

C H A P T E R 3

Determinants

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C H A P T E R 3

Determinants

Section 3.1 The Determinant of a Matrix

2. The determinant of a matrix of order 1 is the entry in the matrix. So, $\det[-3] = -3$.

$$4. \begin{vmatrix} -3 & 1 \\ 5 & 2 \end{vmatrix} = -3(2) - 5(1) = -11$$

$$6. \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 2(3) - 4(-2) = 14$$

$$8. \begin{vmatrix} \frac{1}{3} & 5 \\ 4 & -9 \end{vmatrix} = \frac{1}{3} \cdot (-9) - 5 \cdot 4 = -23$$

$$10. \begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix} = 2(9) - (-6)(-3) = 0$$

16. (a) The minors of the matrix are shown.

$$M_{11} = \begin{vmatrix} 3 & 1 \\ -7 & -8 \end{vmatrix} = -17 \quad M_{12} = \begin{vmatrix} 6 & 1 \\ 4 & -8 \end{vmatrix} = -52 \quad M_{13} = \begin{vmatrix} 6 & 3 \\ 4 & -7 \end{vmatrix} = -54$$

$$M_{21} = \begin{vmatrix} 4 & 2 \\ -7 & -8 \end{vmatrix} = -18 \quad M_{22} = \begin{vmatrix} -3 & 2 \\ 4 & -8 \end{vmatrix} = 16 \quad M_{23} = \begin{vmatrix} -3 & 4 \\ 4 & -7 \end{vmatrix} = 5$$

$$M_{31} = \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = -2 \quad M_{32} = \begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix} = -15 \quad M_{33} = \begin{vmatrix} -3 & 4 \\ 6 & 3 \end{vmatrix} = -33$$

- (b) The cofactors of the matrix are shown.

$$C_{11} = (-1)^2 M_{11} = -17 \quad C_{12} = (-1)^3 M_{12} = 52 \quad C_{13} = (-1)^4 M_{13} = -54$$

$$C_{21} = (-1)^3 M_{21} = 18 \quad C_{22} = (-1)^4 M_{22} = 16 \quad C_{23} = (-1)^5 M_{23} = -5$$

$$C_{31} = (-1)^4 M_{31} = -2 \quad C_{32} = (-1)^5 M_{32} = 15 \quad C_{33} = (-1)^6 M_{33} = -33$$

18. (a) You found the cofactors of the matrix in Exercise 16. Now find the determinant by expanding along the third row.

$$\begin{vmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{vmatrix} = 4C_{31} - 7C_{32} - 8C_{33} = 4(-2) - 7(15) - 8(-33) = 151$$

- (b) Expand along the first column.

$$\begin{vmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{vmatrix} = -3C_{11} + 6C_{12} + 4C_{13} = -3(-17) + 6(18) + 4(-2) = 151$$

$$12. \begin{vmatrix} \lambda - 2 & 0 \\ 4 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4) - 4(0) = \lambda^2 - 6\lambda + 8$$

14. (a) The minors of the matrix are shown.

$$M_{11} = |-5| = 5 \quad M_{12} = 6 = 6$$

$$M_{21} = |1| = 1 \quad M_{22} = 0 = 0$$

- (b) The cofactors of the matrix are shown.

$$C_{11} = (-1)^2 M_{11} = 5 \quad C_{12} = (-1)^3 M_{12} = 6$$

$$C_{21} = (-1)^3 M_{21} = 1 \quad C_{22} = (-1)^4 M_{22} = 0$$

20. Expand along the third row because it has a zero.

$$\begin{vmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{vmatrix} = (-2) \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} - 0 \begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= (-2)(-6) - 0(4) + 1(7)$$

$$= 19$$

22. Expand along the first row because it has two zeros.

$$\begin{vmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{vmatrix} = -3 \begin{vmatrix} 11 & 0 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 7 & 0 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 7 & 11 \\ 1 & 2 \end{vmatrix} = -3(22) = -66$$

24. Expand along the first row.

$$\begin{vmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{vmatrix} = 0.1 \begin{vmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{vmatrix} - 0.2 \begin{vmatrix} -0.3 & 0.2 \\ 0.5 & 0.4 \end{vmatrix} + 0.3 \begin{vmatrix} -0.3 & 0.2 \\ 0.5 & 0.4 \end{vmatrix}$$

$$= 0.1(0) - 0.2(-0.22) + 0.3(-0.22)$$

$$= -0.022$$

26. Expand along the first row.

$$\begin{vmatrix} x & y & 1 \\ -2 & -2 & 1 \\ 1 & 5 & 1 \end{vmatrix} = x \begin{vmatrix} -2 & 1 \\ 5 & 1 \end{vmatrix} - y \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & -2 \\ 1 & 5 \end{vmatrix}$$

$$= x(-7) - y(-3) + (-8)$$

$$= -7x + 3y - 8$$

28. Expand along the first row, because it has two zeros.

$$\begin{vmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 11 & 12 \\ 4 & 1 & -1 & 2 \\ 1 & 5 & 2 & 10 \end{vmatrix} = 3 \begin{vmatrix} 6 & 11 & 12 \\ 1 & -1 & 2 \\ 5 & 2 & 10 \end{vmatrix} + 7 \begin{vmatrix} 2 & 6 & 12 \\ 4 & 1 & 2 \\ 1 & 5 & 10 \end{vmatrix}$$

The determinants of the 3×3 matrices are:

$$\begin{vmatrix} 6 & 11 & 12 \\ 1 & -1 & 2 \\ 5 & 2 & 10 \end{vmatrix} = 6 \begin{vmatrix} -1 & 2 \\ 2 & 10 \end{vmatrix} - 11 \begin{vmatrix} 1 & 2 \\ 5 & 10 \end{vmatrix} + 12 \begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix}$$

$$= 6(-10 - 4) - 11(10 - 10) + 12(2 + 5) = -84 + 84 = 0$$

$$\begin{vmatrix} 2 & 6 & 12 \\ 4 & 1 & 2 \\ 1 & 5 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 5 & 10 \end{vmatrix} - 6 \begin{vmatrix} 4 & 2 \\ 1 & 10 \end{vmatrix} + 12 \begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= 2(10 - 10) - 6(40 - 2) + 12(20 - 1) = 0$$

So, the determinant of the original matrix is $3(0) + 7(0) = 0$.

30. Expand along the first row.

$$\left| \begin{array}{cccc} w & x & y & z \\ 10 & 15 & -25 & 30 \\ -30 & 20 & -15 & -10 \\ 30 & 35 & -25 & -40 \end{array} \right| = w \left| \begin{array}{ccc} 15 & -25 & 30 \\ 20 & -15 & -10 \\ 35 & -25 & -40 \end{array} \right| - x \left| \begin{array}{ccc} 10 & -25 & 30 \\ -30 & -15 & -10 \\ 30 & -25 & -40 \end{array} \right| + y \left| \begin{array}{ccc} 10 & 15 & 30 \\ -30 & 20 & -10 \\ 30 & 35 & -40 \end{array} \right| - z \left| \begin{array}{ccc} 10 & 15 & -25 \\ -30 & 20 & -15 \\ 30 & 35 & -25 \end{array} \right|$$

The determinants of the 3×3 matrices are:

$$\begin{vmatrix} 15 & -25 & 30 \\ 20 & -15 & -10 \\ 35 & -25 & -40 \end{vmatrix} = 15 \begin{vmatrix} -15 & -10 \\ -25 & -40 \end{vmatrix} + 25 \begin{vmatrix} 20 & -10 \\ 35 & -40 \end{vmatrix} + 30 \begin{vmatrix} 20 & -15 \\ 30 & -25 \end{vmatrix}$$

$$\begin{aligned}
 &= 15(600 - 250) + 25(-800 + 350) + 30(-500 + 525) \\
 &= 5250 - 11,250 + 750 \\
 &= -5250
 \end{aligned}$$

$$\begin{vmatrix} 10 & -25 & 30 \\ -30 & -15 & -10 \\ 30 & -25 & -40 \end{vmatrix} = 10 \begin{vmatrix} -15 & -10 \\ -25 & -40 \end{vmatrix} + 25 \begin{vmatrix} -30 & -10 \\ 30 & -40 \end{vmatrix} + 30 \begin{vmatrix} -30 & -15 \\ 30 & -25 \end{vmatrix}$$

$$\begin{aligned}
 &= 10(600 - 250) + 25(1200 + 300) + 30(750 + 450) \\
 &= 3500 + 37,500 + 36,000 \\
 &= 77,000
 \end{aligned}$$

$$\begin{vmatrix} 10 & 15 & 30 \\ -30 & 20 & -10 \\ 30 & 35 & -40 \end{vmatrix} = 10 \begin{vmatrix} 20 & -10 \\ 35 & -40 \end{vmatrix} - 15 \begin{vmatrix} -30 & -10 \\ 30 & -40 \end{vmatrix} + 30 \begin{vmatrix} -30 & 20 \\ 30 & 35 \end{vmatrix}$$

$$\begin{aligned}
 &= 10(-800 + 350) - 15(1200 + 300) + 30(-1050 - 600) \\
 &= -4500 - 22,500 + 49,500 \\
 &\equiv -76,500
 \end{aligned}$$

$$\begin{vmatrix} 10 & 15 & -25 \\ -30 & 20 & -15 \\ 30 & 35 & -25 \end{vmatrix} = 10 \begin{vmatrix} 20 & -15 \\ 35 & -25 \end{vmatrix} - 15 \begin{vmatrix} -30 & -15 \\ 30 & -25 \end{vmatrix} - 25 \begin{vmatrix} -30 & 20 \\ 30 & 35 \end{vmatrix}$$

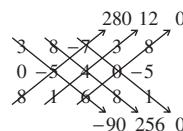
$$\begin{aligned}
 &= 10(-500 + 525) - 15(750 + 450) - 25(-1050 - 600) \\
 &= 250 - 18,000 + 41,250 \\
 &= 23,500
 \end{aligned}$$

So, the determinant is $-5250w - 77,000x - 76,500y - 23,500z$.

- 32.** Expand along the fourth row because it has all zeros.

$$\begin{vmatrix} -4 & 3 & 2 & -1 & -2 \\ 1 & -2 & 7 & -13 & -12 \\ -6 & 2 & -5 & -6 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & -2 & 0 & 9 \end{vmatrix} = 0$$

34. Copy the first two columns and complete the diagonal products as follows.



Add the lower three products and subtract the upper three products to find the determinant.

$$\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix} = -90 + 256 + 0 - 280 - 12 - 0 = -126$$

36. $\begin{vmatrix} 4 & 3 & 2 & 5 \\ 1 & 6 & -1 & 2 \\ -3 & 2 & 4 & 5 \\ 6 & 1 & 3 & -2 \end{vmatrix} = -1098$

38. $\begin{vmatrix} 8 & 5 & 1 & -2 & 0 \\ -1 & 0 & 7 & 1 & 6 \\ 0 & 8 & 6 & 5 & -3 \\ 1 & 2 & 5 & -8 & 4 \\ 2 & 6 & -2 & 0 & 6 \end{vmatrix} = 48,834$

40. The determinant of a triangular matrix is the product of the elements on the main diagonal.

$$\begin{vmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 4(7)(-2) = -56$$

42. The determinant of a triangular matrix is the product of the elements on the main diagonal.

$$\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ 3 & 5 & 3 & 0 \\ -8 & 7 & 0 & -2 \end{vmatrix} = 4\left(\frac{1}{2}\right)(3)(-2) = -12$$

50. $\begin{vmatrix} \lambda - 5 & 3 \\ 1 & \lambda - 5 \end{vmatrix} = (\lambda - 5)(\lambda - 5) - 3(1) = \lambda^2 - 10\lambda + 22$

The determinant is zero when $\lambda^2 - 10\lambda + 22 = 0$. Use the Quadratic Formula to find λ .

$$\begin{aligned} \lambda &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(22)}}{2(1)} \\ &= \frac{10 \pm \sqrt{12}}{2} \\ &= \frac{10 \pm 2\sqrt{3}}{2} \\ &= 5 \pm \sqrt{3} \end{aligned}$$

52.
$$\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda - 2 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 3 \\ 2 & \lambda - 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & \lambda \\ 2 & 2 \end{vmatrix}$$

$$\begin{aligned} &= \lambda(\lambda^2 - 2\lambda - 6) + 1(0 - 2\lambda) \\ &= \lambda^3 - 2\lambda^2 - 8\lambda \\ &= \lambda(\lambda^2 - 2\lambda - 8) \\ &= \lambda(\lambda - 4)(\lambda + 2) \end{aligned}$$

The determinant is zero when $\lambda(\lambda - 4)(\lambda + 2) = 0$. So, $\lambda = 0, 4, -2$.

44. (a) False. The determinant of a triangular matrix is equal to the *product* of the entries on the main diagonal. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

then $\det(A) = 2 \neq 3 = 1 + 2$.

- (b) True. See Theorem 3.1 on page 113.

- (c) True. This is because in a cofactor expansion each cofactor gets multiplied by the corresponding entry. If this entry is zero, the product would be zero independent of the value of the cofactor.

46. $(x - 6)(x + 1) - 3(-2) = 0$

$x^2 - 5x - 6 + 6 = 0$

$x^2 - 5x = 0$

$x(x - 5) = 0$

$x = 0, 5$

48. $(x + 3)(x - 1) - (-4)(1) = 0$

$x^2 + 2x - 3 + 4 = 0$

$x^2 + 2x + 1 = 0$

$(x + 1)^2 = 0$

$x = -1$

54. (a) Take the determinant of the $(n - 1) \times (n - 1)$ matrix that is left after deleting the i th row and j th column.

(b) If $i + j$ is odd, then $C_{ij} = -M_{ij}$. If $i + j$ is even, then $C_{ij} = M_{ij}$.

(c) $|A| = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

56. $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix} = (3x^2)(1) - 1(-3y^2) = 3x^2 + 3y^2$

62.
$$\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)[u^2v(1-w) + u^2vw] + u[uv^2(1-w) + uv^2w]$$

$$= (1-v)(u^2v) + u(uv^2)$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

64. Evaluating the left side yields

$$\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = cwz - cxy.$$

Evaluating the right side yields

$$c \begin{vmatrix} w & x \\ y & z \end{vmatrix} = c(wz - xy) = cwz - cxy.$$

66. Evaluating the left side yields

$$\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = cwx - cwx = 0.$$

68. Expand the left side of the equation along the first row.

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} &= 1 \begin{vmatrix} b & c \\ b^3 & c^3 \end{vmatrix} - 1 \begin{vmatrix} a & c \\ a^3 & c^3 \end{vmatrix} + 1 \begin{vmatrix} a & b \\ a^3 & b^3 \end{vmatrix} \\ &= bc^3 - b^3c - ac^3 + a^3c + ab^3 - a^3b \\ &= b(c^3 - a^3) + b^3(a - c) + ac(a^2 - c^2) \\ &= (c - a)[bc^2 + abc + ba^2 - b^3 - a^2c - ac^2] \\ &= (c - a)[c^2(b - a) + ac(b - a) + b(a - b)(a + b)] \\ &= (c - a)(b - a)[c^2 + ac - ab - b^2] \\ &= (c - a)(b - a)[(c - b)(c + b) + a(c - b)] \\ &= (c - a)(b - a)(c - b)(c + b + a) \\ &= (a - b)(b - c)(c - a)(a + b + c) \end{aligned}$$

70. Expanding along the first row, the determinant of a 4×4 matrix involves four 3×3 determinants. Each of these 3×3 determinants requires 6 triple products. So, there are $4(6) = 24$ quadruple products.

58.
$$\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = (e^{-x})(1-x)e^{-x} - (-e^{-x})(xe^{-x})$$

$$= (1-x)e^{-2x} + xe^{-2x}$$

$$= (1-x+x)(e^{-2x}) = e^{-2x}$$

60.
$$\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix} = x(1 + \ln x) - 1(x \ln x)$$

$$= x + x \ln x - x \ln x = x$$

Section 3.2 Determinants and Elementary Operations

2. Because the second row is a multiple of the first row, the determinant is zero.
4. Because the first and third rows are the same, the determinant is zero.
6. Because the first and third rows are interchanged, the sign of the determinant is changed.
8. Because 3 has been factored out of the third row, the first determinant is 3 times the second one.
10. Because 2 has been factored out of the second column and 3 factored out of the third column, the first determinant is 6 times the second one.
12. Because 6 has been factored out of each row, the first determinant is 6^4 times the second one.
14. Because a multiple of the first row was added to the second row to produce a new second row, the determinants are equal.
16. Because a multiple of the second column was added to the third column to produce a new third column, the determinants are equal.
18. Because the second and third rows are interchanged, the sign of the determinant is changed.
20. Because the fifth column is a multiple of the second column, the determinant is zero.
22. Expand by cofactors along the second row.

$$\begin{vmatrix} -1 & 3 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 2(1 - 2) = -2$$

A graphing utility or a software program gives the same determinant, -2 .

$$\begin{aligned} 24. \quad & \begin{vmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 2 & 0 \\ 4 & 1 & -1 & 0 \\ 3 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 2 & 0 \\ 4 & 1 & -1 & 0 \\ -1 & 0 & 2 & 0 \end{vmatrix} \\ & = \begin{vmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 2 & 0 \\ 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \end{aligned}$$

Because there is an entire row of zeros, the determinant is 0. A graphing utility or a software program gives the same determinant, 0.

$$\begin{aligned} 26. \quad & \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -3 & -2 \end{vmatrix} \\ & = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & 2 \end{vmatrix} = 1(-3)(2) = -6 \end{aligned}$$

$$28. \quad \begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -2 & 0 \end{vmatrix} = 0$$

$$\begin{aligned} 30. \quad & \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & -15 & 20 \end{vmatrix} \\ & = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & 8 \end{vmatrix} \\ & = 3(-5)(8) \\ & = -120 \end{aligned}$$

$$\begin{aligned} 32. \quad & \begin{vmatrix} 9 & -4 & 2 & 5 \\ 2 & 7 & 6 & -5 \\ 4 & 1 & -2 & 0 \\ 7 & 3 & 4 & 10 \end{vmatrix} = \begin{vmatrix} 9 & -4 & 2 & 5 \\ 11 & 3 & 8 & 0 \\ 4 & 1 & -2 & 0 \\ -11 & 11 & 0 & 0 \end{vmatrix} \\ & = \begin{vmatrix} 9 & -4 & 2 & 5 \\ 27 & 7 & 0 & 0 \\ 4 & 1 & -2 & 0 \\ -11 & 11 & 0 & 0 \end{vmatrix} \\ & = (-5) \begin{vmatrix} 27 & 7 & 0 \\ 4 & 1 & -2 \\ -11 & 11 & 0 \end{vmatrix} \\ & = (-5)(2) \begin{vmatrix} 27 & 7 \\ -11 & 11 \end{vmatrix} \\ & = (-10)(11) \begin{vmatrix} 27 & 7 \\ -1 & 1 \end{vmatrix} \\ & = (-110)(27 + 7) \\ & = -3740 \end{aligned}$$

34.
$$\begin{vmatrix} 0 & -4 & 9 & 3 \\ 9 & 2 & -2 & 7 \\ -5 & 7 & 0 & 11 \\ -8 & 0 & 0 & 16 \end{vmatrix} = (-8) \begin{vmatrix} 0 & -4 & 9 & 3 \\ 9 & 2 & -2 & 7 \\ -5 & 7 & 0 & 11 \\ 1 & 0 & 0 & -2 \end{vmatrix}$$

$$= (-8) \begin{vmatrix} 0 & -4 & 9 & 3 \\ 0 & 2 & -2 & 25 \\ 0 & 7 & 0 & 1 \\ 1 & 0 & 0 & -2 \end{vmatrix}$$

$$= -(-8)(1) \begin{vmatrix} -4 & 9 & 3 \\ 2 & -2 & 25 \\ 7 & 0 & 1 \end{vmatrix}$$

$$= (8)[8 + 1575 + 0 + 42 + 0 - 18]$$

$$= 12,856$$

36.
$$\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ -1 & 3 & -8 & -3 & 0 \\ 4 & 7 & -8 & 0 & 0 \\ -5 & 6 & -5 & -6 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 2 & 1 \\ -1 & 3 & -8 & -3 \\ 4 & 7 & -8 & 0 \\ -5 & 6 & -5 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 2 & 1 \\ -4 & 3 & -2 & 0 \\ 4 & 7 & -8 & 0 \\ -11 & 6 & 7 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} -4 & 3 & -2 \\ 4 & 7 & -8 \\ -11 & 6 & 7 \end{vmatrix}$$

$$= - \begin{vmatrix} 0 & 10 & -10 \\ 4 & 7 & -8 \\ -11 & 6 & 7 \end{vmatrix}$$

$$= -10 \begin{vmatrix} 0 & 1 & -1 \\ 4 & 7 & -8 \\ -11 & 6 & 7 \end{vmatrix}$$

$$= -10[(-1)(28 - 88) - 1(24 + 77)]$$

$$= 410$$

38. (a) False. Adding a multiple of one row to another does not change the value of the determinant.
 (b) True. See page 118.
 (c) True. In this case you can transform a matrix into a matrix with a row of zeros, which has zero determinant as can be seen by expanding by cofactors along that row. You achieve this transformation by adding a multiple of one row to another (which does not change the determinant of a matrix).

40.
$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

42.
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

44.
$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} 0 & -a & -a-c-ac \\ 0 & b & -c \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= ac - b(-a - c - ac)$$

$$= ac + ab + bc + abc$$

$$= \frac{abc(ac + ab + bc + abc)}{abc}$$

$$= abc\left(1 + \frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right)$$

46. (a)
$$\begin{vmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= -(1)(4)(-3)$$

$$= 12$$

(b)
$$\begin{vmatrix} a & 0 & 1 \\ 0 & c & 0 \\ b & 0 & -16 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 4 & 0 & -16 \end{vmatrix}$$

Expand by cofactors in the second row.

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 4 & 0 & -16 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 \\ 4 & -16 \end{vmatrix} = -3(-16 - 4) = 60$$

48. If B is obtained from A by multiplying a row of A by a nonzero constant c , then

$$\det(B) = \det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ ca_{i1} & \cdots & ca_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = ca_{i1}C_{i1} + \dots + ca_{in}C_{in} = c(a_{i1}C_{i1} + \dots + a_{in}C_{in}) = c\det(A).$$

Section 3.3 Properties of Determinants

2. (a) $|A| = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$

(b) $|B| = \begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = 5$

(c) $AB = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 26 & -3 \\ 23 & -4 \end{bmatrix}$

(d) $|AB| = \begin{vmatrix} 26 & -3 \\ 23 & -4 \end{vmatrix} = -35$

Notice that $|A||B| = (-7)(5) = -35 = |AB|$.

4. (a) $|A| = \begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{vmatrix} = 0$

(b) $|B| = \begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = -7$

(c) $AB = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 9 \\ 8 & -6 & 3 \\ 6 & -2 & 15 \end{bmatrix}$

(d) $|AB| = \begin{vmatrix} 7 & -4 & 9 \\ 8 & -6 & 3 \\ 6 & -2 & 15 \end{vmatrix} = 0$

Notice that $|A||B| = 0(-7) = 0 = |AB|$.

6. (a) $|A| = \begin{vmatrix} 2 & 4 & 7 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{vmatrix} = 7$

(b) $|B| = \begin{vmatrix} 6 & 1 & -1 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -13$

(c) $AB = \begin{bmatrix} 2 & 4 & 7 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 1 & -1 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 9 & 18 \\ 8 & -3 & -2 & -1 \\ 0 & 0 & 2 & 3 \\ 7 & -1 & -1 & 1 \end{bmatrix}$

(d) $|AB| = \begin{vmatrix} 8 & 10 & 9 & 18 \\ 8 & -3 & -2 & -1 \\ 0 & 0 & 2 & 3 \\ 7 & -1 & -1 & 1 \end{vmatrix} = -91$

Notice that $|A||B| = 7(-13) = -91 = |AB|$.

8. $|A| = \begin{vmatrix} 21 & 7 \\ 28 & -56 \end{vmatrix} = 7^2 \begin{vmatrix} 3 & 1 \\ 4 & -8 \end{vmatrix} = 49(-28) = -1372$

$$\begin{aligned} \mathbf{10.} \quad |A| &= \begin{vmatrix} 4 & 16 & 0 \\ 12 & -8 & 8 \\ 16 & 20 & -4 \end{vmatrix} = 4^3 \begin{vmatrix} 1 & 4 & 0 \\ 3 & -2 & 2 \\ 4 & 5 & -1 \end{vmatrix} \\ &= 4^3 \begin{vmatrix} 1 & 4 & 0 \\ 11 & 8 & 0 \\ 4 & 5 & -1 \end{vmatrix} \\ &= (-64)(-36) = 2304 \end{aligned}$$

$$\begin{aligned} \mathbf{12.} \quad |A| &= \begin{vmatrix} 40 & 25 & 10 \\ 30 & 5 & 20 \\ 15 & 35 & 45 \end{vmatrix} = 5^3 \begin{vmatrix} 8 & 5 & 2 \\ 6 & 1 & 4 \\ 3 & 7 & 9 \end{vmatrix} \\ &= 5^3 \begin{vmatrix} -22 & 0 & -18 \\ 6 & 1 & 4 \\ -39 & 0 & -19 \end{vmatrix} \\ &= 125(-284) = -35,500 \end{aligned}$$

$$\mathbf{14.} \quad |A| = \begin{vmatrix} 0 & 16 & -8 & -32 \\ -16 & 8 & -8 & 16 \\ 8 & -24 & 8 & -8 \\ -8 & 32 & 0 & 32 \end{vmatrix} = 8^4 \begin{vmatrix} 0 & 2 & -1 & -4 \\ -2 & 1 & -1 & 2 \\ 1 & -3 & 1 & -1 \\ -1 & 4 & 0 & 4 \end{vmatrix} = 4096(15) = 61,440$$

$$\mathbf{16. (a)} \quad |A| = \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$\mathbf{(b)} \quad |B| = \begin{vmatrix} 3 & -2 \\ 0 & 0 \end{vmatrix} = 0$$

$$\mathbf{(c)} \quad A + B = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{(d)} \quad |A + B| = \begin{vmatrix} 4 & -4 \\ 1 & 0 \end{vmatrix} = 4$$

Notice that $|A| + |B| = 2 + 0 = 2 \neq |A + B|$.

$$\mathbf{18. (a)} \quad |A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 5$$

$$\mathbf{(b)} \quad |B| = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2(2) = -4$$

$$\mathbf{(c)} \quad A + B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\mathbf{(d)} \quad |A + B| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix} = -2$$

Notice that $|A| + |B| = 5 + (-4) = 1 \neq |A + B|$.

20. Because

$$\begin{vmatrix} 3 & -6 \\ 4 & 2 \end{vmatrix} = 30 \neq 0,$$

the matrix is nonsingular.

22. Because

$$\begin{vmatrix} 14 & 5 & 7 \\ -15 & 0 & 3 \\ 1 & -5 & -10 \end{vmatrix} = 0,$$

the matrix is singular.

24. Because

$$\begin{vmatrix} 0.8 & 0.2 & -0.6 & 0.1 \\ -1.2 & 0.6 & 0.6 & 0 \\ 0.7 & -0.3 & 0.1 & 0 \\ 0.2 & -0.3 & 0.6 & 0 \end{vmatrix} = 0.015 \neq 0,$$

the matrix is nonsingular.

$$26. A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$|A^{-1}| = \frac{1}{3} \left(\frac{1}{6} \right) - \left(-\frac{1}{3} \right) \left(\frac{1}{3} \right) = \frac{1}{18} + \frac{1}{9} = \frac{1}{6}$$

Notice that $|A| = 6$.

$$\text{So, } |A^{-1}| = \frac{1}{|A|} = \frac{1}{6}.$$

$$28. A^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ 2 & -1 & 0 \\ \frac{3}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$|A^{-1}| = \begin{vmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ 2 & -1 & 0 \\ \frac{3}{2} & -1 & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{3}{2} & -1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\text{Notice that } |A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ -3 & 0 & -1 \end{vmatrix} = -2.$$

$$\text{So, } |A^{-1}| = \frac{1}{|A|} = -\frac{1}{2}.$$

$$30. A^{-1} = \begin{bmatrix} 2 & -3 & \frac{7}{2} & 4 \\ 1 & -3 & \frac{3}{2} & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -1 \end{bmatrix}$$

$$|A^{-1}| = \begin{vmatrix} 2 & -3 & \frac{7}{2} & 4 \\ 1 & -3 & \frac{3}{2} & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -1 \end{vmatrix} = \begin{vmatrix} 2 & -3 & \frac{7}{2} & 4 \\ 1 & -3 & \frac{3}{2} & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 2 & -3 & 4 \\ 1 & -3 & 3 \\ 0 & 1 & -1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 0 & 3 & -2 \\ 1 & -3 & 3 \\ 0 & 1 & -1 \end{vmatrix} = \frac{1}{2}$$

$$\text{Notice that } |A| = \begin{vmatrix} 0 & 1 & 0 & 3 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 2 & -2 \\ 1 & -2 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & -2 & -4 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{vmatrix} = 2.$$

$$\text{So, } |A^{-1}| = \frac{1}{|A|} = \frac{1}{2}.$$

32. The coefficient matrix of the system is

$$\begin{bmatrix} 3 & -4 \\ \frac{2}{3} & -\frac{8}{9} \end{bmatrix}$$

Because the determinant of this matrix is zero, the system does not have a unique solution.

34. The coefficient matrix of the system is

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & 2 \end{bmatrix}$$

Because the determinant of this matrix is zero, the system does not have a unique solution.

36. The coefficient matrix of the system is

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Because the determinant of this matrix is 8, and not zero, the system has a unique solution.

38. Find the values of k that make A singular by setting $|A| = 0$.

$$\begin{aligned} |A| &= \begin{vmatrix} k-1 & 2 \\ 2 & k+2 \end{vmatrix} \\ &= (k-1)(k+2) - 4 \\ &= k^2 + k - 6 \\ &= (k+3)(k-2) = 0 \end{aligned}$$

which implies that $k = -3$ or $k = 2$.

40. Find the values of k that make A singular by setting $|A| = 0$. Using the second column in the cofactor expansion, you have

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{vmatrix} = -k \begin{vmatrix} -2 & -k \\ 3 & -4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ -2 & -k \end{vmatrix} \\ &= -k(8+3k) - (-k+4) \\ &= -3k^2 - 7k - 4 \\ &= -(3k+4)(k+1). \end{aligned}$$

So, $|A| = 0$ implies that $k = -\frac{4}{3}$ or $k = -1$.

42. Find the value of k necessary to make A singular by setting $|A| = 0$.

$$|A| = \begin{vmatrix} k & -3 & -k \\ -2 & k & 1 \\ k & 1 & 0 \end{vmatrix} = -k^3 - 2k = 0$$

So, $k = 0$ or $k = \pm\sqrt{2}$.

44. First obtain $|A| = \begin{vmatrix} -4 & 10 \\ 5 & 6 \end{vmatrix} = -74$.

(a) $|A^T| = |A| = -74$

(b) $|A^2| = |A||A| = (-74)^2 = 5476$

(c) $|AA^T| = |A||A^T| = (-74)(-74) = 5476$

(d) $|2A| = 2^2|A| = 4(-74) = -296$

(e) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{(-74)} = -\frac{1}{74}$

46. First obtain $|A| = \begin{vmatrix} 1 & 5 & 4 \\ 0 & -6 & 2 \\ 0 & 0 & -3 \end{vmatrix} = 18$.

(a) $|A^T| = |A| = 18$

(b) $|A^2| = |A||A| = 18^2 = 324$

(c) $|AA^T| = |A||A^T| = (18)(18) = 324$

(d) $|2A| = 2^3|A| = 8(18) = 144$

(e) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{18}$

48. First observe $|A| = \begin{vmatrix} 4 & 1 & 9 \\ -1 & 0 & -2 \\ -3 & 3 & 0 \end{vmatrix} = 3$.

(a) $|A^T| = |A| = 3$

(b) $|A^2| = |A||A| = 9$

(c) $|AA^T| = |A||A^T| = 9$

(d) $|2A| = 2^3|A| = 24$

(e) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{3}$

50. First observe that $|A| = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = -12$.

- (a) $|A^T| = |A| = -12$
- (b) $|A^2| = |A||A| = 144$
- (c) $|AA^T| = |A||A^T| = 144$
- (d) $|2A| = 2^4|A| = -192$
- (e) $|A^{-1}| = \frac{1}{|A|} = -\frac{1}{12}$

52. (a) $|A| = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} = -16 - 24 = -40$

- (b) $|A^T| = |A| = -40$
- (c) $|A^2| = |A||A| = |A|^2 = 1600$
- (d) $|2A| = 2^2|A| = -160$
- (e) $|A^{-1}| = \frac{1}{|A|} = -\frac{1}{40}$

54. $|A| = \begin{vmatrix} \frac{3}{4} & \frac{2}{3} & -\frac{1}{4} \\ \frac{2}{3} & 1 & \frac{1}{3} \\ -\frac{1}{4} & \frac{1}{3} & \frac{3}{4} \end{vmatrix} = -\frac{1}{36}$.

- (a) $|A^T| = |A| = -\frac{1}{36}$
- (b) $|A^2| = |A||A| = \frac{1}{1296}$
- (c) $|2A| = 2^3|A| = -\frac{2}{9}$
- (d) $|A^{-1}| = \frac{1}{|A|} = -36$

62. Expand the determinant on the left

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = (a+b)((a+b)^2 - a^2) - a((a+b)a - a^2) + a(a^2 - a(a+b)) \\ = (a+b)(2ab + b^2) - a(ab) + a(-ab) \\ = 2a^2b + ab^2 + 2ab^2 + b^3 - 2a^2b \\ = b^2(3a + b).$$

64. Because the rows of A all add up to zero, you have

$$|A| = \begin{vmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -3 & 1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 0.$$

56. (a) $|A| = \begin{vmatrix} 6 & 5 & 1 & -1 \\ -2 & 4 & 3 & 5 \\ 6 & 1 & -4 & -2 \\ 2 & 2 & 1 & 3 \end{vmatrix} = -312$

- (b) $|A^T| = |A| = -312$
- (c) $|A^2| = |A||A| = |A|^2 = 97,344$
- (d) $|2A| = 2^4|A| = -4992$
- (e) $|A^{-1}| = \frac{1}{|A|} = -\frac{1}{312}$

58. (a) $|AB| = |A||B| = 4(5) = 20$

- (b) $|2A| = 2^3|A| = 8(4) = 32$
- (c) Because $|A| \neq 0$ and $|B| \neq 0$, A and B are nonsingular.
- (d) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}, |B^{-1}| = \frac{1}{|B|} = \frac{1}{5}$
- (e) $|(AB)^T| = |AB| = 20$

60. Given that AB is singular, then $|AB| = |A||B| = 0$. So, either $|A|$ or $|B|$ must be zero, which implies that either A or B is singular.

66. Calculating the determinant of A by expanding along the first row is equivalent to calculating the determinant of A^T by expanding along the first column. Because the determinant of a matrix can be found by expanding along any row or column, you see that $|A| = |A^T|$.

68. $|A^{10}| = |A|^{10} = 0 \Rightarrow |A| = 0 \Rightarrow A$ is singular.

70. If the order of A is odd, then $(-1)^n = -1$, and the result of Exercise 59 implies that $|A| = -|A|$ or $|A| = 0$.

72. (a) False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $\det(A) = \det(B) = 1 \neq 0 = \det(A + B)$

(b) True. Because $\det(A) = \det(B)$,

$$\begin{aligned} \det(AB) &= \det(A)\det(B) \\ &= \det(A)\det(A) \\ &= \det(AA) \\ &= \det(A^2). \end{aligned}$$

(c) True. See page 129 for “Equivalent Conditions for a Nonsingular Matrix” and Theorem 3.7 on page 128.

74. Because

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$A^{-1} \neq A^T$ and the matrix is not orthogonal.

76. Because

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = A^T,$$

this matrix is orthogonal.

78. Because

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = A^T,$$

this matrix is orthogonal.

80. If $A^T = A^{-1}$, then $|A^T| = |A^{-1}|$ and so

$$|I| = |AA^{-1}| = |A||A^{-1}| = |A||A^T| = |A|^2 = 1 \Rightarrow |A| = \pm 1.$$

$$82. A = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Using a graphing utility, you have

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = A^T.$$

Because $A^{-1} = A^T$, A is an orthogonal matrix.

For this given A , $|A| = 1$.

84. $|SB| = |S||B| = 0|B| = 0 \Rightarrow SB$ is singular.

Section 3.4 Applications of Determinants

2. The matrix of cofactors is

$$\begin{bmatrix} 4 & -0 \\ -0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

So, the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

Because $|A| = -4$, the inverse of A is

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = -\frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

4. The matrix of cofactors is

$$\begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ 2 & -4 & 2 \\ -5 & 1 & 1 \end{bmatrix}$$

So, the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 4 & 2 & -5 \\ -2 & -4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

Because $|A| = -6$, the inverse of A is

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{5}{6} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$

8. The matrix of cofactors is

$$\begin{bmatrix} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{bmatrix}$$

So, the adjoint of A is $\text{adj}(A) = \begin{bmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{bmatrix}$. Because $\det(A) = -3$, the inverse of A is

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

6. The matrix of cofactors is

$$\begin{bmatrix} \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

So, the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Because $\det(A) = 0$, the matrix A has no inverse.

10. The coefficient matrix is

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, \text{ where } |A| = 7.$$

Because $|A| \neq 0$, you can use Cramer's Rule.

$$A_1 = \begin{bmatrix} -10 & -1 \\ -1 & 2 \end{bmatrix}, \quad |A_1| = -21$$

$$A_2 = \begin{bmatrix} 2 & -10 \\ 3 & -1 \end{bmatrix}, \quad |A_2| = 28.$$

The solution is

$$x = \frac{|A_1|}{|A|} = \frac{-21}{7} = -3$$

$$y = \frac{|A_2|}{|A|} = \frac{28}{7} = 4.$$

12. The coefficient matrix is

$$A = \begin{bmatrix} 18 & 12 \\ 30 & 24 \end{bmatrix}, \text{ where } |A| = 72.$$

Because $|A| \neq 0$, you can use Cramer's Rule.

$$A_1 = \begin{bmatrix} 13 & 12 \\ 23 & 24 \end{bmatrix}, \quad |A_1| = 36$$

$$A_2 = \begin{bmatrix} 18 & 13 \\ 30 & 23 \end{bmatrix}, \quad |A_2| = 24$$

The solution is

$$x_1 = \frac{|A_1|}{|A|} = \frac{36}{72} = \frac{1}{2}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{24}{72} = \frac{1}{3}.$$

14. The coefficient matrix is

$$A = \begin{bmatrix} 13 & -6 \\ 26 & -12 \end{bmatrix}, \text{ where } |A| = 0.$$

Because $|A| = 0$, Cramer's Rule cannot be applied. (The system does not have a solution.)

16. The coefficient matrix is

$$A = \begin{bmatrix} -0.4 & 0.8 \\ 0.2 & 0.3 \end{bmatrix}, \text{ where } |A| = -0.28.$$

Because $|A| \neq 0$, you can use Cramer's Rule.

$$A_1 = \begin{bmatrix} 1.6 & 0.8 \\ 0.6 & 0.3 \end{bmatrix}, \quad |A_1| = 0$$

$$A_2 = \begin{bmatrix} -0.4 & 1.6 \\ 0.2 & 0.6 \end{bmatrix}, \quad |A_2| = -0.56$$

The solution is

$$x = \frac{|A_1|}{|A|} = \frac{0}{-0.28} = 0$$

$$y = \frac{|A_2|}{|A|} = \frac{-0.56}{-0.28} = 2.$$

18. The coefficient matrix is

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 5 \\ 8 & -5 & -2 \end{bmatrix}, \text{ where } |A| = -82.$$

Because $|A| \neq 0$, you can use Cramer's Rule.

$$A_1 = \begin{bmatrix} -2 & -2 & 3 \\ 16 & 2 & 5 \\ 4 & -5 & -2 \end{bmatrix}, \quad |A_1| = -410$$

$$A_2 = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 16 & 5 \\ 8 & 4 & -2 \end{bmatrix}, \quad |A_2| = -656$$

$$A_3 = \begin{bmatrix} 4 & -2 & -2 \\ 2 & 2 & 16 \\ 8 & -5 & 4 \end{bmatrix}, \quad |A_3| = 164$$

The solution is

$$x = \frac{|A_1|}{|A|} = \frac{-410}{-82} = 5$$

$$y = \frac{|A_2|}{|A|} = \frac{-656}{-82} = 8$$

$$z = \frac{|A_3|}{|A|} = \frac{164}{-82} = -2.$$

20. The coefficient matrix is

$$A = \begin{bmatrix} 14 & -21 & -7 \\ -4 & 2 & -2 \\ 56 & -21 & 7 \end{bmatrix}, \text{ where } |A| = 1568.$$

Because $|A| \neq 0$, you can use Cramer's Rule.

$$A_1 = \begin{bmatrix} -21 & -21 & -7 \\ 2 & 2 & -2 \\ 7 & -21 & 7 \end{bmatrix}, \quad |A_1| = 1568$$

$$A_2 = \begin{bmatrix} 14 & -21 & -7 \\ -4 & 2 & -2 \\ 56 & 7 & 7 \end{bmatrix}, \quad |A_2| = 3136$$

$$A_3 = \begin{bmatrix} 14 & -21 & -21 \\ -4 & 2 & 2 \\ 56 & -21 & 7 \end{bmatrix}, \quad |A_3| = -1568$$

The solution is

$$x_1 = \frac{|A_1|}{|A|} = \frac{1568}{1568} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{3136}{1568} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-1568}{1568} = -1.$$

22. The coefficient matrix is

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}, \text{ where } |A| = 0.$$

Because $|A| = 0$, Cramer's Rule cannot be applied.

(The system does not have a solution.)

24. The coefficient matrix is

$$A = \begin{bmatrix} -8 & 7 & -10 \\ 12 & 3 & -5 \\ 15 & -9 & 2 \end{bmatrix}.$$

$$A_1 = \begin{bmatrix} -151 & 7 & -10 \\ 86 & 3 & -5 \\ 187 & -9 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -8 & -151 & -10 \\ 12 & 86 & -5 \\ 15 & 187 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -8 & 7 & -151 \\ 12 & 3 & 86 \\ 15 & -9 & 187 \end{bmatrix}$$

Using a graphing utility, $|A| = 1149$, $|A_1| = 11,490$,

$|A_2| = -3447$, and $|A_3| = 5745$.

$$\text{So, } x_1 = \frac{|A_1|}{|A|} = \frac{11,490}{1149} = 10,$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-3447}{1149} = -3, \text{ and } x_3 = \frac{|A_3|}{|A|} = \frac{5745}{1149} = 5.$$

26. The coefficient matrix is

$$A = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 3 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ -2 & -3 & -3 & 0 \end{bmatrix}.$$

$$A_1 = \begin{bmatrix} -8 & -1 & 0 & 1 \\ 24 & 5 & 5 & 0 \\ -6 & 0 & 2 & 1 \\ -15 & -3 & -3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -8 & 0 & 1 \\ 3 & 24 & 5 & 0 \\ 0 & -6 & 2 & 1 \\ -2 & -15 & -3 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & -1 & -8 & 1 \\ 3 & 5 & 24 & 0 \\ 0 & 0 & -6 & 1 \\ -2 & -3 & -15 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & -1 & 0 & -8 \\ 3 & 5 & 5 & 24 \\ 0 & 0 & 2 & -6 \\ -2 & -3 & -3 & -15 \end{bmatrix}$$

Using a graphing utility, $|A| = 1$, $|A_1| = 3$, $|A_2| = 7$, $|A_3| = -4$, and $|A_4| = 2$.

$$\text{So, } x_1 = \frac{|A_1|}{|A|} = \frac{3}{1} = 3, \quad x_2 = \frac{|A_2|}{|A|} = \frac{7}{1} = 7,$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-4}{1} = -4 \text{ and } x_4 = \frac{|A_4|}{|A|} = \frac{2}{1} = 2.$$

28. Draw the altitude from vertex C to side c , then from trigonometry

$$c = a \cos B + b \cos A.$$

Similarly, the other two equations follow by using the other altitudes. Now use Cramer's Rule to solve for $\cos C$ in this system of three equations.

$$\cos C = \frac{\begin{vmatrix} 0 & c & a \\ c & 0 & b \\ b & a & c \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{-c(c^2 - b^2) + a(ac)}{-c(-ba) + b(ac)} = \frac{a^2 + b^2 - c^2}{2ab}.$$

Solving for c^2 you obtain

$$2ab \cos C = a^2 + b^2 - c^2$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

30. Use the formula for area as follows.

$$\text{Area} = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 4 & 2 & 1 \end{vmatrix} = \pm \frac{1}{2}(-8) = 4.$$

32. Use the formula for area as follows.

$$\text{Area} = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \pm \frac{1}{2}(6) = 3$$

34. Use the fact that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} = 2$$

to determine that the three points are not collinear.

36. Use the fact that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -3 & 1 \\ -4 & 7 & 1 \\ 2 & -13 & 1 \end{vmatrix} = 0$$

to determine that the three points are collinear.

38. Find an equation as follows.

$$0 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ -4 & 7 & 1 \\ 2 & 4 & 1 \end{vmatrix} = 3x + 6y - 30$$

So, an equation of the line is $2y + x = 10$.

46. Use the formula for volume as follows.

$$\text{Volume} = \pm \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \pm \frac{1}{6} \begin{vmatrix} 5 & 4 & -3 & 1 \\ 4 & -6 & -4 & 1 \\ -6 & -6 & -5 & 1 \\ 0 & 0 & 10 & 1 \end{vmatrix} = \pm \frac{1}{6}(1386) = 231$$

48. Use the fact that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -2 & -5 & 1 \\ 2 & 6 & 11 & 1 \end{vmatrix} = 0$$

to determine that the four points are coplanar.

50. Use the fact that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 7 & 1 \\ -3 & 6 & 6 & 1 \\ 4 & 4 & 2 & 1 \\ 3 & 3 & 4 & 1 \end{vmatrix} = -1$$

to determine that the four points are not coplanar.

40. Find an equation as follows.

$$0 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ 1 & 4 & 1 \\ 3 & 4 & 1 \end{vmatrix} = 2y - 8$$

So, an equation of the line is $y = 4$.

42. Use the formula for volume as follows.

$$\begin{aligned} \text{Volume} &= \pm \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \\ &= \pm \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & 1 \end{vmatrix} = \frac{1}{6}(3) = \frac{1}{2} \end{aligned}$$

44. Use the formula for volume as follows.

$$\begin{aligned} \text{Volume} &= \pm \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \\ &= \pm \frac{1}{6} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 1 & 1 & 4 & 1 \end{vmatrix} = \pm \frac{1}{6}(24) = 4 \end{aligned}$$

52. Use the fact that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 9 & 1 \\ -1 & -5 & 9 & 1 \\ 1 & -5 & -9 & 1 \\ -1 & -5 & -9 & 1 \end{vmatrix} = 0$$

to determine that the four points are coplanar.

54. Find an equation as follows.

$$\begin{aligned}
 0 &= \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{vmatrix} \\
 &= x \begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} - y \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} + z \begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{vmatrix} \\
 &= 4x - 2y - 2z - 2, \quad \text{or} \quad 2x - y - z = 1
 \end{aligned}$$

56. Find an equation as follows.

$$\begin{aligned}
 0 &= \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z & 1 \\ 1 & 2 & 7 & 1 \\ 4 & 4 & 2 & 1 \\ 3 & 3 & 4 & 1 \end{vmatrix} \\
 &= x \begin{vmatrix} 2 & 7 & 1 \\ 4 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 7 & 1 \\ 4 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & 2 & 1 \\ 4 & 4 & 1 \\ 3 & 3 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 7 \\ 4 & 4 & 2 \\ 3 & 3 & 4 \end{vmatrix} \\
 &= -x - y - z + 10, \quad \text{or} \quad x + y + z = 10
 \end{aligned}$$

58. Find an equation as follows.

$$\begin{aligned}
 0 &= \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z & 1 \\ 3 & 2 & -2 & 1 \\ 3 & -2 & 2 & 1 \\ -3 & -2 & -2 & 1 \end{vmatrix} \\
 &= x \begin{vmatrix} 2 & -2 & 1 \\ -2 & 2 & 1 \\ -2 & -2 & 1 \end{vmatrix} - y \begin{vmatrix} 3 & -2 & 1 \\ 3 & 2 & 1 \\ -3 & -2 & 1 \end{vmatrix} + z \begin{vmatrix} 3 & 2 & 1 \\ 3 & -2 & 1 \\ -3 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & -2 \\ 3 & -2 & 2 \\ -3 & -2 & -2 \end{vmatrix} \\
 &= 16x - 24y - 24z - 48, \quad \text{or} \quad 2x - 3y - 3z = 6
 \end{aligned}$$

60. Cramer's Rule was used correctly.

62. Given the system of linear equations,

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, then the lines must be parallel or coinciding.

64. Following the proof of Theorem 3.10, you have

$$A \operatorname{adj}(A) = |A|I.$$

Now, if A is not invertible, then $|A| = 0$, and $A \operatorname{adj}(A)$ is the zero matrix.

$$\begin{aligned}
 \text{66. } \operatorname{adj}(\operatorname{adj}(A)) &= \operatorname{adj}(|A|A^{-1}) \\
 &= \det(|A|A^{-1})(|A|A^{-1})^{-1} \\
 &= |A|^n |A^{-1}| \frac{1}{|A|} A = |A|^{n-2} A
 \end{aligned}$$

68. Answers will vary. Sample answer:

$$\text{Let } A = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}.$$

$$\operatorname{adj}(A) = \begin{bmatrix} 2 & -1 \\ -3 & -1 \end{bmatrix} \Rightarrow \operatorname{adj}(\operatorname{adj}(A)) = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = |A| \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\text{So, } \operatorname{adj}(\operatorname{adj}(A)) = |A|^{n-2} A.$$

70. Answers will vary. Sample answer:

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{adj}(A^{-1}) = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \Rightarrow (\text{adj}(A))^{-1} = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$$

$$\text{So, } \text{adj}(A^{-1}) = [\text{adj}(A)]^{-1}$$

Review Exercises for Chapter 3

- 2.** Using the formula for the determinant of a 2×2 matrix, you have

$$\begin{vmatrix} 0 & -3 \\ 1 & 2 \end{vmatrix} = 0(2) - (1)(-3) = 3$$

- 4.** Using the formula for the determinant of a 2×2 matrix, you have

$$\begin{vmatrix} -2 & 0 \\ 0 & 3 \end{vmatrix} = (-2)(3) - (0)(0) = -6$$

$$\begin{aligned} \mathbf{10.} \quad & \begin{vmatrix} -15 & 0 & 3 \\ 3 & 9 & -6 \\ 12 & -3 & 6 \end{vmatrix} = 3^3 \begin{vmatrix} -5 & 0 & 1 \\ 1 & 3 & -2 \\ 4 & -1 & 2 \end{vmatrix} = 27 \begin{vmatrix} -5 & 0 & 1 \\ -9 & 3 & 0 \\ 14 & -1 & 0 \end{vmatrix} = 27 \begin{vmatrix} -9 & 3 \\ 14 & -1 \end{vmatrix} \\ & = 27(9 - 42) \\ & = -891 \end{aligned}$$

- 12.** The determinant of a triangular matrix is the product of its diagonal entries. So, the determinant equals $2(1)(3)(-1) = -6$.

$$\begin{aligned} \mathbf{14.} \quad & \begin{vmatrix} 3 & -1 & 2 & 1 \\ -2 & 0 & 1 & -3 \\ -1 & 2 & -3 & 4 \\ -2 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 & 1 \\ -2 & 0 & 1 & -3 \\ 5 & 0 & 1 & 6 \\ 1 & 0 & 0 & 2 \end{vmatrix} \\ & = -(-1) \begin{vmatrix} -2 & 1 & -3 \\ 5 & 1 & 6 \\ 1 & 0 & 2 \end{vmatrix} \\ & = 1 \begin{vmatrix} 1 & -3 \\ 1 & 6 \end{vmatrix} + 2 \begin{vmatrix} -2 & 1 \\ 5 & 1 \end{vmatrix} \\ & = 9 + 2(-7) = -5 \end{aligned}$$

- 6.** The determinant of a triangular matrix is the product of the entries along the main diagonal.

$$\begin{vmatrix} 5 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 5(-1)(1) = -5$$

- 8.** Because the matrix has a column of zeros, the determinant is 0.

$$\begin{aligned} \mathbf{16.} \quad & \begin{vmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 3 & -1 & 2 & -2 \\ 1 & 2 & 0 & 1 & -1 \\ 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & -1 & 1 & -4 & -10 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & -2 & 3 & -4 & -4 \\ 0 & -1 & 1 & 0 & 2 \end{vmatrix} \\ & = - \begin{vmatrix} 1 & -1 & 4 & 10 \\ 0 & 1 & -2 & -5 \\ 0 & 1 & 4 & 16 \\ 0 & 0 & 4 & 12 \end{vmatrix} \\ & = - \begin{vmatrix} 1 & -2 & -5 \\ 0 & 6 & 21 \\ 0 & 4 & 12 \end{vmatrix} \\ & = -(72 - 84) = 12 \end{aligned}$$

$$18. \begin{vmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{vmatrix}$$

$$= 3(-3) \begin{vmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{vmatrix}$$

$$= -9(3) \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = (-27)(-9) = 243$$

20. Because the second and third columns are interchanged, the sign of the determinant is changed.

22. Because a multiple of the first row of the matrix on the left was added to the second row to produce the matrix on the right, the determinants are equal.

26. First find

$$|A| = \begin{vmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{vmatrix} = -1.$$

- (a) $|A^T| = |A| = -1$
 (b) $|A^3| = |A|^3 = (-1)^3 = -1$
 (c) $|A^T A| = |A^T| |A| = (-1)(-1) = 1$
 (d) $|5A| = 5^3 |A| = 125(-1) = -125$

$$28. (a) |A| = \begin{vmatrix} -2 & 1 & 3 \\ 2 & 0 & 4 \\ -1 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -9 & 3 \\ 0 & 10 & 4 \\ -1 & 5 & 0 \end{vmatrix} = (-1) \begin{vmatrix} -9 & 3 \\ 10 & 4 \end{vmatrix} = 66$$

$$(b) |A^{-1}| = \frac{1}{|A|} = \frac{1}{66}$$

$$30. A^{-1} = \frac{1}{74} \begin{bmatrix} 7 & -2 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} \frac{7}{74} & -\frac{1}{37} \\ \frac{1}{37} & \frac{5}{37} \end{bmatrix}$$

$$\begin{aligned} |A^{-1}| &= \frac{7}{74} \left(\frac{5}{37} \right) - \left(\frac{1}{37} \right) \left(-\frac{1}{37} \right) \\ &= \frac{35}{2738} + \frac{1}{1369} = \frac{1}{74} \end{aligned}$$

Notice that $|A| = 74$.

$$\text{So, } |A^{-1}| = \frac{1}{|A|} = \frac{1}{74}.$$

$$24. (a) |A| = \begin{vmatrix} 0 & 1 & 2 \\ 5 & 4 & 3 \\ 7 & 6 & 8 \end{vmatrix} = -15$$

$$(b) |B| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 3 & -2 \end{vmatrix} = 12$$

$$(c) AB = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 4 & 3 \\ 7 & 6 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -4 \\ 14 & 10 & 4 \\ 20 & 25 & -2 \end{bmatrix}$$

$$(d) |AB| = \begin{vmatrix} 1 & 5 & -4 \\ 14 & 10 & 4 \\ 20 & 25 & -2 \end{vmatrix} = -180$$

Notice that $|A||B| = (-15)(12) = -180 = |AB|$

$$32. A^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{2}{3} & \frac{1}{6} & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\left| A^{-1} \right| = \begin{vmatrix} -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{2}{3} & \frac{1}{6} & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -\frac{2}{3} & \frac{1}{6} & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{vmatrix} = -\frac{1}{12}$$

Notice that

$$\left| A \right| = \begin{vmatrix} -1 & 1 & 2 \\ 2 & 4 & 8 \\ 1 & -1 & 0 \end{vmatrix} = 1(8 - 8) - (-1)(-8 - 4) = -12.$$

$$\text{So, } \left| A^{-1} \right| = \frac{1}{\left| A \right|} = -\frac{1}{12}.$$

$$34. (\text{a}) \begin{bmatrix} 1 & 2 & 1 & 4 \\ -3 & 1 & -2 & 1 \\ 2 & 3 & -1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 7 & 1 & 13 \\ 0 & -1 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 7 & 1 & 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -20 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So, $x_3 = -1$, $x_2 = -1 - 3(-1) = 2$, and $x_1 = 4 - (-1) - 2(2) = 1$.

$$(\text{b}) \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 7 & 1 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(c) The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & -2 \\ 2 & 3 & -1 \end{bmatrix}, \text{ where } \left| A \right| = -20.$$

$$A_1 = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & -2 \\ 9 & 3 & -1 \end{bmatrix} \text{ and } \left| A_1 \right| = -20$$

$$A_2 = \begin{bmatrix} 1 & 4 & 1 \\ -3 & 1 & -2 \\ 2 & 9 & -1 \end{bmatrix} \text{ and } \left| A_2 \right| = -40$$

$$A_3 = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 1 & 1 \\ 2 & 3 & 9 \end{bmatrix} \text{ and } \left| A_3 \right| = 20$$

$$\text{So, } x_1 = \frac{-20}{-20} = 1, x_2 = \frac{-40}{-20} = 2, \text{ and } x_3 = \frac{20}{-20} = -1.$$

36. (a) $\begin{bmatrix} 2 & 3 & 5 & 4 \\ 3 & 5 & 9 & 7 \\ 5 & 9 & 13 & 17 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 3 & 5 & 9 & 7 \\ 5 & 9 & 13 & 17 \end{bmatrix}$

$$\begin{aligned} &\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} & 7 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{aligned}$$

So, $x_3 = -1$, $x_2 = 2 - 3(-1) = 5$, and $x_1 = 2 - \frac{5}{2}(-1) - \frac{3}{2}(5) = -3$.

(b) $\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$$\begin{aligned} &\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{aligned}$$

So, $x_1 = -3$, $x_2 = 5$, and $x_3 = -1$.

(c) The coefficient matrix is

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 13 \end{bmatrix} \text{ and } |A| = -4.$$

$$\text{Also, } A_1 = \begin{bmatrix} 4 & 3 & 5 \\ 7 & 5 & 9 \\ 17 & 9 & 13 \end{bmatrix} \text{ and } |A_1| = 12,$$

$$A_2 = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 7 & 9 \\ 5 & 17 & 13 \end{bmatrix} \text{ and } |A_2| = -20,$$

$$A_3 = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 5 & 9 & 17 \end{bmatrix} \text{ and } |A_3| = 4.$$

So, $x_1 = \frac{12}{-4} = -3$, $x_2 = \frac{-20}{-4} = 5$, and $x_3 = \frac{4}{-4} = -1$.

38. Because the determinant of the coefficient matrix is

$$\begin{vmatrix} 2 & -5 \\ 3 & -7 \end{vmatrix} = 1 \neq 0,$$

the system has a unique solution.

40. Because the determinant of the coefficient matrix is

$$\begin{vmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 8 & 6 & 0 \end{vmatrix} = 0,$$

the system does not have a unique solution.

42. Because the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 5 & 3 & 0 & 0 \\ 4 & 2 & 5 & 0 & 0 \\ 0 & 0 & 3 & 8 & 6 \\ 2 & 4 & 0 & 0 & -2 \\ 2 & 0 & -1 & 0 & 0 \end{vmatrix} = -896 \neq 0,$$

the system has a unique solution.

44. (a) $|BA| = |B||A| = 5(-2) = -10$

$$(b) |B^4| = |B|^4 = 5^4 = 625$$

$$(c) |2A| = 2^3|A| = 2^3(-2) = -16$$

$$(d) |(AB)^T| = |AB| = |A||B| = -10$$

$$(e) |B^{-1}| = \frac{1}{|A|} = \frac{1}{5}$$

$$46. \begin{vmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 5 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$10 = 5 + 5$$

$$48. \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = \begin{vmatrix} 0 & 1-a^2 & 1-a & 1-a \\ 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & 0 \\ 0 & 1-a & 0 & a-1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1-a^2 & 1-a & 1-a \\ 1-a & a-1 & 0 \\ 1-a & 0 & a-1 \end{vmatrix}$$

$$= (1-a)^3 \begin{vmatrix} 1+a & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \quad (\text{factoring out } (1-a) \text{ from each row})$$

$$= (1-a)^3 (1(1) - 1(-1-a-1)) \quad (\text{expanding along the third row})$$

$$= (1-a)^3 (a+3)$$

$$50. J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$52. J(u, v) = \begin{vmatrix} e^u \sin v & e^u \cos v \\ e^u \cos v & -e^u \sin v \end{vmatrix} = -e^{2u} \sin^2 v - e^{2u} \cos^2 v = -e^{2u}$$

$$54. J(u, v, w) = \begin{vmatrix} 1 & -1 & 1 \\ 2v & 2u & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2u) + 1(2v) + 1(2v - 2u) = 4v$$

58. Because $|B| \neq 0$, B^{-1} exists, and you can let
 $C = AB^{-1}$, then

$$A = CB \quad \text{and} \quad |C| = |AB^{-1}| = |A||B^{-1}| = |A| \frac{1}{|B|} = 1.$$

56. Use the information given in the table on page 122.

Cofactor expansion would cost:

$$(3,628,799)(0.001) + (6,235,300)(0.003) = \$22,334.70.$$

Row reduction would cost much less:

$$(285)(0.001) + 339(0.003) = \$1.30.$$

60. The matrix of cofactors is given by

$$\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix}.$$

So, the adjoint is

$$\text{adj} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

62. The determinant of the coefficient matrix is

$$\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5 \neq 0.$$

So, the system has a unique solution. Using Cramer's Rule,

$$A_1 = \begin{bmatrix} 0.3 & 1 \\ -1.3 & -1 \end{bmatrix}, |A_1| = 1.0$$

$$A_2 = \begin{bmatrix} 2 & 0.3 \\ 3 & -1.3 \end{bmatrix}, |A_2| = -3.5.$$

So,

$$x = \frac{|A_1|}{|A|} = \frac{1}{-5} = -0.2$$

$$y = \frac{|A_2|}{|A|} = \frac{-3.5}{-5} = 0.7.$$

64. The determinant of the coefficient matrix is

$$\begin{vmatrix} 4 & 4 & 4 \\ 4 & -2 & -8 \\ 8 & 2 & -4 \end{vmatrix} = 0.$$

So, Cramer's Rule does not apply. (The system does not have a solution.)

66. The coefficient matrix is $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -2 & 6 \end{bmatrix}$.

$$A_1 = \begin{bmatrix} -5 & -1 & 1 \\ 10 & 2 & 3 \\ 1 & -2 & 6 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 10 & 3 \\ 5 & 1 & 6 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 4 & -1 & -5 \\ 2 & 2 & 10 \\ 5 & -2 & 1 \end{bmatrix}$$

Using a graphing utility, $|A| = 55$, $|A_1| = -55$, $|A_2| = 165$, and $|A_3| = 110$.

So, $x_1 = |A_1|/|A| = -1$, $x_2 = |A_2|/|A| = 3$, and $x_3 = |A_3|/|A| = 2$.

68. Use the formula for area as follows.

$$\begin{aligned} \text{Area} &= \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm \frac{1}{2} \begin{vmatrix} -4 & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 6 & 1 \end{vmatrix} \\ &= \pm \frac{1}{2}(-6)(-4 - 4) = 24 \end{aligned}$$

70. Find an equation as follows.

$$0 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 2 \begin{vmatrix} x & y & 1 \\ 6 & -1 & 1 \end{vmatrix} = x(6) - y(-4) - 32$$

So, an equation of the line is $2y + 3x = 16$.

72. Find an equation as follows.

$$\begin{aligned} 0 &= \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x & y & z & 1 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 1 & 1 \\ -3 & 2 & 5 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x & y & z \\ 2 & -1 & 1 \\ -3 & 2 & 5 \end{vmatrix} \\ &= x(-7) - y(13) + z(1) = 0. \end{aligned}$$

So, the equation of the plane is $7x + 13y - z = 0$.

74. (a) $a + b + c = 1765$

$$4a + 2b + c = 1855$$

$$9a + 3b + c = 1920$$

(b) The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}, \text{ where } |A| = -2.$$

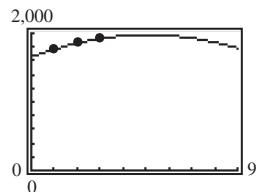
$$A_1 = \begin{bmatrix} 1765 & 1 & 1 \\ 1855 & 2 & 1 \\ 1920 & 3 & 1 \end{bmatrix} \text{ and } |A_1| = 25$$

$$A_2 = \begin{bmatrix} 1 & 1765 & 1 \\ 4 & 1855 & 1 \\ 9 & 1920 & 1 \end{bmatrix} \text{ and } |A_2| = -255$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1265 \\ 4 & 2 & 1855 \\ 9 & 3 & 1920 \end{bmatrix} \text{ and } |A_3| = -3300$$

$$\text{So, } a = \frac{25}{-2} = -12.5, b = \frac{-255}{-2} = 127.5, \text{ and } c = \frac{-3300}{-2} = 1650.$$

(c) $y = -12.5t^2 + 127.5t + 1650$



(d) The function fits the data exactly.

76. (a) True. If either A or B is singular, then $\det(A)$ or $\det(B)$ is zero (Theorem 3.7), but then

$$\det(AB) = \det(A)\det(B) = 0 \neq -1, \text{ which leads to a contradiction.}$$

(b) False. $\det(2A) = 2^3\det(A) = 8 \cdot 5 = 40 \neq 10$.

(c) False. Let A and B be the 3×3 identity matrix I_3 . Then $\det(A) = \det(B) = \det(I_3) = 1$, but

$$\det(A + B) = \det(2I_3) = 2^3 \cdot 1 = 8 \text{ while } \det(A) + \det(B) = 1 + 1 = 2.$$

78. (a) False. The *transpose* of the matrix of cofactors of A is called the adjoint matrix of A .

(b) False. Cramer's Rule requires the determinant of this matrix to be in the *numerator*. The denominator is always $\det(A)$, where A is the coefficient matrix of the system (assuming, of course, that it is nonsingular).

Project Solutions for Chapter 3

1 Stochastic Matrices

$$1. P\mathbf{x}_1 = P \begin{bmatrix} 7 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 4 \end{bmatrix}$$

$$P\mathbf{x}_2 = P \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -.65 \\ .65 \end{bmatrix}$$

$$P\mathbf{x}_3 = P \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.1 \\ .55 \\ .55 \end{bmatrix}$$

$$2. S = \begin{bmatrix} 7 & 0 & -2 \\ 10 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix} \quad S^{-1}PS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .65 & 0 \\ 0 & 0 & .55 \end{bmatrix} = D$$

The entries along D are the corresponding eigenvalues of P .

$$3. S^{-1}PS = D \Rightarrow PS = SD \Rightarrow P = SDS^{-1}. \text{ Then}$$

$$P^n = (SDS^{-1})^n = (SDS^{-1})(SDS^{-1}) \cdots (SDS^{-1}) = SDS^nS^{-1}.$$

$$\text{For } n = 15, D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (0.65)^{15} & 0 \\ 0 & 0 & (0.65)^{15} \end{bmatrix} \Rightarrow P^{15} = SDS^{15}S^{-1} \approx \begin{bmatrix} 0.333 & 0.333 & 0.333 \\ 0.476 & 0.477 & 0.475 \\ 0.191 & 0.190 & 0.192 \end{bmatrix} \Rightarrow P^{15}\mathbf{x} \approx \begin{bmatrix} 0.333 \\ 0.476 \\ 0.191 \end{bmatrix}.$$

2 The Cayley-Hamilton Theorem

$$1. |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda + 1 \end{vmatrix} = \lambda^2 - \lambda - 6$$

$$A^2 - A - 6I = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2. |\lambda I - A| = \begin{vmatrix} \lambda - 6 & 0 & -4 \\ 2 & \lambda - 1 & -3 \\ -2 & 0 & \lambda - 4 \end{vmatrix} = \lambda^3 - 11\lambda^2 + 26\lambda - 16$$

$$A^3 - 11A^2 + 26A - 16I = \begin{bmatrix} 344 & 0 & 336 \\ -36 & 1 & -1 \\ 168 & 0 & 176 \end{bmatrix} - 11 \begin{bmatrix} 44 & 0 & 40 \\ -8 & 1 & 7 \\ 20 & 0 & 24 \end{bmatrix} + 26 \begin{bmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix} - 16 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$3. |\lambda I - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc)$$

$$A^2 - (a + d)A + (ad - bc)I = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} - (a + d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4. $\left(\frac{1}{c_0}\right)(-A^{n-1} - c_{n-1}A^{n-2} - \dots - c_2A - c_1I)A = \frac{1}{c_0}(-A^n - c_{n-1}A^{n-1} - \dots - c_2A^2 - c_1A) = \frac{1}{c_0}(c_0I) = I$

Because $c_0I = -A^n - c_{n-1}A^{n-1} - \dots - c_2A^2 - c_1A$ from the equation, $p(A) = 0$.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 5 \end{vmatrix} = \lambda^2 - 6\lambda - 1$$

$$A^{-1} = \frac{1}{(-1)}(-A + 6I) = A - 6I = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

5. (a) Because $A^2 = 2A + I$ you have $A^3 = 2A^2 + A = 2(2A + I) + A = 5A + 2I$.

$$A^3 = 5 \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & -5 \\ 10 & -3 \end{bmatrix}$$

Similarly, $A^4 = 2A^3 + A^2 = 2(5A + 2I) + (2A + I) = 12A + 5I$. Therefore,

$$A^4 = 12 \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -12 \\ 24 & -7 \end{bmatrix}.$$

Note: This approach is a lot more efficient because you can calculate A^n without calculating all the previous powers of A .

(b) First, calculate the characteristic polynomial of A .

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & -1 \\ -2 & \lambda - 2 & 1 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = \lambda^3 - 4\lambda^2 + 3\lambda + 2.$$

By the Cayley-Hamilton Theorem, $A^3 - 4A^2 + 3A + 2I = 0$ or $A^3 = 4A^2 - 3A - 2I$. Now you can write any positive power A^n as a linear combination of A^2 , A and I . For example,

$$A^4 = 4A^3 - 3A^2 - 2A = 4(4A^2 - 3A - 2I) - 3A^2 - 2A = 13A^2 - 14A - 8I,$$

$$A^5 = 4A^4 - 3A^3 - 2A^2 = 4(13A^2 - 14A - 8I) - 3(4A^2 - 3A - 2I) - 2A^2 = 38A^2 - 47A - 26I.$$

Here

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}, \quad A^2 = AA = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & -2 \\ 2 & 0 & 5 \end{bmatrix}.$$

With this method, you can calculate A^5 directly without calculating A^3 and A^4 first.

$$A^5 = 38A^2 - 47A - 26I = 38 \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & -2 \\ 2 & 0 & 5 \end{bmatrix} - 47 \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} - 26 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 29 \\ 20 & 32 & -29 \\ 29 & 0 & 70 \end{bmatrix}$$

Similarly,

$$A^4 = 13A^2 - 14A - 8I = 13 \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & -2 \\ 2 & 0 & 5 \end{bmatrix} - 14 \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} - 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 12 \\ 11 & 16 & -12 \\ 12 & 0 & 29 \end{bmatrix}$$

$$A^3 = 4A^2 - 3A - 2I = \begin{bmatrix} 2 & 0 & 5 \\ 6 & 8 & -5 \\ 5 & 0 & 12 \end{bmatrix}.$$