Project 2
Bottle Rocket Design

Assigned: Oct 31, 2016
Due: Report submitted to D2L by 11:59 pm, Friday, December 2, 2016.

1. Background and Introduction

1.1. Rocket projects at CU

Designing, building and firing rockets is a lot of fun. Rocket projects at CU started in 2001, and over the years, our seniors have built several hybrid rocket motors and static-test-fired them at the Lockheed Martin plant in Denver. The ultimate goal of these Mach-SR1 projects was to use a relatively safe (compared to a high performance solid rocket) hybrid rocket to carry a payload to the "edge of space", usually defined as 100 km altitude, which is unreachable by conventional means such as aircraft and balloons. Under the sponsorship of United Launch Alliance and guidance of Professor Kantha, the HySoR (*Hy*brid *So*unding *R*ocket) project (http://hysor.wordpress.com/category/overview/) consists of an undergraduate team and a graduate team with the goal of designing and building a hybrid rocket to accomplish the intermediate goal of launching a 2 kg payload to an altitude of 10 km and recovering it successfully.

1.2. Sophomore Design Project

To help our students understand the intricacies of estimating the performance of an aerospace system by building, testing and analyzing a simple system, the Bottle Rocket Lab was started in 2007 in ASEN 2004. The students are tasked with the job of building a simple bottle rocket, determining the sensitivity of its performance to changes in its design parameters, and comparing it with theoretical expectations. A bottle rocket is a very simple rocket and consists of a plastic bottle (typically a two liter soda bottle) filled partially with a liquid (usually water) and pressurized by air. When the stopper is removed, the water is forced out leading to a reactionary force that propels the bottle according to Newton's laws of motion. ASEN 2004 students measure the thrust generated on a static test stand and determine the drag in a wind tunnel for different design options, then use these data to determine the height and range of a bottle rocket launched at an angle to the vertical. Next, they perform a detailed sensitivity analysis of its performance to changes in design parameters and verify the outcome by launching the rocket and measuring the change in height and/or range. The photograph below shows a typical bottle rocket launch in the open field adjacent to the engineering building, where the lab was typically held.

To prepare students for the Bottle Rocket Design and Performance Analysis Lab in ASEN 2004 in spring, a Bottle Rocket Design Project has been formulated to help them understand the functional dependence of bottle rocket performance on the design parameters, such as the volumetric fraction of water in the bottle, the initial pressure of air and the launch angle. Your task is to use the knowledge you have gained to date to develop a MATLAB code to determine the bottle rocket thrust as a function of time, and predict the resulting height and range of the rocket. You are then asked to use the code to explore the parameter space in order to determine which of the parameters affect the height and the range the most and what combination of parameters will allow the rocket to land within 1 meter of an 85 meter marker. You will use this knowledge in spring, when you actually build your bottle rocket, test it in the wind tunnel and on the static test stand, and then launch it taking into account the effect of wind on the rocket trajectory.

1.3. Trajectory of a Bottle Rocket

To determine the bottle rocket trajectory, we need to appeal to Newton's laws of motion. These are essentially laws of conservation of momentum. We can write conservation laws for the two components of momentum in the horizontal (x) and vertical (z) directions, or more conveniently, we can write them in the following form:

$$
\begin{aligned}
m_R \frac{dV}{dt} &= F - D - m_R g_0 \sin\theta \\
V \frac{d\theta}{dt} &= -g_0 \cos\theta \\
\frac{dx}{dt} &= V \cos\theta \\
\frac{dz}{dt} &= V \sin\theta
\end{aligned}
\tag{1}
$$

x the horizontal coordinate
z the vertical coordinate
t time
m_R mass of the rocket
F thrust
D drag
g_0 gravitational acceleration
V velocity (speed) of the rocket
θ angle of the rocket trajectory to the horizontal

Note that m_R, V, θ, F, and D are all functions of time t and θ is in radians ($180^o = \pi$ radians). The drag force is a function of dynamic pressure $q = \frac{1}{2}\rho V^2$, the drag coefficient C_D and the cross-sectional area of the bottle A_B:

$$D = qC_D A_B = \frac{\rho_a}{2} V^2 C_D A_B \qquad (2)$$

The drag coefficient depends on a variety of factors, including the Reynolds number of the flow, any flow separation toward the rear of the bottle, the drag on the stabilizing surfaces and the shape of the nose cone, and is hard to determine theoretically. It is usually measured in a wind tunnel, which ASEN 2004 students do. Depending on how they modified the bottle to fly properly, C_D is varied from 0.3 to 0.5.

The most important term in Eq. (1) is the thrust F. It can be measured on a static test stand, which is what ASEN 2004 students do. It can also be estimated by applying the laws of thermodynamics and aerodynamics you have been taught to the expansion of air in the bottle. We also need to keep track of the mass of the rocket, as water and later air is expelled through its mouth (we will call it the throat in conformity with rocket terminology).

1.4. Bottle Rocket Thermodynamics

Let p^i_{air}, v^i_{air} and T^i_{air} be the initial pressure, volume and temperature of air inside the bottle. Then the initial mass of air in the bottle is $m^i_{air} = \frac{p^i_{air} v^i_{air}}{RT^i_{air}}$ where R = 287 J kg^{-1}K^{-1}. The bottle rocket thrust phase can be divided into two phases: 1. Before the water is exhausted and 2. After the water is exhausted.

1.4.1. Before the water is exhausted

During this phase, the mass of air m_{air} remains constant but the air volume v increases as water is expelled, and therefore the air density is inversely proportional to its volume. We will assume the expansion of air during the rocket operation is isentropic, meaning the process is adiabatic (no heat transfer to or from the air mass) and there are no frictional losses. This is a good approximation and the air pressure p at any future time t is then given by

$$\frac{p}{p^i_{air}} = \left(\frac{v^i_{air}}{v}\right)^{\gamma} \qquad (3)$$

where the specific heat ratio γ =1.4. The mass flow rate of water out the throat of the bottle is

$$\dot{m} = c_d \rho_w A_t V_e \qquad (4)$$

where ρ_w is the density of water, V_e is the velocity of the exhaust, A_t is the throat area and c_d is the discharge coefficient, which is less than 1. The thrust of a rocket is given by

$$F = \dot{m} V_e + (p_e - p_a) A_t \qquad (5)$$

where p_a is the ambient pressure. Since the water is incompressible, applying the Bernoulli equation for incompressible flows,

$$(p - p_a) = \frac{\rho_w}{2} V_e^2 \qquad (6)$$

so that the exhaust velocity

$$V_e = \sqrt{\frac{2\left(p - p_a\right)}{\rho_w}} \tag{7}$$

and since $p_e = p_a$, the thrust

$$F = \dot{m}V_e = 2c_d\left(p - p_a\right)A_t \tag{8}$$

is independent of the liquid density!

Note that pressure p decreases with time as air expands and therefore the thrust decrease with time. The pressure at time t can be computed if the volume is known from Eq. (3). The rate of change of volume of air with time is:

$$\frac{dv}{dt} = c_d A_t V_e = c_d A_t \sqrt{\frac{2\left(p - p_a\right)}{\rho_w}} = c_d A_t \sqrt{\frac{2}{\rho_w}\left[p_0\left(\frac{v_0}{v}\right)^{\gamma} - p_a\right]} \tag{10}$$

from which volume $v(t)$ of air can be determined. Eq. (10) needs to be solved with the initial condition $v = v_{air}^i$ at time $t = 0$. Because of its nonlinear nature, this has to be done numerically using an ODE solver such as the 4th order Runge-Kutta. The integration stops when $v = v_B$, the volume of the bottle, and all the water has been expelled.

Eq. (3) can be used to determine $p(t)$ and using which F and I_{sp} can be determined from Eqs. (8) and (9). The rocket mass changes with time according to

$$\dot{m}_R = -\dot{m} = -\rho_w c_d A_t V_e = -c_d A_t \sqrt{2\rho_w\left(p - p_a\right)} \tag{11}$$

The rocket trajectory can be estimated using Eqs. (1), (2) and (11), given the initial angle θ^i (initial velocity $V^i = 0$). The initial mass of the rocket is

$$m_R^i = m_B + m_{air}^i + m_{water}^i = m_B + \rho_w\left(v_B - v_{air}^i\right) + \left(\frac{p_{air}^i}{RT_{air}^i}\right)v_{air}^i \tag{12}$$

Note that m_B is the mass of the empty bottle, which includes the stabilizing vanes and the nose cone.

1.4.2. After the water is exhausted:

Let p_{end} be the pressure and T_{end} be the temperature of air in the bottle at the time all the water is expelled:

$$p_{end} = p_{air}^i\left(\frac{v_{air}^i}{v_B}\right)^{\gamma}; T_{end} = T_{air}^i\left(\frac{v_{air}^i}{v_B}\right)^{\gamma-1} \tag{13}$$

Once the water is exhausted, the volume of air remains constant but its mass decreases, and therefore the density is proportional to the mass. Again we assume air expands isentropically, until the pressure p drops to the ambient pressure p_a. Then the pressure at any time t is given by

$$\frac{p}{p_{end}} = \left(\frac{m_{air}}{m_{air}^i}\right)^{\gamma} \tag{14}$$

and the corresponding density and temperature are

$$\rho = \frac{m_{air}}{v_B};\ T = \frac{p}{\rho R} \tag{15}$$

Define critical pressure

$$p_* = p\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \tag{16}$$

1. If $p_* > p_a$, the flow is choked (exit Mach number $M_e = 1$) and the exit velocity is

$$V_e = \sqrt{\gamma R T_e} \tag{17}$$

where

$$T_e = \left(\frac{2}{\gamma+1}\right)T \quad \text{and} \quad \rho_e = \frac{p_e}{RT_e} \quad \text{with} \quad p_e = p_* \tag{18}$$

2. If $p_* \le p_a$, the flow is not choked and

$$p_e = p_a \tag{19}$$

The exit Mach number is then obtained from

$$\frac{p}{p_a} = \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\gamma/(\gamma-1)} \tag{20}$$

with

$$\frac{T_e}{T} = \left(1 + \frac{\gamma-1}{2}M_e^2\right),\ \rho_e = \frac{p_a}{RT_e} \tag{21}$$

and the exit velocity is

$$V_e = M_e\sqrt{\gamma R T_e} \tag{22}$$

The thrust in both of the above cases is given by

$$F = \dot{m}_{air}V_e + \left(p_e - p_a\right)A_t \tag{23}$$

where

$$\dot{m}_{air} = c_d \rho_e A_t V_e \tag{24}$$

The rocket mass decreases according to

$$\dot{m}_R = -\dot{m}_{air} = -c_d \rho_e A_t V_e \tag{26}$$

The rocket trajectory can be estimated using Eqs. (1), (2) and (26), with the conditions at the time of the water exhaustion prescribed as the initial conditions.

1.4.3. Ballistic Phase:

Thrust is generated by the bottle rocket until the air pressure in the bottle p falls to the ambient pressure p_a. After that the thrust is zero and the rocket enters its free ballistic phase under the influence of gravity:

$$F = 0 \text{ and } m_R \sim m_B. \quad (27)$$

The ballistic trajectory of the rocket can be determined from Eq. (1) for the rest of the flight (until the rocket hits the ground), with initial conditions corresponding to those at the end of the thrust phase.

As you can see, the bottle rocket flight consists of three distinct phases: 1. From the moment the stopper is removed until the water is exhausted, 2. After the water is exhausted until the air pressure drops to the ambient value and the thrust phase ends, and 3. Ballistic phase. The first two comprise the thrust phase, which is usually a small fraction of the total flight time.

2. Your Assignment: Determine the Rocket Trajectory

The performance of the bottle rocket depends on four parameters: p^i_{air} the initial pressure of air (the limit being the burst pressure of the bottle, with some factor of safety), the initial volume fraction (or equivalently initial mass) of water, the drag coefficient and the launch angle.

Your assignment is to determine the parametric combination of these parameters that will allow the bottle rocket to land within 1 meter of an 85 meter marker, you must also determine which parameter affects range and height most. You will write a report to document your program development and your findings. As a check on your code, you will be given the range and height for a particular parametric combination. You are required to reproduce this result, before exploring the parameter space. You are advised to use consistent units. We prefer SI units.

For submission you are required to write a report, in AIAA format, that should have the following sections:

- Abstract – Summary of what was done and what were the outcomes
- Introduction – Objective of study and relevant background, including the equations to be solved
- Methodology – Approach to complete the objective (e.g. Algorithm developed)
- Results – Tables, graphs and charts
- Discussion – Explanation of what was observed and why
- Conclusion – Summary of findings
- References – List of source that provide information that helped you carry out the study
- Appendix – Documents that are important but not essential to explaining the study and analyzing the results (e.g. MATLAB code)

NB: You should begin this project early since computer programs rarely work the first time you write it and it would take time to fix all the problems. Do not hesitate to see your TA or me if you run into trouble.

3. Suggested Activities

In order to ensure that you get this project done on time the following are some soft-deadlines you could use to keep you on track:

1. Friday 4th Nov – understand the equations for each phase of the rockets flight.

2. Friday 11th Nov – develop the algorithm you would like to employ to determine the trajectory.
3. Friday 18th Nov – completed majority (if not all) of coding, ensuring that your algorithm was checked by either your instructor or your TA. Getting to this point is important since we usually go into Thanksgiving break with the best of intentions but it always ends with us achieving less than we intended.
4. Week of 25th Nov – minor debugging and writing of the report (ideally you should only be writing the report).
5. Friday 2th Dec – submit project to dropbox at the required time, in the required format.

6. References

Anderson, J. D., Jr., **Introduction to Flight, 7th Ed.**, McGraw-Hill (2009).

Sutton, G. and Biblarz, O., **Rocket Propulsion Elements, 8th Ed.**, Wiley (2010).