Numerical Integration

ASEN 3111

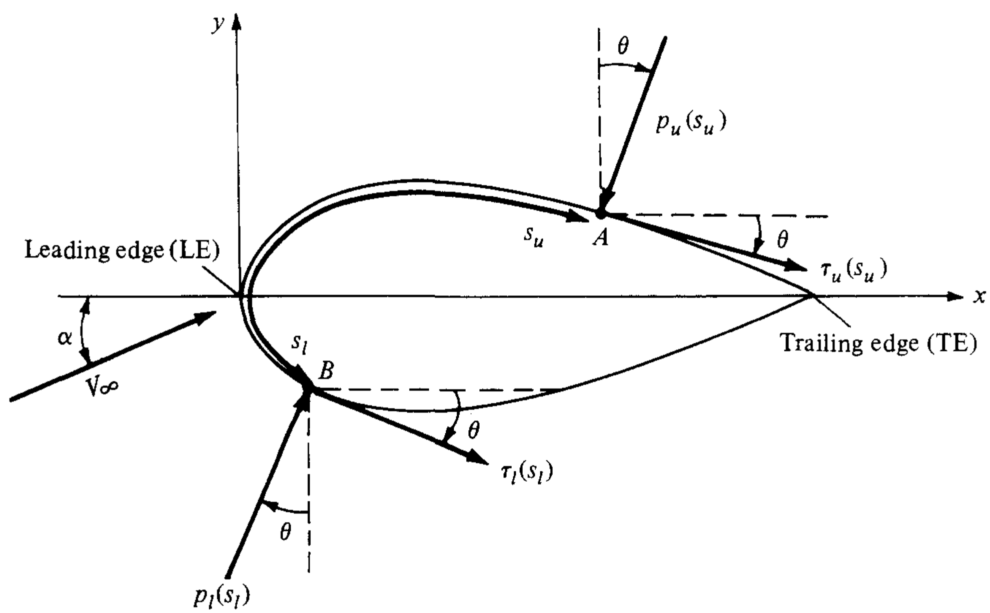
Numerical Integration for Lift and Drag

Question: How do we compute the integrals to the right for arbitrary geometry and pressure and shear stress distributions?

$$\boxed{\begin{aligned} L &= N\cos\alpha - A\sin\alpha \\ D &= N\sin\alpha + A\cos\alpha \end{aligned}}$$

$$\boxed{\begin{aligned} N' &= -\int_{\mathrm{LE}}^{\mathrm{TE}} (p_u\cos\theta + \tau_u\sin\theta)\,ds_u + \int_{\mathrm{LE}}^{\mathrm{TE}} (p_l\cos\theta - \tau_l\sin\theta)\,ds_l \\ A' &= \int_{\mathrm{LE}}^{\mathrm{TE}} (-p_u\sin\theta + \tau_u\cos\theta)\,ds_u + \int_{\mathrm{LE}}^{\mathrm{TE}} (p_l\sin\theta + \tau_l\cos\theta)\,ds_l \end{aligned}}$$

Answer: Via *numerical integration* or *quadrature*!

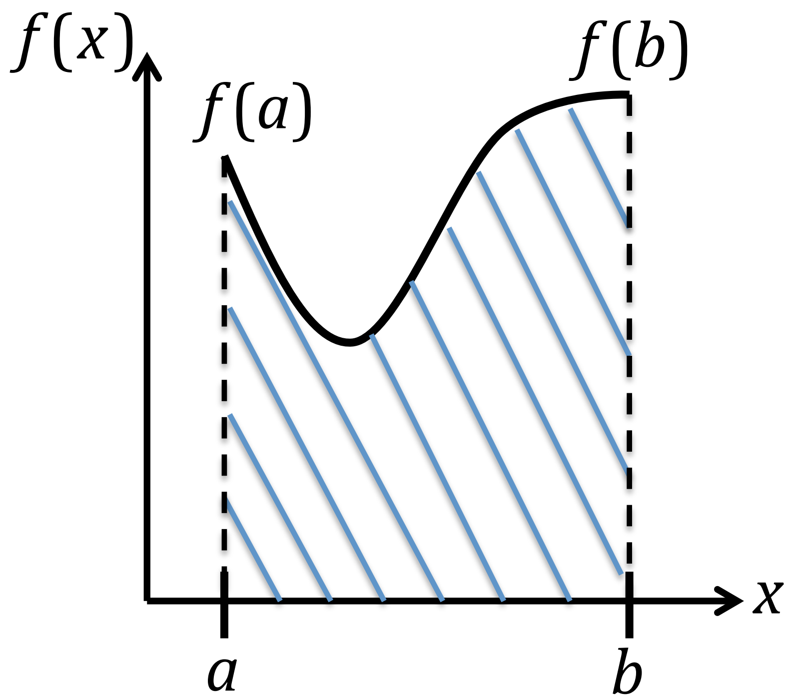


Integration of 1D Definite Integrals

Consider the 1D definite integral:

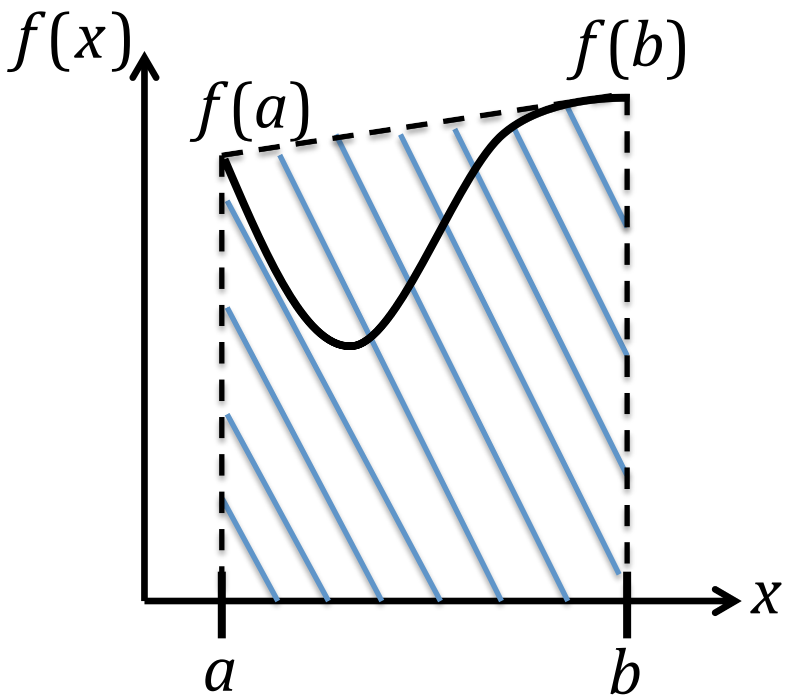
$$\int_a^b f(x)dx$$

We may view this integration as "taking the area under the curve":



The Trapezoidal Rule

The trapezoidal rule approximates the area under the curve as a trapezoid:

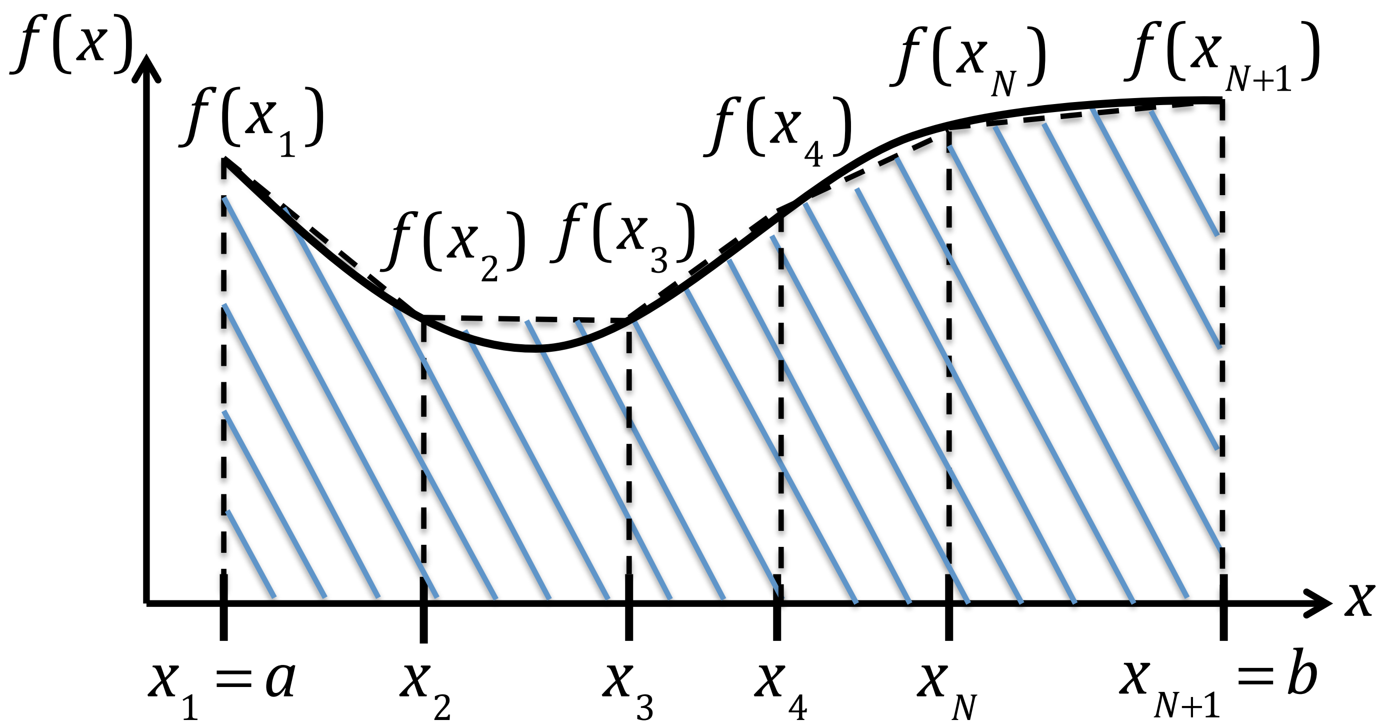


This yields the approximation:

$$\int_a^b f(x)dx \approx \underbrace{(b-a)}_{\text{BASE}}\underbrace{\left[\frac{f(a)+f(b)}{2}\right]}_{\text{HEIGHT}}$$

The Composite Trapezoidal Rule

Often more resolution is needed to approximate the integral. Then, the domain is discretized into N panels or $N + 1$ grid points and the trapezoidal rule is applied to each panel:



The Composite Trapezoidal Rule

The composite trapezoidal rule results from summing the trapezoidal rules over each panel:

$$\int_a^b f(x)dx \approx \sum_{k=1}^{N}\left(x_{k+1} - x_k\right)\left[\frac{f(x_{k+1}) + f(x_k)}{2}\right]$$

The grid points may be arbitrarily spaced provided they are ordered, but they are most often taken to be equi-spaced in which case each panel has the same width.

Error for The Composite Trapezoidal Rule

SECOND-ORDER ACCURACY

$$\left|\int_a^b f(x)dx - \sum_{k=1}^{N}\left(x_{k+1}-x_k\right)\left[\frac{f(x_{k+1})+f(x_k)}{2}\right]\right| \sim O\left(N^{-2}\right)$$

If the number of panels is doubled,
the error decreases by a factor of four!

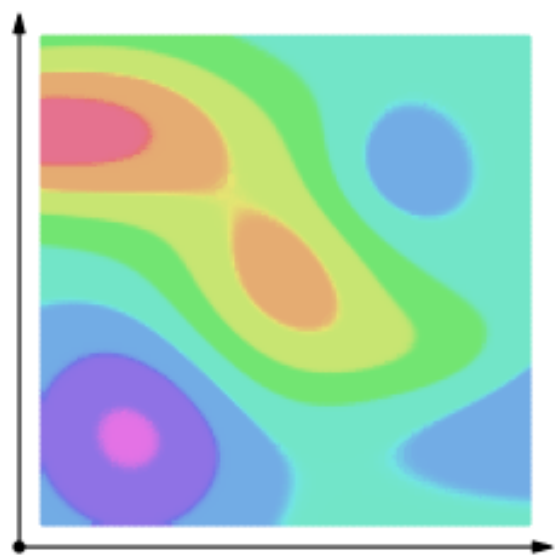
Caveat: The panels must be *"roughly" equi-spaced* and the integrand must be *smooth*.

Composite Trapezoidal Rule for Line Integrals

Now consider a line integral along a curve:

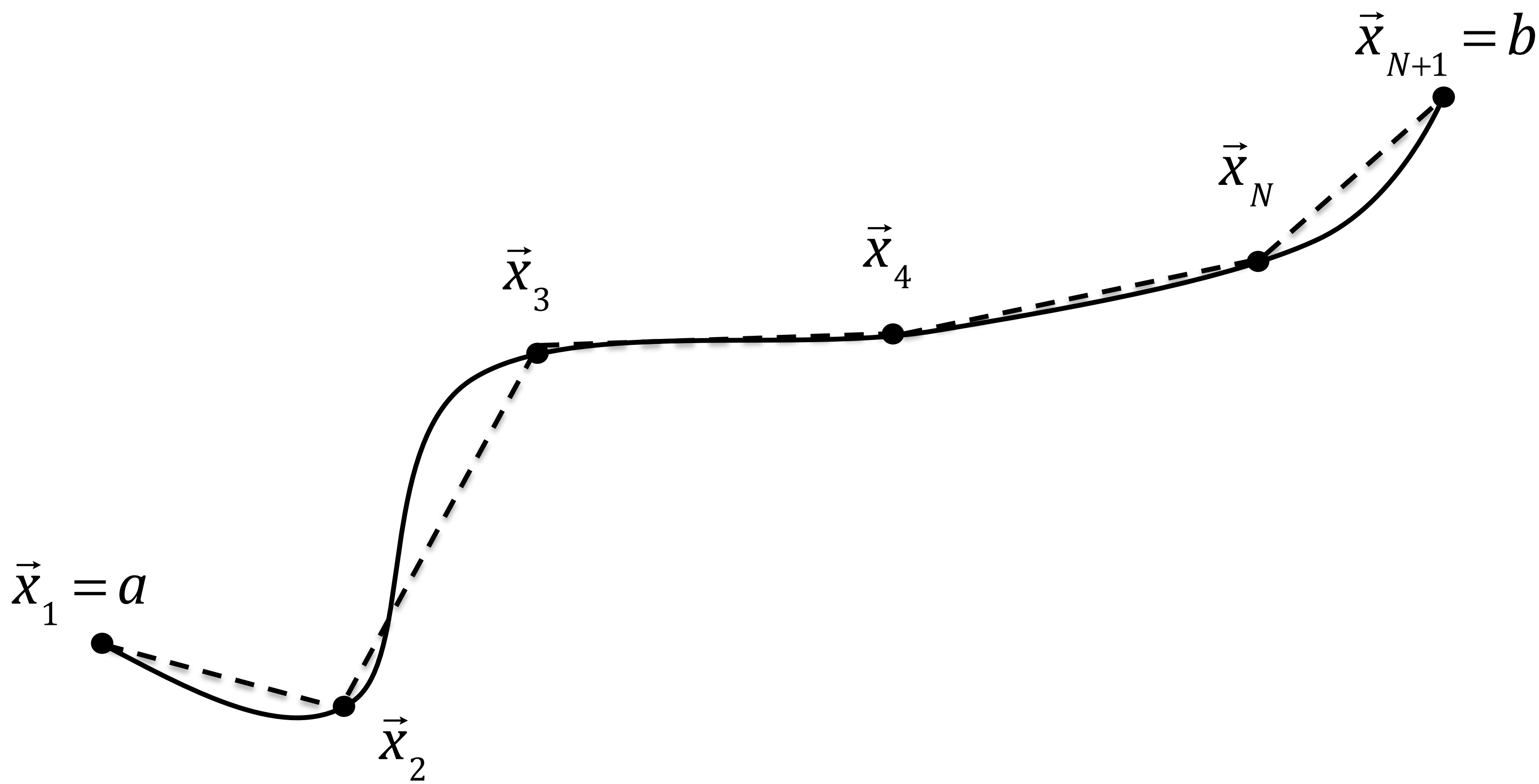
$$\int_C f\,ds$$

We again view this integration as “taking the area under the curve”, but in a slighty different sense:



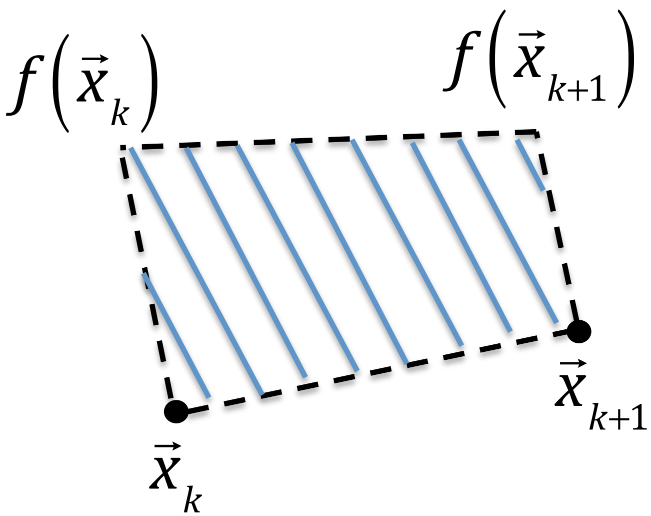
Composite Trapezoidal Rule for Line Integrals

To numerically approximate this integral, we first approximate the curve by using *N* straight panels connecting *N* + 1 grid points:



Composite Trapezoidal Rule for Line Integrals

We then apply the standard trapezoidal rule over each panel:



$$\int_{\vec{x}_k}^{\vec{x}_{k+1}} f(s)\,ds \approx \overbrace{\left|\vec{x}_{k+1} - \vec{x}_k\right|}^{\textbf{\textit{BASE}}} \overbrace{\left[\frac{f(\vec{x}_{k+1}) + f(\vec{x}_k)}{2}\right]}^{\textbf{\textit{HEIGHT}}}$$

resulting in the composite rule:

$$\int_C f\,ds \approx \sum_{k=1}^{N} \left|\vec{x}_{k+1} - \vec{x}_k\right| \left[\frac{f(\vec{x}_{k+1}) + f(\vec{x}_k)}{2}\right]$$

Example: Composite Trapezoidal Rule for Axial and Normal Force Calculations

Now suppose we wish to compute the normal force (per unit span) on an airfoil due to the pressure and shear stress along the upper surface:

$$N_u^{'} = -\int_{LE}^{TE} \left(p_u \cos\theta + \tau_u \sin\theta \right) ds_u$$

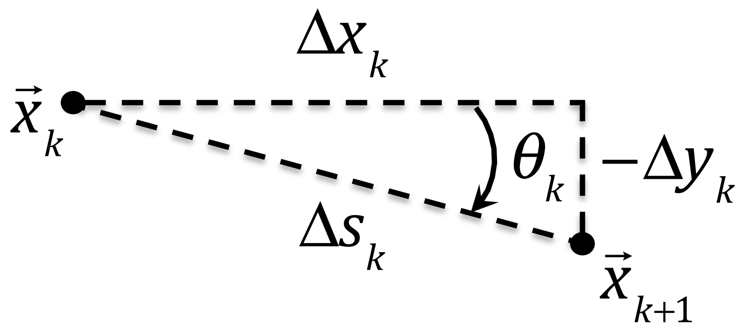
Application of the composite trapezoidal rule yields:

$$N_u^{'} \approx -\sum_{k=1}^{N} \left| \vec{x}_{k+1} - \vec{x}_k \right| \left(\left[\frac{p_u(\vec{x}_{k+1}) + p_u(\vec{x}_k)}{2} \right] \cos\theta_k + \left[\frac{\tau_u(\vec{x}_{k+1}) + \tau_u(\vec{x}_k)}{2} \right] \sin\theta_k \right)$$

What is θ_k?

Example: Composite Trapezoidal Rule for Axial and Normal Force Calculations

θ_k is the angle between the *x*-axis and the k^{th} panel, oriented clockwise:



$$\Delta s_k = \left| \vec{x}_{k+1} - \vec{x}_k \right|$$

$$\Delta x_k = x_{k+1} - x_k$$

$$\Delta y_k = y_{k+1} - y_k$$

Then, we see:

$$\cos\theta_k = \frac{\Delta x_k}{\Delta s_k} \qquad \sin\theta_k = -\frac{\Delta y_k}{\Delta s_k}$$

Example: Composite Trapezoidal Rule for Axial and Normal Force Calculations

With this in mind, our approximation becomes:

$$N_u^{'} \approx \sum_{k=1}^{N} \Delta s_k \left(-\left[\frac{p_u(\vec{x}_{k+1}) + p_u(\vec{x}_k)}{2} \right] \frac{\Delta x_k}{\Delta s_k} + \left[\frac{\tau_u(\vec{x}_{k+1}) + \tau_u(\vec{x}_k)}{2} \right] \frac{\Delta y_k}{\Delta s_k} \right)$$

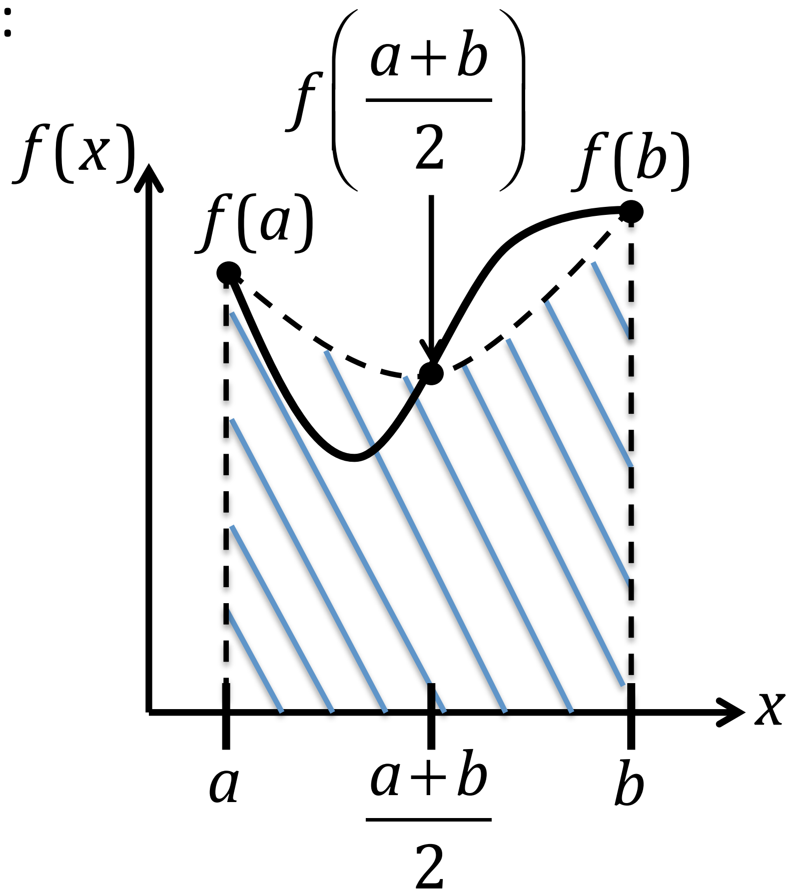
or after simplification:

$$N_u^{'} \approx \sum_{k=1}^{N} \left(-\left[\frac{p_u(\vec{x}_{k+1}) + p_u(\vec{x}_k)}{2} \right] \Delta x_k + \left[\frac{\tau_u(\vec{x}_{k+1}) + \tau_u(\vec{x}_k)}{2} \right] \Delta y_k \right)$$

A similar approach may be utilized to compute the contribution from the lower surface of the airfoil as well as the axial force.

Simpson's Rule

To obtain more accuracy, Simpson's rule approximates the area under the curve by a parabola connecting the curve at the *endpoints* and the *midpoint*:



Simpson's Rule

It can be shown through a bit of tedious algebra that this yields the approximation:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

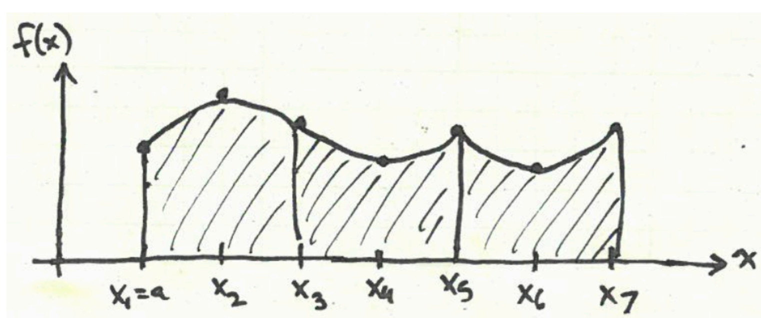
Composite Simpson's Rule

To obtain more resolution, a composite rule may be applied. For $N + 1$ equi-spaced points, where N is even, the rule is:

$$\int_a^b f(x)dx \approx \frac{h}{3}\sum_{k=1}^{N/2}\left[f\left(x_{2k-1}\right)+4f\left(x_{2k}\right)+f\left(x_{2k+1}\right)\right]$$

where:

$$x_k = a+(k-1)h \qquad h=\frac{b-a}{N}$$



Error for Composite Simpson's Rule

FOURTH-ORDER ACCURACY

$$Error \sim O\left(N^{-4}\right)$$

If the number of panels is doubled,
the error decreases by a factor of sixteen!

Composite Simpson's Rule for Line Integrals

For a line integral along a curve, the higher accuracy of Simpson's rule is *lost* is the curve is approximated by panels.

To overcome this issue, it is best to *parameterize* the curve, yielding a definite integral over a straight curve:

$$\int_C f\, ds = \int_{t_1}^{t_2} f\left(\vec{r}(t)\right)\left|\frac{d\vec{r}}{dt}(t)\right| dt$$

Parameterization of the curve!

$$\vec{r} : \left[t_1, t_2\right] \rightarrow C$$
$$\vec{r}\left(t_1\right) = a$$
$$\vec{r}\left(t_2\right) = b$$

Composite Simpson's Rule for Line Integrals

For a line integral along a curve, the higher accuracy of Simpson's rule is *lost* is the curve is approximated by panels.

To overcome this issue, it is best to *parameterize* the curve, yielding a definite integral over a straight curve:

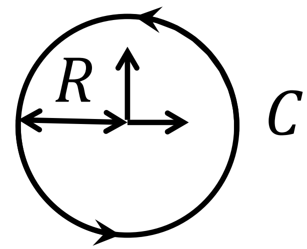
$$\int_C f\,ds = \int_{t_1}^{t_2} f\left(\vec{r}(t)\right)\left|\frac{d\vec{r}}{dt}(t)\right|dt$$

We can then apply the composite Simpson's rule to this definite integral.

Example: Composite Simpson's Rule for Line Integral over a Circle

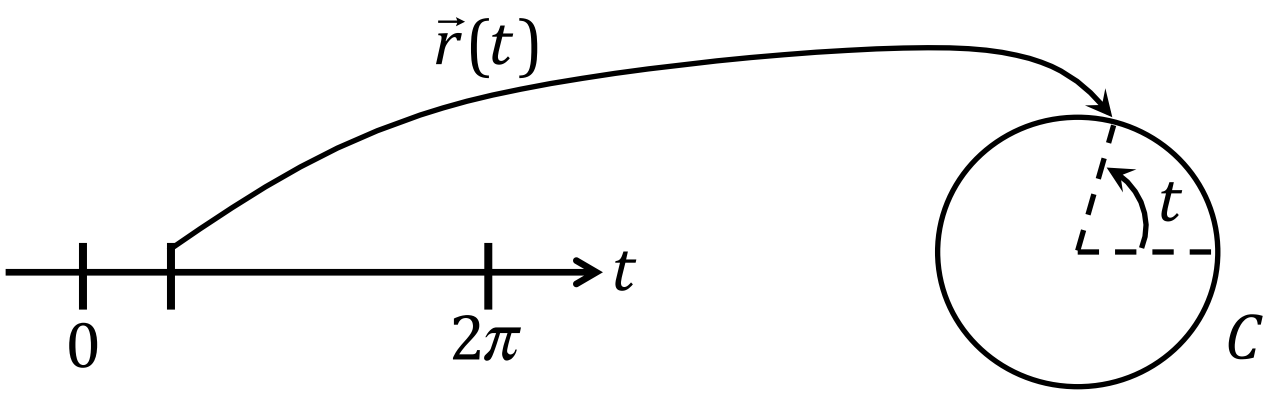
Consider the integral of a function over a circle of radius *R*:

$$\int_C f\,ds$$



We can parameterize the curve as:

$$\vec{r}(t) = R\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$



Example: Composite Simpson's Rule for Line Integral over a Circle

With this parameterization:

$$\int_C f\,ds = \int_{t_1}^{t_2} f\left(\vec{r}(t)\right)\cancelto{R}{\left|\frac{d\vec{r}}{dt}(t)\right|}\,dt$$

Thus the composite Simpson's rule yields the approximation:

$$\int_C f\,ds \approx \frac{h}{3} R \sum_{k=1}^{N/2}\left[f\left(t_{2k-1}\right) + 4f\left(t_{2k}\right) + f\left(t_{2k+1}\right)\right]$$

$$t_k = (k-1)h \qquad\qquad h = \frac{2\pi}{N}$$