ASEN2003
Review Lectures C1 and C2

Control Systems

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Laplace Transform of 2nd Order ODEs

- The standard 2nd order form is:

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2 x(t)=\omega_n^2 u(t)$$

- Taking the Laplace Transform of our standard form equation:

$$L\left[\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2 x(t)\right]=L\left[\omega_n^2 u(t)\right]$$

$$\left[\ s^2X(s)-sx(0)-\dot{x}(0)\ \right]+2\zeta\omega_n\left[sX(s)-x(0)\right]+\omega_n^2X(s)=\omega_n^2U(s)$$

$$\left[\ s^2+2\zeta\omega_n s+\omega_n^2\ \right]X(s)=\omega_n^2U(s)+\left[s+2\zeta\omega_n\ \right]x(0)+\dot{x}(0)$$

- If we assume that the initial conditions are zero (system starts at rest):

$$\left[\ s^2+2\zeta\omega_n s+\omega_n^2\ \right]X(s)=\omega_n^2U(s)$$

Laplace Transform of 2nd Order ODEs

- Therefore,

$$X(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} U(s) = G(s)U(s)$$

- Where G(s) is the *Transfer Function*:

$$G(s) = \frac{X(s)}{U(s)} = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- So, knowing the control *u*(t), we can compute its Laplace Transform *U*(s), and then convolve that with *G*(s) to get *X*(s). Then we simply compute the inverse Laplace Transform:

$$x(t) = L^{-1}\left[X(s)\right] = L^{-1}\left[G(s)U(s)\right]$$

A General 2nd Order System with PD Control

A general *PD control* can be written as:

$$u = -K_P(x - x_R) - K_D(\dot{x} - \dot{x}_R)$$

where we are driving $x \Rightarrow x_R$ and $\dot{x} \Rightarrow \dot{x}_R$. The Laplace Transform of u is:

$$U(s) = -K_P(X - X_R) - sK_D(X - X_R)$$

But we still need to decide what this control looks like in the time domain (step, ramp, etc.). For example, for a step function in the time domain, the LT of a step function is 1/s, so we need to multiply $U(s)$ above by 1/s.

Rigid Arm Dynamics

- The torque or moment produced by the output shaft is what drives the arm, governed by Newton's equations. The angle of the arm (or output shaft) is denoted by, θ_L. The first derivative of θ_L is the speed of the output shaft, ω_L. The second derivative of θ_L is the angular acceleration of the load. The equation of motion is determined by summing the moments about the (fixed) output shaft as follows:

$$J\ddot{\theta}_L + B\dot{\theta}_L + M_L = M_o$$

- Now substituting for the torque exerted by the shaft:

$$J\ddot{\theta}_L + B\dot{\theta}_L + M_L = \frac{K_g K_m}{R_m}\left(V_{in} - K_g K_m \omega_L\right)$$

$$\ddot{\theta}_L + \left(\frac{B}{J} + \frac{K_g^2 K_m^2}{J R_m}\right)\dot{\theta}_L = \frac{K_g K_m}{J R_m} V_{in} - \frac{M_L}{J}$$

Rigid Arm Dynamics

- Ignoring friction and assume no disturbances ($M_L = 0$):

$$\ddot{\theta}_L + \left(\frac{K_g^2 K_m^2}{JR_m}\right)\dot{\theta}_L = \frac{K_g K_m}{JR_m} V_{in}$$

- Taking the Laplace Transform:

$$s^2\Theta_L + s\left(\frac{K_g^2 K_m^2}{JR_m}\right)\Theta_L = \frac{K_g K_m}{JR_m}\mathbf{V}_{in}$$

- Which gives the open loop transfer function, G(s), between $\boldsymbol{V}_{\text{in}}$ and position of the arm:

$$G(s) = \frac{\Theta_L}{V_{in}} = \frac{K_g K_m / JR_m}{s\left(s + K_g^2 K_m^2 / JR_m\right)}$$

Rigid Arm Control

- Now introduce our PD control:

$$V_{in} = K_p\left(\theta_D - \theta_L\right) + K_d\left(\dot{\theta}_D - \dot{\theta}_L\right)$$

- Its Laplace Transform is (assuming we are driving rate to zero):

$$V_{in} = V_{PD} = K_p\left(\Theta_D - \Theta_L\right) - sK_d\Theta_L$$

- Substituting into our Open Loop transfer function and solving for the closed loop transfer function:

$$\bar{G}(s) = \frac{\Theta_L}{\Theta_D} = \frac{K_pK_gK_m/JR_m}{s^2 + \left(K_g^2K_m^2/JR_m + K_dK_gK_m/JR_m\right)s + K_pK_gK_m/JR_m}$$

$$\omega_n^2 = \frac{K_pK_gK_m}{JR_m} \qquad \zeta = \frac{K_g^2K_m^2 + K_dK_gK_m}{2\sqrt{K_pK_gK_mJR_m}}$$

Rigid Arm Control

- We still need to implement this transfer function to control the arm:

$$\Theta_L(s) = \bar{G}(s)\Theta_D(s)$$

- We need to specify what $\Theta_R(s)$ looks like (ramp, step, etc.). For example, if it is a step function, then:

$$\Theta_D(s) = \Theta_D \frac{1}{s}$$

- And then we can get x(t) through the inverse LT:

$$x(t) = L^{-1}\left[\bar{G}(s)\frac{\Theta_D}{s}\right]$$

- The step function gets implemented through Matlab using the **tf** and **step** functions – see the lab document for sample code.