Problem 2 (Ch. 1.7)

Show that if G is finite and H and K are subgroups such that $H \supset K$ then $[G : K] = [G : H][H : K]$.

Solution. Using Langrange's theorem (Theorem 1.5) since G is finite:

$$\begin{aligned}|G| &= |H|[G : H]\\ &= |K|[H : K][G : H]\end{aligned}$$

But Lagrange's theorem also says $\frac{|G|}{|K|} = [G : K]$, thus

$$[G : K] = [G : H][H : K]$$

as desired.

Problem 3 (Ch. 1.7)

Let H_1 and H_2 be subgroups of G. Show that any right coset relative to $H_1 \cap H_2$ is the intersection of a right coset of H_1 with a right coset of H_2. Use this to prove Poincaré's Theorem *that if H_1 and H_2 have finite index in G then so has $H_1 \cap H_2$.*

Solution. Let $x \in H_1 \cap H_2 g$ for an arbitrary $g \in G$. Then $x = h_{12} g$ for some $h_{12} \in H_1 \cap H_2$. But then $x \in H_1 g$ since $h_{12} \in H_1$ and $x \in H_2 g$ since $h_{12} \in H_2$. Thus $x \in H_1 g \cap H_2 g$. Thus $H_1 \cap H_2 g \subset\in H_1 g \cap H_2 g$. Since g was arbitrary, we proved this for an arbitrary right coset relative to $H_1 \cap H_2$, so this is true for all of them.

Now let $x \in H_1 g_1 \cap H_2 g_2$ for arbitrary $g_1, g_2 \in G$. If there does not exist such x for the given g_1, g_2, then our statement is vacuously true. So now, assume that our x exists. Then $x = h_1 g_1 = h_2 g_2$ for some $h_1 \in H_1$, $h_2 \in H_2$. Want to show that $h_1, h_2 \in H_1 \cap H_2$. Note that $h_1 = (h_2 g_2) g_1^{-1} = h_2 (g_2 g_1^{-1})$. But then h_1 is in a right coset relative to H_2. Like wise, $h_2 = h_1 (g_1 g_2^{-1})$, so h_2 is in a right coset relative to H_1.

Problem 4 (Ch. 1.7)

Let G be a finitely generated group, H a subgroup of finite index. Show that H is finitely generated.

Solution. Let $[G : H] = r$. Then $G = H \cup Hg_1 \cup \cdots \cup Hg_{r-1}$. Note that if $S = \{s_1, s_2, \ldots, s_n\}$ is the finite set that generates G, so $G = \langle S \rangle$, then $G = H$... something about H being a group so its closed does something? Every element in G can be written as a product of S, but also as H times some other element in g (are we making use of finite index though?). ff

Problem 5 (Ch. 1.7)

Let H and K be two subgroups of a group G. Show that the set of maps $x \to hxk$, $h \in H$, $k \in K$ is a group of transformatoins of the set G. Show that the orbit of x relative to this group is the set $HxK = \{hxk \mid h \in H, k \in K\}$. This is called the double coset of x relative to the pair (H, K). *Show that if G is finite then $|HxK| = |H|[K\colon x^{-1}Hx \cap K] = |K|[H\colon xKx^{-1} \cap H]$.*

Solution. ff

Problem 3 (Ch. 1.8)

Let G be the group of pairs of real numbers (a, b) $a \neq 0$, with the product $(a, b)(c, d) = (ac, ad + b)$ (exercise 4, p.36). Verify that $K = \{(1, b) \mid b \in \mathbb{R}\}$ is a normal subgroup of G. Show that $G/K \cong (\mathbb{R}^*, \cdot, 1)$ the multiplicative group of non-zero reals.

Solution. ff

Problem 4 (Ch. 1.8)

Show that any subgroup of index two is normal. Hence prove that A_n is normal in S_n.

Solution. If a subgroup H of G has index two, then there exists $g \in G$, $g \notin H$ such that $G = H \cup Hg$.

Problem 5 (Ch. 1.8)

Verify that the intersection of any set of normal subgroups of a group is a normal subgroup. Show if H and K are normal subgroups, then HK is a normal subgroup.

Solution. Let H and K be normal subgroups of G. If $x \in H \cap K$, then $gxg^{-1} \in H$ since $x \in H$ and H is normal, and $gxg^{-1} \in K$ since $x \in K$ and K is normal. Thus $gxg^{-1} \in H \cap K$, thus $H \cap K$ is normal.

ff something to do with that silly paragraph at the end of 1.8