Probability and Random Process (SWE3026)

Random Processes

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Discrete-Time Markov Chains

$X_0, X_1, X_2, X_3, \cdots \leftarrow$ a random process

In real life usually there is dependence between X_i's .

Markov chain:

X_m depends on X_{m-1} but not on the other pervious values.

Usually X_i shows the state of the system at time i .

W.L.G $X_i \in \{0, 1, 2, \cdots\}$

Discrete-Time Markov Chains

Discrete-Time Markov Chains

Consider the random process $\{X_n, n = 0, 1, 2, \cdots\}$, where $R_{X_i} = S \subset \{0, 1, 2, \cdots\}$. We say that this process is a Markov chain if

$$\begin{aligned} P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \cdots, X_0 = i_0) \\ = P(X_{m+1} = j | X_m = i), \end{aligned}$$

for all $m, j, i, i_0, i_1, \cdots i_{m-1}$. If the number of states is finite, e.g., $S = \{0, 1, 2, \cdots, r\}$, we call it a finite Markov chain.

Discrete-Time Markov Chains

Transition probabilities

$$p_{ij} = P(X_{m+1} = j | X_m = i).$$

State Transition Matrix and Diagram

State Transition Matrix:

We often list the transition probabilities in a matrix. The matrix is called the state transition matrix or transition probability matrix and is usually shown by P .

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}.$$

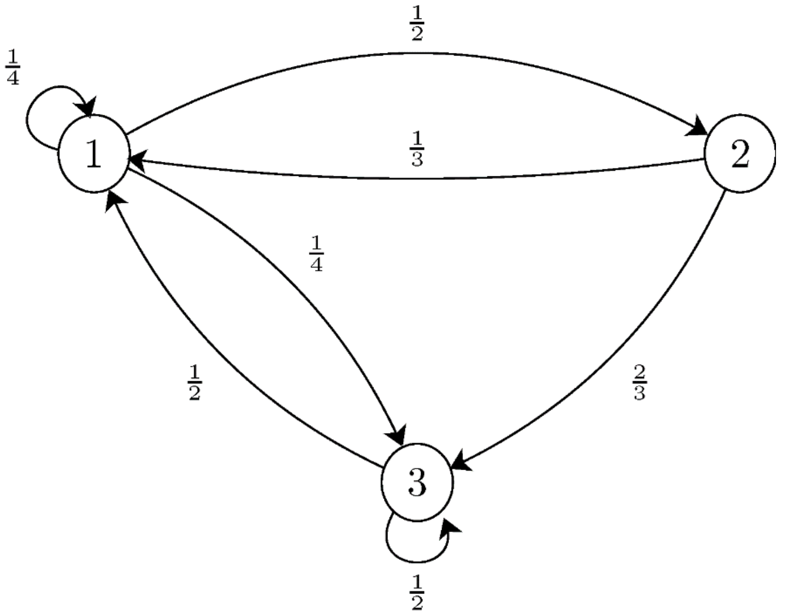
State Transition Matrix and Diagram

State Transition Diagram:

A Markov chain is usually shown by a state transition diagram.

Example.

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$



State Transition Matrix and Diagram

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Example. **Consider the Markov chain shown in pervious example.**

a) **Find** $P(X_4 = 3|X_3 = 2)$.

b) **Find** $P(X_3 = 1|X_2 = 1)$.

c) **If we know** $P(X_0 = 1) = 1/3$, **find** $P(X_0 = 1, X_1 = 2)$.

d) **If we know** $P(X_0 = 1) = 1/3$, **find** $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

Probability Distributions

State Probability Distributions:

Consider a Markov chain $\{X_n, n = 0, 1, 2, ...\}$, where $X_n \in S = \{1, 2, \cdots, r\}$.

$$\pi^{(0)} = \begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & \cdots & P(X_0 = r) \end{bmatrix}.$$

time 0

$$\pi^{(1)} = \begin{bmatrix} P(X_1 = 1) & P(X_1 = 2) & \cdots & P(X_1 = r) \end{bmatrix}.$$

In general:

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 1) & P(X_n = 2) & \cdots & P(X_n = r) \end{bmatrix}.$$

Probability Distributions

If we have $\pi^{(0)}$, how do we find $\pi^{(1)}$?

$$\begin{aligned} P(X_1 = 1) &= \sum_{k=1}^{r} P(X_1 = 1 | X_0 = k) P(X_0 = k) \\ &= \sum_{k=1}^{r} P_{k1} P(X_0 = k) \end{aligned}$$

Probability Distributions

Generally:

$$P(X_1 = j) = \sum_{k=1}^{r} P(X_0 = k) P_{kj}$$

$$\begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & \cdots & P(X_0 = r) \end{bmatrix} \cdot \begin{bmatrix} P_{1j} \\ P_{2j} \\ \cdot \\ \cdot \\ \cdot \\ P_{rj} \end{bmatrix}$$

Probability Distributions

In matrix form:

$$\pi^{(1)} = \begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & \cdots & P(X_0 = r) \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}$$

$$= \pi^{(0)} P.$$

Probability Distributions

More generally:

$$\pi^{(n+1)} = \pi^{(n)} P, \quad \text{for } n = 0, 1, 2, \cdots ;$$

$$\pi^{(n)} = \pi^{(0)} P^n, \quad \text{for } n = 0, 1, 2, \cdots .$$

Probability Distributions

Example. **Consider a system that can be in one of two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by**

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Suppose that the system is in state 0 at time $n = 0$, i.e., $X_0 = 0$.

a) **Draw the state transition diagram.**

b) **Find the probability that the system is in state 1 at time $n = 3$.**

Probability Distributions

n -Step Transition Probabilities:

LOTP

$$p_{ij}^{(2)} = P(X_2 = j | X_0 = i) = \sum_{k \in S} p_{ik} p_{kj}$$

Probability of going from state i to state j in exactly two transitions.

Probability Distributions

Generally:

p_{ij}^{m} **: Probability of going from state i to state j in exactly m transitions.**

$$
\begin{aligned}
p_{ij}^{(m+n)} &= P(X_{m+n} = j | X_0 = i) \\
&= \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}.
\end{aligned}
$$

LOTP

Probability Distributions

Two step transition matrix:

$$P^{(2)} = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} & \cdots & p_{1r}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & \cdots & p_{2r}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1}^{(2)} & p_{r2}^{(2)} & \cdots & p_{rr}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix} = P^2.$$

Probability Distributions

The Chapman-Kolmogorov equation can be written as

$$\begin{aligned} p_{ij}^{(m+n)} &= P(X_{m+n} = j | X_0 = i) \\ &= \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}. \end{aligned}$$

The n-step transition matrix is given by

$$P^{(n)} = P^n, \quad \text{for } n = 1, 2, 3, \cdots .$$

Classification of States

Two states i and j are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words,

$$i \leftrightarrow j \quad \text{means} \quad i \rightarrow j \text{ and } j \rightarrow i.$$

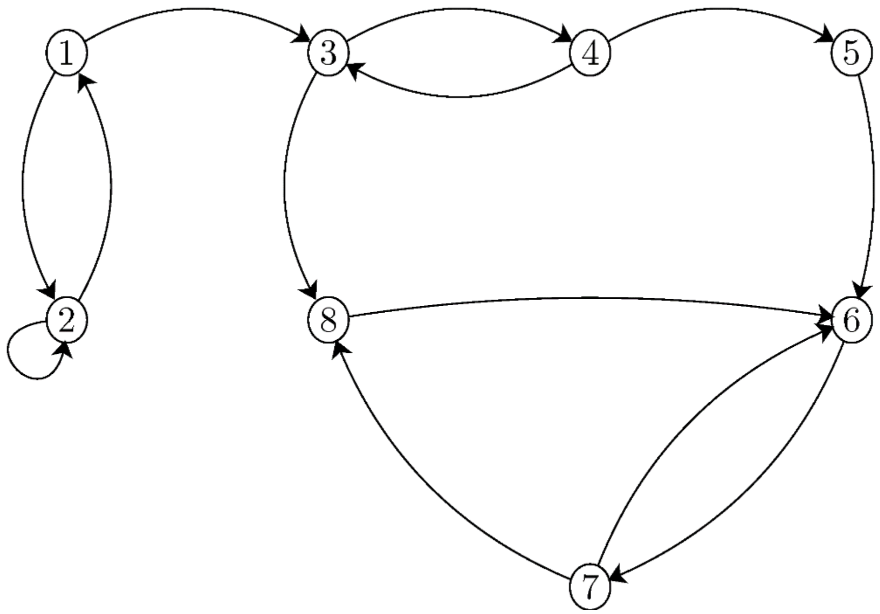
Classification of States

Communication is an equivalence relation. That means that

- **Every state communicates with itself** $i \leftrightarrow i$;
- **If** $i \leftrightarrow j$, **then** $j \leftrightarrow i$;
- **If** $i \leftrightarrow j$ **and** $j \leftrightarrow k$, **then** $i \leftrightarrow k$.

Classification of States

Example. **Consider the Markov chain shown in the following Figure. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the equivalence classes for this Markov chain.**



Classification of States

A Markov chain is said to be irreducible if all states communicate with each other.

Recurrent and transient states:

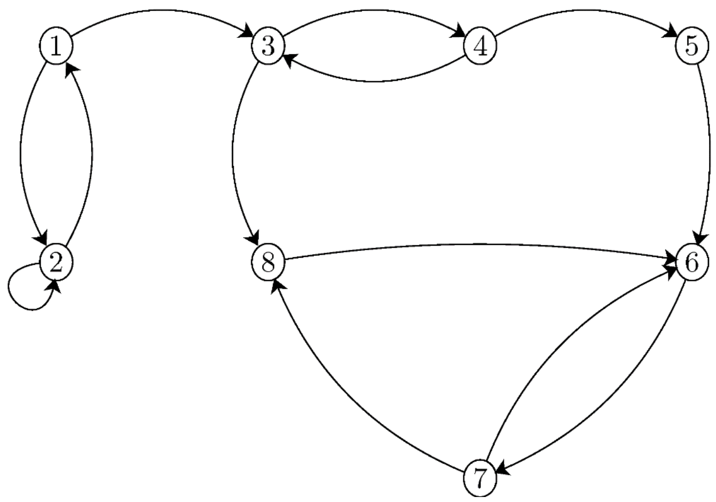
$$f_{ii} = P(X_n = i, \text{ for some } n \geq 1 | X_0 = i).$$

recurrent if $f_{ii} = 1;$

transient if $f_{ii} < 1.$

Classification of States

Consider a discrete-time Markov chain. Let V be the total number of visits to state i .



a. If i is a recurrent state, then

$$P(V = \infty | X_0 = i) = 1.$$

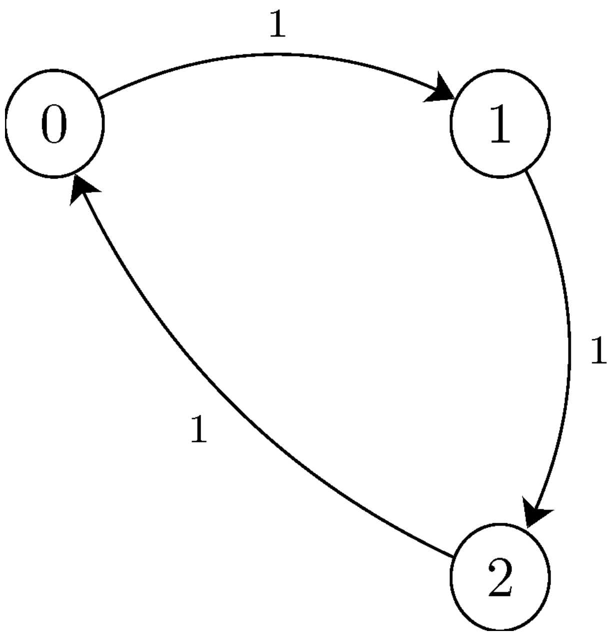
b. If i is a transient state, then

$$V | X_0 = i \ \sim \ Geometric(1 - f_{ii}).$$

Classification of States

Periodicity:

Consider the Markov chain shown in the following Figure. There is a periodic pattern in this chain.



Classification of States

The period of a state i is the largest integer d satisfying the following property: $p_{ii}^{(n)} = 0$, where n is not divisible by d. The period of i is shown by $d(i)$. If $p_{ii}^{(n)} = 0$, for all $n > 0$, then we let $d(i) = \infty$.

- If $d(i) > 1$, we say that state i is periodic.
- If $d(i) = 1$, we say that state i is aperiodic.

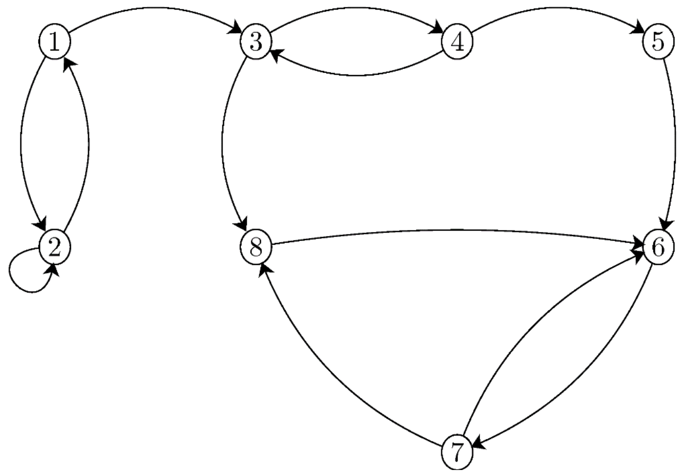
➢ If $i \leftrightarrow j$, then $d(i) = d(j)$.

Classification of States

Consider a finite irreducible Markov chain X_n:

a) If there is a self-transition in the chain ($p_{ii} > 0$ for some i), then the chain is aperiodic.

b) Suppose that you can go from state i to state i in l steps, i.e., $p_{ii}^{(l)} > 0$. Also suppose that $p_{ii}^{(m)} > 0$. If $\gcd(l, m) = 1$, then state i is aperiodic.



Classification of States

c) The chain is aperiodic if and only if there exists a positive integer n such that all elements of the matrix P^n are strictly positive, i.e.,

$$\boldsymbol{p_{ij}^{(n)} > 0, \quad \text{for all } i, j \in S.}$$