CHAPTER 7
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Section 7.1 Eigenvalues and Eigenvectors

2. $A\mathbf{x}_1 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1\mathbf{x}_1$

$A\mathbf{x}_2 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \lambda_2\mathbf{x}_2$

4. $A\mathbf{x}_1 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda_1\mathbf{x}_1$

$A\mathbf{x}_2 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix} = -3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \lambda_2\mathbf{x}_2$

$A\mathbf{x}_3 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ -3 \end{bmatrix} = -3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \lambda_3\mathbf{x}_3$

6. $A\mathbf{x}_1 = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1\mathbf{x}_1$

$A\mathbf{x}_2 = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \lambda_2\mathbf{x}_2$

$A\mathbf{x}_3 = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \lambda_3\mathbf{x}_3$

8. (a) $A(c\mathbf{x}_1) = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} c \\ 2c \\ -c \end{bmatrix} = \begin{bmatrix} 5c \\ 10c \\ -5c \end{bmatrix} = 5\begin{bmatrix} c \\ 2c \\ -c \end{bmatrix} = 5(c\mathbf{x}_1)$

(b) $A(c\mathbf{x}_2) = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2c \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} 6c \\ -3c \\ 0 \end{bmatrix} = -3\begin{bmatrix} -2c \\ c \\ 0 \end{bmatrix} = -3(c\mathbf{x}_2)$

(c) $A(c\mathbf{x}_3) = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 3c \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} -9c \\ 0 \\ -3c \end{bmatrix} = -3\begin{bmatrix} 3c \\ 0 \\ c \end{bmatrix} = -3(c\mathbf{x}_3)$

10. (a) Because $A\mathbf{x} = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 28 \\ 28 \end{bmatrix} = 7\begin{bmatrix} 4 \\ 4 \end{bmatrix}$

x *is* an eigenvector of A (with corresponding eigenvalue 7).

(b) Because $A\mathbf{x} = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 64 \\ -32 \end{bmatrix} = -8\begin{bmatrix} -8 \\ 4 \end{bmatrix}$

x *is* an eigenvector of A (with corresponding eigenvalue -8).

(c) Because $A\mathbf{x} = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 92 \\ -4 \end{bmatrix} \neq \lambda\begin{bmatrix} -4 \\ 8 \end{bmatrix}$

x is *not* an eigenvector of A.

(d) Because $A\mathbf{x} = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -45 \\ 19 \end{bmatrix} \neq \lambda\begin{bmatrix} 5 \\ -3 \end{bmatrix}$

x is *not* an eigenvector of A.

12. (a) Because $A\mathbf{x} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}$ is *not* an eigenvector of A.

(b) Because $A\mathbf{x} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{x}$ *is* an eigenvector (with corresponding eigenvalue 0).

(c) The zero vector is never an eigenvector.

(d) Because $A\mathbf{x} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix}\begin{bmatrix} 2\sqrt{6} - 3 \\ -2\sqrt{6} + 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 + 2\sqrt{6} \\ 4\sqrt{6} \\ 6\sqrt{6} + 12 \end{bmatrix} = \left(4 + 2\sqrt{6}\right)\begin{bmatrix} 2\sqrt{6} - 3 \\ -2\sqrt{6} + 6 \\ 3 \end{bmatrix}$,

$\mathbf{x}$ *is* an eigenvector of A $\left(\text{with corresponding eigenvalue } 4 + 2\sqrt{6}\right)$.

14. Geometrically, multiplying a vector in R^2 by A corresponds to a horizontal shear.

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

The only vectors mapped onto scalar multiples of themselves are those lying on the x-axis.

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = 1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

So, the only eigenvalue is 1, and the corresponding eigenspace is the x-axis.

16. (a) The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 4 \\ 2 & \lambda - 8 \end{vmatrix} = \lambda^2 - 9\lambda = \lambda(\lambda - 9) = 0.$$

(b) The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 9$.

For $\lambda_1 = 0$, $\begin{bmatrix} \lambda_1 - 1 & 4 \\ 2 & \lambda_1 - 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution is $\{(4t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(4, 1)$.

For $\lambda_2 = 9$, $\begin{bmatrix} \lambda_2 - 1 & 4 \\ 2 & \lambda_2 - 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution is $\{(-t, 2t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 9$ is $(-1, 2)$.

18. (a) The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & -4 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda + 2)(\lambda - 1) - 4 = (\lambda + 3)(\lambda - 2) = 0.$$

(b) The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

For $\lambda_1 = -3$, $\begin{bmatrix} \lambda_1 + 2 & -4 \\ -1 & \lambda_2 - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution is $\{(-4t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -3$ is $(-4, 1)$.

For $\lambda_2 = 2$, $\begin{bmatrix} \lambda_2 + 2 & -4 \\ -1 & \lambda_2 - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution is $\{(t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 2$ is $(1, 1)$.

20. (a) The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^2 - \tfrac{1}{4}\lambda - \tfrac{1}{8} = \left(\lambda - \tfrac{1}{2}\right)\left(\lambda + \tfrac{1}{4}\right) = 0.$$

(b) The eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{4}$.

For $\lambda_1 = \frac{1}{2}$, $\begin{bmatrix} \lambda_1 - \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = \frac{1}{2}$ is $(1, 1)$.

For $\lambda_2 = -\frac{1}{4}$, $\begin{bmatrix} \lambda_2 - \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \lambda_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, -2t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = -\frac{1}{4}$ is $(1, -2)$.

22. (a) The characteristic equation is $|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -2 & -1 \\ 0 & \lambda & -2 \\ 0 & -2 & \lambda \end{vmatrix} = (\lambda - 3)(\lambda^2 - 4) = 0.$

(b) The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

For $\lambda_1 = -2$, $\begin{bmatrix} \lambda_1 - 3 & -2 & -1 \\ 0 & \lambda_1 & -2 \\ 0 & -2 & \lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -5 & -2 & -1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, -5t, 5t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -2$ is $(1, -5, 5)$.

For $\lambda_2 = 2$, $\begin{bmatrix} \lambda_2 - 3 & -2 & -1 \\ 0 & \lambda_2 & -2 \\ 0 & -2 & \lambda_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -2 & -1 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(-3t, t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 2$ is $(-3, 1, 1)$.

For $\lambda_3 = 3$, $\begin{bmatrix} \lambda_3 - 3 & -2 & -1 \\ 0 & \lambda_3 & -2 \\ 0 & -2 & \lambda_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -2 & -1 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, 0, 0) : t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 3$ is $(1, 0, 0)$.

24. (a) The characteristic equation is $|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -2 & 3 \\ 3 & \lambda + 4 & -9 \\ 1 & 2 & \lambda - 5 \end{vmatrix} = \lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda - 2)^2 = 0.$

(b) The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$ (repeated).

For $\lambda_1 = 0$, $\begin{bmatrix} \lambda_1 - 3 & -2 & 3 \\ 3 & \lambda_1 + 4 & -9 \\ 1 & 2 & \lambda_1 - 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(-t, 3t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(-1, 3, 1)$.

For $\lambda_2 = 2$, $\begin{bmatrix} \lambda_2 - 3 & -2 & 3 \\ 3 & \lambda_2 + 4 & -9 \\ 1 & 2 & \lambda_2 - 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(-2s + 3t, s, t) : s, t \in R\}$. So, two independent eigenvectors corresponding to $\lambda_2 = 2$ are $(-2, 1, 0)$ and $(3, 0, 1)$.

26. (a) The characteristic equation is $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & \frac{3}{2} & -\frac{5}{2} \\ 2 & \lambda - \frac{13}{2} & 10 \\ -\frac{3}{2} & \frac{9}{2} & \lambda - 8 \end{vmatrix} = \left(\lambda - \frac{29}{2}\right)\left(\lambda - \frac{1}{2}\right)^2 = 0.$

(b) The eigenvalues are $\lambda_1 = \frac{29}{2}, \lambda_2 = \frac{1}{2}$ (repeated).

For $\lambda_1 = \frac{29}{2}$, $\begin{bmatrix} \lambda_1 - 1 & \frac{3}{2} & -\frac{5}{2} \\ 2 & \lambda_1 - \frac{13}{2} & 10 \\ -\frac{3}{2} & \frac{9}{2} & \lambda_1 - 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, -4t, 3t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = \frac{29}{2}$ is $(1, -4, 3)$.

For $\lambda_2 = \frac{1}{2}$, $\begin{bmatrix} \lambda_2 - 1 & \frac{3}{2} & -\frac{5}{2} \\ 2 & \lambda_2 - \frac{13}{2} & 10 \\ -\frac{3}{2} & \frac{9}{2} & \lambda_2 - 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(3s - 5t, s, t) : s, t \in R\}$. So, two eigenvectors corresponding to $\lambda_2 = \frac{1}{2}$ are $(3, 1, 0)$ and $(-5, 0, 1)$.

28. (a) The characteristic equation is $|\lambda I - A| = \begin{vmatrix} \lambda - 5 & 0 & 0 & 0 \\ -1 & \lambda - 4 & 0 & 0 \\ 0 & 0 & \lambda - 1 & -3 \\ 0 & 0 & 0 & \lambda - 4 \end{vmatrix} = (\lambda - 5)(\lambda - 4)^2(\lambda - 1) = 0.$

(b) The eigenvalues are $\lambda_1 = 5, \lambda_2 = 4, \lambda_3 = 1$, and $\lambda_4 = 4$.

For $\lambda_1 = 5$, $\begin{bmatrix} \lambda_1 - 5 & 0 & 0 & 0 \\ -1 & \lambda_1 - 4 & 0 & 0 \\ 0 & 0 & \lambda_1 - 1 & -3 \\ 0 & 0 & 0 & \lambda_1 - 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, t, 0, 0) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 5$ is $(1, 1, 0, 0)$.

For $\lambda_2 = 4$, $\begin{bmatrix} \lambda_2 - 5 & 0 & 0 & 0 \\ -1 & \lambda_2 - 4 & 0 & 0 \\ 0 & 0 & \lambda_2 - 1 & -3 \\ 0 & 0 & 0 & \lambda_2 - 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(0, s, t, t) : s, t \text{ both not } = 0\}$. So, an eigenvector corresponding to $\lambda_2 = 4$ is $(0, 1, 1, 1)$.

For $\lambda_3 = 1$, $\begin{bmatrix} \lambda_3 - 5 & 0 & 0 & 0 \\ -1 & \lambda_3 - 4 & 0 & 0 \\ 0 & 0 & \lambda_3 - 1 & -3 \\ 0 & 0 & 0 & \lambda_2 - 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(0, 0, t, 0) : t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 1$ is $(0, 0, 1, 0)$.

30. Using a graphing utility: $\lambda = -7, 3$

32. Using a graphing utility: $\lambda = \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

34. Using a graphing utility: $\lambda = 0, 1, 2$

36. Using a graphing utility: $\lambda = \frac{1}{5}, \frac{7 \pm \sqrt{105}}{4}$

38. Using a graphing utility: $\lambda = 0, 0, 3, 5$

40. Using a graphing utility: $\lambda = 0, 1, 1, 4$

42. The eigenvalues are the entries on the main diagonal, $-5, 7$, and 3.

44. The eigenvalues are the entries on the main diagonal, $\frac{1}{2}, \frac{5}{4}, 0$, and $\frac{3}{4}$.

46. (a) The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 8 & -16 \\ -1 & \lambda + 2 \end{vmatrix} = (\lambda + 8)(\lambda + 2) - 16 = \lambda(\lambda + 10) = 0$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -10$.

(b) For $\lambda_1 = 0$, $\begin{bmatrix} 8 & -16 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution is $\{(2t, t) : t \in R\}$. So, a basis for the eigenspace is $B_1 = \{(2, 1)\}$.

For $\lambda_2 = -10$, $\begin{bmatrix} -2 & -16 \\ -1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution is $\{(-8t, t) : t \in R\}$. So, a basis for the eigenspace is $B_2 = \{(-8, 1)\}$.

(c) $A' = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}$

48. (a) The characteristic equation is $|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 & -4 \\ -2 & \lambda - 4 & 0 \\ -5 & -5 & \lambda - 6 \end{vmatrix} = \lambda^3 - 13\lambda^2 + 32\lambda - 20$

$$= (\lambda - 1)(\lambda - 2)(\lambda - 10).$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 10$.

(b) For $\lambda_1 = 1$, $\begin{bmatrix} -2 & -1 & -4 \\ -2 & -3 & 0 \\ -5 & -5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The solution is $\{(-3t, 2t, t) : t \in R\}$. So, a basis for the eigenspace is $B_1 = \{(-3, 2, 1)\}$.

For $\lambda_2 = 2$, $\begin{bmatrix} -1 & -1 & -4 \\ -2 & -2 & 0 \\ -5 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The solution is $\{(t, -t, 0) : t \in R\}$. So, a basis for the eigenspace is $B_2 = \{(1, -1, 0)\}$.

For $\lambda_3 = 10$, $\begin{bmatrix} 7 & -1 & -4 \\ -2 & 6 & 0 \\ -5 & -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The solution is $\{(3t, t, 5t) : t \in R\}$. So, a basis for the eigenspace is $B_3 = \{(3, 1, 5)\}$.

(c) $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

50. The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 6 & 1 \\ -1 & \lambda - 5 \end{vmatrix} = \lambda^2 - 11\lambda + 31 = 0.$$

Because

$$A^2 - 11A + 31I = \begin{bmatrix} 6 & -1 \\ 1 & 5 \end{bmatrix}^2 - 11\begin{bmatrix} 6 & -1 \\ 1 & 5 \end{bmatrix} + 31\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 35 & -11 \\ 11 & 24 \end{bmatrix} - \begin{bmatrix} 66 & -11 \\ 11 & 55 \end{bmatrix} + \begin{bmatrix} 31 & 0 \\ 0 & 31 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the theorem holds for this matrix.

52. The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 3 & -1 & 0 \\ 1 & \lambda - 3 & -2 \\ 0 & -4 & \lambda - 3 \end{vmatrix} = \lambda^3 - 3\lambda^2 - 16\lambda = 0.$$

Because

$$A^3 - 3A^2 - 16A = \begin{bmatrix} -3 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}^3 - 3\begin{bmatrix} -3 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}^2 - 16\begin{bmatrix} -3 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 16 & 6 \\ -16 & 96 & 68 \\ -12 & 136 & 99 \end{bmatrix} - 3\begin{bmatrix} 8 & 0 & 2 \\ 0 & 16 & 12 \\ -4 & 24 & 17 \end{bmatrix} - 16\begin{bmatrix} -3 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the theorem holds for this matrix.

54. For the $n \times n$ matrix $A = [a_{ij}]$, the sum of the diagonal entries, or the trace, of A is given by $\sum_{i=1}^{n} a_{ii}$.

Exercise 16: $\lambda_1 = 0, \lambda_2 = 9$

(a) $\sum_{i=1}^{2} \lambda_i = 9 = \sum_{i=1}^{2} a_{ii}$

(b) $|A| = \begin{vmatrix} 1 & -4 \\ -2 & 8 \end{vmatrix} = 0 = 0 \cdot 9 = \lambda_1 \cdot \lambda_2$

Exercise 18: $\lambda_1 = -3$, and $\lambda_2 = 2$

(a) $\sum_{i=1}^{2} \lambda_i = -2 = \sum_{i=1}^{2} a_{ij}$

(b) $|A| = \begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} = -6 = (-3)(2) = \lambda_1 \cdot \lambda_2$

Exercise 20: $\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{4}$

(a) $\sum_{i=1}^{2} \lambda_i = \frac{1}{4} = \sum_{i=1}^{2} a_{ii}$

(b) $|A| = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{8} = \frac{1}{2}\left(-\frac{1}{4}\right) = \lambda_1 \cdot \lambda_2$

Exercise 22: $\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = 3$

(a) $\sum_{i=1}^{3} \lambda_i = 3 = \sum_{i=1}^{3} a_{ii}$

(b) $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -12 = -2 \cdot 2 \cdot 3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

Exercise 24: $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 2$

(a) $\sum_{i=1}^{3} \lambda_i = 4 = \sum_{i=1}^{3} a_{ii}$

(b) $|A| = \begin{vmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{vmatrix} = 0 = 0 \cdot 2 \cdot 2 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

Exercise 26: $\lambda_1 = \frac{29}{2}, \lambda_2 = \frac{1}{2}, \lambda_3 = \frac{1}{2}$

(a) $\sum_{i=1}^{3} \lambda_i = \frac{31}{2} = \sum_{i=1}^{3} a_{ii}$

(b) $|A| = \begin{vmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -2 & \frac{13}{2} & -10 \\ \frac{3}{2} & -\frac{9}{2} & 8 \end{vmatrix} = \frac{29}{8} = \frac{29}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

Exercise 28: $\lambda_1 = 5, \lambda_2 = 4, \lambda_3 = 1, \lambda_4 = 4$

(a) $\sum_{i=1}^{4} \lambda_i = 14 = \sum_{i=1}^{4} a_{ii}$

(b) $|A| = \begin{vmatrix} 5 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$

$= 80 = 5 \cdot 4 \cdot 1 \cdot 4 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4$

56. $\lambda = 0$ is an eigenvalue of $A \Leftrightarrow |0I - A| = 0 \Leftrightarrow |A| = 0.$

58. Observe that $|\lambda I - A^T| = |(\lambda I - A)^T| = |\lambda I - A|.$

Because the characteristic equations of A and A^T are the same, A and A^T must have the same eigenvalues. However, the eigenspaces are not the same.

60. Let $\mathbf{u} = (u_1, u_2)$ be the fixed vector in R^2, and $\mathbf{v} = (v_1, v_2)$. Then $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{u_1v_1 + u_2v_2}{u_1^2 + u_2^2}(u_1, u_2).$

Because $T(1, 0) = \dfrac{u_1}{u_1^2 + u_2^2}(u_1, u_2)$ and $T(0, 1) = \dfrac{u_2}{u_1^2 + u_2^2}(u_1, u_2),$

the standard matrix A of T is $A = \dfrac{1}{u_1^2 + u_2^2}\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}.$

Now,

$$A\mathbf{u} = \frac{1}{u_1^2 + u_2^2}\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{u_1^2 + u_2^2}\begin{bmatrix} u_1^3 + u_1u_2^2 \\ u_1^2u_2 + u_2^3 \end{bmatrix} = \frac{1}{u_1^2 + u_2^2}\begin{bmatrix} u_1(u_1^2 + u_2^2) \\ u_2(u_1^2 + u_2^2) \end{bmatrix} = \frac{u_1^2 + u_2^2}{u_1^2 + u_2^2}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 1\mathbf{u}$$

and

$$A\begin{bmatrix} u_2 \\ -u_1 \end{bmatrix} = \frac{1}{u_1^2 + u_2^2}\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} u_2 \\ -u_1 \end{bmatrix} = \frac{1}{u_1^2 + u_2^2}\begin{bmatrix} u_1^2u_2 - u_1^2u_2 \\ u_1u_2^2 - u_1u_2^2 \end{bmatrix} = \frac{1}{u_1^2 + u_2^2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} u_2 \\ -u_1 \end{bmatrix}.$$

So, $\lambda = 1$ and $\lambda_2 = 0$ are the eigenvalues of A.

62. Let $A^2 = O$ and consider $A\mathbf{x} = \lambda\mathbf{x}$. Then $O = A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$ which implies $\lambda = 0$.

64. (a) $-2, 1, 3$ (repeated)

(b) There are four roots of the characteristic equation, so A has order 4.

(c) When $\lambda = -2, 1,$ or 3, $\lambda I - A$ is singular.

(d) No. Zero is not an eigenvalue of A, so A is nonsingular.

66. The characteristic equation of A is $|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0$ which has no real solution.

68. (a) True. $A\mathbf{x} = \lambda\mathbf{x}$ and $\lambda\mathbf{x}$ is parallel to $\mathbf{x}$ for any real number λ. See discussion on page 348.

(b) False. The set of eigenvectors corresponding to λ together with the zero vector (which is never an eigenvector for any eigenvalue) forms a subspace of R^n. (Theorem 7.1 on page 350).

70. Substituting the value $\lambda = 3$ yields the system

$$\begin{bmatrix} \lambda - 3 & -1 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, 3 has two linearly independent eigenvectors and the dimension of the eigenspace is 2.

72. Substituting the value $\lambda = 3$ yields the system

$$\begin{bmatrix} \lambda - 3 & -1 & -1 \\ 0 & \lambda - 3 & -1 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, 3 has one linearly independent eigenvector, and the dimension of the eigenspace is 1.

74. Because $T\left(e^{-2x}\right) = \frac{d}{dx}\left[e^{-2x}\right] = -2e^{-2x}$, the eigenvalue corresponding to $f(x) = e^{-2x}$ is -2.

76. The standard matrix for T is

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 1 \\ 0 & \lambda + 1 & -2 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 1)^2.$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$ (repeated). The corresponding eigenvectors are found by solving

$$\begin{bmatrix} \lambda_i - 2 & -1 & 1 \\ 0 & \lambda_i + 1 & -2 \\ 0 & 0 & \lambda_i + 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for each λ_i. So, $p_1(x) = 1$ corresponds to $\lambda_1 = 2$, and $p_2(x) = 1 - 3x$ corresponds to $\lambda_2 = -1$.

78. The characteristic equation of A is

$$\begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = \lambda^2 - 2\cos\theta\,\lambda + \left(\cos^2\theta + \sin^2\theta\right) = \lambda^2 - 2\cos\theta\,\lambda + 1.$$

There are real eigenvalues if the discriminant of this quadratic equation in λ is nonnegative:

$$b^2 - 4ac = 4\cos^2\theta - 4 = 4\left(\cos^2\theta - 1\right) \geq 0 \quad\Rightarrow\quad \cos^2\theta = 1 \quad\Rightarrow\quad \theta = 0, \pi.$$

The only rotations that send vectors to multiples of themselves are the identity $(\theta = 0)$ and the $180°$–rotation $(\theta = \pi)$.

80. 0 is the only eigenvalue of a nilpotent matrix. For if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = A\lambda\mathbf{x} = \lambda^2\mathbf{x}$.

So,

$$A^k\mathbf{x} = \lambda^k\mathbf{x} = \mathbf{0} \quad\Rightarrow\quad \lambda^k = 0 \Rightarrow \lambda = 0.$$

Section 7.2 Diagonalization

2. (a) $P^{-1}AP = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

(b) $\lambda_1 = 2, \lambda_2 = 4$

4. (a) $P^{-1}AP = \begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\lambda_1 = -1, \lambda_2 = 2$

6. (a) $P^{-1}AP = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \\ 0 & 0 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.80 & 0.10 & 0.05 & 0.05 \\ 0.10 & 0.80 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.80 & 0.10 \\ 0.05 & 0.05 & 0.10 & 0.80 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.7 \end{bmatrix}$$

(b) $\lambda_1 = -1, \lambda_2 = 0.8, \lambda_3 = 0.7, \lambda_4 = 0.7$

8. The eigenvalues of A are $\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{4}$ (See Exercise 20, Section 7.1). The corresponding eigenvectors $(1, 1)$ and $(1, -2)$ are used to form the columns of P. So,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}.$$

10. The eigenvalues of A are $\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = 3$. From Exercise 22, Section 7.1, the corresponding eigenvectors $(1, -5, 5), (-3, 1, 1)$ and $(1, 0, 0)$ are used to form the columns of P. So,

$$P = \begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 0 & -0.1 & 0.1 \\ 0 & 0.5 & 0.5 \\ 1 & 1.6 & 1.4 \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} 0 & -0.1 & 0.1 \\ 0 & 0.5 & 0.5 \\ 1 & 1.6 & 1.4 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

12. The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2$ (repeated). From Exercise 24, Section 7.1, the corresponding eigenvectors $(-1, 3, 1), (3, 0, 1)$ and $(-2, 1, 0)$ are used to form the columns of P. So,

$$P = \begin{bmatrix} -1 & 3 & -2 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{5}{2} \\ -\frac{3}{2} & -2 & \frac{9}{2} \end{bmatrix}, \text{and}$$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{5}{2} \\ -\frac{3}{2} & -2 & \frac{9}{2} \end{bmatrix}\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

14. The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$ (repeated). Furthermore, there are just two linearly independent eigenvectors of A, $\mathbf{x}_1 = (0, 0, 1)$ and $\mathbf{x}_2 = (1, -2, 4)$. So, A is not diagonalizable.

16. The matrix A has only one eigenvalue, $\lambda = 0$, and a basis for the eigenspace is $\{(1, -2)\}$, So, A does not satisfy Theorem 7.5 and is not diagonalizable.

18. A is triangular, so the eigenvalues are simply the entries on the main diagonal. So, the only eigenvalue is $\lambda = 1$, and a basis for the eigenspace is $\{(0, 1)\}$.

Because A does not have two linearly independent eigenvectors, it does not satisfy Theorem 7.5 and it is not diagonalizable.

20. For eigenvalue $\lambda_1 = 3$, the corresponding eigenvector is $[1, 0, 0]^T$. For the repeated eigenvalue $\lambda_2 = -2$, the corresponding eigenvector is $[2, -5, 0]^T$. So, A does not satisfy Theorem 7.5 (it does not have three linearly independent eigenvectors) and is not diagonalizable.

22. From Exercise 40, Section 7.1, you know that A has only three linearly independent eigenvectors. So, A does not satisfy Theorem 7.5 and is not diagonalizable.

24. The eigenvalue of A is $\lambda = 2$ (repeated). Because A does not have two *distinct* eigenvalues, Theorem 7.6 does not guarantee that A is diagonalizable.

26. The eigenvalues of A are $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = -2$. Because A has three *distinct* eigenvalues, it is diagonalizable by Theorem 7.6.

28. The standard matrix for T is

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 5, \lambda_2 = -3$ (repeated), and corresponding eigenvectors $(1, 2, -1), (3, 0, 1)$ and $(-2, 1, 0)$. Let $B = \{(1, 2, -1), (3, 0, 1), (-2, 1, 0)\}$ and the matrix of T relative to this basis is

$$A' = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

30. The standard matrix for T is

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$, and corresponding eigenvectors $(1, 0, 0), (0, 1, 0)$, and $(-1, -6, 2)$. Let $B = \{(1, x, -1 - 6x + 2x^2)\}$ and the matrix of T relative to this basis is

$$A' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. Let P be the matrix of eigenvectors corresponding to the n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$. Then $P^{-1}AP = D$ is a diagonal matrix $\Rightarrow A = PDP^{-1}$. From Exercise 31, $A^k = PD^kP^{-1}$, which show that the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$.

34. The eigenvalues and corresponding eigenvectors of A are $\lambda_1 = 3, \lambda_2 = -2$ and $\mathbf{x}_1 = (3, 2)$ and $\mathbf{x}_2 = (-1, 1)$. Construct a nonsingular matrix P from the eigenvectors of A,

$$P = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

and find a diagonal matrix B similar to A.

$$B = P^{-1}AP = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Then,

$$A^7 = PB^7P^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3^7 & 0 \\ 0 & (-2)^7 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1261 & 1389 \\ 926 & 798 \end{bmatrix}.$$

36. The eigenvalues and corresponding eigenvectors of A are $\lambda_1 = 5, \lambda_2 = -4$, and $\lambda_3 = 0$, $\mathbf{x}_1 = (-5, 1, 9)$, $\mathbf{x}_2 = (-1, 2, 0)$, and $\mathbf{x}_3 = (5, -2, 2)$. Construct a nonsingular matrix P from the eigenvectors of A.

$$P = \begin{bmatrix} -5 & -1 & 5 \\ 1 & 2 & -2 \\ 9 & 0 & 2 \end{bmatrix}$$

and find a diagonal matrix B similar to A.

$$B = P^{-1}AP = \begin{bmatrix} -\frac{2}{45} & -\frac{1}{45} & \frac{4}{45} \\ \frac{2}{9} & \frac{11}{18} & \frac{1}{18} \\ \frac{1}{5} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{bmatrix} 2 & 3 & -2 \\ -2 & -5 & 0 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} -5 & -1 & 5 \\ 1 & 2 & -2 \\ 9 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then,

$$A^8 = PB^8P^{-1} = P\begin{bmatrix} 390{,}625 & 0 & 0 \\ 0 & 65{,}536 & 0 \\ 0 & 0 & 0 \end{bmatrix}P^{-1} = \begin{bmatrix} 72{,}242 & 3353 & -177{,}252 \\ 11{,}766 & 71{,}419 & 42{,}004 \\ -156{,}250 & -78{,}125 & 312{,}500 \end{bmatrix}.$$

38. (a) True. See Theorem 7.5 on page 360.

(b) False. Matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is diagonalizable (it is already diagonal) but it has only one eigenvalue $\lambda = 2$ (repeated).

40. (a) $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow e^X = I + I + \dfrac{I}{2!} + \dfrac{I}{3!} + \cdots = \begin{bmatrix} 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots & 0 \\ 0 & 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$

(b) $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow e^X = I + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \dfrac{1}{2!}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \dfrac{1}{3!}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} e & 0 \\ e - 1 & 1 \end{bmatrix}$

(c) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow e^X = I + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \dfrac{1}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \dfrac{1}{3!}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \dfrac{1}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cdots$

Because $e = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{3!} + \dfrac{1}{4!}$ and $e^{-1} = 1 - 1 + \dfrac{1}{2} - \dfrac{1}{3!} + \dfrac{1}{4!} - \cdots$, you see that $e^X = \dfrac{1}{2}\begin{bmatrix} e + e^{-1} & e - e^{-1} \\ e - e^{-1} & e + e^{-1} \end{bmatrix}$.

(d) $X = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow e^X = I + \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} + \dfrac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 2^2 \end{bmatrix} + \dfrac{1}{3!}\begin{bmatrix} 2^3 & 0 \\ 0 & -2^3 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-2} \end{bmatrix}$.

42. Assume that A is diagonalizable, $P^{-1}AP = D$, where D is diagonal. Then

$$D^T = \left(P^{-1}AP\right)^T = P^T A^T \left(P^{-1}\right)^T = P^T A^T \left(P^T\right)^{-1}$$

is diagonal, which shows that A^T is diagonalizable.

44. Consider the characteristic equation $|\lambda I - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = 0.$

This equation has real and unequal roots if and only if $(a + d)^2 - 4(ad - bc) > 0$, which is equivalent to $(a - d)^2 > -4bc$. So, A is diagonalizable if $-4bc < (a - d)^2$, and not diagonalizable if $-4bc > (a - d)^2$.

46. From Exercise 80, Section 7.1, you know that zero is the only eigenvalue of the nilpotent matrix A. If A were diagonalizable, then there would exist an invertible matrix P, such that $P^{-1}AP = D$, where D is the zero matrix. So, $A = PDP^{-1} = O$, which is impossible.

48. (a) A is diagonalizable when it is similar to a diagonal matrix D.

(b) A is diagonalizable when it has n linearly independent eigenvectors.

(c) A is diagonalizable when it has n distinct eigenvalues.

50. The only eigenvalue is $\lambda = 0$, and a basis for the eigenspace is $\{(0, 1)\}$. Since the matrix does not have two linearly independent eigenvectors, the matrix is not diagonalizable.

Section 7.3 Symmetric Matrices and Orthogonal Diagonalization

2. Because

$$\begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 11 & 0 & -2 \\ 3 & 0 & 5 & 0 \\ 5 & -2 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 11 & 0 & -2 \\ 3 & 0 & 5 & 0 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

the matrix *is* symmetric.

4. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda & -a & 0 \\ -a & \lambda & -a \\ 0 & -a & \lambda \end{vmatrix} = \lambda\left(\lambda - a\sqrt{2}\right)\left(\lambda + a\sqrt{2}\right).$$

The eigenvalues are $\lambda_1 = -a\sqrt{2}$, $\lambda_2 = 0$, and $\lambda_3 = a\sqrt{2}$. Since the eigenvalues are real, A is diagonalizable. The corresponding eigenvectors are $\left(1, -\sqrt{2}, 1\right)$, $(1, 0, -1)$, and $\left(1, \sqrt{2}, 1\right)$, respectively. So,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -1 & 1 \end{bmatrix} \text{ and}$$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -a\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a\sqrt{2} \end{bmatrix}.$$

6. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -a & -a \\ -a & \lambda - a & -a \\ -a & -a & \lambda - a \end{vmatrix} = \lambda^2(\lambda - 3a).$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3a$. Since the eigenvalues are real, A is diagonalizable. The corresponding eigenvectors are $(-1, 1, 0)$ and $(-1, 0, 1)$ for λ_1 and $(1, 1, 1)$ for λ_2. So,

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3a \end{bmatrix}.$$

8. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 3 \end{vmatrix} = (\lambda - 3)^2 = 0.$$

Therefore, the eigenvalue is $\lambda = 3$. The multiplicity of $\lambda = 3$ is 2, so the dimension of the corresponding eigenspace is 2 (by Theorem 7.7).

10. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & -1 \\ -1 & \lambda - 2 & -1 \\ -1 & -1 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 1)^2(\lambda - 4) = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. The multiplicity of $\lambda_1 = 1$ is 2, so the dimension of the corresponding eigenspace is 2 (by Theorem 7.7). The dimension of the eigenspace corresponding to $\lambda_2 = 4$ is 1.

12. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda & -4 & -4 \\ -4 & \lambda - 2 & 0 \\ -4 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 6)(\lambda + 6)\lambda = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -6$ and $\lambda_3 = 0$. The dimension of the eigenspace corresponding of each eigenvalue is 1.

14. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 3)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$. The dimension of the eigenspace corresponding to $\lambda_1 = 0$ is 1. The multiplicity of $\lambda_2 = 3$ is 2, so the dimension of the corresponding eigenspace is 2 (by Theorem 7.7).

16. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 1 & -2 & 0 & 0 \\ -2 & \lambda + 1 & 0 & 0 \\ 0 & 0 & \lambda + 1 & -2 \\ 0 & 0 & -2 & \lambda + 1 \end{vmatrix} = (\lambda - 1)^2(\lambda + 3)^2.$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -3$. The dimension of the corresponding eigenspace of each eigenvalue is 2 (by Theorem 7.7).

18. The characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 0 & 0 & 0 \\ 1 & \lambda - 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda - 1 & 1 \\ 0 & 0 & 0 & 1 & \lambda - 1 \end{vmatrix} = \lambda^2(\lambda - 2)^2(\lambda - 1).$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 1$. The dimensions of the corresponding eigenspaces are 2, 2, and 1, respectively (by Theorem 7.7).

20. Because $PP^T = \begin{bmatrix} \frac{4}{9} & -\frac{4}{9} \\ \frac{4}{9} & \frac{3}{9} \end{bmatrix}\begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & \frac{3}{9} \end{bmatrix} = \begin{bmatrix} \frac{32}{81} & \frac{4}{81} \\ \frac{4}{81} & \frac{25}{81} \end{bmatrix} \neq I_2$,

P is not orthogonal.

22. Because

$$PP^T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$P^T = P^{-1}$ and P is orthogonal. Letting

$p_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$ and $p_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$ produces

$p_1 \cdot p_2 = 0$ and $\|p_1\| = \|p_2\| = 1$. So, $\{p_1, p_2\}$ is an orthonormal set.

24. Because $PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$P^T = P^{-1}$ and P is orthogonal. Letting

$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ produces

$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0$ and $\|p_1\| = \|p_2\| = \|p_3\| = 1$. So, $\{p_1, p_2, p_3\}$ is an orthonormal set.

26. Because

$$PP^T = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}\begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} = I_3,$$ $P^T = P^{-1}$ and P is orthogonal.

Letting $p_1 = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $p_3 = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}$ produces

$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0$ and $\|p_1\| = \|p_2\| = \|p_3\| = 1$.

So, $\{p_1, p_2, p_3\}$ is an orthonormal set.

28. Because $PP^T = \begin{bmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} 4 & -1 & 1 \\ -1 & 0 & 4 \\ -4 & -17 & -1 \end{bmatrix} = \begin{bmatrix} 33 & 64 & 4 \\ 64 & 290 & 16 \\ 4 & 16 & 18 \end{bmatrix} \neq I_3$,

P is not orthogonal.

30. Because $PP^T = \begin{bmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{6} \\ 0 & \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{53}{36} & 0 & \frac{9\sqrt{5}-4}{36} \\ 0 & \frac{4}{5} & -\frac{2}{5} \\ \frac{9\sqrt{5}-4}{36} & -\frac{2}{5} & \frac{91}{180} \end{bmatrix} \neq I_3,$

P is not orthogonal.

32. Because $PP^T = \begin{bmatrix} \frac{1}{10}\sqrt{10} & 0 & 0 & -\frac{3}{10}\sqrt{10} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{10}\sqrt{10} & 0 & 0 & \frac{1}{10}\sqrt{10} \end{bmatrix} \begin{bmatrix} \frac{1}{10}\sqrt{10} & 0 & 0 & \frac{3}{10}\sqrt{10} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{10}\sqrt{10} & 0 & 0 & \frac{1}{10}\sqrt{10} \end{bmatrix} = I_4$, $P^T = P^{-1}$ and P is orthogonal. Letting

$p_1 = \begin{bmatrix} \frac{1}{10}\sqrt{10} \\ 0 \\ 0 \\ \frac{3}{10}\sqrt{10} \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $p_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $p_4 = \begin{bmatrix} -\frac{3}{10}\sqrt{10} \\ 0 \\ 0 \\ \frac{1}{10}\sqrt{10} \end{bmatrix}$ produces

$p_1 \cdot p_2 = p_1 \cdot p_3 = p_1 \cdot p_4 = p_2 \cdot p_3 = p_2 \cdot p_4 = p_3 \cdot p_4 = 0$ and $\|p_1\| = \|p_2\| = \|p_3\| = \|p_4\| = 1$. So, $\{p_1, p_2, p_3, p_4\}$ is an orthonormal set.

34. The characteristic polynomial of A is

$|\lambda I - A| = \begin{vmatrix} \lambda + 1 & 2 \\ 2 & \lambda - 2 \end{vmatrix} = (\lambda + 2)(\lambda - 3).$

The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Every eigenvector corresponding to $\lambda_1 = -2$ is of the form $x_1 = (2t, t)$, and every eigenvector corresponding to $\lambda_2 = 3$ is of the form $x_2 = (s, -2s)$.

$x_1 \cdot x_2 = 2st - 2st = 0$

So, x_1 and x_2 are orthogonal.

36. The matrix is diagonal, so the eigenvalues are $\lambda_1 = 3, \lambda_2 = -3,$ and $\lambda_3 = 2$. Every eigenvector corresponding to $\lambda_1 = 3$ is of the form $x_1 = (t, 0, 0)$, every eigenvector corresponding to $\lambda_2 = -3$ is of the form $x_2 = (0, s, 0)$, and every eigenvector corresponding to $\lambda_3 = 2$ is of the form $x_3 = (0, 0, u)$.

$x_1 \cdot x_2 = x_1 \cdot x_3 = x_2 \cdot x_3 = 0$

So, $\{x_1, x_2, x_3\}$ is an orthogonal set.

38. The characteristic polynomial of A is

$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda + 1 \end{vmatrix} = \lambda^2(\lambda - 1)$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$. Every eigenvector corresponding to $\lambda_1 = 0$ is of the form $x_1 = (0, 0, 0)$ and $x_2 = (0, 0, 0)$, and every eigenvector corresponding to $\lambda_2 = 1$ is of the form $x_3 = (0, t, 0)$.

$x_1 \cdot x_2 = x_1 \cdot x_3 = x_2 \cdot x_3 = 0$

So, $\{x_1, x_2, x_3\}$ is an orthogonal set.

40. The matrix is not symmetric, so it is not orthogonally diagonalizable.

42. The matrix is symmetric, so it is orthogonally diagonalizable.

44. The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 6$, with corresponding eigenvectors $(1, -1)$ and $(1, 1)$, respectively. Normalize each eigenvector to form the columns of P. Then

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } P^T AP = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$

46. The eigenvalues of A are $\lambda_1 = -1$ (repeated) and $\lambda_2 = 2$, with corresponding eigenvectors $(-1, 0, 1)$, $(-1, 1, 0)$ and $(1, 1, 1)$, respectively. Use Gram–Schmidt Orthonormalization process to orthonormalize the two eigenvectors corresponding to $\lambda_1 = -1$.

$$(-1, 0, 1) \to \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$(-1, 1, 0) - \frac{1}{2}(-1, 0, 1) = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right) \to \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

Normalizing the third eigenvector corresponding to $\lambda_2 = 2$, you can form the columns of P. So,

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and

$$P^T AP = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

48. The eigenvalues of A are $\lambda_1 = 5$, $\lambda_2 = 0$, $\lambda_3 = -5$, with corresponding eigenvectors $(3, 5, 4)$, $(-4, 0, 3)$ and $(3, -5, 4)$ respectively. Normalize each eigenvector to form the columns of P. Then

$$P = \frac{1}{10}\begin{bmatrix} 3\sqrt{2} & -8 & 3\sqrt{2} \\ 5\sqrt{2} & 0 & -5\sqrt{2} \\ 4\sqrt{2} & 6 & 4\sqrt{2} \end{bmatrix}$$

and

$$P^T AP = \frac{1}{10}\begin{bmatrix} 3\sqrt{2} & 5\sqrt{2} & 4\sqrt{2} \\ -8 & 0 & 6 \\ 3\sqrt{2} & -5\sqrt{2} & 4\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \frac{1}{10}\begin{bmatrix} 3\sqrt{2} & -8 & 3\sqrt{2} \\ 5\sqrt{2} & 0 & -5\sqrt{2} \\ 4\sqrt{2} & 6 & 4\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

50. The characteristic polynomial of A, $|\lambda I - A| = (\lambda - 8)(\lambda + 4)^2$, yields the eigenvalues $\lambda_1 = 8$ and $\lambda_2 = -4$. λ_1 has a multiplicity of 1 and λ_2 has a multiplicity of 2. An eigenvector for λ_1 is $\mathbf{v}_1 = (1, 1, 2)$, which normalizes to

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right).$$

Two eigenvectors for λ_2 are $\mathbf{v}_2 = (-1, 1, 0)$ and $\mathbf{v}_3 = (-2, 0, 1)$. Note that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$, as guaranteed by Theorem 7.9. The eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$, however, are not orthogonal to each other. To find two orthonormal eigenvectors for λ_2, use the Gram-Schmidt process as follows.

$$\mathbf{w}_2 = \mathbf{v}_2 = (-1, 1, 0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 = (-1, -1, 1)$$

These vectors normalize to

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right).$$

The matrix P has $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as its column vectors.

$$P = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \text{ and } P^TAP = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} -2 & 2 & 4 \\ 2 & -2 & 4 \\ 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

52. The eigenvalues of A are $\lambda_1 = 0$ (repeated) and $\lambda_2 = 2$ (repeated). The eigenvectors corresponding to $\lambda_1 = 0$ are $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$, while those corresponding to $\lambda_2 = 2$ are $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. Normalizing these eigenvectors to form P, you have

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

and

$$P^TAP = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

54. (a) False. The fact that a matrix P is invertible does *not* imply $P^{-1} = P^T$, only that P^{-1} exists. The definition of orthogonal matrix (page 370) requires that a matrix P is invertible *and* $P^{-1} = P^T$. For example,

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

is invertible $\left(|A| \neq 0\right)$ but $A^{-1} \neq A^T$.

(b) True. See Theorem 7.10, page 373.

56. Suppose $P^{-1}AP = D$ is diagonal, with λ the only eigenvalue. Then

$$A = PDP^{-1} = P(\lambda I)P^{-1} = \lambda I.$$

58. (a) Yes. $A = A^T$

(b) and (c) Yes, by Theorem 7.7, page 368.

(d) The multiplicity of each eigenvalue is 1, so the dimensions of the corresponding eigenspaces are 1.

(e) No. The columns do not form an orthonormal set.

(f) Yes, by Theorem 7.9, page 372.

(g) Yes, by Theorem 7.10, page 373.

60. $$A^TA = \begin{bmatrix} 1 & 4 \\ -3 & -6 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 17 & -27 & 6 \\ -27 & 45 & -12 \\ 6 & -12 & 5 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -3 & -6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 24 \\ 24 & 53 \end{bmatrix}$$

Both products are symmetric.

Section 7.4 Applications of Eigenvalues and Eigenvectors

2. $$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 4 \\ \frac{1}{16} & 0 \end{bmatrix}\begin{bmatrix} 160 \\ 160 \end{bmatrix} = \begin{bmatrix} 640 \\ 10 \end{bmatrix}$$

$$\mathbf{x}_3 = L\mathbf{x}_2 = \begin{bmatrix} 0 & 4 \\ \frac{1}{16} & 0 \end{bmatrix}\begin{bmatrix} 640 \\ 10 \end{bmatrix} = \begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

The eigenvalues are $\frac{1}{2}$ and $-\frac{1}{2}$. Choosing the positive eigenvalue, $\lambda = \frac{1}{2}$, you find the corresponding eigenvector by row-reducing $\lambda I - L = \frac{1}{2}I - L$.

$$\begin{bmatrix} \frac{1}{2} & -4 \\ -\frac{1}{16} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -8 \\ 0 & 0 \end{bmatrix}$$

So, an eigenvector is $(8, 1)$, and the stable age distribution vector is $\mathbf{x} = t\begin{bmatrix} 8 \\ 1 \end{bmatrix}$.

4. $$x_2 = Lx_1 = \begin{bmatrix} 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 4 \end{bmatrix}$$

$$x_3 = Lx_2 = \begin{bmatrix} 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 16 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 2 \end{bmatrix}$$

The eigenvalues of L are $0, 1,$ and -1. Choosing the positive eigenvalue, let $\lambda = 1$. A corresponding eigenvector is found by row-reducing $1I - L$.

$$\begin{bmatrix} 1 & -2 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, an eigenvector is $(4, 2, 1)$ and a stable age distribution vector is $x = t\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

6. $x_2 = Lx_1 = \begin{bmatrix} 0 & 6 & 4 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 240 \\ 12 \\ 24 \\ 12 \\ 12 \end{bmatrix}$

$$x_3 = Lx_2 = \begin{bmatrix} 0 & 6 & 4 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 240 \\ 12 \\ 24 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 168 \\ 120 \\ 12 \\ 12 \\ 6 \end{bmatrix}$$

The eigenvalues of L are -1, 0, and 2. Choosing the positive eigenvalue, let $\lambda = 2$. A corresponding eigenvector is found by row-reducing $2I - L$.

$$\begin{bmatrix} 2 & -6 & -4 & 0 & 0 \\ -\frac{1}{2} & 2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -128 \\ 0 & 1 & 0 & 0 & -32 \\ 0 & 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, an eigenvector is $(128, 32, 16, 4, 1)$ and a stable age distribution vector is

$$x = t\begin{bmatrix} 128 \\ 32 \\ 16 \\ 4 \\ 1 \end{bmatrix}.$$

8. Construct the age transition matrix.

$$A = \begin{bmatrix} 3 & 6 & 3 \\ 0.8 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

The current age distribution vector is

$$x_1 = \begin{bmatrix} 120 \\ 120 \\ 120 \end{bmatrix}.$$

In 1 year, the age distribution vector will be

$$x_2 = Ax_1 = \begin{bmatrix} 3 & 6 & 3 \\ 0.8 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 120 \\ 120 \\ 120 \end{bmatrix} = \begin{bmatrix} 1440 \\ 96 \\ 30 \end{bmatrix}.$$

In 2 years, the age distribution vector will be

$$x_3 = Ax_2 = \begin{bmatrix} 3 & 6 & 3 \\ 0.8 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 1440 \\ 96 \\ 30 \end{bmatrix} = \begin{bmatrix} 4986 \\ 1152 \\ 24 \end{bmatrix}.$$

10. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$, with corresponding eigenvector $(2, 1)$ and $(-2, 1)$, respectively. Then A can be diagonalized as follows

$$P^{-1}AP = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D.$$

So, $A = PDP^{-1}$ and $A^n = PD^nP^{-1}$.

If n is even, $D^n = I$ and $A^n = I$. If n is odd, $D^n = D$ and $A^n = PDP^{-1} = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} = A$. So, $A^n\mathbf{x}_1$ does not approach a limit as n approaches infinity.

12. The solution to the differential equation $y' = ky$ is $y = Ce^{kt}$. So, $y_1 = C_1e^{-5t}$ and $y_2 = C_2e^{4t}$.

14. The solution to the differential equation $y' = ky$ is $y = Ce^{kt}$. So, $y_1 = C_1e^{1/2t}$ and $y_2 = C_2e^{1/8t}$.

16. The solution to the differential equation $y' = ky$ is $y = Ce^{kt}$. So, $y_1 = C_1e^{5t}$, $y_2 = C_2e^{-2t}$, and $y_3 = C_3e^{-3t}$.

18. The solution to the differential equation $y' = ky$ is $y = Ce^{kt}$. So, $y_1 = C_1e^{-2/3t}$, $y_2 = C_2e^{-3/5t}$, and $y_3 = C_3e^{-8t}$.

20. The solution to the differential equation $y' = ky$ is $y = Ce^{kt}$.

So, $y_1 = C_1e^{-0.1t}$, $y_2 = C_2e^{-\frac{7}{4}t}$, $y_3 = C_3e^{-2\pi t}$, and $y_4 = C_4e^{\sqrt{5}t}$.

22. This system has the matrix form

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{y}.$$

The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 9$, with corresponding eigenvectors $(4, 1)$ and $(-1, 2)$, respectively. So, you can diagonalize A using a matrix P whose columns are the eigenvectors of A.

$$P = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$$

The solution of the system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ is $w_1 = C_1$ and $w_2 = C_2e^{9t}$. Return to the original system by applying the substitution $\mathbf{y} = P\mathbf{w}$.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4w_1 - w_2 \\ w_1 + 2w_2 \end{bmatrix}$$

So, the solution is

$y_1 = 4C_1 - C_2e^{9t}$

$y_2 = C_1 + 2C_2e^{9t}$.

24. This system has the matrix form

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{y}.$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$, with corresponding eigenvectors $(1, -1)$ and $(-1, 2)$, respectively. So, you can diagonalize A using a matrix P whose columns are the eigenvectors of A.

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

The solution of the system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ is $w_1 = C_1e^{2t}$ and $w_2 = C_2e^{3t}$. Return to the original system by applying the substitution $\mathbf{y} = P\mathbf{w}$.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 - w_2 \\ -w_1 + 2w_2 \end{bmatrix}$$

So, the solution is

$$\begin{aligned} y_1 &= C_1e^{2t} - C_2e^{3t} \\ y_2 &= -C_1e^{2t} + 2C_2e^{3t}. \end{aligned}$$

26. This system has the matrix form

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\mathbf{y}.$$

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 3$, with corresponding eigenvectors $(-1, 1, 1)$, $(0, 1, -1)$ and $(2, 1, 1)$, respectively. So, you can diagonalize A using a matrix P whose columns are the eigenvectors.

$$P = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The solution of the system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ is $w_1 = C_1$, $w_2 = C_2e^t$ and $w_3 = C_3e^{3t}$. Return to the original system by applying the substitution $\mathbf{y} = P\mathbf{w}$.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -w_1 + 2w_3 \\ w_1 + w_2 + w_3 \\ w_1 + w_2 + w_3 \end{bmatrix}$$

So, the solution is

$$\begin{aligned} y_1 &= -C_1 + 2C_3e^{3t} \\ y_2 &= C_1 + C_2e^t + C_3e^{3t} \\ y_3 &= C_1 - C_2e^t + C_3e^{3t}. \end{aligned}$$

28. This system has the matrix form

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\mathbf{y}.$$

The eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 3$ and $\lambda_3 = 1$, with corresponding eigenvectors $(1, 0, 0)$, $(0, 1, 0)$ and $(1, -6, 3)$, respectively. So, you can diagonalize A using a matrix P whose columns are the eigenvectors of A.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -6 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution of the system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ is $w_1 = C_1e^{-2t}$, $w_2 = C_2e^{3t}$ and $w_3 = C_3e^t$. Return to the original system by applying the substitution $\mathbf{y} = P\mathbf{w}$.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w + w_3 \\ w_2 - 6w_3 \\ 3w_3 \end{bmatrix}$$

So, the solution is

$$\begin{aligned} y_1 &= C_1e^{-2t} + C_3e^t \\ y_2 &= C_2e^{3t} - 6C_3e^t \\ y_3 &= 3C_3e^t. \end{aligned}$$

30. Because

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{y}$$

the system represented by $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= y_1 + y_2. \end{aligned}$$

Note that

$$y_1' = C_1e^t \cos t - C_1e^t \sin t + C_2e^t \sin t + C_2e^t \cos t = y_1 - y_2$$

and

$$y_2' = -C_2e^t \cos t + C_2e^t \sin t + C_1e^t \sin t + C_1e^t \cos t = y_1 + y_2.$$

32. Because

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\mathbf{y}$$ the system represented by $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_1 - 3y_2 + 3y_3. \end{aligned}$$

Note that

$$\begin{aligned} y_1' &= C_1e^t + C_2te^t + C_2e^t + C_3t^2e^t + 2C_3te^t = y_2 \\ y_2' &= (C_1 + C_2)e^t + (C_2 + 2C_3)te^t + (C_2 + 2C_3)e^t + C_3t^2e^t + 2C_3te^t = y_3 \\ y_3' &= (C_1 + 2C_2 + 2C_3)e^t + (C_2 + 4C_3)te^t + (C_2 + 4C_3)e^t + C_3t^2e^t + 2C_3te^t \\ &= \left(C_1e^t + C_2te^t + C_3t^2e^t\right) - 3\left((C_1 + C_2)e^t + (C_2 + 2C_3)te^t + C_3t^2e^t\right) \\ &\quad + 3\left((C_1 + 2C_2 + 2C_3)e^t + (C_2 + 4C_3)te^t + C_3t^2e^t\right) \\ &= y_1 - 3y_2 + 3y_3. \end{aligned}$$

34. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

36. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 12 & -\frac{5}{2} \\ -\frac{5}{2} & 0 \end{bmatrix}.$$

38. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ -2 & 20 \end{bmatrix}.$$

40. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 6$, with corresponding eigenvectors $\mathbf{x}_1 = (1, 1)$ and $\mathbf{x}_2 = (-1, 1)$, respectively. Using unit vectors in the direction of $\mathbf{x}_1$ and $\mathbf{x}_2$ to form the columns of P, you have

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad P^TAP = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}.$$

42. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 4$, with corresponding eigenvectors $\mathbf{x}_1 = (1, \sqrt{3})$ and $\mathbf{x}_2 = (-\sqrt{3}, 1)$, respectively. Using unit vectors in the direction of $\mathbf{x}_1$ and $\mathbf{x}_2$ to form the columns of P, you have

$$P = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad P^T AP = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

44. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 17 & 16 \\ 16 & -7 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -15$ and $\lambda_2 = 25$, with corresponding eigenvectors $\mathbf{x}_1 = (1, -2)$ and $\mathbf{x}_2 = (2, 1)$, respectively. Using unit vectors in the direction of $\mathbf{x}_1$ and $\mathbf{x}_2$ to form the columns of P, you have

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad P^T AP = \begin{bmatrix} -15 & 0 \\ 0 & 25 \end{bmatrix}.$$

46. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

This matrix has eigenvalues of -1 and 3, and corresponding unit eigenvectors $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, respectively. So, let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad P^T AP = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

This implies that the rotated conic is a hyperbola with equation $-(x')^2 + 3(y')^2 = 9$.

48. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 16 & -17 \end{bmatrix}.$$

This matrix has eigenvalues of -25 and 15, with corresponding unit eigenvectors $\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ respectively. Let

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad P^T AP = \begin{bmatrix} -25 & 0 \\ 0 & 15 \end{bmatrix}.$$

This implies that the rotated conic is a hyperbola with equation $-25(x')^2 + 15(y')^2 - 50 = 0$.

50. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}.$$

This matrix has eigenvalues of 4 and 12, and corresponding unit eigenvectors $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, respectively. So, let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}.$$

This implies that the rotated conic is an ellipse. Furthermore,

$$\begin{aligned} [d \quad e]P &= \begin{bmatrix} 10\sqrt{2} & 26\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= [-16 \quad 36] = [d' \quad e'], \end{aligned}$$

so the equation in the $x'y'$-coordinate system is

$$4(x')^2 + 12(y')^2 - 16x' + 36y' + 31 = 0.$$

52. The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}.$$

The eigenvalues of A are 4 and 6, with corresponding unit eigenvectors $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, respectively. So, let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}.$$

This implies that the rotated conic is an ellipse. Furthermore,

$$[d \quad e]P = \begin{bmatrix} 10\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= [10 \quad -10] = [d' \quad e'],$$

so the equation in the $x'y'$-coordinate system is

$$4(x')^2 + 6(y')^2 + 10x' + 10y' = 0.$$

54. The matrix of the quadratic form is $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$

The eigenvalues of A are 1, 1 and 4, with corresponding unit eigenvectors $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, respectively. Then let

$$P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ and } P^T AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

So, the equation of the rotated quadratic surface is

$$(x')^2 + (y')^2 + 4(z')^2 - 1 = 0 \quad \text{(ellipsoid)}.$$

56. The matrix of the quadratic form is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

The eigenvalues of A are 0, 1, and 2, with corresponding eigenvectors $(-1, 1, 0), (0, 0, 1)$, and $(1, 1, 0)$, respectively.
Then let

$$P = \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix} \text{ and } P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

So, the equation of the rotated quadratic surface is

$$(y')^2 + 2(z')^2 - 8 = 0.$$

58. The quadratic form f can be written using matrix notation as

$$f(x_1, x_2) = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2] \begin{bmatrix} 11 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Verify that the eigenvalues of $A = \begin{bmatrix} 11 & 0 \\ 0 & 4 \end{bmatrix}$ are $\lambda_1 = 11$ and $\lambda_2 = 4$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So, the constrained maximum of 11 occurs when $(x_1, x_2) = (1, 0)$ and the constrained minimum of 4 occurs when $(x_1, x_2) = (0, 1)$.

60. To find the maximum and minimum values of $z = -5x^2 + 9y^2$ subject to the constraint $x^2 + 9y^2 = 9$, you cannot use the Constrained Optimization Theorem directly because the constraint is not $\|\mathbf{x}\|^2 = 1$. However, with the change of variables $x = 3x'$ and $y = y'$,

the problem becomes finding the maximum and minimum values of

$$z = -45(x')^2 + 9(y')^2$$

subject to the constraint $(x')^2 + (y')^2 = 1$. Verify that the maximum value of 9 occurs when $(x', y') = (0, 1)$, or $(x, y) = (0, 1)$, and the minimum value of -45 occurs when $(x', y') = (1, 0)$, or $(x, y) = (3, 0)$.

62. The quadratic form f can be written using matrix notation as

$$f(x_1, x_2) = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Verify that the eigenvalues of $A = \begin{bmatrix} 5 & 6 \\ 6 & 0 \end{bmatrix}$ are $\lambda_1 = 9$ and $\lambda_2 = -4$, with corresponding eigenvectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

So, the constrained maximum of 9 occurs when

$(x_1, x_2) = \dfrac{1}{\sqrt{13}}(3, 2) = \left(\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}}\right)$ and the constrained minimum of -4 occurs when

$$(x_1, x_2) = \frac{1}{\sqrt{13}}(-2, 3) = \left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right).$$

64. To find the maximum and minimum values of $z = 9xy$ subject to the constraint $9x^2 + 16y^2 = 144$, you cannot use the Constrained Optimization Theorem directly because the constraint is not $\|\mathbf{x}\|^2 = 1$. However, with the change of variables

$x = 4x'$ and $y = 3y'$,

the problem becomes finding the maximum and minimum values of

$z = 108x'y'$

subject to the constraint $(x')^2 + (y')^2 = 1$. Verify that the maximum value of 54 occurs when $(x', y') = (1, 1)$, or $(x, y) = (4, 3)$, and the minimum value of -54 occurs when $(x', y') = (-1, 1)$, or $(x, y) = (-4, 3)$.

66. The quadratic form f can be written using matrix notation as

$$f(x, y, z) = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Verify that the eigenvalues of A are $\lambda_1 = 3$ (repeated) and $\lambda_2 = -6$, with corresponding eigenvectors

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

So, the constrained maximum of 3 occurs when

$(x, y, z) = \dfrac{1}{\sqrt{5}}(-2, 0, 1) = \left(\dfrac{-2}{\sqrt{5}}, 0, \dfrac{1}{\sqrt{5}}\right)$ and

$(x, y, z) = \dfrac{-1}{\sqrt{5}}(2, 1, 0) = \left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}, 0\right)$, and the minimum of -6 occurs when

$$(x, y, z) = \frac{1}{3}(1, -2, 2) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right).$$

68. (a) To model population growth, use the average number of offspring for each age class and the probabilities of surviving to the next age class to form the age transition matrix A. The initial age distribution vector $\mathbf{x}_1$ is used to find $\mathbf{x}_2$ by the formula $\mathbf{x}_n = A\mathbf{x}_{n-1}$. An eigenvector corresponding to a positive eigenvalue of A is a stable age distribution vector.

(b) To solve a system of first order linear differential equations find the coefficient matrix A for the system, then find a matrix P that diagonalizes A. Solve the system $\mathbf{w}' = P^{-1}AP_{\mathbf{w}}$ to find $\mathbf{w}$, and then $P_{\mathbf{w}}$ is the solution of the original system.

(c) To use the Principal Axes Theorem to perform a rotation of axes, find the matrix A of the quadratic form of the conic or quadric surface. The eigenvalues of A are the coefficients of the squared terms in the rotated system.

(d) Write the quadratic form then apply the Constrained Optimization Theorem.

Review Exercises for Chapter 7

2. (a) The characteristic equation of A is given by

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ 4 & \lambda + 2 \end{vmatrix} = \lambda^2 = 0.$$

(b) The eigenvalue of A is $\lambda = 0$ (repeated).

(c) To find the eigenvectors corresponding to $\lambda = 0$, solve the matrix equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix,

$$\begin{bmatrix} -2 & -1 & : & 0 \\ 4 & 2 & : & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

you see that a basis for the eigenspace is $\{(-1, 2)\}$.

4. (a) The characteristic equation of A is given by

$$|\lambda I - A| = \begin{vmatrix} \lambda + 4 & -1 & -2 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda + 4)(\lambda - 1)(\lambda - 3) = 0.$$

(b) The eigenvalues of A are $\lambda_1 = -4$, $\lambda_2 = 1$, and $\lambda_3 = 3$.

(c) To find the eigenvectors corresponding to $\lambda_1 = -4$, solve the matrix equation $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix,

$$\begin{bmatrix} 0 & -1 & -2 & \vdots & 0 \\ 0 & -5 & -1 & \vdots & 0 \\ 0 & 0 & -7 & \vdots & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

you see that a basis for the eigenspace $\lambda_1 = -4$ is $\{(1, 0, 0)\}$. Similarly, solve $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 1$, and see that $\{(1, 5, 0)\}$ is a basis for the eigenspace of $\lambda_2 = 1$. Finally, solve $(\lambda_3 I - A)\mathbf{x} = \mathbf{0}$ for $\lambda_3 = 3$, and determine that $\{(5, 7, 14)\}$ is a basis for its eigenspace.

6. (a) The characteristic equation of A is given by

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & -4 \\ 0 & \lambda - 1 & 2 \\ -1 & 0 & \lambda + 2 \end{vmatrix} = (\lambda + 3)(\lambda - 1)(\lambda - 2) = 0.$$

(b) The eigenvalues of A are $\lambda_1 = -3$, $\lambda_2 = 1$, and $\lambda_3 = 2$.

(c) To find the eigenvector corresponding to $\lambda_1 = -3$, solve the matrix equation $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$.

Row-reducing the augmented matrix,

$$\begin{bmatrix} -4 & 0 & -4 & \vdots & 0 \\ 0 & -4 & 2 & \vdots & 0 \\ -1 & 0 & -1 & \vdots & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & -\frac{1}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

you can see that a basis for the eigenspace of $\lambda_1 = -3$ is $\{(-2, 1, 2)\}$.

Similarly, solve $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 1$, and see that $\{(0, 1, 0)\}$ is a basis for the eigenspace of $\lambda_2 = 1$. Finally, solve $(\lambda_3 I - A)\mathbf{x} = \mathbf{0}$ for $\lambda_3 = 2$, and see that $\{(4, -2, 1)\}$ is a basis for its eigenspace.

8. (a) $|\lambda I - A| = (\lambda - 1)(\lambda - 2)(\lambda - 4)^2 = 0$

(b) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$ (repeated)

(c) A basis for the eigenspace of $\lambda_1 = 1$ is $\{(-1, 0, 1, 0)\}$.

A basis for the eigenspace of $\lambda_2 = 2$ is $\{(-2, 1, 1, 0)\}$.
A basis for the eigenspace of $\lambda_3 = 4$ is $\{(2, 3, 1, 0), (0, 0, 0, 1)\}$.

10. The eigenvalues of A are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{3}$, the corresponding eigenvectors $(3, 4)$ and $(-1, 2)$ are used to form the columns of P. So,

$$P = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{bmatrix}, \text{ and}$$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

12. The eigenvalues of A are the solutions of

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 2 & -2 \\ 2 & \lambda & 1 \\ -2 & 1 & \lambda \end{vmatrix} = (\lambda + 1)^2(\lambda - 5) = 0.$$

Therefore, the eigenvalues are -1 (repeated) and 5. The corresponding eigenvectors are solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

So, $(1, 1, -1)$ and $(2, 5, 1)$ are eigenvectors corresponding to $\lambda_1 = -1$, while $(2, -1, 1)$ corresponds to $\lambda_2 = 5$. Now form P from these eigenvectors and note that

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad\text{and}\quad P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

14. The eigenvalues of A are the solutions of

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 1 & -1 \\ 2 & \lambda - 3 & 2 \\ 1 & -1 & \lambda \end{vmatrix}$$

$$= (\lambda - 1)^2(\lambda - 3) = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1$ (repeated) and $\lambda_2 = 3$. The corresponding eigenvectors are solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$. So, $(-1, 0, 1)$ and $(1, 1, 0)$ are eigenvectors corresponding to $\lambda_1 = 1$, while $(-1, 2, 1)$ corresponds to $\lambda_2 = 3$. Now form P from these eigenvectors and note that

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad\text{and}\quad P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

16. Consider the characteristic equation $|\lambda I - A| = \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = \lambda^2 - 2\cos\theta \cdot \lambda + 1 = 0.$

The discriminant of this quadratic equation in λ is $b^2 - 4ac = 4\cos^2\theta - 4 = -4\sin^2\theta.$

Because $0 < \theta < \pi$, this discriminant is always negative, and the characteristic equation has no real roots.

18. The eigenvalue is $\lambda = -1$ (repeated). To find its corresponding eigenspace, solve $(\lambda I - A)\mathbf{x} = \mathbf{0}$ with $\lambda = -1$.

$$\left[\begin{array}{cc:c} \lambda + 1 & -2 & 0 \\ 0 & \lambda + 1 & 0 \end{array}\right] = \left[\begin{array}{cc:c} 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Because the eigenspace is only one-dimensional, the matrix A is not diagonalizable.

20. The eigenvalues are $\lambda = -2$ (repeated) and $\lambda = 4$. Because the eigenspace corresponding to $\lambda = -2$ is only one-dimensional, the matrix is not diagonalizable.

22. The eigenvalues of B are 5 and 3 with corresponding eigenvectors $(-1, 1)$ and $(-1, 2)$, respectively. Form the columns of P from the eigenvectors of B. So,

$$P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \quad\text{and}$$

$$P^{-1}BP = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = A.$$

Therefore, A and B are similar.

24. The eigenvalues of B are 1 and -2 (repeated) with corresponding eigenvectors $(-1, -1, 1)$, $(1, 1, 0)$, and $(1, 0, 1)$, respectively. Form the columns of P from the eigenvectors of B. So,

$$P = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad\text{and}$$

$$P^{-1}BP = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 & -3 \\ 3 & -5 & -3 \\ -3 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = A.$$

Therefore, A and B are similar.

26. Because

$$A^T = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \end{bmatrix} = A$$

A is symmetric. Furthermore, the column vectors of A form an orthonormal set. So, A is both symmetric and orthogonal.

28. Because

$$A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = A,$$

A is symmetric. However, column 3 is not a unit vector, so A is *not* orthogonal.

30. Because

$$A^T = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} \neq A$$

A is *not* symmetric. However, the column vectors form an orthonormal set, so A *is* orthogonal.

32. Because

$$A^T = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & 0 \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} = A$$

A is symmetric. Because the column vectors of A do not form an orthonormal set, A is *not* orthogonal.

34. The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 4 & 2 \\ 2 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 5).$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$. Every eigenvector corresponding to $\lambda_1 = 0$ is of the form $x_1 = (t, 2t)$, and every eigenvector corresponding to $\lambda_2 = 5$ is of the form $x_2 = (2s, -s)$.

$x_1 \cdot x_2 = 2st - 2st = 0$

So, x_1 and x_2 are orthogonal.

36. The matrix is diagonal, so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$. Every eigenvector corresponding to $\lambda_1 = 2$ is of the form $x_1 = (t_1, t_2, 0)$, and every eigenvector corresponding to $\lambda_2 = 5$ is of the form $x_2 = (0, 0, s)$.

$x_1 \cdot x_2 = 0$

So, x_1 and x_2 are orthogonal.

38. The matrix is not symmetric, so it is not orthogonally diagonalizable.

40. The matrix is symmetric, so it is orthogonally diagonalizable.

42. The eigenvalues of A are 17 and -17, with corresponding unit eigenvectors $\left(\frac{5}{\sqrt{34}}, \frac{3}{\sqrt{34}}\right)$ and $\left(-\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right)$, respectively.

Form the columns of P from the eigenvectors of A.

$$P = \begin{bmatrix} \frac{5}{\sqrt{34}} & -\frac{3}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} & \frac{5}{\sqrt{34}} \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} \frac{5}{\sqrt{34}} & \frac{3}{\sqrt{34}} \\ -\frac{3}{\sqrt{34}} & \frac{5}{\sqrt{34}} \end{bmatrix} \begin{bmatrix} 8 & 15 \\ 15 & -8 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{34}} & -\frac{3}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} & \frac{5}{\sqrt{34}} \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 0 & -17 \end{bmatrix}$$

44. The eigenvalues of A are -3, 0, and b, with corresponding unit eigenvectors $(0, 1, 0)$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$.

Form the columns of P from the eigenvectors of A.

$$P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 0 & -3 & 0 \\ -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

46. The eigenvalues of A are 3, -1, and 5, with corresponding eigenvectors

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), (0, 0, 1).$

Form the columns of P from the eigenvectors of A

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

48. The eigenvalues of A are $-\frac{1}{2}$ and 1. The eigenvectors corresponding to $\lambda = 1$ are $\mathbf{x} = t(2, 1)$. By choosing $t = \frac{1}{3}$, you find the steady state probability vector for A to be $\mathbf{v} = \left(\frac{2}{3}, \frac{1}{3}\right)$. Note that

$$A\mathbf{v} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \mathbf{v}.$$

50. The eigenvalues of A are $\frac{1}{5}$ and 1. The eigenvectors corresponding to $\lambda = 1$ are $\mathbf{x} = t(1, 3)$. By choosing $t = \frac{1}{4}$, you can find the steady state probability vector for A to be $\mathbf{v} = \left(\frac{1}{4}, \frac{3}{4}\right)$. Note that

$$A\mathbf{v} = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \mathbf{v}.$$

52. The eigenvalues of A are -0.2060, 0.5393 and 1. The eigenvectors corresponding to $\lambda = 1$ are $\mathbf{x} = t(2, 1, 2)$. By choosing $t = \frac{1}{5}$, find the steady state probability vector for A to be $\mathbf{v} = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$. Note that

$$A\mathbf{v} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} = \mathbf{v}.$$

54. The eigenvalues of A are $\frac{1}{10}$, $\frac{1}{5}$, and 1. The eigenvectors corresponding to $\lambda = 1$ are $\mathbf{x} = t(3, 1, 5)$. By choosing $t = \frac{1}{9}$, you can find the steady state probability vector for A to be $\mathbf{v} = \left(\frac{1}{3}, \frac{1}{9}, \frac{5}{9}\right)$. Note that

$$A\mathbf{v} = \begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.0 \\ 0.5 & 0.5 & 0.6 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{9} \\ \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{9} \\ \frac{5}{9} \end{bmatrix} = \mathbf{v}.$$

56. Show by induction that for the $n \times n$ matrix $\lambda I_n - A$,

$$\left|\lambda I_n - A\right| = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & -1 \\ a_0 & a_1 & a_2 & \cdots & \lambda + a_{n-1} \end{vmatrix} = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

For $\left|\lambda I_1 - A\right| = \lambda + a_0$ and for $n = 2$,

$$\left|\lambda I_2 - A\right| = \begin{vmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{vmatrix} = \lambda^2 + a_1\lambda + a_0.$$

Assuming the property for n, you see that

$$\left|\lambda I_{n+1} - A\right| = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & -1 \\ a_0 & a_1 & a_2 & \cdots & \lambda + a_n \end{vmatrix} = (\lambda + a_n)\lambda^n + \left|\lambda I_n - A\right| = \lambda^{n+1} + a_n\lambda^n + \cdots + a_0.$$

Showing the property is valid for $n + 1$. You can now evaluate the characteristic equation of A as follows.

$$\left|\lambda I_n - A\right| = \begin{vmatrix} \lambda & -1 & 0 & \ldots & 0 \\ 0 & \lambda & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \vdots & -1 \\ a_0 & a_1 & a_2 & \vdots & \lambda + a_{n-1} \end{vmatrix} = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_1\lambda + a_0.$$

58. From the form $p(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, you have $a_3 = 2$, $a_2 = -7$, $a_1 = -120$, and $a_0 = 189$. This implies that the companion matrix of p is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{189}{2} & 60 & \frac{7}{2} \end{bmatrix}.$$

The eigenvalues of A are $\frac{3}{2}$, 9, and -7, the zeros of p.

60. The characteristic equation of A is $|\lambda I - A| = \lambda^3 - 20\lambda^2 + 128\lambda - 256 = 0$.

Using $A^3 - 20A^2 + 128A - 256I_3 = 0$, you can find the powers of A by the process below.

$$A^3 = 20A^2 - 128A + 256I_3$$

$$A^4 = 20A^3 - 128A^2 + 256A$$

$$\begin{aligned} A^3 &= 20A^2 - 128A + 256I_3 \\ &= 20\begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix} - 128\begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix} + 256\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1520 & 960 & -720 \\ -480 & -640 & 1440 \\ -240 & -960 & 2000 \end{bmatrix} - \begin{bmatrix} 1152 & 512 & -384 \\ -256 & 0 & 768 \\ -128 & -512 & 1408 \end{bmatrix} + \begin{bmatrix} 256 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 256 \end{bmatrix} \\ &= \begin{bmatrix} 624 & 448 & -336 \\ -224 & -384 & 672 \\ -112 & -448 & 848 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^4 &= 20A^3 - 128A^2 + 256A \\ &= 20\begin{bmatrix} 624 & 448 & -336 \\ -224 & -384 & 672 \\ -112 & -448 & 848 \end{bmatrix} - 128\begin{bmatrix} 76 & 48 & -36 \\ -24 & -32 & 72 \\ -12 & -48 & 100 \end{bmatrix} + 256\begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix} \\ &= \begin{bmatrix} 12{,}480 & 8960 & -6720 \\ -4480 & -7680 & 13{,}440 \\ -2240 & -8960 & 16{,}960 \end{bmatrix} - \begin{bmatrix} 9728 & 6144 & -4608 \\ -3072 & -4096 & 9216 \\ -1536 & -6144 & 12{,}800 \end{bmatrix} + \begin{bmatrix} 2304 & 1024 & -768 \\ -512 & 0 & 1536 \\ -256 & -1024 & 2816 \end{bmatrix} \\ &= \begin{bmatrix} 5056 & 3840 & -2880 \\ -1920 & -3584 & 5760 \\ -960 & -3840 & 6976 \end{bmatrix} \end{aligned}$$

62. $(A + cI)\mathbf{x} = A\mathbf{x} + cI\mathbf{x} = \lambda\mathbf{x} + c\mathbf{x} = (\lambda + c)\mathbf{x}$. So, $\mathbf{x}$ is an eigenvector of $(A + cI)$ with eigenvalue $(\lambda + c)$.

64. (a) The eigenvalues of A are 3 and 1, with corresponding eigenvectors $(1, 1)$ and $(1, -1)$. Letting these eigenvectors form the columns of P, you can diagonalize A.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = D$$

So, $A = PDP^{-1} = P\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}P^{-1}$. Letting $B = P\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}P^{-1} = \frac{1}{2}\begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{bmatrix}$

you have $B = \left(P\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}P^{-1}\right)^2 = P\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}^2 P^{-1} = P\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}p^{-1} = A.$

(b) In general, let $A = PDP^{-1}$, D diagonal with positive eigenvalues on the diagonal. Let D' be the diagonal matrix consisting of the square roots of the diagonal entries of D. Then if $B = PD'P^{-1}$,

$$B^2 = (PD'P^{-1})(PD'P^{-1}) = P(D')^2 P^{-1} = PDP^{-1} = A.$$

66. The eigenvalues of A are $a + b$ and $a - b$, with corresponding unit eigenvectors $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, respectively. So, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Note that

$$P^{-1}AP = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}.$$

68. (a) A is diagonalizable if and only if $a = b = c = 0$.

(b) If exactly two of a, b, and c are zero, then the eigenspace of 2 has dimension 3. If exactly one of a, b, c is zero, then the dimension of the eigenspace is 2. If none of a, b, c is zero, the eigenspace is dimension 1.

70. (a) True. See Theorem 7.2 on page 432.

(b) False. See remark after the "Definitions of Eigenvalue and Eigenvector" on page 426. If $\mathbf{x} = \mathbf{0}$ is allowed to be an eigenvector, then the definition of eigenvalue would be meaningless, because $A\mathbf{0} = \lambda\mathbf{0}$ for *all* real numbers λ.

(c) True. See page 459.

72. The population after one transition is

$$\mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ \frac{3}{4} & 0 \end{bmatrix}\begin{bmatrix} 32 \\ 32 \end{bmatrix} = \begin{bmatrix} 32 \\ 24 \end{bmatrix}$$

and after two transitions is

$$\mathbf{x}_3 = \begin{bmatrix} 0 & 1 \\ \frac{3}{4} & 0 \end{bmatrix}\begin{bmatrix} 32 \\ 24 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \end{bmatrix}.$$

The eigenvalues of A are $\pm\frac{\sqrt{3}}{2}$. Choose the positive eigenvalue and find the corresponding eigenvector to be $\left(2, \sqrt{3}\right)$, and the stable age distribution vector is

$$\mathbf{x} = t\begin{bmatrix} 2 \\ \sqrt{3} \end{bmatrix}$$

74. The population after one transition is

$$\mathbf{x}_2 = \begin{bmatrix} 0 & 2 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 240 \\ 240 \\ 240 \end{bmatrix} = \begin{bmatrix} 960 \\ 120 \\ 0 \end{bmatrix}$$

and after two transitions is

$$\mathbf{x}_3 = \begin{bmatrix} 0 & 2 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 960 \\ 120 \\ 0 \end{bmatrix} = \begin{bmatrix} 240 \\ 480 \\ 0 \end{bmatrix}.$$

The positive eigenvalue 1 has corresponding eigenvector $(2, 1, 0)$, and the stable distribution vector is $\mathbf{x} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

76. Construct the age transition matrix.

$$A = \begin{bmatrix} 4 & 8 & 2 \\ 0.75 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$$

The current age distribution vector is $\mathbf{x}_1 = \begin{bmatrix} 120 \\ 120 \\ 120 \end{bmatrix}$.

In one year, the age distribution vector will be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 4 & 8 & 2 \\ 0.75 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}\begin{bmatrix} 120 \\ 120 \\ 120 \end{bmatrix} = \begin{bmatrix} 1680 \\ 90 \\ 72 \end{bmatrix}.$$

In two years, the age distribution vector will be

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 4 & 8 & 2 \\ 0.75 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}\begin{bmatrix} 1680 \\ 90 \\ 72 \end{bmatrix} = \begin{bmatrix} 7584 \\ 1260 \\ 54 \end{bmatrix}.$$

78. The matrix corresponds to the system $\mathbf{y}' = A\mathbf{y}$ is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This matrix has eigenvalues 1 and -1, with corresponding eigenvectors $(1, 1)$ and $(1, -1)$. So, a matrix P that diagonalizes A is

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The system represented by $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has solutions $w_1 = C_1e^t$ and $w_2 = C_2e^{-t}$. Substitute $\mathbf{y} = P\mathbf{w}$ and obtain

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} C_1e^t + C_2e^{-t} \\ C_1e^t - C_2e^{-t} \end{bmatrix}$$

which yields the solutions

$$\begin{aligned} y_1 &= C_1e^t + C_2e^{-t} \\ y_2 &= C_1e^t - C_2e^{-t}. \end{aligned}$$

80. The matrix corresponding to the system $\mathbf{y}' = A\mathbf{y}$ is

$$A = \begin{bmatrix} 6 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of A are 6, 3, and 1, with corresponding eigenvectors $(1, 0, 0)$, $(1, 3, 0)$, and $(-3, 5, 10)$. So, you can diagonalize A by forming P.

$$P = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & 5 \\ 0 & 0 & 10 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system represented by $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has solutions $w_1 = C_1e^{6t}$, $w_2 = C_2e^{3t}$, and $w_3 = C_3e^t$. Substitute $\mathbf{y} = P\mathbf{w}$ and obtain

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & 5 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 - 3w_3 \\ 3w_2 + 5w_3 \\ 10w_3 \end{bmatrix}$$

which yields the solution

$$\begin{aligned} y_1 &= C_1e^{6t} + C_2e^{3t} - 3C_3e^t \\ y_2 &= 3C_2e^{3t} + 5C_3e^t \\ y_3 &= 10C_3e^t. \end{aligned}$$

82. (a) The matrix of the quadratic form is

$$A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 2 \end{bmatrix}.$$

(b) The eigenvalues are $\frac{1}{2}$ and $\frac{5}{2}$, with corresponding unit eigenvectors $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

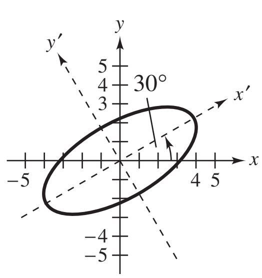
Use these eigenvectors to form the columns of P.

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \text{ and } P^TAP = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix}$$

(c) This implies that the equation of the rotated conic is

$$\frac{1}{2}(x')^2 + \frac{5}{2}(y')^2 = 10, \text{ an ellipse.}$$

(d)



84. (a) The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = \begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}.$$

(b) The eigenvalues are 0 and 25, with corresponding unit eigenvectors $\left(\frac{4}{5}, \frac{3}{5}\right)$ and $\left(-\frac{3}{5}, \frac{4}{5}\right)$. Use these eigenvectors to form the columns of P.

$$P = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \quad \text{and} \quad P^T AP = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}$$

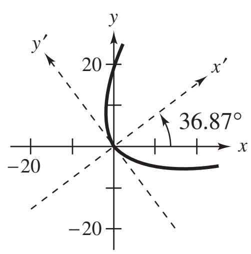
This implies that the equation of the rotated conic is a parabola.

(c) Furthermore,

$$\begin{bmatrix} d & e \end{bmatrix} P = \begin{bmatrix} -400 & -300 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} -500 & 0 \end{bmatrix} = \begin{bmatrix} d' & e' \end{bmatrix}$$

so the equation in the $x'y'$-coordinate system is $25(y')^2 - 500x' = 0$.

(d)



86. To find the maximum and minimum values of $z = x_1x_2$ subject to the constraint $25x_1^2 + 4x_2^2 = 100$, you cannot use the Constrained Optimization Theorem directly because the constraint is not $\|\mathbf{x}\|^2 = 1$. However, with the change of variables

$x_1 = 2x$ and $x_2 = 5y$

the problem becomes finding the maximum and minimum values of

$z = 10xy$

subject to the constraint $x^2 + y^2 = 1$. Verify that the maximum value of 5 occurs when $(x, y) = (0, 1)$, or $(x_1, x_2) = (0, 5)$, and the minimum value of -5 occurs when $(x, y) = (0, -1)$, or $(x_1, x_2) = (0, -5)$.

88. The quadratic form f can be written using matrix notation as

$$f(x, y) = \mathbf{x}^T A\mathbf{x}$$
$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -11 & 5 \\ 5 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Verify that the eigenvalues of $A = \begin{bmatrix} -11 & 5 \\ 5 & -11 \end{bmatrix}$ are $\lambda_1 = -16$ and $\lambda_2 = -6$, with corresponding eigenvalues $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So, the constrained maximum of -6 occurs when

$(x, y) = \frac{1}{\sqrt{2}}(1, 1)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and constrained minimum of -16 occurs when

$(x, y) = \frac{1}{\sqrt{2}}(-1, 1)\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$

Project Solutions for Chapter 7

1 Population Growth and Dynamical Systems (I)

1. $A = \begin{bmatrix} 0.5 & 0.6 \\ -0.4 & 3.0 \end{bmatrix}$, $\lambda_1 = 0.6, \mathbf{w}_1 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$\lambda_2 = 2.9, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$P = \begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix}$, $P^{-1} = \frac{1}{23}\begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 0.6 & 0 \\ 0 & 2.9 \end{bmatrix}$

$\mathbf{w}_1 = C_1e^{0.6t}, \quad \mathbf{w}_2 = C_2e^{2.9t}, \quad \mathbf{y} = P\mathbf{w}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C_1e^{0.6t} \\ C_2e^{2.9t} \end{bmatrix} = \begin{bmatrix} 6C_1e^{0.6t} + C_2e^{2.9t} \\ C_1e^{0.6t} + 4C_2e^{2.9t} \end{bmatrix}$$

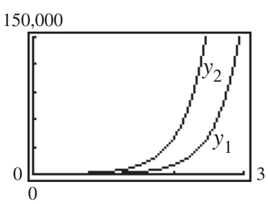
$$\left.\begin{aligned} y_1(0) &= 36 \Rightarrow 6C_1 + C_2 = 36 \\ y_2(0) &= 121 \Rightarrow C_1 + 4C_2 = 121 \end{aligned}\right\}$$

So, $C_1 = 1, C_2 = 30$ and

$$\begin{aligned} y_1 &= 6e^{0.6t} + 30e^{2.9t} \\ y_2 &= e^{0.6t} + 120e^{2.9t}. \end{aligned}$$

2. No, neither species disappears. As $t \to \infty$, $y_1 \to 30e^{2.9t}$ and $y_2 \to 120e^{2.9t}$.

3.



4. As $t \to \infty$, $y_1 \to 30e^{2.9t}$, $y_2 \to 120e^{2.9t}$, and $\dfrac{y_2}{y_1} = 4$.

5. The population y_2 ultimately disappears around $t = 1.6$.

2 The Fibonacci Sequence

1.

$x_1 = 1$	$x_4 = 3$	$x_7 = 13$	$x_{10} = 55$
$x_2 = 1$	$x_5 = 5$	$x_8 = 21$	$x_{11} = 89$
$x_3 = 2$	$x_6 = 8$	$x_9 = 34$	$x_{12} = 144$

2. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} x_{n-1} + x_{n-2} \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$. x_n generated from $\begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix}$

3. $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \end{bmatrix}$

In general, $A^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix}$ or $A^{n-2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$.

4. $\begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \quad \Rightarrow \quad \lambda = \dfrac{1 \pm \sqrt{5}}{2}$

$\lambda_1 = \dfrac{1 + \sqrt{5}}{2}$ eigenvector: $\begin{bmatrix} 2 \\ -1 + \sqrt{5} \end{bmatrix}$

$\lambda_2 = \dfrac{1 - \sqrt{5}}{2}$ eigenvector: $\begin{bmatrix} 2 \\ -1 - \sqrt{5} \end{bmatrix}$

$$P = \begin{bmatrix} 2 & 2 \\ -1 + \sqrt{5} & -1 - \sqrt{5} \end{bmatrix}$$

$$P^{-1} = \frac{1}{4\sqrt{5}}\begin{bmatrix} 1 + \sqrt{5} & 2 \\ -1 + \sqrt{5} & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

5.

$$\begin{aligned}
P^{-1}AP &= D \\
P^{-1}A^{n-2}P &= D^{n-2} \\
A^{n-2} &= PD^{n-2}P^{-1} \\
&= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & 2 \\ -1 + \sqrt{5} & -1 - \sqrt{5} \end{bmatrix}\begin{bmatrix} \left(\dfrac{1 + \sqrt{5}}{2}\right)^{n-2} & 0 \\ 0 & \left(\dfrac{1 - \sqrt{5}}{2}\right)^{n-2} \end{bmatrix}\begin{bmatrix} 1 + \sqrt{5} & 2 \\ -1 + \sqrt{5} & -2 \end{bmatrix} \\
&= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2(\lambda_1)^{n-2} & 2(\lambda_2)^{n-2} \\ (-1 + \sqrt{5})(\lambda_1)^{n-2} & (-1 - \sqrt{5})(\lambda_2)^{n-2} \end{bmatrix}\begin{bmatrix} 1 + \sqrt{5} & 2 \\ -1 + \sqrt{5} & -2 \end{bmatrix} \\
&= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2(1 + \sqrt{5})(\lambda_1)^{n-2} + 2(-1 + \sqrt{5})(\lambda_2)^{n-2} & 4(\lambda_1)^{n-2} - 4\lambda_2{}^{n-2} \\ +4\lambda_1{}^{n-2} - 4\lambda_2{}^{n-2} & 2(-1 + \sqrt{5})\lambda_1{}^{n-2} + 2(1 + \sqrt{5})\lambda_2{}^{n-2} \end{bmatrix}
\end{aligned}$$

$$A^{n-2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \Rightarrow$$

$$\begin{aligned}
x_n &= \frac{1}{4\sqrt{5}}\left[2(1 + \sqrt{5})\lambda_1{}^{n-2} + 2(-1 + \sqrt{5})\lambda_2{}^{n-2} + 4\lambda_1{}^{n-2} - 4\lambda_2{}^{n-2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\lambda_1^n - \lambda_2^n\right]
\end{aligned}$$

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right]$$

$$x_1 = \frac{1}{\sqrt{5}}\left(\sqrt{5}\right) = 1$$

$$x_2 = \frac{1}{\sqrt{5}}\left[\frac{6 + 2\sqrt{5}}{4} - \frac{6 - 2\sqrt{5}}{4}\right] = 1$$

$$x_3 = \frac{1}{\sqrt{5}}\left[\frac{6 + 2\sqrt{5}}{4} \cdot \frac{1 + \sqrt{5}}{2} - \frac{6 - 2\sqrt{5}}{4} \cdot \frac{1 - \sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left[\frac{16 + 8\sqrt{5}}{8} - \frac{16 - 8\sqrt{5}}{8}\right] = 2$$

6. $x_{10} = 55, x_{20} = 6765$

7. For example, $\dfrac{x_{20}}{x_{19}} = \dfrac{6765}{4181} = 1.618\ldots.$

The quotients seem to be approaching a fixed value near 1.618.

8. Let the limit be $\dfrac{x_n}{x_{n-1}} = b$. Then for large $n, n \to \infty$.

$$b \approx \frac{x_n}{x_{n-1}} = \frac{x_{n-1} + x_{n-2}}{x_{n-1}} \approx 1 + \frac{1}{b} \quad \Rightarrow \quad b^2 - b - 1 = 0 \quad \Rightarrow \quad b = \frac{1 \pm \sqrt{5}}{2}$$

Taking the positive value, $b = \dfrac{1 + \sqrt{5}}{2} \approx 1.618$.