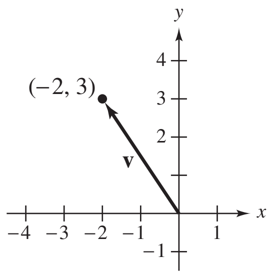
CHAPTER 4
Vector Spaces

C H A P T E R 4
Vector Spaces

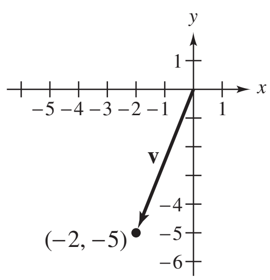
Section 4.1 Vectors in R^n

2. $\mathbf{v} = (-6, 3)$

4.



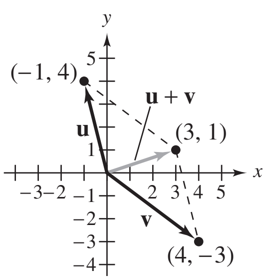
6.



8. $\mathbf{u} + \mathbf{v} = (-1, 4) + (4, -3)$

$= (-1 + 4, 4 - 3)$

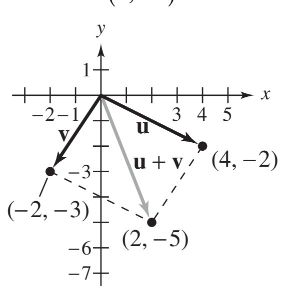
$= (3, 1)$



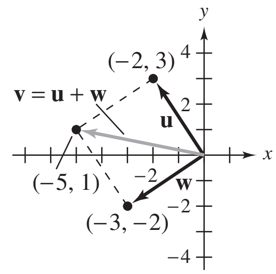
10. $\mathbf{u} + \mathbf{v} = (4, -2) + (-2, -3)$

$= (4 - 2, -2 - 3)$

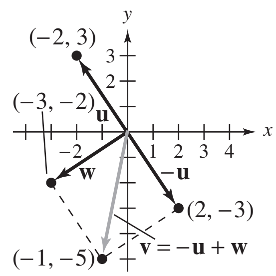
$= (2, -5)$



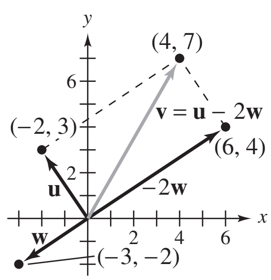
12. $\mathbf{v} = \mathbf{u} + \mathbf{w} = (-2, 3) + (-3, -2) = (-5, 1)$



14. $\mathbf{v} = -\mathbf{u} + \mathbf{w} = -(-2, 3) + (-3, -2) = (-1, -5)$



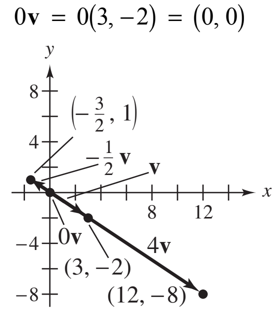
16. $\mathbf{v} = \mathbf{u} - 2\mathbf{w} = (-2, 3) - 2(-3, -2) = (4, 7)$



18. (a) $4\mathbf{v} = 4(3, -2) = (12, -8)$

(b) $-\frac{1}{2}\mathbf{v} = -\frac{1}{2}(3, -2) = \left(-\frac{3}{2}, 1\right)$

(c) $0\mathbf{v} = 0(3, -2) = (0, 0)$



20. $\mathbf{u} - \mathbf{v} + 2\mathbf{w} = (1, 2, 3) - (2, 2, -1) + 2(4, 0, -4)$
$= (-1, 0, 4) + (8, 0, -8) = (7, 0, -4)$

22. $5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w} = 5(1, 2, 3) - 3(2, 2, -1) - \frac{1}{2}(4, 0, -4)$
$= (5, 10, 15) - (6, 6, -3) - (2, 0, -2)$
$= (-3, 4, 20)$

24. $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$ implies that
$3\mathbf{z} = -2\mathbf{u} - \mathbf{v} + \mathbf{w}.$

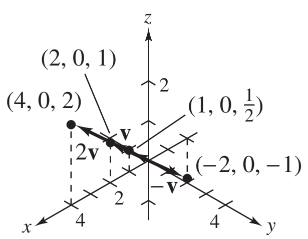
So,

$3\mathbf{z} = -2(1, 2, 3) - (2, 2, -1) + (4, 0, -4)$
$= (-2, -4, -6) - (2, 2, -1) + (4, 0, -4) = (0, -6, -9).$
$\mathbf{z} = \frac{1}{3}(0, -6, -9) = (0, -2, -3).$

26. (a) $-\mathbf{v} = -(2, 0, 1) = (-2, 0, -1)$

(b) $2\mathbf{v} = 2(2, 0, 1) = (4, 0, 2)$

(c) $\frac{1}{2}\mathbf{v} = \frac{1}{2}(2, 0, 1) = \left(1, 0, \frac{1}{2}\right)$



28. (a) Because $(6, -4, 9) \neq c\left(\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}\right)$ for any c, $\mathbf{u}$ is *not* a scalar multiple of $\mathbf{z}$.

(b) Because $\left(-1, \frac{4}{3}, -\frac{3}{2}\right) = -2\left(\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}\right)$, $\mathbf{v}$ is a scalar multiple of $\mathbf{z}$.

30. (a) $\mathbf{u} - \mathbf{v} = (0, 4, 3, 4, 4) - (6, 8, -3, 3, -5)$
$= (-6, -4, 6, 1, 9)$

(b) $2(\mathbf{u} + 3\mathbf{v}) = 2[(0, 4, 3, 4, 4) + 3(6, 8, -3, 3, -5)]$
$= 2[(0, 4, 3, 4, 4) + (18, 24, -9, 9, -15)]$
$= 2(18, 28, -6, 13, -11)$
$= (36, 56, -12, 26, -22)$

(c) $2\mathbf{v} - \mathbf{u} = 2(6, 8, -3, 3, -5) - (0, 4, 3, 4, 4)$
$= (12, 16, -6, 6, -10) - (0, 4, 3, 4, 4)$
$= (12, 12, -9, 2, -14)$

32. (a) $\mathbf{u} - \mathbf{v} = (6, -5, 4, 3) - \left(-2, \frac{5}{3}, -\frac{4}{3}, -1\right)$
$= \left(6 + 2, -5 - \frac{5}{3}, 4 + \frac{4}{3}, 3 + 1\right)$
$= \left(8, -\frac{20}{3}, \frac{16}{3}, 4\right)$

(b) $2(\mathbf{u} + 3\mathbf{v}) = 2\left[(6, -5, 4, 3) + 3\left(-2, \frac{5}{3}, -\frac{4}{3}, -1\right)\right]$
$= 2[(6, -5, 4, 3) + (-6, 5, -4, -3)]$
$= 2(6 - 6, -5 + 5, 4 - 4, 3 - 3)$
$= 2(0, 0, 0, 0)$
$= (0, 0, 0, 0)$

(c) $2\mathbf{v} - \mathbf{u} = 2\left(-2, \frac{5}{3}, -\frac{4}{3}, -1\right) - (6, -5, 4, 3)$
$= \left(-4, \frac{10}{3}, -\frac{8}{3}, -2\right) - (6, -5, 4, 3)$
$= \left(-10, \frac{25}{3}, -\frac{20}{3}, -5\right)$

34. Using a graphing utility with
$\mathbf{u} = (1, 2, -3, 1)$, $\mathbf{v} = (0, 2, -1, -2)$, and
$\mathbf{w} = (2, -2, 1, 3)$, you have the following.

(a) $\mathbf{v} + 3\mathbf{w} = (6, -4, 2, 7)$

(b) $2\mathbf{w} - \frac{1}{2}\mathbf{u} = \left(\frac{7}{2}, -5, \frac{7}{2}, \frac{11}{2}\right)$

(c) $\frac{1}{2}(4\mathbf{v} - 3\mathbf{u} + \mathbf{w}) = \left(-\frac{1}{2}, 0, 3, -4\right)$

36. $\mathbf{w} + \mathbf{u} = -\mathbf{v}$
$\mathbf{w} = -\mathbf{v} - \mathbf{u}$
$= -(0, 2, 3, -1) - (1, -1, 0, 1)$
$= (-1, -1, -3, 0)$

38. $\mathbf{w} + 3\mathbf{v} = -2\mathbf{u}$
$\mathbf{w} = -2\mathbf{u} - 3\mathbf{v}$
$= -2(1, -1, 0, 1) - 3(0, 2, 3, -1)$
$= (-2, 2, 0, -2) - (0, 6, 9, -3)$
$= (-2, -4, -9, 1)$

40. $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = 0$
$\mathbf{w} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$
$= \frac{2}{3}(-6, 0, 2, 0) + \frac{1}{3}(5, -3, 0, 1)$
$= \left(-4, 0, \frac{4}{3}, 0\right) + \left(\frac{5}{3}, -1, 0, \frac{1}{3}\right)$
$= \left(-\frac{7}{3}, -1, \frac{4}{3}, \frac{1}{3}\right)$

42. The equation

$$a\mathbf{u} + b\mathbf{w} = \mathbf{v}$$

$$a(1, 2) + b(1, -1) = (0, 3)$$

yields the system

$$a + b = 0$$

$$2a - b = 3.$$

Solving this system produces $a = 1$ and $b = -1$.
So, $\mathbf{v} = \mathbf{u} - \mathbf{w}$.

44. The equation

$$a\mathbf{u} + b\mathbf{w} = \mathbf{v}$$

$$a(1, 2) + b(1, -1) = (1, -1)$$

yields the system

$$a + b = 1$$

$$2a - b = -1.$$

Solving this system produces $a = 0$ and $b = 1$.
So, $\mathbf{v} = \mathbf{w} = 0\mathbf{u} + 1\mathbf{w}$.

46. The equation

$$a\mathbf{u} + b\mathbf{w} = \mathbf{v}$$

$$a(1, 2) + b(1, -1) = (1, -4)$$

yields the system

$$a + b = 1$$

$$2a - b = -4.$$

Solving this system produces $a = -1$ and $b = 2$.
So, $\mathbf{v} = -\mathbf{u} + 2\mathbf{w}$.

48. The equation

$$a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 = \mathbf{v}$$

$$a(1, 3, 5) + b(2, -1, 3) + c(-3, 2, -4) = (-1, 7, 2)$$

yields the system

$$a + 2b - 3c = -1$$

$$3a - b + 2c = 7$$

$$5a + 3b - 4c = 2.$$

Solving this system you discover that there is no solution. So, $\mathbf{v}$ cannot be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$.

50. The equation

$$a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 = \mathbf{v}$$

$$a(2, 1, 1, 2) + b(-3, 3, 4, -5) + c(-6, 3, 1, 2) = (7, 2, 5, -3)$$

yields the system

$$2a - 3b - 6c = 7$$

$$a + 3b + 3c = 2$$

$$a + 4b + c = 5$$

$$2a - 5b + 2c = -3.$$

Solving this system produces $a = 2, b = 1,$ and $c = -1$.

So, $\mathbf{v} = 2\mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3$.

52. The equation

$$a\begin{bmatrix}1\\7\\4\end{bmatrix} + b\begin{bmatrix}2\\8\\5\end{bmatrix} = \begin{bmatrix}3\\9\\7\end{bmatrix}$$

yields the system

$$a + 2b = 3$$

$$7a + 8b = 9$$

$$4a + 5b = 7.$$

Because the system has no solution, it is not possible to write the third column as a linear combination of the first two columns.

54. Write a matrix using the given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_5$ as columns and augment this matrix with $\mathbf{v}$ as a column.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 5 \\ 1 & 1 & 2 & 2 & 1 & 8 \\ -1 & 2 & 0 & 0 & 2 & 7 \\ 2 & -1 & 1 & 1 & -1 & -2 \\ 1 & 1 & 2 & -4 & 2 & 4 \end{bmatrix}$$

The reduced row-echelon form for A is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

So, $\mathbf{v} = -\mathbf{u}_1 + \mathbf{u}_2 + 2\mathbf{u}_3 + \mathbf{u}_4 + 2\mathbf{u}_5$.

Verify the solution by showing that

$$-(1, 1, -1, 2, 1) + (2, 1, 2, -1, 1) + 2(1, 2, 0, 1, 2) + (0, 2, 0, 1, -4) + 2(1, 1, 2, -1, 2) = (5, 8, 7, -2, 4).$$

56. The equation

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$$

$$a(1, 0, 1) + b(-1, 1, 2) + c(0, 1, 3) = (0, 0, 0)$$

yields the homogeneous system

$$\begin{aligned} a - b &= 0 \\ b + c &= 0 \\ a + 2b + 3c &= 0. \end{aligned}$$

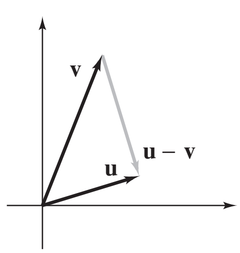
Solving this system produces $a = -t, b = -t$, and $c = t$, where t is any real number.

Letting $t = -1$, you obtain $a = 1, b = 1, c = -1$, and so, $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

58. (a) True. See page 155.

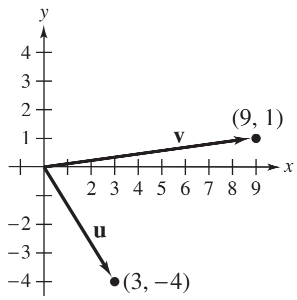
(b) False. The zero vector is the additive identity.

60. You can describe vector subtraction $\mathbf{u} - \mathbf{v}$ as follows.



Or, write subtraction in terms of addition, $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}.$

62. (a)



(b) $\mathbf{u} + \mathbf{v} = (3, -4) + (9, 1) = (12, -3)$

(c) $2\mathbf{v} - \mathbf{u} = 2(9, 1) - (3, -4) = (18, 2) - (3, -4) = (15, 6)$

(d) The equation

$$a\mathbf{u} + b\mathbf{v} = \mathbf{w}$$
$$a(3, -4) + b(9, 1) = (39, 0)$$

yields the system

$$3a + 9b = 39$$
$$-4a + b = 0.$$

Solving this system produces $a = 1$ and $b = 4$. So, $\mathbf{w} = \mathbf{u} + 4\mathbf{v}$.

64. Prove each of the ten properties.

(1) $\mathbf{u} + \mathbf{v} = (u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$ is a vector in R^n.

(2) $$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)\\ &= (v_1 + u_1, \ldots, v_n + u_n)\\ &= (v_1, \ldots, v_n) + (u_1, \ldots, u_n) = \mathbf{v} + \mathbf{u}\end{aligned}$$

(3) $$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \left[(u_1, \ldots, u_n) + (v_1, \ldots, v_n)\right] + (w_1, \ldots, w_n)\\ &= (u_1 + v_1, \ldots, u_n + v_n) + (w_1, \ldots, w_n)\\ &= ((u_1 + v_1) + w_1, \ldots, (u_n + v_n) + w_n)\\ &= (u_1 + (v_1 + w_1), \ldots, u_n + (v_n + w_n))\\ &= (u_1, \ldots, u_n) + (v_1 + w_1, \ldots, v_n + w_n)\\ &= (u_1, \ldots, u_n) + \left[(v_1, \ldots, v_n) + (w_1, \ldots, w_n)\right]\\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

(4) $\mathbf{u} + \mathbf{0} = (u_1, \ldots, u_n) + (0, \ldots, 0) = (u_1 + 0, \ldots, u_n + 0) = (u_1, \ldots, u_n) = \mathbf{u}$

(5) $$\begin{aligned}\mathbf{u} + (-\mathbf{u}) &= (u_1, \ldots, u_n) + (-u_1, \ldots, -u_n)\\ &= (u_1 - u_1, \ldots, u_n - u_n) = (0, \ldots, 0) = \mathbf{0}\end{aligned}$$

(6) $c\mathbf{u} = c(u_1, \ldots, u_n) = (cu_1, \ldots, cu_n)$ is a vector in R^n.

(7) $$\begin{aligned}c(\mathbf{u} + \mathbf{v}) &= c\left[(u_1, \ldots, u_n) + (v_1, \ldots, v_n)\right] = c(u_1 + v_1, \ldots, u_n + v_n)\\ &= (c(u_1 + v_1), \ldots, c(u_n + v_n)) = (cu_1 + cv_1, \ldots, cu_n + cv_n)\\ &= (cu_1, \ldots, cu_n) + (cv_1, \ldots, cv_n)\\ &= c(u_1, \ldots, u_n) + c(v_1, \ldots, cv_n) = c\mathbf{u} + c\mathbf{v}\end{aligned}$$

$$\begin{aligned}(8)\ (c+d)\mathbf{u} &= (c+d)(u_1,\ldots,u_n) = ((c+d)u_1,\ldots,(c+d)u_n)\\ &= (cu_1+du_1,\ldots,cu_n+du_n)\\ &= (cu_1,\ldots,cu_n)+(du_1,\ldots,du_n)\\ &= c\mathbf{u}+d\mathbf{u}\end{aligned}$$

$$\begin{aligned}(9)\ c(d\mathbf{u}) &= c(d(u_1,\ldots,u_n)) = c(du_1,\ldots,du_n) = (c(du_1),\ldots,c(du_n))\\ &= ((cd)u_1,\ldots,(cd)u_n) = (cd)(u_1,\ldots,u_n) = (cd)\mathbf{u}\end{aligned}$$

$$(10)\ 1\mathbf{u} = 1(u_1,\ldots,u_n) = (1u_1,\ldots,1u_1) = (u_1,\ldots,u_n) = \mathbf{u}$$

66. (a) Additive identity
(b) Distributive property
(c) Add $-c\mathbf{0}$ to both sides.
(d) Additive inverse and associative property
(e) Additive inverse
(f) Additive identity

68. (a) Additive inverse
(b) Transitive property
(c) Add **v** to both sides.
(d) Associative property
(e) Additive inverse
(f) Additive identity

Section 4.2 Vector Spaces

2. The additive identity of $C[-1, 0]$ is the zero function, $f(x) = 0, -1 \le x \le 0.$

4. The additive identity of $M_{5,1}$ is the 5×1 zero matrix

$$\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}.$$

6. The additive identity of $M_{2,2}$ is the 2×2 zero matrix

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

8. In $C(-\infty, \infty)$, the additive inverse of $f(x)$ is $-f(x)$.

10. In $M_{1,4}$, the additive inverse of $\begin{bmatrix}v_1 & v_2 & v_3 & v_4\end{bmatrix}$ is $\begin{bmatrix}-v_1 & -v_2 & -v_3 & -v_4\end{bmatrix}$.

12. The additive inverse of

$$\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{bmatrix} \text{ is}$$

$$\begin{bmatrix}-a_{11} & -a_{12} & -a_{13} & -a_{14} & -a_{15}\\ -a_{21} & -a_{22} & -a_{23} & -a_{24} & -a_{25}\\ -a_{31} & -a_{32} & -a_{33} & -a_{34} & -a_{35}\\ -a_{41} & -a_{42} & -a_{43} & -a_{44} & -a_{45}\\ -a_{51} & -a_{52} & -a_{53} & -a_{54} & -a_{55}\end{bmatrix}.$$

14. $M_{1,1}$ with the standard operations is a vector space. All ten vector space axioms hold.

16. This set is *not* a vector space. The set is not closed under addition or scalar multiplication. For example, $(-x^5 + x^4) + (x^5 - x^3) = x^4 - x^3$ is not a fifth-degree polynomial.

18. This set is *not* a vector space. Axiom 1 fails. For example, given $f(x) = x + 1$ and $g(x) = -x - 1$, $f(x) + g(x) = 0$ is not of the form $ax + b$, where $a, b \ne 0$.

20. This set is *not* a vector space. Axiom 1 fails. For example, given $f(x) = x^2$ and $g(x) = -x^2 + x$, $f(x) + g(x) = x$ is not a quadratic function.

22. This set is *not* a vector space. Axiom 6 fails. A counterexample is $-2(4, 1) = (-8, -2)$ is not in the set because $x < 0, y < 0$.

24. This set is a vector space. All ten vector space axioms hold.

26. This set is *not* a vector space. The set is not closed under addition nor scalar multiplication. A counterexample is

$$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}.$$

Each matrix on the left is in the set, but the sum is not in the set.

28. This set is *not* a vector space. Axiom 1 fails. For example,

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}.$$

Each matrix on the left is in the set, but the matrix on the right is not.

30. This set is a vector space. All ten vector space axioms hold.

32. This set is *not* a vector space. The set is not closed under addition nor scalar multiplication. A counterexample is

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}2&0\\0&0\end{bmatrix}.$$

Each matrix on the left is nonsingular, and the sum is not.

34. This set is a vector space. All ten vector space axioms hold.

36. This set is a vector space. All ten vector space axioms hold.

38. This set is *not* a vector space because Axiom 5 fails. The additive identity is (1, 1) and so (0, 0) has no additive inverse. Axioms 7 and 8 also fail.

40. Verify the ten axioms in the definition of vector space.

(1) $\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\begin{bmatrix}v_1&v_2\\v_3&v_4\end{bmatrix}=\begin{bmatrix}u_1+v_1&u_2+v_2\\u_3+v_3&u_4+v_4\end{bmatrix}$ is in $M_{2,2}$.

(2)
$$\begin{aligned}\mathbf{u}+\mathbf{v}&=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\begin{bmatrix}v_1&v_2\\v_3&v_4\end{bmatrix}=\begin{bmatrix}u_1+v_1&u_2+v_2\\u_3+v_3&u_4+v_4\end{bmatrix}\\&=\begin{bmatrix}v_1+u_1&v_2+u_2\\v_3+u_3&v_4+u_4\end{bmatrix}=\begin{bmatrix}v_1&v_2\\v_3&v_4\end{bmatrix}+\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}=\mathbf{v}+\mathbf{u}\end{aligned}$$

(3)
$$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w})&=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\left(\begin{bmatrix}v_1&v_2\\v_3&v_4\end{bmatrix}+\begin{bmatrix}w_1&w_2\\w_3&w_4\end{bmatrix}\right)\\&=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\begin{bmatrix}v_1+w_1&v_2+w_2\\v_3+w_3&v_4+w_4\end{bmatrix}\\&=\begin{bmatrix}u_1+(v_1+w_1)&u_2+(v_2+w_2)\\u_3+(v_3+w_3)&u_4+(v_4+w_4)\end{bmatrix}\\&=\begin{bmatrix}(u_1+v_1)+w_1&(u_2+v_2)+w_2\\(u_3+v_3)+w_3&(u_4+v_4)+w_4\end{bmatrix}\\&=\begin{bmatrix}u_1+v_1&u_2+v_2\\u_3+v_3&u_4+v_4\end{bmatrix}+\begin{bmatrix}w_1&w_2\\w_3&w_4\end{bmatrix}\\&=\left(\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\begin{bmatrix}v_1&v_2\\v_3&v_4\end{bmatrix}\right)+\begin{bmatrix}w_1&w_2\\w_3&w_4\end{bmatrix}=(\mathbf{u}+\mathbf{v})+\mathbf{w}\end{aligned}$$

(4) The zero vector is

$\mathbf{0}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$. So,

$$\mathbf{u}+\mathbf{0}=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}=\mathbf{u}.$$

(5) For every

$\mathbf{u}=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}$, you have $-\mathbf{u}=\begin{bmatrix}-u_1&-u_2\\-u_3&-u_4\end{bmatrix}$.

$$\begin{aligned}\mathbf{u}+(-\mathbf{u})&=\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}+\begin{bmatrix}-u_1&-u_2\\-u_3&-u_4\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&0\end{bmatrix}\\&=\mathbf{0}\end{aligned}$$

(6) $c\mathbf{u} = c\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = \begin{bmatrix} cu_1 & cu_2 \\ cu_3 & cu_4 \end{bmatrix}$ is in $M_{2,2}$.

(7) $$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c\left(\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} + \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}\right) = c\begin{bmatrix} u_1 + v_1 & u_2 + v_2 \\ u_3 + v_3 & u_4 + v_4 \end{bmatrix} \\ &= \begin{bmatrix} c(u_1 + v_1) & c(u_2 + v_2) \\ c(u_3 + v_3) & c(u_4 + v_4) \end{bmatrix} = \begin{bmatrix} cu_1 + cv_1 & cu_2 + cv_2 \\ cu_3 + cv_3 & cu_4 + cv_4 \end{bmatrix} \\ &= \begin{bmatrix} cu_1 & cu_2 \\ cu_3 & cu_4 \end{bmatrix} + \begin{bmatrix} cv_1 & cv_2 \\ cv_3 & cv_4 \end{bmatrix} = c\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} + c\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \\ &= c\mathbf{u} + c\mathbf{v} \end{aligned}$$

(8) $$\begin{aligned} (c + d)\mathbf{u} &= (c + d)\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = \begin{bmatrix} (c + d)u_1 & (c + d)u_2 \\ (c + d)u_3 & (c + d)u_4 \end{bmatrix} \\ &= \begin{bmatrix} cu_1 + du_1 & cu_2 + du_2 \\ cu_3 + du_3 & cu_4 + du_4 \end{bmatrix} = \begin{bmatrix} cu_1 & cu_2 \\ cu_3 & cu_4 \end{bmatrix} + \begin{bmatrix} du_1 & du_2 \\ du_3 & du_4 \end{bmatrix} \\ &= c\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} + d\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = c\mathbf{u} + d\mathbf{u} \end{aligned}$$

(9) $$\begin{aligned} c(d\mathbf{u}) &= c\left(d\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}\right) = c\begin{bmatrix} du_1 & du_2 \\ du_3 & du_4 \end{bmatrix} = \begin{bmatrix} c(du_1) & c(du_2) \\ c(du_3) & c(du_4) \end{bmatrix} \\ &= \begin{bmatrix} (cd)u_1 & (cd)u_2 \\ (cd)u_3 & (cd)u_4 \end{bmatrix} = (cd)\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = (cd)\mathbf{u} \end{aligned}$$

(10) $1(\mathbf{u}) = 1\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = \begin{bmatrix} 1u_1 & 1u_2 \\ 1u_3 & 1u_4 \end{bmatrix} = \mathbf{u}$

42. (a) Axiom 10 fails. For example,

$1(2, 3, 4) = (2, 3, 0) \neq (2, 3, 4)$.

(b) Axiom 4 fails because there is no zero vector. For example,

$(2, 3, 4) + (x, y, z) = (0, 0, 0) \neq (2, 3, 4)$ for all choices of (x, y, z).

(c) Axiom 7 fails. For example,

$2[(1, 1, 1) + (1, 1, 1)] = 2(3, 3, 3) = (6, 6, 6)$

$2(1, 1, 1) + 2(1, 1, 1) = (2, 2, 2) + (2, 2, 2) = (5, 5, 5)$.

So, $c(\mathbf{u} + \mathbf{v}) \neq c\mathbf{u} + c\mathbf{v}$.

(d) $$\begin{aligned} (x_1, y_1, z_1) + (x_2, y_2, z_2) &= (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) \\ c(x, y, z) &= (cx + c - 1, cy + c - 1, cz + c - 1) \end{aligned}$$

This is a vector space. Verify the 10 axioms.

(1) $(x_1, y_1, z_1) + (x_2, y_2, z_2) \in R^3$

(2) $$\begin{aligned} (x_1, y_1, z_1) + (x_2, y_2, z_2) &= (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) \\ &= (x_2 + x_1 + 1, y_2 + y_1 + 1, z_2 + z_1 + 1) \\ &= (x_2, y_2, z_2) + (x_1, y_1, z_1) \end{aligned}$$

(3) $(x_1, y_1, z_1) + [(x_2, y_2, z_2) + (x_3, y_3, z_3)]$

$$\begin{aligned}
&= (x_1, y_1, z_1) + (x_2 + x_3 + 1, y_2 + y_3 + 1, z_2 + z_3 + 1) \\
&= (x_1 + (x_2 + x_3 + 1) + 1, y_1 + (y_2 + y_3 + 1) + 1, z_1 + (z_2 + z_3 + 1) + 1) \\
&= ((x_1 + x_2 + 1) + x_3 + 1, (y_1 + y_2 + 1) + y_3 + 1, (z_1 + z_2 + 1) + z_3 + 1) \\
&= (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) + (x_3, y_3, z_3) \\
&= [(x_1, y_1, z_1) + (x_2, y_2, z_2)] + (x_3, y_3, z_3)
\end{aligned}$$

(4) $\mathbf{0} = (-1, -1, -1)$: $(x, y, z) + (-1, -1, -1) = (x - 1 + 1, y - 1 + 1, z - 1 + 1)$
$$= (x, y, z)$$

(5) $-(x, y, z) = (-x - 2, -y - 2, -z - 2)$:

$$\begin{aligned}
(x, y, z) + (-(x, y, z)) &= (x, y, z) + (-x - 2, -y - 2, -z - 2) \\
&= (x - x - 2 + 1, y - y - 2 + 1, z - z - 2 + 1) \\
&= (-1, -1, -1) \\
&= \mathbf{0}
\end{aligned}$$

(6) $c(x, y, z) \in R^3$

(7) $c((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$$\begin{aligned}
&= c(x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1) \\
&= (c(x_1 + x_2 + 1) + c - 1, c(y_1 + y_2 + 1) + c - 1, c(z_1 + z_2 + 1) + c - 1) \\
&= (cx_1 + c - 1 + cx_2 + c - 1 + 1, cy_1 + c - 1 + cy_2 + c - 1 + 1, cz_1 + c - 1 + cz_2 + c - 1 + 1) \\
&= (cx_1 + c - 1, cy_1 + c - 1, cz_1 + c - 1) + (cx_2 + c - 1, cy_2 + c - 1, cz_2 + c - 1) \\
&= c(x_1, y_1, z_1) + c(x_2, y_2, z_2)
\end{aligned}$$

(8) $$\begin{aligned}
(c + d)(x, y, z) &= ((c + d)x + c + d - 1, (c + d)y + c + d - 1, (c + d)z + c + d - 1) \\
&= (cx + c - 1 + dx + d - 1 + 1, cy + c - 1 + dy + d - 1 + 1, cz + c - 1 + dz + d - 1 + 1) \\
&= (cx + c - 1, cy + c - 1, cz + c - 1) + (dx + d - 1, dy + d - 1, dz + d - 1) \\
&= c(x, y, z) + d(x, y, z)
\end{aligned}$$

(9) $$\begin{aligned}
c(d(x, y, z)) &= c(dx + d - 1, dy + d - 1, dz + d - 1) \\
&= (c(dx + d - 1) + c - 1, c(dy + d - 1) + c - 1, c(dz + d - 1) + c - 1) \\
&= ((cd)x + cd - 1, (cd)y + cd - 1, (cd)z + cd - 1) \\
&= (cd)(x, y, z)
\end{aligned}$$

(10) $$\begin{aligned}
1(x, y, z) &= (1x + 1 - 1, 1y + 1 - 1, 1z + 1 - 1) \\
&= (x, y, z)
\end{aligned}$$

Note: In general, if V is a vector space and a is a constant vector, then the set V together with the operations

$$\begin{aligned}
u \oplus v &= (u + a) + (v + a) - a \\
c * u &= c(u + a) - a
\end{aligned}$$

is also a vector space. Letting $a = (1, 1, 1) \in R^3$ gives the above example.

44. Let $\mathbf{u}$ be an element of the vector space V. Then $-\mathbf{u}$ is the additive inverse of $\mathbf{u}$. Assume, to the contrary, that $\mathbf{v}$ is another additive inverse of $\mathbf{u}$. Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= 0 \\ -\mathbf{u} + \mathbf{u} + \mathbf{v} &= -\mathbf{u} + 0 \\ 0 + \mathbf{v} &= -\mathbf{u} + 0 \\ \mathbf{v} &= -\mathbf{u}. \end{aligned}$$

46. (a) A set on which vector addition and scalar multiplication are defined is a vector space when the following properties hold.

1. $\mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} \in V$ such that $\mathbf{u} + 0 = \mathbf{u}$ for all $\mathbf{u} \in V$.
5. If $\mathbf{u} \in V$, then $-\mathbf{u} \in V$ and $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. If $\mathbf{u} \in V, c \in R, c\mathbf{u} \in V$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

(b) The set of all polynomials of degree 6 or less is a vector space.
The set of all sixth-degree polynomials is not a vector space.

48. R^∞ is a vector space. Verify the ten vector space axioms.

(1) $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots)$ is in R^∞.

(2) $\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3, \ldots) + (v_1, v_2, v_3, \ldots) = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots) = (v_1 + u_1, v_2 + u_2, v_3 + u_3, \ldots) = \mathbf{v} + \mathbf{u}$

(3)
$$\begin{aligned} \mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2, u_3, \ldots) + (v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots) \\ &= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3), \ldots) \\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3, \ldots) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots) + (w_1, w_2, w_3, \ldots) \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \end{aligned}$$

(4) The zero vector is

$$\mathbf{0} = (0, 0, 0, \ldots)$$

$$\mathbf{u} + \mathbf{0} = (u_1, u_2, u_3, \ldots) + (0, 0, 0, \ldots) = (u_1, u_2, u_3, \ldots).$$

(5) The additive inverse of $\mathbf{u}$ is

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots)$$

$$\mathbf{u} + (-\mathbf{u}) = (u_1 + (-u_1), u_2 + (-u_2), u_3 + (-u_3), \ldots) = (0, 0, 0, \ldots) = \mathbf{0}.$$

(6) $c\mathbf{u} = (cu_1, cu_2, cu_3, \ldots)$ is in the set.

(7)
$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c(u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots) \\ &= (c(u_1 + v_1), c(u_2 + v_2), c(u_3 + v_3), \ldots) \\ &= (cu_1 + cv_1, cu_2 + cv_2, cu_3 + cv_3, \ldots) \\ &= (cu_1, cu_2, cu_3, \ldots) + (cv_1, cv_2, cv_3, \ldots) \\ &= c\mathbf{u} + c\mathbf{v} \end{aligned}$$

(8) $(c + d)\mathbf{u} = ((c + d)u_1, (c + d)u_2, (c + d)u_3, \ldots) = (cu_1 + du_1, cu_2 + du_2, cu_3 + du_3, \ldots) = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = c(du_1, du_2, du_3, \ldots) = (c(du_1), c(du_2), c(du_3), \ldots) = ((cd)u_1, (cd)u_2, (cd)u_3, \ldots) = (cd)\mathbf{u}$

(10) $1\mathbf{u} = (1u_1, 1u_2, 1u_3, \ldots) = (u_1, u_2, u_3, \ldots) = \mathbf{u}$

50. (a) True. For a set with two operations to be a vector space, *all* ten axioms must be satisfied. Therefore, if one of the axioms fails, then this set cannot be a vector space.

(b) False. The first axiom is not satisfied, because $x + (1 - x) = 1$ is not a polynomial of degree 1, but is a sum of polynomials of degree 1.

(c) True. This set is a vector space because all ten vector space axioms hold.

52. $(-1)\mathbf{v} + 1(\mathbf{v}) = (-1 + 1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$. Also, $-\mathbf{v} + \mathbf{v} = \mathbf{0}$. So, $(-1)\mathbf{v}$ and $-\mathbf{v}$ are both additive inverses of $\mathbf{v}$. Because the additive inverse of a vector is unique, $(-1)\mathbf{v} = -\mathbf{v}$.

Section 4.3 Subspaces of Vector Spaces

2. Because W is nonempty and $W \subset R^3$, you need only check that W is closed under addition and scalar multiplication. Given

$(x_1, y_1, 4x_1 - 5y_1)$ and $(x_2, y_2, 4x_2 - 5y_2)$,

it follows that

$$(x_1, y_1, 4x_1 - 5y_1) + (x_2, y_2, 4x_2 - 5y_2) = (x_1 + x_2, y_1 + y_2, 4(x_1 + x_2) - 5(y_1 + y_2)) \in W.$$

Furthermore, for any real number c and $(x, y, 4x - 5y) \in W$, it follows that

$$c(x, y, 4x - 5y) = (cx, cy, 4(cx) - 5(cy)) \in W.$$

4. Because W is nonempty and $W \subset M_{3,2}$, you need only check that W is closed under addition and scalar multiplication. Given

$$\begin{bmatrix} a_1 & b_1 \\ a_1 - 2b_1 & 0 \\ 0 & c_1 \end{bmatrix} \in W \text{ and } \begin{bmatrix} a_2 & b_2 \\ a_2 - 2b_2 & 0 \\ 0 & c_2 \end{bmatrix} \in W$$

it follows that

$$\begin{bmatrix} a_1 & b_1 \\ a_1 - 2b_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ a_2 - 2b_2 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ (a_1 + a_2) - 2(b_1 + b_2) & 0 \\ 0 & c_1 + c_2 \end{bmatrix} \in W.$$

Furthermore, for any real number d,

$$d\begin{bmatrix} a & b \\ a - 2b & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} da & db \\ da - 2db & 0 \\ 0 & dc \end{bmatrix} \in W.$$

6. Recall from calculus that differentiability implies continuity. So, $W \subset V$. Furthermore, because W is nonempty, you need only check that W is closed under addition and scalar multiplication. Given differentiable functions f and g on $[-1, 1]$, it follows that $f + g$ is differentiable on $[-1, 1]$ and so $f + g \in W$. Also, for any real number c and for any differentiable function $f \in W$, cf is differentiable, and therefore $cf \in W$.

8. The vectors in W are of the form $(2, a)$. This set is *not* closed under addition or scalar multiplication. For example,

$$(2, 1) + (2, 1) = (4, 2) \notin W$$

and

$$2(2, 1) = (4, 2) \notin W.$$

10. This set is not closed under scalar multiplication. For example,

$$\tfrac{1}{2}(4, 3) = \left(2, \tfrac{3}{2}\right) \notin W.$$

12. This set is not closed under addition. For example, consider $f(x) = -x + 1$ and $g(x) = x + 2$, and $f(x) + g(x) = 3 \notin W$.

14. This set is not closed under addition. For example, $(3, 4, 5) + (5, 12, 13) = (8, 16, 18) \notin W$.

16. This set is not closed under addition. For instance,
$$\begin{bmatrix} 2 \\ 0 \\ 12 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 15 \end{bmatrix} \notin W.$$

18. This set is not closed under addition or scalar multiplication. For example,
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W$$
$$2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W.$$

20. The vectors in W are of the form (a, a^2). This set is not closed under addition or scalar multiplication. For example,
$$(3, 9) + (2, 4) = (5, 13) \notin W$$
and
$$2(3, 9) = (6, 18) \notin W.$$

22. This set is *not* a subspace because it is not closed under scalar multiplication.

24. This set is a subspace of $C(-\infty, \infty)$ because it is closed under addition and scalar multiplication.

26. This set is *not* a subspace because it is not closed under addition or scalar multiplication.

28. This set is *not* a subspace of $C(-\infty, \infty)$ because it is not closed under addition or scalar multiplication.

30. This set *is* a subspace because it is closed under addition and scalar multiplication.

32. This set *is* a subspace of $M_{m,n}$ because it is closed under addition and scalar multiplication.

34. This set is *not* a subspace because it is not closed under addition or scalar multiplication.

36. This set *is not* a subspace because it is not closed under addition.

38. W is *not* a subspace of R^3. For example, $(0, 0, 4) \in W$ and $(1, 1, 4) \in W$, but $(0, 0, 4) + (1, 1, 4) = (1, 1, 8) \notin W$, so W is not closed under addition.

40. W is a subspace of R^3. Note first that $W \subset R^3$ and W is nonempty. If $(s_1, t_1, s_1 + t_1)$ and $(s_2, t_2, s_2 + t_2)$ are in W, then their sum is also in W.
$$(s_1, t_1, s_1 + t_1) + (s_2, t_2, s_2 + t_2) = (s_1 + s_2, t_1 + t_2, (s_1 + s_2) + (t_1 + t_2)) \in W.$$
Furthermore, if c is any real number,
$$c(s, t, s + t) = (cs, ct, cs = ct) \in W.$$

42. W is *not* a subspace of R^3. For example, $(1, 1, 1) \in W$ and $(1, 1, 1) \in W$, but their sum, $(2, 2, 2) \notin W$. So, W is not closed under addition.

44. (a) False. Zero subspace and the whole vector space are not *proper* subspaces, even though they are subspaces.

(b) True. Because W must itself be a vector space under inherited operations, it must contain an additive identity.

(c) True. See Theorem 4.5, part 1 on page 168.

(d) True. See Definition of Subspace, page 168.

46. Example 5 showed that $W_i \subset W_j$ for $i \le j$. To show W_i is a subspace, show that it is closed under addition and scalar multiplication.

W_4: If f and g are integrable, $f + g$ and cf are integrable. So, W_4 *is* a subspace.

W_3: The sum of two continuous functions is continuous, and a continuous function multiplied by a constant is continuous. So, W_3 *is* a subspace.

W_2: If y_1 and y_2 are differentiable, $y_1 + y_2$ and cy_1 are differentiable. So, W_2 *is* a subspace.

W_1: The sum of two polynomials is a polynomial, and a polynomial multiplied by a constant is a polynomial. So, W_1 *is* a subspace.

So, W_i *is* a subspace of W_j for $i \le j$.

48. S is a subspace of $C[0, 1]$. S is nonempty because the zero function is in S. If $f_1, f_2 \in S$, then

$$\begin{aligned}\int_0^1 (f_1 + f_2)(x)dx &= \int_0^1 [f_1(x) + f_2(x)]dx \\ &= \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx \\ &= 0 + 0 = 0 \quad \Rightarrow \quad f_1 + f_2 \in S.\end{aligned}$$

If $f \in S$ and $c \in R$, then

$$\int_0^1 (cf)(x)dx = \int_0^1 cf(x)dx = c\int_0^1 f(x)dx = c0 = 0 \Rightarrow cf \in S.$$

So, S is closed under addition and scalar multiplication.

50. The commutative, associative, and distributive properties in the larger vector space still hold for a subset of the larger space. If the set is closed under addition and scalar multiplication, the remaining axioms for a vector space are satisfied, and the subset is a subspace.

52. Because W is not empty (for example, $\mathbf{x} \in W$) you need only check that W is closed under addition and scalar multiplication. Let

$$a_1\mathbf{x} + b_1\mathbf{y} + c_1\mathbf{z} \in W,$$
$$a_2\mathbf{x} + b_2\mathbf{y} + c_2\mathbf{z} \in W.$$

Then

$$(a_1\mathbf{x} + b_1\mathbf{y} + c_1\mathbf{z}) + (a_2\mathbf{x} + b_2\mathbf{y} + c_2\mathbf{z}) =$$
$$(a_1\mathbf{x} + a_2\mathbf{x}) + (b_1\mathbf{y} + b_2\mathbf{y}) + (c_1\mathbf{z} + c_2\mathbf{z}) =$$
$$(a_1 + a_2)\mathbf{x} + (b_1 + b_2)\mathbf{y} + (c_1 + c_2)\mathbf{z} \in W.$$

Similarly, if $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} \in W$ and $d \in$ R, then

$$d(a\mathbf{x} + b\mathbf{y} + c\mathbf{z}) = da\mathbf{x} + db\mathbf{y} + dc\mathbf{z} \in W.$$

54. Because W is not empty you need only check that W is closed under addition and scalar multiplication. Let $c \in R$ and $\mathbf{x}, \mathbf{y}, \in W$. Then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. So,

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0},$$
$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}.$$

Therefore, $\mathbf{x} + \mathbf{y} \in W$ and $c\mathbf{x} \in W$.

56. Let $V = R^2$. Consider

$$W = \{(x, 0) : x \in \mathbf{R}\}, \qquad U = \{(0, y) : y \in R\}.$$

Then $W \cup U$ is *not* a subspace of V, because it is not closed under addition. Indeed, $(1, 0), (0, 1) \in W \cup U$, but $(1, 1)$ (which is the sum of these two vectors) is not.

58. (a) $V + W$ is nonempty because $\mathbf{0} = \mathbf{0} + \mathbf{0} \in V + W$.

Let $\mathbf{u}_1, \mathbf{u}_2 \in V + W$. Then $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{w}_1, \mathbf{u}_2 = \mathbf{v}_2 + \mathbf{w}_2$, where $\mathbf{v}_i \in V$ and $\mathbf{w}_i \in W$. So,

$$\mathbf{u}_1 + \mathbf{u}_2 = (\mathbf{v}_1 + \mathbf{w}_1) + (\mathbf{v}_2 + \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \in V + W.$$

For scalar c,

$$c\mathbf{u}_1 = c(\mathbf{v}_1 + \mathbf{w}_1) = c\mathbf{v}_1 + c\mathbf{w}_1 \in V + W.$$

(b) If $V = \{(x, 0)\colon\ x \text{ is a real number}\}$ and $W = \{(0, y)\colon\ y \text{ is a real number}\}$, then $V + W = R^2$.

Section 4.4 Spanning Sets and Linear Independence

2. (a) Solving the equation

$$c_1(1, 2, -2) + c_2(2, -1, 1) = (-4, -3, 3)$$

for c_1 and c_2 yields the system

$$\begin{aligned} c_1 + 2c_2 &= -4 \\ 2c_1 - c_2 &= -3 \\ -2c_1 + c_2 &= 3. \end{aligned}$$

The solution of this system is $c_1 = -2$ and $c_1 = -1$. So, $\mathbf{z}$ can be written as a linear combination of the vectors in S.

(b) Proceed as in (a), substituting $(-2, -6, 6)$ for $(1, -5, -5)$. So, the system to be solved is

$$\begin{aligned} c_1 + 2c_2 &= -2 \\ 2c_1 - c_2 &= -6 \\ -2c_1 + c_2 &= 6. \end{aligned}$$

The solution of this system is $c_1 = -\frac{14}{5}$ and $c_2 = \frac{2}{5}$. So, **v** can be written as a linear combination of the vectors in S.

(c) Proceed as in (a), substituting $(-1, -22, 22)$ for $(1, -5, -5)$. So, the system to be solved is

$$\begin{aligned} c_1 + 2c_2 &= -1 \\ 2c_1 - c_2 &= -22 \\ -2c_1 + c_2 &= 22. \end{aligned}$$

The solution of this system is $c_1 = -9$ and $c_2 = 4$. So, **w** can be written as a linear combination of the vectors in S.

(d) Proceed as in (a), substituting $(1, -5, -5)$ for $(-4, -3, 3)$, which yields the system

$$\begin{aligned} c_1 + 2c_2 &= 1 \\ 2c_1 - c_2 &= -5 \\ -2c_1 + c_2 &= -5. \end{aligned}$$

This system has no solution. So, **u** cannot be written as a linear combination of the vectors in S.

4. (a) Solving the equation

$$c_1(6, -7, 8, 6) + c_2(4, 6, -4, 1) = (2, 19, -16, -4)$$

for c_1 and c_2 yields the system

$$\begin{aligned} 6c_1 + 4c_2 &= 2 \\ -7c_1 + 6c_2 &= 19 \\ 8c_1 - 4c_2 &= -16 \\ 6c_1 + c_2 &= -4. \end{aligned}$$

The solution of this system is $c_1 = -1$ and $c_2 = 2$. So, **u** can be written as a linear combination of the vectors in S.

(b) Proceed as in (a), substituting $\left(\frac{49}{2}, \frac{99}{4}, -14, \frac{19}{2}\right)$ for $(-42, 113, -112, -60)$, which yields the system

$$\begin{aligned} 6c_1 + 4c_2 &= \tfrac{49}{2} \\ -7c_1 + 6c_2 &= \tfrac{99}{4} \\ 8c_1 - 4c_2 &= -14 \\ 6c_1 + c_2 &= \tfrac{19}{2}. \end{aligned}$$

The solution of this system is $c_1 = \frac{3}{4}$ and $c_2 = 5$. So, **v** can be written as a linear combination of the vectors in S.

(c) Proceed as in (a), substituting $\left(-4, -14, \frac{27}{2}, \frac{53}{8}\right)$ for $(-42, 113, -112, -60)$, which yields the system

$$\begin{aligned} 6c_1 + 4c_2 &= -4 \\ -7c_1 + 6c_2 &= -14 \\ 8c_1 - 4c_2 &= \tfrac{27}{2} \\ 6c_1 + c_2 &= \tfrac{53}{8}. \end{aligned}$$

This system has no solution. So, **w** cannot be written as a linear combination of the vectors in S.

(d) Proceed as in (a), substituting $\left(8, 4, -1, \frac{17}{4}\right)$ for $(-42, 113, -112, -60)$, which yields the system

$$\begin{aligned} 6c_1 + 4c_2 &= 8 \\ -7c_1 + 6c_2 &= 4 \\ 8c_1 - 4c_2 &= -1 \\ 6c_1 + c_2 &= \tfrac{17}{4}. \end{aligned}$$

The solution of this system is $c_1 = \frac{1}{2}$ and $c_2 = \frac{5}{4}$. So, **z** can be written as a linear combination of vectors in S.

6. From the vector equation

$$c_1\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 5\\ 1 & -2\end{bmatrix} = \begin{bmatrix}6 & 2\\ 9 & 11\end{bmatrix}$$

you obtain the linear system

$$\begin{aligned} 2c_1 &= 6\\ -3c_1 + 5c_2 &= 2\\ 4c_1 + c_2 &= 9\\ c_1 - 2c_2 &= 11. \end{aligned}$$

This system is inconsistent, and so the matrix is not a linear combination of A and B.

8. From the vector equation

$$c_1\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 5\\ 1 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

you obtain the trivial combination

$$0\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix} + 0\begin{bmatrix}0 & 5\\ 1 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0A + 0B.$$

10. Let $\mathbf{u} = (u_1, u_2)$ be any vector in R^2. Solving the equation

$$c_1(-1, 1) + c_2(3, 1) = (u_1, u_2)$$

for c_1 and c_2 yields the system

$$\begin{aligned} -c_1 + 3c_2 &= u_1\\ c_1 + c_2 &= u_2. \end{aligned}$$

The system has a unique solution because the determinant of the coefficient matrix is nonzero. So, S spans R^2.

12. Let $\mathbf{u} = (u_1, u_2)$ be any vector in R^2. Solving the equation

$$c_1(2, 0) + c_2(0, 1) = (u_1, u_2)$$

for c_1 and c_2 yields the system

$$\begin{aligned} 2c_1 &= u_1\\ c_2 &= u_2. \end{aligned}$$

The system has a unique solution because the determinant of the coefficient matrix is nonzero. So, S spans R^2.

14. S does not span R^2 because only vectors of the form $t(1, 1)$ are in span(S). For example, $(0, 1)$ is not in span(S). S spans a line in R^2.

16. Let $\mathbf{u} = (u_1, u_2)$ be any vector in R^2. Solving the equation

$$c_1(0, 2) + c_2(1, 4) = (u_1, u_2)$$

for c_1 and c_2 yields the system

$$\begin{aligned} c_2 &= u_1\\ 2c_1 + 4c_2 &= u_2. \end{aligned}$$

The system has a unique solution because the determinant of the coefficient matrix is nonzero. So, S spans R^2.

18. Let $\mathbf{u} = (u_1, u_2)$ be any vector in R^2. Solving the equation

$$c_1(-1, 2) + c_2(2, -1) + c_3(1, 1) = (u_1, u_2)$$

for $c_1, c_2,$ and c_3 yields the system

$$\begin{aligned} -c_1 + 2c_2 + c_3 &= u_1\\ 2c_1 - c_2 + c_3 &= u_2. \end{aligned}$$

This system is equivalent to

$$\begin{aligned} c_1 - 2c_2 - c_3 &= -u_1\\ 3c_2 + 3c_3 &= 2u_1 + u_2. \end{aligned}$$

So, for any $\mathbf{u} = (u_1, u_2)$ in R^2, you can take $c_3 = 0, c_2 = (2u_1 + u_2)/3$, and $c_1 = 2c_2 - u_1 = (u_1 + 2u_2)/3$. So, S spans R^2.

20. Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in R^3. Solving the equation

$$c_1(5, 6, 5) + c_2(2, 1, -5) + c_3(0, -4, 1) = (u_1, u_2, u_3)$$

for $c_1, c_2,$ and c_3 yields the system

$$\begin{aligned} 5c_1 + 2c_2 &= u_1\\ 6c_1 + c_2 - 4c_3 &= u_2\\ 5c_1 - 5c_2 + c_3 &= u_3. \end{aligned}$$

This system has a unique solution because the determinant of the coefficient matrix is non zero. So, S spans R^3.

22. Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in R^3. Solving the equation

$$c_1(1, 0, 1) + c_2(1, 1, 0) + c_3(0, 1, 1) = (u_1, u_2, u_3)$$

for $c_1, c_2,$ and c_3 yields the system

$$\begin{aligned} c_1 + c_2 &= u_1\\ c_2 + c_3 &= u_2\\ c_1 + c_3 &= u_3. \end{aligned}$$

This system has a unique solution because the determinant of the coefficient matrix is nonzero. So, S spans R^3.

24. This set does not span R^3. Notice that the third and fourth vectors are spanned by the first two.

$$(4, 0, 5) = 2(1, 0, 3) + (2, 0, -1)$$
$$(2, 0, 6) = 2(1, 0, 3)$$

So, S spans a plane in R^3.

26. Let $a_0 + a_1x + a_2x^2 + a_3x^3$ be any vector in P_3. Solving the equation

$$c_1(x^2 - 2x) + c_2(x^3 + 8) + c_3(x^3 - x^2) + c_4(x^2 - 4) = a_0 + a_1x + a_2x^2 + a_3x^3$$

for $c_1, c_2, c_3,$ and c_4 yields the system

$$\begin{aligned} c_2 + c_3 &= a_3 \\ c_1 - c_3 + c_4 &= a_2 \\ -2c_1 &= a_1 \\ 8c_2 - 4c_4 &= a_0. \end{aligned}$$

This system has a unique solution because the determinant of the coefficient matrix is nonzero. So, S spans P_3.

28. The set is linearly dependent because

$$(3, -6) + 3(-1, 2) = 0.$$

30. This set is linearly dependent because

$$-3(1, 0) + (1, 1) + (2, -1) = (0, 0).$$

32. Because $(-1, 3, 2)$ is not a scalar multiple of $(6, 2, 1)$, the set is linearly independent.

34. Because these vectors are multiples of each other, the set S is linearly dependent.

36. From the vector equation

$$c_1(-4, -3, 4) + c_2(1, -2, 3) + c_3(6, 0, 0) = \mathbf{0}$$

you obtain the homogenous system

$$\begin{aligned} -4c_1 + c_2 + 6c_3 &= 0 \\ -3c_1 - 2c_2 &= 0 \\ 4c_1 + 3c_2 &= 0. \end{aligned}$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$. So, the set S is linearly independent.

38. From the vector equation

$$c_1(4, -3, 6, 2) + c_2(1, 8, 3, 1) + c_3(3, -2, -1, 0) = (0, 0, 0,0)$$

you obtain the homogeneous system

$$\begin{aligned} 4c_1 + c_2 + 3c_3 &= 0 \\ -3c_1 + 8c_2 - 2c_3 &= 0 \\ 6c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + c_2 &= 0. \end{aligned}$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$. So, the set S is linearly independent.

40. This set is linearly independent because

$$5(4, 1, 2, 3) - 7(3, 2, 1, 4) + 3(1, 5, 5, 9) - 2(1, 3, 9, 7) = (0, 0, 0, 0).$$

42. From the vector equation

$$c_1(x^2 - 1) + c_2(2x + 5) = 0 + 0x + 0x^2$$

you obtain the homogenous system

$$\begin{aligned} -c_1 + 5c_2 &= 0 \\ 2c_2 &= 0 \\ c_1 &= 0. \end{aligned}$$

This system has only the trivial solution. So, the set is linearly independent.

44. From the vector equation

$$c_1(x^2) + c_2(x^2 + 1) = 0 + 0x + 0x^2$$

you obtain the homogenous system

$$\begin{aligned} c_2 &= 0 \\ 0 &= 0 \\ c_1 + c_2 &= 0. \end{aligned}$$

This system has only the trivial solution. So, the set is linearly independent.

46. From the vector equation

$$c_1(-2 - x) + c_2(2 + 3x + x^2) + c_3(6 + 5x + x^2) = 0 + 0x + 0x^2$$

you obtain the homogenous system

$$\begin{aligned} -2c_1 + 2c_2 + 6c_3 &= 0 \\ -c_1 + 3c_2 + 5c_3 &= 0. \\ c_2 + c_3 &= 0 \end{aligned}$$

This system has infinitely many solutions. For example, $c_1 = 2, c_2 = -1$, and $c_3 = 1$. So, S is linearly dependent.

48. From the vector equation

$$c_1(7 - 4x + 4x^2) + c_2(6 + 2x - 3x^2) + c_3(20 - 6x + 5x^2) = 0 + 0x + 0x^2$$

you obtain the homogenous system

$$\begin{aligned} 7c_1 + 6c_2 + 20c_3 &= 0 \\ -4c_1 + 2c_2 - 6c_3 &= 0. \\ 4c_1 - 3c_2 + 5c_3 &= 0 \end{aligned}$$

This system has infinitely many solutions. For example, $c_1 = 2, c_2 = 1$, and $c_3 = -1$. So, S is linearly dependent.

50. From the vector equation

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

you obtain the homogeneous system

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0. \end{aligned}$$

So, the set is linearly independent.

52. The set is linearly dependent because

$$2\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} + 3\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix}.$$

54. One example of a nontrivial linear combination of vectors in S whose sum is the zero vector is

$$(2, 4) + 2(-1, -2) + 0(0, 6) = (0, 0).$$

Solving this equation for $(2, 4)$ yields

$$(2, 4) = -2(-1, -2) + 0(0, 6).$$

56. One example of a nontrivial linear combination of vectors in S whose sum is the zero vector is

$$2(1, 2, 3, 4) - (1, 0, 1, 2) - (1, 4, 5, 6) = (0, 0, 0, 0).$$

Solving this equation for $(1, 4, 5, 6)$ yields

$$(1, 4, 5, 6) = 2(1, 2, 3, 4) - (1, 0, 1, 2).$$

58. (a) From the vector equation

$$c_1(t, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

you obtain the homogeneous system

$$\begin{aligned} tc_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0. \end{aligned}$$

Because $c_2 = c_3 = 0$, the set will be linearly independent if $t \neq 0$.

(b) Proceeding as in (a), you obtain the homogeneous system

$$\begin{aligned} tc_1 + tc_2 + tc_3 &= 0 \\ tc_1 + c_2 &= 0 \\ tc_1 + c_3 &= 0. \end{aligned}$$

The coefficient matrix will have a nonzero determinant if $2t^2 - t \neq 0$. That is, the set will be linearly independent if $t \neq 0$ or $t \neq \frac{1}{2}$.

60. (a) Because $(-2, 4) = -2(1, -2)$, S is linearly dependent.

(b) Because $2(1, -6, 2) = (2, -12, 4)$, S is linearly dependent.

(c) Because $(0, 0) = 0(1, 0)$, S is linearly dependent.

62. The matrix $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as well. So, both sets of vectors span R^3.

64. (a) False. A set is *linearly dependent* if and only if one of the vectors of this set can be written as a linear combination of the others.

(b) True. See "Definition of a Spanning Set of a Vector Space," page 177.

66. The matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which shows that the equation

$c_1(1, 2, 3) + c_2(3, 2, 1) + c_3(0, 0, 1)$

only has the trivial solution. So, the three vectors are linearly independent. Furthermore, the vectors span R^3 because the coefficient matrix of the linear system

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

is nonsingular.

68. If S_1 is linearly dependent, then for some $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v} \in S_1$, $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. So, in S_2, you have $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$, which implies that S_2 is linearly dependent.

70. Because $\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_n, c$ not all zero, such that

$c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n + c\mathbf{v} = \mathbf{0}$.

But, $c \neq 0$ because $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ are linearly independent. So,

$$c\mathbf{v} = -c_1\mathbf{u}_1 - \cdots - c_n\mathbf{u}_n \Rightarrow \mathbf{v} = \frac{-c_1}{c}\mathbf{u}_1 - \cdots - \frac{c_n}{c}\mathbf{u}_n.$$

72. Suppose $\mathbf{v}_k = c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1}$. For any vector $\mathbf{u} \in V$,

$$\begin{aligned} \mathbf{u} &= d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1} + d_k\mathbf{v}_k \\ &= d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1} + d_k(c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1}) \\ &= (d_1 + c_1 d_k)\mathbf{v}_1 + \cdots + (d_{k-1} + c_{k-1}d_k)\mathbf{v}_{k-1} \end{aligned}$$

which shows that $\mathbf{u} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$.

74. The vectors are linearly dependent because

$$(\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{u} - \mathbf{w}) = \mathbf{0}.$$

76. On $[0, 1]$, $f_2(x) = |x| = x = \frac{1}{3}(3x)$

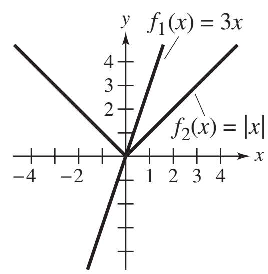
$= \frac{1}{3}f_1(x)$

$\Rightarrow \{f_1, f_2\}$ dependent.

On $[-1, 1]$, f_1 and f_2 are not multiples of each other.

$f_2(x) \neq cf_1(x)$ for $-1 \leq x < 0$, that is

$f(x) = |x| \neq \frac{1}{3}(3x)$ for $-1 \leq x \leq 0$.



Section 4.5 Basis and Dimension

2. There are four vectors in the standard basis for R^4.

$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

4. There are four vectors in the standard basis for $M_{4,1}$.

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

6. There are three vectors in the standard basis for P_2.

$\{1, x, x^2\}$

8. S is linearly dependent and does not span R^2.

10. S does not span R^2, although it is linearly independent.

12. A basis for R^2 can only have two vectors. Because S has three vectors, it is not a basis for R^2.

14. S is linearly dependent and does not span R^2.

16. A basis for R^3 contains three linearly independent vectors. Because

$-1(2, 1, -2) + (-2, -1, 2) + (4, 2, -4) = (0, 0, 0)$

S is linearly dependent and is, therefore, not a basis for R^3.

18. S does not span R^3, although it is linearly independent.

20. S is linearly dependent and does not span R^3.

22. S is not a basis because it has too many vectors. A basis for R^3 can only have three vectors.

24. S is not a basis because it has too many vectors. A basis for P_2 can only have three vectors.

26. S does not span P_2, although S is linearly independent. For example, $1 + x + x^2 \notin \text{span}(S)$.

28. S is not a basis because the vectors are linearly dependent. For example,
$-(1 - 2x + x^2) + (3 - 6x + 3x^2) + (-2 + 4x - 2x^2) = 0 + 0x + 0x^2$. Also, S does not span P_2.

30. S is not a basis because the vectors are linearly dependent.

$$1(-3 + 6x) + 1(3x^2) + 3(1 - 2x - x^2) = 0$$

32. S is not a basis because the vectors are linearly dependent.

For example, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

Also, S does not span $M_{2,2}$.

34. S does not span $M_{2,2}$, although it is linearly independent.

36. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are multiples of each other, they do not form a basis for R^2.

38. Because $\{\mathbf{v}_1, \mathbf{v}_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2.

40. Because the vectors in S are not scalar multiples of one another, they are linearly independent. Because S consists of exactly two linearly independent vectors, it is a basis for R^2.

42. S does not span R^3, although it is linearly independent. So, S is not a basis for R^3.

44. This set contains the zero vector, and is therefore linearly dependent.

$$1(0, 0, 0) + 0(1, 5, 6) + 0(6, 2, 1) = (0, 0, 0)$$

So, S is not a basis for R^3.

46. To determine if the vectors of S are linearly independent, find the solution of
$c_1(1, 0, 0, 1) + c_2(0, 2, 0, 2) + c_3(1, 0, 1, 0) + c_4(0, 2, 2, 0) = (0, 0, 0, 0)$.

Because the corresponding linear system has nontrivial solutions (for instance, $c_1 = 2, c_2 = -1, c_3 = -2,$ and $c_4 = 1$), the vectors are linearly dependent, and S is not a basis for R^4.

48. Form the equation

$$c_1(4t - t^2) + c_2(5 + t^3) + c_3(5 + 3t) + c_4(-3t^2 + 2t^3) = 0$$

which yields the homogeneous system

$$\begin{aligned} c_2 + 2c_4 &= 0 \\ -c_1 - 3c_4 &= 0 \\ 4c_1 + 3c_3 &= 0 \\ 5c_2 + 5c_3 &= 0. \end{aligned}$$

This system has only the trivial solution. So, S consists of exactly four linearly independent vectors. Therefore, S is a basis for P_3.

50. Form the equation

$$c_1(-1 + t^3) + c_2(2t^2) + c_3(3 + t) + c_4(5 + 2t + 2t^2 + t^3) = 0$$

which yields the homogeneous system

$$\begin{aligned} c_1 + c_4 &= 0 \\ 2c_2 + 2c_4 &= 0 \\ c_3 + 2c_4 &= 0 \\ -c_1 + 3c_3 + 5c_4 &= 0. \end{aligned}$$

This system has nontrivial solutions (for instance, $c_1 = 1, c_2 = 1, c_3 = 2,$ and $c_4 = -1$). Therefore, S is not a basis for P_3 because the vectors are linearly dependent.

52. Form the equation

$$c_1\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix} + c_2\begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix} + c_3\begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix} + c_4\begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which yields the homogeneous system

$$\begin{aligned} c_1 + 2c_2 + 4c_3 + 12c_4 &= 0 \\ 2c_1 - 7c_2 - 9c_3 - 16c_4 &= 0 \\ -5c_1 + 6c_2 + 11c_3 + 17c_4 &= 0 \\ 4c_1 + 2c_2 + 12c_3 + 42c_4 &= 0. \end{aligned}$$

Because this system has nontrivial solutions (for instance, $c_1 = 2, c_2 = -1, c_3 = 3,$ and $c_4 = -1$), the set is linearly dependent, and is not a basis for $M_{2,2}$.

54. Form the equation

$$c_1(1, 0, 0) + c_2(1, 1, 0) + c_3(1, 1, 1) = (0, 0, 0)$$

which yields the homogeneous system

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0. \end{aligned}$$

This system has only the trivial solution, so S is a basis for R^3. Solving the system

$$\begin{aligned} c_1 + c_2 + c_3 &= 8 \\ c_2 + c_3 &= 3 \\ c_3 &= 8 \end{aligned}$$

yields $c_1 = 5, c_2 = -5,$ and $c_3 = 8$. So,

$$\mathbf{u} = 5(1, 0, 0) - 5(1, 1, 0) + 8(1, 1, 1) = (8, 3, 8).$$

56. Form the equation

$$c_1\left(\tfrac{2}{3}, \tfrac{5}{2}, 1\right) + c_2\left(1, \tfrac{3}{2}, 0\right) + c_3(2, 12, 6) = (0, 0, 0)$$

which yields the homogeneous system

$$\begin{aligned} \tfrac{2}{3}c_1 + c_2 + 2c_3 &= 0 \\ \tfrac{5}{2}c_1 + \tfrac{3}{2}c_2 + 12c_3 &= 0 \\ c_1 \qquad + 6c_3 &= 0. \end{aligned}$$

Because this system has nontrivial solutions (for instance, $c_1 = 6, c_2 = -2,$ and $c_3 = -1$), the vectors are linearly dependent. So, S is not a basis for R^3.

58. Because a basis for R has one linearly independent vector, the dimension of R is 1.

60. Because a basis for P_4 has five linearly independent vectors, the dimension of P_4 is 5.

62. Because a basis for $M_{3,2}$ has six linearly independent vectors, the dimension of $M_{3,2}$ is 6.

64. Because a basis for P_{2m-1} has $2m$ linearly independent vectors, the dimension for P_{2m-1} is $2m$.

66. One basis for the vector space of all 3×3 symmetric matrices is

$$\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}.$$

Because this basis has 6 vectors, the dimension is 6.

68. Although there are four subsets of S that contain three vectors, only three of them are bases for R^3.

$$\{(1, 3, -2), (-4, 1, 1), (2, 1, 1)\}, \{(1, 3, -2), (-2, 7, -3), (2, 1, 1)\}, \{(-4, 1, 1), (-2, 7, -3), (2, 1, 1)\}$$

The set $\{(1, 3, -2), (-4, 1, 1), (-2, 7, -3)\}$ is linearly dependent.

70. You can add any vector that is not in the span of $S = \{(1, 0, 2), (0, 1, 1)\}$. For instance, the set $\{(1, 0, 2), (0, 1, 1), (1, 0, 0)\}$ is a basis for R^3.

72. (a) W is a line through the origin (the y-axis).

(b) A basis for W is $\{(0, 1)\}$.

(c) The dimension of W is 1.

74. (a) W is a plane through the origin.

(b) A basis for W is $\{(2, 1, 0), (-1, 0, 1)\}$, obtained by letting $s = 1, t = 0$, and then $s = 0, t = 1$.

(c) The dimension of W is 2.

76. (a) A basis for W $\{(5, -3, 1, 1)\}$.

(b) The dimension of W is 1.

78. (a) A basis for W is $\{(1, 0, 1, 2), (4, 1, 0, -1)\}$.

(b) The dimension of W is 2.

80. (a) True. See Theorem 4.10, page 189, and "Definition of Dimension of a Vector Space," page 191.

(b) False. A set of $n - 1$ vectors could be linearly dependent. For instance, they can all be multiples of each other.

82. (1) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set of vectors. Suppose, by way of contradiction, that S does not span V. Then there exists $\mathbf{v} \in V$ such that $\mathbf{v} \notin \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. So, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is linearly independent, which is impossible by Theorem 4.10. So, S does span V, and therefore is a basis.

(2) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ span V. Suppose, by way of contradiction, that S is linearly dependent. Then, some $\mathbf{v}_i \in S$ is a linear combination of the other vectors in S. Without loss of generality, you can assume that $\mathbf{v}_n$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, and therefore, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ spans V. But, $n - 1$ vectors span a vector space of at most dimension $n - 1$, a contradiction. So, S is linearly independent, and therefore a basis.

84. (a) Since the dimension of R^3 is three, any basis must have exactly three vectors. S_1 cannot span R^3.

(b) Four vectors in R^3 must form a linearly dependent set.

(c) If S_3 is linearly independent, it will be a basis for R^3.

86. Let the number of vectors in S be n. If S is linearly independent, then you are done. If not, some $\mathbf{v} \in S$ is a linear combination of other vectors in S. Let $S_1 = S - \mathbf{v}$. Note that $\text{span}(S) = \text{span}(S_1)$ because $\mathbf{v}$ is a linear combination of vectors in S_1. You now consider spanning set S_1. If S_1 is linearly independent, you are done. If not, repeat the process of removing a vector, which is a linear combination of other vectors in S_1, to obtain spanning set S_2. Continue this process with S_2. Note that this process would terminate because the original set S is a finite set and each removal produces a spanning set with fewer vectors than the previous spanning set. So, in at most $n - 1$ steps, the process would terminate leaving you with minimal spanning set, which is linearly independent and is contained in S.

Section 4.6 Rank of a Matrix and Systems of Linear Equations

2. (a) $(6, 5, -1)$

(b) $[6], [5], [-1]$

4. (a) $(0, 3, -4), (4, 0, -1), (-6, 1, 1)$

(b) $\begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$

6. (a) A basis for the row space is $\{(0, 1, -2)\}$.

(b) Because this matrix is already row-reduced, the rank is 1.

8. (a) A basis for the row space is $\left\{\left(1, \frac{5}{2}\right)\right\}$.

(b) Because this matrix row reduces to

$$\begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the rank of the matrix is 1.

10. (a) A basis for the row space is $\left\{\left(1, 0, \frac{4}{5}\right), \left(0, 1, \frac{1}{5}\right)\right\}$.

(b) Because this matrix row reduces to

$$\begin{bmatrix} 1 & 0 & \frac{4}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

the rank of the matrix is 2.

12. (a) A basis for the row space is $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$.

(b) Because this matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the rank of the matrix is 5.

14. Use $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to form the rows of matrix A. Then write A in row-echelon form.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 3 & -9 \\ 0 & 1 & 5 \end{bmatrix} \begin{matrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{matrix} \rightarrow B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{matrix}$$

So, the nonzero row vectors of B

$\mathbf{w}_1 = (1, 0, 0)$, $\mathbf{w}_2 = (0, 1, 0)$, and $\mathbf{w}_3 = (0, 0, 1)$

form a basis for the row space of A. That is, they form a basis for the subspace spanned by S.

16. Use $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to form the rows of matrix A. Then write A in row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{matrix} \rightarrow B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \end{matrix}$$

So, the nonzero row vectors of B

$\mathbf{w}_1 = (1, 0, 0)$ and $\mathbf{w}_2 = (0, 1, 1)$

form a basis for the row space of A. That is, they form a basis for the subspace spanned by S.

18. Begin by forming the matrix whose rows are vectors in S.

$$\begin{bmatrix} 6 & -3 & 6 & 34 \\ 3 & -2 & 3 & 19 \\ 8 & 3 & -9 & 6 \\ -2 & 0 & 6 & -5 \end{bmatrix}$$

This matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, a basis for span(S) is

$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$.

$(\text{span}(S) = R^4)$

20. Begin by forming the matrix whose rows are the vectors in S.

$$\begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix}$$

This matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, a basis for span(S) is

$\{(1, 0, 0, 3), (0, 1, 0, -13), (0, 0, 1, -19)\}$.

22. (a) A basis for the column space is $\{[1]\}$.

(b) Because this matrix is already row-reduced, the rank is 1.

24. (a) Row-reducing the transpose of the original matrix produces

$$\begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

So, a basis for the column space is $\left\{\left(1, 0, -\frac{2}{5}\right), \left(0, 1, \frac{3}{5}\right)\right\}$.

Equivalently, a basis for the column space consists of columns 1 and 2 of the original matrix

$$\left\{\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 20 \\ -5 \\ -11 \end{bmatrix}\right\}.$$

(b) Because this matrix row reduces to

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

the rank of the matrix is 2.

26. (a) Row reducing the transpose of the original matrix produces

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, a basis for the column space is

$\{(1, 0, 0, 0, 0),$
$(0, 1, 0, 0, 0),$
$(0, 0, 1, 0, 0),$
$(0, 0, 0, 1, 0),$
$(0, 0, 0, 0, 1)\}$

(b) Because this matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the rank of the matrix is 5.

28. Solving the system $A\mathbf{x} = \mathbf{0}$ yields only the trivial solution $\mathbf{x} = (0, 0)$. So, the dimension of the solution space is 0. The solution space consists of the zero vector itself.

30. Solving the system $A\mathbf{x} = \mathbf{0}$ yields solutions of the form $(-4s - 2t, s, t)$, where s and t are any real numbers. The dimension of the solution space is 2, and a basis is $\{[-4, 1, 0]^T, [-2, 0, 1]^T\}$.

32. Solving the system $A\mathbf{x} = \mathbf{0}$ yields solutions of the form $(-4t, t, 0)$, where t is any real number. The dimension of the solution space is 1, and a basis is $\{[-4, 1, 0]^T\}$.

34. Solving the system $A\mathbf{x} = \mathbf{0}$ yields solutions of the form $(2s - t, s, t)$, where s and t are any real numbers. The dimension of the solution space is 2, and a basis is $\{[-1, 0, 1]^T, [2, 1, 0]^T\}$.

36. Solving the system $A\mathbf{x} = \mathbf{0}$ yields solutions of the form $\begin{bmatrix} t \\ 16t \end{bmatrix}$, where t is any real number. The dimension of the solution space is 1, and a basis is $\left\{\begin{bmatrix} 1 \\ 16 \end{bmatrix}\right\}$.

38. Solving the system $A\mathbf{x} = \mathbf{0}$ yields solutions of the form $(2s - 5t, -s + t, s, t)$, where s and t are any real numbers. The dimension of the solution set is 2, and a basis is $\{[-5, 1, 0, 1]^T, [2, -1, 1, 0]^T\}$.

40. The only solution of the system $A\mathbf{x} = \mathbf{0}$ is the trivial solution. So, the solution space is $\{[0, 0, 0, 0]^T\}$ whose dimension is 0.

42. (a) $\text{rank}(A) = \text{rank}(B) = 3$

$\text{nullity}(A) = n - r = 5 - 3 = 2$

(b) Choosing $x_3 = s$ and $x_5 = t$ as the free variables, you have

$x_1 = -s - t$
$x_2 = 2s - 3t$
$x_3 = s$
$x_4 = 5t$
$x_5 = t.$

A basis for nullspace is $\{(-1, 2, 1, 0, 0), (-1, -3, 0, 5, 1)\}$.

(c) A basis for the row space of A (which is equal to the row space of B) is $\{(1, 0, 1, 0, 1), (0, 1, -2, 0, 3), (0, 0, 0, 1, -5)\}$.

(d) A basis for the column space A (which is *not* the same as the column space of B) is $\{(-2, 1, 3, 1), (-5, 3, 11, 7), (0, 1, 7, 5)\}$.

(e) Linearly dependent

(f) (i) and (iii) are linearly independent, while (ii) is linearly dependent.

44. (a) This system yields solutions of the form $(2s - 3t, s, t)$, where s and t are any real numbers and a basis for the solution space is $\{(2, 1, 0), (-3, 0, 1)\}$.

(b) The dimension of the solution space is 2.

46. (a) This system yields solutions of the form $\left(\frac{5}{8}t, -\frac{15}{8}t, \frac{9}{8}t, t\right)$, where t is any real number. A basis for the solution space is $\left\{\left(\frac{5}{8}, -\frac{15}{8}, \frac{9}{8}, 1\right)\right\}$ or $\left\{(5, -15, 9, 8)\right\}$.

(b) The dimension of the solution space is 1.

48. (a) This system yields solutions of the form $\left(-t + 2s - r, -4t - 8s - \frac{1}{3}r, r, s, t\right)$, where r, s, and t are any real numbers. A basis for the solution space is

$$\left\{(-1, -4, 0, 0, 1), (2, -8, 0, 1, 0), \left(-1, -\tfrac{1}{3}, 1, 0, 0\right)\right\}.$$

(b) The dimension of the solution space is 3.

50. The system $A\mathbf{x} = \mathbf{b}$ is consistent because its augmented matrix reduces to

$$\begin{bmatrix} 1 & 2 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions of $A\mathbf{x} = \mathbf{b}$ are of the form $(-1 - 2s + 4t, s, t)$, where s and t are any real numbers. That is,

$$x = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ c \\ 1 \end{bmatrix},$$

where

$$xp = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_n = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

52. (a) The system $A\mathbf{x} = \mathbf{b}$ is consistent because its augmented matrix reduces to

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The solutions of $A\mathbf{x} = \mathbf{b}$ are of the form $(4 + 2t, t, 0)$, where t is any real number. That is,

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

where

$$\mathbf{x}_p = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_h = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

54. This system $A\mathbf{x} = \mathbf{b}$ is inconsistent because its augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

56. (a) The system $A\mathbf{x} = \mathbf{b}$ is consistent because its augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 4 & -5 & 6 & 0 \\ 0 & 1 & 2 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The solutions of the system are of the form $(-6t + 5s - 4r, 1 - 4t - 2s - 2r, r, s, t)$, where r, s, and t are any real numbers. That is,

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where

$$\mathbf{x}_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_h = r\begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

58. The vector $\mathbf{b}$ is not in the column space of A because the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

60. The vector $\mathbf{b}$ is in the column space of A if the equation $A\mathbf{x} = \mathbf{b}$ is consistent. Because $A\mathbf{x} = \mathbf{b}$ has the solution

$$\mathbf{x} = \begin{bmatrix} -\frac{5}{4} \\ \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix},$$

$\mathbf{b}$ is in the column space of A. Furthermore,

$$\mathbf{b} = -\frac{5}{4}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

62. The vector $\mathbf{b}$ is in the column space of A if the equation $A\mathbf{x} = \mathbf{b}$ is consistent. Because $A\mathbf{x} = \mathbf{b}$ has the solution

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix},$$

$\mathbf{b}$ is in the column space of A. Furthermore,

$$\mathbf{b} = -\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \\ -25 \end{bmatrix}.$$

64. Many examples are possible. For instance,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

rank 1 rank 1 rank 0

66. Let $\left[a_{ij}\right] = A$ be an $m \times n$ matrix in row-echelon form. The nonzero row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_k$ of A have the form (if the first column of A is not all zero)

$$\mathbf{r}_1 = \left(e_{11}, \ldots, e_{1p}, \ldots, e_{1q}, \ldots\right)$$
$$\mathbf{r}_2 = \left(0, \ldots, 0, e_{2p}, \ldots, e_{2q}, \ldots\right)$$
$$\mathbf{r}_3 = \left(0, \ldots, 0, 0, \ldots, 0, e_{3q}, \ldots\right)$$

and so forth, where e_{11}, e_{2p}, e_{3q} denote leading ones. Then the equation

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_k\mathbf{r}_k = \mathbf{0}$$

implies that

$$c_1e_{11} = 0, c_1e_{1p} + c_2e_{2p} = 0, c_1e_{1q} + c_2e_{2q} + c_3e_{3q} = 0$$

and so forth. You can conclude in turn that $c_1 = 0$, $c_2 = 0, \ldots, c_k = 0$, and so the row vectors are linearly independent.

68. Suppose that the three points are collinear. If they are on the same vertical line, then $x_1 = x_2 = x_3$. So, the matrix has two equal columns, and its rank is less than 3. Similarly, if the three points lie on the nonvertical line $y = mx + b$, you have

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & mx_1 + b & 1 \\ x_2 & mx_2 + b & 1 \\ x_3 & mx_3 + b & 1 \end{bmatrix}.$$

Because the second column is a linear combination of the first and third columns, this determinant is zero, and the rank is less than 3.

On the other hand, if the rank of the matrix

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

is less than 3, then the determinant is zero, which implies that the three points are collinear.

70. For $n = 2$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has rank 2.

For $n = 3$, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ has rank 2.

In general, for $n \geq 2$, the rank is 2, because rows $3, \ldots, n$, are linear combinations of the first two rows. For example, $R_3 = 2R_2 - R_1$.

72. Let

$$\mathbf{x} \in N(A) \Rightarrow A\mathbf{x} = \mathbf{0} \Rightarrow A^TA\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} \in N\left(A^TA\right).$$

74. (a) True. See Theorem 4.13, page 196.

(b) False. The dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ for $m \times n$ matrix of rank r is $n - r$. See Theorem 4.17, page 202.

76. (a) True. The columns of A become rows of the transpose, A^T. So, the columns of A span the same space as the rows of A^T.

(b) True. The rows of A become columns of the transpose, A^T. So, the rows of A span the same space as the columns of A^T.

78. (a) The row space and column space of a matrix have the same dimension, so the column space has a dimension of 2.

(b) 2

(c) $(\text{rank}) + (\text{nullity}) = (\text{number of columns})$, so the nullity is 3.

(d) 3

80. Let A and B be $2m \times n$ row equivalent matrices. The dependency relationships among the columns of A can be expressed in the form $A\mathbf{x} = \mathbf{0}$, while those of B in the form $B\mathbf{x} = \mathbf{0}$. Because A and B are row-equivalent, $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution sets, and therefore the same dependency relationships.

Section 4.7 Coordinates and Change of Basis

2. $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

4. $\begin{bmatrix} -6 \\ 12 \\ -4 \\ 9 \\ -8 \end{bmatrix}$

6. Because $[\mathbf{x}]_B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, you can write

$\mathbf{x} = -(-2, 3) + 4(3, -2) = (14, -11)$

which implies that the coordinates of $\mathbf{x}$ relative to the standard basis S are $[\mathbf{x}]_S = \begin{bmatrix} 14 \\ -11 \end{bmatrix}$.

8. Because $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, you can write

$\mathbf{x} = 2\left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right) + 0\left(3, 4, \frac{7}{2}\right) + 4\left(-\frac{3}{2}, 6, 2\right) = \left(-\frac{9}{2}, 29, 11\right)$

which implies that the coordinates of $\mathbf{x}$ relative to the standard basis S are $[\mathbf{x}_S] = \begin{bmatrix} -\frac{9}{2} \\ 29 \\ 11 \end{bmatrix}$.

10. Because $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$, you can write

$\mathbf{x} = -2(4, 0, 7, 3) + 3(0, 5, -1, -1) + 4(-3, 4, 2, 1) + 1(0, 1, 5, 0) = (-20, 32, -4, -5)$

which implies that the coordinates of $\mathbf{x}$ relative to the standard basis S are

$[\mathbf{x}]_S = \begin{bmatrix} -20 \\ 32 \\ -4 \\ -5 \end{bmatrix}$.

12. Begin by writing $\mathbf{x}$ as a linear combination of the vectors in B.

$\mathbf{x} = (-17, 22) = c_1(-5, 6) + c_2(3, -2)$

Equating corresponding components yields the following system of linear equations.

$$\begin{aligned} -5c_1 + 3c_2 &= -17 \\ 6c_1 - 2c_2 &= 22 \end{aligned}$$

The solution of this system is $c_1 = 4$ and $c_2 = 1$. So, $\mathbf{x} = 4(-5, 6) + (3, -2)$ and $[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

14. Begin by writing **x** as a linear combination of the vectors in B.

$$\mathbf{x} = \left(3, -\tfrac{1}{2}, 8\right) = c_1\left(\tfrac{3}{2}, 4, 1\right) + c_2\left(\tfrac{3}{4}, \tfrac{5}{2}, 0\right) + c_3\left(1, \tfrac{1}{2}, 2\right)$$

Equating corresponding components yields the following system of linear equations.

$$\begin{aligned} \tfrac{3}{2}c_1 + \tfrac{3}{4}c_2 + c_3 &= 3 \\ 4c_1 + \tfrac{5}{2}c_2 + \tfrac{1}{2}c_3 &= -\tfrac{1}{2} \\ c_1 + 2c_3 &= 8 \end{aligned}$$

The solution of this system is $c_1 = 2, c_2 = -4,$ and $c_3 = 3$. So, $\mathbf{x} = 2\left(\tfrac{3}{2}, 4, 1\right) - 4\left(\tfrac{3}{4}, \tfrac{5}{2}, 0\right) + 3\left(1, \tfrac{1}{2}, 2\right)$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$

16. Begin by writing **x** as a linear combination of the vectors in B.

$$\mathbf{x} = (0, -20, 7, 15) = c_1(9, -3, 15, 4) + c_2(3, 0, 0, 1) + c_3(0, -5, 6, 8) + c_4(3, -4, 2, -3)$$

Equating corresponding components yields the following system of linear equations.

$$\begin{aligned} 9c_1 + 3c_2 + 3c_4 &= 0 \\ -3c_1 - 5c_3 - 4c_4 &= -20 \\ 15c_1 + 6c_3 + 2c_4 &= 7 \\ 4c_1 + c_2 + 8c_3 - 3c_4 &= 15 \end{aligned}$$

The solution of this system is $c_1 = -1, c_2 = 1, c_3 = 3,$ and $c_4 = 2$.

So, $(0, -20, 7, 15) = -1(9, -3, 15, 4) + 1(3, 0, 0, 1) + 3(0, -5, 6, 8) + 2(3, -4, 2, -3)$ and $[\mathbf{x}]_B = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$

18. Begin by forming the matrix

$$[B' \;\; B] = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 1 & 6 & 0 & 1 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$[I_2 \;\; P^{-1}] = \begin{bmatrix} 1 & 0 & 6 & -5 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} 6 & -5 \\ -1 & 1 \end{bmatrix}.$$

20. Begin by forming the matrix

$$[B' \;\; B] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Because this matrix is already in the form $[I_2 \;\; P^{-1}]$, you see that the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

22. Begin by forming the matrix

$$[B' \;\; B] = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$[I_3 \;\; P^{-1}] = \begin{bmatrix} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix}.$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}.$$

24. Begin by forming the matrix

$$[B' \ \ B] = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 5 \\ 0 & 1 & 0 & 3 & -1 & 6 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}.$$

Because this matrix is already in the form $\left[I_3 \ P^{-1}\right]$, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & -1 & 6 \\ 2 & 2 & 1 \end{bmatrix}.$$

26. Begin by forming the matrix

$$[B' \ \ B] = \begin{bmatrix} 1 & -1 & -2 & 3 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$\left[I_2 \ \ P^{-1}\right] = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{5}{2} & -2 \end{bmatrix}.$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{5}{2} & -2 \end{bmatrix}.$$

28. Begin by forming the matrix

$$\left[B^1 \ B\right] = \begin{bmatrix} 3 & -3 & 2 & -2 \\ -3 & -3 & -2 & -2 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$\left[I_2 \ P^{-1}\right] = \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{2}{3} \end{bmatrix}.$$

So, the transition matrix from B to B^1 is $P^{-1} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}.$

30. Begin by forming the matrix

$$[B' \ \ B] = \begin{bmatrix} 2 & 0 & -3 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$\left[I_3 \ \ P^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{14}{27} & -\frac{1}{27} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{27} & \frac{4}{27} \end{bmatrix}.$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} 0 & -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{14}{27} & -\frac{1}{27} \\ -\frac{1}{3} & -\frac{2}{27} & \frac{4}{27} \end{bmatrix}.$$

32. Begin by forming the matrix

$$[B' \ \ B] = \begin{bmatrix} 1 & 0 & -1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 2 & 1 & 2 \\ -1 & 2 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

and then use Gauss-Jordan elimination to produce

$$\left[I_3 \ \ P^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & \frac{27}{11} & \frac{8}{11} & \frac{12}{11} \\ 0 & 1 & 0 & \frac{19}{11} & \frac{15}{11} & \frac{6}{11} \\ 0 & 0 & 1 & -\frac{6}{11} & -\frac{3}{11} & \frac{1}{11} \end{bmatrix}.$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} \frac{27}{11} & \frac{8}{11} & \frac{12}{11} \\ \frac{19}{11} & \frac{15}{11} & \frac{6}{11} \\ -\frac{6}{11} & -\frac{3}{11} & \frac{1}{11} \end{bmatrix}.$$

34. Begin by forming the matrix

$$[B' \ \ B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$\left[I_4 \ \ P^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

36. Begin by forming the matrix

$$\begin{bmatrix} B' & B \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & -2 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and then use Gauss-Jordan elimination to produce

$$\begin{bmatrix} I_5 & P^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{12}{157} & \frac{32}{157} & \frac{5}{314} & \frac{10}{157} & -\frac{7}{157} \\ 0 & 1 & 0 & 0 & 0 & \frac{45}{157} & -\frac{37}{157} & -\frac{99}{314} & -\frac{41}{157} & \frac{13}{157} \\ 0 & 0 & 1 & 0 & 0 & -\frac{17}{157} & \frac{7}{157} & \frac{3}{157} & \frac{12}{157} & \frac{23}{157} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{157} & \frac{47}{314} & \frac{287}{628} & \frac{103}{314} & -\frac{25}{314} \\ 0 & 0 & 0 & 0 & 1 & -\frac{4}{157} & \frac{31}{314} & \frac{49}{628} & -\frac{59}{314} & \frac{57}{314} \end{bmatrix}$$

So, the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} \frac{12}{157} & \frac{32}{157} & \frac{5}{314} & \frac{10}{157} & -\frac{7}{157} \\ \frac{45}{157} & -\frac{37}{157} & -\frac{99}{314} & -\frac{41}{157} & \frac{13}{157} \\ -\frac{17}{157} & \frac{7}{157} & \frac{3}{157} & \frac{12}{157} & \frac{23}{157} \\ -\frac{1}{157} & \frac{47}{314} & \frac{287}{628} & \frac{103}{314} & -\frac{25}{314} \\ -\frac{4}{157} & \frac{31}{314} & \frac{49}{628} & -\frac{59}{314} & \frac{57}{314} \end{bmatrix}.$$

38. (a) $\begin{bmatrix} B' & B \end{bmatrix} = \begin{bmatrix} 1 & 32 & 2 & 6 \\ 1 & 31 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -126 & -90 \\ 0 & 1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} I & P^{-1} \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -126 & -90 \\ 4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} B & B' \end{bmatrix} = \begin{bmatrix} 2 & 6 & 1 & 32 \\ -2 & 3 & 1 & 31 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{6} & -5 \\ 0 & 1 & \frac{2}{9} & 7 \end{bmatrix} = \begin{bmatrix} I & P \end{bmatrix} \Rightarrow P = \begin{bmatrix} -\frac{1}{6} & -5 \\ \frac{2}{9} & 7 \end{bmatrix}.$

(c) $PP^{-1} = \begin{bmatrix} -\frac{1}{6} & -5 \\ \frac{2}{9} & 7 \end{bmatrix}\begin{bmatrix} -126 & -90 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $[\mathbf{x}]_B = P[\mathbf{x}]_{B'} = \begin{bmatrix} -\frac{1}{6} & -5 \\ \frac{2}{9} & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{14}{3} \\ -\frac{59}{9} \end{bmatrix}$

40. (a) $\begin{bmatrix} B' & B \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} I & P^{-1} \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} B & B' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 1 & -1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & P \end{bmatrix} \Rightarrow P = \begin{bmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 & 0 \end{bmatrix}.$

(c) $PP^{-1} = \begin{bmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $[\mathbf{x}]_B = P[\mathbf{x}]_{B'} = \begin{bmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}$

42. (a) $[B' \ B] = \begin{bmatrix} 1 & 4 & -2 & 1 & 2 & -4 \\ 2 & 1 & 5 & 3 & -5 & 2 \\ -2 & -4 & 8 & 4 & 2 & -6 \end{bmatrix}$

$$[I \ P^{-1}] = \begin{bmatrix} 1 & 0 & 0 & -\frac{11}{16} & -\frac{55}{16} & -\frac{73}{16} \\ 0 & 1 & 0 & \frac{25}{32} & \frac{45}{32} & -\frac{83}{32} \\ 0 & 0 & 1 & \frac{23}{32} & \frac{3}{32} & -\frac{29}{32} \end{bmatrix}$$

So, $P^{-1} = \begin{bmatrix} -\frac{11}{16} & -\frac{55}{16} & \frac{73}{16} \\ \frac{25}{32} & \frac{45}{32} & -\frac{83}{32} \\ \frac{23}{32} & \frac{3}{32} & -\frac{29}{32} \end{bmatrix}.$

(b) $[B \ B'] = \begin{bmatrix} 1 & 2 & -4 & 1 & 4 & -2 \\ 3 & -5 & 2 & 2 & 1 & 5 \\ 4 & 2 & -6 & -2 & -4 & 8 \end{bmatrix}$

$$[I \ P] = \begin{bmatrix} 1 & 0 & 0 & -\frac{33}{13} & -\frac{86}{13} & \frac{80}{13} \\ 0 & 1 & 0 & -\frac{37}{13} & -\frac{85}{13} & \frac{57}{13} \\ 0 & 0 & 1 & -\frac{30}{13} & -\frac{77}{13} & \frac{55}{13} \end{bmatrix}$$

So, $P = \begin{bmatrix} -\frac{33}{13} & -\frac{86}{13} & \frac{80}{13} \\ -\frac{37}{13} & -\frac{85}{13} & \frac{57}{13} \\ -\frac{30}{13} & -\frac{77}{13} & \frac{55}{13} \end{bmatrix}.$

(c) Using a graphing utility, you have $PP^{-1} = I$.

(d) $[\mathbf{x}]_B = P[\mathbf{x}]_{B'} = P\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{193}{13} \\ \frac{151}{13} \\ \frac{140}{13} \end{bmatrix}$

44. (a) $[B^1 \ B] = \begin{bmatrix} 3 & -3 & 0 & 1 & -9 & 1 \\ 0 & 3 & -3 & -1 & 1 & 9 \\ 3 & 0 & 3 & 9 & 1 & -1 \end{bmatrix}$

$$[I \ P^{-1}] = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{7}{6} & \frac{3}{2} \\ 0 & 1 & 0 & \frac{7}{6} & \frac{11}{6} & \frac{7}{6} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} & -\frac{11}{6} \end{bmatrix}$$

So, the transition matrix from B to B^1 is

$$P^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{7}{6} & \frac{3}{2} \\ \frac{7}{6} & \frac{11}{6} & \frac{7}{6} \\ \frac{3}{2} & \frac{3}{2} & -\frac{11}{6} \end{bmatrix}.$$

(b) $[B \ B^1] = \begin{bmatrix} 1 & -9 & 1 & 3 & -3 & 0 \\ -1 & 1 & 9 & 0 & 3 & -3 \\ 9 & 1 & -1 & 3 & 0 & 3 \end{bmatrix}$

$$[I \ P] = \begin{bmatrix} 1 & 0 & 0 & \frac{69}{185} & -\frac{3}{370} & \frac{3}{10} \\ 0 & 1 & 0 & -\frac{21}{74} & \frac{27}{74} & 0 \\ 0 & 0 & 1 & \frac{27}{370} & \frac{108}{370} & -\frac{3}{10} \end{bmatrix}$$

(c) Using a graphing utility, you have $P P^{-1} = I$.

(d) $[\mathbf{x}]_B = P[\mathbf{x}]_{B^1} = P\begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{567}{370} \\ -\frac{3}{74} \\ \frac{339}{185} \end{bmatrix}$

46. The standard basis for P_3 is $S = \{1, x, x^2, x^3\}$ and because $p = -2(1) - 3(x) + 0(x^2) + 4(x^3)$ it follows that

$$[p]_S = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 4 \end{bmatrix}.$$

48. The standard basis for P_3 is $S = \{1, x, x^2, x^3\}$ and because $p = 4(1) + 11(x) + 1(x^2) + 2(x^3)$ it follows that

$$[p]_S = \begin{bmatrix} 4 \\ 11 \\ 1 \\ 2 \end{bmatrix}.$$

50. The standard basis in $M_{3,1}$ is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and because

$$X = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

it follows that

$$[X]_S = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

52. The standard basis in $M_{3,1}$ is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and because

$$X = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

it follows that

$$[X]_S = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}.$$

54. (a) $[B' \;\; B] = [B' \;\; I_n] \Rightarrow \left[I_n \;\; (B')^{-1}\right] = \left[I_n \;\; P^{-1}\right]$

$\Rightarrow (B')^{-1} = P^{-1}$

(b) $[B' \;\; B] = [I_n \;\; B] \Rightarrow B = P^{-1}$

(c) $[B \;\; B'] = [I_n \;\; B'] \Rightarrow B' = P$

(d) $[B \;\; B'] = [B \;\; I_n] \Rightarrow \left[I_n \;\; B^{-1}\right] = [I_n \;\; P]$

$\Rightarrow B^{-1} = P$

56. (a) True. If P is the transition matrix from B^1 to B, then $P[\mathbf{x}]_{B^1} = [\mathbf{x}]_B$. Multiplying both sides by P^{-1} you see that $[\mathbf{x}]_{B^1} = P^{-1}[\mathbf{x}]_B$ matrix from B to B^1.

(b) True. See discussion before Example 5, page 214.

(c) False. $[p]_S = [-3 \;\; 1 \;\; 5]^T$.

58. Let P be the transition matrix from B'' to B' and let Q be the transition matrix from B' to B. Then for any vector **x,** the coordinate matrices with respect to these bases are related as follows.

$[\mathbf{x}]_{B'} = P[\mathbf{x}]_{B''}$ and $[\mathbf{x}]_B = Q[\mathbf{x}]_{B'}$

Then the transition matrix from B'' to B is QP because

$[\mathbf{x}]_B = Q[\mathbf{x}]_{B'} = QP[\mathbf{x}]_{B''}$.

So, the transition matrix from B to B'', which is the inverse of the transition matrix from B'' to B, is equal to

$(QP)^{-1} = P^{-1}Q^{-1}$.

Section 4.8 Applications of Vector Spaces

2. (a) If $y = e^x$, then $y''' = e^x$ and $y''' + y = 2e^x \neq 0$. So, e^x *is not* a solution of the equation.

(b) If $y = e^{-x}$, then $y''' = -e^{-x}$ and $y''' + y = 0$. So, e^{-x} *is* a solution of the equation.

(c) If $y = e^{-2x}$, then $y''' = -8e^{-2x}$ and $y''' + y = -7e^{-2x} \neq 0$. So, e^{-2x} *is not* a solution of the equation.

(d) If $y = 2e^{-x}$, then $y''' = -2e^{-x}$ and $y''' + y = 0$. So, $2e^{-x}$ *is* a solution of the equation.

4. (a) If $y = e^{3x}$, then $y' = 3e^{3x}$ and $y'' = 9e^{3x}$. So, $y'' - 6y' + 9y = 9e^{3x} - 6(3e^{3x}) + 9(e^{3x}) = 0$ and e^{3x} is a solution.

(b) If $y = xe^{3x}$, then $y' = (3x + 1)e^{3x}$ and $y'' = (9x + 6)e^{3x}$. So,

$y'' - 6y' + 9y = (9x + 6)e^{3x} - 6(3x + 1)e^{3x} + 9xe^{3x} = 0$ and xe^{3x} is a solution.

(c) If $y = x^2e^{3x}$, then $y' = (3x^2 + 2x)e^{3x}$ and $y'' = (9x^2 + 12x + 2)e^{3x}$. So,

$y'' - 6y' + 9y = (9x^2 + 12x + 2)e^{3x} - 6(3x^2 + 2x)e^{3x} + 9x^2e^{3x} \neq 0.$

So, x^2e^{3x} is *not* a solution of the equation.

(d) If $y = (x + 3)e^{3x}$, then $y' = (3x + 10)e^{3x}$ and $y'' = (9x + 33)e^{3x}$. So,

$y'' - 6y' + 9y = (9x + 33)e^{3x} - 6(3x + 10)e^{3x} + 9(x + 3)e^{3x} = 0$ and $(x + 3)e^{3x}$ is a solution.

6. (a) If $y = 3\cos x$, $y^{(4)} = 3\cos x$ and $y^{(4)} - 16y = -45\cos x \neq 0$. So, $3\cos x$ *is not* a solution of the equation.

(b) If $y = 3\cos 2x$, then $y^{(4)} = 48\cos 2x$ and $y^{(4)} - 16y = 0$. So, $3\cos 2x$ *is* a solution of the equation.

(c) If $y = e^{-2x}$, then $y^{(4)} = 16e^{-2x}$ and $y^{(4)} - 16y = 0$. So, e^{-2x} *is* a solution of the equation.

(d) If $y = 3e^{2x} - 4\sin 2x$, then $y^{(4)} = 48e^{2x} - 64\sin 2x$ and $y^{(4)} - 16y = 0$. So, $3e^{2x} - 4\sin 2x$ *is* a solution of the equation.

8. (a) If $y = e^{x-x^2}$, then $y' = (1 - 2x)e^{x-x^2}$ and $y' + (2x - 1)y = 0$. So, e^{x-x^2} *is* a solution of the equation.

(b) If $y = 2e^{x-x^2}$, then $y' = (2 - 4x)e^{x-x^2}$ and $y' + (2x - 1)y = 0$. So, $2e^{x-x^2}$ *is* a solution of the equation.

(c) If $y = 3e^{x-x^2}$, then $y' = (3 - 6x)e^{x-x^2}$ and $y' + (2x - 1)y = 0$. So, $3e^{x-x^2}$ *is* a solution of the equation.

(d) If $y = 4e^{x-x^2}$, then $y' = (4 - 8x)e^{x-x^2}$ and $y' + (2x - 1)y = 0$. So, $4e^{x-x^2}$ *is* a solution of the equation.

10. (a) If $y = x$, then $y' = 1$ and $y'' = 0$. So, $xy'' + 2y' = x(0) + 2(1) \neq 0$, and $y = x$ *is not* a solution.

(b) If $y = \frac{1}{x}$, then $y' = -\frac{1}{x^2}$ and $y'' = \frac{2}{x^3}$. So, $xy'' + 2y' = x\left(\frac{2}{x^3}\right) + -2\left(-\frac{1}{x^2}\right) = 0$, and $y = \frac{1}{x}$ *is* a solution.

(c) If $y = xe^x$, then $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$. So, $xy'' + 2y' = x(xe^x + 2e^x) + 2(xe^x + e^x) \neq 0$, and $y = xe^x$ *is not* a solution.

(d) If $y = xe^{-x}$, then $y' = e^{-x} - xe^{-x}$ and $y'' = xe^{-x} - 2e^{-x}$. So, $xy'' + 2y' = x(xe^{-x} - 2e^{-x}) + 2(e^{-x} - xe^{-x}) \neq 0$, and $y = xe^{-x}$ *is not* a solution.

12. (a) If $y = 3e^{x^2}$, then $y' = 6xe^{x^2}$. So, $y' - 2xy = 6xe^{x^2} - 2x(3e^{x^2}) = 0$, and $y = 3e^{x^2}$ *is* a solution.

(b) If $y = xe^{x^2}$, then $y' = 2x^2e^{x^2} + e^{x^2}$. So, $y' - 2xy = 2x^2e^{x^2} + e^{x^2} - 2x(xe^{x^2}) \neq 0$, and $y = xe^{x^2}$ *is not* a solution.

(c) If $y = x^2e^x$, then $y' = x^2e^x + 2xe^x$. So, $y' - 2xy = x^2e^x + 2xe^x - 2x(x^2e^x) \neq 0$, and $y = x^2e^x$ *is not* a solution.

(d) If $y = xe^{-x}$, then $y' = e^{-x} - xe^{-x}$. So, $y' - 2xy = e^{-x} - xe^{-x} - 2x(xe^{-x}) \neq 0$, and $y = xe^{-x}$ *is not* a solution.

14. $W(e^{3x}, \sin 2x) = \begin{vmatrix} e^{3x} & \sin 2x \\ 3e^{3x} & 2\cos 2x \end{vmatrix}$

$= 2e^{3x}\cos 2x - 3e^{3x}\sin 2x$

16. $W\left(e^{x^2}, e^{-x^2}\right) = \begin{vmatrix} e^{x^2} & e^{-x^2} \\ 2xe^{x^2} & -2xe^{-x^2} \end{vmatrix} = -4x$

18. $W(x, -\sin x, \cos x) = \begin{vmatrix} x & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \\ 0 & \sin x & -\cos x \end{vmatrix} = x$

20. $W(x, e^{-x}, e^x) = \begin{vmatrix} x & e^{-x} & e^x \\ 1 & -e^{-x} & e^x \\ 0 & e^{-x} & e^x \end{vmatrix} = -2x$

22. $W\left(x^2, e^{x^2}, x^2e^x\right) = \begin{vmatrix} x^2 & e^{x^2} & x^2e^x \\ 2x & 2xe^{x^2} & 2xe^x + x^2e^x \\ 2 & 2e^{x^2} + 4x^2e^{x^2} & 2e^x + 4xe^x + x^2e^x \end{vmatrix} = -2x^2e^{x^2+x}\left(2x^4 - x^3 - 3x^2 + x + 3\right)$

24. $W\left(x, x^2, e^x, e^{-x}\right) = \begin{vmatrix} x & x^2 & e^x & e^{-x} \\ 1 & 2x & e^x & -e^{-x} \\ 0 & 2 & e^x & e^{-x} \\ 0 & 0 & e^x & -e^{-x} \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 & 1 \\ 1 & 2x & 1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} x & x^2 & 2 & 1 \\ 1 & 2x & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1\left(4x^2 + 4 - 2x^2\right) = -2x^2 - 4$

26.
$$\begin{aligned} W\left(x, e^x, \sin x, \cos x\right) &= \begin{vmatrix} x & e^x & \sin x & \cos x \\ 1 & e^x & \cos x & -\sin x \\ 0 & e^x & -\sin x & -\cos x \\ 0 & e^x & -\cos x & \sin x \end{vmatrix} \\ &= \begin{vmatrix} x & 2e^x & 0 & 0 \\ 1 & 2e^x & 0 & 0 \\ 0 & e^x & -\sin x & -\cos x \\ 0 & e^x & -\cos x & \sin x \end{vmatrix} \\ &= x\begin{vmatrix} 2e^x & 0 & 0 \\ e^x & -\sin x & -\cos x \\ e^x & -\cos x & \sin x \end{vmatrix} - 1\begin{vmatrix} 2e^x & 0 & 0 \\ e^x & -\sin x & -\cos x \\ e^x & -\cos x & \sin x \end{vmatrix} \\ &= 2xe^x\left(-\sin^2 x - \cos^2 x\right) - 2e^x\left(-\sin^2 x - \cos^2 x\right) \\ &= -2xe^x + 2e^x \end{aligned}$$

28. First calculate the Wronskian of the two functions.

$$W\left(e^{ax}, xe^{ax}\right) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & (ax+1)e^{ax} \end{vmatrix} = (ax + 1)e^{2ax} - axe^{2ax} = e^{2ax}$$

Because $W\left(e^{ax}, xe^{ax}\right) \neq 0$ and the functions are solutions to $y'' - 2ay' + a^2y = 0$, they are linearly independent.

30. First, calculate the Wronskian of the two functions

$$\begin{aligned} W\left(e^{ax}\cos bx, e^{ax}\sin bx\right) &= \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ e^{ax}(a\cos bx - b\sin bx) & e^{ax}(a\sin bx + b\cos bx) \end{vmatrix} \\ &= be^{2ax} \neq 0, \qquad \text{because } b \neq 0 \end{aligned}$$

Because these functions satisfy the differential equation $y'' - 2ay' + \left(a^2 + b^2\right)y = 0$, they are linearly independent.

32. (a) $y = e^{2x} \sin x \Rightarrow y' = (\cos x + 2 \sin x)e^{2x},\ y'' = (4 \cos x + 3 \sin x)e^{2x} \Rightarrow y'' - 4y' + 5y = 0$

$y = e^{2x} \cos x \Rightarrow y' = (2 \cos x - \sin x)e^{2x},\ y'' = (3 \cos x - 4 \sin x)e^{2x} \Rightarrow y'' - 4y' + 5y = 0$

(b) Because $W\left(e^{2x} \sin x, e^{2x} \cos x\right) = \begin{vmatrix} e^{2x} \sin x & e^{2x} \cos x \\ (\cos x + 2 \sin x)e^{x} & (2 \cos x - \sin x)e^{2x} \end{vmatrix}$

$= e^{4x} \neq 0,$

the set is linearly independent.

(c) $y = C_1 e^{2x} \sin x + C_2 e^{2x} \cos x$

34. (a) $y = 1 \Rightarrow y''' = y'' = y' = 0$

$\Rightarrow y''' + 4y' = 0$

$y = 2 \cos 2x \Rightarrow y' = -4 \sin 2x, y'' = -8 \cos 2x, y''' = 16 \sin 2x$

$\Rightarrow y''' + 4y' = 0$

$y = 2 + \cos 2x \Rightarrow y' = -2 \sin 2x, y'' = -4 \cos 2x, y''' = 8 \sin 2x$

$\Rightarrow y''' + 4y' = 0$

(b) Because

$$W(1, 2 \cos 2x, 2 + \cos 2x) = \begin{vmatrix} 1 & 2 \cos 2x & 2 + \cos 2x \\ 0 & -4 \sin 2y & -2 \sin 2x \\ 0 & -8 \cos 2x & -4 \cos 2x \end{vmatrix}$$

$$= 16 \sin 2x \cos 2x - 16 \sin 2x \cos 2x$$

$$= 0,$$

the set is linearly dependent.

36. (a) $y = e^{-x} \Rightarrow y' = -e^{-x}, y'' = e^{-x}, y''' = -e^{-x} \Rightarrow y''' + 3y'' + 3y' + y = 0$

$y = xe^{-x} \Rightarrow y' = (1 - x)e^{-x}, y'' = (x - 2)e^{-x}, y''' = (3 - x)e^{-x} \Rightarrow y''' + 3y'' + 3y' + y = 0$

$y = x^2 e^{-x} \Rightarrow y' = \left(2x - x^2\right)e^{-x}, y'' = \left(x^2 - 4x + 2\right)e^{-x}, y''' = \left(-x^2 + 6x - 6\right)e^{-x} \Rightarrow y''' + 3y'' + 3y' + y = 0$

(b) Because

$$W\left(e^{-x}, xe^{-x}, x^2 e^{-x}\right) = \begin{vmatrix} e^{-x} & xe^{-x} & x^2 e^{-x} \\ -e^{-x} & (1 - x)e^{-x} & \left(2x - x^2\right)e^{-x} \\ e^{-x} & (x - 2)e^{-x} & \left(x^2 - 4x + 2\right)e^{-x} \end{vmatrix}$$

$$= e^{-3x} \begin{vmatrix} 1 & x & x^2 \\ -1 & 1 - x & 2x - x^2 \\ 1 & x - 2 & x^2 - 4x + 2 \end{vmatrix}$$

$$= e^{-3x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & -2 & -4x + 2 \end{vmatrix}$$

$$= 2e^{-3x} \neq 0,$$

the set is linearly independent.

(c) $y = C_1 e^{-x} + C_2 xe^{-x} + C_3 x^2 e^{-x}$

38. (a) $y = 1 \Rightarrow y'' = y''' = y^{(4)} = 0 \Rightarrow y^{(4)} - 2y''' + y'' = 0$

$y = x \Rightarrow y'' = y''' = y^{(4)} = 0 \Rightarrow y^{(4)} - 2y''' + y'' = 0$

$y = e^x \Rightarrow y'' = y''' = y^{(4)} = e^x \Rightarrow y^{(4)} - 2y''' + y'' = 0$

$y = xe^x \Rightarrow y'' = (x + 2)e^x, y''' = (x + 3)e^x, y^{(4)} = (x + 4)e^x \Rightarrow y^{(4)} - 2y''' + y'' = 0$

(b) Because

$$W(1, x, e^x, xe^x) = \begin{vmatrix} 1 & x & e^x & xe^x \\ 0 & 1 & e^x & (x+1)e^x \\ 0 & 0 & e^x & (x+2)e^x \\ 0 & 0 & e^x & (x+3)e^x \end{vmatrix} = \begin{vmatrix} e^x & (x+2)e^x \\ e^x & (x+3)e^x \end{vmatrix} = e^{2x}(x+3) - e^{2x}(x+2) = e^{2x} \neq 0,$$

the set is linearly independent.

(c) $y = C_1 + C_2x + C_3e^x + C_4xe^x$

40. Proving that $\{y_1, y_2\}$ is linearly independent if and only if $W(y_1, y_2) \neq 0$ is equivalent to proving that $\{y_1, y_2\}$ is linearly dependent if and only if $W(y_1, y_2) = 0$.

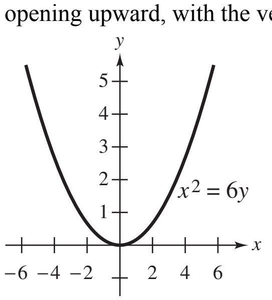
To prove one direction, assume $\{y_1, y_2\}$ is linearly dependent. By the Corollary to Theorem 4.8 on page 183, one of the functions is a scalar multiple of the other. So, $y_1 = cy_2$. Then

$$W(y_1, y_2) = W(y_1, cy_1) = \begin{vmatrix} y_1 & cy_1 \\ y_1' & cy_1' \end{vmatrix} = 0.$$

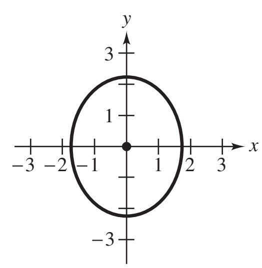
To prove the other direction, assume $W(y_1, y_2) = 0$. Then the column vectors $\begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$ and $\begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$ are linearly dependent (see Summary of Equivalent Conditions for Square Matrices, page 204). So, $\begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = c\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} \Rightarrow y_1 = cy_2$, and $\{y_1, y_2\}$ is linearly dependent.

42. No. For instance, consider the nonhomogeneous differential equation $y'' = 1$. Clearly, $y = x^2/2$ is a solution, whereas the scalar multiple $2(x^2/2)$ is not.

44. The graph of the equation $x^2 = 6y$ is a parabola opening upward, with the vertex at the origin.

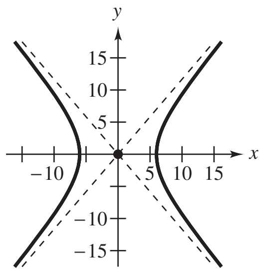


46. The graph of the equation $\dfrac{x^2}{3} + \dfrac{y^2}{5} = 1$ is an ellipse centered at the origin with major axis falling along the y-axis.



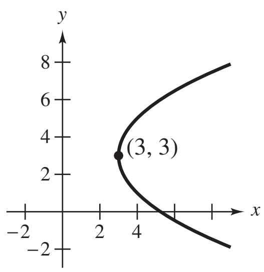
$5x^2 + 3y^2 - 15 = 0$

48. The graph of the equation is a hyperbola centered at the origin with transverse axis along the x-axis.



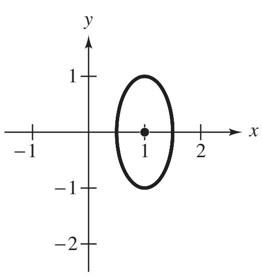
$$\frac{x^2}{36} - \frac{y^2}{49} = 1$$

50. The graph of the equation $(y - 3)^2 = 4(x - 3)$ is a parabola opening to the right, with the vertex at $(3, 3)$.



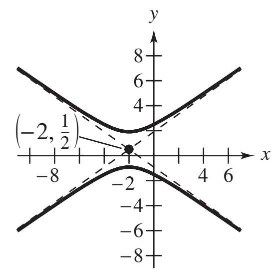
$$y^2 - 6y - 4x + 21 = 0$$

52. The graph of the equation $\dfrac{(x - 1)^2}{\frac{1}{4}} + y^2 = 1$ is an ellipse with the center at $(1, 0)$.



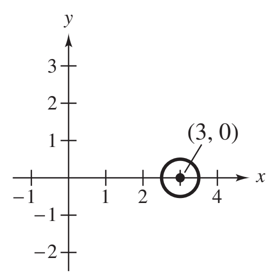
$$4x^2 + y^2 - 8x + 3 = 0$$

54. The graph of the equation $\dfrac{\left(y - \frac{1}{2}\right)^2}{2} - \dfrac{(x + 2)^2}{4} = 1$ is a hyperbola centered at $\left(-2, \frac{1}{2}\right)$, with a vertical transverse axis.



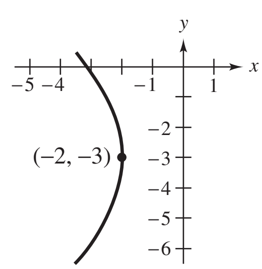
$$4y^2 - 2x^2 - 4y - 8x - 15 = 0$$

56. The graph of the equation $(x - 3)^2 + y^2 = \frac{1}{4}$ is a circle with the center at $(3, 0)$ and a radius of $\frac{1}{2}$.



$$4y^2 + 4x^2 - 24x + 35 = 0$$

58. The graph of the equation $(y + 3)^2 = 4(-2)(x + 2)$ is a parabola that opens to the left, with vertex at $(-2, -3)$.



$$y^2 + 8x + 6y + 25 = 0$$

60. $-2x^2 + 3xy + 2y^2 + 3 = 0$

$$\cot 2\theta = \frac{a - c}{b} = -\frac{4}{3} \Rightarrow \theta \approx -18.43°$$

Matches graph (b).

62. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

$$\cot 2\theta = \frac{a - c}{b} = \frac{1 - 4}{-4} = \frac{3}{4} \Rightarrow \theta \approx 26.57°$$

Matches graph (d).

64. Begin by finding the rotation angle θ, where

$\cot 2\theta = \dfrac{a-c}{b} = \dfrac{0-0}{1} = 0$, implying that

$\theta = \pi/4$.

So, $\sin\theta = 1/\sqrt{2}$ and $\cos\theta = 1/\sqrt{2}$. By substituting

$x = x'\cos\theta - y'\sin\theta = 1/\sqrt{2}(x'-y')$ and

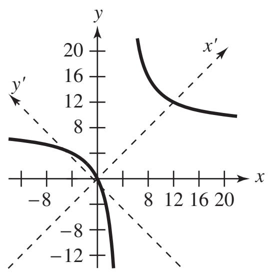
$y = x'\sin\theta + y'\cos\theta = 1/\sqrt{2}(x'+y')$ into

$xy - 8x - 4y = 0$ and simplifying, you obtain

$$\frac{(x')^2}{2} - \frac{12x'}{\sqrt{2}} - \frac{(y')^2}{2} + \frac{4y'}{\sqrt{2}} = 0.$$

In standard form, $\dfrac{\left(x'-6\sqrt{2}\right)^2}{64} - \dfrac{\left(y'-2\sqrt{2}\right)^2}{64} = 1$.

This is the equation of a hyperbola with a transverse axis along the x'-axis.



66. Begin by finding the rotation angle θ, where

$\cot 2\theta = \dfrac{a-c}{b} = \dfrac{1-1}{2} = 0 \Rightarrow \theta = \dfrac{\pi}{4}$.

So, $\sin\theta = \dfrac{1}{\sqrt{2}}$ and $\cos\theta = \dfrac{1}{\sqrt{2}}$. By substituting

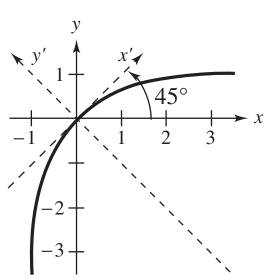
$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x'-y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x'+y')$$

into

$x^2 + 2xy + y^2 - 8x + 8y = 0$ and simplifying, you obtain $(x')^2 = -4\sqrt{2}y'$ or $y' = \dfrac{-1}{4\sqrt{2}}(x')^2$, which is a parabola.



68. Begin by finding the rotation angle θ, where

$\cot 2\theta = \dfrac{5-5}{-2} = 0$, implying that $\theta = \dfrac{\pi}{4}$.

So, $\sin\theta = 1/\sqrt{2}$ and $\cos\theta = 1/\sqrt{2}$. By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x'-y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x'+y')$$

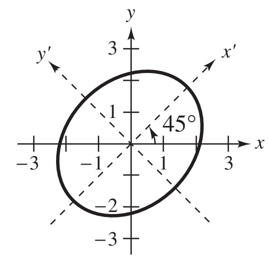
into

$5x^2 - 2xy + 5y^2 - 24 = 0$ and simplifying, you obtain

$4(x')^2 + 6(y')^2 - 24 = 0$.

In standard form, $\dfrac{(x')^2}{6} + \dfrac{(y')^2}{4} = 1$.

This is the equation of an ellipse with major axis along the x'-axis.



70. Begin by finding the rotation angle θ, where

$\cot 2\theta = \dfrac{a-c}{b} = \dfrac{5-5}{-6} = 0$, implying that $\theta = \dfrac{\pi}{4}$.

So, $\sin\theta = 1/\sqrt{2}$ and $\cos\theta = 1/\sqrt{2}$. By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x'-y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x'+y')$$

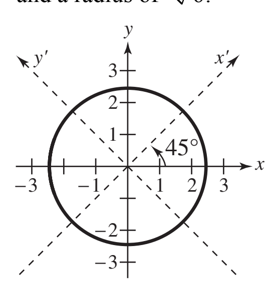
into

$5x^2 - 6xy + 5y^2 - 12 = 0$ and simplifying, you obtain

$2(x')^2 + 2(y')^2 - 12 = 0$.

In standard form, $(x')^2 + (y')^2 = 6$.

This is an equation of a circle with the center at $(0, 0)$ and a radius of $\sqrt{6}$.



72. Begin by finding the rotation angle θ, where

$$\cot 2\theta = \frac{a-c}{b} = \frac{7-5}{-2\sqrt{3}} = \frac{-1}{\sqrt{3}} \Rightarrow 2\theta = \frac{2\pi}{3},$$

implying that $\theta = \frac{\pi}{3}$.

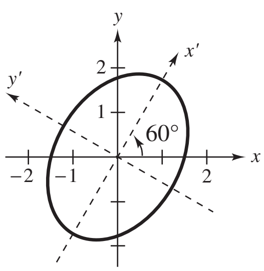
So, $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$. By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y'$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'$$

into $7x^2 - 2\sqrt{3}xy + 5y^2 = 16$ and simplifying, you obtain $\frac{(x')^2}{4} + \frac{(y')^2}{2} = 1$, which is an ellipse with major axis along the x'-axis.



74. Begin by finding the rotation angle θ, where

$$\cot 2\theta = \frac{1-3}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}, \text{ implying that } \theta = \frac{\pi}{3}.$$

So, $\sin\theta = \sqrt{3}/2$ and $\cos\theta = 1/2$. By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{2}\left(x' - \sqrt{3}y'\right)$$

and

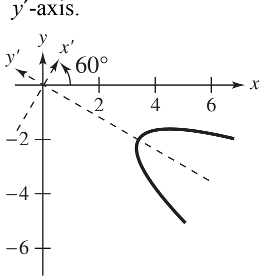
$$y = x'\sin\theta + y'\cos\theta = \frac{1}{2}\left(\sqrt{3}x' + y'\right)$$

into $x^2 + 2\sqrt{3}xy + 3y^2 - 2\sqrt{3}x + 2y + 16 = 0$ and simplifying, you obtain

$$4(x')^2 + 4y' + 16 = 0.$$

In standard form, $y' + 4 = -(x')^2$.

This is the equation of a parabola with axis on the y'-axis.



76. Begin by finding the rotation angle θ, where

$$\cot 2\theta = \frac{a-c}{b} = \frac{5-5}{-2} = 0, \text{ implying that } \theta = \frac{\pi}{4}.$$

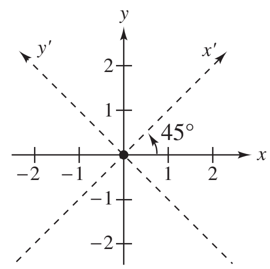
So, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}}$. By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x' + y') \text{ into}$$

$5x^2 - 2xy + 5y^2 = 0$ and simplifying, you obtain $4(x')^2 + 6(y')^2 = 0$, which is a single point, $(0, 0)$.



78. Begin by finding the rotation angle θ, where

$$\cot 2\theta = \frac{a-c}{b} = \frac{1-1}{-10} = 0, \text{ implying that } \theta = \frac{\pi}{4}.$$

So, $\sin\theta = 1/\sqrt{2}$ and $\cos\theta = 1/\sqrt{2}$. By substituting

$$x = x'\cos\theta - y\sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

and

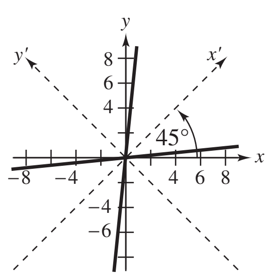
$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x' + y')$$

into

$x^2 - 10xy + y^2 = 0$ and simplifying, you obtain

$$6(y')^2 - 4(x')^2 = 0.$$

The graph of this equation is two lines $y' = \pm\frac{\sqrt{6}}{3}x'$.



80. Let θ satisfy $\cot 2\theta = (a - c)/b$. Substitute $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ into the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. To show that the xy-term will be eliminated, analyze the first three terms under this substitution.

$$\begin{aligned} ax^2 + bxy + cy^2 &= a(x'\cos\theta - y'\sin\theta)^2 + b(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + c(x'\sin\theta + y'\cos\theta)^2 \\ &= a(x')^2\cos^2\theta + a(y')^2\sin^2\theta - 2ax'y'\cos\theta\sin\theta \\ &\quad + b(x')^2\cos\theta\sin\theta + bx'y'\cos^2\theta - bx'y'\sin^2\theta - b(y')^2\cos\theta\sin\theta \\ &\quad + c(x')^2\sin^2\theta + c(y')^2\cos^2\theta + 2cx'y'\sin\theta\cos\theta. \end{aligned}$$

So, the new xy-terms are

$$\begin{aligned} -2ax'y'\cos\theta\sin\theta + bx'y'(\cos^2\theta - \sin^2\theta) + 2cx'y'\sin\theta\cos\theta &= x'y'[-a\sin 2\theta + b\cos 2\theta + c\sin 2\theta] \\ &= -x'y'[(a - c)\sin 2\theta - b\cos 2\theta]. \end{aligned}$$

But, $\cot 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta} = \dfrac{a - c}{b} \Rightarrow b\cos 2\theta = (a - c)\sin 2\theta$, which shows that the coefficient is zero.

82. (a) Set up the Wronskian with the given solutions and their derivatives. Then find the determinant. If the determinant is nonzero, the solutions are linearly independent.

(b) Use the substitutions $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, where θ is found by using the coefficients of the original equation in the formula $\cot 2\theta = \dfrac{a - c}{b}$.

Review Exercises for Chapter 4

2. (a) $\mathbf{u} + \mathbf{v} = (-1, 2, 1) + (0, 1, 1) = (-1, 3, 2)$

(b) $2\mathbf{v} = 2(0, 1, 1) = (0, 2, 2)$

(c) $\mathbf{u} - \mathbf{v} = (-1, 2, 1) - (0, 1, 1) = (-1, 1, 0)$

(d) $$\begin{aligned} 3\mathbf{u} - 2\mathbf{v} &= 3(-1, 2, 1) - 2(0, 1, 1) \\ &= (-3, 6, 3) - (0, 2, 2) = (-3, 4, 1) \end{aligned}$$

4. (a) $\mathbf{u} + \mathbf{v} = (0, 1, -1, 2) + (1, 0, 0, 2) = (1, 1, -1, 4)$

(b) $2\mathbf{v} = 2(1, 0, 0, 2) = (2, 0, 0, 4)$

(c) $\mathbf{u} - \mathbf{v} = (0, 1, -1, 2) - (1, 0, 0, 2) = (-1, 1, -1, 0)$

(d) $$\begin{aligned} 3\mathbf{u} - 2\mathbf{v} &= 3(0, 1, -1, 2) - 2(1, 0, 0, 2) \\ &= (0, 3, -3, 6) - (2, 0, 0, 4) = (-2, 3, -3, 2) \end{aligned}$$

6. $$\begin{aligned} \mathbf{x} &= \tfrac{1}{3}[-2\mathbf{u} + \mathbf{v} - 2\mathbf{w}] \\ &= \tfrac{1}{3}[-2(1, -1, 2) + (0, 2, 3) - 2(0, 1, 1)] \\ &= \tfrac{1}{3}[(-2, 2, -4) + (0, 0, 1)] \\ &= \tfrac{1}{3}(-2, 2, -3) = \left(-\tfrac{2}{3}, \tfrac{2}{3}, -1\right) \end{aligned}$$

8. $$\begin{aligned} 3\mathbf{u} + 2\mathbf{x} &= \mathbf{w} - \mathbf{v} \\ 2\mathbf{x} &= -3\mathbf{u} - \mathbf{v} + \mathbf{w} \\ \mathbf{x} &= -\tfrac{3}{2}\mathbf{u} - \tfrac{1}{2}\mathbf{v} + \tfrac{1}{2}\mathbf{w} \\ &= -\tfrac{3}{2}(1, -1, 2) - \tfrac{1}{2}(0, 2, 3) + \tfrac{1}{2}(0, 1, 1) \\ &= \left(-\tfrac{3}{2}, \tfrac{3}{2}, -3\right) - \left(0, 1, \tfrac{3}{2}\right) + \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right) \\ &= \left(-\tfrac{3}{2} - 0 + 0, \tfrac{3}{2} - 1 + \tfrac{1}{2}, -3 - \tfrac{3}{2} + \tfrac{1}{2}\right) \\ &= \left(-\tfrac{3}{2}, 1, -4\right) \end{aligned}$$

10. To write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, solve the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{v}$$

for c_1, c_2, and c_3. This vector equation corresponds to the system

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 4 \\ 2c_1 \qquad\qquad &= 4 \\ 3c_1 + c_2 \qquad &= 5. \end{aligned}$$

The solution of this system is $c_1 = 2$, $c_2 = -1$, and $c_3 = 0$. So, $\mathbf{v} = 2\mathbf{u}_1 - \mathbf{u}_2$.

12. To write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$, solve the equation

$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{v}$

for c_1, c_2, and c_3. This vector equation corresponds to the system of linear equations

$$\begin{aligned} c_1 - c_2 \qquad &= 4 \\ -2c_1 + 2c_2 - c_3 &= -13 \\ c_1 + 3c_2 - c_3 &= -5 \\ c_1 + 2c_2 - c_3 &= -4. \end{aligned}$$

The solution of this system is $c_1 = 3, c_2 = -1$, and $c_3 = 5$. So, $\mathbf{v} = 3\mathbf{u}_1 - \mathbf{u}_2 + 5\mathbf{u}_3$.

14. The zero vector is the zero polynomial $p(x) = 0$. The additive inverse of a vector in P_8 is

$-\left(a_0 + a_1x + a_2x^2 + \cdots + a_8x^8\right) = -a_0 - a_1x - a_2x^2 - \cdots - a_8x^8.$

16. The zero vector is

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

The additive inverse of

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is $\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}.$

18. W is not a subspace of R^2. For instance, $(2, 1) \in W$ and $(3, 2) \in W$, but their sum $(5, 3) \in W$. So, W is not closed under addition (nor scalar multiplication).

20. W is not a subspace of R^2. For instance $(1, 3) \in W$ and $(2, 12) \in W$, but their sum $(3, 15) \notin W$. So, W is not closed under addition (nor scalar multiplication).

22. W is not a subspace of R^3, because it is not closed under scalar multiplication. For instance $(1, 1, 1) \in W$ and $-2 \in R$, but $-2(1, 1, 1) = (-2, -2, -2) \notin W$.

24. Because W is a nonempty subset of $C[-1, 1]$, you need only check that W is closed under addition and scalar multiplication. If f and g are in W, then

$f(-1) = g(-1) = 0$, and

$(f + g)(-1) = f(-1) + g(-1) = 0$, which implies that $f + g \in W$. Similarly, if c is a scalar, then $cf(-1) = c0 = 0$, which implies that $cf \in W$. So, W is a subspace of $C[-1, 1]$.

26. (a) W is a subspace of R^3, because W is nonempty $\left((0, 0, 0) \in W\right)$ and W is closed under addition and scalar multiplication.

For if (x_1, x_2, x_3) and (y_1, y_2, y_3) are in W, then $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$. Because $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ satisfies $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = 0$, W is closed under addition. Similarly, $c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$ satisfies $cx_1 + cx_2 + cx_3 = 0$, showing that W is closed under scalar multiplication.

(b) W is not closed under addition or scalar multiplication, so it is not a subspace of R^3. For example, $(1, 0, 0) \in W$, and yet $2(1, 0, 0) = (2, 0, 0) \notin W$.

28. (a) To find out whether S spans R^3, form the vector equation

$$c_1(4, 0, 1) + c_2(0, -3, 2) + c_3(5, 10, 0) = (u_1, u_2, u_3).$$

This yields the system of equations

$$\begin{aligned} 4c_1 \qquad\qquad + \; 5c_3 &= u_1 \\ -3c_2 + 10c_3 &= u_2 \\ c_1 + 2c_2 \qquad\qquad &= u_3. \end{aligned}$$

This system has a unique solution for every (u_1, u_2, u_3) because the determinant of the coefficient matrix is not zero. So, S spans R^3.

(b) Solving the same system in (a) with $(u_1, u_2, u_3) = (0, 0, 0)$ yields the trivial solution. So, S is linearly independent.

(c) Because S is linearly independent and spans R^3, it is a basis for R^3.

30. (a) To find out whether S spans R^3, form the vector equation

$$c_1(2, 0, 1) + c_2(2, -1, 1)) + c_3(4, 2, 0) = (u_1, u_2, u_3).$$

This yields the system of linear equations

$$\begin{aligned} 2c_1 + 2c_2 + 4c_3 &= u_1 \\ -c_2 + 2c_3 &= u_2 \\ c_1 + c_2 \qquad\quad &= u_3. \end{aligned}$$

This system has a unique solution for every (u_1, u_2, u_3) because the determinant of the coefficient matrix is not zero. So, S spans R^3.

(b) Solving the same system in part (a) with $(u_1, u_2, u_3) = (0, 0, 0)$ yields the trivial solution. So, S is linearly independent.

(c) Because S is linearly independent and S spans R^3, it is a basis for R^3.

32. (a) The set

$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\begin{aligned} \mathbf{u} &= u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) \\ &= (u_1, u_2, u_3). \end{aligned}$$

(b) S is not linearly independent because

$$2(1, 0, 0) - (0, 1, 0) + 0(0, 0, 1) = (2, -1, 0).$$

(c) S is not a basis for R^3 because S is not linearly independent.

34. S has three vectors, so you need only check that S is linearly independent.

Form the vector equation

$$c_1(1) + c_2(t) + c_3(1 + t^2) = 0 + 0t + 0t^2$$

which yields the homogeneous system of linear equations

$$\begin{aligned} c_1 \qquad + c_3 &= 0 \\ c_2 \qquad &= 0 \\ c_3 &= 0. \end{aligned}$$

This system has only the trivial solution. So, S is linearly independent and S is a basis for P_2.

36. S has four vectors, so you need only check that S is linearly independent.

Form the vector equation

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + c_3\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which yields the homogeneous system of linear equations

$$\begin{aligned} c_1 - c_2 + 2c_3 + c_4 &= 0 \\ c_3 + c_4 &= 0 \\ c_2 + c_3 \qquad &= 0 \\ c_1 + c_2 \qquad + c_4 &= 0. \end{aligned}$$

This system has only the trivial solution. So, S is linearly independent and S is a basis for $M_{2,2}$.

38. (a) The system given by $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $(0, 0)$. So, the solution space is $\{(0, 0)\}$, which does not have a basis.

(b) The nullity is 0.

Note that $\text{rank}(A) + \text{nullity}(A) = 2 + 0 = 2 = n$.

(c) The rank of A is 2 (the number of nonzero row vectors in the reduced row-echelon matrix).

40. (a) The system given by $A\mathbf{x} = \mathbf{0}$ has solutions of the form $(2t, 5t, t, t)$, where t is any real number. So, a basis for the solution space of $A\mathbf{x} = \mathbf{0}$ is $\{(2, 5, 1, 1)\}$.

(b) The nullity of A is 1.

Note that $\text{rank}(A) + \text{nullity}(A) = 3 + 1 = 4 = n$.

(c) The rank of A is 3 (the number of nonzero row vectors in the reduced row-echelon matrix).

42. (a) The system given by $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $(0, 0, 0, 0)$. So, the solution space is $\{(0, 0, 0, 0)\}$, which does not have a basis.

(b) The nullity is 0.

Note that $\text{rank}(A) + \text{nullity}(A) = 4 + 0 = 4 = n$.

(c) The rank of A is 4 (the number of nonzero row vectors in the reduced row-echelon matrix).

44. (a) Using Gauss-Jordan elimination, the matrix reduces to

$$\begin{bmatrix} 1 & 0 & \frac{26}{11} \\ 0 & 1 & \frac{8}{11} \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the rank is 2.

(b) A basis for the row space is $\left\{\left(1, 0, \frac{26}{11}\right), \left(0, 1, \frac{8}{11}\right)\right\}$.

46. (a) Using Gauss-Jordan elimination, the matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the rank is 3.

(b) A basis for the row space is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

48. (a) This system has solutions of the form $\left(1 - \frac{3}{2}s - \frac{1}{2}t + 2r, s, t, r\right)$, where r, s, and t are any real numbers. A basis for the solution space is $\{(-3, 2, 0, 0), (-1, 0, 2, 0), (2, 0, 0, 1)\}$.

(b) The dimension of the solution space is 3, the number of vectors in a basis for the solution space.

50. (a) This system has solutions of the form $\left(0, -\frac{3}{2}t, -t, t\right)$, where t is any real number. A basis for the solution space is $\left\{\left(0, -\frac{3}{2}, -1, 1\right)\right\}$.

(b) The dimension of the solution space is 1, the number of vectors in a basis.

52. Because $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, write $\mathbf{x}$ as

$\mathbf{x} = 1(2, 0) + 1(3, 3) = (5, 3)$. Because $(5, 3) = 5(1, 0) + 3(0, 1)$, the coordinate vector of $\mathbf{x}$ relative to the standard basis is

$$[\mathbf{x}]_S = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

54. Because $[\mathbf{x}]_B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, write $\mathbf{x}$ as

$\mathbf{x} = 4(2, 4) - 7(-1, 1) = (15, 9)$. Because $(15, 9) = 15(1, 0) + 9(0, 1)$, the coordinate vector of $\mathbf{x}$ relative to the standard basis is

$$[\mathbf{x}]_S = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

56. Because $[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$, write $\mathbf{x}$ as

$\mathbf{x} = 4(1, 0, 1) + 0(0, 1, 0) + 2(0, 1, 1) = (4, 2, 6)$.

Because $(4, 2, 6) = 4(1, 0, 0) + 2(0, 1, 0) + 6(0, 0, 1)$, the coordinate vector of $\mathbf{x}$ relative to the standard basis is

$$[\mathbf{x}]_S = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}.$$

58. To find $[\mathbf{x}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, solve the equation

$c_1(2, 2) + c_2(0, -1) = (-1, 2)$.

The resulting system of linear equations is

$$\begin{aligned} 2c_1 \phantom{{}- c_2} &= -1 \\ 2c_1 - c_2 &= 2 \end{aligned}$$

So, $c_1 = -\frac{1}{2}$ and $c_2 = -3$, and you have

$$[\mathbf{x}]_{B'} = \begin{bmatrix} -\frac{1}{2} \\ -3 \end{bmatrix}.$$

60. To find $[\mathbf{x}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, solve the equation

$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(1, 1, 1) = (4, -2, 9)$.

Forming the corresponding linear system, the solution is $c_1 = -5$, $c_2 = -11$, and $c_3 = 9$. So,

$$[\mathbf{x}]_{B'} = \begin{bmatrix} -5 \\ -11 \\ 9 \end{bmatrix}.$$

62. To find $[\mathbf{x}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$, solve the equation

$$c_1(1, -1, 2, 1) + c_2(1, 1, -4, 3) + c_3(1, 2, 0, 3) + c_4(1, 2, -2, 0) = (5, 3, -6, 2).$$

The resulting system of linear equations is

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 5 \\ -c_1 + c_2 + 2c_3 + 2c_4 &= 3 \\ 2c_1 - 4c_2 \qquad\quad - 2c_4 &= -6 \\ c_1 + 3c_2 + 3c_3 \qquad &= 2. \end{aligned}$$

So, $c_1 = 2, c_2 = 1, c_3 = -1$, and $c_4 = 3$, and you have

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

64. Begin by forming

$$[B' \ \ B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 0 & -1 & 1 \end{bmatrix}.$$

Then use Gauss-Jordan elimination to obtain

$$[I_2 \ \ P^{-1}] = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}.$$

So,

$$P^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}.$$

66. Begin by forming

$$[B' \ \ B] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then use Gauss-Jordan elimination to obtain

$$[I_3 \ \ P^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & \frac{3}{2} & \frac{3}{2} \end{bmatrix}.$$

So,

$$P^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} \end{bmatrix}.$$

68. Begin by forming

$$[B^1 \ B] = \begin{bmatrix} 1 & -2 & 1 & 1 & 3 & 3 \\ -1 & 1 & 0 & 1 & 4 & 3 \\ \frac{2}{3} & 0 & -\frac{1}{3} & 1 & 3 & 4 \end{bmatrix}.$$

Then use Gauss-Jordan elimination to obtain

$$[I_3 \ P^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 6 & 20 & 21 \\ 0 & 1 & 0 & 7 & 24 & 24 \\ 0 & 0 & 1 & 9 & 31 & 30 \end{bmatrix}.$$

So,

$$P^{-1} = \begin{bmatrix} 6 & 20 & 21 \\ 7 & 24 & 24 \\ 9 & 31 & 30 \end{bmatrix}.$$

70. (a) $[B' \ \ B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 1 \end{bmatrix} = [I \ \ P^{-1}]$

(b) $[B \ \ B'] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} = [I \ \ P]$

(c) $P^{-1}P = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

72. (a) $[B' \;\; B] = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 1 \\ -1 & 2 & 2 & 1 & 1 & -1 \\ 2 & -1 & 2 & -1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix} = [I \;\; P^{-1}]$

(b) $[B \;\; B'] = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} = [I \;\; P]$

(c) $P^{-1}P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} -2 & 1 & -2 \\ 2 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$

74. (a) Because W is a nonempty subset of V, you need to only check that W is closed under addition and scalar multiplication. If $f, g \in W$, then $f' = 4f$ and $g' = 4g$. So,

$(f + g)' = f' + g' = 4f + 4g = 4(f + g),$

which shows that $f + g \in W$. Finally, if c is a scalar, then $(cf)' = (cf') = c(4f) = 4(cf)$, which implies that $cf \in W$.

(b) V is not closed under addition nor scalar multiplication. For instance, let $f = e^x - 1 \in U$. Note that $2f = 2e^x - 2 \notin U$ because

$(2f)' = 2e^x \neq (2f) + 1 = 2e^x - 1.$

76. Suppose, on the contrary, that A and B are linearly dependent. Then $B = cA$ for some scalar c. So, $(cA)^T = B^T = -B$, which implies that $cA = -B$. So, $B = O$, a contradiction.

78. Because $-(\mathbf{v}_1 - 2\mathbf{v}_2) - (2\mathbf{v}_2 - 3\mathbf{v}_3) = 3\mathbf{v}_3 - \mathbf{v}_1$, the set is linearly dependent.

80. S is a nonempty subset of R^n, so you need only show closure under addition and scalar multiplication. Let $\mathbf{x}, \mathbf{y} \in S$. Then $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$. So,

$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x} + \mathbf{y})$, which implies that $\mathbf{x} + \mathbf{y} \in S$. Finally, for any scalar c, $A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$, which implies that $c\mathbf{x} \in S$.

If $\lambda = 3$, then solve for $\mathbf{x}$ in the equation $A\mathbf{x} = \lambda\mathbf{x} = 3\mathbf{x}$, or $A\mathbf{x} - 3\mathbf{x} = \mathbf{0}$, or $(A - 3I_3)\mathbf{x} = \mathbf{0}$.

$$\left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to this homogeneous system is $x_1 = t, x_2 = 0$, and $x_3 = 0$, where t is any real number. So, a basis for S is $\{(1, 0, 0)\}$, and the dimension of S is 1.

82. From Exercise 81, you see that a set of functions $\{f_1, \ldots, f_n\}$ can be linearly independent in $C[a, b]$ and linearly dependent in $C[c, d]$, where $[a, b]$ and $[c, d]$ are different domains.

84. (a) False. This set is not closed under addition or scalar multiplication:

$(0, 1, 1) \in W$, but $2(0, 1, 1) = (0, 2, 2)$ is not in W.

(b) True. See "Definition of Basis," on page 186.

(c) False. For example, let $A = I_3$ be the 3×3 identity matrix. It is invertible and the rows of A form the standard basis for R3 and, in particular, the rows of A are linearly independent.

86. (a) True. It is a nonempty subset of R^2, and it is closed under addition and scalar multiplication.

(b) False. These operations only preserve the linear relationships among the columns.

88. (a) Because $y' = y'' = y''' = y^{(4)} = e^x$, you have

$y^{(4)} - y = e^x - e^x = 0.$

Therefore, e^x is a solution.

(b) Because $y' = -e^{-x}, y'' = e^{-x}, y''' = -e^{-x}$, and $y^{(4)} = e^{-x}$, you have

$y^{(4)} - y = e^{-x} - e^{-x} = 0.$

Therefore e^{-x} is a solution.

(c) Because $y' = -\sin x, y'' = -\cos x, y''' = \sin x$, and $y^{(4)} = \cos x$, you have

$y^{(4)} - y = \cos x - \cos x = 0.$

Therefore, $\cos x$ is a solution.

(d) Because $y' = \cos x, y'' = -\sin x, y''' = -\cos x$, and $y^{(4)} = \sin x$, you have

$y^{(4)} - y = \sin x - \sin x = 0.$

Therefore, $\sin x$ is a solution.

90. (a) Because $y'' = -25 \cos 5x - 25 \sin 5x$, you have

$$\begin{aligned} y'' + 25y &= -25 \cos 5x - 25 \sin 5x + 25(\sin 5x + \cos 5x) \\ &= -25 \cos 5x - 25 \sin 5x + 25 \sin 5x + 25 \cos 5x \\ &= 0 \end{aligned}$$

Therefore, $\sin 5x + \cos 5x$ is a solution.

(b) Because $y'' = -5 \sin x - 5 \cos x$, you have

$$\begin{aligned} y'' + 25y &= -5 \sin x - 5 \cos x + 25(5 \sin x + 5 \cos x) \\ &= -5 \sin x - 5 \cos x + 125 \sin x + 125 \cos x \\ &= 120 \sin x + 120 \cos x \\ &\neq 0 \end{aligned}$$

Therefore, $5 \sin x + 5 \cos x$ is *not* a solution.

(c) Because $y'' = -25 \sin 5x$, you have

$$\begin{aligned} y'' + 25y &= -25 \sin 5x + 25(\sin 5x) \\ &= -25 \sin 5x + 25 \sin 5x \\ &= 0 \end{aligned}$$

Therefore, $\sin 5x$ is a solution.

(d) Because $y'' = -25 \cos 5x$, you have

$$\begin{aligned} y'' + 25y &= -25 \cos 5x + 25(\cos 5x) \\ &= -25 \cos 5x + 25 \cos 5x \\ &= 0 \end{aligned}$$

Therefore, $\cos 5x$ is a solution.

92. $W(2, x^2, 3 + x) = \begin{vmatrix} 2 & x^2 & 3 + x \\ 0 & 2x & 1 \\ 0 & 2 & 0 \end{vmatrix}$

$= -4$

94. $W(x, \sin^2 x, \cos^2 x) = \begin{vmatrix} x & \sin^2 x & \cos^2 x \\ 1 & 2 \sin x \cos x & -2 \sin x \cos x \\ 0 & 4 \cos^2 x - 2 & 2 - 4 \cos^2 x \end{vmatrix} = 4 \cos^2 x - 2$

96. (a) $y = e^{-3x} \Rightarrow y' = -3e^{-3x}, y'' = 9e^{-3x} \Rightarrow y'' + 6y' + 9y = 0$

$y = 3e^{-3x} \Rightarrow y' = -9e^{-3x}, y'' = 27e^{-3x} \Rightarrow y'' + 6y' + 9y = 0$

(b) The Wronskian of this set is

$W(e^{-3x}, 3e^{-3x}) = \begin{vmatrix} e^{-3x} & 3e^{-3x} \\ -3x^{-3x} & -9e^{-3x} \end{vmatrix} = -9e^{-6x} + 9e^{-6x} = 0 = 0.$

Because $W(e^{-3x}, 3e^{-3x}) = 0$, the set is linearly dependent.

98. (a) $y = \sin 3x \Rightarrow y'' = -9 \sin 3x \Rightarrow y'' + 9y = 0$

$y = \cos 3x \Rightarrow y'' = -9 \cos 3x \Rightarrow y'' + 9y = 0$

(b) The Wronskian of this set is

$W(\sin 3x, \cos 3x) = \begin{vmatrix} \sin 3x & \cos 3x \\ 3 \cos 3x & -3 \sin 3x \end{vmatrix} = -3 \sin^2 3x - 3 \cos^2 3x = -3.$

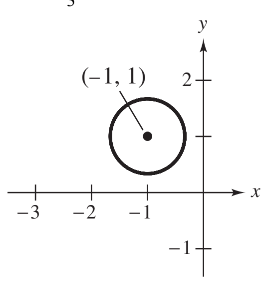
Because $W(\sin 3x, \cos 3x) \neq 0$ the set is linearly independent.

(c) $y = C_1 \sin 3x + C_2 \cos 3x$

100. Begin by completing the square.

$$9x^2 + 18x + 9y^2 - 18y = -14$$
$$9(x^2 + 2x + 1) + 9(y^2 - 2y + 1) = -14 + 9 + 9$$
$$(x + 1)^2 + (y - 1)^2 = \tfrac{4}{9}$$

This is the equation of a circle centered at $(-1, 1)$ with a radius of $\frac{2}{3}$.

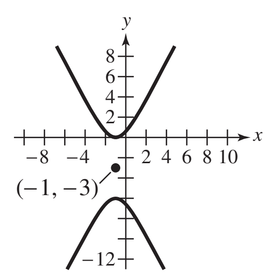


$9x^2 + 9y^2 + 18x - 18y + 14 = 0$

102. Begin by completing the square.

$$4x^2 + 8x - y^2 - 6y = -4$$
$$4(x^2 + 2x + 1) - (y^2 + 6y + 9) = -4 + 4 - 9$$
$$\frac{(y + 3)^2}{9} - \frac{(x + 1)^2}{\frac{9}{4}} = 1$$

This is the equation of a hyperbola centered at $(-1, -3)$.



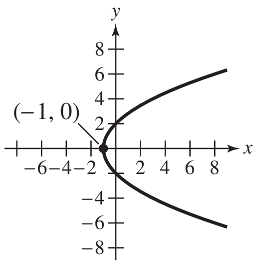
$4x^2 - y^2 + 8x - 6y + 4 = 0$

104. $y^2 - 4x - 4 = 0$

$$y^2 = 4x + 4$$

$$y^2 = 4(x + 1)$$

This is the equation of a parabola with vertex $(-1, 0)$.



$y^2 - 4x - 4 = 0$

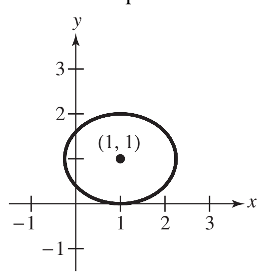
106. Begin by completing the square.

$$16x^2 - 32x + 25y^2 - 50y = -16$$

$$16(x^2 - 2x + 1) + 25(y^2 - 2y + 1) = -16 + 16 + 25$$

$$\frac{(x-1)^2}{\frac{25}{16}} + (y-1)^2 = 1$$

This is the equation of an ellipse centered at (1, 1).



$16x^2 + 25y^2 - 32x - 50y + 16 = 0$

108. From the equation

$$\cot 2\theta = \frac{a-c}{b} = \frac{9-9}{4} = 0,$$

you find that the angle of rotation is $\theta = \frac{\pi}{4}$. Therefore,

$\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}}$.

By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x' + y')$$

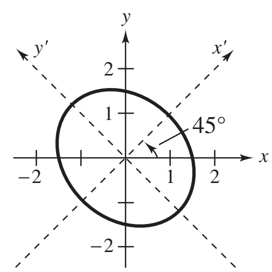
into $9x^2 + 4xy + 9y^2 - 20 = 0$, you obtain

$$11(x')^2 + 7(y')^2 = 20.$$

In standard form,

$$\frac{(x')^2}{\frac{20}{11}} + \frac{(y')^2}{\frac{20}{7}} = 1$$

which is the equation of an ellipse with major axis along the y'-axis.

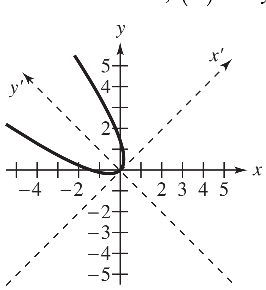


110. From the equation $\cot 2\theta = \frac{a-c}{b} = \frac{1-1}{2} = 0$, you find the angle of rotation to be $\theta = \frac{\pi}{4}$.

Therefore, $\sin\theta = \frac{\sqrt{2}}{2}$ and $\cos\theta = \frac{\sqrt{2}}{2}$. By substituting $x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{2}}{2}(x' - y')$ and

$y = x'\sin\theta + y'\cos\theta = \frac{\sqrt{2}}{2}(x' + y')$ into $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y = 0$, you obtain $2(x')^2 - 2y' = 0$.

In standard form, $(x')^2 = y'$ which is the equation of a parabola with vertex $(0, 0)$.



Project Solutions for Chapter 4

1 Solutions of Linear Systems

1. Because $(-2, -1, 1, 1)$ is a solution of $A\mathbf{x} = \mathbf{0}$, so is any multiple $-2(-2, -1, 1, 1) = (4, 2, -2, -2)$ because the solution space is a subspace.

2. The solutions of $A\mathbf{x} = \mathbf{0}$ form a subspace, so any linear combination $2\mathbf{x}_1 - 3\mathbf{x}_2$ of solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ is again a solution.

3. Let the first system be $A\mathbf{x} = \mathbf{b}_1$. Because it is consistent, $\mathbf{b}_1$ is in the column space of A. The second system is $A\mathbf{x} = \mathbf{b}_2$, and $\mathbf{b}_2$ is a multiple of $\mathbf{b}_1$, so it is in the column space of A as well. So, the second system is consistent.

4. $2\mathbf{x}_1 - 3\mathbf{x}_2$ is *not* a solution (unless $\mathbf{b} = 0$). The set of solutions to a nonhomogeneous system is not a subspace. If $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$, then

$A(2\mathbf{x}_1 - 3\mathbf{x}_2) = 2A\mathbf{x}_1 - 3A\mathbf{x}_2 = 2\mathbf{b} - 3\mathbf{b} = -\mathbf{b} \neq \mathbf{b}$.

5. Yes, $\mathbf{b}_1$ and $\mathbf{b}_2$ are in the column space of A, therefore so is $\mathbf{b}_1 + \mathbf{b}_2$.

2 Direct Sum

1. Basis for U: $\{(1, 0, 1), (0, 1, -1)\}$

Basis for W: $\{(1, 0, 1)\}$

Basis for Z: $\{(1, 1, 1)\}$

$U + W = U$ because $W \subseteq U$

$U + Z = R^3$ because $\{(1, 0, 1), (0, 1, -1), (1, 1, 1)\}$ is a basis for R^3.

$W + Z = \text{span}\{(1, 0, 1), (1, 1, 1)\} = \text{span}\{(1, 0, 1), (0, 1, 0)\}$

2. Suppose $\mathbf{u}_1 + \mathbf{w}_1 = \mathbf{u}_2 + \mathbf{w}_2$, which implies $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{w}_2 - \mathbf{w}_1$.

Because $\mathbf{u}_1 - \mathbf{u}_2 \in U \cap W$ and $\mathbf{w}_2 - \mathbf{w}_1 \in U \cap W$, and $U \cap W = \{\mathbf{0}\}$, $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

$U \oplus Z$ and $W \oplus Z$ are direct sums.

3. Let $\mathbf{v} \in V$, then $\mathbf{v} = \mathbf{u} + \mathbf{w}, \mathbf{u} \in U, \mathbf{w} \in W$. Then $\mathbf{v} = (c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k) + (d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m)$,

and $\mathbf{v}$ is in the span of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{w}_1, \ldots, \mathbf{w}_m\}$. To show that this set is linearly independent, suppose

$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k + d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m = \mathbf{0}$

$\Rightarrow c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = -(d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m)$

But $U \cap W \neq \{\mathbf{0}\} \Rightarrow c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$ and $d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m = \mathbf{0}$.

Because $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are linearly independent,

$c_1 = \cdots = c_k = 0$ and $d_1 = \cdots = d_m = 0$.

4. Basis for U: $\{(1, 0, 0), (0, 0, 1)\}$

Basis for W: $\{(0, 1, 0), (0, 0, 1)\}$

$U + W$ is spanned by $\{(1, 0, 0), (0, 0, 1), (0, 1, 0)\} \Rightarrow U + W = R^3$. This is not a direct sum because $(0, 0, 1) \in U \cap W$.

$\dim U = 2, \dim W = 2, \dim(U \cap W) = 1$

$$\begin{array}{ccccccc} \dim U & + & \dim W & = & \dim(U + W) & + & \dim(U \cap W). \\ 2 & + & 2 & = & 3 & + & 1 \end{array}$$

In general, $\dim U + \dim W = \dim(U + W) + \dim(U \cap W)$.

5. No, $\dim U + \dim W = 2 + 2 = 4$, then $\dim(U + W) + \dim(U \cap W) = \dim(U + W) = 4$, which is impossible in R^3.