C H A P T E R 5
Inner Product Spaces

CHAPTER 5
Inner Product Spaces

Section 5.1 Length and Dot Product in R^n

2. $\|\mathbf{v}\| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

4. $\|\mathbf{v}\| = \sqrt{2^2 + 0^2 + (-5)^2 + 5^2} = \sqrt{54} = 3\sqrt{6}$

6. (a) $\|\mathbf{u}\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$

(b) $\|\mathbf{v}\| = \sqrt{2^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{17}{4}} = \frac{1}{2}\sqrt{17}$

(c) $\|\mathbf{u} + \mathbf{v}\| = \|(3, 0)\| = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$

8. (a) $\|\mathbf{u}\| = \sqrt{0^2 + 1^2 + (-1)^2 + 2^2} = \sqrt{6}$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 3^2 + 0^2} = \sqrt{11}$

(c) $\|\mathbf{u} + \mathbf{v}\| = \|(1, 2, 2, 2)\|$

$= \sqrt{1^2 + 2^2 + 2^2 + 2^2} = \sqrt{13}$

10. (a) A unit vector **v** in the direction of **u** is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2^2 + (-2)^2}}(2, -2) = \frac{\sqrt{2}}{4}(2, -2) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

$$\|\mathbf{v}\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

(b) A unit vector in the direction opposite that of **u** is given by

$$-\mathbf{v} = -\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

$$\|\mathbf{v}\| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

12. (a) A unit vector **v** in the direction of **u** is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{(-1)^2 + 3^2 + 4^2}}(-1, 3, 4) = \frac{1}{\sqrt{26}}(-1, 3, 4) = \left(-\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}\right).$$

$$\|\mathbf{v}\| = \sqrt{\left(-\frac{1}{\sqrt{26}}\right)^2 + \left(\frac{3}{\sqrt{26}}\right)^2 + \left(\frac{4}{\sqrt{26}}\right)^2} = \sqrt{\frac{1}{26} + \frac{9}{26} + \frac{16}{26}} = 1$$

(b) A unit vector in the direction opposite that of **u** is given by

$$-\mathbf{v} = -\left(-\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}\right) = \left(\frac{1}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, -\frac{4}{\sqrt{26}}\right).$$

$$\|-\mathbf{v}\| = \sqrt{\left(\frac{1}{\sqrt{26}}\right)^2 + \left(-\frac{3}{\sqrt{26}}\right)^2 + \left(-\frac{4}{\sqrt{26}}\right)^2} = \sqrt{\frac{1}{26} + \frac{9}{26} + \frac{16}{26}} = 1$$

14. First find a unit vector in the direction of **u**.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{(-1)^2 + 1^2}}(-1, 1) = \frac{1}{\sqrt{2}}(-1, 1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Then **v** is four times this vector.

$$\mathbf{v} = 4\frac{\mathbf{u}}{\|\mathbf{u}\|} = 4\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right) = \left(-2\sqrt{2}, 2\sqrt{2}\right)$$

16. First find a unit vector in the direction of **u**.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{0 + 4 + 1 + 1}}(0, 2, 1, -1) = \frac{1}{\sqrt{6}}(0, 2, 1, -1)$$

Then **v** is three times this vector.

$$\mathbf{v} = 3\frac{1}{\sqrt{6}}(0, 2, 1, -1) = \left(0, \frac{6}{\sqrt{6}}, \frac{3}{\sqrt{6}}, -\frac{3}{\sqrt{6}}\right)$$

18. Solve the equation for c as follows.

$$\|c(1, 2, 3)\| = 1$$
$$|c|\|(1, 2, 3)\| = 1$$
$$|c| = \frac{1}{\|(1, 2, 3)\|} = \frac{1}{\sqrt{14}}$$
$$c = \pm\frac{1}{\sqrt{14}}$$

20.
$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \|(-4, 2, 6)\| \\ &= \sqrt{(-4)^2 + 2^2 + 6^2} \\ &= 2\sqrt{14} \end{aligned}$$

22.
$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \|(-1, 0, -3, 0)\| \\ &= \sqrt{(-1)^2 + 0^2 + (-3)^2 + 0^2} \\ &= \sqrt{10} \end{aligned}$$

24. (a) $\mathbf{u} \cdot \mathbf{v} = (-1)(2) + (2)(-2) = -2 - 4 = -6$

(b) $\mathbf{v} \cdot \mathbf{v} = 2(2) + (-2)(-2) = 4 + 4 = 8$

(c) $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = (-1)(-1) + (2)(2) = 1 + 4 = 5$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -6(2, -2) = (-12, 12)$

(e) $\mathbf{u} \cdot (5\mathbf{v}) = 5(\mathbf{u} \cdot \mathbf{v}) = 5(-6) = -30$

26. (a)
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 4(0) + 0(2) + (-3)(5) + (5)(4) \\ &= 0 + 0 - 15 + 20 \\ &= 5 \end{aligned}$$

(b)
$$\begin{aligned} \mathbf{v} \cdot \mathbf{v} &= 0(0) + 2(2) + 5(5) + 4(4) \\ &= 0 + 4 + 25 + 16 \\ &= 45 \end{aligned}$$

(c)
$$\begin{aligned} \|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} &= 4(4) + 0(0) + (-3)(-3) + 5(5) \\ &= 16 + 0 + 9 + 25 \\ &= 50 \end{aligned}$$

(d)
$$\begin{aligned} (\mathbf{u} \cdot \mathbf{v})\mathbf{v} &= 5(0, 2, 5, 4) \\ &= (0, 10, 25, 20) \end{aligned}$$

(e) $\mathbf{u} \cdot (5\mathbf{v}) = 5(\mathbf{u} \cdot \mathbf{v}) = 5(5) = 25$

28.
$$\begin{aligned} (3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v}) &= 3\mathbf{u} \cdot (\mathbf{u} - 3\mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} - 3\mathbf{v}) \\ &= 3\mathbf{u} \cdot \mathbf{u} - 9\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + 3\mathbf{v} \cdot \mathbf{v} \\ &= 3\mathbf{u} \cdot \mathbf{u} - 10\mathbf{u} \cdot \mathbf{v} + 3\mathbf{v} \cdot \mathbf{v} \\ &= 3(8) - 10(7) + 3(6) \\ &= -28 \end{aligned}$$

30. $\mathbf{u} = \left(-1, \frac{1}{2}, \frac{1}{4}\right)$ and $\mathbf{v} = \left(0, \frac{1}{4}, -\frac{1}{2}\right)$

(a) $\|\mathbf{u}\| = 1.1456$ and $\|\mathbf{v}\| = 0.5590$

(b) $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = (0, 0.4472, -0.8944)$

(c) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = (0.8729, -0.4364, -0.2182)$

(d) $\mathbf{u} \cdot \mathbf{v} = 0$

(e) $\mathbf{u} \cdot \mathbf{u} = 1.3125$

(f) $\mathbf{v} \cdot \mathbf{v} = 0.3125$

32. $\mathbf{u} = \left(-1, \sqrt{3}, 2\right)$ and $\mathbf{v} = \left(\sqrt{2}, -1, -\sqrt{2}\right)$

(a) $\|\mathbf{u}\| = 2.8284$ and $\|\mathbf{v}\| = 2.2361$

(b) $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = (0.6325, -0.4472, -0.6325)$

(c) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = (0.3536, -0.6124, -0.7071)$

(d) $\mathbf{u} \cdot \mathbf{v} = -5.9747$

(e) $\mathbf{u} \cdot \mathbf{u} = 8$

(f) $\mathbf{v} \cdot \mathbf{v} = 5$

34. (a) $\|\mathbf{u}\| = \sqrt{6}, \|\mathbf{v}\| = \sqrt{3}$

(b) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3} - \frac{\sqrt{6}}{6}\right)$

(c) $-\frac{\mathbf{u}}{\|\mathbf{u}\|} = \left(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{3}}{3}\right)$

(d) $\mathbf{u} \cdot \mathbf{v} = -2$

(e) $\mathbf{u} \cdot \mathbf{u} = 6$

(f) $\mathbf{v} \cdot \mathbf{v} = 3$

36. You have

$\mathbf{u} \cdot \mathbf{v} = -1(1) + 0(1) = -1,$

$\|\mathbf{u}\| = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$, and

$\|\mathbf{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$. So,

$$\begin{aligned} |\mathbf{u} \cdot \mathbf{v}| &\le \|\mathbf{u}\|\|\mathbf{v}\| \\ |-1| &\le 1\sqrt{2} \\ 1 &\le \sqrt{2}. \end{aligned}$$

38. You have

$$\mathbf{u}\cdot\mathbf{v} = 1(0) - 1(1) + 0(-1) = -1,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}, \text{ and}$$

$$\|\mathbf{v}\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}. \text{ So,}$$

$$\begin{aligned}|\mathbf{u}\cdot\mathbf{v}| &\le \|\mathbf{u}\|\|\mathbf{v}\|\\ |-1| &\le \sqrt{2}\cdot\sqrt{2}\\ 1 &\le 2.\end{aligned}$$

40. The cosine of the angle θ between **u** and **v** is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-4)(5) + (1)(0)}{\sqrt{(-4)^2 + 1^2}\sqrt{5^2 + 0^2}} = \frac{-20}{5\sqrt{17}} = \frac{-4}{\sqrt{17}}$$

So, $\theta = \cos^{-1}\left(-\frac{4}{\sqrt{17}}\right) \approx 2.897$ radians $(165.96°)$.

42. The cosine of the angle θ between **u** and **v** is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\cos\frac{\pi}{3}\left(\cos\frac{\pi}{4}\right) + \sin\frac{\pi}{3}\left(\sin\frac{\pi}{4}\right)}{\sqrt{\left(\cos\frac{\pi}{3}\right)^2 + \left(\sin\frac{\pi}{3}\right)^2}\sqrt{\left(\cos\frac{\pi}{4}\right)^2 + \left(\sin\frac{\pi}{4}\right)^2}} = \frac{\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}{1\cdot 1} = \cos\left(\frac{\pi}{12}\right).$$

So, $\theta = \frac{\pi}{12}$ radians $(15°)$.

44. The cosine of the angle θ between **u** and **v** is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2(-3) + 3(2) + 1(0)}{\|\mathbf{u}\|\|\mathbf{v}\|} = 0.$$

So, $\theta = \frac{\pi}{2}$ radians $(90°)$.

46. The cosine of the angle θ between **u** and **v** is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1(-1) - 1(2) + 0(-1) + 1(0)}{\sqrt{1^2 + (-1)^2 + 0^2 + 1^2}\sqrt{(-1)^2 + 2^2 + (-1)^2 + 0^2}} = \frac{-3}{\sqrt{3}\sqrt{6}} = -\frac{3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

So, $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) \approx \frac{3\pi}{4}$ radians $(135°)$.

48. Because $\mathbf{u}\cdot\mathbf{v} = (4, 3)\cdot\left(\frac{1}{2}, -\frac{2}{3}\right) = 2 - 2 = 0$, the vectors **u** and **v** are orthogonal.

50. Because $\mathbf{u}\cdot\mathbf{v} = 1(0) - 1(-1) = 1 \ne 0$, the vectors **u** and **v** are not orthogonal. Moreover, because one is not a scalar multiple of the other, they are not parallel.

52. Because $\mathbf{u}\cdot\mathbf{v} = 0(1) + (3)(-8) + (-4)(-6) = 0$, the vectors **u** and **v** are orthogonal.

54. Because

$$\mathbf{u}\cdot\mathbf{v} = 4(-2) + \tfrac{3}{2}\left(-\tfrac{3}{4}\right) + (-1)\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\left(-\tfrac{1}{4}\right) = -\tfrac{39}{4} \ne 0,$$

the vectors are not orthogonal. Moreover, because one vector is a scalar multiple of the other, they are parallel.

56.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 0\\ (11, 2)\cdot(v_1, v_2) &= 0\\ 11v_1 + 2v_2 &= 0\end{aligned}$$

So, $\mathbf{v} = (-2t, 11t)$, where t is any real number.

58.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 0\\ (4, -1, 0)\cdot(v_1, v_2, v_3) &= 0\\ 4v_1 + (-1)v_2 + 0v_3 &= 0\\ 4v_1 - v_2 &= 0\end{aligned}$$

So, $\mathbf{v} = (t, 4t, s)$, where s and t are any real numbers.

60. Because $\mathbf{u} + \mathbf{v} = (-1, 1) + (2, 0) = (1, 1)$, you have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\|\\ \|(1, 1)\| &\le \|(-1, 1)\| + \|(2, 0)\|\\ \sqrt{2} &\le \sqrt{2} + 2.\end{aligned}$$

62. Because $\mathbf{u} + \mathbf{v} = (1, -1, 0) + (0, 1, 2) = (1, 0, 2)$, you have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\|\\ \|(1, 0, 2)\| &\le \|(1, -1, 0)\| + \|(0, 1, 2)\|\\ \sqrt{5} &\le \sqrt{2} + \sqrt{5}.\end{aligned}$$

64. First note that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, because $\mathbf{u}\cdot\mathbf{v} = (3, -2)\cdot(4, 6) = 0.$

Then note

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\\ \|(7, 4)\|^2 &= \|(3, -2)\|^2 + \|(4, 6)\|^2\\ 65 &= 13 + 52\\ 65 &= 65.\end{aligned}$$

66. First note that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, because $\mathbf{u}\cdot\mathbf{v} = (4, 1, -5)\cdot(2, -3, 1) = 0.$

Then note

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\\ \|(6, -2, -4)\|^2 &= \|(4, 1, -5)\|^2 + \|(2, -3, 1)\|^2\\ 56 &= 42 + 14\\ 56 &= 56.\end{aligned}$$

68. (a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}(-1)(2) + (2)(-2)\end{bmatrix} = \begin{bmatrix}-2 & -4\end{bmatrix} = -6$

(b) $\mathbf{v}\cdot\mathbf{v} = \mathbf{v}^T\mathbf{v} = \begin{bmatrix}2 & -2\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}2(2) + (-2)(-2)\end{bmatrix} = \begin{bmatrix}4 + 4\end{bmatrix} = 8$

(c) $\|\mathbf{u}\|^2 = \mathbf{u}^T\mathbf{u} = \begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}(-1)(-1) + 2(2)\end{bmatrix} = \begin{bmatrix}1 + 4\end{bmatrix} = 5$

(d) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v} = (\mathbf{u}^T\mathbf{v})\mathbf{v} = \left(\begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix}\right)\begin{bmatrix}2\\-2\end{bmatrix} = -6\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}-12\\12\end{bmatrix}$

(e) $\mathbf{u}\cdot(5\mathbf{v}) = 5(\mathbf{u}^T\mathbf{v}) = 5\left(\begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix}\right) = 5(-6) = -30$

70. (a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix}4 & 0 & -3 & 5\end{bmatrix}\begin{bmatrix}0\\2\\5\\4\end{bmatrix} = \begin{bmatrix}4(0) + 0(2) + (-3)(5) + (5)(4)\end{bmatrix} = \begin{bmatrix}0 + 0 - 15 + 20\end{bmatrix} = 5$

(b) $\mathbf{v}\cdot\mathbf{v} = \mathbf{v}^T\mathbf{v} = \begin{bmatrix}0 & 2 & 5 & 4\end{bmatrix}\begin{bmatrix}0\\2\\5\\4\end{bmatrix} = \begin{bmatrix}0(0) + 2(2) + 5(5) + 4(4)\end{bmatrix} = \begin{bmatrix}0 + 4 + 25 + 16\end{bmatrix} = 45$

(c) $\|\mathbf{u}\|^2 = \mathbf{u}^T\mathbf{u} = \begin{bmatrix}4 & 0 & -3 & 5\end{bmatrix}\begin{bmatrix}4\\0\\-3\\5\end{bmatrix} = \begin{bmatrix}4(4) + 0(0) + (-3)(-3) + 5(5)\end{bmatrix} = \begin{bmatrix}16 + 0 + 9 + 25\end{bmatrix} = 50$

(d) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v} = (\mathbf{u}^T\mathbf{v})\mathbf{v} = \left(\begin{bmatrix}4 & 0 & -3 & 5\end{bmatrix}\begin{bmatrix}0\\2\\5\\4\end{bmatrix}\right)\begin{bmatrix}0\\2\\5\\4\end{bmatrix} = 5\begin{bmatrix}0\\2\\5\\4\end{bmatrix} = \begin{bmatrix}0\\10\\25\\20\end{bmatrix}$

(e) $\mathbf{u}\cdot(5\mathbf{v}) = 5(\mathbf{u}^T\mathbf{v}) = 5\left(\begin{bmatrix}4 & 0 & -3 & 5\end{bmatrix}\begin{bmatrix}0\\2\\5\\4\end{bmatrix}\right) = 5(5) = 25$

72. Because $\mathbf{u} \cdot \mathbf{v} = -\sin\theta\sin\theta + \cos\theta(-\cos\theta) + 1(0)$

$$= -(\sin\theta)^2 - (\cos\theta)^2$$
$$= -(\sin^2\theta + \cos^2\theta)$$
$$= -1 \neq 0,$$

the vectors **u** and **v** are not orthogonal. Moreover, because one is not a scalar multiple of the other, they are not parallel.

74. (a) False. The unit vector in the direction of **v** is given by $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

(b) False. If $\mathbf{u} \cdot \mathbf{v} < 0$ then the angle between them lies between $\dfrac{\pi}{2}$ and π, because

$$\cos\theta < 0 \Rightarrow \frac{\pi}{2} < \theta < \pi.$$

76. (a) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{u}$ is meaningless because $\mathbf{u} \cdot \mathbf{v}$ is a scalar.

(b) $c \cdot (\mathbf{u} \cdot \mathbf{v})$ is meaningless because c is a scalar, as well as $\mathbf{u} \cdot \mathbf{v}$.

78. $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) = (8, 15), (\mathbf{v}_2, -\mathbf{v}_1) = (15, -8)$

$$(8, 15) \cdot (15, -8) = 8(15) + 15(-8) = 120 - 120 = 0$$

So, $(\mathbf{v}_2, -\mathbf{v}_1)$ is orthogonal to **v**.

Answers will vary. Sample answer:

Two unit vectors orthogonal to **v:**

$$\begin{aligned} -1(15, -8) = (-15, 8)\text{: } (8, 15) \cdot (-15, 8) &= 8(-15) + 15(8) \\ &= -120 + 120 \\ &= 0 \\ 3(15, -8) = (45, -24)\text{: } (8, 15) \cdot (45, -24) &= 8(45) + (15)(-24) \\ &= 360 - 360 \\ &= 0 \end{aligned}$$

80.
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (4600, 4290, 5250) \cdot (499.99, 199.99, 99.99) \\ &= 4600(499.99) + 4290(199.99) + 5250(99.99) \\ &= \$3,682,858.60 \end{aligned}$$

This represents the total revenue earned from selling the three models of cellular phones.

82. Let $\mathbf{v} = (t, t, t)$ be the diagonal of the cube, and $\mathbf{u} = (t, t, 0)$ the diagonal of one of its sides. Then,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2t^2}{(\sqrt{2}\,t)(\sqrt{3}\,t)} = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

and $\theta = \cos^{-1}\left(\dfrac{\sqrt{6}}{3}\right) \approx 35.26°$.

84.
$$\begin{aligned} \tfrac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \tfrac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2 &= \tfrac{1}{4}\left[(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})\right] \\ &= \tfrac{1}{4}\left[\mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} - (\mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v})\right] \\ &= \tfrac{1}{4}[4\mathbf{u} \cdot \mathbf{v}] = \mathbf{u} \cdot \mathbf{v} \end{aligned}$$

86. If $\mathbf{u}$ and $\mathbf{v}$ have the same direction, then $\mathbf{u} = c\mathbf{v}$, $c > 0$, and

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| = \|c\mathbf{v} + \mathbf{v}\| &= (c + 1)\|\mathbf{v}\| \\ &= c\|\mathbf{v}\| + \|\mathbf{v}\| = \|c\mathbf{v}\| + \|\mathbf{v}\| \\ &= \|\mathbf{u}\| + \|\mathbf{v}\|.\end{aligned}$$

On the other hand, if

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| &= \|\mathbf{u}\| + \|\mathbf{v}\|, \text{ then} \\ \|\mathbf{u} + \mathbf{v}\|^2 &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 \\ (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| \\ \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| \\ 2\mathbf{u} \cdot \mathbf{v} &= 2\|\mathbf{u}\|\|\mathbf{v}\|\end{aligned}$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = 1 \quad \Rightarrow \quad \theta = 0 \quad \Rightarrow \quad \mathbf{u} \text{ and } \mathbf{v} \text{ have the same direction.}$$

88. (a) When $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $(\theta = 90°)$.

(b) When $\mathbf{u} \cdot \mathbf{v} > 0$, the vectors form an acute angle for $\theta \left(0° \le \theta < 90° \text{ or } 0 \le \theta < \dfrac{\pi}{2}\right)$.

(c) When $\mathbf{u} \cdot \mathbf{v} < 0$, the vectors form an obtuse angle for $\theta \left(90° < \theta \le 180° \text{ or } \dfrac{\pi}{2} < \theta \le \pi\right)$.

Section 5.2 Inner Product Spaces

2. 1. Since the product of real numbers is commutative,
$\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + 9u_2v_2 = v_1u_1 + 9v_2u_2 = \langle \mathbf{v}, \mathbf{u}\rangle$.

2. Let $\mathbf{w} = (w_1, w_2)$. Then,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle &= u_1(v_1 + w_1) + 9u_2(v_2 + w_2) \\ &= u_1v_1 + u_1w_1 + 9u_2v_2 + 9u_2w_2 \\ &= u_1v_1 + 9u_2v_2 + u_1w_1 + 9u_2w_2 \\ &= \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle.\end{aligned}$$

3. If c is any scalar, then
$c\langle \mathbf{u}, \mathbf{v}\rangle = c(u_1v_1 + 9u_2v_2) = (cu_1)v_1 + 9(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v}\rangle$.

4. Since the square of a real number is nonnegative, $\langle \mathbf{v}, \mathbf{v}\rangle = v_1^2 + 9v_2^2 \ge 0$. Moreover, this expression is equal to zero if and only if $\mathbf{v} = \mathbf{0}$ (that is, if and only if $v_1 = v_2 = 0$).

4. 1. Since the product of real numbers is commutative,
$\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_2 + u_2v_1 + u_1v_2 + 2u_2v_2 = 2v_2u_1 + v_1u_2 + v_2u_1 + 2v_2u_2 = \langle \mathbf{v}, \mathbf{u}\rangle$.

2. Let $\mathbf{w} = (w_1, w_2)$. Then,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle &= 2u_1(v_2 + w_2) + u_2(v_1 + w_1) + u_1(v_2 + w_2) + 2u_2(v_2 + w_2) \\ &= 2u_1v_2 + 2u_1w_2 + u_2v_1 + u_2w_1 + u_1v_2 + u_1w_2 + 2u_2v_2 + 2u_2w_2 \\ &= 2u_1v_2 + u_2v_1 + u_1v_2 + 2u_2v_2 + 2u_1w_2 + u_2w_1 + u_1w_2 + 2u_2w_2 \\ &= \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle.\end{aligned}$$

3. If c is any scalar, then
$c\langle \mathbf{u}, \mathbf{v}\rangle = c(2u_1v_2 + u_2v_1 + u_1v_2 + 2u_2v_2) = 2(cu_1)v_2 + (cu_2)v_1 + (cu_1)v_2 + 2(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v}\rangle$.

4. Since the square of a real number is nonnegative, $\langle \mathbf{v}, \mathbf{v} \rangle = 2v_2^2 + v_1^2 + v_2^2 + 2v_2^2 \geq 0$. Moreover, this expression is equal to zero if and only if $\mathbf{v} = \mathbf{0}$ (that is, if and only if $v_1 = v_2 = 0$).

6. 1. Since the product of real numbers is commutative,
$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3 = v_1u_1 + 2v_2u_2 + v_3u_3 = \langle \mathbf{v}, \mathbf{u} \rangle$.

2. Let $\mathbf{w} = (w_1, w_2, w_3)$. Then,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2) + u_3(v_3 + w_3) \\ &= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2 + u_3v_3 + u_3w_3 \\ &= u_1v_1 + 2u_2v_2 + u_3v_3 + u_1w_1 + 2u_2w_2 + u_3w_3 \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle.\end{aligned}$$

3. If c is any scalar, then
$c\langle \mathbf{u}, \mathbf{v} \rangle = c(u_1v_1 + 2u_2v_2 + u_3v_3) = (cu_1)v_1 + 2(cu_2)v_2 + (cu_3)v_3 = \langle c\mathbf{u}, \mathbf{v} \rangle$.

4. Since the square of a real number is nonnegative, $\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 + 2v_2^2 + v_3^2 \geq 0$. Moreover, this expression is equal to zero if and only if $\mathbf{v} = \mathbf{0}$ (that is, if and only if $v_1 = v_2 = v_3 = 0$).

8. 1. Since the product of real numbers is commutative,
$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + \frac{1}{4}u_2v_2 + \frac{1}{2}u_3v_3 = \frac{1}{2}v_1u_1 + \frac{1}{4}v_2u_2 + \frac{1}{2}v_3u_3 = \langle \mathbf{v}, \mathbf{u} \rangle$.

2. Let $\mathbf{w} = (w_1, w_2, w_3)$. Then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= \tfrac{1}{2}u_1(v_1 + w_1) + \tfrac{1}{4}u_2(v_2 + w_2) + \tfrac{1}{2}u_3(v_3 + w_3) \\ &= \tfrac{1}{2}u_1v_1 + \tfrac{1}{2}u_1w_1 + \tfrac{1}{4}u_2v_2 + \tfrac{1}{4}u_2w_2 + \tfrac{1}{2}u_3v_3 + \tfrac{1}{2}u_3w_3 \\ &= \tfrac{1}{2}u_1v_1 + \tfrac{1}{4}u_2v_2 + \tfrac{1}{2}u_3v_3 + \tfrac{1}{2}u_1w_1 + \tfrac{1}{4}u_2w_2 + \tfrac{1}{2}u_3w_3 \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle.\end{aligned}$$

3. If c is any scalar, then
$c\langle \mathbf{u}, \mathbf{v} \rangle = c\left(\frac{1}{2}u_1v_1 + \frac{1}{4}u_2v_2 + \frac{1}{2}u_3v_3\right) = \frac{1}{2}(cu_1)v_1 + \frac{1}{4}(cu_2)v_2 + \frac{1}{2}(cu_3)v_3 = \langle c\mathbf{u}, \mathbf{v} \rangle$.

4. Since the square of a real number is nonnegative, $\langle \mathbf{v}, \mathbf{v} \rangle = \frac{1}{2}v_1^2 + \frac{1}{4}v_2^2 + \frac{1}{2}v_3^2 \geq 0$. Moreover, this expression is equal to zero if and only if $\mathbf{v} = \mathbf{0}$ (that is, if and only if $v_1 = v_2 = 0$).

10. The product $\langle \mathbf{u}, \mathbf{v} \rangle$ is not an inner product because Axiom 4 is not satisfied. For example, let $\mathbf{v} = (1, 1)$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = (1)(1) - 6(1)(1) = -5$, which is less than zero.

12. The product $\langle \mathbf{u}, \mathbf{v} \rangle$ is not an inner product because it is not commutative. For example, if $\mathbf{u} = (1, 2)$, and $\mathbf{v} = (2, 3)$, then $\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(3) - 2(2) = 5$ while $\langle \mathbf{v}, \mathbf{u} \rangle = 3(2)(2) - 3(1) = 9$.

14. The product $\langle \mathbf{u}, \mathbf{v} \rangle$ is not an inner product because nonzero vectors can have a norm of zero. For example, if $\mathbf{v} = (1, 1, 0)$, then $\langle (1, 1, 0), (1, 1, 0) \rangle = 0$.

16. The product $\langle \mathbf{u}, \mathbf{v} \rangle$ is not an inner product because Axiom 2 is not satisfied. For example, let $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (1, 1, 1)$, and $\mathbf{w} = (2, 1, 2)$.

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = 2(1)(0) + 3(3)(2) + 0(3) = 18$$

$$\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle = 2(1)(0) + 3(1)(1) + 0(1) + 2(1)(0) + 3(2)(1) + 0(2) = 9$$

So, $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle \neq \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

18. (a) $\langle \mathbf{u}, \mathbf{v} \rangle = -1(6) + 1(8) = 2$

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

(c) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{6^2 + 8^2} = 10$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-7, -7)\| = \sqrt{(-7)^2 + (-7)^2} = 7\sqrt{2}$

20. (a) $\langle \mathbf{u}, \mathbf{v} \rangle = 0(-1) + 2(-6)(1) = -12$

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{0(0) + 2(-6)(-6)} = 6\sqrt{2}$

(c) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{(-1)(-1) + 2(1)(1)} = \sqrt{3}$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(1, -7)\| = \sqrt{1(1) + 2(-7)(-7)} = \sqrt{99} = 3\sqrt{11}$

22. (a) $\langle \mathbf{u}, \mathbf{v} \rangle = 0(1) + 1(2) + 2(0) = 2$

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$

(c) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-1, -1, 2)\| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}$

24. (a) $\langle \mathbf{u}, \mathbf{v} \rangle = (1)(2) + 2(1)(5) + (1)(2) = 14$

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{(1)^2 + 2(1)^2 + (1)^2} = 2$

(c) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{(2)^2 + 2(5)^2 + (2)^2} = \sqrt{58}$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(1, 1, 1) - (2, 5, 2)\| = \|(-1, -4, -1)\| = \sqrt{(-1)^2 + 2(-4)^2 + (-1)^2} = \sqrt{34}$

26. (a) $\langle \mathbf{u}, \mathbf{v} \rangle = 1(2) + (-1)(1) + 2(0) + 0(-1) = 1$

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{1^2 + (-1)^2 + 2^2 + 0^2} = \sqrt{6}$

(c) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{2^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{6}$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-1, -2, 2, 1)\| = \sqrt{(-1)^2 + (-2)^2 + 2^2 + 1^2} = \sqrt{10}$

28. 1. Since the product of real numbers within a matrix is commutative,

$$\begin{aligned}\langle A, B \rangle &= 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22} \\ &= 2b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + 2b_{22}a_{22} \\ &= \langle B, A \rangle.\end{aligned}$$

2. Let $W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$. Then,

$$\begin{aligned}\langle A, B + W \rangle &= 2a_{11}(b_{11} + w_{11}) + a_{12}(b_{12} + w_{12}) + a_{21}(b_{21} + w_{21}) + 2a_{22}(b_{22} + w_{22}) \\ &= 2a_{11}b_{11} + 2a_{11}w_{11} + a_{12}b_{12} + a_{12}w_{12} + a_{21}b_{21} + a_{21}w_{21} + 2a_{22}b_{22} + 2a_{22}w_{22} \\ &= 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22} + 2a_{11}w_{11} + a_{12}w_{12} + a_{21}w_{21} + 2a_{22}w_{22} \\ &= \langle A, B \rangle + \langle A, W \rangle.\end{aligned}$$

3. If c is any scalar, then

$$\begin{aligned} c\langle A, B\rangle &= c(2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22}) \\ &= 2(ca_{11})b_{11} + (ca_{12})b_{12} + (ca_{21})b_{21} + 2(ca_{22})b_{22} \\ &= \langle CA, B\rangle \end{aligned}$$

4. Since the square of a real number is nonnegative, $\langle B, B\rangle = 2b_{11}^2 + b_{12}^2 + b_{21}^2 + 2b_{22}^2 \geq 0.$ Moreover, this expression is equal to zero if and only if $B = 0$ (that is, if and only if $b_{11} = b_{12} = b_{21} = b_{22} = 0$).

30. (a) $\langle A, B\rangle = 2(1)(0) + (0)(1) + (0)(1) + 2(1)(0) = 0$

(b) $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{2(1)^2 + 0^2 + 0^2 + 2(1)^2} = 2$

(c) $\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{2 \cdot 0^2 + 1^2 + 1^2 + 2 \cdot 0^2} = \sqrt{2}$

(d) Use the fact that $d(A, B) = \|A - B\|$. Because

$A - B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, you have

$\langle A - B, A - B\rangle = 2(1)^2 + (-1)^2 + (-1)^2 + 2(1)^2 = 6.$

$d(A, B) = \sqrt{\langle A - B, A - B\rangle} = \sqrt{6}$

32. (a) $\langle A, B\rangle = 2(1)(1) + (0)(0) + (0)(1) + 2(-1)(-1) = 4$

(b) $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{2(1)^2 + 0^2 + 0^2 + 2(-1)^2} = \sqrt{4} = 2$

(c) $\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{2(1)^2 + 0^2 + 1^2 + 2(-1)^2} = \sqrt{5}$

(d) Use the fact that $d(A, B) = \|A - B\|$. Because $A - B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, you have

$\langle A - B, A - B\rangle = 2(0)^2 + 0^2 + (-1)^2 + 2(0)^2 = 1.$

$d(A, B) = \sqrt{\langle A - B, A - B\rangle} = \sqrt{1} = 1$

34. 1. Since the product of real numbers is commutative,

$\langle p, q\rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n = b_0a_0 + b_1a_1 + \cdots + b_na_n = \langle q, p\rangle.$

2. Let $\mathbf{w} = w_0 + w_1x + \cdots + w_nx^n$, then

$$\begin{aligned} \langle p, q + w\rangle &= a_0(b_0 + w_0) + a_1(b_1 + w_1)x + \cdots + a_n(b_n + w_n)x^n \\ &= a_0b_0 + a_0w_0 + a_1b_1x + a_1w_1x + \cdots + a_nb_nx^n + a_nw_nx^n \\ &= a_0b_0 + a_1b_1x + \cdots + a_nb_nx^n + a_0w_0 + a_1w_1x + \cdots + a_nw_nx^n \\ &= \langle p, q\rangle + \langle p, w\rangle. \end{aligned}$$

3. If c is any scalar, then

$$\begin{aligned} c\langle p, q\rangle &= c\left(a_0b_0 + a_1b_1x + \cdots + a_nb_nx^n\right) \\ &= (ca_0)b_0 + (ca_1)b_1x + \cdots + (ca_n)b_nx^n \\ &= \langle cp, q\rangle. \end{aligned}$$

4. Since the square of a real number is nonnegative, $\langle q, q\rangle = b_0^2 + b_1^2x^2 + \cdots + b_n^2x^{2n} \geq 0.$ Moreover, this expression is equal to zero if and only if $q = 0$ (that is, if and only if $q_0 = \cdots = q_n = 0$).

36. (a) $\langle p, q\rangle = 1(1) + 1(0) + \frac{1}{2}(2) = 2$

(b) $\|p\|^2 = \langle p, p\rangle = 1^2 + 1^2 + \left(\frac{1}{2}\right)^2 = \frac{9}{4}$

$\|p\| = \sqrt{\langle p, p\rangle} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

(c) $\|q\|^2 = \langle q, q\rangle = 1^2 + 0^2 + 2^2 = 5$

$\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{5}$

(d) Use the fact that $d(p, q) = \|p - q\|$. Because

$p - q = x - \frac{3}{2}x^2$, you have

$\langle p - q, p - q\rangle = 0^2 + 1^2 + \left(-\frac{3}{2}\right)^2 = \frac{13}{4}.$

$d(p, q) = \sqrt{\langle p - q, p - q\rangle} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$

38. (a) $\langle p, q\rangle = 1(0) + (-3)(-1) + 1(2) = 5$

(b) $\|p\|^2 = \langle p, p\rangle = 1^2 + (-3)^2 + 1^2 = 11$

$\|p\| = \sqrt{\langle p, p\rangle} = \sqrt{11}$

(c) $\|q\|^2 = \langle q, q\rangle = 0^2 + (-1)^2 + 2^2 = 5$

$\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{5}$

(d) Use the fact that $d(p, q) = \|p - q\|$ Because

$p - q = 1 - 2x - x^2$, you have

$\langle p - q, p - q\rangle = 1^2 + (-2)^2 + (-1)^2 = 6.$

$d(p, q) = \sqrt{\langle p - q, p - q\rangle} = \sqrt{6}$

40. (a) $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} (-x)(x^2 - x + 2)dx = \int_{-1}^{1} (-x^3 + x^2 - 2x)dx = \left[-\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-1}^{1} = \frac{2}{3}$

(b) $\|f\|^2 = \langle f, f\rangle = \int_{-1}^{1} (-x)(-x)dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

$\|f\| = \sqrt{\frac{2}{3}}$

(c) $\|g\|^2 = \langle g, g\rangle = \int_{-1}^{1} (x^2 - x + 2)^2 dx = \int_{-1}^{1} (x^4 - 2x^3 + 5x^2 - 4x + 4)dx = \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{5x^3}{3} - 2x^2 + 4x\right]_{-1}^{1} = \frac{176}{15}$

$\|g\| = \sqrt{\frac{176}{15}}$

(d) Use the fact that $d(f, g) = \|f - g\|$. Because $f - g = -x - (x^2 - x + 2) = -x^2 - 2$, you have

$\langle f - g, f - g\rangle = \langle -x^2 - 2, -x^2 - 2\rangle = \int_{-1}^{1} (x^4 + 4x^2 + 4)dx = \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right]_{-1}^{1} = \frac{166}{15}.$

$d(f, g) = \sqrt{\langle f - g, f - g\rangle} = \sqrt{\frac{166}{15}}.$

42. (a) $\langle f, g\rangle = \int_{-1}^{1} xe^{-x}dx = \left.-e^{-x}(x + 1)\right]_{-1}^{1} = -2e^{-1} + 0 = -\frac{2}{e}$

(b) $\|f\|^2 = \langle f, f\rangle = \int_{-1}^{1} x^2dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$\|f\| = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

(c) $\|g\|^2 = \langle g, g\rangle = \int_{-1}^{1} e^{-2x}dx = \left.-\frac{e^{-2x}}{2}\right]_{-1}^{1} = \frac{1}{2}(-e^{-2} + e^2)$

$\|g\| = \sqrt{\frac{1}{2}(-e^{-2} + e^2)}$

(d) Use the fact that $d(f, g) = \|f - g\|$. Because $f - g = x - e^{-x}$, you have

$$\begin{aligned}\langle f - g, f - g\rangle &= \int_{-1}^{1} \left(x - e^{-x}\right)^2 dx \\ &= \int_{-1}^{1} \left(x^2 - 2e^{-x} + e^{-2x}\right)dx \\ &= \left[\frac{x^3}{3} + 2e^{-x}(x + 1) - \frac{e^{-2x}}{2}\right]_{-1}^{1} \\ &= \frac{2}{3} + 4e^{-1} - \frac{e^{-2}}{2} + \frac{e^2}{2}.\end{aligned}$$

$$d(f, g) = \sqrt{\langle f - g, f - g\rangle} = \sqrt{\frac{2}{3} + 4e^{-1} - \frac{e^{-2}}{2} + \frac{e^2}{2}}$$

44. Because $\langle \mathbf{u}, \mathbf{v}\rangle = (3)\left(\frac{1}{3}\right) + (-1)(1) = 0$, the angle between **u** and **v** is $\frac{\pi}{2}$.

46. Because $\langle \mathbf{u}, \mathbf{v}\rangle = 2\left(\frac{1}{4}\right)(2) + (-1)(1) = 0$, the angle between **u** and **v** is $\frac{\pi}{2}$.

48. Because $\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{(0)(3) + (1)(-2) + (-2)(1)}{\sqrt{(0)^2 + (1)^2 + (-2)^2}\sqrt{(3)^2 + (-1)^2 + (1)^2}} = \dfrac{-4}{\sqrt{5} \cdot \sqrt{14}} = -\dfrac{4}{\sqrt{70}}$,

the angle between **u** and **v** is $\cos^{-1}\left(-\frac{4}{\sqrt{70}}\right) \approx 2.069$ radians (118.56°).

50. Because $\dfrac{\langle p, q\rangle}{\|p\|\|q\|} = \dfrac{(1)(0) + 2(0)(1) + (1)(-1)}{\sqrt{(1)^2 + 2(0)^2 + (1)^2}\sqrt{(0)^2 + 2(1)^2 + (-1)^2}} = \dfrac{-1}{\sqrt{2}\sqrt{3}} = -\dfrac{1}{\sqrt{6}}$,

the angle between p and q is $\cos^{-1}\left(-\frac{1}{\sqrt{6}}\right) \approx 1.991$ radians (114.09°).

52. First compute

$$\langle f, g\rangle = \langle 1, x^2\rangle = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}$$

$$\|f\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} 1\, dx = x\Big]_{-1}^{1} = 2 \Rightarrow \|f\| = \sqrt{2}$$

$$\|g\|^2 = \langle x^2, x^2\rangle = \int_{-1}^{1} x^4\, dx = \frac{x^5}{5}\Big]_{-1}^{1} = \frac{2}{5} \Rightarrow \|g\| = \sqrt{\frac{2}{5}}.$$

So,

$$\frac{\langle f, g\rangle}{\|f\|\|g\|} = \frac{2/3}{\sqrt{2}\sqrt{2/5}} = \frac{\sqrt{5}}{3}$$

and the angle between f and g is $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right) \approx 0.73$ radians (41.81°).

54. (a) To verify the Cauchy-Schwarz Inequality, observe

$$\begin{aligned}
\left|\langle \mathbf{u}, \mathbf{v}\rangle\right| &\le \|\mathbf{u}\|\,\|\mathbf{v}\| \\
\left|(-1)(1) + (1)(-1)\right| &\le \sqrt{(-1)^2 + (1)^2} \cdot \sqrt{(1)^2 + (-1)^2} \\
\left|-2\right| &\le \sqrt{2} \cdot \sqrt{2} \\
2 &\le 2.
\end{aligned}$$

(b) To verify the Triangle Inequality, observe

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\| \\
\sqrt{(0)^2 + (0)^2} &\le \sqrt{(-1)^2 + (1)^2} + \sqrt{(1)^2 + (-1)^2} \\
0 &\le \sqrt{2} + \sqrt{2} \\
0 &\le 2\sqrt{2}.
\end{aligned}$$

56. (a) To verify the Cauchy-Schwarz Inequality, observe

$$\begin{aligned}
\left|\langle \mathbf{u}, \mathbf{v}\rangle\right| &\le \|\mathbf{u}\|\,\|\mathbf{v}\| \\
\left|(1)(1) + (0)(2) + (2)(0)\right| &\le \sqrt{(1)^2 + (0)^2 + (2)^2} \cdot \sqrt{(1)^2 + (2)^2 + (0)^2} \\
\left|1\right| &\le \sqrt{5} \cdot \sqrt{5} \\
1 &\le 5.
\end{aligned}$$

(b) To verify the Triangle Inequality, observe

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\| \\
\sqrt{(2)^2 + (2)^2 + (2)^2} &\le \sqrt{(1)^2 + (0)^2 + (2)^2} + \sqrt{(1)^2 + (2)^2 + (0)^2} \\
\sqrt{12} &\le \sqrt{5} + \sqrt{5} \\
2\sqrt{3} &\le 2\sqrt{5}.
\end{aligned}$$

58. (a) To verify the Cauchy-Schwarz Inequality, observe

$$\begin{aligned}
\left|\langle p, q\rangle\right| &\le \|p\|\,\|q\| \\
\left|(0)(1) + 2(1)(0) + (0)(-1)\right| &\le \sqrt{(0)^2 + 2(1)^2 + (0)^2} \cdot \sqrt{(1)^2 + 2(0)^2 + (-1)^2} \\
\left|0\right| &\le \sqrt{2} \cdot \sqrt{2} \\
0 &\le 2.
\end{aligned}$$

(b) To verify the Triangle Inequality, observe

$$\begin{aligned}
\|p + q\| &\le \|p\| + \|q\| \\
\sqrt{(1)^2 + 2(1)^2 + (-1)^2} &\le \sqrt{(0)^2 + 2(1)^2 + (0)^2} + \sqrt{(1)^2 + 2(0)^2 + (-1)^2} \\
\sqrt{4} &\le \sqrt{2} + \sqrt{2} \\
2 &\le 2\sqrt{2}.
\end{aligned}$$

60. (a) To verify the Cauchy-Schwarz Inequality, observe

$$\begin{aligned}
\left|\langle A, B\rangle\right| &\le \|A\|\,\|B\| \\
\left|(0)(1) + (1)(1) + (2)(2) + (-1)(-2)\right| &\le \sqrt{(0)^2 + (1)^2 + (2)^2 + (-1)^2} \cdot \sqrt{(1)^2 + (1)^2 + (2)^2 + (-2)^2} \\
\left|7\right| &\le \sqrt{6} \cdot \sqrt{10} \\
7 &\le \sqrt{60} \\
7 &\le 7.746.
\end{aligned}$$

(b) To verify the Triangle Inequality, observe

$$\|A + B\| \le \|A\| + \|B\|$$

$$\sqrt{(1)^2 + (2)^2 + (4)^2 + (-3)^2} \le \sqrt{(0)^2 + (1)^2 + (2)^2 + (-1)^2} + \sqrt{(1)^2 + (1)^2 + (2)^2 + (-2)^2}$$

$$\sqrt{30} \le \sqrt{6} + \sqrt{10}$$

$$5.477 \le 5.612.$$

62. (a) To verify the Cauchy-Schwarz Inequality, observe

$$\langle f, g\rangle = \langle x, \cos \pi x\rangle = \int_0^2 x \cos \pi x dx = \left[\frac{\cos \pi x}{\pi^2} + \frac{x \sin \pi x}{\pi}\right]_0^2 = 0$$

$$\|f\|^2 = \langle x, x\rangle = \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} \Rightarrow \|f\| = \frac{2\sqrt{6}}{3}$$

$$\|g\|^2 = \langle \cos \pi x, \cos \pi x\rangle = \int_0^2 \cos^2 \pi dx$$

$$= \int_0^2 \frac{1 + \cos^2 \pi x}{2} dx = \left[\frac{1}{2}x + \frac{\sin 2\pi x}{4x}\right]_0^2 = 1 \Rightarrow \|g\| = 1$$

and observe that

$$|\langle f, g\rangle| \le \|f\|\|g\|$$

$$0 \le \frac{2\sqrt{6}}{3}(1).$$

(b) To verify the Triangle Inequality, observe

$$\|f + g\|^2 = \|x + \cos \pi x\|^2 = \int_0^2 (x + \cos \pi x) dx = \left[\frac{x^2}{2} + \frac{\sin \pi x}{\pi}\right]_0^2 = 2$$

$\Rightarrow \|f + g\| = \sqrt{2}.$

So, $\|f + g\| \le \|f\| + \|g\|$

$$\sqrt{2} \le \frac{2\sqrt{6}}{3} + 1.$$

64. (a) To verify the Cauchy-Schwarz Inequality, compute

$$\langle f, g\rangle = \langle x, e^{-x}\rangle = \int_0^1 xe^{-x}\, dx = -e^{-x}(x + 1)\Big]_0^1 = 1 - 2e^{-1}$$

$$\|f\|^2 = \langle x, x\rangle = \int_0^1 x^2 dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3} \Rightarrow \|f\| = \frac{\sqrt{3}}{3}$$

$$\|g\|^2 = \langle e^{-x}, e^{-x}\rangle = \int_0^1 e^{-2x} dx = -\frac{e^{-2x}}{2}\Big]_0^1 = -\frac{e^{-2}}{2} + \frac{1}{2} \Rightarrow \|g\| = \sqrt{-\frac{e^{-2}}{2} + \frac{1}{2}}$$

and observe that

$$|\langle f, g\rangle| \le \|f\|\|g\|$$

$$|1 - 2e^{-1}| \le \left(\frac{\sqrt{3}}{3}\right)\left(\sqrt{-\frac{e^{-2}}{2} + \frac{1}{2}}\right)$$

$$0.264 \le 0.380.$$

(b) To verify the Triangle Inequality, compute

$$\|f + g\|^2 = \langle x + e^{-x}, x + e^{-x} \rangle = \int_0^1 \left(x + e^{-x}\right)^2 dx = \left[-2e^{-x}(x + 1) - \frac{e^{-2x}}{2} + \frac{x^3}{3}\right]_0^1$$

$$= \left[-4e^{-1} - \frac{e^{-2}}{2} + \frac{1}{3}\right] - \left[-2 - \frac{1}{2} + 0\right]$$

$$= -4e^{-1} - \frac{e^{-2}}{2} + \frac{17}{6} \Rightarrow \|f + g\| = \sqrt{-4e^{-1} - \frac{e^{-2}}{2} + \frac{17}{6}}$$

and observe that

$$\|f + g\| \le \|f\| + \|g\|$$

$$\sqrt{-4e^{-1} - \frac{e^{-2}}{2} + \frac{17}{6}} \le \frac{\sqrt{3}}{3} + \sqrt{-\frac{e^{-2}}{2} + \frac{1}{2}}$$

$$1.138 \le 1.235.$$

66. The functions $f(x) = x$ and $g(x) = \frac{1}{2}(3x^2 - 1)$ are orthogonal because

$$\langle f, g \rangle = \int_{-1}^1 x\frac{1}{2}(3x^2 - 1)dx = \frac{1}{2}\int_{-1}^1 (3x^3 - x)dx = \left[\frac{1}{2}\left(\frac{3x^4}{4} - \frac{x^2}{2}\right)\right]_{-1}^1 = 0.$$

68. The functions $f(x) = 1$ and $g(x) = \cos(2nx)$ are orthogonal because $\langle f, g \rangle = \int_0^\pi \cos(2nx)dx = \left[\frac{1}{2n}\sin(2nx)\right]_0^\pi = 0.$

70. (a) $$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} = \frac{(-3)(6) + (-1)(3)}{6^2 + 3^2}(6, 3)$$

$$= -\frac{7}{15}(6, 3)$$

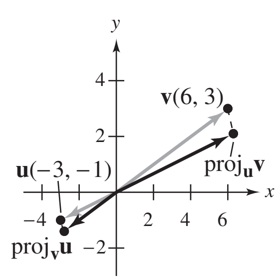
$$= \left(-\frac{14}{5}, -\frac{7}{5}\right)$$

(b) $$\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u} = \frac{6(-3) + 3(-1)}{(-3)^2 + (-1)^2}(-3, -1)$$

$$= -\frac{21}{10}(-3, -1)$$

$$= \left(\frac{63}{10}, \frac{21}{10}\right)$$

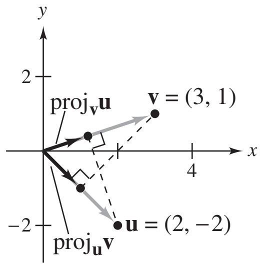
(c)



72. (a) $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v} = \dfrac{(2)(3) + (-2)(1)}{(3)^2 + (1)^2}(3, 1) = \dfrac{4}{10}(3, 1) = \left(\dfrac{6}{5}, \dfrac{2}{5}\right)$

(b) $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\langle \mathbf{v}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = \dfrac{(3)(2) + (1)(-2)}{(2)^2 + (-2)^2}(2, -2) = \dfrac{4}{8}(2, -2) = (1, -1)$

(c)



74. (a) $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v} = \dfrac{(1)(-1) + (2)(2) + (-1)(-1)}{(-1)^2 + (2)^2 + (-1)^2}(-1, 2, -1) = \dfrac{4}{6}(-1, 2, -1) = \left(-\dfrac{2}{3}, \dfrac{4}{3}, -\dfrac{2}{3}\right)$

(b) $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\langle \mathbf{v}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = \dfrac{(-1)(1) + (2)(2) + (-1)(-1)}{(1)^2 + (2)^2 + (-1)^2}(1, 2, -1) = \dfrac{4}{6}(1, 2, -1) = \left(\dfrac{2}{3}, \dfrac{4}{3}, -\dfrac{2}{3}\right)$

76. (a) $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v} = \dfrac{(-1)(2) + (4)(-1) + (-2)(2) + (3)(-1)}{(2)^2 + (-1)^2 + (2)^2 + (-1)^2}(2, -1, 2, -1)$

$$= \frac{-13}{10}(2, -1, 2, -1) = \left(-\frac{13}{5}, \frac{13}{10}, -\frac{13}{5}, \frac{13}{10}\right)$$

(b) $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\langle \mathbf{v}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = \dfrac{(2)(-1) + (-1)(4) + (2)(-2) + (-1)(3)}{(-1)^2 + (4)^2 + (-2)^2 + (3)^2}(-1, 4, -2, 3)$

$$= \frac{-13}{30}(-1, 4, -2, 3) = \left(\frac{13}{30}, -\frac{26}{15}, \frac{13}{15}, -\frac{13}{10}\right)$$

78. The inner products $\langle f, g\rangle$ and $\langle g, g\rangle$ are as follows.

$$\langle f, g\rangle = \int_{-1}^{1}(x^3 - x)(2x - 1)\,dx = \int_{-1}^{1}(2x^4 - x^3 - 2x^2 + x)\,dx = \left[\frac{2x^5}{5} - \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1} = -\frac{8}{15}$$

$$\langle g, g\rangle = \int_{-1}^{1}(2x - 1)^2\,dx = \int_{-1}^{1}(4x^2 - 4x + 1)\,dx = \left[\frac{4x^3}{3} - 2x^2 + x\right]_{-1}^{1} = \frac{14}{3}$$

So, the projection of f onto g is $\text{proj}_{\mathbf{g}}f = \dfrac{\langle f, g\rangle}{\langle g, g\rangle}g = \dfrac{-8/15}{14/3}(2x - 1) = -\dfrac{4}{35}(2x - 1)$.

80. The inner products $\langle f, g\rangle$ and $\langle g, g\rangle$ are as follows.

$$\langle f, g\rangle = \int_{0}^{1} xe^{-x}\,dx = \left[-e^{-x}(x + 1)\right]_{0}^{1} = -2e^{-1} + 1$$

$$\langle g, g\rangle = \int_{0}^{1} e^{-2x}\,dx = \left[\frac{-e^{-2x}}{2}\right]_{0}^{1} = \frac{-e^{-2}}{2} + \frac{1}{2} = \frac{1 - e^{-2}}{2}$$

So, the projection of f onto g is $\text{proj}_{g}f = \dfrac{\langle f, g\rangle}{\langle g, g\rangle}g = \dfrac{-2e^{-1} + 1}{\frac{1 - e^{-2}}{2}}(e^{-x}) = \dfrac{-4e^{-1} + 2}{1 - e^{-2}}(e^{-x}) = \dfrac{-4e^{-x-1} + 2e^{-x}}{1 - e^{-2}}$.

82. The inner product $\langle f, g\rangle$ is

$$\langle f, g\rangle = \int_{-\pi}^{\pi} \sin 2x \sin 3x\, dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos x - \cos 5x)\, dx = \frac{1}{2}\left(\sin x - \frac{\sin 5x}{5}\right)\Bigg]_{-\pi}^{\pi} = 0$$

which implies that $\text{proj}_{\mathbf{g}} f = 0$.

84. The inner product $\langle f, g\rangle$ is $\langle f, g\rangle = \int_{-\pi}^{\pi} x \cos 2x\, dx = \left[\frac{\cos 2x}{4} + \frac{x \sin 2x}{2}\right]_{-\pi}^{\pi} = \frac{1}{4} - \frac{1}{4} = 0$

which implies that $\text{proj}_g f = 0$.

86. (a) False. The norm of a vector $\mathbf{u}$ is defined as a square root of $\langle \mathbf{u}, \mathbf{u}\rangle$.

(b) False. The angle between $a\mathbf{v}$ and $\mathbf{v}$ is zero if $a > 0$ and it is π if $a < 0$.

88.
$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle \\ &= \big(\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle\big) + \big(\langle \mathbf{u}, \mathbf{u}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle\big) \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2\end{aligned}$$

90. To prove that $\mathbf{u} - \text{proj}_v\mathbf{u}$ is orthogonal to $\mathbf{v}$, you calculate their inner product as follows

$$\langle \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \text{proj}_{\mathbf{v}}\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \left\langle \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}, \mathbf{v}\right\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\langle \mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle = 0.$$

92. You have from the definition of inner product $\langle \mathbf{u}, c\mathbf{v}\rangle = \langle c\mathbf{v}, \mathbf{u}\rangle = c\langle \mathbf{v}, \mathbf{u}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$.

94. Let $W = \{(c, 2c, 3c) : c \in R\}$. Then

$W^{\perp} = \{\mathbf{v} \in R^3 : \mathbf{v} \cdot (c, 2c, 3c) = 0\} = \{(x, y, z) \in R^3 : (x, y, z) \cdot (1, 2, 3) = 0\}$.

You need to solve $x + 2y + 3z = 0$. Choosing y and z as free variables, you obtain the solution $x = -2t - 3s, y = t, z = s$ for any real numbers t and s. Therefore,

$W^{\perp} = \{t(-2, 1, 0) + s(-3, 0, 1) : t, s \in R\} = \text{span}\{(-2, 1, 0), (-3, 0, 1)\}$.

96. (a) All four axioms of the definition of an inner product must be satisfied.

(i) $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$

(ii) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$

(iii) $c\langle \mathbf{u}, \mathbf{v}\rangle = \langle c\mathbf{u}, \mathbf{v}\rangle$

(iv) $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(b) To find an orthogonal projection, find $\langle \mathbf{u}, \mathbf{v}\rangle$ and $\langle \mathbf{v}, \mathbf{v}\rangle$, and have $\mathbf{v} \neq 0$ so that

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}.$$

98. Let $\mathbf{u} = (x, y)$. Then $\|\mathbf{u}\| = \sqrt{c_1x^2 + c_2y^2} = 1$. Since the equation of the graph is $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$, $c_1 = \frac{1}{4}$ and $c_2 = \frac{1}{9}$.

100. Let $\mathbf{u} = (x, y)$. Then $\|\mathbf{u}\| = \sqrt{c_1x^2 + c_2y^2} = 1$. Since the equation of the graph is $\frac{1}{25}x^2 + \frac{1}{9}y^2 = 1$, $c_1 = \frac{1}{25}$ and $c_2 = \frac{1}{9}$.

Section 5.3 Orthonormal Bases: Gram-Schmidt Process

2. (a) The set is *not* orthogonal since
$(-3, 5) \cdot (4, 0) = (-3)(4) + 5(0) = -12 \neq 0.$

(b) The set is *not* orthonormal since it is *not* orthogonal.

(c) Because the two vectors are not scalar multiples of each other, by the Corollary to Theorem 4.8 they are linearly independent. By Theorem 4.12, they are a basis for R^2.

4. (a) The set is orthogonal since $(2, 1) \cdot \left(\frac{1}{3}, -\frac{2}{3}\right) = 2\left(\frac{1}{3}\right) + 1\left(-\frac{2}{3}\right) = 0.$

(b) The set is *not* orthonormal since $\|(2, 1)\| = \sqrt{2^2 + 1^2} = \sqrt{5} \neq 1.$

(c) Because the vectors are not scalar multiples of each other, by the Corollary to Theorem 4.8 they are linearly independent. By Theorem 4.12, they form a basis for R^2.

6. (a) The set is orthogonal since $(2, -4, 2) \cdot (0, 2, 4) = 0 - 8 + 8 = 0,$ $(2, -4, 2) \cdot (-10, -4, 2) = -20 + 16 + 4 = 0,$ and $(0, 2, 4) \cdot (-10, -4, 2) = 0 - 8 + 8 = 0.$

(b) The set is *not* orthonormal since $\|(2, -4, 2)\| = \sqrt{2^2 + (-4)^2 + 2^2} = \sqrt{24} \neq 1.$

(c) Because the three vectors do not lie in the same plane, they span R^3. By Theorem 4.12, they form a basis for R^3.

8. (a) The set is orthogonal since $\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{-\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) = 0,$ $\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right) = 0,$ and $\left(\frac{-\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right) = 0.$

(b) The set is orthonormal since $\left\|\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1,$ $\left\|\left(\frac{-\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right)\right\| = \sqrt{\frac{1}{6} + \frac{2}{3} + \frac{1}{6}} = 1,$ and $\left\|\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)\right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1.$

(c) Because the three vectors do not lie in the same plane, they span R^3. By Theorem 4.12, they form a basis for R^3.

10. (a) The set is orthogonal since $(-6, 3, 2, 1) \cdot (2, 0, 6, 0) = -12 + 12 = 0.$

(b) The set is *not* orthonormal since $\|(-6, 3, 2, 1)\| = \sqrt{36 + 9 + 4 + 1} = \sqrt{50} \neq 1.$

(c) Since there aren't enough vectors, the set is *not* a basis for R^4.

12. (a) The set is orthogonal since

$$\left(\frac{\sqrt{10}}{10}, 0, 0, \frac{3\sqrt{10}}{10}\right) \cdot (0, 0, 1, 0) = 0,$$

$$\left(\frac{\sqrt{10}}{10}, 0, 0, \frac{3\sqrt{10}}{10}\right) \cdot (0, 1, 0, 0) = 0,$$

$$\left(\frac{\sqrt{10}}{10}, 0, 0, \frac{3\sqrt{10}}{10}\right) \cdot \left(\frac{-3\sqrt{10}}{10}, 0, 0, \frac{\sqrt{10}}{10}\right) = \frac{-3}{10} + \frac{3}{10} = 0,$$

$$(0, 0, 1, 0) \cdot (0, 1, 0, 0) = 0,$$

$$(0, 0, 1, 0) \cdot \left(\frac{-3\sqrt{10}}{10}, 0, 0, \frac{\sqrt{10}}{10}\right) = 0,$$

and $(0, 1, 0, 0) \cdot \left(\frac{-3\sqrt{10}}{10}, 0, 0, \frac{\sqrt{10}}{10}\right) = 0.$

(b) The set is orthonormal since $\left\|\left(\frac{\sqrt{10}}{10}, 0, 0, \frac{3\sqrt{10}}{10}\right)\right\| = \frac{1}{10} + \frac{9}{10} = 1$, $\|(0, 0, 1, 0)\| = 1$, $\|(0, 1, 0, 0)\| = 1$, and

$\left\|\left(\frac{-3\sqrt{10}}{10}, 0, 0, \frac{\sqrt{10}}{10}\right)\right\| = \frac{9}{10} + \frac{1}{10} = 1.$

(c) By the Corollary to Theorem 5.10, the set of four vectors is a basis for R^4.

14. (a) The set is orthogonal since $(2, -5) \cdot (10, 4) = 20 - 20 = 0$.

(b) Since $\|(2, -5)\| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ and $\|(10, 4)\| = \sqrt{10^2 + 4^2} = 2\sqrt{29}$, normalizing the set produces an orthonormal set.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{29}}(2, -5) = \left(\frac{2\sqrt{29}}{29}, -\frac{5\sqrt{29}}{29}\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{2\sqrt{29}}(10, 4) = \left(\frac{5\sqrt{29}}{29}, \frac{2\sqrt{29}}{29}\right)$$

16. (a) The set is orthogonal since $\left(\frac{6}{13}, -\frac{2}{13}, \frac{3}{13}\right) \cdot \left(\frac{2}{13}, \frac{6}{13}, 0\right) = \frac{12}{13} - \frac{12}{13} + 0 = 0$.

(b) Since $\left\|\frac{6}{13}, -\frac{2}{13}, \frac{3}{13}\right\| = \sqrt{\left(\frac{6}{13}\right)^2 + \left(-\frac{2}{13}\right)^2 + \left(\frac{3}{13}\right)^2} = \sqrt{\frac{49}{169}} = \frac{7}{13}$ and

$\left\|\left(\frac{2}{13}, \frac{6}{13}, 0\right)\right\| = \sqrt{\left(\frac{2}{13}\right)^2 + \left(\frac{6}{13}\right)^2 + 0^2} = \sqrt{\frac{40}{169}} = \frac{2\sqrt{10}}{13}$, normalizing the set produces an orthonormal set.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{13}{7}\left(\frac{6}{13}, -\frac{2}{13}, \frac{3}{13}\right) = \left(\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{13}{2\sqrt{10}}\left(\frac{2}{13}, \frac{6}{13}, 0\right) = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0\right)$$

18. The set $\{(\sin\theta, \cos\theta), (\cos\theta, -\sin\theta)\}$ is orthogonal because

$(\sin\theta, \cos\theta) \cdot (\cos\theta, -\sin\theta) = \sin\theta\cos\theta - \cos\theta\sin\theta = 0.$

Furthermore, the set is orthonormal because

$\|(\sin\theta, \cos\theta)\| = \sin^2\theta + \cos^2\theta = 1$

$\|(\cos\theta, -\sin\theta)\| = \cos^2\theta + (-\sin\theta)^2 = 1.$

So, the set forms an orthonormal basis for R^2.

20. Use Theorem 5.11 to find the coordinates of $\mathbf{w} = (4, -3)$ relative to B.

$$(4, -3) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right) = \frac{4\sqrt{3}}{3} - \frac{3\sqrt{6}}{3}$$

$$(4, -3) \cdot \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}\right) = -\frac{4\sqrt{6}}{3} - \frac{3\sqrt{3}}{3}$$

So, $[\mathbf{w}]_B = \begin{bmatrix} \frac{4\sqrt{3}}{3} - \sqrt{6} \\ -\frac{4\sqrt{6}}{3} - \sqrt{3} \end{bmatrix}$.

22. Use Theorem 5.11 to find the coordinates of $\mathbf{w} = (3, -5, 11)$ relative to B.

$$(3, -5, 11) \cdot (1, 0, 0) = 3$$
$$(3, -5, 11) \cdot (0, 1, 0) = -5$$
$$(3, -5, 11) \cdot (0, 0, 1) = 11$$

So, $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$.

24. Use Theorem 5.11 to find the coordinates of $\mathbf{w} = (2, -1, 4, 3)$ relative to B.

$$(2, -1, 4, 3) \cdot \left(\tfrac{5}{13}, 0, \tfrac{12}{13}, 0\right) = \tfrac{10}{13} + \tfrac{48}{13} = \tfrac{58}{13}$$
$$(2, -1, 4, 3) \cdot (0, 1, 0, 0) = -1$$
$$(2, -1, 4, 3) \cdot \left(-\tfrac{12}{13}, 0, \tfrac{5}{13}, 0\right) = -\tfrac{24}{13} + \tfrac{20}{13} = -\tfrac{4}{13}$$
$$(2, -1, 4, 3) \cdot (0, 0, 0, 1) = 3$$

So, $[\mathbf{w}]_B = \left[\tfrac{58}{13} \quad -1 \quad -\tfrac{4}{13} \quad 3\right]^T$.

26. First, orthogonalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (-1, 2)$$
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 = (1, 0) - \frac{(-1)(1) + 2(0)}{(-1)^2 + 2^2}(-1, 2) = (1, 0) + \frac{1}{5}(-1, 2) = \left(\frac{4}{5}, \frac{2}{5}\right)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{(-1)^2 + 2^2}}(-1, 2) = \left(-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$
$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2}}\left(\frac{4}{5}, \frac{2}{5}\right) = \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$$

So, the orthonormal basis is $B' = \left\{\left(-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right), \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)\right\}$.

28 First, orthogonalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (4, -3)$$
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1$$
$$= (3, 2) - \frac{3(4) + 2(-3)}{4^2 + (-3)^2}(4, -3)$$
$$= (3, 2) - \frac{6}{25}(4, -3)$$
$$= \left(\frac{51}{25}, \frac{68}{25}\right)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{4^2 + (-3)^2}}(4, -3) = \left(\frac{4}{5}, -\frac{3}{5}\right)$$
$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{\left(\frac{51}{25}\right)^2 + \left(\frac{68}{25}\right)^2}}\left(\frac{51}{25}, \frac{68}{25}\right) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

So, the orthonormal basis is $B' = \left\{\left(\frac{4}{5}, -\frac{3}{5}\right), \left(\frac{3}{5}, \frac{4}{5}\right)\right\}$.

30. First, orthogonalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, 0, 0)$$
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1$$
$$= (1, 1, 1) - \frac{1}{1}(1, 0, 0)$$
$$= (0, 1, 1)$$
$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2$$
$$= (1, 1, -1) - \frac{1}{1}(1, 0, 0) - \frac{0}{2}(0, 1, 1)$$
$$= (0, 1, -1)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (1, 0, 0)$$
$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{2}}(0, 1, 1) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{2}}(0, 1, -1) = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

So, the orthonormal basis is

$$\left\{(1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

32. First, orthogonalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (0, 1, 2)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 = (2, 0, 0) - 0(0, 1, 2) = (2, 0, 0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 = (1, 1, 1) - \frac{3}{5}(0, 1, 2) - \frac{2}{4}(2, 0, 0) = \left(0, \frac{2}{5}, -\frac{1}{5}\right)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{5}}(0, 1, 2) = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{2}(2, 0, 0) = (1, 0, 0)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \sqrt{5}\left(0, \frac{2}{5}, -\frac{1}{5}\right) = \left(0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$$

So, the orthonormal basis is $\left\{\left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), (1, 0, 0), \left(0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)\right\}$.

34. First, orthogonalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (3, 4, 0, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 = (-1, 1, 0, 0) - \frac{1}{25}(3, 4, 0, 0) = \left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$= (2, 1, 0, -1) - \frac{10}{25}(3, 4, 0, 0) - \frac{-\frac{7}{5}}{\frac{49}{25}}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)$$

$$= (2, 1, 0, -1) - \left(\frac{6}{5}, \frac{8}{5}, 0, 0\right) + \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right) = (0, 0, 0, -1)$$

$$\mathbf{w}_4 = \mathbf{v}_4 - \frac{\langle \mathbf{v}_4, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_4, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \frac{\langle \mathbf{v}_4, \mathbf{w}_3 \rangle}{\langle \mathbf{w}_3, \mathbf{w}_3 \rangle}\mathbf{w}_3$$

$$= (0, 1, 1, 0) - \frac{4}{25}(3, 4, 0, 0) - \frac{\frac{21}{25}}{\frac{49}{25}}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right) - 0(0, 0, 0, -1)$$

$$= (0, 1, 1, 0) - \left(\frac{12}{25}, \frac{16}{25}, 0, 0\right) - \left(-\frac{12}{25}, \frac{9}{25}, 0, 0\right) = (0, 0, 1, 0)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{5}(3, 4, 0, 0) = \left(\frac{3}{5}, \frac{4}{5}, 0, 0\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{5}{7}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right) = \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = (0, 0, 0, -1)$$

$$\mathbf{u}_4 = \mathbf{w}_4 = (0, 0, 1, 0)$$

So, the orthonormal basis is $\left\{\left(\frac{3}{5}, \frac{4}{5}, 0, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right), (0, 0, 0, -1), (0, 0, 1, 0)\right\}$.

36. Because there is just one vector, you simply need to normalize it.

$$\mathbf{u}_1 = \frac{1}{\sqrt{2^2 + (-9)^2 + 6^2}}(2, -9, 6) = \frac{1}{11}(2, -9, 6) = \left(\frac{2}{11}, -\frac{9}{11}, \frac{6}{11}\right)$$

38. First, orthogonalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, 3, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 = \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 = (3, 0, -3) - \frac{3}{10}(1, 3, 0) = \left(\frac{27}{10}, -\frac{9}{10}, -3\right)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{10}}(1, 3, 0) = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{10}{3\sqrt{190}}\left(\frac{27}{10}, -\frac{9}{10}, -3\right) = \left(\frac{9}{\sqrt{190}}, -\frac{3}{\sqrt{190}}, -\frac{10}{\sqrt{190}}\right)$$

40. First, normalize each vector in B.

$$\mathbf{w}_1 = \mathbf{v}_1 = (7, 24, 0, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 = (0, 0, 1, 1) - 0(7, 24, 0, 0) = (0, 0, 1, 1)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2 = (0, 0, 1, -2) - 0(7, 24, 0, 0) - \frac{-1}{2}(0, 0, 1, 1) = \left(0, 0, \frac{3}{2}, -\frac{3}{2}\right)$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{25}(7, 24, 0, 0) = \left(\frac{7}{25}, \frac{24}{25}, 0, 0\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{2}}(0, 0, 1, 1) = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{3/\sqrt{2}}\left(0, 0, \frac{3}{2}, -\frac{3}{2}\right) = \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

So, the orthonormal basis is

$$\left\{\left(\frac{7}{25}, \frac{24}{25}, 0, 0\right), \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

42. The set $\left\{\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)\right\}$ from Exercise 41 is not orthonormal using the Euclidean inner product because

$$\left\|\left(\frac{2}{3}, -\frac{1}{3}\right)\right\| = \sqrt{\frac{4}{9} + \frac{1}{9}} = \frac{\sqrt{5}}{3} \neq 1.$$

44. $\langle 1, 1\rangle = \int_{-1}^{1} 1\, dx = x\Big]_{-1}^{1} = 1 - (-1) = 2$

46. $\langle x^2, x\rangle = \int_{-1}^{1} x^2 x\, dx = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big]_{-1}^{1} = \frac{1}{4} - \left(\frac{1}{4}\right) = 0$

48. $\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \right\rangle = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right)dx$

$$= \int_{-1}^{1}\left(x^4 - \frac{1}{3}x^2 - \frac{1}{3}x^2 + \frac{1}{9}\right)dx$$

$$= \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx$$

$$= \frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x\Big]_{-1}^{1}$$

$$= \left[\frac{1}{5}(1)^5 - \frac{2}{9}(1)^3 + \frac{1}{9}(1)\right] - \left[\frac{1}{5}(-1)^5 - \frac{2}{9}(-1)^3 + \frac{1}{9}(-1)\right]$$

$$= \frac{8}{45}$$

50. The solutions of the homogeneous system are of the form $(-3s + 3t, s, t)$, where s and t are any real numbers. So, a basis for the solution space is $\{(-3, 1, 0), (3, 0, 1)\}$.

To find an orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$, use the alternative form of the Gram-Schmidt orthonormalization process, as shown below.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\right)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1$$

$$= (3, 0, 1) - \left[(3, 0, 1) \cdot \left(-\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\right)\right]\left(-\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\right)$$

$$= \left(\frac{3}{10}, \frac{9}{10}, 1\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{10}{\sqrt{190}}\left(\frac{3}{10}, \frac{9}{10}, 1\right) = \left(\frac{3\sqrt{190}}{190}, \frac{9\sqrt{190}}{190}, \frac{\sqrt{190}}{19}\right)$$

So, an orthonormal basis for the solution space is

$$\left\{\left(-\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}, 0\right), \left(\frac{3\sqrt{190}}{190}, \frac{9\sqrt{190}}{190}, \frac{\sqrt{190}}{19}\right)\right\}.$$

52. The solutions of the homogeneous system are of the form $(s + t, 0, s, t)$, where s and t are any real numbers. So, a basis for the solution space is $\{(1, 0, 1, 0), (1, 0, 0, 1)\}$.

To find an orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$, use the alternative form of the Gram-Schmidt orthonormalization process as shown.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}(1, 0, 1, 0) = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right)$$

$$\begin{aligned}\mathbf{u}_2 &= \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1\rangle \mathbf{u}_1 \\ &= (1, 0, 0, 1) - \left[(1, 0, 0, 1) \cdot \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right)\right]\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right) \\ &= \left(\frac{1}{2}, 0, -\frac{1}{2}, 1\right)\end{aligned}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{3/2}}\left(\frac{1}{2}, 0, -\frac{1}{2}, 1\right) = \left(\frac{\sqrt{6}}{6}, 0, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)$$

So, an orthonormal basis for the solution space is $\left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right), \left(\frac{\sqrt{6}}{6}, 0, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)\right\}$.

54. The solutions of the homogenous system are of the form $(-r - t, -s, r, s, t)$, where r, s, and t are any real numbers. So, a basis for the solution space is $\{(-1, 0, 1, 0, 0), (0, -1, 0, 1, 0), (-1, 0, 0, 0, 1)\}$.

To find an orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, use the alternative form of the Gram-Schmidt orthonormalization process as shown.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}(-1, 0, 1, 0, 0) = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right)$$

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1\rangle \mathbf{u}_1 \\ &= (0, -1, 0, 1, 0) - \left[(0, -1, 0, 1, 0) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right)\right]\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right) \\ &= (0, -1, 0, 1, 0)\end{aligned}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1, 0) = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$$

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1\rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2\rangle \mathbf{u}_2 \\ &= (-1, 0, 0, 0, 1) - \left[(-1, 0, 0, 0, 1) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right)\right]\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right) \\ &\qquad - \left[(-1, 0, 0, 0, 1) \cdot \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)\right]\left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right) \\ &= \left(-\frac{1}{2}, 0, -\frac{1}{2}, 0, 1\right)\end{aligned}$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{3/2}}\left(-\frac{1}{2}, 0, -\frac{1}{2}, 0, 1\right) = \left(-\frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}, 0, \sqrt{\frac{2}{3}}\right)$$

So, an orthonormal basis of the solution space is

$$\left\{\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right), \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}, 0, \sqrt{\frac{2}{3}}\right)\right\}.$$

56. (a) True. See definition on page 254.
(b) True. See Theorem 5.10 on page 257.

58. Let $p_1(x) = x^2$, $p_2(x) = 2x + x^2$, and $p_3(x) = 1 + 2x + x^2$.

Then, because $\langle p_1, p_2\rangle = 0(0) + 0(2) + 1(1) = 1 \neq 0$, the set is not orthogonal. Orthogonalize the set as follows.

$$\mathbf{w}_1 = p_1 = x^2$$
$$\mathbf{w}_2 = p_2 - \frac{\langle p_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 = x^2 + 2x - \frac{0(0) + 2(0) + 1(1)}{0^2 + 0^2 + 1^2}x^2 = 2x$$
$$\mathbf{w}_3 = p_3 - \frac{\langle p_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2 - \frac{\langle p_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1$$
$$= 1 + 2x + x^2 - \frac{1(0) + 2(2) + 1(0)}{0^2 + 2^2 + 0^2}(2x) - \frac{1(0) + 2(0) + 1(1)}{0^2 + 0^2 + 1^2}x^2$$
$$= 1 + 2x + x^2 - 2x - x^2 = 1$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{0^2 + 0^2 + 1^2}}x^2 = x^2$$
$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{0^2 + 2^2 + 0^2}}(2x) = x$$
$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{1^2 + 0^2 + 0^2}}(1) = 1$$

So, the orthonormal set is $\{x^2, x, 1\}$.

60. Let $p_1(x) = \dfrac{5x + 12x^2}{13}$, $p_2(x) = \dfrac{12x - 5x^2}{13}$, and $p_3(x) = 1$. Then $\langle p_1, p_2\rangle = \dfrac{60}{169} - \dfrac{60}{169} = 0$, $\langle p_1, p_3\rangle = 0$, and $\langle p_2, p_3\rangle = 0$.

Furthermore,

$$\|p_1\| = \sqrt{\frac{25 + 144}{169}} = 1,\ \|p_2\| = \sqrt{\frac{25 + 144}{169}}, \text{ and } \|p_3\| = 1.$$

So, $\{p_1, p_2, p_3\}$ is an orthonormal set.

62. Let $p(x) = \sqrt{2}(-1 + x^2)$ and $q(x) = \sqrt{2}(2 + x + x^2)$. Because $\langle p, q\rangle = \sqrt{2}\sqrt{2} + 0(\sqrt{2}) + (-\sqrt{2})(2\sqrt{2}) = -2 \neq 0$, the set is not orthogonal.

Orthogonalize the set as follows.

$$\mathbf{w}_1 = p = \sqrt{2}(x^2 - 1)$$
$$\mathbf{w}_2 = q - \frac{\langle q, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 = \sqrt{2}(2 + x + x^2) - \frac{-2}{4}\left(\sqrt{2}(x^2 - 1)\right) = \frac{3\sqrt{2}}{2} + \sqrt{2}x + \frac{3\sqrt{2}}{2}x^2$$

Then, normalize the vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{2}\sqrt{2}(-1 + x^2) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}x^2$$
$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{11}}\left(\frac{3\sqrt{2}}{2} + \sqrt{2}x + \frac{3\sqrt{2}}{2}x^2\right) = \frac{3}{\sqrt{22}} + \frac{2}{\sqrt{22}}x + \frac{3}{\sqrt{22}}x^2$$

So, the orthonormal set is $\left\{-\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}x^2, \dfrac{3}{\sqrt{22}} + \dfrac{2}{\sqrt{22}}x + \dfrac{3}{\sqrt{22}}x^2\right\}$.

64. Let $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ be an arbitrary linear combination of vectors in S. Then

$$\begin{aligned}\langle \mathbf{w}, \mathbf{v}\rangle &= \langle \mathbf{w}, c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n\rangle \\ &= \langle \mathbf{w}, c_1\mathbf{v}_1\rangle + \cdots + \langle \mathbf{w}, c_n\mathbf{v}_n\rangle \\ &= c_1\langle \mathbf{w}, \mathbf{v}_1\rangle + \cdots + c_n\langle \mathbf{w}, \mathbf{v}_n\rangle = c_1 \cdot 0 + \cdots + c_n \cdot 0 = 0.\end{aligned}$$

Because $c_1, \ldots, c_n$ are arbitrary real numbers, you conclude that **w** is orthogonal to *any* linear combination of vectors in S.

66. Let $\mathbf{v} \in W \cap W^{\perp}$. Then $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w}$ in W. In particular, since $\mathbf{v} \in W$, $\mathbf{v} \cdot \mathbf{v} = 0$, which implies that $\mathbf{v} = \mathbf{0}$.

68. $A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$N(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

$N(A^T) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$R(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}\right\}$

$R(A^T) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$

$N(A) = R(A^T)^{\perp}$ and $N(A^T) = R(A)^{\perp}$

70. To form an orthonormal basis B' for V, follow these steps:

(i) Begin with a basis for the inner product space. It need not be orthogonal nor consist of unit vectors.

(ii) Convert the given basis to an orthogonal basis.

(iii) Normalize each vector in the orthogonal basis to form an orthonormal basis.

Section 5.4 Mathematical Models and Least Squares Analysis

2. The system

$$\begin{aligned} c_0 \qquad &= 0 \\ c_0 + 3c_1 &= 1 \\ c_0 + 4c_1 &= 2 \end{aligned}$$

has no solution. The points are *not* collinear.

4. The system

$$\begin{aligned} c_0 - c_1 &= 5 \\ c_0 + c_1 &= -1 \\ c_0 + c_1 &= -4 \end{aligned}$$

has no solution. The points are *not* collinear.

6. Orthogonal: $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}^T \cdot \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$

8. Not orthogonal: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}^T \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix} = -6 \neq 0$

10. (a) $S = \operatorname{span}\left\{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right\} \Rightarrow S^{\perp} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$

(b) $S \oplus S^{\perp} = R^3$

12. (a) $S = \text{span}\left\{\begin{bmatrix}0\\1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\0\\2\end{bmatrix}\right\} \Rightarrow S^{\perp} = \text{span}\left\{\begin{bmatrix}1\\-4\\-2\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\0\\1\end{bmatrix}\right\}$

(b) $S \oplus S^{\perp} = R^5$

14. (a) Because $S = \left\{[x, y, 0, 0, z]^T\right\}$,

$$S^{\perp} = \text{span}\left\{\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\right\}.$$

(b) Since $S = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$, you can see that $S \oplus S^{\perp} = R^5$.

16. The orthogonal complement of

$$S^{\perp} = \text{span}\left\{\begin{bmatrix}1\\-4\\-2\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\0\\1\end{bmatrix}\right\}$$

is

$$\left(S^{\perp}\right)^{\perp} = S = \text{span} = \left\{\begin{bmatrix}0\\1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\0\\2\end{bmatrix}\right\}.$$

18. Using the Gram-Schmidt process, an orthogonal basis for S is $\left\{\begin{bmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

$$\text{proj}_S\mathbf{v} = (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2 + (\mathbf{u}_3 \cdot \mathbf{v})\mathbf{u}_3$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\\ 0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{5}\\ \frac{2}{5}\\ 1\\ 1\end{bmatrix}$$

20. Using the Gram-Schmidt process, an orthonormal basis for S is

$$\left\{\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}, \begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{bmatrix}\right\}.$$

$$\text{proj}_S\mathbf{v} = (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2 + (\mathbf{u}_3 \cdot \mathbf{v})\mathbf{u}_3$$

$$= 5\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix} + \frac{-1}{\sqrt{2}}\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{bmatrix} + 0\begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{5}{2}\\ 2\\ 3\\ \frac{5}{2}\end{bmatrix}$$

22. $A = \begin{bmatrix}0 & -1 & 1\\ 1 & 2 & 0\\ 1 & 1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$

$$A^T = \begin{bmatrix}0 & 1 & 1\\ -1 & 2 & 1\\ 1 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$N(A) = \text{span}\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$$

$$N(A^T) = \text{span}\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$$

$$R(A) = \text{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}$$

$$R(A^T) = \text{span}\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}\right\}$$

24. $A = \begin{bmatrix}1 & 0 & -1\\ 0 & -1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$

$$A^T = \begin{bmatrix}1 & 0 & 1 & 1\\ 0 & -1 & 1 & 0\\ -1 & 1 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$N(A) = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$N(A^T) = \text{span}\left\{\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\right\}$$

$$R(A) = \text{span}\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$$

$$R(A^T) = \text{span}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\} \quad \left(R(A^T) = R^3\right)$$

26. $A^T A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$

$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 3 & 0 & 3 & 5 \\ 0 & 3 & 1 & -1 \\ 3 & 1 & 4 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{7}{6} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} \frac{7}{6} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

28. $A^T A = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ 4 & 11 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$A^T \mathbf{b} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 6 & 4 & 0 & 1 \\ 4 & 11 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{50} \\ 0 & 1 & 0 & \frac{4}{25} \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} \frac{3}{50} \\ \frac{4}{25} \\ 0 \end{bmatrix}$

30. $A^T A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 14 \end{bmatrix}$

$A^T \mathbf{b} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

The normal equations are

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

$$\begin{bmatrix} 2 & 4 \\ 4 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

The solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{17}{6} \\ \frac{7}{6} \end{bmatrix}.$$

Finally, the projection of $\vec{\mathbf{b}}$ onto S is

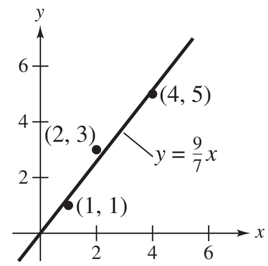
$$A\mathbf{x} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{17}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ -\frac{5}{3} \\ \frac{2}{3} \end{bmatrix}.$$

32. $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$

$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix}$

$\begin{bmatrix} 3 & 7 & 9 \\ 7 & 21 & 27 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{9}{7} \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ \frac{9}{7} \end{bmatrix}$

line: $y = \frac{9}{7}x$

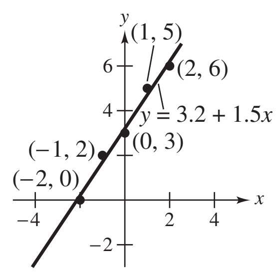


34. $A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 15 \end{bmatrix}$

$\begin{bmatrix} 5 & 0 & 16 \\ 0 & 10 & 15 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3.2 \\ 0 & 1 & 1.5 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 3.2 \\ 1.5 \end{bmatrix}$

line: $y = 3.2 + 1.5x$



36. $A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix}$

$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ \frac{3}{2} \\ \frac{5}{2} \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{37}{2} \\ \frac{95}{2} \end{bmatrix}$

$$\begin{bmatrix} 4 & 6 & 14 & 10 \\ 6 & 14 & 36 & \frac{37}{2} \\ 14 & 36 & 98 & \frac{95}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{39}{20} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} \frac{39}{20} \\ -\frac{4}{5} \\ \frac{1}{2} \end{bmatrix}$$

Quadratic Polynomial: $y = \frac{39}{20} - \frac{4}{5}x + \frac{1}{2}x^2$

38. $A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$

$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ \frac{7}{2} \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{31}{2} \\ -17 \\ 27 \end{bmatrix}$

$$\begin{bmatrix} 5 & 0 & 10 & \frac{31}{2} \\ 0 & 10 & 0 & -17 \\ 10 & 0 & 34 & 27 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{257}{70} \\ 0 & 1 & 0 & -\frac{17}{10} \\ 0 & 0 & 1 & -\frac{2}{7} \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} \frac{257}{70} \\ -\frac{17}{10} \\ -\frac{2}{7} \end{bmatrix}$$

Quadratic Polynomial: $y = \frac{257}{70} - \frac{17}{10}x - \frac{2}{7}x^2$

40. Substitute the data points $(8, 29.3)$, $(9, 32.0)$, $(10, 32.5)$, $(11, 32.7)$, $(12, 31.7)$, and $(13, 31.2)$ into the quadratic polynomial $y = c_0 + c_1t + c_2t^2$. You then obtain the system of linear equations

$$\begin{aligned} c_0 + 8c_1 + 64c_2 &= 29.3 \\ c_0 + 9c_1 + 81c_2 &= 32.0 \\ c_0 + 10c_1 + 100c_2 &= 32.5 \\ c_0 + 11c_1 + 121c_2 &= 32.7 \\ c_0 + 12c_1 + 144c_2 &= 31.7 \\ c_0 + 13c_1 + 169c_2 &= 31.2. \end{aligned}$$

This produces the least squares problem

$$A\mathbf{t} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 8 & 64 \\ 1 & 9 & 81 \\ 1 & 10 & 100 \\ 1 & 11 & 121 \\ 1 & 12 & 144 \\ 1 & 13 & 169 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 29.3 \\ 32.0 \\ 32.5 \\ 32.7 \\ 31.7 \\ 31.2 \end{bmatrix}.$$

The normal equations are

$$A^TA\mathbf{t} = A^T\mathbf{b}$$

$$\begin{bmatrix} 6 & 53 & 679 \\ 53 & 579 & 6497 \\ 679 & 6497 & 89{,}595 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 189.4 \\ 1668.1 \\ 21{,}511.5 \end{bmatrix}.$$

and the solution is

$$\mathbf{t} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -13.2 \\ 8.50 \\ -0.393 \end{bmatrix}.$$

The least squares quadratic is $y = -13.2 + 8.50t - 0.393t^2$. Substitute the same data points into the cubic polynomial $y = c_0 + c_1t + c_2t^2 + c_3t^3$. You then obtain the system of linear equations

$$\begin{aligned}
c_0 + 8c_1 + 64c_2 + 512c_3 &= 29.3\\
c_0 + 9c_1 + 81c_2 + 729c_3 &= 32.0\\
c_0 + 10c_1 + 100c_2 + 1000c_3 &= 32.5\\
c_0 + 11c_1 + 121c_2 + 1331c_3 &= 32.7\\
c_0 + 12c_1 + 144c_2 + 1728c_3 &= 31.7\\
c_0 + 13c_1 + 169c_2 + 2197c_3 &= 31.2.
\end{aligned}$$

This produces the least squares problem

$$A\mathbf{t} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \\ 1 & 13 & 169 & 2197 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 29.3 \\ 32.0 \\ 32.5 \\ 32.7 \\ 31.7 \\ 31.2 \end{bmatrix}.$$

The normal equations are

$$A^TA\mathbf{t} = A^T\mathbf{b}$$

$$\begin{bmatrix} 6 & 63 & 679 & 7497 \\ 63 & 679 & 7497 & 84{,}595 \\ 679 & 7497 & 84{,}595 & 972{,}993 \\ 7497 & 84{,}595 & 972{,}993 & 11{,}377{,}939 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 189.4 \\ 1993.1 \\ 21{,}511.5 \\ 237.677.3 \end{bmatrix}$$

and the solution is

$$\mathbf{t} = \begin{bmatrix} -123.7 \\ 41.07 \\ -3.543 \\ 0.1000 \end{bmatrix}.$$

The least squares regression cubic is $y = -123.7 + 41.07t - 3.543t^2 + 0.1000t^3$.

2018 (quadratic):

$y = -13.2 + 8.50(18) - 0.393(18)^2 \approx \$\,12.5$ billion

2018 (cubic):

$y = -123.7 + 41.07(18) - 3.543(18)^2 + 0.1000(18)^3 \approx \$\,50.8$ billion

Because the original data increased from 2008 to 2013 with the revenue leveling off in 2012, you can expect the revenue to increase or stay about the same for future years. Because the cubic polynomial predicts the revenue to be about \$ 50.8 billion in 2018, this model is more accurate for predicting future revenues.

42. The vector $A\mathbf{x}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ for a given vector $\mathbf{b}$ is $A\mathbf{x} = proj_s\mathbf{b}$, where $S = R(A)$. Since $A\mathbf{x} - \mathbf{b} = \text{proj}_s\mathbf{b} - \mathbf{b}$, $(A\mathbf{x} - \mathbf{b}) \in S^\perp$. Then $(A\mathbf{x} - \mathbf{b}) \in N(A^T)$, because $S^\perp = R(A)^\perp N(A^T)$. So

$$\begin{aligned}
A^T(A\mathbf{x} - \mathbf{b}) &= \mathbf{0}\\
A^TA\mathbf{x} - A^T\mathbf{b} &= \mathbf{0}\\
A^TA\mathbf{x} &= A^T\mathbf{b}.
\end{aligned}$$

These equations are used to find $\mathbf{b}$ and solve the least squares problem.

44. (a) False. They are orthogonal subspaces of R^m not R^n.

(b) True. See the "Definition of Orthogonal Complement" on page 266.

(c) True. See page 265 for the definition of the "Least Squares Problem."

46. Let S be a subspace of R^n and $S^\perp$ its orthogonal complement. $S^\perp$ contains the zero vector. If $\mathbf{v}_1, \mathbf{v}_2 \in S^\perp$, then for all $\mathbf{w} \in S$,

$(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 \in S^\perp$

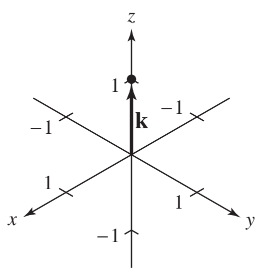
and for any scalar c,

$(c\mathbf{v}_1) \cdot \mathbf{w} = c(\mathbf{v}_1 \cdot \mathbf{w}) = c\mathbf{0} = \mathbf{0} \Rightarrow c\mathbf{v}_1 \in S^\perp$.

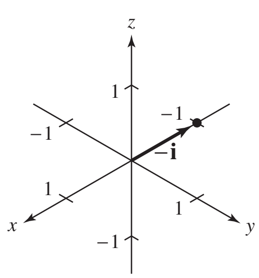
48. Let $\mathbf{x} \in S_1 \cap S_2$, where $R^n = S_1 \oplus S_2$. Then $\mathbf{x} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 \in S_1$ and $\mathbf{v}_2 \in S_2$. But, $\mathbf{x} \in S_1 \Rightarrow \mathbf{x} = \mathbf{x} + \mathbf{0}$, $\mathbf{x} \in S_1$, $\mathbf{0} \in S_2$, and $\mathbf{x} \in S_2 \Rightarrow \mathbf{x} = \mathbf{0} + \mathbf{x}$, $\mathbf{0} \in S_1$, $\mathbf{x} \in S_2$. So, $\mathbf{x} = \mathbf{0}$ by the uniqueness of direct sum representation.

Section 5.5 Applications of Inner Product Spaces

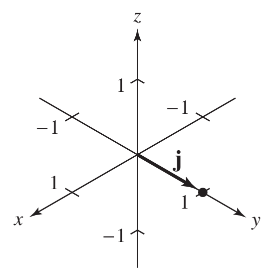
2. $$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + \mathbf{k} = \mathbf{k}$$



4. $$\mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = -\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = -\mathbf{i}$$



6. $$\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{k} = 0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = \mathbf{j}$$



8. (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$$= \mathbf{i}\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= \mathbf{i}(0 - 0) - \mathbf{j}(6 - 1) + \mathbf{k}(0 - 0)$$

$$= 0\mathbf{i} - 5\mathbf{j} + 0\mathbf{k} = -5\mathbf{j}$$

(b) $\mathbf{v} \times \mathbf{u} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

$$= \mathbf{i}\begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$$

$$= \mathbf{i}(0 - 0) - \mathbf{j}(1 - 6) + \mathbf{k}(0 - 0)$$

$$= 0\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} = 5\mathbf{j}$$

(c) $\mathbf{v} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix}$

$$= \mathbf{i}\begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= \mathbf{i}(0 - 0) - \mathbf{j}(3 - 3) + \mathbf{k}(0 - 0)$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

10. (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

$$= \mathbf{i}\begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= \mathbf{i}(-2 + 2) - \mathbf{j}(2 + 2) + \mathbf{k}(2 + 2)$$

$$= 0\mathbf{j} - 4\mathbf{j} + 4\mathbf{k} = -4\mathbf{j} + 4\mathbf{k}$$

(b) $\mathbf{v} \times \mathbf{u} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

$$= \mathbf{i}\begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \mathbf{i}(-2 + 2) - \mathbf{j}(-2 - 2) + \mathbf{k}(-2 - 2)$$

$$= 0\mathbf{j} + 4\mathbf{j} - 4\mathbf{k} = 4\mathbf{j} - 4\mathbf{k}$$

(c) $\mathbf{v} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$$= \mathbf{i}\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$$

$$= \mathbf{i}(2 - 2) - \mathbf{j}(2 - 2) + \mathbf{k}(2 - 2)$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

12. (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & -3 \\ 3 & -3 & 3 \end{vmatrix}$

$$= \mathbf{i}\begin{vmatrix} -3 & -3 \\ -3 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -3 \\ 3 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -3 \\ 3 & -3 \end{vmatrix}$$

$$= \mathbf{i}(-9 - 9) - \mathbf{j}(9 + 9) + \mathbf{k}(-9 + 9) = -18\mathbf{i} - 18\mathbf{j}$$

(b) $\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 3 \\ 3 & -3 & -3 \end{vmatrix}$

$$= \mathbf{i}\begin{vmatrix} -3 & 3 \\ -3 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 3 \\ 3 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -3 \\ 3 & -3 \end{vmatrix}$$

$$= \mathbf{i}(9 + 9) - \mathbf{j}(-9 - 9) + \mathbf{k}(-9 + 9) = 18\mathbf{i} + 18\mathbf{j}$$

(c) $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 3 \\ 3 & -3 & 3 \end{vmatrix}$

$$= \mathbf{i}\begin{vmatrix} -3 & 3 \\ -3 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -3 \\ 3 & -3 \end{vmatrix}$$

$$= \mathbf{i}(-9 + 9) - \mathbf{j}(9 - 9) + \mathbf{k}(-9 + 9)$$

$$= \mathbf{0}$$

14. (a) $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 9 & -3 \\ 4 & 6 & -5 \end{bmatrix}$

$= \mathbf{i}\begin{vmatrix} 9 & -3 \\ 6 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & -3 \\ 4 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 9 \\ 4 & 6 \end{vmatrix}$

$= \mathbf{i}(-45 + 18) - \mathbf{j}(10 + 12) + \mathbf{k}(-12 - 36)$

$= -27\mathbf{i} - 22\mathbf{j} - 48\mathbf{k}$

(b) $\mathbf{v} \times \mathbf{u} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -5 \\ -2 & 9 & -3 \end{bmatrix}$

$= \mathbf{i}\begin{vmatrix} 6 & -5 \\ 9 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & -5 \\ -2 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & 6 \\ -2 & 9 \end{vmatrix}$

$= \mathbf{i}(-18 + 45) - \mathbf{j}(-12 - 10) + \mathbf{k}(36 + 12)$

$= 27\mathbf{i} + 22\mathbf{j} + 48\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -5 \\ 4 & 6 & -5 \end{bmatrix}$

$= \mathbf{i}\begin{vmatrix} 6 & -5 \\ 6 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & -5 \\ 4 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & 6 \\ 4 & 6 \end{vmatrix}$

$= \mathbf{i}(-30 + 30) - \mathbf{j}(-20 + 20) + \mathbf{k}(24 - 24)$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

16. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = -3\mathbf{i} - \mathbf{j} - \mathbf{k} = (-3, -1, -1)$

Furthermore, $\mathbf{u} \times \mathbf{v} = (-3, -1, -1)$ is orthogonal to both $(-1, 1, 2)$ and $(0, 1, -1)$ because $(-3, -1, -1) \cdot (-1, 1, 2) = 0$ and $(-3, -1, -1) \cdot (0, 1, -1) = 0$.

18. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 4 & 2 & 0 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 8\mathbf{k} = (-2, 4, -8)$

Furthermore, $\mathbf{u} \times \mathbf{v} = (-2, 4, -8)$ is orthogonal to both $(-2, 1, 1)$ and $(4, 2, 0)$ because $(-2, 4, -8) \cdot (-2, 1, 1) = 0$ and $(-2, 4, -8) \cdot (4, 2, 0) = 0$.

20. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 0 \\ 3 & 2 & -2 \end{vmatrix} = -2\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} = (-2, 8, 5)$

Furthermore, $\mathbf{u} \times \mathbf{v} = (-2, 8, 5)$ is orthogonal to both $(4, 1, 0)$ and $(3, 2, -2)$ because $(-2, 8, 5) \cdot (4, 1, 0) = 0$ and $(-2, 8, 5) \cdot (3, 2, -2) = 0$.

22. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} = (1, 3, 1)$

Furthermore, $\mathbf{u} \times \mathbf{v} = (1, 3, 1)$ is orthogonal to both $(2, -1, 1)$ and $(3, -1, 0)$ because $(1, 3, 1) \cdot (2, -1, 1) = 0$ and $(1, 3, 1) \cdot (3, -1, 0) = 0$.

24. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ -1 & 3 & -2 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)$

Furthermore, $\mathbf{u} \times \mathbf{v} = (1, 1, 1)$ is orthogonal to both $(1, -2, 1)$ and $(-1, 3, -2)$ because $(1, 1, 1) \cdot (1, -2, 1) = 0$ and $(1, 1, 1) \cdot (-1, 3, -2) = 0$.

26. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 19 & -12 \\ 5 & -19 & 12 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = (0, 0, 0)$

Furthermore, $\mathbf{u} \times \mathbf{v} = (0, 0, 0)$ is orthogonal to both $(-5, 19, -12)$ and $(5, -19, 12)$ because

$(0, 0, 0) \cdot (-5, 19, -12) = 0$ and

$(0, 0, 0) \cdot (5, -19, 12) = 0$.

28. Using a graphing utility,

$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (7, 1, 3)$.

Check if $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$:

$\mathbf{w} \cdot \mathbf{u} = (7, 1, 3) \cdot (1, 2, -3) = 7 + 2 - 9 = 0$

$\mathbf{w} \cdot \mathbf{v} = (7, 1, 3) \cdot (-1, 1, 2) = -7 + 1 + 6 = 0$

30. Using a graphing utility,

$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (0, 9, 0)$.

Check if $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$:

$\mathbf{w} \cdot \mathbf{u} = (0, 9, 0) \cdot (2, 0, -1) = 0 + 0 + 0 = 0$

$\mathbf{w} \cdot \mathbf{v} = (0, 9, 0) \cdot (-1, 0, -4) = 0 + 0 + 0 = 0$

32. Using a graphing utility,

$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (0, 5, 5)$.

Check if $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$:

$\mathbf{w} \cdot \mathbf{u} = (0, 5, 5) \cdot (3, -1, 1) = 0 - 5 + 5 = 0$

$\mathbf{w} \cdot \mathbf{v} = (0, 5, 5) \cdot (2, 1, -1) = 0 + 5 - 5 = 0$

34. Using a graphing utility,

$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (-8, 16, -2).$

Check if **w** is orthogonal to both **u** and **v**:

$\mathbf{w} \cdot \mathbf{u} = (-8, 16, -2) \cdot (4, 2, 0) = -32 + 32 + 0 = 0$

$\mathbf{w} \cdot \mathbf{v} = (-8, 16, -2) \cdot (1, 0, -4) = -8 + 0 + 8 = 0$

36. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -2 \end{vmatrix} = (2, 7, 1)$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{54} = 3\sqrt{6}$

$\text{Unit vector} = \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{3\sqrt{6}}(2, 7, 1) = \dfrac{\sqrt{6}}{18}(2, 7, 1)$

38. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 1 & 0 & -3 \end{vmatrix} = -6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{36 + 9 + 4} = 7$

$\text{Unit vector} = \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\dfrac{6}{7}\mathbf{i} + \dfrac{3}{7}\mathbf{j} - \dfrac{2}{7}\mathbf{k}$

40. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -14 & 5 \\ 14 & 28 & -15 \end{vmatrix} = 70\mathbf{i} + 175\mathbf{j} + 392\mathbf{k}$

$$\begin{aligned}\|\mathbf{u} \times \mathbf{v}\| &= \sqrt{70^2 + 175^2 + 392^2} \\ &= \sqrt{189{,}189} = 21\sqrt{429}\end{aligned}$$

$$\begin{aligned}\text{Unit vector} &= \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{21\sqrt{249}}(70, 175, 392) \\ &= \frac{1}{3\sqrt{429}}(10, 25, 56) \\ &= \frac{\sqrt{429}}{1287}(10, 25, 56)\end{aligned}$$

48. $(4, 0, 3) - (1, -2, 0) = (3, 2, 3)$

$(2, 2, 3) - (-1, 0, 0) = (3, 2, 3)$

$(2, 2, 3) - (4, 0, 3) = (-2, 2, 0)$

$(-1, 0, 0) - (1, -2, 0) = (-2, 2, 0)$

$\mathbf{u} = (3, 2, 3)$ and $\mathbf{v} = (-2, 2, 0)$

Because

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 3 \\ -2 & 2 & 0 \end{vmatrix} = -6\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} = (-6, -6, 10),$

the area of the parallelogram is

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-6)^2 + (-6)^2 + 10^2} = \sqrt{172} = 2\sqrt{43}$ square units.

42. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{vmatrix} = 6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{36 + 36 + 9} = 9$

$$\begin{aligned}\text{Unit vector} &= \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{9}(6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) \\ &= \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\end{aligned}$$

44. Because

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\mathbf{i} + \mathbf{k} = (-1, 0, 1),$

the area of the parallelogram is

$\|\mathbf{u} \times \mathbf{v}\| = \|(-1, 0, 1)\| = \sqrt{2}$ square units.

46. Because

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 3\mathbf{k} = (0, 0, 3),$

the area of the parallelogram is

$\|(0, 0, 3)\| = 3$ square units.

50. $(0, 1, 2) - (2, -3, 4) = (-2, 4, -2)$

$(0, 1, 2) - (-1, 2, 0) = (1, -1, 2)$

Because

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = (6, 2, -2),$$

the area of the triangle is

$$A = \tfrac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \tfrac{1}{2}\sqrt{6^2 + 2^2 + (-2)^2} = \tfrac{1}{2}\sqrt{44} = \sqrt{11}.$$

52. Because

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{i} = (-1, 0, 0),$$

the triple scalar product of **u**, **v**, and **w** is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (-1, 0, 0) \cdot (-1, 0, 0) = 1.$$

54. Because

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\mathbf{i} = (3, 0, 0),$$

the triple scalar product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (2, 0, 1) \cdot (3, 0, 0) = 6.$$

56. $$c(\mathbf{u} \times \mathbf{v}) = c\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = c\mathbf{u} \times \mathbf{v} = c\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ cv_1 & cv_2 & cv_3 \end{vmatrix} = \mathbf{u} \times c\mathbf{v}$$

58. $$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0,$$ because two rows are the same.

60. $$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1 - \cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\left(1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\right) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

62. (a) Because

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = (2, 1, -1),$$

The volume is given by

$$\begin{aligned} |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| &= |(1, 1, 0) \cdot (2, 1, -1)| \\ &= 1(2) + 1(1) + 0(-1) = 3 \text{ cubic units.} \end{aligned}$$

(b) Because

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} - \mathbf{k} = (1, 1, -1),$$

The volume is given by

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |(1, 1, 0) \cdot (1, 1, -1)| = 2 \text{ cubic units.}$$

(c) $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 0 & -2 \\ 3 & 0 & 2 \end{vmatrix} = 0 - 2(6) + 2(0) = -12$$

The volume is given by $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 12$ cubic units.

(d) $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$
$$= 1(2) - 2(-1 - 4) - 1(0 - 4) = 16$$

The volume is given by $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 16$ cubic units.

64. $$\begin{aligned} \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1 - \cos^2\theta} \\ &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\ &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\ &= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2} \\ &= \|\mathbf{u} \times \mathbf{v}\| \end{aligned}$$

66. (a) $$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \mathbf{u} \times \left[(v_2w_3 - w_2v_3)\mathbf{i} - (v_1w_3 - w_1v_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}\right] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ (v_2w_3 - w_2v_3) & (w_1v_3 - v_1w_3) & (v_1w_2 - v_2w_1) \end{vmatrix} \\ &= \left[(u_2(v_1w_2 - v_2w_1)) - u_3(w_1v_3 - v_1w_3)\right]\mathbf{i} - \left[u_1(v_1w_2 - v_2w_1) - u_3(v_2w_3 - w_2v_3)\right]\mathbf{j} \\ &\quad + \left[u_1(w_1v_3 - v_1w_3) - u_2(v_2w_3 - w_2v_3)\right]\mathbf{k} \\ &= (u_2w_2v_1 + u_3w_3v_1 - u_2v_2w_1 - u_3v_3w_1, u_1w_1v_2 + u_3w_3v_2 - u_1v_1w_2 - u_3v_3w_2, \\ &\quad u_1w_1v_3 + u_2w_2v_3 - u_1v_1w_3 - u_2v_2w_3) \\ &= (u_1w_1 + u_2w_2 + u_3w_3)(v_1, v_2, v_3) - (u_1v_1 + u_2v_2 + u_3v_3)(w_1, w_2, w_3) \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \end{aligned}$$

(b) Let

$\mathbf{u} = (1, 0, 0), \mathbf{v} = (0, 1, 0)$ and $\mathbf{w} = (1, 1, 1).$

Then

$\mathbf{v} \times \mathbf{w} = (1, 0, -1)$ and $\mathbf{u} \times \mathbf{v} = (0, 0, 1).$

So

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (1, 0, 0) \times (1, 0, -1) = (0, 1, 0),$

while

$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (0, 0, 1) \times (1, 1, 1) = (-1, 1, 0),$

which are not equal.

68. (a) The standard basis for P_1 is $\{1, x\}$. Applying the Gram-Schmidt orthonormalization process produces the orthonormal basis

$$B = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{\frac{1}{\sqrt{3}}, \frac{1}{3}(2x - 5)\right\}.$$

The least squares approximating function is given by $g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2$.

Find the inner products

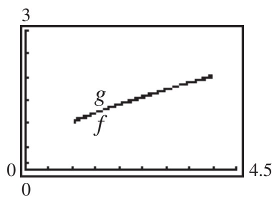
$$\langle f_1, \mathbf{w}_1\rangle = \int_1^4 \sqrt{x}\frac{1}{\sqrt{3}}dx = \frac{2}{3\sqrt{3}}x^{3/2}\Big]_1^4 = \frac{14}{3\sqrt{3}}$$

$$\langle f, \mathbf{w}_2\rangle = \int_1^4 \sqrt{x}\left(\frac{1}{3}\right)(2x - 5)dx = \left[\frac{4}{15}x^{5/2} - \frac{10}{9}x^{3/2}\right]_1^4 = \frac{22}{45}$$

and conclude that

$$g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 = \frac{14}{3\sqrt{3}}\frac{1}{\sqrt{3}} + \frac{22}{45}\left(\frac{1}{3}\right)(2x - 5) = \frac{44}{135}x + \frac{20}{27} = \frac{4}{135}(25 + 11x).$$

(b)



70. (a) The standard basis for P_1 is $\{1, x\}$. Applying the Gram-Schmidt orthonormalization process produces the orthonormal basis

$$B = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{1, \sqrt{3}(2x - 1)\right\}.$$

The least squares approximating function is then given by $g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2$.

Find the inner products

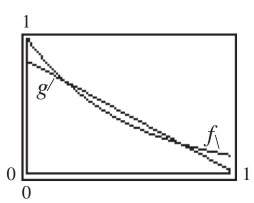
$$\langle f, \mathbf{w}_1\rangle = \int_0^1 e^{-2x}dx = -\tfrac{1}{2}e^{-2x}\Big]_0^1 = -\tfrac{1}{2}\left(e^{-2} - 1\right)$$

$$\langle f, \mathbf{w}_2\rangle = \int_0^1 e^{-2x}\sqrt{3}(2x - 1)dx = -\sqrt{3} \times e^{-2x}\Big]_0^1 = -\sqrt{3}e^{-2}$$

and conclude that

$$g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 = -\tfrac{1}{2}\left(e^{-2} - 1\right) - \sqrt{3}e^{-2}\left(\sqrt{3}(2x - 1)\right) = -6e^{-2}x + \tfrac{1}{2}\left(5e^{-2} + 1\right) \approx -0.812x + 0.8383.$$

(b)



72. (a) The standard basis for P_1, is $\{1, x\}$. Applying the Gram-Schmidt orthonormalization process produces the orthonormal basis

$$B = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{\frac{\sqrt{2\pi}}{\pi}, \frac{\sqrt{6\pi}}{\pi^2}(4x - \pi)\right\}.$$

The least squares approximating function is then given by $g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2$.

Find the inner products

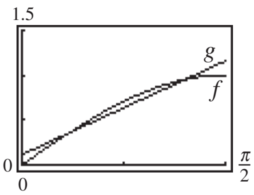
$$\langle f, \mathbf{w}_1\rangle = \int_0^{\pi/2} (\sin x)\left(\frac{\sqrt{2\pi}}{\pi}\right)dx = -\frac{\sqrt{2\pi}}{\pi}\cos x\Big]_0^{\pi/2} = \frac{\sqrt{2\pi}}{\pi}$$

$$\langle f, \mathbf{w}_2\rangle = \int_0^{\pi/2} (\sin x)\left[\frac{\sqrt{6\pi}}{\pi^2}(4x - \pi)\right]dx = \frac{\sqrt{6\pi}}{\pi^2}\left[-4x\cos x + 4\sin x + \pi\cos x\right]_0^{\pi/2} = \frac{\sqrt{6\pi}}{\pi^2}(4 - \pi)$$

and conclude that

$$\begin{aligned} g(x) &= \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 \\ &= \frac{\sqrt{2\pi}}{\pi}\left(\frac{\sqrt{2\pi}}{\pi}\right) + \frac{\sqrt{6\pi}}{\pi^2}(4 - \pi)\left[\frac{\sqrt{6\pi}}{\pi^2}(4x - \pi)\right] \\ &= \frac{2}{\pi} + \frac{6}{\pi^3}(4 - \pi)(4x - \pi) \\ &= \frac{24(4 - \pi)}{\pi^3}x - \frac{8(3 - \pi)}{\pi^2} \approx 0.6644x + 0.1148. \end{aligned}$$

(b)



74. (a) The standard basis for P_2 is $\{1, x, x^2\}$. Applying the Gram-Schmidt orthonormalization process produces the orthonormal basis

$$B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{\frac{1}{\sqrt{3}}, \frac{1}{3}(2x - 5), \frac{2\sqrt{5}}{3\sqrt{3}}\left(x^2 - 5x + \frac{11}{2}\right)\right\}.$$

The least squares approximating function is then given by $g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \langle f, \mathbf{w}_3\rangle\mathbf{w}_3$.

Find the inner products

$$\langle f, \mathbf{w}_1\rangle = \int_1^4 \sqrt{x}\frac{1}{\sqrt{3}}dx = \frac{14}{3\sqrt{3}} \text{ (see Exercise 51)}$$

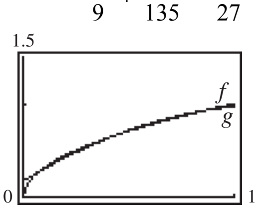
$$\langle f, \mathbf{w}_2\rangle = \int_1^4 \sqrt{x}\frac{1}{3}(2x - 5)dx = \frac{22}{45} \text{ (see Exercise 51)}$$

$$\langle f, \mathbf{w}_3\rangle = \int_1^4 \sqrt{x}\frac{2\sqrt{5}}{3\sqrt{3}}\left(x^2 - 5x + \frac{11}{2}\right)dx = \frac{2\sqrt{5}}{3\sqrt{3}}\int_1^4\left(x^{5/2} - 5x^{3/2} + \frac{11}{2}x^{1/2}\right)dx = \frac{2\sqrt{5}}{3\sqrt{3}}\left[\frac{2}{7}x^{7/2} - 2x^{5/2} + \frac{11}{3}x^{3/2}\right]_1^4 = \frac{-2\sqrt{5}}{63\sqrt{3}}$$

and conclude that g is given by

$$\begin{aligned} g(x) &= \frac{14}{3\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) + \frac{22}{45}\left(\frac{1}{3}(2x - 5)\right) - \frac{2\sqrt{5}}{63\sqrt{3}} \cdot \frac{2\sqrt{5}}{3\sqrt{3}}\left(x^2 - 5x + \frac{11}{2}\right) \\ &= \frac{14}{9} + \frac{44x}{135} - \frac{22}{27} - \frac{20}{567}x^2 + \frac{100}{567}x - \frac{110}{567} = -\frac{20}{567}x^2 + \frac{1424}{2835}x + \frac{310}{567}. \end{aligned}$$

(b)



76. (a) The standard basis for P_2 is $\{1, x, x^2\}$. Applying the Gram-Schmidt orthonormalization process produces the orthonormal basis $B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{\dfrac{1}{\sqrt{\pi}}, \dfrac{2\sqrt{3}}{\pi^{3/2}}x, \dfrac{6\sqrt{5}}{\pi^{5/2}}\left(x^2 - \dfrac{\pi^2}{12}\right)\right\}$.

The least squares approximating function is then given by $g(x) = \langle f, \mathbf{w}\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \langle f, \mathbf{w}_3\rangle\mathbf{w}_3$.

Find the inner products

$$\langle f, \mathbf{w}_1\rangle = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{\pi}}\cos x\,dx = \frac{\sin x}{\sqrt{\pi}}\Bigg]_{-\pi/2}^{\pi/2} = \frac{2}{\sqrt{\pi}}$$

$$\langle f, \mathbf{w}_2\rangle = \int_{-\pi/2}^{\pi/2} \frac{2\sqrt{3}}{\pi^{3/2}}x\cos x\,dx = \left[\frac{2\sqrt{3}\cos x}{\pi^{3/2}} + \frac{2\sqrt{3}x\sin x}{\pi^{3/2}}\right]_{-\pi/2}^{\pi/2} = 0$$

$$\langle f, \mathbf{w}_3\rangle = \int_{-\pi/2}^{\pi/2} \frac{6\sqrt{5}}{\pi^{3/2}}\left(x^2 - \frac{\pi^2}{12}\right)\cos x\,dx = \left[\frac{12\sqrt{5}x\cos x}{\pi^{5/2}} + \frac{\sqrt{5}\left(12x^2 - \pi^2 - 24\right)\sin x}{2\pi^{5/2}}\right]_{-\pi/2}^{\pi/2} = \frac{2\sqrt{5}\left(\pi^2 - 12\right)}{\pi^{5/2}}$$

and conclude that

$$\begin{aligned}
g(x) &= \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \langle f, \mathbf{w}_3\rangle\mathbf{w}_3 \\
&= \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{1}{\sqrt{\pi}}\right) + (0)\left(\frac{2\sqrt{3}}{\pi^{3/2}}x\right) + \left(\frac{2\sqrt{5}\left(\pi^2 - 12\right)}{\pi^{5/2}}\right)\left(\frac{6\sqrt{5}}{\pi^{5/2}}\left(x^2 - \frac{\pi^2}{12}\right)\right) \\
&= \frac{2}{\pi} + \frac{60\pi^2 - 720}{\pi^5}\left(x^2 - \frac{\pi^2}{12}\right) = \left(\frac{60\left(\pi^2 - 12\right)}{\pi^5}\right)x^2 + \frac{60 - 3\pi^2}{\pi^3} \approx -0.4177x^2 + 0.9802.
\end{aligned}$$

78. The fourth order Fourier approximation of $f(x) = \pi - x$ is of the form

$$g(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + a_3\cos 3x + b_3\sin 3x + a_4\cos 4x + b_4\sin 4x.$$

In Exercise 67, you determined a_0 and the general form of the coefficients a_j and b_j.

$$\begin{aligned}
a_0 &= 0 \\
a_j &= 0, \quad j = 1, 2, 3, \ldots \\
b_j &= \frac{2}{j}, \quad j = 1, 2, 3, \ldots
\end{aligned}$$

So, the approximation is $g(x) = 2\sin x + \sin 2x + \dfrac{2}{3}\sin 3x + \dfrac{1}{2}\sin 4x$.

80. The fourth order Fourier approximation of $f(x) = (x - \pi)^2$ is of the form

$$g(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + a_3\cos 3x + b_3\sin 3x + a_4\cos 4x + b_4\sin 4x.$$

In Exercise 69, you determined a_0 and the general form of the coefficients a_j and b_j.

$$\begin{aligned}
a_0 &= \frac{2\pi^2}{3} \\
a_j &= \frac{4}{j^2}, \quad j = 1, 2, \ldots \\
b_j &= 0, \quad j = 1, 2, \ldots
\end{aligned}$$

So, the approximation is $g(x) = \dfrac{\pi^2}{3} + 4\cos x + \cos 2x + \dfrac{4}{9}\cos 3x + \dfrac{1}{4}\cos 4x$.

82. The second order Fourier approximation of $f(x) = e^{-x}$ is of the form

$$g(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x.$$

In Exercise 71, you found that

$$a_0 = \left(1 - e^{-2\pi}\right)/\pi$$
$$a_1 = \left(1 - e^{-2\pi}\right)/2\pi$$
$$b_1 = \left(1 - e^{-2\pi}\right)/2\pi.$$

So, you need to determine a_2 and b_2.

$$a_2 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos 2x \, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cos 2x \, dx$$
$$= \frac{1}{\pi}\left[\frac{1}{5}\left(-e^{-x} \cos 2x + 2e^{-x} \sin 2x\right)\right]_0^{2\pi} = \frac{1}{5\pi}\left(1 - e^{-2\pi}\right)$$
$$b_2 = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin 2x \, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x} \sin 2x \, dx$$
$$= \frac{1}{\pi}\left[\frac{1}{5}\left(-e^{-x} \sin 2x - 2e^{-x} \cos 2x\right)\right]_0^{2\pi} = \frac{2}{5\pi}\left(1 - e^{-2\pi}\right)$$

So, the approximation is

$$g(x) = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{2\pi} \cos x + \frac{1 - e^{-2\pi}}{2\pi} \sin x + \frac{1 - e^{-2\pi}}{5\pi} \cos 2x + \frac{1 - e^{-2\pi}}{5\pi} 2 \sin 2x$$
$$= \frac{1}{10\pi}\left(1 - e^{-2\pi}\right)\left(5 + 5 \cos x + 5 \sin x + 2 \cos 2x + 4 \sin 2x\right).$$

84. The second order Fourier approximation of $f(x) = e^{-2x}$ is of the form

$$g(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \cos 2x.$$

In Exercise 73, you found that

$$a_0 = \frac{1 - e^{-4\pi}}{2\pi}$$
$$a_1 = 2\left(\frac{1 - e^{-4\pi}}{5\pi}\right)$$
$$b_1 = \frac{1 - e^{-4\pi}}{5\pi}.$$

So, you need to determine a_2 and b_2.

$$a_2 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos 2x \, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-2x} \cos 2x \, dx = \left[\frac{1}{4\pi} e^{-2x} \sin 2x - \frac{1}{4\pi} e^{-2x} \cos 2x\right]_0^{2\pi} = \frac{1 - e^{-4\pi}}{4\pi}$$
$$b_2 = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin 2x \, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-2x} \sin 2x \, dx = \left[-\frac{1}{4\pi} e^{-2x} \sin 2x - \frac{1}{4\pi} e^{-2x} \cos 2x\right]_0^{2\pi} = \frac{1 - e^{-4\pi}}{4\pi}$$

So, the approximation is

$$g(x) = \frac{1 - e^{-4\pi}}{4\pi} + 2\left(\frac{1 - e^{-4\pi}}{5\pi}\right) \cos x + \frac{1 - e^{-4\pi}}{5\pi} \sin x + \frac{1 - e^{-4\pi}}{4\pi} \cos 2x + \frac{1 - e^{-4\pi}}{4\pi} \sin 2x$$
$$= 5\left(\frac{1 - e^{-4\pi}}{20\pi}\right) + 8\left(\frac{1 - e^{-4\pi}}{20\pi}\right) \cos x + 4\left(\frac{1 - e^{-4\pi}}{20\pi}\right) \sin x + 5\left(\frac{1 - e^{-4\pi}}{20\pi}\right) \cos 2x + 5\left(\frac{1 - e^{-4\pi}}{20\pi}\right) \sin 2x$$
$$= \left(\frac{1 - e^{-4\pi}}{20\pi}\right)\left(5 + 8 \cos x + 4 \sin x + 5 \cos 2x + 5 \sin 2x\right).$$

86. The fourth order Fourier approximation of $f(x) = 1 + x$ is of the form

$$g(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + a_4 \cos 4x + b_4 \sin 4x.$$

In Exercise 71, you found that

$a_0 = 2 + 2\pi$

$a_j = 0, j = 1, 2, \ldots$

$b_j = \frac{-2}{j}, j = 1, 2, \ldots$

So, the approximation is $g(x) = (1 + \pi) - 2 \sin x - \sin 2x - \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x$.

88. Because $f(x) = \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$, you see that the fourth order Fourier approximation is simply $g(x) = \frac{1}{2} - \frac{1}{2} \cos 2x$.

90. Because

$$a_0 = \frac{2\pi^2}{3}, a_j = \frac{4}{j^2} (j = 1, 2, \ldots), b_j = 0 (j = 1, 2, \ldots),$$

the nth order Fourier approximation is

$$g(x) = \frac{\pi^2}{3} + 4 \cos x + \cos 2x + \frac{4}{9} \cos 3x + \frac{4}{16} \cos 4x + \cdots + \frac{4}{n^2} \cos nx.$$

92. (a) If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then the cross product of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

(b) For a continuous function f on $[a_1 b]$ and a finite-dimensional subspace W of $C[a_1 b]$, the least squares approximating function of f with respect to W is given by $g = \langle f, \mathbf{w}_1 \rangle + \langle f, \mathbf{w}_2 \rangle \mathbf{w}_2 + \cdots + \langle f, \mathbf{w}_n \rangle \mathbf{w}_n$, where $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for W.

(c) On the interval $[0, 2\pi]$, the least squares approximation of a continuous function f with respect to the vector space spanned by $\{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\}$ is $g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$, where the Fourier coefficients $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos jx \, dx, \ j = 1, 2, \ldots n$$

$$b_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin jx \, dx, \ j = 1, 2, \ldots n.$$

Review Exercises for Chapter 5

2. (a) $\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

(b) $\|\mathbf{v}\| = \sqrt{2^2 + 3^2} = \sqrt{13}$

(c) $\mathbf{u} \cdot \mathbf{v} = -1(2) + 2(3) = 4$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-3, -1)\| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

4. (a) $\|\mathbf{u}\| = \sqrt{(-3)^2 + 2^2 + (-2)^2} = \sqrt{17}$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$

(c) $\mathbf{u} \cdot \mathbf{v} = -3(1) + 2(3) + (-2)(5) = -7$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-4, -1, -7)\|$

$$= \sqrt{(-4)^2 + (-1)^2 + (-7)^2} = \sqrt{66}$$

6. (a) $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = \sqrt{9} = 3$

(b) $\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 0^2 + 2^2} = \sqrt{9} = 3$

(c) $\mathbf{u} \cdot \mathbf{v} = 1(2) + (-2)(-1) + 2(0) + (0)(2) = 4$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

$$= \|(-1, -1, 2, -2)\|$$

$$= \sqrt{(-1)^2 + (-1)^2 + 2^2 + (-2)^2} = \sqrt{10}$$

8. (a) $\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 0^2 + 1^2 + 1^2} = \sqrt{4} = 2$

(b) $\|\mathbf{v}\| = \sqrt{0^2 + 1^2 + (-2)^2 + 2^2 + 1^2} = \sqrt{10}$

(c) $\mathbf{u} \cdot \mathbf{v} = 1(0) + (-1)(1) + 0(-2) + 1(2) + 1(1) = 2$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

$$= \|(1, -2, 2, -1, 0)\|$$

$$= \sqrt{1^2 + (-2)^2 + 2^2 + (-1)^2} = \sqrt{10}$$

10. The norm of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + (-4)^2 + 1^2} = 3\sqrt{2}.$$

So, a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3\sqrt{2}}(-1, -4, 1)$$

$$= \left(-\frac{1}{3\sqrt{2}}, -\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right).$$

12. The norm of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{0^2 + 2^2 + (-1)^2} = \sqrt{5}.$$

So, a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{5}}(0, 2, -1) = \left(0, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right).$$

14. Solve the equation for c as follows.

$$\|c(2, 2, -1)\| = 3$$

$$|c|\|(2, 2, -1)\| = 3$$

$$|c|\sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$|c|3 = 3 \Rightarrow c = \pm 1$$

16. The cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1(0) + (-1)(1)}{\sqrt{1^2 + (-1)^2}\sqrt{0^2 + 1^2}} = \frac{-1}{\sqrt{2}\sqrt{1}} = \frac{-1}{\sqrt{2}}$$

which implies that $\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$ radians $(135°)$.

18. The cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\cos\frac{\pi}{6}\cos\frac{5\pi}{6} + \sin\frac{\pi}{6}\sin\frac{5\pi}{6}}{\sqrt{\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}}\sqrt{\cos^2\frac{5\pi}{6} + \sin^2\frac{5\pi}{6}}} = \frac{\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} = \frac{-\frac{1}{2}}{\sqrt{1} \cdot \sqrt{1}} = -\frac{1}{2}$$

which implies that $\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ radians $(120°)$.

20. The cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{4+1}{\sqrt{17}\sqrt{20}} = \frac{\sqrt{85}}{34}$$

which implies that $\theta = \cos^{-1}\left(\frac{\sqrt{85}}{34}\right) \approx 1.18$ radians $(67.7°)$.

22. A vector $\mathbf{v} = (v_1, v_2, v_3)$ that is orthogonal to $\mathbf{u}$ must satisfy the equation $\mathbf{u}\cdot\mathbf{v} = v_1 - 2v_2 + v_3 = 0$.

This equation has solutions of the form $\mathbf{v} = (2s - t, s, t)$, where s and t are any real numbers.

24. A vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ that is orthogonal to $\mathbf{u}$ must satisfy the equation $\mathbf{u}\cdot\mathbf{v} = 0v_1 + v_2 + 2v_3 - v_4 = 0$.

This equation has solutions of the form $\mathbf{v} = \left(r, s, \frac{1}{2}t - \frac{1}{2}s, t\right)$, where r, s, and t are any real numbers.

26. (a) $\langle\mathbf{u}, \mathbf{v}\rangle = 2(0)\left(\frac{4}{3}\right) + (3)(1) + 2\left(\frac{1}{3}\right)(-3) = 1$

(b) $$\begin{aligned} d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \sqrt{\langle\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle} \\ &= \sqrt{2\left(-\frac{4}{3}\right)^2 + 2^2 + 2\left(\frac{10}{3}\right)^2} \\ &= \frac{\sqrt{268}}{3} = \frac{2}{3}\sqrt{67} \end{aligned}$$

28. Verify the Triangle Inequality as follows.

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\| \\ \left\|\left(\frac{4}{3}, 4, -\frac{8}{3}\right)\right\| &\le \sqrt{9 + 2\left(\frac{1}{9}\right)} + \sqrt{2\left(\frac{16}{9}\right) + 1 + 18} \\ \sqrt{2\left(\frac{4}{3}\right)^2 + 4^2 + 2\left(-\frac{8}{3}\right)^2} &\le 3.037 + 4.749 \\ 5.812 &\le 7.786 \end{aligned}$$

Verify the Cauchy-Schwarz Inequality as follows.

$$\begin{aligned} |\langle\mathbf{u}, \mathbf{v}\rangle| &\le \|\mathbf{u}\|\|\mathbf{v}\| \\ \left|(3)(1) + 2\left(\frac{1}{3}\right)(-3)\right| &\le (3.037)(4.749) \\ 1 &\le 14.423 \end{aligned}$$

30. (a) $\langle f, g\rangle = \int_0^1 x\,4x^2\,dx = x^4\Big]_0^1 = 1$

(b) The vectors are not orthogonal.

(c) Because $\|f\| = \sqrt{\frac{1}{3}}$ and $\|g\| = \frac{4}{\sqrt{5}}$, verify the Cauchy-Schwarz Inequality as follows

$$\begin{aligned} |\langle f, g\rangle| &\le \|f\|\|g\| \\ 1 &\le \sqrt{\frac{1}{3}}\left(\frac{4}{\sqrt{5}}\right) \approx 1.0328. \end{aligned}$$

32. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\begin{aligned} \text{proj}_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \\ &= \frac{2(0) + 3(4)}{0^2 + 4^2}(0, 4) \\ &= \frac{12}{16}(0, 4) \\ &= (0, 3). \end{aligned}$$

34. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\begin{aligned} \text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \\ &= \frac{2(7) + (-1)(6)}{7^2 + 6^2}(7, 6) \\ &= \frac{8}{85}(7, 6) \\ &= \left(\frac{56}{85}, \frac{48}{85}\right). \end{aligned}$$

36. The projection of **u** onto **v** is given by

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} = \mathbf{v}$$
$$= \frac{(-1)(4) + 3(0) + 1(5)}{4^2 + 0^2 + 5^2}(4, 0, 5)$$
$$= \frac{1}{41}(4, 0, 5)$$
$$= \left(\frac{4}{41}, 0, \frac{5}{41}\right).$$

38. Orthogonalize the vectors in B.

$$\mathbf{w}_1 = (3, 4)$$
$$\mathbf{w}_2 = (1, 2) - \frac{11}{25}(3, 4) = \left(-\frac{8}{25}, \frac{6}{25}\right)$$

Then normalize each vector.

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{5}(3, 4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$
$$\mathbf{u}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 = \frac{1}{2/5}\left(-\frac{8}{25}, \frac{6}{25}\right) = \left(-\frac{4}{5}, \frac{3}{5}\right)$$

So, an orthonormal basis for R^2 is $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(-\frac{4}{5}, \frac{3}{5}\right)\right\}$.

40. Orthogonalize the vectors in B.

$$\mathbf{w}_1 = (0, 0, 2)$$
$$\mathbf{w}_2 = (0, 1, 1) - \tfrac{2}{4}(0, 0, 2) = (0, 1, 0)$$
$$\mathbf{w}_3 = (1, 1, 1) - \tfrac{2}{4}(0, 0, 2) - \tfrac{1}{1}(0, 1, 0) = (1, 0, 0)$$

Then normalize each vector to obtain the orthonormal basis for R^3.

$\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

42. (a) To find $\mathbf{x} = (-3, 4, 4)$ as a linear combination of the vectors in $B = \{(-1, 2, 2), (1, 0, 0)\}$ solve the vector equation

$$c_1(-1, 2, 2) + c_2(1, 0, 0) = (-3, 4, 4).$$

The solution to the corresponding system of equations is $c_1 = 2$ and $c_2 = -1$.

So, $[\mathbf{x}]_B = (2, -1)$, and you can write

$$(-3, 4, 4) = 2(-1, 2, 2) - (1, 0, 0).$$

(b) To apply the Gram-Schmidt orthonormalization process, first orthogonalize each vector in B.

$$\mathbf{w}_1 = (-1, 2, 2)$$
$$\mathbf{w}_2 = (1, 0, 0) - \frac{-1}{9}(-1, 2, 2) = \left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right)$$

Then normalize $\mathbf{w}_1$ and $\mathbf{w}_2$ as follows

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{3}(-1, 2, 2) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$
$$\mathbf{u}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 = \frac{1}{2\sqrt{2}/3}\left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right) = \left(\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right).$$

So, $B' = \left\{\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)\right\}$.

(c) The coordinates of **x** relative to B' are found by calculating

$$\langle \mathbf{x}, \mathbf{u}_1\rangle = (-3, 4, 4)\cdot\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{19}{3}$$
$$\langle \mathbf{x}, \mathbf{u}_2\rangle = (-3, 4, 4)\cdot\left(\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right) = \frac{-4}{3\sqrt{2}}.$$

So,

$$(-3, 4, 4) = \frac{19}{3}\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) - \frac{4}{3\sqrt{2}}\left(\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right).$$

44. These functions are orthogonal because $\langle f, g\rangle = \int_{-1}^{1} \sqrt{1-x^2}\,2x\sqrt{1-x^2}\,dx = \int_{-1}^{1} (2x - 2x^3)dx = \left[x^2 - \frac{x^4}{2}\right]_{-1}^{1} = 0.$

46. (a) $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 (x+2)(15x-8)\,dx = \int_0^1 (15x^2 + 22x - 16)dx = \left[5x^3 + 11x^2 - 16x\right]_0^1 = 0$

(b) $\langle -4f, g\rangle = -4\langle f, g\rangle = -4(0) = 0$

(c) $\|f\|^2 = \langle f, f\rangle = \int_0^1 (x+2)^2 dx = \int_0^1 (x^2 + 4x + 4)dx = \left[\frac{x^3}{3} + 2x^2 + 4x\right]_0^1 = \frac{19}{3}$

$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{19}{3}}$

(d) Because f and g are already orthogonal, you only need to normalize them. You know $\|f\| = \sqrt{\frac{19}{3}}$ and so you compute $\|g\|$.

$\|g\|^2 = \langle g, g\rangle = \int_0^1 (15x-8)^2 dx = \int_0^1 (225x^2 - 240x + 64)dx = \left[75x^3 - 120x^2 + 64x\right]_0^1 = 19$

$\|g\| = \sqrt{19}$

So,

$$\mathbf{u}_1 = \frac{1}{\|f\|}f = \frac{1}{\sqrt{\frac{19}{3}}}(x+2) = \sqrt{\frac{3}{19}}(x+2)$$

$$\mathbf{u}_2 = \frac{1}{\|g\|}g = \frac{1}{\sqrt{19}}(15x-8).$$

The orthonormal set is

$$B' = \left\{\left(\sqrt{\frac{3}{19}}x + 2\sqrt{\frac{3}{19}}\right), \left(\frac{15}{\sqrt{19}}x - \frac{8}{\sqrt{19}}\right)\right\}.$$

48. The solution space of the homogeneous system consists of vectors of the form $(-t, s, s, t)$, where s and t are any real numbers. So, a basis for the solution space is $B = \{(-1, 0, 0, 1), (0, 1, 1, 0)\}$. Because these vectors are orthogonal, and their length is $\sqrt{2}$, you normalize them to obtain the orthonormal basis

$$\left\{\left(-\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right), \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)\right\}.$$

50.

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v}) \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 \end{aligned}$$

52. Use the Triangle Inequality

$\|\mathbf{u}+\mathbf{w}\| \le \|\mathbf{u}\| + \|\mathbf{w}\|$ with $\mathbf{w} = \mathbf{v} - \mathbf{u}$

$\|\mathbf{u}+\mathbf{w}\| = \|\mathbf{u} + (\mathbf{v}-\mathbf{u})\| = \|\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}-\mathbf{u}\|$

and so, $\|\mathbf{v}\| - \|\mathbf{u}\| \le \|\mathbf{v}-\mathbf{u}\|$. By symmetry, you also have $\|\mathbf{u}\| - \|\mathbf{v}\| \le \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\|$.

So, $\big|\|\mathbf{u}\| - \|\mathbf{v}\|\big| \le \|\mathbf{u}-\mathbf{v}\|$. To complete the proof, first observe that the Triangle Inequality implies that

$\|\mathbf{u}-\mathbf{w}\| \le \|\mathbf{u}\| + \|-\mathbf{w}\| = \|\mathbf{u}\| + \|\mathbf{w}\|$. Letting $\mathbf{w} = \mathbf{u}+\mathbf{v}$, you have

$\|\mathbf{u}-\mathbf{w}\| = \|\mathbf{u} - (\mathbf{u}+\mathbf{v})\| = \|-\mathbf{v}\| = \|\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{u}+\mathbf{v}\|$ and so $\|\mathbf{v}\| - \|\mathbf{u}\| \le \|\mathbf{u}+\mathbf{v}\|$.

Similarly, $\|\mathbf{u}\| - \|\mathbf{v}\| \le \|\mathbf{u}+\mathbf{v}\|$, and $\big|\|\mathbf{u}\| - \|\mathbf{v}\|\big| \le \|\mathbf{u}+\mathbf{v}\|$. In conclusion, $\big|\|\mathbf{u}\| - \|\mathbf{v}\|\big| \le \|\mathbf{u} \pm \mathbf{v}\|$.

54. Extend the V-basis $\{(0, 1, 0, 1), (0, 2, 0, 0)\}$ to a basis of R^4.

$B = \{(0, 1, 0, 1), (0, 2, 0, 0), (1, 0, 0, 0), (0, 0, 1, 0)\}$

Now, $(1, 1, 1, 1) = (0, 1, 0, 1) + (1, 0, 1, 0) = \mathbf{v} + \mathbf{w}$

where $\mathbf{v} \in V$ and $\mathbf{w}$ is orthogonal to every vector in V.

56.
$$\begin{aligned}(x_1 + x_2 + \cdots + x_n)^2 &= (x_1 + x_2 + \cdots + x_n)(x_1 + x_2 + \cdots + x_n)\\ &= (x_1, \ldots, x_n) \cdot (x_1, \ldots, x_n) + (x_2, \ldots, x_n, x_1) \cdot (x_1, \ldots, x_n)\\ &\quad + \cdots + (x_n, x_1, \ldots, x_{n-1}) \cdot (x_1, \ldots, x_n)\\ &\le (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}\\ &\quad + (x_2^2 + \cdots + x_n^2 + x_1^2)^{\frac{1}{2}}(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} + \cdots\\ &\quad + (x_n^2 + x_1^2 + \cdots + x_{n-1}^2)^{\frac{1}{2}}(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}\\ &= n(x_1^2 + \cdots + x_n^2)\end{aligned}$$

58. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a dependent set of vectors, and assume $\mathbf{u}_k$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$, which are linearly independent. The Gram-Schmidt process will orthonormalize $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$, but then $\mathbf{u}_k$ will be a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$.

60. An orthonormal basis for S is

$$\left\{\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\right\}$$

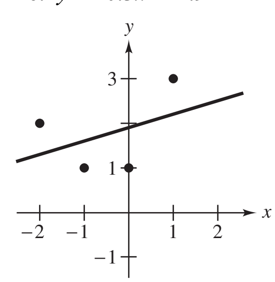
$$\begin{aligned}\operatorname{proj}_{\mathbf{s}}\mathbf{v} &= (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2\\ &= \left(-\frac{2}{\sqrt{2}}\right)\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left(-\frac{2}{\sqrt{2}}\right)\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.\end{aligned}$$

62. $$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$A^TA\mathbf{x} = A^T\mathbf{b} \Rightarrow \mathbf{x} = \begin{bmatrix} 1.9 \\ 0.3 \end{bmatrix}$$

line: $y = 0.3x + 1.9$



64. Substitute the data points $(6, 15.3), (7, 15.4), (8, 15.1), (9, 15.4), (10, 16.1),$ $(11, 16.7), (12, 17.9),$ and $(13, 19.3)$ into the linear polynomial $y = c_0 + c_1 t$. You obtain the system of linear equations

$$\begin{aligned} c_0 + 6c_1 &= 15.3 \\ c_0 + 7c_1 &= 15.4 \\ c_0 + 8c_1 &= 15.1 \\ c_0 + 9c_1 &= 15.4 \\ c_0 + 10c_1 &= 16.1 \\ c_0 + 11c_1 &= 16.7 \\ c_0 + 12c_1 &= 17.9 \\ c_0 + 13c_1 &= 19.3. \end{aligned}$$

This produces the least squares problem

$$A\mathbf{t} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.3 \\ 15.4 \\ 15.1 \\ 15.4 \\ 16.1 \\ 16.7 \\ 17.9 \\ 19.3 \end{bmatrix}.$$

The normal equations are

$$A^T A\mathbf{t} = A^T\mathbf{b}$$

$$\begin{bmatrix} 8 & 76 \\ 76 & 764 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 131.2 \\ 1269.4 \end{bmatrix}$$

and the solution is

$$\mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 11.2 \\ 0.55 \end{bmatrix}.$$

So, the least squares linear equation is $y = 11.2 + 0.55t$.

Substitute the same data points into the quadratic polynomial $y = c_0 + c_1 t + c_2 t^2$. You then obtain the system of linear equations

$$\begin{aligned} c_0 + 6c_1 + 36c_2 &= 15.3 \\ c_0 + 7c_1 + 49c_2 &= 15.4 \\ c_0 + 8c_1 + 64c_2 &= 15.1 \\ c_0 + 9c_1 + 81c_2 &= 15.4 \\ c_0 + 10c_1 + 100c_2 &= 16.1 \\ c_0 + 11c_1 + 121c_2 &= 16.7 \\ c_0 + 12c_1 + 144c_2 &= 17.9 \\ c_0 + 13c_1 + 169c_2 &= 19.3. \end{aligned}$$

This produces the least squares problem

$$A\mathbf{t} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 6 & 36 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \\ 1 & 9 & 81 \\ 1 & 10 & 100 \\ 1 & 11 & 121 \\ 1 & 12 & 144 \\ 1 & 13 & 169 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15.3 \\ 15.4 \\ 15.1 \\ 15.4 \\ 16.1 \\ 16.7 \\ 17.9 \\ 19.3 \end{bmatrix}.$$

The normal equations are

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 8 & 76 & 764 \\ 76 & 764 & 8056 \\ 764 & 8056 & 88{,}292 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 131.2 \\ 1269.4 \\ 12{,}989.2 \end{bmatrix}$$

and the solution is

$$\mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 22.6 \\ -2.01 \\ 0.135 \end{bmatrix}.$$

The least squares regression quadratic is $y = 22.6 - 2.01t + 0.135t^2$.

2018 (linear):

$y = 11.2 + 0.55(18) \approx 21.1$ million

2018 (quadratic):

$y = 22.6 - 2.01(18) + 0.135(18)^2 \approx 30.2$ million

Because the original data increased from 2006 to 2013, you expect the production to continue to increase. Because the predicted value given by the quadratic polynomial is greater than the actual value for 2013, this model is more accurate for predicting future petroleum productions.

66. The cross product is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} + \mathbf{k} = (-2, -1, 1).$$

Furthermore, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ because

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + (1)(-1) + 1(1) = 0$$

and

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0(-2) + 1(-1) + 1(1) = 0.$$

68. The cross product is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} = (1, 1, 2).$$

Furthermore, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ because

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 2(1) + 0(1) + (-1)(2) = 0$$

and

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1(1) + 1(1) + (-1)(2) = 0.$$

70. Because

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 0 \\ 3 & 4 & -1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k} = (1, -1, -1),$$

the volume is

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |(1, 2, 1) \cdot (1, -1, -1)| = |-2| = 2.$$

72. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 3 & 3 \\ 3 & 0 & 3 \end{vmatrix} = 1(9) + 3(-9) = -9$$

$$\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |-9| = 9 \text{ cubic units}$$

74. Because $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$, you see that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\sin\theta = 1$, which means $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\|$.

76. (a) The standard basis for P_1 is $\{1, x\}$. In the interval $[0, 2]$, the Gram-Schmidt orthonormalization process yields the orthonormal basis $\left\{\dfrac{1}{\sqrt{2}}, \dfrac{\sqrt{3}}{\sqrt{2}}(x - 1)\right\}$.

Because

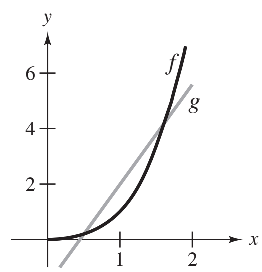
$$\langle f, \mathbf{w}_1 \rangle = \int_0^2 x^3 \frac{1}{\sqrt{2}} dx = \frac{4}{\sqrt{2}}$$

$$\begin{aligned}\langle f, \mathbf{w}_2 \rangle &= \int_0^2 x^3 \frac{\sqrt{3}}{\sqrt{2}}(x - 1)\, dx \\ &= \frac{\sqrt{3}}{\sqrt{2}} \int_0^2 (x^4 - x^3)\, dx \\ &= \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{x^5}{5} - \frac{x^4}{5}\right)\Bigg]_0^2 \\ &= \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{32}{5} - 4\right) \\ &= \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{12}{5}\right),\end{aligned}$$

g is given by

$$\begin{aligned}g(x) &= \langle f, \mathbf{w}_1 \rangle + \langle f, \mathbf{w}_2 \rangle \mathbf{w}_2 \\ &= \frac{4}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{12}{5}\right)\frac{\sqrt{3}}{\sqrt{2}}(x - 1) \\ &= \frac{18}{5}x - \frac{8}{5}.\end{aligned}$$

(b)



78. (a) The standard basis for P_1 is $\{1, x\}$. In the interval $[0, \pi]$ the Gram-Schmidt orthonormalization process yields the orthonormal basis $\left\{\frac{1}{\sqrt{\pi}}, \frac{\sqrt{3}}{\pi^{3/2}}(2x - \pi)\right\}$.

Because

$$\langle f, \mathbf{w}_1\rangle = \int_0^\pi \sin x \cos x\left(\frac{1}{\sqrt{\pi}}\right) dx = 0$$

$$\langle f, \mathbf{w}_2\rangle = \int_0^\pi \sin x \cos x\left(\frac{\sqrt{3}}{\pi^{3/2}}\right)(2x - \pi)\ dx$$

$$= -\frac{\sqrt{3}}{2\pi^{1/2}},$$

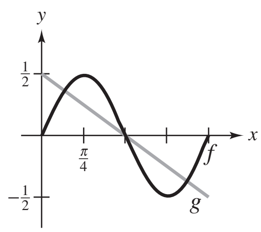
g is given by

$$g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2$$

$$= 0\left(\frac{1}{\sqrt{\pi}}\right) + \left(-\frac{\sqrt{3}}{2\pi^{1/2}}\right)\left(\frac{\sqrt{3}}{\pi^{3/2}}(2x - \pi)\right)$$

$$= -\frac{3x}{\pi^2} + \frac{3}{2\pi}.$$

(b)



80. (a) The standard basis for P_2 is $\{1, x, x^2\}$. In the interval $[1, 2]$, the Gram-Schmidt orthonormalization process yields the orthonormal basis $\left\{1, 2\sqrt{3}\left(x - \frac{3}{2}\right), \frac{30}{\sqrt{5}}\left(x^2 - 3x + \frac{13}{6}\right)\right\}$.

Because

$$\langle f, \mathbf{w}_1\rangle = \int_0^2 \frac{1}{x}\, dx = \ln 2$$

$$\langle f, \mathbf{w}_2\rangle = \int_1^2 \frac{1}{x}2\sqrt{3}\left(x - \frac{3}{2}\right)dx = 2\sqrt{3}\int_1^2\left(1 - \frac{3}{2x}\right)dx = 2\sqrt{3}\left(1 - \frac{3}{2}\ln 2\right)$$

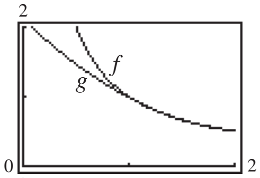
$$\langle f, \mathbf{w}_3\rangle = \int_1^2 \frac{1}{x}\frac{30}{\sqrt{5}}\left(x^2 - 3x + \frac{13}{6}\right)dx = \frac{30}{\sqrt{5}}\int_1^2\left(x - 3 + \frac{13}{6x}\right)dx = \frac{30}{\sqrt{5}}\left(\frac{13}{6}\ln 2 - \frac{3}{2}\right),$$

g is given by $g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \langle f, \mathbf{w}_3\rangle\mathbf{w}_3$

$$= (\ln 2) + 2\sqrt{3}\left(1 - \frac{3}{2}\ln 2\right)2\sqrt{3}\left(x - \frac{3}{2}\right) + \frac{30}{\sqrt{5}}\left(\frac{13}{6}\ln 2 - \frac{3}{2}\right)\frac{30}{\sqrt{5}}\left(x^2 - 3x + \frac{13}{6}\right)$$

$$= \ln 2 + 12\left(1 - \frac{3}{2}\ln 2\right)\left(x - \frac{3}{2}\right) + 180\left(\frac{13}{6}\ln 2 - \frac{3}{2}\right)\left(x^2 - 3x + \frac{13}{6}\right) = .3274x^2 - 1.459x + 2.1175.$$

(b)



82. Find the coefficients as follows

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(jx)dx = \frac{1}{\pi}\left[\frac{1}{j^2}\cos(jx) + \frac{x}{j}\sin(jx)\right]_{-\pi}^{\pi} = 0,\ j = 1, 2 \ldots$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(jx) = \frac{1}{\pi}\left[\frac{1}{j^2}\sin(jx) - \frac{x}{j}\cos(jx)\right]_{-\pi}^{\pi} = -\frac{2}{j}\cos(\pi j)\ j = 1, 2, \ldots$$

So, the approximation is $g(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x = 2\sin x - \sin 2x$.

84. (a) True. See note following Theorem 5.17, page 278.

(b) True. See Theorem 5.18, part 3, page 279.

(c) True. See discussion starting on page 285.

Project Solutions for Chapter 5

1 The *QR*-factorization

1. (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} .7071 & .4082 \\ 0 & .8165 \\ .7071 & -.4082 \end{bmatrix}\begin{bmatrix} 1.4142 & 0.7071 \\ 0 & 1.2247 \end{bmatrix} = QR$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} .5774 & -.7071 \\ 0 & 0 \\ .5774 & 0 \\ .5774 & .7071 \end{bmatrix}\begin{bmatrix} 1.7321 & 1.7321 \\ 0 & 1.4142 \end{bmatrix} = QR$

(c) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} .5 & -.5 & -.7071 \\ .5 & .5 & 0 \\ .5 & .5 & 0 \\ .5 & -.5 & .7071 \end{bmatrix}\begin{bmatrix} 2 & 2 & -.5 \\ 0 & 2 & .5 \\ 0 & 0 & .7071 \end{bmatrix} = QR$

2. The normal equations simplify using $A = QR$ as follows

$$\begin{aligned} A^T A\mathbf{x} &= A^T\mathbf{b} \\ (QR)^T QR\mathbf{x} &= (QR)^T\mathbf{b} \\ R^T Q^T QR\mathbf{x} &= R^T Q^T\mathbf{b} \\ R^T R\mathbf{x} &= R^T Q^T\mathbf{b} \quad (Q^T Q = I) \\ R\mathbf{x} &= Q^T\mathbf{b}. \end{aligned}$$

Because R is upper triangular, only back-substitution is needed.

3. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} .7071 & .4082 \\ 0 & .8165 \\ .7071 & -.4082 \end{bmatrix} \begin{bmatrix} 1.4142 & 0.7071 \\ 0 & 1.2247 \end{bmatrix} = QR.$

$$R\mathbf{x} = Q^T\mathbf{b} \begin{bmatrix} 1.4142 & 0.7071 \\ 0 & 1.2247 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} .7071 & 0 & .7071 \\ .4082 & .8165 & -.4082 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.4142 \\ 0.8165 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1.3333 \\ 0.6667 \end{bmatrix}$$

2 Orthogonal Matrices and Change of Basis

1. $P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \neq P^T$

2. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$

3. If $P^{-1} = P^T$, then $P^TP = I \Rightarrow$ columns of P are pairwise orthogonal.

4. If P is orthogonal, then $P^{-1} = P^T$ by definition of orthogonal matrix. Then $\left(P^{-1}\right)^{-1} = \left(P^T\right)^{-1} = \left(P^{-1}\right)^T$. The last equality holds because $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ for any invertible matrix A. So, P^{-1} is orthogonal.

5. No. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not orthogonal. The product of orthogonal matrices is orthogonal. If $P^{-1} = P^T$ and $Q^{-1} = Q^T$, then $(PQ)^{-1} = Q^{-1}P^{-1} = Q^TP^T = (PQ)^T$.

6. $\|P\mathbf{x}\| = (P\mathbf{x})^T P\mathbf{x} = \mathbf{x}^T P^T P\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|$

7. Let

$$P = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

be the change of basis matrix from B' to B. Because P is orthogonal, lengths are preserved.