Problem 1

A set $S \subseteq \mathbb{R}$ is called **dense in** $\mathbb{R}$ *whenever this property holds:*

for each nonempty real interval (a,b), one has $S \cap (a,b) \neq \emptyset$.

(a). Define $f\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ by $f(m,n) = m + n\sqrt{2}$. Prove that f is one-to-one.

(b). Let $S = \{m + n\sqrt{2}\colon m,n \in \mathbb{Z}\}$. Prove that $S \cap (0,1)$ is infinite.

(c). Prove that for each $\varepsilon > 0, S \cap (0,\varepsilon) \neq \emptyset$.

(d). Prove that S is dense in $\mathbb{R}$.

(a). *Solution.* Let $f(m_1,n_1) = f(m_2,n_2)$. Then $m_1 + n_1\sqrt{2} = m_2 + n_2\sqrt{2} \implies m_1 - m_2 = (n_2 - n_1)\sqrt{2} \implies \frac{m_1-m_2}{=} n_2 - n_1\sqrt{2}$. For the sake of contradiction, assume that $m_1 \neq m_2$ or $n_1 \neq n_2$. If $n_1 \neq n_2$, then we have $\frac{m_1-m_2}{n_2-n_1} = \sqrt{2}$. But the value on the left of the inequality is a rational number, and we know $\sqrt{2} \notin \mathbb{Q}$, thus we get a contradiction. We must have then $n_1 = n_2$. If $m_1 \neq m_2$, we have $m_1 - m_2 \neq 0$, however $(n_2 - n_1)\sqrt{2} = 0$, thus our equality is broken, another contradiction. Thus we must have that $m_1 = m_2$ as well. But then the original values we mapped from $\mathbb{Z}\times\mathbb{Z}$ are the same, $(m_1,n_1) = (m_2,n_2)$, showing that f is one-to-one.

(b). *Solution.* Let $n \in \mathbb{Z}$ be arbitrary. Then we claim that there exists $m \in \mathbb{Z}$ such that $0 < m - n\sqrt{2} < 1$. This implies $(m - n\sqrt{2}) \in S \cap (0,1)$ (n is negative here, but this is fine because if $n \in \mathbb{Z}$, so is $-n$). Given our inequality, we can rerrange it to get

$$n\sqrt{2} < m < 1 + n\sqrt{2}$$

We claim that this inequality is preserved when we choose m to be the smallest integer greater than $n\sqrt{2}$. By definition, $m > n\sqrt{2}$. Furthermore, we know that $n\sqrt{2} \notin \mathbb{Z}$ (since $\sqrt{2}$ is irrational) thus, $m - n\sqrt{2} < m - (m-1)$ (ie. $n\sqrt{2}$ is closer to m than $m-1$); if this were not true, $m-1$ would be closer to m than $n\sqrt{2}$, in otherwords $m - 1 > n\sqrt{2}$, but this contradicts that m is the smallest integer greater than $n\sqrt{2}$. But this inequality gives $m < 1 + n\sqrt{2}$. Thus, our chosen m satisfies our bounds, and so $m - n\sqrt{2} \in S \cap (0,1)$. But since this works for arbitrary n, and there are infinitely many integers, there are infinitely many n, m such that $m - n\sqrt{2} \in S \cap (0,1)$. Each m, n corresponds to a unique element in S (by part (a)), thus $S \cap (0,1)$ contains infintly many elements.

(c). *Solution.* Let $\varepsilon > 0$. Corollary (a) of the Archimedes property from the lecture notes states that there exists some $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. Furthermore, since $0 < j < 2^j$ for all j, we have that $0 < 2^{-j} < j^{-1}$, thus $0 < 2^{-j} < \varepsilon$.

Now, see that $0 < \sqrt{2} - 1 < 2^{-1}$, since $1.5^2 > 2$. Furthermore, since these are both positive values, exponentiating them to some natural number will preserve the inequality. Thus if we exponentiate them by j, we see $0 < (\sqrt{2}-1)^j < 2^{-j} < \varepsilon$. By binomial theorem, we have

$$0 < \sum_{k=0}^{j} \binom{j}{k} (\sqrt{2})^k (-1)^{j-k} < \varepsilon$$

Note that if k is even, $(\sqrt{2})^k = 2^{k/2}$ which is an integer, and if j is odd, $(\sqrt{2})^k = 2^{(k-1)/2}\sqrt{2}$ which is a multiple of $\sqrt{2}$. Regardless then, since $\binom{j}{k}$ is always an integer, we have that our summation is just the sum of an integer and a multiple of $\sqrt{2}$. Specifically, if

$$m = \sum_{\substack{k=0 \\ k \text{ even}}}^{j} \binom{j}{k} 2^{k/2} (-1)^{j-k}, \quad n = \sum_{\substack{k=0 \\ k \text{ odd}}}^{j} \binom{j}{k} 2^{(k-1)/2} (-1)^{j-k}$$

Then $m, n \in \mathbb{Z}$, and $0 < m + n\sqrt{2} < \varepsilon$. Thus $m + n\sqrt{2} \in S \cap (0,\varepsilon)$ and so the set is not empty.

(d). *Solution.* Let a, b be given, such that $a, b \in \mathbb{R}$, $a < b$ (WLOG). We know that there exists $m, n \in \mathbb{Z}$ such that $0 < m + n\sqrt{2} < (b-a)/2$ since $(b-a)/2 > 0$ (by part (c)). Let $k = \lceil a/(m+n\sqrt{2}) \rceil$. Then $k(m+n\sqrt{2})$ is of size ff and at least, of size ff. But since k is an integer, km, kn are integers, and ff inequalities, thus $km + kn\sqrt{2} \in S \cap (a, b)$, so the intersection is nonempty. This confirms that S is dnese in $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ be given and $b > a$ (WLOG). We let $x, y \in \mathbb{R}$ be arbitrary such that $x < y$. Note that by the density of the rationals (proven in class), we have that $m \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that

$$\begin{aligned} x < \frac{m}{k} < y &\implies kx < m < yx \\ &\implies \sqrt{2}kx < m < \sqrt{2}ky \\ &\implies \sqrt{2}kx + m < m + n\sqrt{2} < \sqrt{2}ky + n \end{aligned}$$

Let a, b be given, such that $a, b \in \mathbb{R}$, $0 < a < b$ (WLOG). Let $\varepsilon = \frac{b-a}{3}$, then by part (c), there exists $s \in S$ such that $0 < s < \varepsilon$. By the Archimedian property, there exists a natural number k such that $ks > a$. By the well-ordering property, we can choose k that is the smallest element that satisfies the inequality. If $ks < a + \frac{b-a}{3}$, then we are done. Otherwise, $ks > a + \frac{b-a}{3}$. We can subtract a smaller value on the left side of the inequality than the right and preserve the inequality, so we get $ks - s = (k-1)s > a$, but this contradicts that k is the smallest such integer, thus we cannot have $ks > a + \frac{b-a}{3}$. Thus we have shown that when a, b positive, we have that $S \cap (a, b) \neq \emptyset$.

We consider now the cases when a, b are not both positive. If $a \leq 0$ and $b \geq 0$ (where $b \neq a$), then we let $\varepsilon = b$, and then by part (c), we have $S \cap (a, b) \neq \emptyset$. Now let a, b be both negative. From the first part of this proof, we know that there is an $s \in S$ such that $s \in (-a, -b)$ (since both values are now positive), thus $-s \in (a, b)$, and obviously $-s \in S$ for any $s \in S$. Thus, $S \cap (a, b) \neq \emptyset$ for all cases.

Problem 2

For each $x \in \mathbb{R}$, evaluate

$$f(x) := \lim_{n\to\infty} \frac{1}{1+nx}$$

Use the ε, N definition of a limit to prove your answer.

Solution. Let $a_n = \frac{1}{1+nx}$ where x is given (so $f(x) = \lim_{n\to\infty} a_n$). Either $x > 0$, $x < 0$ or $x = 0$. We deal with these cases in turn. We claim that

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Let $x > 0$. Let $\varepsilon > 0$. We let $N = \max\{\lfloor \frac{1-\varepsilon}{\varepsilon x} \rfloor, 1\}$. Then for $n > N$, we have $n > \frac{1-\varepsilon}{\varepsilon x} \implies nx > \frac{1}{\varepsilon} - 1$. Then $\frac{1}{1+nx} < \varepsilon$. But all of the terms on the left are positive anyway, we can just take their absolute value to get $|\frac{1}{1+nx}| < \varepsilon$, thus $\lim_{n\to\infty} \frac{1}{1+nx} = 0$ when $x > 0$.

Now let $x < 0$. Let $\varepsilon > 0$. We let $N = \max\{\lfloor \frac{1-\varepsilon}{-\varepsilon x} \rfloor, 1, \lceil \frac{1}{x} \rceil\}$. Then for $n > N$, we have $n > \frac{1-\varepsilon}{-\varepsilon x} \implies nx < 1 - \frac{1}{\varepsilon}$. Then $\varepsilon > \frac{1}{1-nx}$. But since $1 - nx > 0$, we have that the numerator and denominator are positive, and thus $\varepsilon > \left|\frac{1}{1+nx}\right|$.

Finally, let $x = 0$. Let $\varepsilon > 0$. We let $N = 1$. Then for $n > N$, we have $|a_n - 1| = |1 - 1| = 0 < \varepsilon$. Thus $\lim_{n\to\infty} \frac{1}{1+nx} = 1$ when $x = 0$.

Problem 3

Given a real sequence $(a_n)_n$ with $a_n \to A$ as $n \to \infty$, present direct ε, N-proofs that $a_n^3 \to A^3$ and $a_n^{1/3} \to A^{1/3}$ as $n \to \infty$. (Assume $A \in \mathbb{R}$.)

Solution. We know that $a_n \to A$ as $n \to \infty$, thus for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n - A| \leq \frac{\varepsilon}{|(|A|+1)^2 - |A|A + A^2|}$.

We first seek to prove that $a_n^3 \to A^3$ as $n \to \infty$. Recall that if $x_n \to x$ as $n \to \infty$, then $|x_n| \leq |x| + 1$ (by the course notes), we have

$$\begin{aligned}
|a_n^3 - A^3| &= |a_n - A||a_n^2 - a_n A + A^2| \\
&\leq |a_n - A||(|A|+1)^2 - |A|A + A^2| \\
&< |(|A|+1)^2 - |A|A + A^2| \left(\frac{\varepsilon}{|(|A|+1)^2 - |A|A + A^2|} \right) \\
&= \varepsilon
\end{aligned}$$

But this is sufficient to show that $a_n^3 \to A^3$ as $n \to \infty$.

We now seek to prove that $a_n^{1/3} \to A^{1/3}$ as $n \to \infty$. Let $\varepsilon > 0$. Consider first when $A \neq 0$. Since $a_n \to A$ as $n \to \infty$, for all $\varepsilon > 0$, we have $N \in \mathbb{N}$ such that for all $n > N$,

$$|a_n - A| < \min \left\{ \frac{A}{2}, \left(\left(\frac{A}{2}\right)^{2/3} + \left(\frac{A}{2}\right)^{1/3} A^{1/3} + A^{2/3} \right) \varepsilon \right\}$$

But when $|a_n - A| < \frac{A}{2}$, we have $\frac{A}{2} < a_n$. Thus

$$(a_n)^{2/3} + (a_n)^{1/3} A^{1/3} + A^{2/3} > \left(\frac{A}{2}\right)^{2/3} + \left(\frac{A}{2}\right)^{1/3} A^{1/3} + A^{2/3}$$

This gives us the useful inequality

$$\frac{|a_n - A|}{(a_n)^{2/3} + (a_n)^{1/3} A^{1/3} + A^{2/3}} < \frac{|a_n - A|}{\left(\frac{A}{2}\right)^{2/3} + \left(\frac{A}{2}\right)^{1/3} A^{1/3} + A^{2/3}} \tag{1}$$

Now see that, using the difference of cubes formula, we have

$$|a_n - A| = |(a_n)^{1/3} - A^{1/3}||(a_n)^{2/3} + (a_n)^{1/3} A^{1/3} + A^{2/3}|$$

which, when combined with (1), gives

$$\begin{aligned}
|a_n^{1/3} - A^{1/3}| &= \frac{|a_n - A|}{(a_n)^{2/3} + (a_n)^{1/3} A^{1/3} + A^{2/3}} \\
&< \frac{|a_n - A|}{\left(\frac{A}{2}\right)^{2/3} + \left(\frac{A}{2}\right)^{1/3} A^{1/3} + A^{2/3}} \\
&< \frac{\left(\frac{A}{2}\right)^{2/3} + \left(\frac{A}{2}\right)^{1/3} A^{1/3} + A^{2/3}}{\left(\frac{A}{2}\right)^{2/3} + \left(\frac{A}{2}\right)^{1/3} A^{1/3} + A^{2/3}} \varepsilon \\
&= \varepsilon
\end{aligned}$$

which shows the convergence as desired.

When $A = 0$, we simply pick N such that for all $n > N$, $|a_n| < \varepsilon^3$. But then $|a_n^{1/3}| = |a_n|^{1/3} < \varepsilon$. Thus, $a_n^{1/3} \to A^{1/3}$ as $n \to \infty$.

Problem 4

(a). *Prove: For any real M, m and b obeying $M > m$, there is some real R for which*

$$Mx > mx + b \qquad \forall x > R$$

(That is easy, but it sets the conceptual stage for the next part.)

(b). *Suppose $(y_n)_{n\in\mathbb{N}}$ is a real sequence with the property that (y_n/n) converges to some number M. Prove that for every real $m \in (-\infty, M)$ and $b \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that*

$$y_n > mn + b \qquad \forall n > N$$

(c). *True or False (with proof or counterexample):*

$$\text{If } \tfrac{y_n}{n} \to M \text{ as } n \to \infty, \text{ then } |y_n - Mn| \to 0.$$

(a). *Solution.* We give $R = \frac{b}{M-m}$. Let $x > R$. Then $(M-m)x > b \implies Mx > mx + b$ as desired.

(b). *Solution.* We have that for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|y_n/n - M| < M - m$. From part (a), we can choose a natural $N_2 > \frac{b}{M-m}$ where $m \in (-\infty, M)$ such that for all $n > N_2$, we have that for all $b \in \mathbb{R}$, $Mn > mn + b$.

Now let $N = \max\{N_1, N_2\}$, then we have that for $n > N$, $|y_n/n| < M - m$, so

$$\frac{y_n}{n} > 2M - m > M + \frac{b}{n} > m + \frac{b}{n}$$

but multiplying by n gives us

$$y_n > mn + b$$

for all $n > N$, as desired.

(c). *Solution.* False, we provide a counterexample. Let $y_n = 1$. Then $y_n/n = 1/n \to 0$, however $|y_n - 0n| = 1$, which obviously does not go to 0.

Problem 5

Let α and β be positive real numbers. Prove that $\lim_{n\to\infty}(\alpha^n + \beta^n)^{1/n} = \max\{\alpha, \beta\}$.

Solution. Let $\beta < \alpha$ (WLOG) be positive real numbers. Then, $\alpha = \max\{\alpha, \beta\}$. It is true that $\alpha^n < \alpha^n + \beta^n < 2\alpha^n$, so $\alpha < (\alpha^n + \beta^n)^{1/n} < 2^{1/n}\alpha$. Clearly, $\alpha \to \alpha$ as $n \to \infty$ since there is no dependence on n. By theorem 3.3(c) of Rudin, we have $\lim_{n\to\infty} 2^{1/n}\alpha = (\lim_{n\to\infty} 2^{1/n})(\lim_{n\to\infty} \alpha)$. The left limit goes to 1 by theorem 3.20(b) of Rudin, and the right limit is again just α. Thus $\lim_{n\to\infty} = \alpha$. Thus, by squeeze theorem, we have that $\lim_{n\to\infty}(\alpha^n + \beta^n)^{1/n} = \alpha$, which is the max, so we are done.

Problem 6

(a). *Let (x_n) be a sequence of positive real numbers obeying*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$$

Show that there exist $r \in (0,1)$ and $C > 0$ for which $0 < x_n < Cr^n$ holds for all n sufficiently large. Use this to prove that $\lim_{n\to\infty} x_n = 0$.

(b). *Prove that if $x_n \to 0$, then the sequence $y_n = 1/x_n$ cannot converge.*

(c). *Use (a) and (b) to test for converge: $\left(\frac{10^n}{n!}\right)$, $\left(\frac{2^n}{n}\right)$, and $\left(\frac{2^{3n}}{3^{2n}}\right)$.*

[Detailed $\varepsilon - N$ arguments are expected in (a)-(b), but not in (c).]

(a). *Solution.* :(

(b). *Solution.* We are going to assume that $x_n \neq 0$ for all n so that y_n is always well-defined, but we will make some comments about this at the end.

For all $\varepsilon \in \mathbb{N}$, by the convergence of x_n to 0, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, we have $|x_n| < \varepsilon$. Then we choose N such that for all $n > N$, $|x_n| < \frac{1}{k}$ where $0 < k \in \mathbb{R}$. But then taking the reciprocal, we have

$$\frac{1}{|x_n|} = |y_n| > k$$

This is true for all for any $n > N$, thus y_n does not converge to k. Furthermore, since k was arbitrary, y_n does not converge to any value.

If we consider when $x_n = 0$ finitely many times, we can apply a similar proof for the elements that are not 0 (if N is the greatest n such that $x_N = 0$, then we restrict our focus to $n > N$). However, if we will always have a next n such that $x_n = 0$ (there are infinitely many zeros), then our sequence will always have problems with well-definedness, and so we cannot make a well-formed statement about it.

(c). *Solution.* :(

Problem 7

Let (x_n) and (y_n) be real sequences. Prove: If $(x_n y_n)$ converges, and $y_n \to +\infty$ as $n \to \infty$, then $x_n \to 0$ as $n \to \infty$.

[For "$y_n \to +\infty$," see Rudin, Definition 3.15, p. 55; note also the following paragraph.]

Solution. By the hypothesis of the question, we require that for all $\varepsilon > 0$, there exists some $N_1 \in \mathbb{N}$ such that for all $n_1 \geq N_1$, $|y_{n_1} x_{n_1} - L| < \varepsilon$. Now, since $y_n \to +\infty$, we have that for all $M \in \mathbb{N}$, there exists some $N_2 \in \mathbb{N}$ such that for all $n_2 \geq N_2$, we have that $y_n \geq M$. Now let $N = \max\{N_1, N_2\}$, then for all $n \geq N$, we have both of our conditions from above, thus $|Mx_n| \leq |y_n x_n - L| < \varepsilon$. Rearranging, this means we must have $|M||x_n| < \varepsilon$ In order to preserve this inequality, we must have that $|x_n| < \frac{\varepsilon}{|M|}$. But since ε and M were both arbitrary, and $\frac{\varepsilon}{|M|} > 0$ since the numerator and denominator are positive, this means $\varepsilon' = \frac{\varepsilon}{|M|} > 0$ is arbitrary as well. And so for all $n > N$, $|x_n - 0| < \varepsilon'$, so $x_n \to 0$ as $n \to \infty$.

Problem 8

Given a real-valued sequence $a_1, a_2, \ldots,$ consider the corresponding sequence of averages.

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n}, \qquad n = 1, 2, 3, \ldots$$

(a). *Prove: If $a_n \to a$ as $n \to \infty$ (with $a \in \mathbb{R}$), then also $s_n \to a$ as $n \to \infty$.*

(b). *Proof or Counterexample: If $s_n \to a$ as $n \to \infty$ (with $a \in \mathbb{R}$), then also $a_n \to a$ as $n \to \infty$.*

(c). *Repeat parts (a)-(b), after changing "(with $a \in \mathbb{R}$)" to "(with $a = +\infty$)" in both parts.*

(a). *Solution.* Let $\varepsilon > 0$ be arbitrary. Let N_1 be the integer such that for all $n_1 \geq N$, we have that $|a_n - a| < \frac{\varepsilon}{3}$. Now choose $N = \max\{N_1, \lceil \frac{3A}{\varepsilon} \rceil\}$ where $A = |a_1 + \cdots + a_{N_1} - aN_1|$. If we let $n \geq N$, we have that

$$\begin{aligned}
|s_n - a| &= \left| \frac{a_1 + \cdots + a_n - an}{n} \right| && (2)\\
&= \left| \frac{a_1 - a + a_2 - a + \cdots + a_n - a}{n} \right| && (3)\\
&\leq \left| \frac{a_1 + \cdots + a_{N_1} - aN_1}{n} \right| + \left| \frac{a_{N_1+1} - a + \cdots + a_n - a}{n} \right| && (4)\\
&\leq \frac{A}{n} + \frac{|a_{N_1+1} - a| + \cdots + |a_n - a|}{n} && (5)\\
&\leq \frac{A}{n} + \frac{(n - N_1)\frac{\varepsilon}{3}}{n} && (6)\\
& && (7)
\end{aligned}$$

Note that $\frac{A}{n} \leq A / \lceil \frac{3A}{\varepsilon} \rceil \leq A/(3A/\varepsilon) = \frac{\varepsilon}{3}$. Also see that $\frac{(n-N_1)\frac{\varepsilon}{3}}{n} \leq \frac{\varepsilon}{3}$ (since $0 \leq \frac{n-N_1}{n} < 1$ because $n \geq N_1$ (always nonnegative) and $0 \leq n - N_1 < n$). Thus

$$|s_n - a| \leq \frac{2\varepsilon}{3} < \varepsilon$$

as desired.

(b). *Solution.* This is not true, we provide the counterexample: $a_n = (-1)^n$. Note that $s_n \to 0$ as $n \to \infty$, since if n is even, $s_n = 0$, and if n is odd, $s_n = \frac{-1}{n}$. Both of these converge to 0 as $n \to \infty$, and they encompass all the s_n, so $s_n \to \infty$ as $n \to \infty$. Now note a_n does not go to a as $n \to \infty$, since we can provide $\varepsilon = \frac{1}{2}$, and regardless of a, there are always terms two terms, say a_n, a_{n+1}, which are at least a distance of 1 apart.

(c). *Solution.* Proof of (a) :(

We provide a counterexample for (b):

$$a_n = \begin{cases} n/2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Note that $s_n \to +\infty$ as $n \to \infty$, since $s_n = \frac{1+2+\cdots+n}{2n} = \frac{n+1}{4}$, which obviously goes to $+\infty$ as $n \to \infty$ (we can always find N given M, namely $N = 3M$). However, a_n does not go to $+\infty$ as $n \to \infty$, since it is not true for all M that we can find N such that for all $n > N$, $a_n > M$: take $M = 1$. We will always have $a_n = 0$ when n is odd, so $a_n < M$.