Probability and Random Process (SWE3026)

Continuous and Mixed Random Variables

JinYeong Bak
jy.bak@skku.edu
College of Computing, SKKU

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

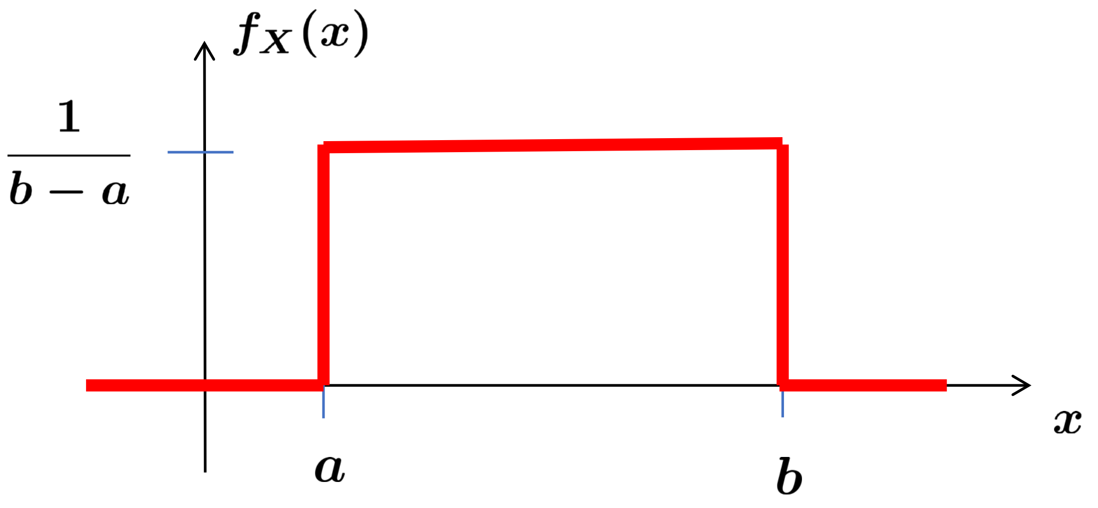
Special Distributions

- **Uniform Distribution:** $X \sim Uniform(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$$EX = \frac{a+b}{2},$$

$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 \\ &= \frac{(b-a)^2}{12}. \end{aligned}$$



Special Distributions

Example. **Let** $X \sim Uniform(-1, 3),$

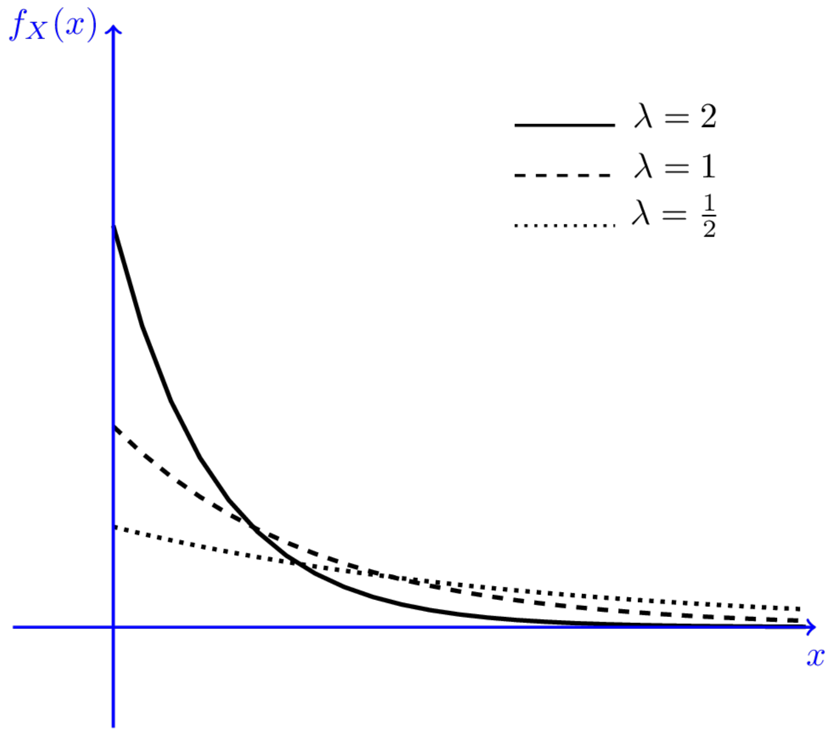
$$E[X] = \frac{3 + (-1)}{2} = 1,$$

$$\mathrm{Var}(X) = \frac{(3 - (-1))^2}{12} = \frac{16}{12} = \frac{4}{3}.$$

Special Distributions

- **Exponential Distribution:** $X \sim Exponential(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases}$$
$$= \lambda e^{-\lambda x}\, u(x)$$



Where $u(x)$ denote the unit step function.

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Special Distributions

We find the CDF using the equation

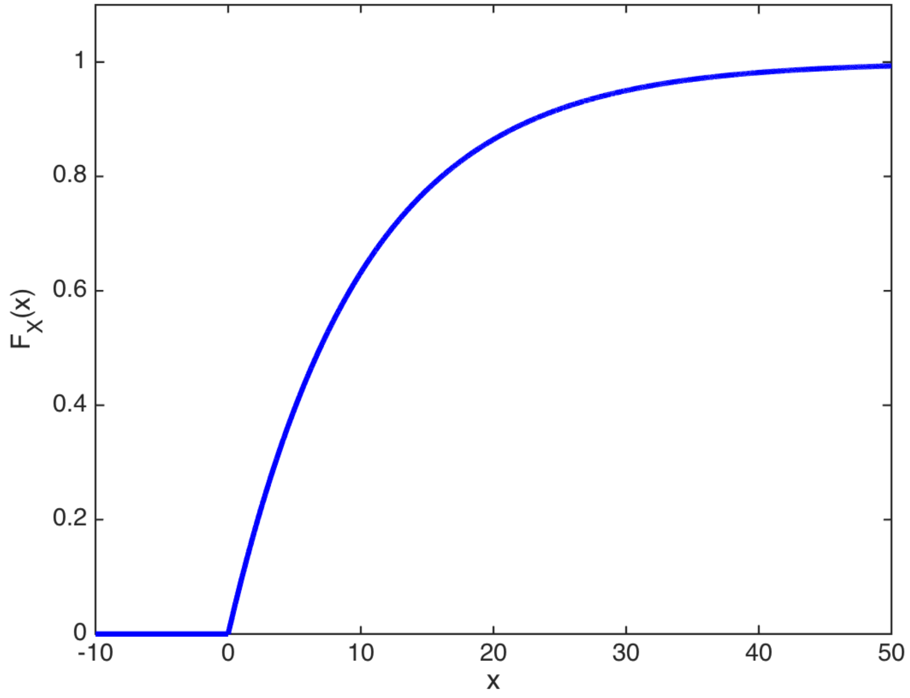
$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}.$$

So

$$F_X(x) = \left(1 - e^{-\lambda x}\right) u(x).$$

Special Distributions

Example. **CDF of** $Exponential(\frac{1}{10})$



Special Distributions

Expected value

$$
\begin{aligned}
EX &= \int_0^\infty x\lambda e^{-\lambda x} dx \\
&= \frac{1}{\lambda} \int_0^\infty y e^{-y} dy \qquad\qquad \text{choosing } y = \lambda x \\
&= \frac{1}{\lambda} \Big[-e^{-y} - y e^{-y} \Big]_0^\infty \\
&= \frac{1}{\lambda}.
\end{aligned}
$$

Special Distributions

$\mathbf{Var}(X):$ $$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}.$$

Thus, we obtain

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Special Distributions

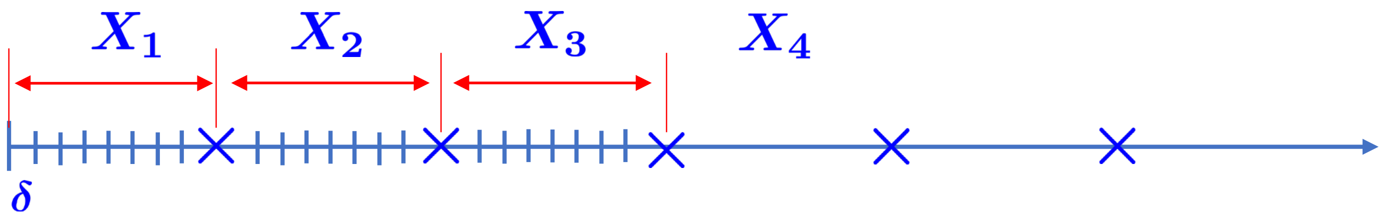
If $X \sim Exponential(\lambda)$, then

$$\boldsymbol{EX = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}.}$$

Special Distributions

If X is exponential with parameter $\lambda > 0$, then X is a **memoryless** **random variable, that is**

$$P(X > x + a \mid X > a) = P(X > x), \qquad \text{for } a, x \geq 0.$$



Special Distributions

Proof:

$$\begin{aligned} P(X > x + a | X > a) &= \frac{P\big(X > x + a, X > a\big)}{P(X > a)} = \frac{P(X > x + a)}{P(X > a)} \\ &= \frac{1 - F_X(x + a)}{1 - F_X(a)} = \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} \\ &= P(X > x). \end{aligned}$$

Special Distributions

Exponential random variables are often used to model times between arrivals at service centers (these are called interarrival times) – the memoryless property says that no matter how long you have waited for an arrival ($X > a$), the probability that you will have to wait for more than x more time instants is always $P(X > x)$ (that is, it doesn't depend on a).

Special Distributions

- **Normal (Gaussian) Distribution:**

  The normal distribution is by far the most important probability distribution. One of the main reasons for that is the Central Limit Theorem (CLT).

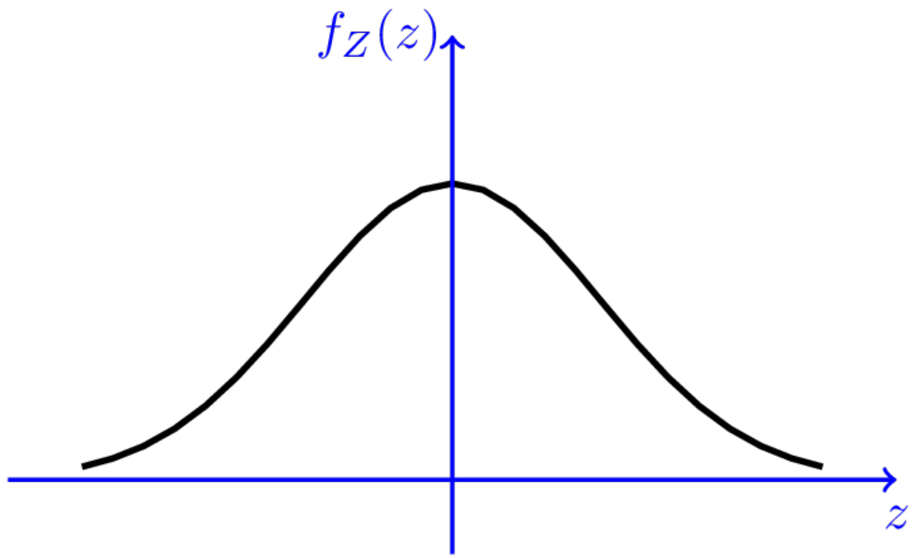
Special Distributions

As we'll see later: The Central Limit Theorem says that random variables that are formed as sums of many independent random variables have distributions that are approximately Gaussian (e.g., the random voltage caused by electronic noise that results from the summed contributions of a very large number of randomly-located charge carriers).

Special Distributions

- **Standard normal random variable:** $\boldsymbol{Z \sim N(0,1)}$

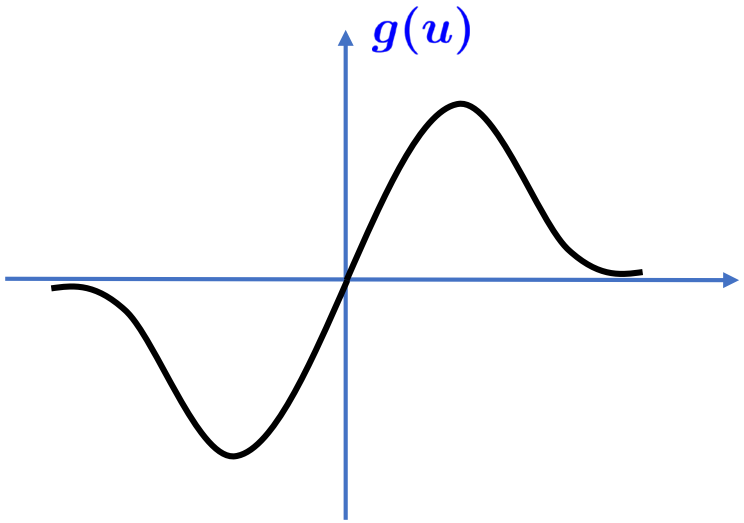
$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad \text{for all } z \in \mathbb{R}.$$



Special Distributions

Expected value of the standard normal:

$$E[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u e^{-\frac{u^2}{2}}}_{g(u)} du = 0,$$



Since $g(u)$ is an odd function.

Special Distributions

Variance of the standard normal:

$$\mathrm{Var}(Z) = E[Z^2] - (\underset{\substack{\| \\ 0}}{EZ})^2 = E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} du = 1.$$

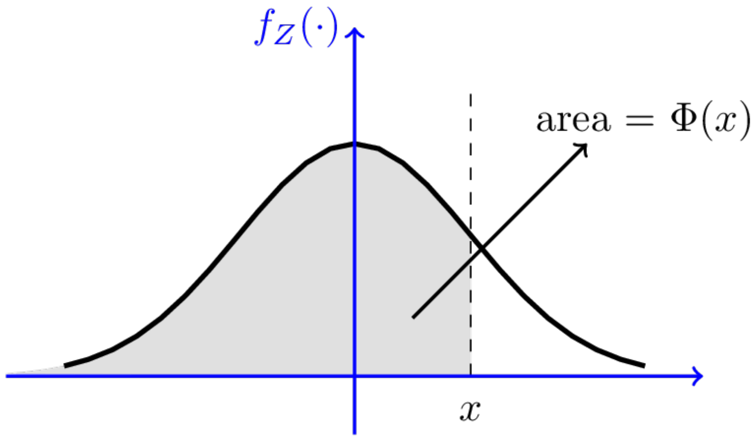
Integration by parts.

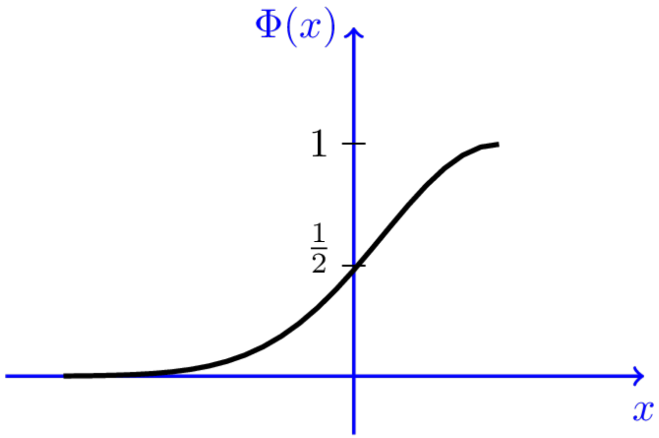
$$Z \sim N(0,1) \;\Rightarrow \begin{cases} E[Z] = 0 \\ \mathrm{Var}(Z) = 1 \end{cases}$$

Special Distributions

CDF of the standard normal: The CDF of the standard normal distribution is denoted by the Φ function.

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$





Special Distributions

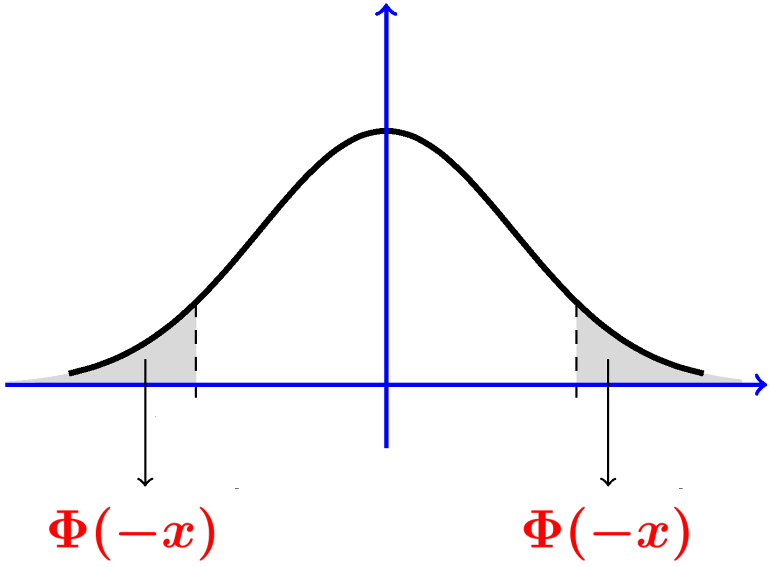
Here are some properties of the $\mathbf{\Phi}$ function:

1) $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0;$

2) $\Phi(0) = \frac{1}{2};$

3) $\Phi(-x) = 1 - \Phi(x), \text{ for all } x \in \mathbb{R};$

4) $\Phi(-x) + \Phi(x) = 1.$



Special Distributions

The standard normal CDF $\Phi(z)$ can't be evaluated in closed form (except at a few specific values of z), but numerical approximations are widely available

Special Distributions

- **General Normal random variables:**

$$X = \sigma Z + \mu, \qquad \text{where } \sigma > 0,\ \mu \in \mathbb{R}.$$

$$Z \sim N(0,1)$$

$$\Rightarrow\ X \sim N(\mu, \sigma^2).$$

$$EX = E[\sigma Z + \mu] = \sigma \underbrace{EZ}_{0} + \mu = \mu, \quad \text{(linearity of expectation)}$$

Special Distributions

- **General Normal random variables:**

$$\mathrm{Var}(X) = \mathrm{Var}(\sigma Z + \mu) = \sigma^2 \underbrace{\mathrm{Var}(Z)}_{1} = \sigma^2.$$

$$X \sim N(\mu, \sigma^2) \;\Rightarrow\; \begin{cases} E[X] = \mu \\ \mathrm{Var}(X) = \sigma^2 \end{cases}$$

Special Distributions

CDF and PDF of Normal random variables ($X \sim N(\mu, \sigma^2)$):

$$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x)$$
$$= P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Special Distributions

If $X \sim N(\mu, \sigma^2)$, $E[X] = \mu$; $\mathrm{Var}(X) = \sigma^2$,

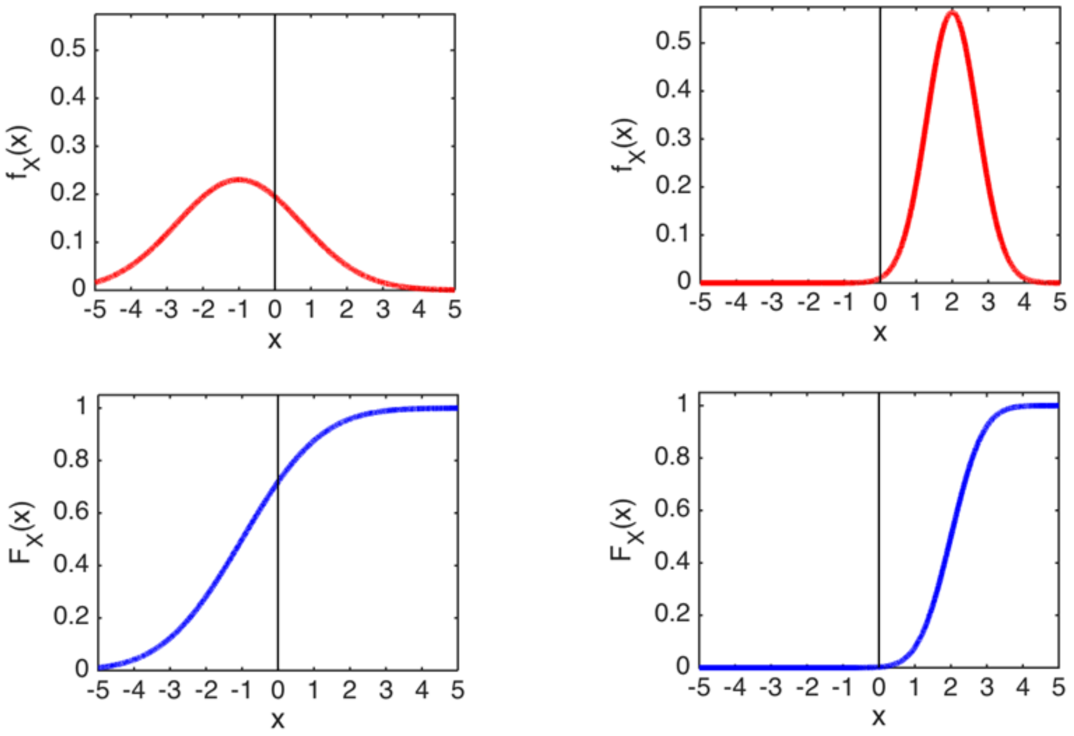
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

$$F_X(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$P(a < X \leq b) = F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Special Distributions

Example. **PDF (top) and CDF (bottom) of $N(-1, 3)$ random variable (left) and $N(2, 0.5)$ random variable (right):**



Special Distributions

Example. **Suppose that** $X \sim N(\mu, \sigma^2)$. **Then:**

$$\begin{aligned}
P(|X-\mu| \le \sigma) &= P(\mu-\sigma \le X \le \mu+\sigma) \\
&= \Phi\left(\frac{\mu+\sigma-\mu}{\sigma}\right) - \Phi\left(\frac{\mu-\sigma-\mu}{\sigma}\right) \\
&= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827
\end{aligned}$$

Special Distributions

$$\begin{aligned}
P(|X-\mu| \le 3\sigma) &= P(\mu - 3\sigma \le X \le \mu + 3\sigma) \\
&= \Phi\left(\frac{\mu + 3\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 3\sigma - \mu}{\sigma}\right) \\
&= \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.9973
\end{aligned}$$

Special Distributions

- **So, the probability that a Gaussian random variable takes a value within 1 standard deviation from its mean is 0.6827.**

- **The probability that a Gaussian random variable takes a value within 3 standard deviations from its mean is 0.9973.**

Special Distributions

Theorem. If $X \sim N(\mu_X, \sigma_X^2)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = a\mu_X + b, \quad \sigma_Y^2 = a^2\sigma_X^2.$$

$$E[Y] = E[aX + b] = aEX + b = a\mu + b,$$

$$\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) = a^2\sigma^2.$$

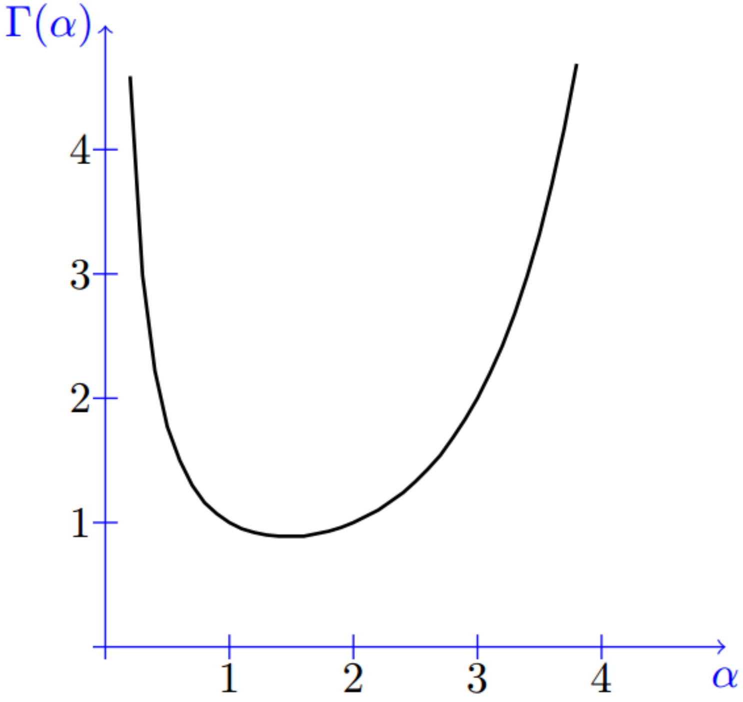
Gamma Distribution

Gamma function: $X \sim Gamma(\alpha, \lambda)$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x, \quad \text{for } \alpha > 0.$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \text{for } \alpha > 0.$$

$$\Gamma(n) = (n-1)!, \ \text{for } n \in \mathbb{N};$$



Gamma Distribution

Gamma Distribution:

$$f_X(x) = \frac{\lambda^{\alpha} x^{\alpha - 1} e^{-\lambda x}}{\Gamma(\alpha)} \qquad x > 0$$

If $\alpha = 1$ **:**

- $f_X(x) = \lambda e^{-\lambda x} \qquad x > 0$
- $Gamma(1, \lambda) = Exponential(\lambda)$

Gamma Distribution

If $X_1, X_2, \cdots, X_n$ are independent, then

$$X_1, X_2, \cdots, X_n \sim Exponential(\lambda)$$

$$X_1 + X_2 + \cdots + X_n \sim Gamma(n, \lambda)$$