CHAPTER 6
Linear Transformations

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Section 6.1 Introduction to Linear Transformations

2. (a) The image of **v** is

$$T(0, 4) = (0, 2(4) - 0, 4) = (0, 8, 4).$$

(b) If $T(v_1, v_2) = (v_1, 2v_2 - v_1, v_2) = (2, 4, 3)$, then

$$\begin{aligned} v_1 &= 2 \\ 2v_2 - v_1 &= 4 \\ v_2 &= 3 \end{aligned}$$

which implies that $v_1 = 2$ and $v_2 = 3$. So, the preimage of $\mathbf{w}$ is $(2, 3)$.

4. (a) The image of **v** is

$$T(2, 3, 0) = (3 - 2, 2 + 3, 2(2)) = (1, 5, 4).$$

(b) If $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1) = (-11, -1, 10)$,

then

$$\begin{aligned} v_2 - v_1 &= -11 \\ v_1 + v_2 &= -1 \\ 2v_1 &= 10 \end{aligned}$$

which implies that $v_1 = 5$ and $v_2 = -6$. So, the preimage of **w** is $\{(5, -6, t) : t \text{ is any real number}\}$.

6. (a) The image of **v** is

$$T(2, 1, 4) = (2(2) + 1, 2 - 1) = (5, 1).$$

(b) If $T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2) = (-1, 2)$, then

$$\begin{aligned} 2v_1 + v_2 &= -1 \\ v_1 - v_2 &= 2, \end{aligned}$$

which implies that $v_1 = \frac{1}{3}$, $v_2 = -\frac{5}{3}$, and $v_3 = t$, where t is any real number. So, the preimage of **w** is $\left\{\left(\frac{1}{3}, -\frac{5}{3}, t\right) : t \text{ is any real number}\right\}$.

8. (a) The image of **v** is

$$T(2, 4) = \left(\frac{\sqrt{3}}{2}(2) - \frac{1}{2}(4), 2 - 4, 4\right) = \left(\sqrt{3} - 2, -2, 4\right).$$

(b) If $T(v_1, v_2) = \left(\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2, v_1 - v_2, v_2\right) = \left(\sqrt{3}, 2, 0\right)$,

then

$$\begin{aligned} \frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2 &= \sqrt{3} \\ v_1 - v_2 &= 2 \\ v_2 &= 0 \end{aligned}$$

which implies that $v_1 = 2$ and $v_2 = 0$. So, the preimage of **w** is $(2, 0)$.

10. T is *not* a linear transformation because it does not preserve addition nor scalar multiplication.

For example,

$$\begin{aligned} T(1, 1) + T(1, 1) &= (1, 1) + (1, 1) \\ &= (2, 2) \neq (2, 4) = T(2, 2). \end{aligned}$$

12. T is *not* a linear transformation because it does not preserve addition. For example,

$$\begin{aligned} T(1, 1, 1) + T(1, 1, 1) &= (2, 2, 2) + (2, 2, 2) \\ &= (4, 4, 4) \\ &\neq (3, 3, 3) = T(2, 2, 2). \end{aligned}$$

14. T is *not* a linear transformation because it does not preserve addition nor scalar multiplication. For example,

$$\begin{aligned} T(1, 1) + T(1, 1) &= (1, 1, 1) + (1, 1, 1) \\ &= (2, 2, 2) \neq (4, 4, 4) = T(2, 2). \end{aligned}$$

16. T preserves addition.

$$\begin{aligned} T(A_1) + T(A_2) &= T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= a_1 + b_1 + c_1 + d_1 + a_2 + b_2 + c_2 + d_2 \\ &= (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) + (d_1 + d_2) = T(A_1 + A_2) \end{aligned}$$

T preserves scalar multiplication.

$$T(kA) = ka + kb + kc + kd = k(a + b + c + d) = kT(A)$$

Therefore, T is a linear transformation.

18. T is *not* a linear transformation. T does *not* preserve addition.

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = b_1^2 + b_2^2 \neq (b_1 + b_2)^2 = T(A_1 + A_2)$$

20. Let A and B be two elements of $M_{3,3}$ (two 3×3 matrices) and let c be a scalar. First

$$T(A + B) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix}(A + B) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix}A + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix}B = T(A) + T(B)$$

by Theorem 2.3, part 2 and

$$T(cA) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix}(cA) = c\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix}A = cT(A)$$

by Theorem 2.3, part 4. So, T is a linear transformation.

22. T preserves addition.

$$\begin{aligned} T(a_0 + a_1x + a_2x^2) + T(b_0 + b_1x + b_2x^2) &= (a_1 + 2a_2x) + (b_1 + 2b_2x) \\ &= (a_1 + b_1) + 2(a_2 + b_2)x \\ &= T((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2) \end{aligned}$$

T preserves scalar multiplication.

$$T(c(a_0 + a_1x + a_2x^2)) = T(ca_0 + ca_1x + ca_2x^2) = ca_1 + 2ca_2x = c(a_1 + 2a_2x) = cT(a_0 + a_1x + a_2x^2)$$

Therefore, T *is* a linear transformation.

24. Because $(2, 0) = \frac{2}{3}(1, 2) - \frac{4}{3}(-1, 1)$, you have

$$\begin{aligned} T(2, 0) &= T\left[\tfrac{2}{3}(1, 2) - \tfrac{4}{3}(-1, 1)\right] \\ &= \tfrac{2}{3}T(1, 2) - \tfrac{4}{3}T(-1, 1) \\ &= \tfrac{2}{3}(1, 0) - \tfrac{4}{3}(0, 1) \\ &= \left(\tfrac{2}{3}, -\tfrac{4}{3}\right) \end{aligned}$$

Similarly, $(0, 3) = (1, 2) + (-1, 1)$, which gives

$$\begin{aligned} T(0, 3) &= T[(1, 2) + (-1, 1)] \\ &= T(1, 2) + T(-1, 1) \\ &= (1, 0) + (0, 1) \\ &= (1, 1). \end{aligned}$$

26. Because $(2, -1, 0)$ can be written as

$$(2, -1, 0) = 2(1, 0, 0) - 1(0, 1, 0) + 0(0, 0, 1),$$

you can use Property 4 of Theorem 6.1 to write

$$\begin{aligned} T(2, -1, 0) &= 2T(1, 0, 0) - T(0, 1, 0) + 0T(0, 0, 1) \\ &= 2(2, 4, -1) - (1, 3, -2) + (0, 0, 0) \\ &= (3, 5, 0). \end{aligned}$$

28. Because $(-2, 4, -1)$ can be written as

$$(-2, 4, -1) = -2(1, 0, 0) + 4(0, 1, 0) - 1(0, 0, 1),$$

you can use Property 4 of Theorem 6.1 to write

$$\begin{aligned} T(-2, 4, -1) &= -2T(1, 0, 0) + 4T(0, 1, 0) - T(0, 0, 1) \\ &= -2(2, 4, -1) + 4(1, 3, -2) - (0, -2, 2) \\ &= (0, 6, -8). \end{aligned}$$

30. Because $(0, 2, -1)$ can be written as

$(0, 2, -1) = \frac{3}{2}(1, 1, 1) - \frac{1}{2}(0, -1, 2) - \frac{3}{2}(1, 0, 1)$, you can use Property 4 of Theorem 6.1 to write

$$\begin{aligned} T(0, 2, -1) &= \tfrac{3}{2}T(1, 1, 1) - \tfrac{1}{2}T(0, -1, 2) - \tfrac{3}{2}T(1, 0, 1) \\ &= \tfrac{3}{2}(2, 0, -1) - \tfrac{1}{2}(-3, 2, -1) - \tfrac{3}{2}(1, 1, 0) \\ &= \left(3, -\tfrac{5}{2}, -1\right). \end{aligned}$$

32. Because $(-2, 1, 0)$ can be written as

$(-2, 1, 0) = 2(1, 1, 1) + (0, -1, 2) - 4(1, 0, 1)$, you can use Property 4 of Theorem 6.1 to write

$$\begin{aligned} T(-2, 1, 0) &= 2T(1, 1, 1) + T(0, -1, 2) - 4T(1, 0, 1) \\ &= 2(2, 0, -1) + (-3, 2, -1) - 4(1, 1, 0) \\ &= (-3, -2, -3). \end{aligned}$$

34. Because the matrix has 2 columns, the dimension of R^n is 2. Because the matrix has 3 rows, the dimension of R^m is 3. So, $T: R^2 \to R^3$.

36. Because the matrix has five columns, the dimension of R^n is 5. Because the matrix has two rows, the dimension of R^m is 2. So, $T: R^5 \to R^2$.

38. Because the matrix has five columns, the dimension of R^n is 5. Because the matrix has three rows, the dimension of R^m is 3. So, $T: R^5 \to R^3$.

40. (a) $T(2, 4) = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 4 \end{bmatrix} = (10, 12, 4)$

(b) The preimage of $(-1, 2, 2)$ is given by solving the equation

$$T(v_1, v_2) = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

for $\mathbf{v} = (v_1, v_2)$. The equivalent system of linear equations

$$\begin{aligned} v_1 + 2v_2 &= -1 \\ -2v_1 + 4v_2 &= 2 \\ -2v_1 + 2v_2 &= 2 \end{aligned}$$

has the solution $v_1 = -1$ and $v_2 = 0$. So, $(-1, 0)$ is the preimage of $(-1, 2, 2)$ under T.

(c) Because the system of linear equations represented by the equation

$$\begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has no solution, $(1, 1, 1)$ has no preimage under T.

42. (a) $T(1, 0, -1, 3, 0) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix} = (7, -5).$

(b) The preimage of $(-1, 8)$ is determined by solving the equation $T(v_1, v_2, v_3, v_4, v_5) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}.$

The equivalent system of linear equations has the solution $v_1 = 5 + 2r + \frac{7}{2}s + 4t$, $v_2 = r$, $v_3 = 4 + \frac{1}{2}s$, $v_4 = s$, and $v_5 = t$, where *r, s,* and *t* are any real numbers. So, the preimage is given by the set of vectors $\left\{\left(5 + 2r + \frac{7}{2}s + 4t, r, 4 + \frac{1}{2}s, s, t\right) : r, s, t \text{ are real numbers}\right\}$.

44. (a) $T(0, 1, 0, 1, 0) = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (4, 0, 4)$

(b) The preimage of $(0, 0, 0)$ is determined by solving the equation as shown.

$$T(v_1, v_2, v_3, v_4, v_5) = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = (0, 0, 0)$$

The equivalent system of linear equations has the solution $v_1 = -t$, $v_2 = -s$, $v_3 = 0$, $v_4 = s$, and $v_5 = t$, where s and t are any real numbers. So, the preimage is given by the set of vectors $\{(-t, -s, 0, s, t)\}$.

(c) The preimage of $(1, -1, 2)$ is determined by solving the equation as shown.

$$T(v_1, v_2, v_3, v_4, v_5) = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = (1, -1, 2)$$

The equivalent system of linear equations has the solution $v_1 = -3 - t$, $v_2 = \frac{1}{2} - s$, $v_3 = 2$, $v_4 = s$, and $v_5 = t$, where s and t are real numbers. So, the preimage is given by the set of vectors $\left\{\left(-3 - t, \frac{1}{2} - s, 2, s, t\right)\right\}$.

46. If $\theta = 45°$, then T is given by

$$T(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$$
$$= \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right).$$

Solving $T(x, y) = \mathbf{v} = (1, 1)$, you have

$$\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = 1 \quad \text{and} \quad \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 1.$$

So, $x = \sqrt{2}$ and $y = 0$, and the preimage of $\mathbf{v}$ is $(\sqrt{2}, 0)$.

48. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \end{bmatrix}$

You then obtain the following system of equations.

$$12a - 5b = 13$$
$$12b + 5a = 0$$

Solving the second equation for a gives $a = \frac{-12}{5}b$.

Substituting this back into the first equation produces

$$12\left(\frac{-12}{5}b\right) - 5b = 13$$
$$\frac{-144}{5}b - 5b = 13$$
$$\frac{-169}{5}b = 13$$
$$b = \frac{-5}{13}.$$

Substituting $b = \frac{-5}{13}$ into $a = \frac{-12}{5}b$ you obtain

$$a = \frac{12}{13}.$$

50. If $\mathbf{v} = (x, y, z)$ is a vector into R^3, then $T(\mathbf{v}) = (0, y, z)$. In other words, T maps every vector in R^3 to its orthogonal projection in the yz-plane.

52. T is a linear transformation.

T preserves addition.

$$\begin{aligned} T(A + B) &= (A + B)X - X(A + B) \\ &= AX + BX - XA - XB \\ &= (AX - XA) + (BX - XB) \\ &= T(A) + T(B) \end{aligned}$$

T preserves scalar multiplication.

$$\begin{aligned} T(cA) &= (cA)X - X(cA) \\ &= c(AX) - c(XA) \\ &= c(AX - XA) \\ &= cT(A) \end{aligned}$$

54. T is *not* a linear transformation.

Consider $A = I_n$. Then $T(A) = 1$, but $T(2A) = 2^n \neq 2T(A)$.

56. $T\left(\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\right) = T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & -1 \\ 7 & 4 \end{bmatrix}$

58. This statement is true because D_x is a linear transformation and therefore preserves addition and scalar multiplication.

60. This statement is false because $\cos \frac{x}{2} \neq \frac{1}{2} \cos x$ for all x.

62. If $D_x(g(x)) = e^x$, then $g(x) = e^x + C$.

64. If $D_x(g(x)) = \frac{1}{x}$, then $g(x) = \ln x + C$.

66. Solve the equation $\int_0^1 p(x)dx = 1$ for $p(x)$ in P_2.

$$\int_0^1 (a_0 + a_1x + a_2x^2)dx = 1 \Rightarrow \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3}\right]_0^1 = 1 \Rightarrow a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 1.$$

Letting $a_2 = -3b$ and $a_1 = -2a$ be free variables, $a_0 = 1 + a + b$, and $p(x) = (1 + a + b) - 2ax - 3bx^2$.

68. (a) False. This function does not preserve addition nor scalar multiplication. For example, $f(3x) = 27x^3 \neq 3f(x)$.

(b) False. If $f: R \to R$ is given by $f(x) = ax + b$ for some $a, b \in R$, then it preserves addition and scalar multiplication if and only if $b = 0$.

70. (a)
$$\begin{aligned} T(x, y) &= T[x(1, 0) + y(0, 1)] \\ &= xT(1, 0) + yT(0, 1) \\ &= x(0, 1) + y(1, 0) = (y, x) \end{aligned}$$

(b) T is a reflection about the line $y = x$.

72. Use the result of Exercise 71(a) as follows.

$$T(3, 4) = \left(\frac{3 + 4}{2}, \frac{3 + 4}{2}\right) = \left(\frac{7}{2}, \frac{7}{2}\right)$$

$$\begin{aligned} T(T(3, 4)) &= T\left(\frac{7}{2}, \frac{7}{2}\right) \\ &= \left(\frac{1}{2}\left(\frac{7}{2} + \frac{7}{2}\right), \frac{1}{2}\left(\frac{7}{2} + \frac{7}{2}\right)\right) = \left(\frac{7}{2}, \frac{7}{2}\right) \end{aligned}$$

T is projection onto the line $y = x$.

74. To show that $T : V \to W$ is a linear transformation, show that $T : V \to W$ preserves addition and scalar multiplication by using the definition:

(1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

and

(2) $T(c\mathbf{u}) = cT(\mathbf{u})$,

where c is any nonzero constant.

76. (a) Because $T(0, 0) = (-h, -k) \neq (0, 0)$, a translation cannot be a linear transformation.

(b) $T(0, 0) = (0 - 2, 0 + 1) = (-2, 1)$

$T(2, -1) = (2 - 2, -1 + 1) = (0, 0)$

$T(5, 4) = (5 - 2, 4 + 1) = (3, 5)$

(c) Because $T(x, y) = (x - h, y - k) = (x, y)$ implies $x - h = x$ and $y - k = y$, a translation has no fixed points.

78. There are many possible examples. For instance, let $T: R^3 \to R^3$ be given by $T(x, y, z) = (0, 0, 0)$. Then if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is any set of linearly independent vectors, their images $T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)$ form a dependent set.

80. Let T be defined by $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_0 \rangle$. Then because

$T(\mathbf{v} + \mathbf{w}) = \langle \mathbf{v} + \mathbf{w}, \mathbf{v}_0 \rangle = \langle \mathbf{v}, \mathbf{v}_0 \rangle + \langle \mathbf{w}, \mathbf{v}_0 \rangle = T(\mathbf{v}) + T(\mathbf{w})$

and $T(c\mathbf{v}) = \langle c\mathbf{v}, \mathbf{v}_0 \rangle = c\langle \mathbf{v}, \mathbf{v}_0 \rangle = cT(\mathbf{v})$, T is a linear transformation.

82. Because

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= \langle \mathbf{u} + \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{u} + \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n \\ &= \left(\langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1\right) + \cdots + \left(\langle \mathbf{u}, \mathbf{w}_n \rangle \mathbf{w}_n + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n\right) \\ &= \left(\langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{u}, \mathbf{w}_n \rangle \mathbf{w}_n\right) + \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and

$T(c\mathbf{u}) = \langle c\mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle c\mathbf{u}, \mathbf{w}_n \rangle \mathbf{w}_n = c\langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + c\langle \mathbf{u}, \mathbf{w}_n \rangle \mathbf{w}_n = c\left[\langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{u}, \mathbf{w}_n \rangle \mathbf{w}_n\right] = cT(\mathbf{u})$,

T is a linear transformation.

84. Suppose first that T is a linear transformation. Then $T(a\mathbf{u} + b\mathbf{v}) = T(a\mathbf{u}) + T(b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$.

Second, suppose $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$. Then $T(\mathbf{u} + \mathbf{v}) = T(1\mathbf{u} + 1\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

and $T(c\mathbf{u}) = T(c\mathbf{u} + \mathbf{0}) = cT(\mathbf{u}) + T(\mathbf{0}) = cT(\mathbf{u})$.

Section 6.2 The Kernel and Range of a Linear Transformation

2. $T : R^3 \to R^3, T(x, y, z) = (x, 0, z)$

The kernel consists of all vectors lying on the y-axis. That is, $\ker(T) = \{(0, y, 0) : y \text{ is a real number}\}$.

4. $T : R^3 \to R^3, T(x, y, z) = (-z, -y, -x)$

Solving the equation $T(x, y, z) = (-z, -y, -x) = (0, 0, 0)$ yields that trivial solution $x = y = z = 0$. So, $\ker(T) = \{(0, 0, 0)\}$.

6. $T : P_2 \to R, T(a_0 + a_1x + a_2x^2) = a_0$

Solving the equation $T(a_0 + a_1x + a_2x^2) = a_0 = 0$ yields solutions of the form $a_0 = 0$ and a_1 and a_2 are any real numbers. So, $\ker(T) = \{a_1x + a_2x^2 : a_1, a_2 \in R\}$.

8. $T : P_3 \to P_2$,

$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$

Solving the equation

$T(a_0 + a_1 + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2 = 0$

yields solutions of the form $a_1 = a_2 = a_3 = 0$ and a_0 any real number. So, $\ker(T) = \{a_0 : a_0 \in R\}$.

10. $T : R^2 \to R^2, T(x, y) = (x - y, y - x)$

Solving the equation $T(x, y) = (x - y, y - x) = (0, 0)$ yields solutions of the form $x = y$. So, $\ker(T) = \{(x, x) : x \in R\}$.

12. (a) $\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{array}{r} x_1 + 2x_2 = 0 \\ -3x_1 - 6x_2 = 0 \end{array} \Rightarrow \begin{array}{r} x_1 + 2x_2 = 0 \\ 0 = 0 \end{array}$$

Using the parameter $t = x_2$ produces the family of solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So, $\ker(T) = \{t(-2, 1): t \text{ is a real number}\}$

$= \operatorname{span}\{(-2, 1)\}.$

(b) Transpose A and find the equivalent row-echelon form.

$$A^T = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

So, $\operatorname{range}(T) = \{t(1, -3): t \text{ is a real number}\}.$

$= \operatorname{span}\{(1, -3)\}.$

14. (a) Because

$$T(\mathbf{x}) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has solutions of the form $\left(-2t, -\frac{1}{2}t, t\right)$ where t is any real number,

$\ker(T) = \left\{t\left(-2, -\frac{1}{2}, 1\right): t \text{ is a real number}\right\}$

$= \operatorname{span}\left\{\left(-2, -\frac{1}{2}, 1\right)\right\}.$

$\left\{\left(-2, -\frac{1}{2}, 1\right)\right\}.$

(b) Transpose A and find the equivalent reduced row-echelon form

$$A^T = \begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So, range (T) is $\{(1, 0), (0, 1)\} = R^2$.

16. (a) Because

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has only the trivial solution $x_1 = x_2 = 0$, $\ker(T) = \{(0, 0)\}$.

(b) Transpose A and find the equivalent reduced row-echelon form.

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

So, range(T) = $\operatorname{span}\left\{\left(1, 0, \frac{1}{3}\right), \left(0, 1, \frac{1}{3}\right)\right\}$.

18. (a) Because

$$T(\mathbf{x}) = \begin{bmatrix} -1 & 3 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solutions of the form $(-10s - 4t, -15s - 24t, 13s + 16t, 9s, 9t)$,

$\ker(T) = \operatorname{span}\{(-10, -15, 13, 9, 0), (-4, -24, 16, 0, 9)\}$.

(b) Transpose A and find the equivalent reduced row-echelon form.

$$A^T = \begin{bmatrix} -1 & 2 & 2 \\ 3 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, range (T) = $\operatorname{span}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} = R^3$.

20. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So,

$\ker(T) = \{(2t, -3t) : t \text{ is any real number}\}.$

(b) $\text{nullity}(T) = \dim(\ker(T)) = 1$

(c) Transpose T and find the equivalent reduced row-echelon form.

$$A^T = \begin{bmatrix} 3 & -9 \\ 2 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

So, $\text{range}(T) = \{(t, -3t) : t \text{ is any real number}\}.$

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 1$

22. (a) Because $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = (0, 0)$, the kernel of T is $\{(0, 0)\}$.

(b) $\text{nullity}(T) = \dim(\ker(T)) = 0$

(c) Transpose A and find the equivalent row-echelon form.

$$A^T = \begin{bmatrix} 4 & 0 & 2 \\ 1 & 0 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $\text{range}(T) = \{(t, 0, s) : s, t \in R\}.$

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 2$

24. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So,

$\ker(T) = \{(5t, t) : t \in R\}.$

(b) $\text{nullity}(T) = \dim(\ker(T)) = 1$

(c) Transpose A and find its equivalent row-echelon form.

$$A^T = \begin{bmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

So, $\text{range}(T) = \{(t, -5t) : t \in R\}.$

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 1$

26. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So,

$\ker(T) = \{(0, t, 0) : t \in R\}.$

(b) $\text{nullity}(T) = \dim(\ker(T)) = 1$

(c) Transpose A and find its equivalent row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\text{range}(T) = \{(t, 0, s) : s, t \in R\}.$

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 2$

28. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So,

$\ker(T) = \{(-t, 0, t): t \text{ is any real number}\}.$

(b) $\text{nullity}(T) = \dim(\ker(T)) = 1$

(c) Transpose A and find its equivalent reduced row-echelon form.

$$A^T = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\text{range}(T) = \{(s, 0, s), (0, t, 0): s \text{ and } t \text{ are any real numbers}\}.$

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 2$

30. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So,

$\ker(T) = \{(t, -t, s, -s) : s, t \in R\}$.

(b) $\text{nullity}(T) = \dim(\ker(T)) = 2$

(c) Transpose A and find its equivalent row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $\text{range}(T) = R^2$.

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 2$

32. (a) The kernel of T is given by the solution to the equation $T(\mathbf{x}) = \mathbf{0}$. So,

$\ker(T) = \{(-t - s - 2r, 6t - 2s, r, s, t) : r, s, t \in R\}$.

(b) $\text{nullity}(T) = \dim(\ker(T)) = 3$

(c) Transpose A and find its equivalent row-echelon form.

$$A^T = \begin{bmatrix} 3 & 4 & 2 \\ -2 & 3 & -3 \\ 6 & 8 & 4 \\ -1 & 10 & -4 \\ 15 & -14 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 17 & 0 & 18 \\ 0 & 17 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\text{range}(T) = \{(17s, 17t, 18s - 5t) : s, t \in R\}$.

(d) $\text{rank}(T) = \dim(\text{range}(T)) = 2$

34. Because $\text{rank}(T) + \text{nullity}(T) = 3$, and you are given $\text{rank}(T) = 1$, then $\text{nullity}(T) = 2$. So, the kernel of T is a plane, and the range is a line.

36. Because $\text{rank}(T) + \text{nullity}(T) = 3$, and you are given $\text{rank}(T) = 3$, then $\text{nullity}(T) = 0$. So, the kernel of T is the single point $\{(0, 0, 0)\}$, and the range is all of R^3.

38. The kernel of T is determined by solving $T(x, y, z) = (-x, y, z) = (0, 0, 0)$, which implies that the kernel is the single point $\{(0, 0, 0)\}$. From the equation $\text{rank}(T) + \text{nullity}(T) = 3$, you see that the rank of T is 3. So, the range of T is all of R^3.

40. The kernel of T is determined by solving $T(x, y, z) = (x, y, 0) = (0, 0, 0)$, which implies that $x = y = 0$. So, the nullity of T is 1, and the kernel is a line (the z-axis). The range of T is found by observing that $\text{rank}(T) + \text{nullity}(T) = 3$. That is, the range of T is 2-dimensional, the xy-plane in R^3.

42. $\text{rank}(T) + \text{nullity}(T) = \dim R^4 \Rightarrow \text{nullity}(T) = 4 - 0 = 4$

44. $\text{rank}(T) + \text{nullity}(T) = \dim P_3 \Rightarrow \text{nullity}(T) = 4 - 2 = 2$

46. $\text{rank}(T) + \text{nullity}(T) = \dim M_{3,3} \Rightarrow \text{nullity}(T) = 9 - 6 = 3$

48. Because $|A| = -1 \neq 0$, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, $\ker(T) = \{(0, 0)\}$ and T is one-to-one (by Theorem 6.6). Furthermore, because $\text{rank}(T) = \dim(R^2) - \text{nullity}(T) = 2 - 0 = 2 = \dim(R^2)$, T is onto (by Theorem 6.7).

50. Because $|A| = -24 \neq 0$, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, $\ker(T) = \{(0, 0, 0)\}$ and T is one-to-one (by Theorem 6.6). Furthermore, because $\text{rank}(T) = \dim R^3 - \text{nullity}(T) = 3 - 0 = 3 = \dim(R^3)$, T is onto (by Theorem 6.7).

52. The matrix representation of $T : R^2 \to R^2$ is given by $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

The matrix in row-echelon form is $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

So, you have the following.

$\dim(\text{domain}) = 2, \text{rank}(T) = 1, \text{nullity}(T) = 1$

Because the rank of T is not equal to the dimension of R^2, T is not onto. Because $\ker(T) \neq \{\mathbf{0}\}$, T is not one-to one.

54. $A = \begin{bmatrix} -1 & 3 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{10}{9} & \frac{4}{9} \\ 0 & 1 & 0 & \frac{5}{3} & \frac{8}{3} \\ 0 & 0 & 1 & \frac{-13}{9} & \frac{-16}{9} \end{bmatrix}$

So, you have the following.

$\dim(\text{domain}) = 5, \text{rank}(T) = 3, \text{nullity}(T) = 2$

Because the rank of T is equal to the dimension of R^3, T is onto. Because $\ker(T) \neq \{\mathbf{0}\}$, T is not one-to-one.

56. The vector spaces isomorphic to R^6 are those whose dimension is six. That is, (a) $M_{2,3}$ (d) $M_{6,1}$ (e) P_5 and (g) $\{(x_1, x_2, x_3, 0, x_5, x_6, x_7) : x_i \in R\}$ are isomorphic to R^6.

58. Solve the equation $T(p) = \int_0^1 p(x)dx = \int_0^1 (a_0 + a_1x + a_2x^2)dx = 0$ yielding $a_0 + a_1/2 + a_2/3 = 0$.

Letting $a_2 = -3b, a_1 = -2a$, you have $a_0 = -a_1/2 - a_2/3 = a + b$, and $\ker(T) = \{(a + b) - 2ax - 3bx^2 : a, b \in R\}$.

60. $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(a) $\dim(R^4) = 4$

(b) $\dim(R^3) = 3$

(c) $x_1 + 5x_3 = 0 \to x_1 = -5x_3$

$x_2 + 2x_3 = 0 \to x_2 = -2x_3$

$x_4 = 0$

So, $\ker(T) = \{(-5x_3, -2x_3, x_3, 0)\}$ and $\dim(\ker(T)) = 1$.

(d) T is not one-to-one since the $\ker(T) \neq \{\mathbf{0}\}$.

(e) $\text{rank}(T) = 3$

$= \dim(R^3)$

So, T is onto by Theorem 6.7.

(f) T is *not* an isomorphism since it is not one-to-one.

62. If T is onto, then $m \geq n$.

If T is one-to-one, then $m \leq n$.

64. Theorem 6.9 tells you that if $M_{m,n}$ and $M_{j,k}$ are of the same dimension then they are isomorphic. So, you can conclude that $mn = jk$.

66. (a) False. A concept of a dimension of a linear transformation does not exist.

(b) True. See discussion on page 315 before Theorem 6.6.

(c) True. Because $\dim(P_1) = \dim(R^2) = 2$ and any two vector spaces of equal finite dimension are isomorphic (Theorem 6.9 on page 317).

68. From Theorem 6.5,

$\text{rank}(T) + \text{nullity}(T) = n = \text{dimension of } V$. T is one-to-one if and only if $\text{nullity}(T) = 0$ if and only if $\text{rank}(T) = \text{dimension of } V$.

70. $T^{-1}(U)$ is nonempty because $T(\mathbf{0}) = \mathbf{0} \in U \Rightarrow \mathbf{0} \in T^{-1}(U)$.

Let $\mathbf{v}_1, \mathbf{v}_2 \in T^{-1}(U) \Rightarrow T(\mathbf{v}_1) \in U$ and $T(\mathbf{v}_2) \in U$. Because U is a subspace of W,

$T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in U \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 \in T^{-1}(U)$.

Let $\mathbf{v} \in T^{-1}(U)$ and $c \in R \Rightarrow T(\mathbf{v}) \in U$. Because U is a subspace of W, $cT(\mathbf{v}) = T(c\mathbf{v}) \in U \Rightarrow c\mathbf{v} \in T^{-1}(U)$.

If $U = \{\mathbf{0}\}$, then $T^{-1}(U)$ is the kernel of T.

Section 6.3 Matrices for Linear Transformations

2. Because

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-4\end{bmatrix} \quad\text{and}\quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\1\\1\end{bmatrix},$$

the standard matrix for T is $A = \begin{bmatrix}2 & -3\\1 & -1\\-4 & 1\end{bmatrix}$.

4. Because

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\0\\4\end{bmatrix} \quad\text{and}\quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-5\end{bmatrix},$$

the standard matrix for T is

$$\begin{bmatrix}5 & 1\\0 & 0\\4 & -5\end{bmatrix}.$$

6. Because

$$T\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ T\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

and $T\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$,

the standard matrix for T is $A = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$.

8. Because

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\2\\0\end{bmatrix} \quad\text{and}\quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\0\\2\end{bmatrix},$$

the standard matrix for T is

$$A = \begin{bmatrix}1 & 1\\1 & -1\\2 & 0\\0 & 2\end{bmatrix}.$$

So, $T(\mathbf{v}) = \begin{bmatrix}1 & 1\\1 & -1\\2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}0\\6\\6\\-6\end{bmatrix}$

and $T(3, -3) = (0, 6, 6, -6)$.

10. $T(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_2 - x_4, x_3 - x_1, x_2 + x_4)$

The standard matrix is

$$A = \begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & -1\\-1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix}.$$

The image of $\mathbf{v}$ is

$$A\mathbf{v} = \begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & -1\\-1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\\-2\end{bmatrix} = \begin{bmatrix}-2\\4\\2\\0\end{bmatrix}.$$

So, $T(\mathbf{v}) = (-2, 4, 2, 0)$.

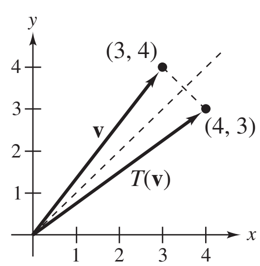
12. (a) The matrix of a reflection in the line $y = x$, $T(x, y) = (y, x)$, is given by $A = \left[T(1, 0) \vdots T(0, 1)\right] = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

(b) The image of $\mathbf{v} = (3, 4)$ is given by

$$A\mathbf{v} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}.$$

So, $T(3, 4) = (4, 3)$.

(c)



14. (a) The matrix of a reflection in the x-axis, $T(x, y) = (x, -y)$, is given by

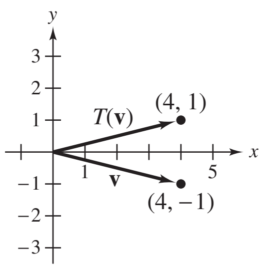
$$A = \left[T(1, 0) \vdots T(0, 1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (4, -1)$ is given by

$$A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

So, $T(4, -1) = (4, 1)$.

(c)



16. (a) The counterclockwise rotation of $120°$ is given by

$$\begin{aligned} T(x, y) &= \left(\cos(120)x - \sin(120)y, \sin(120)x + \cos(120)y\right) \\ &= \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y\right). \end{aligned}$$

So, the matrix is

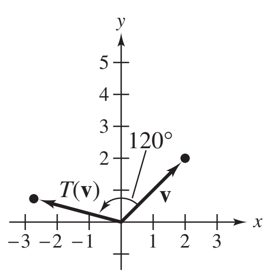
$$A = \left[T(1, 0) \vdots T(0, 1)\right] = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (2, 2)$ is given by

$$A\mathbf{v} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 - \sqrt{3} \\ \sqrt{3} - 1 \end{bmatrix}.$$

So, $T(2, 2) = \left(-1 - \sqrt{3}, \sqrt{3} - 1\right)$.

(c)



18. (a) The clockwise rotation of $30°$ is given by

$$T(x, y) = \left(\cos(-30)x - \sin(-30)y, \sin(-30)x + \cos(-30)y\right)$$
$$= \left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y, -\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right).$$

So, the matrix is

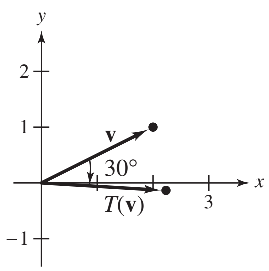
$$A = \left[T(1, 0) \vdots T(0, 1)\right] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (2, 1)$ is given by

$$A\mathbf{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + \frac{1}{2} \\ -1 + \frac{\sqrt{3}}{2} \end{bmatrix}.$$

So, $T(2, 1) = \left(\sqrt{3} + \frac{1}{2}, -1 + \frac{\sqrt{3}}{2}\right)$.

(c)



20. (a) The matrix of a reflection through the yz-coordinate plane is given by

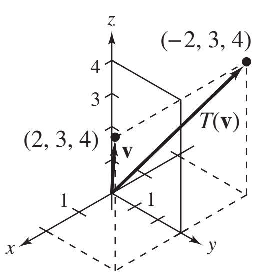
$$A = \left[T(1, 0, 0) \vdots T(0, 1, 0) \vdots T(0, 0, 1)\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (2, 3, 4)$ is given by

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}.$$

So, $T(2, 3, 4) = (-2, 3, 4)$.

(c)



22. (a) The reflection of a vector $\mathbf{v}$ through $\mathbf{w}$ is given by

$$T(\mathbf{v}) = 2\operatorname{proj}_{\mathbf{w}}\mathbf{v} - \mathbf{v}$$

$$T(x, y) = 2\frac{3x + y}{10}(3, 1) - (x, y)$$

$$= \left(\frac{4}{5}x + \frac{3}{5}y, \frac{3}{5}x - \frac{4}{5}y\right).$$

The standard matrix for T is

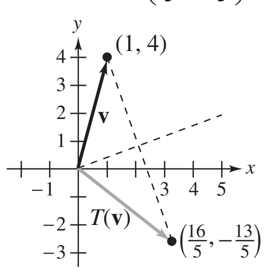
$$A = \left[T(1, 0) \vdots T(0, 1)\right] = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (1, 4)$ is

$$A\mathbf{v} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{16}{5} \\ -\frac{13}{5} \end{bmatrix}.$$

So, $T(1, 4) = \left(\frac{16}{5}, -\frac{13}{5}\right)$.

(c)



24. (a) The standard matrix for T is

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -5 & 0 \\ 0 & 1 & -3 \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (3, 13, 4)$ is

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -5 & 0 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ -56 \\ 1 \end{bmatrix}.$$

So, $T(3, 13, 4)$ is $(17, -56, 1)$.

(c) Using a graphing utility or a computer software program to perform the multiplication in part (b) gives the same results.

26. (a) The standard matrix for T is

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The image of $\mathbf{v} = (0, 1, -1, 1)$ is

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}.$$

So, $T(0, 1, -1, 1) = (2, 1, -3, 0)$.

(c) Using a graphing utility or a computer software program to perform the multiplication in part (b) gives the same result.

28. The standard matrices for T_1 and T_2 are

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The standard matrix for $T = T_2 \circ T_1$ is

$$A = A_2 A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A_2$$

and the standard matrix for $T' = T_1 \circ T_2$ is

$$A' = A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A_2.$$

30. The standard matrices for T_1 and T_2 are

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The standard matrix for $T = T_2 \circ T_1$ is

$$A_2 A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

and the standard matrix for $T' = T_1 \circ T_2$ is

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. The standard matrix for T is

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Because $|A| = 0$, A is not invertible, and so T is not invertible.

34. The standard matrix for T is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Because $|A| = -2 \neq 0$, A is invertible.

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

So, $T^{-1}(x, y) = \left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y\right)$.

36. The standard matrix for T is

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Because $|A| = -1 \neq 0$, A is invertible. Calculate A^{-1} by Gauss-Jordan elimination

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and conclude that

$$T^{-1}(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, x_2, x_4, x_3 - x_4).$$

38. (a) The standard matrix for T is

$$A' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

and the image of $\mathbf{v}$ under T is

$$A'\mathbf{v} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

So, $T(\mathbf{v}) = (-2, -2)$.

(b) The image of each vector in B is as follows.

$$T(1, 1, 1) = (0, 0) = 0(1, 1) + 0(2, 1)$$
$$T(1, 1, 0) = (0, 1) = -1(1, 1) + (1, 2)$$
$$T(0, 1, 1) = (-1, 0) = -2(1, 1) + (1, 2)$$

So, $[T(1, 1, 1)]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [T(1, 1, 0)]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $[T(0, 1, 1)]_{B'} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

which implies that $A = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$.

Then, because $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

So, $T(\mathbf{v}) = -2(1, 1) + 0(1, 2) = (-2, 2)$.

40. (a) The standard matrix for T is

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

and the image of $\mathbf{v} = (4, -3, 1, 1)$ under T is

$$A'\mathbf{v} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \Rightarrow T(\mathbf{v}) = (3, -3).$$

(b) Because

$$T(1, 0, 0, 1) = (2, 0) = 0(1, 1) + (2, 0)$$
$$T(0, 1, 0, 1) = (2, 1) = (1, 1) + \tfrac{1}{2}(2, 0)$$
$$T(1, 0, 1, 0) = (2, -1) = -(1, 1) + \tfrac{3}{2}(2, 0)$$
$$T(1, 1, 0, 0) = (2, -1) = -(1, 1) + \tfrac{3}{2}(2, 0),$$

The matrix for T relative to B and B' is $A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$.

Because $\mathbf{v} = (4, -3, 1, 1) = \frac{7}{2}(1, 0, 0, 1) - \frac{5}{2}(0, 1, 0, 1) + (1, 0, 1, 0) - \frac{1}{2}(1, 1, 0, 0)$, you have

$$\left[T(\mathbf{v})\right]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} \frac{7}{2} \\ -\frac{5}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

So, $T(\mathbf{v}) = -3(1, 1) + 3(2, 0) = (3, -3)$.

42. (a) The standard matrix for T is $A' = \begin{bmatrix} 3 & -13 \\ 1 & -4 \end{bmatrix}$ and the image of $\mathbf{v} = (4, 8)$ under T is

$$A'\mathbf{v} = \begin{bmatrix} 3 & -13 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -92 \\ -28 \end{bmatrix} \Rightarrow T(\mathbf{v}) = (-92, -28).$$

(b) Because

$$T(2, 1) = (-7, -2) = -(2, 1) - (5, 1)$$
$$T(5, 1) = (2, 1)$$

the matrix for T relative to B and B' is $A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$.

Because $\mathbf{v} = (4, 8) = 12(2, 1) - 4(5, 1)$, you have $\left[T(\mathbf{v})\right]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ -4 \end{bmatrix} = \begin{bmatrix} -12 \\ -16 \end{bmatrix}$.

So, $T(\mathbf{v}) = -12(2, 1) - 16(5, 1) = (-92, -28)$.

44. The image of each vector in B is $T(1) = x^2, T(x) = x^3, T(x^2) = x^4$.

So, the matrix of T relative to B and B' is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

46. The image of each vector in B is as follows.

$$D(e^{2x}) = 2e^{2x}$$
$$D(xe^{2x}) = e^{2x} + 2xe^{2x}$$
$$D(x^2e^{2x}) = 2xe^{2x} + 2x^2e^{2x}$$

So, the matrix of T relative to B is $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

48. Because $5e^{2x} - 3xe^{2x} + x^2e^{2x} = 5(e^{2x}) - 3(xe^{2x}) + 1(x^2e^{2x})$,

$$A[\mathbf{v}]_B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix} \Rightarrow D_x(5e^{2x} - 3xe^{2x} + x^2e^{2x}) = 7e^{2x} - 4xe^{2x} + 2x^2e^{2x}.$$

50. (a) Let $T : R^n \to R^m$ be a linear transformation such that, for the standard basis vectors *ei* of R^n,

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix whose n columns correspond to $T(e_i)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the standard matrix of T.

(b) Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be linear transformations with standard matrices A_1 and A_2, respectively. The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix A for T is given by the matrix product $A = A_2A_1$.

(c) To find the inverse of a linear transformation T, first find the standard matrix A of T. Then find the inverse of A using the techniques shown in Section 2.3.

(d) To find the transformation matrix relative to nonstandard basis, first find $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$. Then determine the coordinate matrices relative to B'. Finally, form the matrix T relative to B and B' by using the coordinate matrices as columns to produce $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

52. Because $T(\mathbf{v}) = k\mathbf{v}$ for all $\mathbf{v} \in R^n$, the standard matrix for T is the $n \times n$ diagonal matrix

$$\begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & & \vdots \\ \vdots & & k & 0 \\ 0 & \cdots & 0 & k \end{bmatrix}.$$

54. (a) True. See discussion, under "Composition of Linear Transformations," pages 323–324.

(b) False. See Example 3, page 324.

56. $(1 \Rightarrow 2)$: Let T be invertible. If $T(\mathbf{v}_1) = T(\mathbf{v}_2)$, then $T^{-1}(T(\mathbf{v}_1)) = T^{-1}(T(\mathbf{v}_2))$ and $\mathbf{v}_1 = \mathbf{v}_2$, so T is one-to-one. T is onto because for any $\mathbf{w} \in R^n$, $T^{-1}(\mathbf{w}) = \mathbf{v}$ satisfies $T(\mathbf{v}) = \mathbf{w}$.

$(2 \Rightarrow 1)$: Let T be an isomorphism. Define T^{-1} as follows: Because T is onto, for any $\mathbf{w} \in R^n$, there exists $\mathbf{v} \in R^n$ such that $T(\mathbf{v}) = \mathbf{w}$. Because T is one-to-one, this $\mathbf{v}$ is unique. So, define the inverse of T by $T^{-1}(\mathbf{w}) = \mathbf{v}$ if and only if $T(\mathbf{v}) = \mathbf{w}$.

Finally, the corollaries to Theorems 6.3 and 6.4 show that 2 and 3 are equivalent.

If T is invertible, $T(\mathbf{x}) = A\mathbf{x}$ implies that $T^{-1}(T(\mathbf{x})) = \mathbf{x} = A^{-1}(A\mathbf{x})$ and the standard matrix of T^{-1} is A^{-1}.

58. $\mathbf{b}$ is in the range of the linear transformation $T: R^n \to R^m$ given by $T(\mathbf{x}) = A\mathbf{x}$ if and only if $\mathbf{b}$ is in the column space of A.

Section 6.4 Transition Matrices and Similarity

2. The standard matrix for T is $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$. Furthermore, the transition matrix P from B' to the standard basis B, and its inverse are $P = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$. Therefore, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -\frac{11}{4} & -4 \end{bmatrix}$$

4. The standard matrix for T is $A = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$. Furthermore, the transition matrix P from B' to the standard basis B, and its inverse, are $P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$. Therefore, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ -20 & -11 \end{bmatrix}$$

6. The standard matrix for T is $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. Furthermore, the transition matrix P from B' to the standard basis B, and its inverse, are $P = \begin{bmatrix} 12 & 13 \\ -13 & -12 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -\frac{12}{25} & -\frac{13}{25} \\ \frac{13}{25} & \frac{12}{25} \end{bmatrix}$. Therefore, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} -\frac{12}{25} & -\frac{13}{25} \\ \frac{13}{25} & \frac{12}{25} \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 12 & 13 \\ -13 & -12 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

8. The standard matrix for T is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Furthermore, the transition matrix P from B' to the standard basis B, and its inverse, are $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Therefore, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

10. The standard matrix for T is $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. Furthermore, the transition matrix P from B' to the standard basis B, and its inverse, are $P = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{15} & \frac{1}{15} \\ \frac{1}{5} & \frac{4}{15} & \frac{3}{15} \end{bmatrix}$. Therefore, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{15} & \frac{1}{15} \\ \frac{1}{5} & \frac{4}{15} & \frac{2}{15} \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{5} & \frac{2}{5} & 1 \\ -\frac{1}{15} & -\frac{19}{15} & \frac{1}{3} \\ -\frac{2}{15} & -\frac{8}{15} & -\frac{1}{3} \end{bmatrix}.$$

12. The standard matrix for T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$. Furthermore, the transition matrix P from B' to the standard basis B, and its inverse, are $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Therefore, the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

14. (a) The transition matrix P from B' to B is found by row-reducing $[B \vdots B']$ to $[I \vdots P]$.

$$[B \vdots B'] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 3 & -1 & 1 \end{array}\right] \quad \Rightarrow \quad [I \vdots P] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{1}{5} \end{array}\right]$$

So, $P = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$.

(b) The coordinate matrix for $\mathbf{v}$ relative to B is $[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Furthermore, the image of $\mathbf{v}$ under T relative to B is $[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$.

(c) The matrix of T relative to B' is $A' = P^{-1}AP = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}$.

(d) The image of $\mathbf{v}$ under T relative to B' is $P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -14 \end{bmatrix}$.

You can also find the image of $\mathbf{v}$ under T relative to B' by $A'[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -14 \end{bmatrix}$.

16. (a) The transition matrix P from B' to B is found by row-reducing $[B \vdots B']$ to $[I \vdots P]$.

$$P = \begin{bmatrix} -1 & -5 \\ 0 & -3 \end{bmatrix}$$

(b) The coordinate matrix for $\mathbf{v}$ relative to B is $[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} -1 & -5 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 19 \\ 12 \end{bmatrix}$.

Furthermore, the image of $\mathbf{v}$ under T relative to B is $[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 19 \\ 12 \end{bmatrix} = \begin{bmatrix} 50 \\ -12 \end{bmatrix}$.

(c) The matrix of T relative to B' is $A' = P^{-1}AP = \begin{bmatrix} -1 & \frac{5}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -5 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 18 \\ 0 & -1 \end{bmatrix}$

(d) The image of $\mathbf{v}$ under T relative to B' is $P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & \frac{5}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 50 \\ -12 \end{bmatrix} = \begin{bmatrix} -70 \\ 4 \end{bmatrix}$.

You can also find the image of $\mathbf{v}$ under T relative to B' by $A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 18 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -70 \\ 4 \end{bmatrix}$.

18. (a) The transition matrix P from B' to B is found by row-reducing $[B \quad B']$ to $[I \quad P]$.

$$[B \quad B'] = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [I \quad P]$$

So, $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(b) The coordinate matrix for $\mathbf{v}$ relative to B is $[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$.

Furthermore, the image of $\mathbf{v}$ under T relative to B is

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{4} \\ \frac{19}{4} \end{bmatrix}.$$

(c) The matrix of T relative to B' is $A' = P^{-1}AP$.

$$A' = P^{-1}AP = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -1 & -\frac{5}{4} \\ \frac{1}{4} & 2 & -\frac{1}{4} \\ \frac{5}{4} & 1 & \frac{15}{4} \end{bmatrix}$$

(d) The image of $\mathbf{v}$ under T relative to B' is

$$P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{11}{4} \\ \frac{19}{4} \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} \\ \frac{9}{4} \\ \frac{29}{4} \end{bmatrix}.$$

You can also find the image of $\mathbf{v}$ under T relative to B' by

$$A'[\mathbf{v}]_{B'} = \begin{bmatrix} \frac{1}{4} & -1 & -\frac{5}{4} \\ \frac{1}{4} & 2 & -\frac{1}{4} \\ \frac{5}{4} & 1 & \frac{15}{4} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} \\ \frac{9}{4} \\ \frac{29}{4} \end{bmatrix}.$$

20. A is similar to A' since

$$A' = P^{-1}AP = \begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}.$$

22. A is similar to A' since

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

24. The transition matrix from B' to the standard matrix has columns consisting of the vectors in B'.

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

and it follows that

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

So, the matrix for T relative to B' is

$$\begin{aligned} A' &= P^{-1}AP \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

26. First, note that A and B are similar.

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 2 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 7 & 10 \\ 10 & 8 & 10 \\ -18 & -12 & -17 \end{bmatrix} \end{aligned}$$

Now,

$$\begin{aligned} |B| &= \begin{vmatrix} 11 & 7 & 10 \\ 10 & 8 & 10 \\ -18 & -12 & -17 \end{vmatrix} \\ &= 11(-16) - 7(10) + 10(24) \\ &= -6 = |A|. \end{aligned}$$

28. Because $B = P^{-1}AP$, and $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$,

you have
$$\begin{aligned} B^4 &= P^{-1}A^4P \\ &= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -74 & -225 \\ 30 & 91 \end{bmatrix}. \end{aligned}$$

30. If $B = P^{-1}AP$ and A is an idempotent matrix, then

$$\begin{aligned} B^2 &= \left(P^{-1}AP\right)^2 \\ &= \left(P^{-1}AP\right)\left(P^{-1}AP\right) \\ &= P^{-1}A^2P \\ &= P^{-1}AP \\ &= B, \end{aligned}$$

which shows that B is an idempotent matrix.

32. If $A\mathbf{x} = \mathbf{x}$ and $B = P^{-1}AP$, then $PB = AP$ and $PBP^{-1} = A$. So, $PBP^{-1}\mathbf{x} = A\mathbf{x} = \mathbf{x}$.

34. Because A and B are similar, they represent the same linear transformation with respect to different bases. So, the range is the same, and so is the rank.

36. If A is nonsingular, then so is $P^{-1}AP = B$, and

$$\begin{aligned} B &= P^{-1}AP \\ B^{-1} &= \left(P^{-1}AP\right)^{-1} = P^{-1}A^{-1}\left(P^{-1}\right)^{-1} = P^{-1}A^{-1}P \end{aligned}$$

which shows that A^{-1} and B^{-1} are similar.

38. Because $B = P^{-1}AP$, you have $AP = PB$, as follows.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & b_{nn} \end{bmatrix}$$

So,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{bmatrix} = b_{ii}\begin{bmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{bmatrix}$$

for $i = 1, 2, \ldots, n$.

40. (a) There are two ways to get from the coordinate matrix $[\mathbf{v}]_{B'}$ to the coordinate matrix $[T(\mathbf{v})]_{B'}$. One way is direct, using the matrix A' to obtain $A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$. The second way is indirect, using the matrices P, A, and P^{-1} to obtain $P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$.

(b) To determine if two square matrices A and A' are similar, the equation $A' = P^{-1}AP$ must hold true for some invertible matrix P.

42. (a) True. See discussion, page 330, and note that $A' = P^{-1}AP \Rightarrow PA'P^{-1} = PP^{-1}APP^{-1} = A$.

(b) False. Unless it is a diagonal matrix, see Example 5, page 333.

Section 6.5 Applications of Linear Transformations

2. The standard matrix for T is $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \Rightarrow T(5, 2) = (-5, 2)$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix} \Rightarrow T(-1, -6) = (1, -6)$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix} \Rightarrow T(a, 0) = (-a, 0)$

(d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \Rightarrow T(0, b) = (0, b)$

(e) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c \\ -d \end{bmatrix} = \begin{bmatrix} -c \\ -d \end{bmatrix} \Rightarrow T(c, -d) = (-c, -d)$

(f) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -f \\ g \end{bmatrix} \Rightarrow T(f, g) = (-f, g)$

4. The standard matrix for T is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(a) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow T(-1, 2) = (-2, 1)$

(b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \Rightarrow T(2, 3) = (-3, -2)$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -a \end{bmatrix} \Rightarrow T(a, 0) = (0, -a)$

(d) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} -b \\ 0 \end{bmatrix} \Rightarrow T(0, b) = (-b, 0)$

(e) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} e \\ -d \end{bmatrix} = \begin{bmatrix} d \\ -e \end{bmatrix} \Rightarrow T(e, -d) = (d, -e)$

(f) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -f \\ g \end{bmatrix} = \begin{bmatrix} -g \\ f \end{bmatrix} \Rightarrow T(-f, g) = (-g, f)$

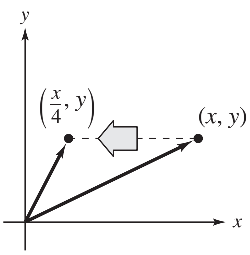
6. (a) $$\begin{aligned} T(x, y) &= xT(1, 0) + yT(0, 1) \\ &= x(1, 1) + y(0, 1) \\ &= (x, x + y) \end{aligned}$$

(b) T is vertical shear.

8. $T(x, y) = \left(\dfrac{x}{4}, y\right)$

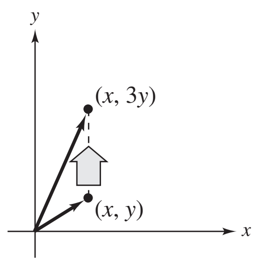
(a) Identify T as a horizontal contraction from its standard matrix $A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$.

(b)



10. (a) Identify T as a vertical expansion from its standard matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

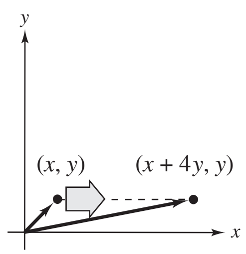
(b)



12. $T(x, y) = (x + 4y, y)$

(a) Identify T as a horizontal shear from its standard matrix $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

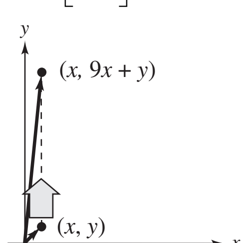
(b)



14. $T(x, y) = (x, 9x + y)$

(a) Identify T as a vertical shear from its matrix $A = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}$.

(b)



16. The reflection in the x-axis is given by $T(x, y) = (x, -y)$. If (x, y) is a fixed point, then $T(x, y) = (x, y) = (x, -y)$ which implies that $y = 0$.
So, the set of fixed points is $\{(t, 0) : t \text{ is real}\}$

18. The reflection in the line $y = -x$ is given by $T(x, y) = (-y, -x)$. If (x, y) is a fixed point then $T(x, y) = (x, y) = (-y, -x)$ which implies $-x = y$.
So, the set of fixed points is $\{(t, -t) : t \text{ is real}\}$

20. A horizontal expansion has the standard matrix $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ where $k > 1$.
A fixed point of T satisfies the equation

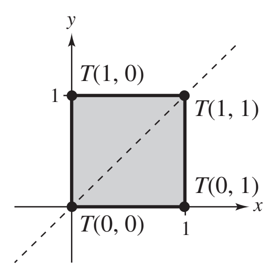
$$T(\mathbf{v}) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} kv_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{v}$$

So the fixed points of T are $\{\mathbf{v} = (0, t) \text{: } t \text{ is a real number}\}$.

22. A vertical shear has the form $T(x, y) = (x, y + kx)$. If (x, y) is a fixed point, then $T(x, y) = (x, y) = (x, y + kx)$ which implies that $x = 0$. So the set of fixed points is $\{(0, t) : t \text{ is real}\}$.

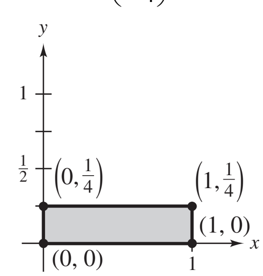
24. Find the image of each vertex under $T(x, y) = (y, x)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (0, 1),$
$T(1, 1) = (1, 1), \quad T(0, 1) = (1, 0)$



26. Find the image of each vertex under $T(x, y) = \left(x, \dfrac{y}{4}\right)$.

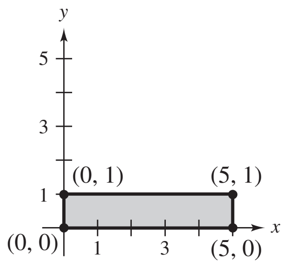
$T(0, 0) = (0, 0), \quad T(1, 0) = (1, 0),$
$T(1, 1) = \left(1, \dfrac{1}{4}\right), \quad T(0, 1) = \left(0, \dfrac{1}{4}\right)$



28. Find the image of each vertex under $T(x, y) = (5x, y)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (5, 0),$

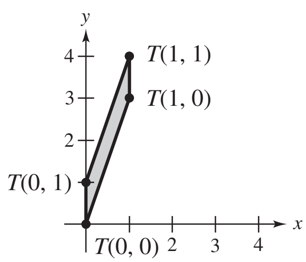
$T(1, 1) = (5, 1), \quad T(0, 1) = (0, 1)$



30. Find the image of each vertex under $T(x, y) = (x, y + 3x)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (1, 3),$

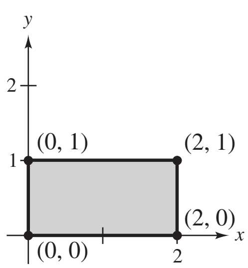
$T(1, 1) = (1, 4), \quad T(0, 1) = (0, 1)$



32. Find the image of each vertex under $T(x, y) = (y, x)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (0, 1),$

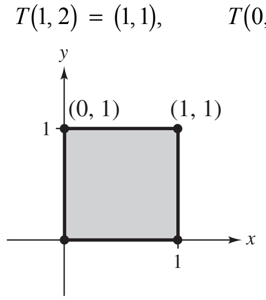
$T(1, 2) = (2, 1), \quad T(0, 2) = (2, 0)$



34. Find the image of each vertex under $T(x, y) = \left(x, \frac{1}{2}y\right)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (1, 0),$

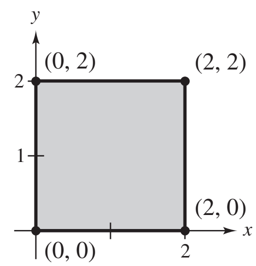
$T(1, 2) = (1, 1), \quad T(0, 2) = (0, 1)$



36. Find the image of each vertex under $T(x, y) = (2x, y)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (2, 0),$

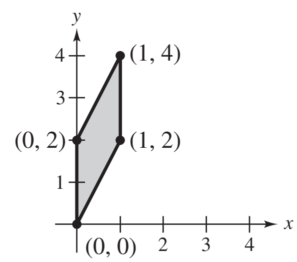
$T(1, 2) = (2, 2), \quad T(0, 2) = (0, 2)$



38. Find the image of each vertex under $T(x, y) = (x, y + 2x)$.

$T(0, 0) = (0, 0), \quad T(1, 0) = (1, 2),$

$T(1, 2) = (1, 4), \quad T(0, 2) = (0, 2)$

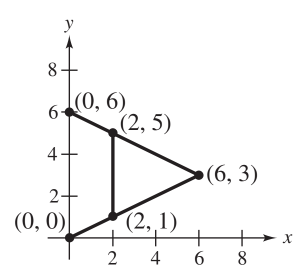


40. Find the image of each vertex under $T(x, y) = (y, x)$.

(a) $T(0, 0) = (0, 0), \quad T(1, 2) = (2, 1),$

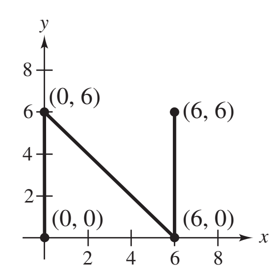
$T(3, 6) = (6, 3), \quad T(5, 2) = (2, 5)$

$T(6, 0) = (0, 6)$



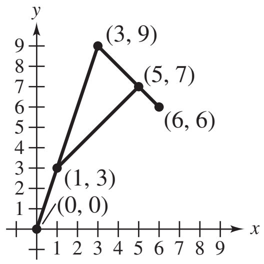
(b) $T(0, 0) = (0, 0), \quad T(0, 6) = (6, 0),$

$T(6, 6) = (6, 6), \quad T(6, 0) = (0, 6)$

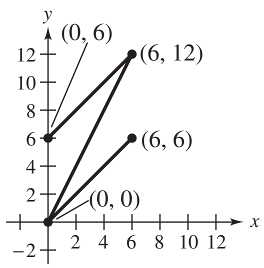


42. Find the image of each vertex under $T(x, y) = (x, x + y)$.

(a) $T(0, 0) = (0, 0)$, $T(1, 2) = (1, 3)$, $T(3, 6) = (3, 9)$, $T(5, 2) = (5, 7)$, $T(6, 0) = (6, 6)$

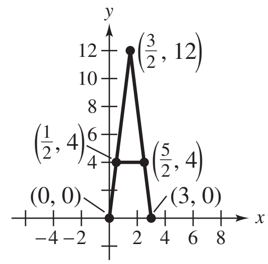


(b) $T(0, 0) = (0, 0)$, $T(0, 6) = (0, 6)$, $T(6, 6) = (6, 12)$, $T(6, 0) = (6, 6)$

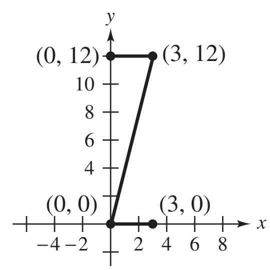


44. Find the image of each vertex under $T(x, y) = \left(\frac{1}{2}x, 2y\right)$.

(a) $T(0, 0) = (0, 0)$, $T(1, 2) = \left(\frac{1}{2}, 4\right)$, $T(3, 6) = \left(\frac{3}{2}, 12\right)$, $T(5, 2) = \left(\frac{5}{2}, 4\right)$, $T(6, 0) = (3, 0)$



(b) $T(0, 0) = (0, 0)$, $T(0, 6) = (0, 12)$, $T(6, 6) = (3, 12)$, $T(6, 0) = (3, 0)$



46. The linear transformation defined by A is a vertical shear.

48. The linear transformation defined by A is a vertical contraction.

50. The linear transformation defined by A is a reflection in the y-axis followed by a horizontal contraction.

52. Because $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ represents a vertical expansion, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents a reflection in the line $x = y$, A is a vertical expansion followed by a reflection in the line $x = y$.

54. (a) The linear transformation of $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ represents a reflection in the y-axis.

(b) The linear transformation of $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a reflection in the x-axis.

(c) The linear transformation of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents a reflection in the line $y = x$.

(d) The linear transformation of $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, where $k > 1$, represents a horizontal expansion.

(e) The linear transformation of $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, where $0 < k < 1$, represents a horizontal contraction.

(f) The linear transformation of $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, where $k > 1$, represents a vertical expansion.

(g) The linear transformation of $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, where $0 < k < 1$, is represented by a vertical contraction.

(h) The linear transformation of $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear.

(i) The linear transformation of $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ represents a vertical shear.

(j) The linear transformation of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ represents a rotation about the x-axis.

(k) The linear transformation of $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ represents a rotation about the y-axis.

(l) The linear transformation of $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ represents a rotation about the z-axis.

56. A rotation of $60°$ about the x-axis is given by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60° & -\sin 60° \\ 0 & \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

58. A rotation of $120°$ about the x-axis is given by the matrix

$$A = \begin{bmatrix} 1 & -0 & 0 \\ 0 & \cos 120° & -\sin 120° \\ 0 & \sin 120° & \cos 120° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

60. Using the matrix obtained in Exercise 56, you find

$$T(1, 1, 1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\left(1 - \sqrt{3}\right)}{2} \\ \frac{\left(1 + \sqrt{3}\right)}{2} \end{bmatrix}.$$

62. Using the matrix obtained in Exercise 58, you find

$$T(1, 1, 1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\left(-1 - \sqrt{3}\right)}{2} \\ \frac{\left(-1 + \sqrt{3}\right)}{2} \end{bmatrix}.$$

64. The indicated tetrahedron is produced by a $-90°$ rotation about the z-axis.

66. The indicated tetrahedron is produced by a $180°$ rotation about the z-axis.

68. The indicated tetrahedron is produced by a $180°$ rotation about the x-axis.

70. The matrix is $\begin{bmatrix} \cos 60° & 0 & \sin 60° \\ 0 & 1 & 0 \\ -\sin 60° & 0 & \cos 60° \end{bmatrix}\begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}.$

$$T(1, 1, 1) = \left(\frac{3\sqrt{3} - 1}{4}, \frac{\sqrt{3} + 1}{2}, \frac{\sqrt{3} - 1}{4}\right)$$

72. The matrix is

$$\begin{bmatrix} \cos 135° & -\sin 135° & 0 \\ \sin 135° & \cos 135° & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 120° & -\sin 120° \\ 0 & \sin 120° & \cos 120° \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

$$T(1, 1, 1) = \left(\frac{\sqrt{6} - \sqrt{2}}{4}, \frac{\sqrt{6} + 3\sqrt{2}}{4}, \frac{\sqrt{3} - 1}{2}\right)$$

Review Exercises for Chapter 6

2. (a) $T(\mathbf{v}) = T(4, -1) = (3, -2)$

(b) $T(v_1, v_2) = (v_1 + v_2, 2v_2) = (8, 4)$

$$v_1 + v_2 = 8$$
$$2v_2 = 4$$
$$v_1 = 6, v_2 = 2$$

Preimage of **w** is $(6, 2)$.

4. (a) $T(v) = T(-2, 1, 2) = (-1, 3, 2)$

(b) $T(v_1, v_2, v_3) = (v_1 + v_2, v_2 + v_3, v_3) = (0, 1, 2)$

$$v_1 + v_2 = 0$$
$$v_2 + v_3 = 1$$
$$v_3 = 2$$
$$v_2 = -1, v_1 = 1$$

Preimage of **w** is $(1, -1, 2)$.

6. (a) $T(\mathbf{v}) = T(2, -3) = 7$

(b) The preimage of **w** is given by solving the equation $T(v_1, v_2) = 2v_1 - v_2 = 4$.

The resulting linear equation $2v_1 - v_2 = 4$ has the solutions $v_1 = \frac{t + 4}{2}$, where t is any real number. So, the preimage of **w** is

$$\left\{\left(\frac{t + 4}{2}, t\right): t \text{ is any real number}\right\}.$$

8. T preserves addition.

$$\begin{aligned} T(x_1, y_1) + T(x_2, y_2) &= (x_1 + y_1) + (x_2 + y_2) \\ &= (x_1 + x_2) + (y_1 + y_2) \\ &= T(x_1 + x_2, y_1 + y_2) \end{aligned}$$

T preserves scalar multiplication.

$$cT(x, y) = c(x + y) = (cx) + (cy) = T(cx, cy)$$

So, T is a linear transformation with standard matrix $\begin{bmatrix} 1 & 1 \end{bmatrix}$.

10. T does not preserve addition or scalar multiplication, so, T is *not* a linear transformation.
A counterexample is

$$\begin{aligned} T(1, 1) + T(1, 0) &= (4, 1) + (4, 0) \\ &= (8, 1) \neq (5, 1) = T(2, 1). \end{aligned}$$

12. $T(x, y) = (x + y, y)$

$$\begin{aligned} T(x_1, y_1) + T(x_2, y_2) &= (x_1 + y_1, y_1) + (x_2 + y_2, y_2) \\ &= x_1 + y_1 + x_2 + y_2, y_1 + y_2 \\ &= (x_1 + x_2) + (y_1 + y_2), y_1 + y_2 \end{aligned}$$

So, T preserves addition.

$$cT(x, y) = c(x + y, y) = cx + cy, cy = T(cx, cy)$$

So, T preserves scalar multiplication.
So, T is a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

14. T does not preserve addition or scalar multiplication, and so, T is *not* a linear transformation. A counterexample is

$$\begin{aligned} -2T(3, -3) &= -2(|3|, |-3|) = (-6, -6) \neq (6, 6) \\ &= T(-6, 6) = T(-2(3), -2(-3)). \end{aligned}$$

16. T preserves addition.

$$\begin{aligned}
T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= (x_1 - x_2, x_2 - x_3, x_3 - x_1) + (y_1 - y_2, y_2 - y_3, y_3 - y_1) \\
&= (x_1 - x_2 + y_1 - y_2, x_2 - x_3 + y_2 - y_3, x_3 - x_1 + y_3 - y_1) \\
&= ((x_1 + y_1) - (x_2 + y_2), (x_2 + y_2) - (x_3 + y_3), (x_3 + y_3) - (x_1 + y_1)) \\
&= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)
\end{aligned}$$

T preserves scalar multiplication.

$$\begin{aligned}
cT(x_1, x_2, x_3) &= c(x_1 - x_2, x_2 - x_3, x_3 - x_1) \\
&= (c(x_1 - x_2), c(x_2 - x_3), c(x_3 - x_1)) \\
&= (cx_1 - cx_2, cx_2 - cx_3, cx_3 - cx_1) \\
&= T(cx_1, cx_2, cx_3)
\end{aligned}$$

So, T is a linear transformation with standard matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

18. T preserves addition.

$$\begin{aligned}
T(x_1, y_1, z_1) + T(x_2, y_2, z_2) &= (x, 0, -y_1) + (x_2, 0, -y_2) \\
&= (x_1 + x_2, 0, -(y_1 + y_2)) \\
&= T(x_1 + x_2, y_1 + y_2, z_1 + z_2)
\end{aligned}$$

T preserves scalar multiplication.

$$cT(x, y, z) = c(x, 0, -y) = (cx, 0, -cy) = T(cx, cy, cz)$$

So, T is a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

20. Because $(0, 1, 1) = (1, 1, 1) - (1, 0, 0)$, you have

$$\begin{aligned}
T(0, 1, 1) &= T(1, 1, 1) - T(1, 0, 0) \\
&= 1 - 3 \\
&= -2.
\end{aligned}$$

22. Because $(2, 4) = 2(1, -1) + 3(0, 2)$, you have

$$\begin{aligned}
T(2, 4) &= 2T(1, -1) + 3T(0, 2) \\
&= 2(2, -3) + 3(0, 8) \\
&= (4, -6) + (0, 24) \\
&= (4, 18).
\end{aligned}$$

24. (a) Because A is a 2×3 matrix, it maps R^3 into R^2, $(n = 3, m = 2)$.

(b) Because $T(\mathbf{v}) = A\mathbf{v}$ and

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix},$$

it follows that $T(5, 2, 2) = (7, 7)$.

(c) The preimage of $\mathbf{w}$ is given by the solution to the equation $T(v_1, v_2, v_3) = \mathbf{w} = (4, 2)$.

The equivalent system of linear equations

$$\begin{aligned}
v_1 + 2v_2 - v_3 &= 4 \\
v_1 \qquad + v_3 &= 2
\end{aligned}$$

has the solution

$\{(2 - t, 1 + t, t) : t \text{ is a real number}\}$.

26. (a) Because A is a 2×2 matrix, it maps R^2 into R^2 $(n = 2, m = 2)$.

(b) Because $T(\mathbf{v}) = A\mathbf{v}$ and

$$A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix}, \text{ it follows that}$$

$T(8, 4) = (20, 4)$.

(c) The preimage of $\mathbf{w}$ is given by the solution to the equation $T(v_1, v_2) = \mathbf{w} = (5, 2)$.

The equivalent system of linear equations

$$\begin{aligned} 2v_1 + v_2 &= 5 \\ v_2 &= 2, v_1 = \tfrac{3}{2} \end{aligned}$$

has the solution $\left(\frac{3}{2}, 2\right)$.

28. (a) Because A is a 3×2 matrix, it maps R^2 into R^3 $(n = 2, m = 3)$.

(b) Because $T(\mathbf{v}) = A\mathbf{v}$ and

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -18 \end{bmatrix}, \text{ it follows that}$$

$T(3, 5) = (-3, 5, -18)$.

(c) The preimage of $\mathbf{w}$ is given by the solution to the equation $T(v_1, v_2) = \mathbf{w} = (5, 2, -1)$.

The equivalent system of linear equations

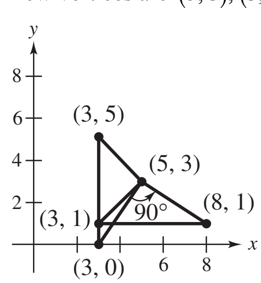
$$\begin{aligned} -v_1 &= 5 \\ v_2 &= 2 \\ -v_1 - 3v_2 &= -1 \end{aligned}$$

has the solution $v_1 = -5$ and $v_2 = 2$. So, the preimage is $(-5, 2)$.

30. If you translate the vertex $(5, 3)$ back to the origin $(0, 0)$, then the other vertices $(3, 5)$ and $(3, 0)$ are translated to $(-2, 2)$ and $(-2, -3)$, respectively. The rotation of $90°$ is given by the matrix in Exercise 29, and you have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Translating back to the original coordinate system, the new vertices are $(5, 3)$, $(3, 1)$ and $(8, 1)$.



32. (a) The standard matrix for T is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Solving $A\mathbf{v} = \mathbf{0}$ yields the solution $\mathbf{v} = \mathbf{0}$. So, $\ker(T) = \{(0, 0, 0)\}$.

(b) Because $\ker(T)$ is dimension 0, $\text{range}(T)$ must be all of R^3.

34. (a) The standard matrix for T is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solving $A\mathbf{v} = \mathbf{0}$ yields the solution $\{(t, -t, t) : t \in R\}$. So, $\ker(T)$ is $\{(1, -1, 1)\}$.

(b) Use Gauss-Jordan elimination to reduce A^T as follows.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors form a basis for the range of T, $\{(1, 0, 1), (0, 1, -1)\}$.

36. To find the kernel of T, row reduce A.

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) $\ker(T) = \{(0, 0)\}$

(b) $\dim(\ker(T)) = \text{nullity}(T) = 0$

(c) $A^T = \begin{bmatrix} -1 & 0 & -2 \\ 2 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$\text{range}(T)$ is span $\{(1, 0, 2), (0, 1, 2)\}$.

(d) $\dim(\text{range}(T)) = \text{rank}(T) = 2$

38. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) $\ker(T) = \{(0, 0, 0)\}$

(b) $\dim(\ker(T)) = \text{nullity}(T) = 0$

(c) $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\text{range}(T)$ is span $\{(1, 0, 0), (0, 1, 0)\ (0, 0, 1)\}$

(d) $\dim(\text{range}(T)) = 3$

40. $\text{Rank}(T) = \dim P_5 - \text{nullity}(T) = 6 - 4 = 2$

42. $\text{nullity}(T) = \dim(M_{3,3}) - \text{rank}(T) = 9 - 5 = 4$

44. The standard matrix for T is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, you have

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A.$$

46. The standard matrix for T, relative to $B = \{1, x, x^2, x^3\}$, is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, you have

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

48. The standard matrix for T_1 and T_2 are

$$A_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3 & 1 \end{bmatrix}.$$

The standard matrix for $T = T_1 \circ T_2$ is

$$A = A_2A_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = [7]$$

and the standard matrix for $T' = T_2 \circ T_1$ is

$$A' = A_1A_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}.$$

50. The standard matrix for T is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

A is invertible and its inverse is given by

$$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

52. The standard matrix for T is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Because A is *not* invertible, T has no inverse.

54. (a) Because $|A| = 1 \neq 0$, $\ker(T) = \{(0, 0)\}$ and T is one-to-one.

(b) Because $\text{rank}(A) = 2$, T is onto.

(c) The transformation is one-to-one and onto, and is, therefore, invertible.

56. (a) Because $|A| = 40 \neq 0$, $\ker(T) = \{(0, 0, 0)\}$, and T is one-to-one.

(b) Because $\text{rank}(A) = 3$, T is onto.

(c) The transformation is one-to-one and onto, and therefore invertible.

58. (a) The standard matrix for T is

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

so it follows that

$$A\mathbf{v} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Rightarrow T(\mathbf{v}) = (6, 0).$$

(b) The image of each vector in B is as follows.

$$T(2, 1) = (2, 0) = -2(-1, 0) + 0(2, 2)$$

$$T(-1, 0) = (0, 0) = 0(-1, 0) + 0(2, 2)$$

Therefore, the matrix for T relative to B and B' is

$$A' = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Because $\mathbf{v} = (-1, 3) = 3(2, 1) + 7(-1, 0)$,

$$[\mathbf{v}]_B = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \text{and} \quad A'[\mathbf{v}]_B = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}.$$

So, $T(\mathbf{v}) = -6(-1, 0) + 0(2, 2) = (6, 0)$.

60. The standard matrix for T is

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The transition matrix from B' to B, the standard matrix, is P

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix A' for T relative to B' is

$$A' = P^{-1}AP$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Because, $A' = P^{-1}AP$, it follows that A and A' are similar.

62. Since $A' = P^{-1}AP$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

A and A' are similar.

64. (a) Because $T(\mathbf{v}) = \text{proj}_{\mathbf{u}}\mathbf{v}$ where $\mathbf{u} = (4, 3)$, you have

$$T(\mathbf{v}) = \frac{4x + 3y}{25}(4, 3).$$

So,

$$T(1, 0) = \left(\frac{16}{25}, \frac{12}{25}\right) \quad \text{and} \quad T(0, 1) = \left(\frac{12}{25}, \frac{9}{25}\right)$$

and the standard matrix for T is

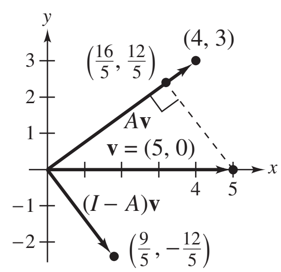
$$A = \frac{1}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}.$$

(b) $$(I - A)^2 = \left(\frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}\right)^2 = \frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix} = I - A.$$

(c) $$A\mathbf{v} = \frac{1}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{16}{5} \\ \frac{12}{5} \end{bmatrix}$$

$$(I - A)\mathbf{v} = \frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{5} \\ -\frac{12}{5} \end{bmatrix}$$

(d)



66. Suppose $\mathbf{b} = \mathbf{0}$. Then $T(\mathbf{v}) = A\mathbf{v}$. $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$

$cT(\mathbf{v}) = c(A\mathbf{v}) = (cA)\mathbf{v} = T(c\mathbf{v})$

So, $T : R^2 \to R^2$ is a linear transformation.

Suppose T is a linear transformation. Then $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) + \mathbf{b}$ and $T(\mathbf{u}) + T(\mathbf{v}) = (A\mathbf{u} + \mathbf{b}) + (A\mathbf{v} + \mathbf{b})$.

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \\ A(\mathbf{u} + \mathbf{v}) + \mathbf{b} &= (A\mathbf{u} + \mathbf{b}) + (A\mathbf{v} + \mathbf{b}) \\ A\mathbf{u} + A\mathbf{v} + \mathbf{b} &= A\mathbf{u} + A\mathbf{v} + 2\mathbf{b} \\ \mathbf{b} &= 2\mathbf{b} \\ 0 &= \mathbf{b} \end{aligned}$$

68. (a) Let $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $S + T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\text{rank}(S + T)$

$= \text{rank}(S) + \text{rank}(T)$.

(b) Let $S = T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $S + T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\text{rank}(S + T)$

$= 1 < 2 = \text{rank}(S) + \text{rank}(T)$.

70. (a) Let $\mathbf{v} \in \text{kernel}(T)$, which implies that $T(\mathbf{v}) = \mathbf{0}$. Clearly $(S \circ T)(\mathbf{v}) = \mathbf{0}$ as well, which shows that $\mathbf{v} \in \text{kernel}(S \circ T)$.

(b) Let $\mathbf{w} \in W$. Because $S \circ T$ is onto, there exists $\mathbf{v} \in V$ such that $(S \circ T)(\mathbf{v}) = \mathbf{w}$. So, $S(T(\mathbf{v})) = \mathbf{w}$, and S is onto.

72. Compute the images of the basis vectors under D_x.

$$\begin{aligned} D_x(1) &= 0 \\ D_x(x) &= 1 \\ D_x(\sin x) &= \cos x \\ D_x(\cos x) &= -\sin x \end{aligned}$$

So, the matrix of D_x relative to this basis is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The range of D_x is spanned by $\{x, \sin x, \cos x\}$, whereas the kernel is spanned by $\{1\}$.

74. First compute the effect of T on the basis $\{1, x, x^2, x^3\}$.

$$\begin{aligned} T(1) &= 1 \\ T(x) &= 1 + x \\ T(x^2) &= 2x + x^2 \\ T(x^3) &= 3x^2 + x^3 \end{aligned}$$

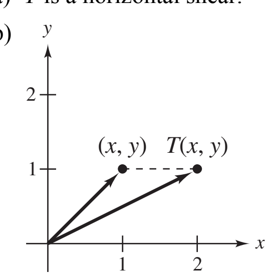
The standard matrix for T is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because the $\text{rank}(A) = 4$, the $\text{rank}(T) = 4$ and $\text{nullity}(T) = 0$.

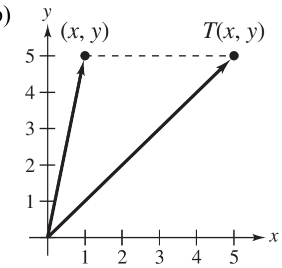
76. (a) T is a horizontal shear.

(b)



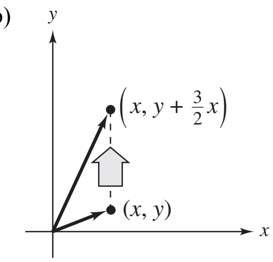
78. (a) T is a horizontal expansion.

(b)

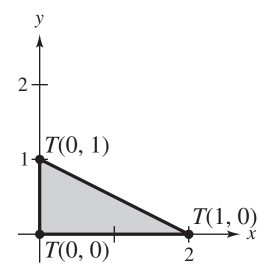


80. (a) T is a vertical shear.

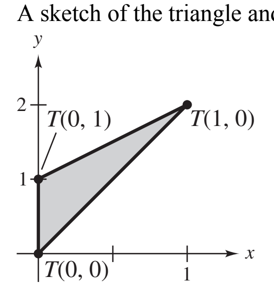
(b)



82. The image of each vertex is $T(0, 0) = (0, 0)$, $T(1, 0) = (2, 0)$, $T(0, 1) = (0, 1)$. A sketch of the triangle and its image follows.



84. The image of each vertex is $T(0, 0) = (0, 0)$, $T(1, 0) = (1, 2)$, $T(0, 1) = (0, 1)$. A sketch of the triangle and its image follows.



86. The transformation is a vertical shear $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ followed by a vertical expansion $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

88. A rotation of $90°$ about the x-axis is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Because $A\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$,

the image of $(1, -1, 1)$ is $(1, -1, -1)$.

90. A rotation of $30°$ about the y-axis is given by

$$A = \begin{bmatrix} \cos 30° & 0 & \sin 30° \\ 0 & 1 & 0 \\ -\sin 30° & 0 & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Because

$$A\mathbf{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} + \frac{1}{2} \\ -1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix},$$

the image of $(1, -1, 1)$ is $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}, -1, -\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$.

94. A rotation of $60°$ about the x-axis is given by

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60° & -\sin 60° \\ 0 & \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

while a rotation of $60°$ about the z-axis is given by

$$A_2 = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the pair of rotations is given by

$$A_2A_1 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

92. A rotation of $120°$ about the y-axis is given by

$$A_1 = \begin{bmatrix} \cos 120° & 0 & \sin 120° \\ 0 & 1 & 0 \\ -\sin 120° & 0 & \cos 120° \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

while a rotation of $45°$ about the z-axis is given by

$$A_2 = \begin{bmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the pair of rotations is given by

$$A_2A_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

96. The standard matrix for T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Therefore, T is given by $T(x, y, z) = (x, -z, y)$. The image of each vertex is as follows.

$T(0, 0, 0) = (0, 0, 0)$

$T(1, 1, 0) = (1, 0, 1)$

$T(0, 0, 1) = (0, -1, 0)$

$T(1, 1, 1) = (1, -1, 1)$

$T(1, 0, 0) = (1, 0, 0)$

$T(0, 1, 0) = (0, 0, 1)$

$T(1, 0, 1) = (1, -1, 0)$

$T(0, 1, 1) = (0, -1, 1)$

98. The standard matrix for T is

$$\begin{bmatrix} \cos 120^\circ & -\sin 120^\circ & 0 \\ \sin 120^\circ & \cos 120^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, T is given by

$T(x, y, z) = \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y, z\right)$. The image of each vertex is as follows.

$T(0, 0, 0) = (0, 0, 0)$

$T(1, 0, 0) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$

$T(1, 1, 0) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}, 0\right)$

$T(0, 1, 0) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right)$

$T(0, 0, 1) = (0, 0, 1)$

$T(1, 0, 1) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\right)$

$T(1, 1, 1) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}, 1\right)$

$T(0, 1, 1) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1\right)$

100. (a) True. The statement is true because if T is a reflection $T(x, y) = (x, -y)$, then the standard matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) True. The statement is true because the linear transformation $T(x, y) = (x, ky)$ has the standard matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$

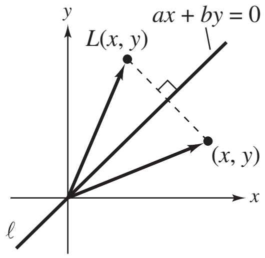
102. (a) True. D_x is a linear transformation because it preserves addition and scalar multiplication. Further, $D_x(P_n) = P_{n-1}$ because for all natural numbers $i \geq 1$, $D_x(x^i) = ix^{i-1}$.

(b) False. If T is a linear transformation $V \to W$, then kernel of T is defined to be a set of $\mathbf{v} \in V$, such that $T(\mathbf{v}) = \mathbf{0}_W$.

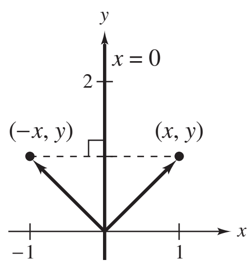
(c) True. If $T = T_2 \circ T_1$ and A_i is the standard matrix for T_i, $i = 1, 2$, then the standard matrix for T is equal $A_2 A_1$ by Theorem 6.11 on page 323.

Project Solutions for Chapter 6

1 Reflections in the Plane-I

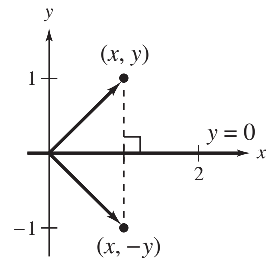


1.



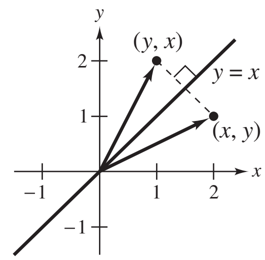
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3.



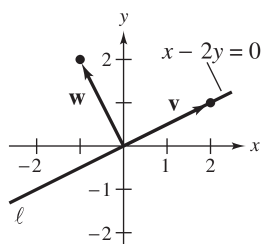
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4. $\mathbf{v} = (2, 1) \qquad B = \{\mathbf{v}, \mathbf{w}\}$

$\mathbf{w} = (-1, 2)$

$L(\mathbf{v}) = \mathbf{v} \qquad L(\mathbf{w}) = -\mathbf{w} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = A$

$B' = \{\mathbf{e}_1, \mathbf{e}_2\}$ standard basis



A is a matrix of L relative to basis B.

$A' = P^{-1}AP$ matrix of L relative to the standard basis B'.

$$[B' \,\vdots\, B] \to [I \,\vdots\, P^{-1}] \Rightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \Rightarrow P = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A' = P^{-1}AP = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

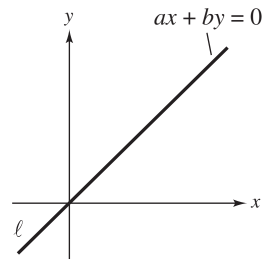
$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

5. $\mathbf{v} = (-b, a)$ $\qquad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\mathbf{w} = (a, b)$



$$P^{-1} = \begin{bmatrix} -b & a \\ a & b \end{bmatrix}$$

$$P = \frac{1}{a^2 + b^2}\begin{bmatrix} -b & +a \\ +a & +b \end{bmatrix}$$

$$A' = P^{-1}AP = \begin{bmatrix} -b & a \\ a & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P = \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix}\begin{bmatrix} -b & a \\ a & b \end{bmatrix}\frac{1}{a^2 + b^2} = \frac{1}{a^2 + b^2}\begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix}$$

6. $3x + 4y = 0 \qquad A' = \dfrac{1}{3^2 + 4^2}\begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix}$

$$\frac{1}{25}\begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -75 \\ -100 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -100 \\ 75 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -24 \cdot 5 \\ -7 \cdot 5 \end{bmatrix} = \begin{bmatrix} -\frac{24}{5} \\ -\frac{7}{5} \end{bmatrix}$$

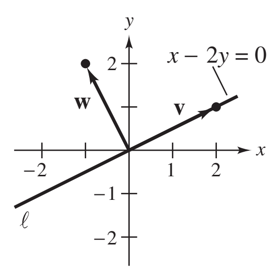
2 Reflections in the Plane-II

1. $\mathbf{v} = (0, 1) \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\mathbf{v} = (1, 0) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\mathbf{v} = (2, 1) \qquad B = \{\mathbf{v}, \mathbf{w}\}$

$\mathbf{w} = (-1, 2)$

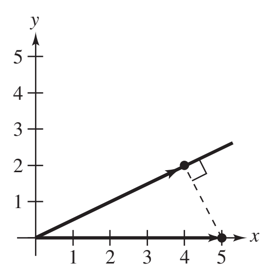


$$\left.\begin{aligned} \text{proj}_{\mathbf{v}}\mathbf{v} &= \mathbf{v} \\ \text{proj}_{\mathbf{v}}\mathbf{w} &= \mathbf{0} \end{aligned}\right\} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \; P = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$A' = P^{-1}AP =$ matrix of L relative to standard basis.

$$= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\frac{1}{5} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$



$$\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

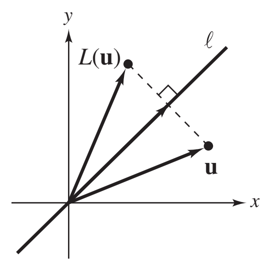
4. $\mathbf{v} = (-b, a)$ $\qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\mathbf{w} = (a, b)$

$$P^{-1} = \begin{bmatrix} -b & a \\ a & b \end{bmatrix} \qquad P = \frac{1}{a^2 + b^2}\begin{bmatrix} -b & a \\ a & b \end{bmatrix}$$

$$A' = P^{-1}AP = \frac{1}{a^2 + b^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix}$$

5. $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{1}{2}(\mathbf{u} + L(\mathbf{u})) \quad \Rightarrow \quad L(\mathbf{u}) = 2\text{proj}_{\mathbf{v}}\mathbf{u} - \mathbf{u}$



$$\begin{aligned} L &= 2\,\text{proj}_{\mathbf{v}} - I \\ &= 2\frac{1}{a^2 + b^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{a^2 + b^2}\left(\begin{bmatrix} 2b^2 & -2ab \\ -2ab & 2a^2 \end{bmatrix} - \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}\right) \\ &= \frac{1}{a^2 + b^2}\begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix} \end{aligned}$$