Problem 2 (Ch. 1.7)

Show that if G is finite and H and K are subgroups such that $H \supset K$ then $[G:K] = [G:H][H:K]$.

Solution. Using Langrange's theorem (Theorem 1.5) since G is finite:

$$\begin{aligned}|G| &= |H|[G:H] \\ &= |K|[H:K][G:H]\end{aligned}$$

But Lagrange's theorem also says $\frac{|G|}{|K|} = [G:K]$, thus

$$[G:K] = [G:H][H:K]$$

as desired.

Problem 3 (Ch. 1.7)

Let H_1 and H_2 be subgroups of G. Show that any right coset relative to $H_1 \cap H_2$ is the intersection of a right coset of H_1 with a right coset of H_2. Use this to prove Poincaré's Theorem *that if H_1 and H_2 have finite index in G then so has $H_1 \cap H_2$.*

Solution. Let $x \in (H_1 \cap H_2)g$ for an arbitrary $g \in G$. Then $x = h_{12}g$ for some $h_{12} \in H_1 \cap H_2$. But then $x \in H_1 g$ since $h_{12} \in H_1$ and $x \in H_2 g$ since $h_{12} \in H_2$. Thus $x \in H_1 g \cap H_2 g$. Thus $H_1 \cap H_2 g \subseteq H_1 g \cap H_2 g$ since x was arbitrary.

Now let $x \in H_1 g \cap H_2 g$ for the same $g \in G$ as before. If these sets are disjoint, then $H_1 g \cap H_2 g = \emptyset \subseteq (H_1 \cap H_2)g$. then our statement is vacuously true. So now, assume that our x exists. Then $x = h_1 g = h_2 g$ for some $h_1 \in H_1$, $h_2 \in H_2$. Note that $h_1 = h_2 g g^{-1} = h_2$. Thus $h_1 \in H_2$, so $h_1 \in H_1 \cap H_2$. Thus $x \in (H_1 \cap H_2)g$. So $H_1 g \cap H_2 g \subseteq (H_1 \cap H_2)g$ since x was arbitrary. Therefore $(H_1 \cap H_2)g = H_1 g \cap H_2 g$, and since the coset relative to $H_1 \cap H_2$ (arbitrary g), this is true for all right cosets relative to $H_1 \cap H_2$.

Problem 4 (Ch. 1.7)

Let G be a finitely generated group, H a subgroup of finite index. Show that H is finitely generated.

Solution. Let $[G:H] = r$. Then $G = Hx_1 \sqcup Hx_2 \sqcup \cdots \sqcup Hx_r$ where $x_1 = 1$. Note that if $S = \{s_1, s_2, \ldots, s_n\}$ is the finite set that generates G, so $G = \langle S \rangle$. Note that without loss of generality, we can include s_i^{-1} in S for all i (since the group generated by it already included it, and the set is still finite, since we just double the size of our set none of the inverses were in it before). Note that for any i, j, $x_i g_j = u_{ij} x_{i'}$ for some $u_{ij} \in H$, since we must have that $x_i g_j$ is in some coset (namely $Hx_{i'}$).

We claim that H is generated by $\{u_{ij}\}$, thus it is finitely generated. Let $h = g_{i_1} g_{i_2} \cdots g_{i_l} \in H$, where $g_{ij} \in H$. Thus

$$\begin{aligned}h &= (x_1 g_{i_1}) g_{i_2} \cdots g_{i_l} \\ &= (u_{1i_1} x_{1'}) g_{i_2} \cdots g_{i_l} \\ &= u_{1i_1} (u_{2i_2} x_{2'}) \cdots g_{i_l} \\ &\vdots \\ &= u_{1i_1} u_{2i_2} \cdots u_{li_l} x_{l'} \in H = Hx_1\end{aligned}$$

Thus $x_{l'} = x_1 = 1$, and thus H is generated by $\{u_{ij}\}$.

Problem 5 (Ch. 1.7)

Let H and K be two subgroups of a group G. Show that the set of maps $x \to hxk$, $h \in H$, $k \in K$ is a group of transformations of the set G. Show that the orbit of x relative to this group is the set $HxK = \{hxk \mid h \in H, k \in K\}$. This is called the double coset of x relative to the pair (H, K). *Show that if G is finite then $|HxK| = |H|[K\colon x^{-1}Hx \cap K] = |K|[H\colon xKx^{-1} \cap H]$.*

Solution. Let $\alpha\colon x \to hxk$, $\beta\colon x \to h'xk'$ and $\gamma\colon x \to h''xk''$, where $h, h', h'' \in H$ and $k, k', k'' \in K$. First note that group is closed (under composition), since $(\beta\alpha)(x) = h'hxkk'$, and $h'h \in H$, $kk' \in K$ since H,K are groups, so $\beta\alpha$ is in the set of transformations as well. Further

$$(\gamma\beta)\alpha(x) = \gamma\beta(hxk) = \gamma(h'hxkk') = \gamma(\beta\alpha)(x)$$

so the operation is associative. Also, there exists an identity in H and K, so define $1(x) = 1_H x 1_K$, and see $1\alpha(x) = 1_H hxk 1_K = hxk = \alpha(x) = h1_H x 1_K x = \alpha 1(x)$. Finally, for any h, k, there exist inverses $h^{-1} \in H$, $k^{-1} \in K$, so define $\alpha^{-1}(x) = h^{-1}xk^{-1}$ and see $\alpha^{-1}\alpha(x) = h^{-1}hxkk^{-1} = 1_H x 1_K = 1(x)$ and $\alpha\alpha^{-1}(x) = hh^{-1}xk^{-1}k = 1_H x 1_K = 1(x)$. Thus, since our choice of h, k (and other elements) were abitrary, these properties hold for any map of the form $x \to hxk$, thus this forms a group of transformations of the set G.

The orbit of x relative to this group is $\{\alpha(x) \mid \alpha \in \text{our group of transformations}\} = \{hxk \mid h \in H, k \in K\}$. But this is the definition of HxK, thus HxK is the orbit of x relative to this group of transformations.

Now, let G be finite of order r. Note that the subgroups of G must also be finite then. By Lagrange's theorem, we have

$$\frac{|HxK|}{|H|} = [H : HxK]$$

hmm... I ran out of time

Problem 3 (Ch. 1.8)

Let G be the group of pairs of real numbers (a, b) $a \neq 0$, with the product $(a, b)(c, d) = (ac, ad + b)$ (exercise 4, p.36). Verify that $K = \{(1, b) \mid b \in \mathbb{R}\}$ is a normal subgroup of G. Show that $G/K \cong (\mathbb{R}^*, \cdot, 1)$ the multiplicative group of non-zero reals.

Solution. If $(a, b) = g \in G$, one can verify that $g^{-1} = (1/a, -b/a)$. Let $(1, c) = k \in K$. See that

$$g^{-1}kg = (1/a, -b/a)(1, c)(a, b) = (\frac{1}{a}, -b/a)(a, b + c) = (1, c/a) \in K$$

and since $g \in G$, $k \in K$ were arbitrary, this shows that K is normal in G.

Note that $G/K =$ the set of cosets of the form $K, Kg, Kg_2 \ldots$ I think just do an explicit bijection. I think we map $x \in \mathbb{R}$ to some g such that $kg = (x, *)$ for any k. Then $g = (x, 1)$.

We provide a map $\phi\colon (\mathbb{R}^*, \cdot, 1) \to G/K$ by $x \mapsto K(x, 0)$. We show that this is a bijection by providing an explicit inverse, namely $\phi^{-1}\colon K(a, b) \mapsto a$. Note that this map is well-defined, ie. it maps to the same $a \in \mathbb{R}^*$, irrespective of the representative chosen: every element in $K(a, b)$ is of the form $k(a, b)$ where $k \in K$. Let $k = (1, b')$, then $(1, b')(a, b) = (a, b + b')$, which would get mapped to a as well. To show that ϕ respects the group operation, if $x, y \in \mathbb{R}^*$, we have $\phi(x)\phi(y) = K(x, 0)K(y, 0)$; but for any $k, k' \in K$, $k(x, 0)k'(y, 0) = k(x, 0)k'(x, 0)^{-1}(x, 0)(y, 0) = kk''(x, 0)(y, 0)$ (where $k'' \in K$), since K is normal in G. This is true for any $k \in K$, thus $K(x, 0)K(y, 0) = K(x, 0)(y, 0)$. But then

$$\begin{aligned}
\phi(x)\phi(y) &= K(x, 0)(y, 0) \\
&= K(xy, x) \\
&= K(xy, 0) && \text{since } (xy, x)^{-1} = (1, -x) \in K \\
&= \phi(xy)
\end{aligned}$$

so we have shown they are isomorphic.

Problem 4 (Ch. 1.8)

Show that any subgroup of index two is normal. Hence prove that A_n is normal in S_n.

Solution. If a subgroup H of G has index two, then for any $g \in G$, $g \notin H$ we have $G = H \sqcup Hg$. Recall (from Jacobson) that $Hg = g^{-1}H$. Then $g^{-1}Hg = Hgg$. Note that $g^2 \in H$. For if $g^2 \notin H$, then $g^2 \in Hg$, (since the index is two, so there are no other cosets), and so $Hg = Hg^2$. But multiplying by g^{-1} on both sides gives $H = Hg$, which contradicts the assumption that $H \neq Hg$. Thus $g^{-1}Hg = Hg^2 = H$. In otherwords, $g^{-1}hg \in H$ for all $h \in H$.

Now, if $g \in h$, $g^{-1}hg \in H$ since H is a group, and is closed. Thus, regardless if $g \in H$ or $g \notin H$, we have $g^{-1}hg \in H$, thus H is normal in G.

Note $|S_n|/|A_n| = [S_n : A_n] = 2$, thus A_n is normal in S_n.

Problem 5 (Ch. 1.8)

Verify that the intersection of any set of normal subgroups of a group is a normal subgroup. Show if H and K are normal subgroups, then HK is a normal subgroup.

Solution. Let H and K be normal subgroups of G. If $x \in H \cap K$, then $gxg^{-1} \in H$ since $x \in H$ and H is normal, and $gxg^{-1} \in K$ since $x \in K$ and K is normal. Thus $gxg^{-1} \in H \cap K$, thus $H \cap K$ is normal.

Now let $x \in HK = \{hk \colon h \in H, k \in K\}$, say $x = hk$. Then $gxg^{-1} = ghkg^{-1} = ghg^{-1}gkg^{-1} = h'k' \in HK$ where $h' \in H$, $k' \in K$, since $ghg^{-1} \in H$ by the normality of H, and $gkg^{-1} \in K$ by the normality of K. Thus HK is normal.