{1/n} \geq 1$, thus $n(a^{1/n} - 1) \geq 0$, and so x_n is bounded below. Furthermore, we claim that (x_n) is monotonically decreasing: $x_{n+1} = (n+1)(a^{1/(n+1)} - 1) = n(a^{1/(n+1)} - 1) + a^{1/(n+1)} - 1$. That is $x_n - x_{n+1} \geq 0$. We see

$$\begin{aligned}
x_n - x_{n+1} &= n\left(a^{1/n} - 1\right) - (n+1)\left(a^{1/(n+1)} - 1\right) \\
&\geq n^2\left(a^{1/n^2} - 1\right) - (n+1)\left(a^{1/(n+1)} - 1\right) \\
&=
\end{aligned}$$

I can't figure this one out (I'm pretty sure we have to use the inequalities from part (a) to somehow rearrange to get the inequality, like using one the less than direction on the positive term and greater than direction on the negative term, but it wasn't working for me in time)... but if we just assume it's monotonically decreasing, we are done: since it is bounded and monotonically decreasing, it converges.

(c). *Solution.* The sequence is bounded above by 0, since $0 < a < 1 = 1^n \implies a^{1/n} < 1$, thus $n(a^{1/n} - 1) < 0$. Furthermore, it is bounded below, and monotonically decreasing (ran out of time, same as above). Thus the series converges.

(d). *Solution.* We have

$$\begin{aligned}
L(ab) &= \lim_{n\to\infty} n((ab)^{1/n} - 1) \\
&= \lim_{n\to\infty} n(a^{1/n}b^{1/n} - 1) \\
&= \lim_{n\to\infty} n/2(-(a^{1/n} - b^{1/n})^2 - 2 + a^{2/n} + b^{2/n}) \\
&=
\end{aligned}$$

ran out of time; I was trying to get a_n and b_n to seperate, and then we would have $L(a) + L(b)$, but I couldn't get rid of a square.