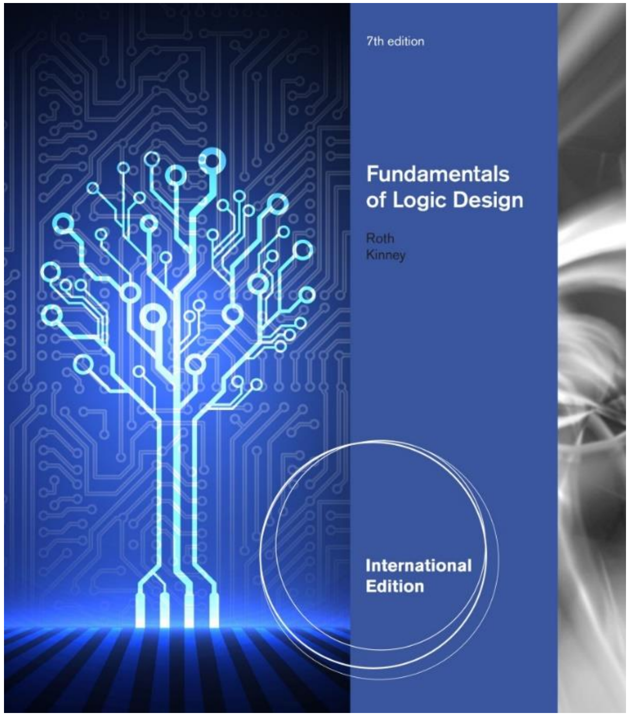

Boolean Algebra II

Contents

1. Multiplying out and Factoring Expressions
2. Exclusive-OR and Equivalence Operations
3. The Consensus Theorem
4. Algebraic Simplification of Switching Expressions
5. Proving Validity of an Equation

Objectives

- Apply the laws and theorems of Boolean algebra to the manipulation of algebraic expressions to simplifying an expression, finding the complement of an expression and multiplying out and factoring an expression.
- Prove any of the theorems using a truth table or give an algebraic proof.
- Define the exclusive-OR and equivalence operations. State, prove, and use the basic theorems that concern these operations.
- Use the consensus theorem to delete terms from and add terms to a switching algebra expression.
- Given an equation, prove algebraically that it is valid or show that it is not valid.

Multiplying Out and Factoring Expressions

- Given an expression in product-of-sums form, the corresponding sum-of-products expression can be obtained by multiplying out, using the two distributive laws:

$$X(Y + Z) = XY + XZ \tag{3-1}$$

$$(X + Y)(X + Z) = X + YZ \tag{3-2}$$

- In addition, the following theorem is very useful for factoring and multiplying out:

$$\boxed{(X + Y)(X' + Z) = XZ + X'Y \qquad (3\text{-}3)}$$

A good way to practically check is to substitute 0's or 1's for some of the variables.
If $X = 0$, $(0+Y)(1+Z) = Y = 0 \cdot Z + 1 \cdot Y$
If $X = 1$, $(1+Y)(0+Z) = Z = 1 \cdot Z + 0 \cdot Y$
Another way to prove this equation is using Consensus Theorem:
$(X+Y)(X'+Z) = XX'+XZ+X'Y+YZ = 0+XZ+X'Y+YZ = XZ+X'Y$

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Multiplying Out and Factoring Expressions

- In general, when we multiply out an expression, we should use (3-3) along with (3-1) and (3-2).
- To avoid generating unnecessary terms when multiplying out, (3-2) and (3-3) should generally be applied before (3-1), and terms should be grouped to expedite their application.

Example 1:

Example

$$(A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$$
$$= (A + B + C'D)(A + B + E)[AC + A'(D' + E)]$$
$$= (A + B + C'DE)(AC + A'D' + A'E)$$
$$= \underline{AC} + \cancel{\underline{ABC}} + A'BD' + A'BE + A'C'DE \qquad (3\text{-}4)$$

Absorption Theorem ($X+XY=X$)

What theorem was used to eliminate ABC? (*Hint:* let $X = AC$.)

In this example, if the ordinary distributive law (3-1) had been used to multiply out the expression by brute force, 162 terms would have been generated, and 158 of these terms would then have to be eliminated. 3x3x3x3x2 = 162 terms

Example 2:

- By repeatedly applying (3-1), (3-2), and (3-3), any expression can be converted to a product-of-sums form.

Example of Factoring

$$AC + A'BD' + A'BE + A'C'DE$$

$$= \underbrace{AC}_{XZ} + \underbrace{A'}_{X'}(\underbrace{BD' + BE + C'DE}_{Y})$$ $XZ+X'Y = (X+Y)(X'+Z)$

$$= (A + BD' + BE + C'DE)(A' + C)$$

$$= [\underbrace{A + C'DE}_{X} + B(\underbrace{D'}_{Y} + \underbrace{E}_{Z})](A' + C)$$ $X+YZ = (X+Y)(X+Z)$

$$= (A + B + C'DE)(A + \cancel{C'DE} + D' + E)(A' + C)$$ Absorption Theorem ($X+XY=X$)

$$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C) \quad (3\text{-}5)$$

This is the same expression we started with in (3-4).

Exclusive-OR and Equivalent Operations

Exclusive-OR (XOR) Operation:
(A) XOR truth table, (B) XOR definition,
(C) XOR logic symbol.

- For **exclusive-OR**, $X \oplus Y = 1$ if and only if (iff) $X = 1$ or $Y = 1$, but not both.
- The ordinary OR operation is sometimes called **inclusive-OR** because $X + Y = 1$ iff $X = 1$ or $Y = 1$, or both.
- Exclusive OR can be expressed in terms of AND and OR. Because $X \oplus Y = 1$ iff X is 0 and Y is 1 or X is 1 and Y is 0, we can write

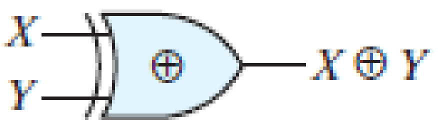
$$\boxed{X \oplus Y = X'Y + XY'} \qquad (3\text{-}6)$$

A)

X	Y	$X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

B) $0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$

$1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$

C)


Exclusive-OR and Equivalent Operations

Theorems of Exclusive-OR:

$$X \oplus 0 = X \tag{3-8}$$
$$X \oplus 1 = X' \tag{3-9}$$
$$X \oplus X = 0 \tag{3-10}$$
$$X \oplus X' = 1 \tag{3-11}$$
$$X \oplus Y = Y \oplus X \text{ (commutative law)} \tag{3-12}$$
$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z \text{ (associative law)} \tag{3-13}$$
$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive law)} \tag{3-14}$$
$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y' \tag{3-15}$$

Exclusive-NOR gate → Equivalence gate

Proof)

$$X \oplus Y = XY' + X'Y = Y'X + YX' = Y \oplus X \tag{3-12}$$
$$(X \oplus Y)' = (XY' + X'Y)' = (X' + Y)(X + Y') = X'Y' + XY$$
$$= X' \oplus Y = XY + X'Y' = X \oplus Y' \tag{3-15}$$

Exclusive-OR and Equivalent Operations

Equivalence Operations:

(A) Equivalence operation truth table,
(B) Definition,
(C) Logic symbols, equivalence and exclusive-NOR gates.

- From the definition of equivalence, $(X \equiv Y) = 1$ iff $X=Y$. Because $(X \equiv Y) = 1$ iff $X=Y=1$ or $X=Y=0$, we can write

$$\boxed{(X \equiv Y) = XY + X'Y'}$$

- Equivalence is the complement of exclusive OR, i.e., $(X \oplus Y)' = XY + X'Y'$

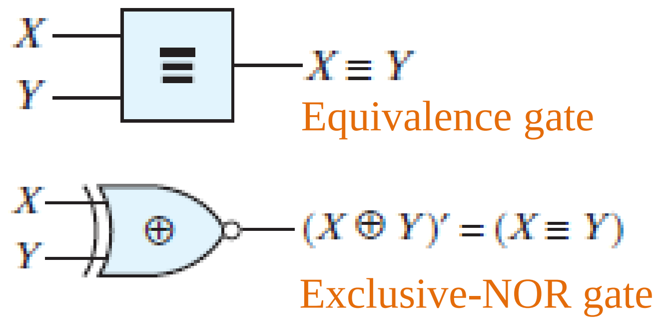
A)

X	Y	$X \equiv Y$
0	0	1
0	1	0
1	0	0
1	1	1

B) $(0 \equiv 0) = 1$ $\quad (0 \equiv 1) = 0$

$(1 \equiv 0) = 0$ $\quad (1 \equiv 1) = 1$

C)



Equivalence Operations:

- For this operation, the output will be 1 iff *X* and *Y* have the same value. So,

$$(X \equiv Y) = XY + X'Y' \qquad (3\text{-}17)$$

- Equivalence is the complement of the exclusive-OR operation.

DeMorgan's law

$$\begin{aligned}(X \oplus Y)' &= (X'Y + XY')' = (X + Y')(X' + Y) \\ &= XY + X'Y' = (X \equiv Y)\end{aligned} \qquad (3\text{-}18)$$

The Consensus Theorem

Consensus Theorem:

- The **consensus theorem** is used to cancel out redundant terms in an expression and is stated below:

$$XY + X'Z + YZ = XY + X'Z \qquad (3\text{-}20)$$

Consensus term

- The term that is eliminated can be called the **consensus term**.
- The dual form of the consensus theorem is:

$$(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z) \qquad (3\text{-}21)$$

Consensus term

Example 3:

Example

$$A'C'D + \underline{A'BD} + \cancel{BCD} + \underline{ABC} + ACD' \qquad (3\text{-}22)$$

First, we eliminate BCD as shown. (Why can it be eliminated?) Consensus Theorem

Now that BCD has been eliminated, it is no longer there, and it *cannot* be used to eliminate another term. Checking all pairs of terms shows that no additional terms can be eliminated by the consensus theorem.

Now we start over again:

$$\underline{A'C'D} + \cancel{A'BD} + \underline{BCD} + \cancel{ABC} + \underline{ACD'} \qquad (3\text{-}23)$$

This time, we do not eliminate BCD; instead we eliminate two other terms by the consensus theorem. After doing this, observe that BCD can no longer be eliminated. Note that the expression reduces to four terms if BCD is eliminated first, but that it can be reduced to three terms if BCD is not eliminated.

The Consensus Theorem

Example 3 (Continued):

Sometimes it is impossible to directly reduce an expression to a minimum number of terms by simply eliminating terms. It may be necessary to first add a term using the consensus theorem and *then* use the added term to eliminate other terms. For example, consider the expression

$$F = ABCD + B'CDE + A'B' + BCE'$$

If we compare every pair of terms to see if a consensus term can be formed, we find that the only consensus terms are $ACDE$ (from $ABCD$ and $B'CDE$) and $A'CE'$ (from $A'B'$ and BCE'). Because neither of these consensus terms appears in the original expression, we cannot directly eliminate any terms using the consensus theorem. However, if we first add the consensus term $ACDE$ to F, we get

$$F = \cancel{ABCD} + \cancel{B'CDE} + A'B' + BCE' + ACDE$$

Then, we can eliminate $ABCD$ and $B'CDE$ using the consensus theorem, and F reduces to

$$F = A'B' + BCE' + ACDE$$

The term $ACDE$ is no longer redundant and cannot be eliminated from the final expression.

Ways to simplify switching expressions:

- In addition to multiplying out and factoring, three basic ways of simplifying switching functions are Combining Terms, Eliminating Terms and Eliminating Literals

1. *Combining terms.* Use the theorem $\boxed{XY + XY' = X}$ to combine two terms. For example,

$$abc'd' + abcd' = abd' \qquad [X = abd', Y = c] \qquad (3\text{-}24)$$

When combining terms by this theorem, the two terms to be combined should contain exactly the same variables, and exactly one of the variables should appear complemented in one term and not in the other. Because $X + X = X$, a given term may be duplicated and combined with two or more other terms. For example,

$$ab'c + abc + a'bc = ab'c + abc + abc + a'bc = ac + bc$$

The theorem still can be used, of course, when X and Y are replaced with more complicated expressions. For example,

$$(\overset{Y}{a + bc})(\overset{X}{d + e'}) + \overset{Y'}{a'(b' + c')}(\overset{X}{d + e'}) = \overset{X}{d + e'}$$

$$[X = d + e', Y = a + bc, Y' = a'(b' + c')]$$

Algebraic Simplification of Switching Expressions

2. *Eliminating terms.* Use the theorem $\boxed{X + XY = X}$ to eliminate redundant terms if possible; then try to apply the consensus theorem ($XY + X'Z + YZ = XY + X'Z$) to eliminate any consensus terms. For example,

$$a'b + a'bc = a'b \qquad [X = a'b]$$

$$a'bc' + bcd + a'bd = a'bc' + bcd \qquad [X = c, Y = bd, Z = a'b] \qquad (3\text{-}25)$$

3. *Eliminating literals.* Use the theorem $\boxed{X + X'Y = X + Y}$ to eliminate redundant literals. Simple factoring may be necessary before the theorem is applied.

Example 4:

Example

$$\begin{aligned} A'B + A'B'C'D' + ABCD' &= A'(B + B'C'D') + ABCD' \\ &= A'(B + C'D') + ABCD' \\ &= B(A' + ACD') + A'C'D' \\ &= B(A' + CD') + A'C'D' \\ &= A'B + BCD' + A'C'D' \qquad (3\text{-}26) \end{aligned}$$

4. *Adding Redundant Terms*: such as adding $X \cdot X'$, multiplying $(X + X')$, etc.

Example 5:

Example

$$WX + XY + X'Z' + WY'Z' \quad \text{(add } WZ' \text{ by consensus theorem)}$$
$$= WX + XY + X'Z' + WY'Z' + WZ' \quad \text{(eliminate } WY'Z'\text{)}$$
$$= WX + XY + X'Z' + WZ' \quad \text{(eliminate } WZ'\text{)}$$
$$= WX + XY + X'Z' \qquad (3\text{-}27)$$

Example 6:

Example

$$\underbrace{A'B'C'D' + A'BC'D'}_{\text{① } A'C'D'} + A'BD + \cancel{A'BC'D} + ABCD + ACD' + B'CD'$$

$X+X'=1$ ① $A'C'D'$ ② $X+XY=X$

$$= A'C'D' + BD(A' + AC) + ACD' + B'CD'$$

③ $X(Y+Z)=XY+XZ$

$$= A'C'D' + A'BD + \underbrace{BCD + ACD'}_{+ABC} + B'CD'$$

$+ABC$ ④ Consensus Theorem

$$= A'C'D' + \underbrace{A'BD + \cancel{BCD} + \cancel{ACD'} + \overbrace{B'CD' + ABC}^{\text{consensus } ACD'}}_{\text{consensus } BCD}$$
$$= A'C'D' + A'BD + B'CD' + ABC \qquad (3\text{-}28)$$

What theorems were used in steps 1, 2, 3, and 4?

Algebraic Simplification of Switching Expressions

Example 7:

- For a product-of-sums form instead of a sum-of-products form, the duals of the preceding theorems should be applied.

Example

$$\underbrace{(A'+B'+C')(A'+B'+C)}_{\textcircled{1}\ (A'+B')}(B'+C)\underline{(A+C)(\cancel{A+B+C})}$$

$(X+Y)(X+Y')=X$ ② $X(X+Y)=X$

$$= (A'+B')(\cancel{B'+C})(A+C) = (A'+B')(A+C) \qquad (3\text{-}29)$$

③ Consensus Theorem

What theorems were used in steps 1, 2, and 3?

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Proving the Validity of an Equation

Proving an equation is valid:

- To determine if an equation is valid, meaning valid for all combinations of values of the variables, several methods can be used:
  1. Construct a truth table and evaluate both sides of the equation for all combinations of values of the variables.
  2. Manipulate one side of the equation by applying various theorems until it is identical with the other side.
  3. Reduce both sides of the equation independently to the same expression.
  4. It is permissible to perform the same operation on both sides of the equation provided that the operation is reversible.
     - ✓ Complementation of both sides – Permissible
     - ✓ *Multiplication of both sides – Not Permissible* since multiplication is not reversible because division is not defined for Boolean algebra
     - ✓ *Addition of both sides – Not Permissible* since addition is not reversible because subtraction is not defined for Boolean algebra

Proving the Validity of an Equation

To prove an equation is NOT valid:

- To prove that an equation is *not* valid, it is sufficient to show one combination of values of the variables for which the two sides of the equation have different values.
  1. First reduce both sides to a sum of products (or a product of sums).
  2. Compare the two sides of the equation to see how they differ.
  3. Then try to add terms to one side of the equation that are present on the other side.
  4. Finally try to eliminate terms from one side that are not present on the other.
- Whatever method is used, frequently compare both sides of the equation and let the difference between them serve as a guide for what steps to take next.

Proving the Validity of an Equation

Example 8: Manipulate one side of the equation by applying various theorems until it is identical with the other side.

Show that

$$A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC$$

Starting with the left side, we first add consensus terms, then combine terms, and finally eliminate terms by the consensus theorem.

$$A'BD' + BCD + ABC' + AB'D$$

$$= A'BD' + BCD + ABC' + AB'D + BC'D + A'BC + ABD$$

(add consensus of $A'BD'$ and ABC')

(add consensus of $A'BD'$ and BCD)

(add consensus of BCD and ABC')

$$= AD + A'BD' + BCD + ABC' + BC'D' + A'BC = BC'D' + AD + A'BC$$

(eliminate consensus of $BC'D'$ and AD)

(eliminate consensus of AD and $A'BC$)

(eliminate consensus of $BC'D'$ and $A'BC$) (3-30)

Proving the Validity of an Equation

Example 9: Reduce both sides of the equation independently to the same expression.

Show that the following equation is valid:

$$A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$$
$$= ABCD + A'C'D' + ABD + ABCD' + BC'D$$

First, we will reduce the left side:

$A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$

(eliminate $A'BC'D$ using absorption)

$= (A' + BC)(A + C'D') + BC'D + A'BC'$

(multiply out using (3-3))

$= ABC + A'C'D' + BC'D + A'BC'$

(eliminate $A'BC'$ by consensus)

$= ABC + A'C'D' + BC'D$

Now we will reduce the right side:

$= ABCD + A'C'D' + ABD + ABCD' + BC'D$

(combine $ABCD$ and $ABCD'$)

$= ABC + A'C'D' + ABD + BC'D$

(eliminate ABD by consensus)

$= ABC + A'C'D' + BC'D$

Because both sides of the original equation were independently reduced to the same expression, the original equation is valid.

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Proving the Validity of an Equation

Cancellation Laws in Boolean Algebra:

The cancellation law for ordinary algebra, where:

$$\text{If } x+y = x+z, \quad \text{then} \quad y = z$$

Does not hold for Boolean algebra. (Let x=1, y=0, z=1; then, 1+0 = 1+1, but 0 $\neq$ 1)

The cancellation law for multiplication, where:

$$\text{If } xy = xz, \quad \text{then} \quad y = z$$

Does not hold for Boolean algebra. (Let x=0, y=0, z=1; then $0 \cdot 0 = 0 \cdot 1$, but 0 $\neq$ 1)

However, the converses of these hold true for Boolean algebra:

$$\text{If } y = z, \quad \text{then} \quad x+y = x+z$$

$$\text{If } y = z, \quad \text{then} \quad xy = xz$$