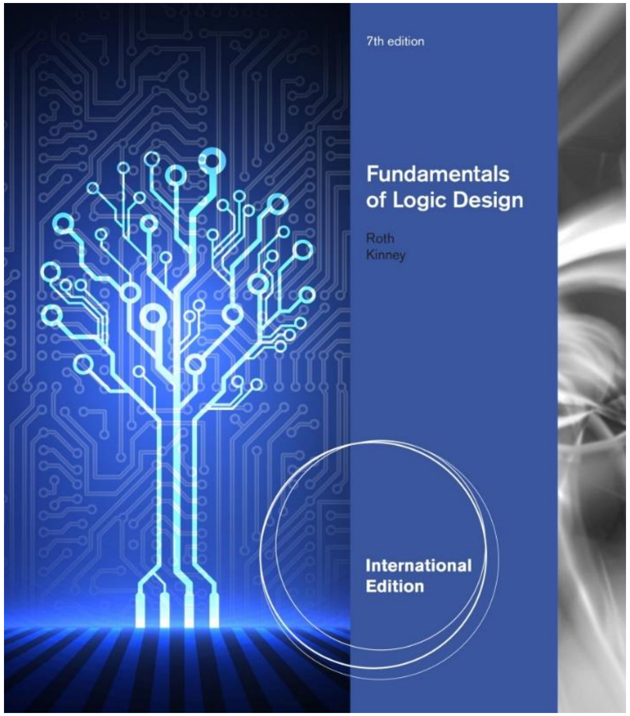


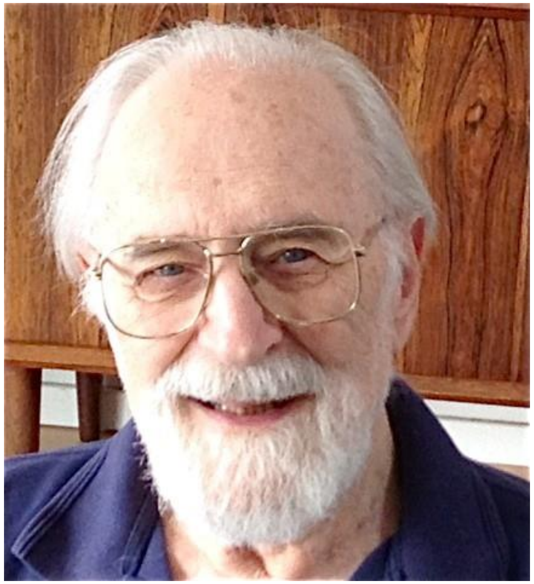
Karnaugh Maps

성균관대학교
SUNG KYUN KWAN UNIVERSITY

Dr. Maurice Karnaugh

Maurice Karnaugh

- Born in October 4, 1924 in New York
- Studied mathematics and physics at City College of New York (1944~1948)
- Transferred to Yale University to complete his B.Sc. (1949), M. Sc. (1950), and Ph.D. (1952) in Physics
- Worked at Bell Labs (1952~1966), developing the Karnaugh map (1954)
- Worked at IBM's Federal Systems Division (1966~1970)
- Held an adjunct position at New York University Tandon School of Engineering (1980~1999)
- IEEE Fellow (1976)
- Famous for Karnaugh map, PCM encoding, magnetic logic circuits and coding, etc.



Maurice Karnaugh (1924~)

Contents

Objectives

- Given a function (completely or incompletely specified) of three to five variables, plot it on a Karnaugh map. The function may be given in minterm, maxterm, or algebraic form.
- Determine the essential prime implicants of a function from a map.
- Obtain the minimum sum-of-products or minimum product-of-sums form of a function from the map.
- Determine all of the prime implicants of a function from a map.
- Understand the relation between operations performed using the map and the corresponding algebraic operations.

Minimum Forms of Switching Functions

Minimum Sum-of-Products:

- A **minimum sum-of-products** expression for a function is defined as a sum of product terms which (a) has a minimum number of terms and (b) of all those expressions which have the same minimum number of terms, has a minimum number of literals.
- It corresponds directly to a minimum two-level gate circuit which has a minimum number of gates and gate inputs.

How to Find a Minimum Sum-of-Products Given a Minterm Expansion:

- Combine terms by using the uniting theorem $XY + XY' = X$. *Do this repeatedly* to eliminate as many literals as possible. A given term may be used more than once because $X + X = X$.
- Eliminate redundant terms by using the consensus theorem or other theorems.

Minimum Forms of Switching Functions

Definition:

- Implicant
  - ✓ Any single 1 or any group of 1's which can be combined together on a map of the function *F* represents a product term which is called an *implicant* of *F*.
- Prime Implicant
  - ✓ A product term implicant is called a *prime implicant* if it cannot be combined with another term to eliminate a variable.
- Essential Prime Implicant
  - ✓ If a minterm is covered by only one prime implicant, that prime implicant is said to be *essential*, and <u>it must be included in the minimum sum of products</u>.

- Minimum sum-of-products form of the function *F* can be represented as
  - ✓ *F* = Essential Prime Implicant(s) + Prime Implicant(s)

Minimum Forms of Switching Functions

Example 1: Find a minimum sum-of-products expression for

$$F(a, b, c) = \Sigma\, m\,(0, 1, 2, 5, 6, 7)$$

$$F = a'b'c' + a'b'c + a'bc' + ab'c + abc' + abc$$ Implicants

$$= a'b' + b'c + bc' + ab$$ Prime Implicants

None of the terms in the above expression can be eliminated by consensus. However, combining terms in a different way leads directly to a minimum sum of products:

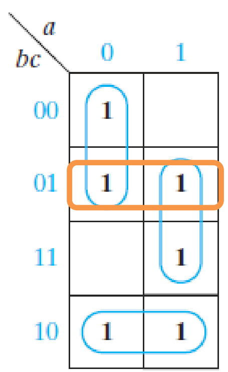
$$F = a'b'c' + a'b'c + a'bc' + ab'c + abc' + abc$$

$$= a'b' + bc' + ac \qquad (5\text{-}2)$$

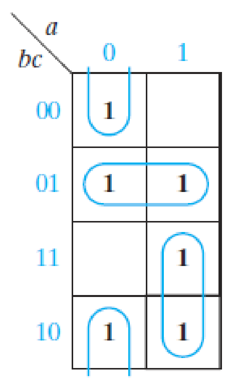
Minterm Expansion
Uniting Theorem
$XY' + XY = X$

If the uniting theorem is applied to all possible pairs of minterms, six two-literal products are obtained: $a'b'$, $a'c'$, $b'c$, bc', ac, ab. Then, the consensus theorem can be applied to obtain a second minimal solution:

$$a'c' + b'c + ab \qquad (5\text{-}3)$$



$F = a'b' + bc' + ac$



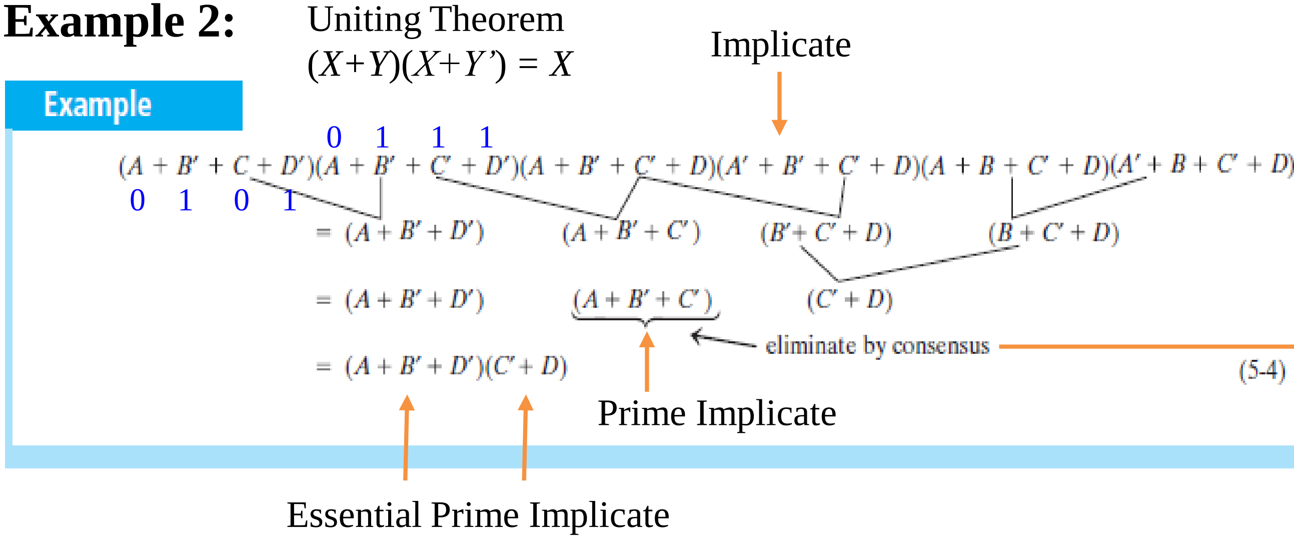
$F = a'c' + b'c + ab$

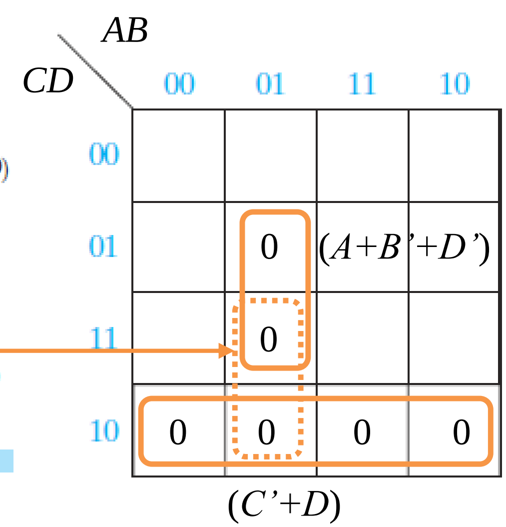
Minimum Product-of-Sums:

- A **minimum product-of-sums** expression for a function is defined as a product of sum terms which (a) has a minimum number of terms and (b) of all those expressions which have the same minimum number of terms, has a minimum number of literals.
- Given a maxterm expansion, the minimum product of sums can often be obtained by a procedure similar to that used in the minimum sum-of-products case, except that the uniting theorem $(X + Y)(X + Y') = X$ is used to combine terms.

Example 2:

Maxterm Expansion
Uniting Theorem
$(X+Y)(X+Y') = X$

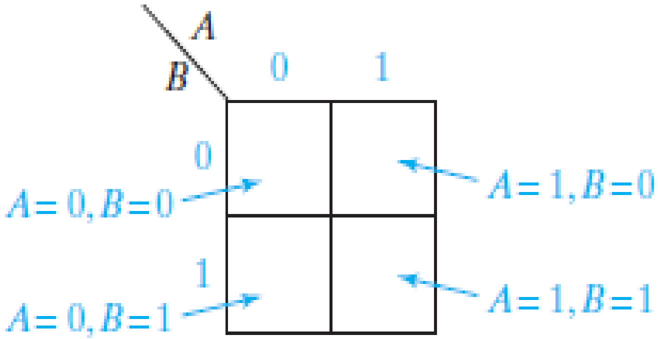




Two or Three-Variable Karnaugh Maps

Karnaugh Maps:

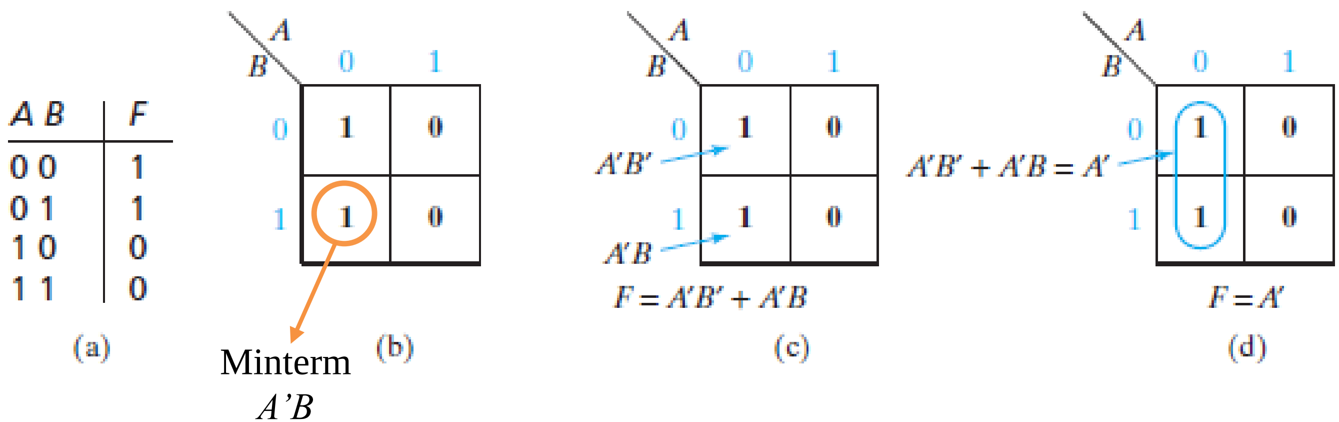
- A **Karnaugh map** is a systematic way of simplifying switching functions and lead directly to minimum cost two-level circuits composed of AND and OR gates.
- It specifies the value of the function for every combination of values of the independent variables.
- Two variable K-map



Two or Three-Variable Karnaugh Maps

Two Variable Karnaugh Maps:

- Note that the value of *F* for *A* = *B* = 0 is plotted in the upper left square, and the other map entries are plotted in a similar way in the figure below.
- Each 1 on the map corresponds to a minterm of *F*. For example, a 1 in square 01 indicates that *A'B* is a minterm.
- Minterms in adjacent squares of the map can be combined since they differ in only one variable.



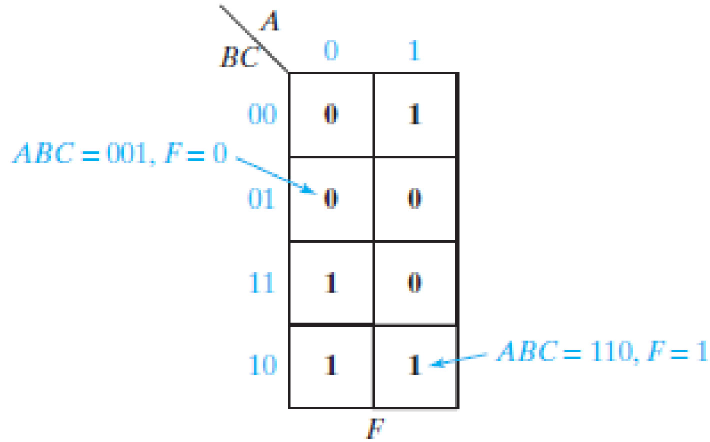
Two or Three-Variable Karnaugh Maps

Three-Variable Karnaugh Maps:

- A three-variable Karnaugh map can be plotted in a similar way to the two-variable map.
- The value of one variable, *A*, is listed on the top and the vales of the other two, *B* and *C*, are listed on the side.

A B C	F
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	1
1 0 1	0
1 1 0	1
1 1 1	0

(a)



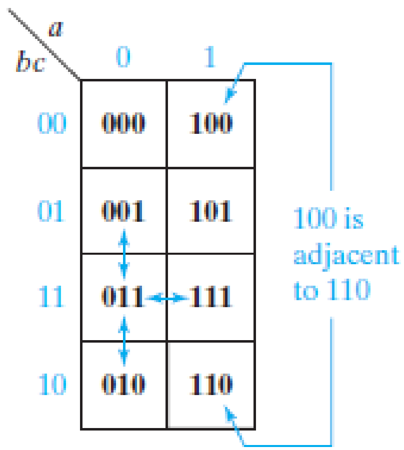
(b)

Two or Three-Variable Karnaugh Maps

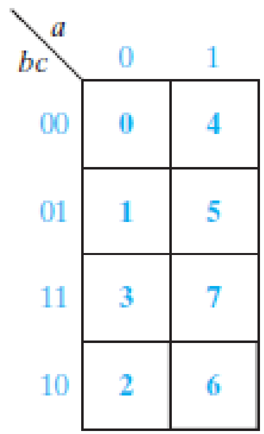
Locations of Minterms on a Karnaugh Map:

- Minterms in adjacent squares of the map differ in only one variable and therefore can be combined using the uniting theorem $XY + XY' = X$.

FIGURE 5-3 Location of Minterms on a Three-Variable Karnaugh Map



(a) Binary notation



(b) Decimal notation

Two or Three-Variable Karnaugh Maps

Mapping Minterm and Maxterm Expressions on Karnaugh Maps:

- Given the minterm or maxterm expansion of a function, it can be mapped on a Karnaugh map as follows:

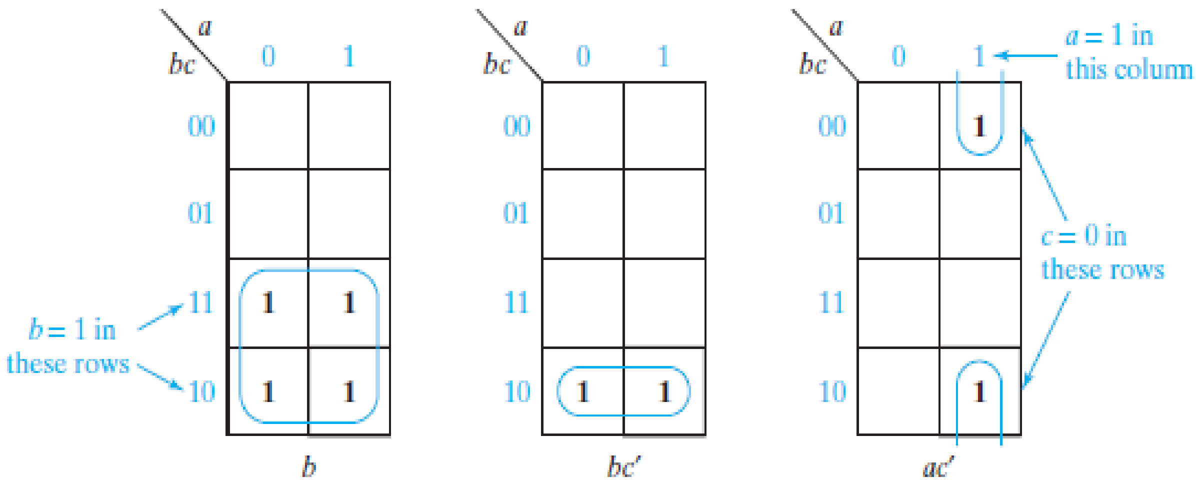
FIGURE 5-4
Karnaugh Map of
$F(a, b, c) =$
$\Sigma\, m(1, 3, 5) =$
$\Pi\, M(0, 2, 4, 6, 7)$

$bc \backslash a$	0	1
00	0 (0)	0 (4)
01	1 (1)	1 (5)
11	1 (3)	0 (7)
10	0 (2)	0 (6)

Plotting Product Terms:

- To plot the term *b*, 1's are entered in the four squares of the map where *b* = 1 as shown below:

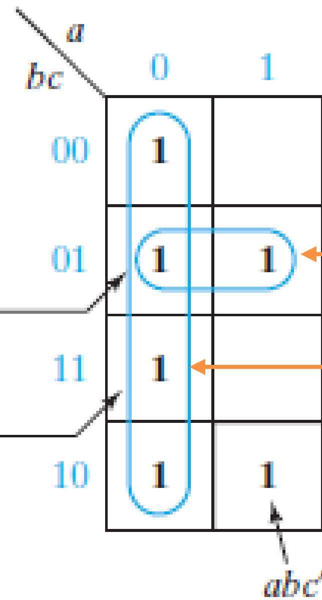
FIGURE 5-5
Karnaugh Maps for Product Terms



Plotting A Karnaugh Map using an Expression in Algebraic form:

- Given $f(a,b,c) = abc' + b'c + a'$, we would plot the map:

1. The term abc' is 1 when $a = 1$ and $bc = 10$, so we place a 1 in the square which corresponds to the $a = 1$ column and the $bc = 10$ row of the map.
2. The term $b'c$ is 1 when $bc = 01$, so we place 1's in both squares of the $bc = 01$ row of the map.
3. The term a' is 1 when $a = 0$, so we place 1's in all the squares of the $a = 0$ column of the map. (Note: Since there already is a 1 in the abc = 001 square, we do not have to place a second 1 there because $x + x = x$.)

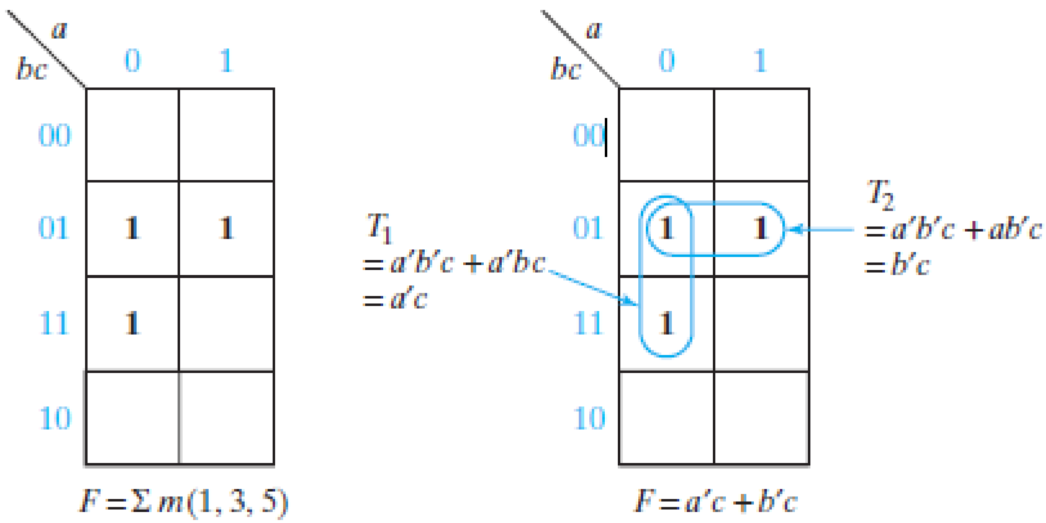


$a'b'c + ab'c = b'c$

$(a'b'c'+a'b'c)+(a'bc+a'bc')$
$= a'b' + a'b$
$= a'$

Simplifying Expressions:

FIGURE 5-6
Simplification of a Three-Variable Function

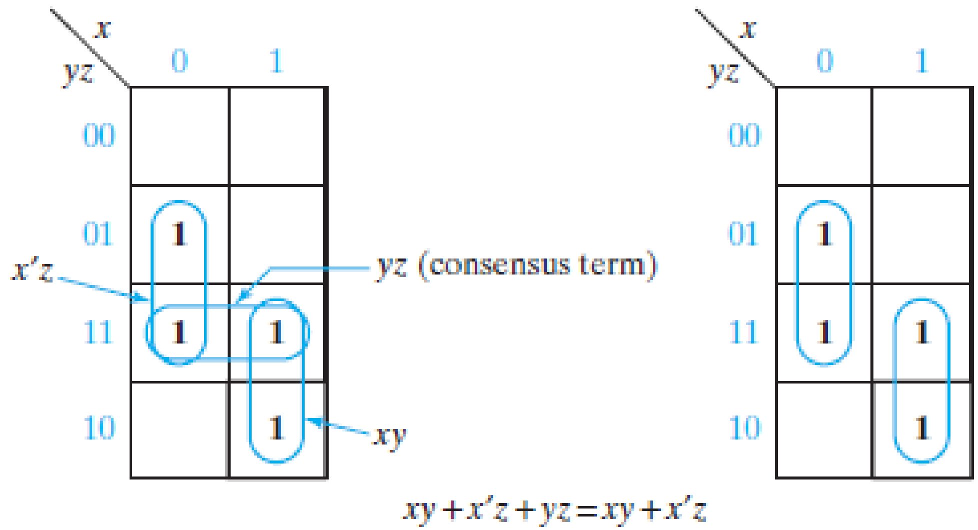


(a) Plot of minterms

(b) Simplified form of F

Consensus Theorem in Karnaugh Maps:

FIGURE 5-8
Karnaugh Maps that Illustrate the Consensus Theorem

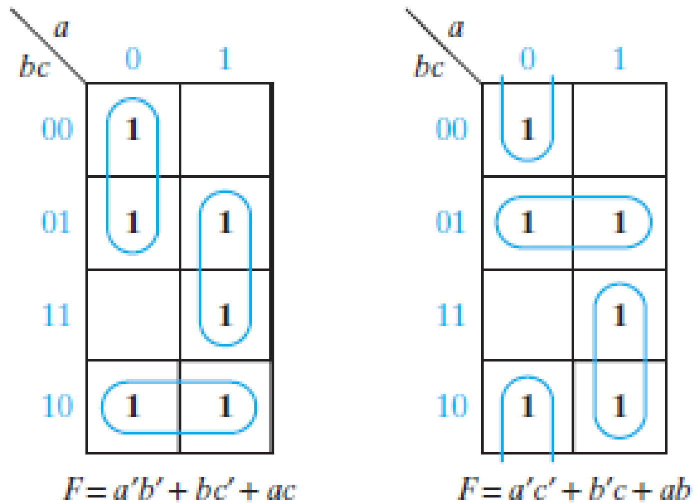


Two or Three-Variable Karnaugh Maps

If a function has two or more minimum sum-of-products forms, all of these forms can be determined from a map.

Figure 5-9 shows the two minimum solutions for $F = \Sigma\, m(0, 1, 2, 5, 6, 7)$.

FIGURE 5-9
Function with Two Minimum Forms



Four-Variable Karnaugh Maps

Location of terms on a Four-Variable K-map:

- m_5 (0101) could combine with m_1 (0001), m_4 (0100), m_{13} (0111), m_7 (1101) because it differs in only one variable from each of the other minterms.
- The definition of adjacent squares must be extended so that not only are <u>top and bottom</u> rows as in the three-variable map, but the <u>first and last columns</u> are also adjacent.
- Four corner terms (m_0, m_8, m_2, m_{10}) combine to give $b'd'$

FIGURE 5-10 Location of Minterms on Four-Variable Karnaugh Map

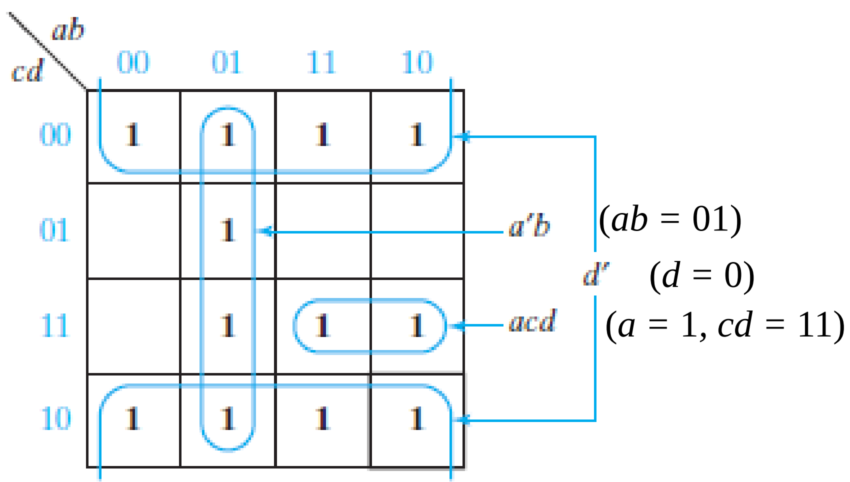
CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Four-Variable Karnaugh Maps

Plotting functions on a Four-Variable Karnaugh Map:

- This is accomplished in the same way as for two- or three-variable Karnaugh maps.
- "1"s are plotted for whichever values of the variables would result in the expression yielding "1".
- $F(a, b, c, d) = acd + a'b + d'$
  - ✓ The first term is 1 when $a = c = d = 1$, so we place 1's in the two squares which are in the $a = 1$ column and $cd = 11$ row.
  - ✓ The term $a'b$ is 1 when $ab = 01$, so we place four 1's in the $ab = 01$ column.
  - ✓ d' is 1 when $d = 0$, so we place eight 1's in the two rows for which $d = 0$.

FIGURE 5-11 Plot of $acd + a'b + d'$

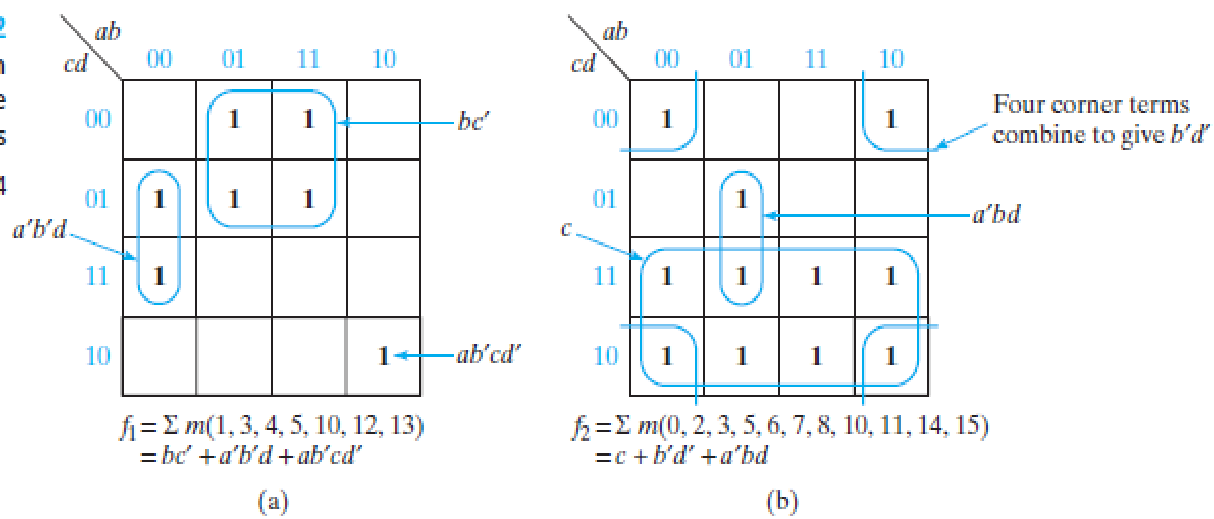


Four-Variable Karnaugh Maps

Simplifying Expressions in Four-Variable Karnaugh Maps:

- Minterms can be combined in groups of <u>two</u>, <u>four</u>, or <u>eight</u> to eliminate one, two, or three variables, respectively.
- $f_1 = \sum m(1, 3, 4, 5, 10, 12, 13)$
  - ✓ The pair of 1's in the *ab* = 00 column and also in the *d* = 1 rows represents *a'b'd*.
  - ✓ The group of four 1's in the *b* = 1 columns and *c* = 0 rows represents *bc'*.
- $f_2 = \sum m(0, 2, 3, 5, 6, 7, 8, 10, 11, 14, 15)$
  - ✓ Note that the four corner 1's span the *b* = 0 columns and *d* = 0 rows and, therefore, can be combined to form the term *b'd'*
  - ✓ The group of eight 1's covers both rows where *c* = 1 and, therefore, represents the term *c*.
  - ✓ The pair of 1's which is looped on the map represents the term *a'bd* because it is in the *ab* = 01 column and spans the *d* = 1 rows.

FIGURE 5-12 Simplification of Four-Variable Functions



Four-Variable Karnaugh Maps

Expressions with "don't cares":

- "Don't cares" are noted as *X*'s in Karnaugh maps.
- When choosing terms to form the minimum sum of products, all the 1's must be covered, but the X's are only used if they will simplify the resulting expression.
- The only don't care term used in forming the simplified expression is m_{13}

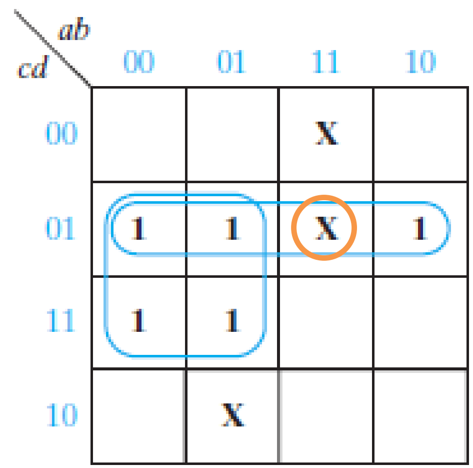


FIGURE 5-13 Simplification of an Incompletely Specified Function

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$$f = \Sigma\, m(1, 3, 5, 7, 9) + \Sigma\, d(6, 12, 13)$$
$$= a'd + c'd$$

From SOP to POS form using Karnaugh Maps:

- For the function specified below as *f*, the process of finding the product-of-sums from the sum-of-products is shown.

$$f = x'z' + wyz + w'y'z' + x'y$$

First, the 1's of f are plotted in Figure 5-14. Then, from the 0's,

$$f' = y'z + wxz' + w'xy$$

and the minimum product of sums for f is

$$f = (y + z')(w' + x' + z)(w + x' + y')$$

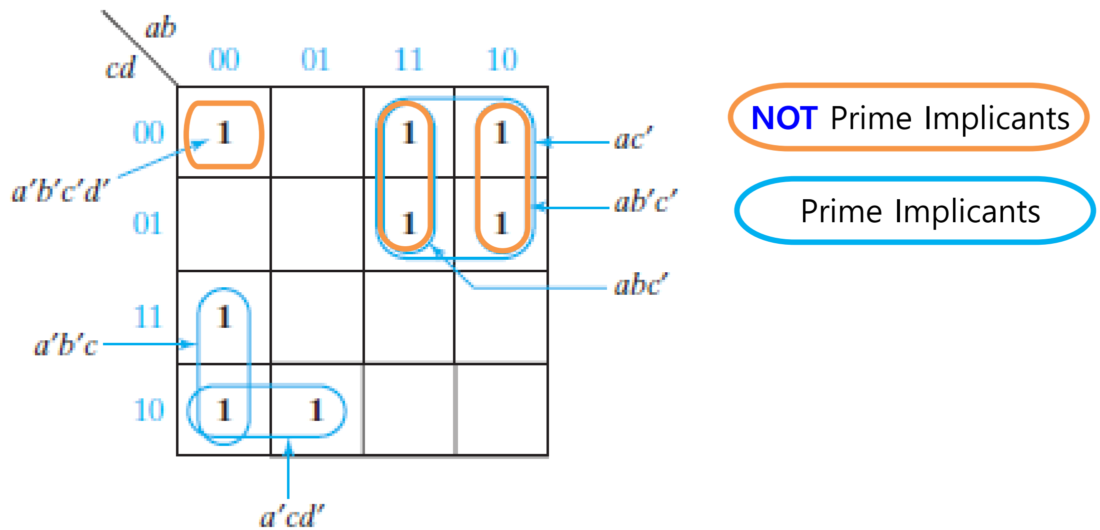
$yz \backslash wx$	00	01	11	10
00	1	1	0	1
01	0	0	0	0
11	1	0	1	1
10	1	0	0	1

Prime Implicants:

- A product term implicant is called a **prime implicant** if it cannot be combined with another term to eliminate a variable.

❑ *a'b'c*, *a'cd'*, and *ac'* are prime implicants
❑ *a'b'c'd'*, *abc'*, and *ab'c'* are not prime implicants

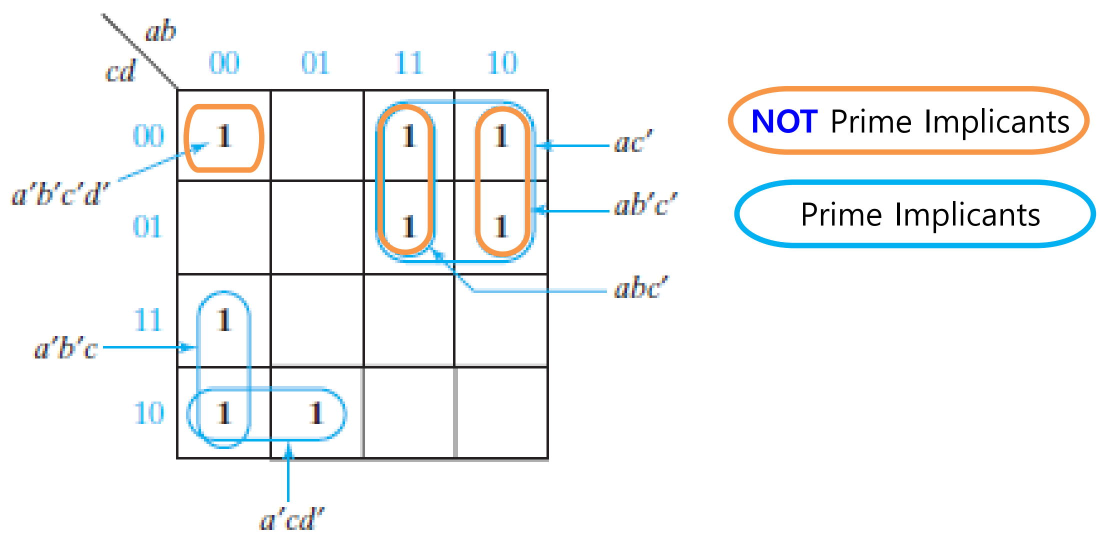
A sum-of-products expression containing a term which is not a prime implicant cannot be minimum.



Determination of Minimum Expressions Using Essential Prime Implicants

Prime Implicants:

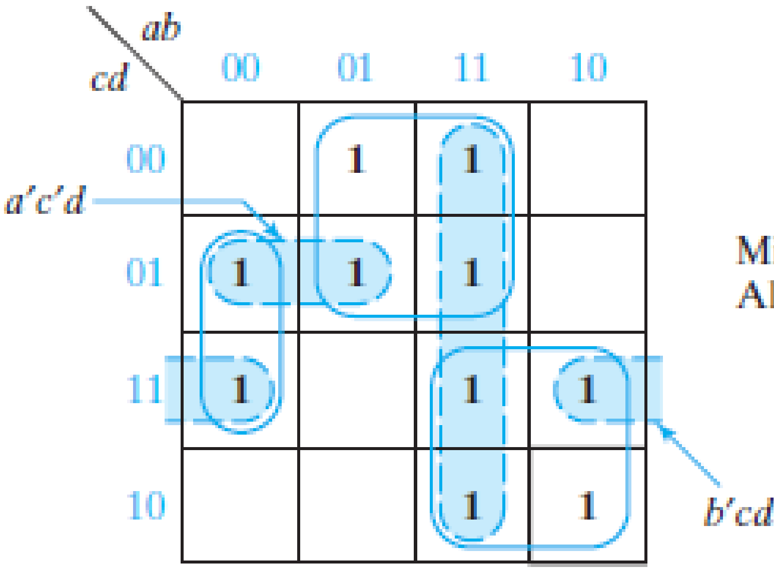
- A single 1 on a map represents a prime implicant if it is not adjacent to any other 1's.
- Two adjacent 1's on a map form a prime implicant if they are not contained in a group of four 1's
- Four adjacent 1's on a map form a prime implicant if they are not contained in a group of eight 1's



Determination of Minimum Expressions Using Essential Prime Implicants

Determination of All Prime Implicants:

The minimum solution may not include all prime implicants, as shown below:



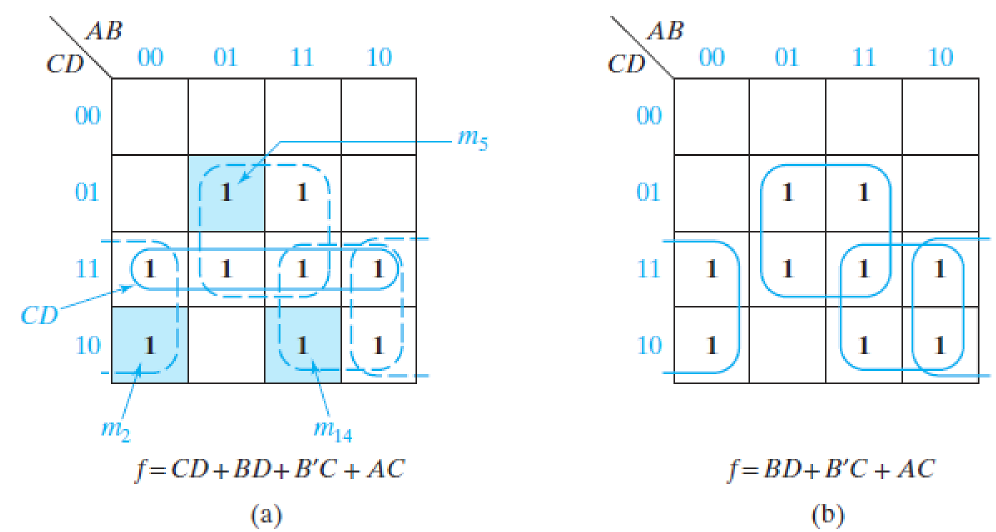
Minimum solution: $F = a'b'd + bc' + ac$

All prime implicants: $a'b'd$, bc', ac, $a'c'd$, ab, $b'cd$

Essential Prime Implicants:

- If a minterm is covered by only one prime implicant, that prime implicant is said to be **essential**, and it must be included in the minimum sum of products.
  - ✓ m_2 is covered only by prime implicant $B'C$
  - ✓ m_5 is covered only by prime inplicant BD
  - ✓ m_{14} is covered only by prime implicant AC
  - ✓ m_3 is covered by both $B'C$ and CD → *CD is not an essential prime implicants*
- In order to find a minimum sum of products from a map, we should first loop all of the essential prime implicants.

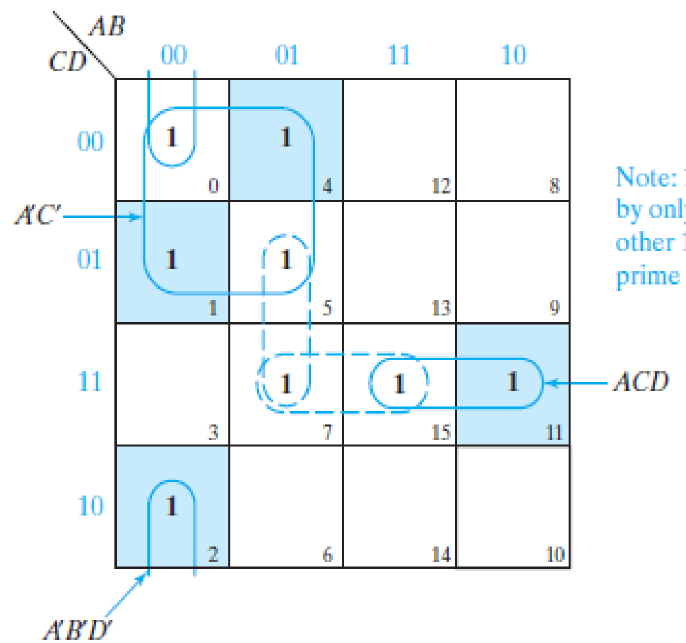
FIGURE 5-17



Finding Essential Prime Implicants:

- Sometimes essential prime implicants can be found by inspection.
- Other times, we must look at all squares adjacent to that minterm. If the given minterm and all of the 1's adjacent to it are covered by a single term, then that term is an *essential prime implicant.*
- If all of the 1's adjacent to a given minterm are *not covered by a single term, then* we cannot say whether these prime implicants are essential or not without checking the other minterms.

FIGURE 5-18



Note: 1's shaded in blue are covered by only one prime implicant. All other 1's are covered by at least two prime implicants.

[1]This statement is proved in Appendix D.

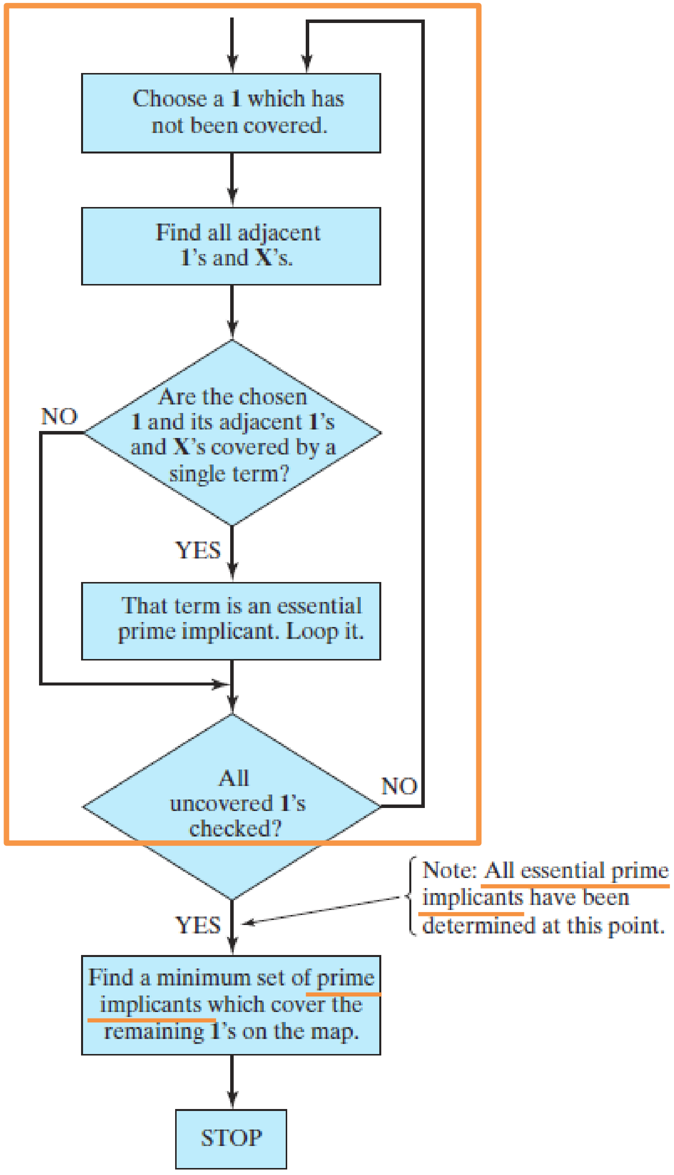
Determination of Minimum Expressions Using Essential Prime Implicants

Procedure to Obtain a Minimum Sum of Products from a Karnaugh Map:

1. Choose a minterm (a 1) which has not yet been covered.
2. Find all 1's and *X*'s adjacent to that minterm. (Check the *n adjacent squares on* an *n-variable map*.)
3. If a single term covers the minterm and all of the adjacent 1's and *X*'s, then that term is an essential prime implicant, so select that term. (Note that "don't-care" terms are treated like 1's in steps 2 and 3 but not in step 1.)
4. Repeat steps 1, 2, and 3 until all essential prime implicants have been chosen.
5. Find a minimum set of prime implicants which cover the remaining 1's on the map. If there is more than one such set, choose a set with a minimum number of literals.

FIGURE 5-19 Flowchart for Determining a Minimum Sum of Products Using a Karnaugh Map

Essential Prime Implicant



Alternative method to obtain minimum product-of-sum expressions for a function *f* :

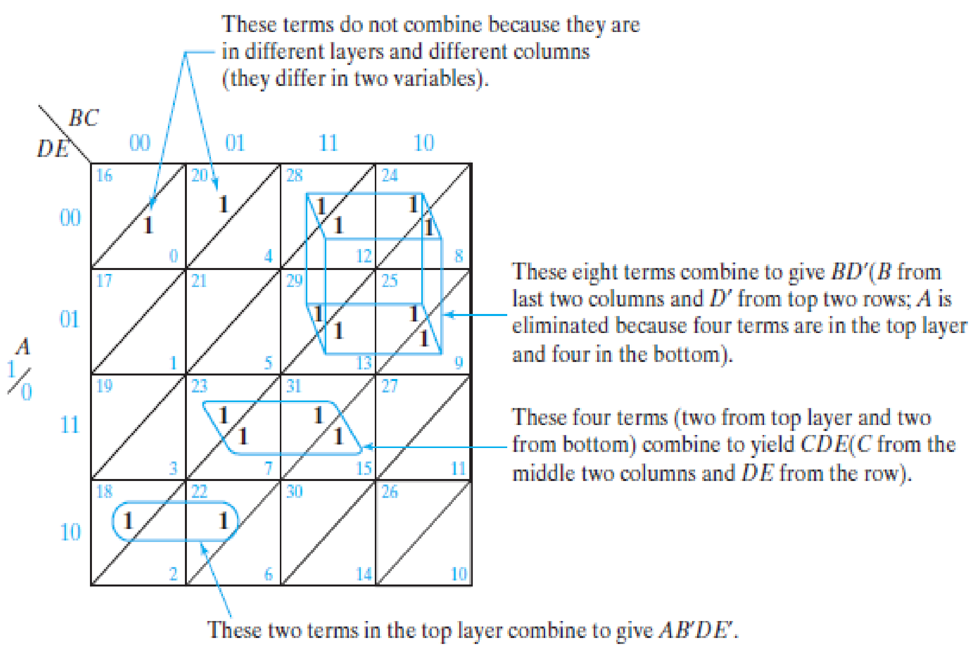
- Find minimum a sum-of-products expression for *f'*, and then complement *f'* to obtain a minimum product-of-sums expression for *f*.
- Alternatively, we can perform the dual of the procedure for finding minimum sum of products.
- Let *S* be a sum term. If every input combination for which *S* = 0, *f* is also 0, then *S* can be a term in a product-of-sums expression for *F*.
- We will call such a sum term an *implicate* of *f*.
- Implicate S is a *prime implicate* if it cannot be combined with any other implicate to eliminate a literal from *S*.
- <u>The prime implicates of *f* can be found by looping the largest groups of adjacent zeros on the Karnaugh map for *f*.</u>
- If a prime implicate is the only prime implicate covering a maxterm (zero) of *f*, then it is an *essential prime implicate* and must be included in any minimum product-of-sums expression for *f*.

Five-Variable Karnaugh Maps

Five-Variable Karnaugh Maps:

- A five-variable map can be constructed in three dimensions by placing one four-variable map on top of a second one.
- Terms in the bottom layer are numbered 0 through 15 and corresponding terms in the top layer are numbered 16 through 31, so that terms in the bottom layer contain *A' and those in the top layer contain A.*
- *To represent* the map in two dimensions, we will divide each square in a four-variable map by a diagonal line and place terms in the bottom layer below the line and terms in the top layer above the line.

FIGURE 5-21
A Five-Variable Karnaugh Map

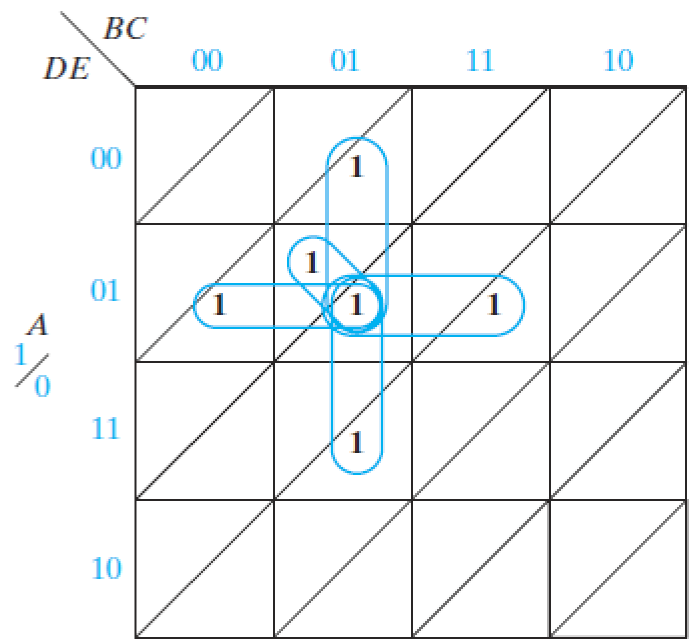


Five-Variable Karnaugh Maps

Checking for adjacency:

- Each term can be adjacent to exactly five other terms, four in the same layer and one in the other layer.
- Each term should be checked against the five possible adjacent squares.
- In general, the number of adjacent squares is equal to the number of variables

FIGURE 5-22



Five-Variable Karnaugh Maps

Example of Five-Variable Karnaugh Map:

- $F(A, B, C, D, E) = \sum m(0, 1, 4, 5, 13, 15, 20, 21, 22, 23, 24, 26, 28, 30, 31)$
- Prime implicant P_1 is chosen first because all of the 1's adjacent to minterm 0 are convered by P_1. (Essential Prime Implicant)
- Prime implicant P_2 is chosen next because all of the 1's adjacent to minterm 24 are convered by P_2. (Essential Prime Implicant)
- If we choose prime implicants P_3 and P_4 next, the remaining two 1's can be covered by two different groups of four, $A'D'C$ and $B'CD'$.
- Thus, $F = P_1 + P_2 + P_3 + P_4 + A'B'C$, or $F = P_1 + P_2 + P_3 + P_4 + B'CD'$

FIGURE 5-23

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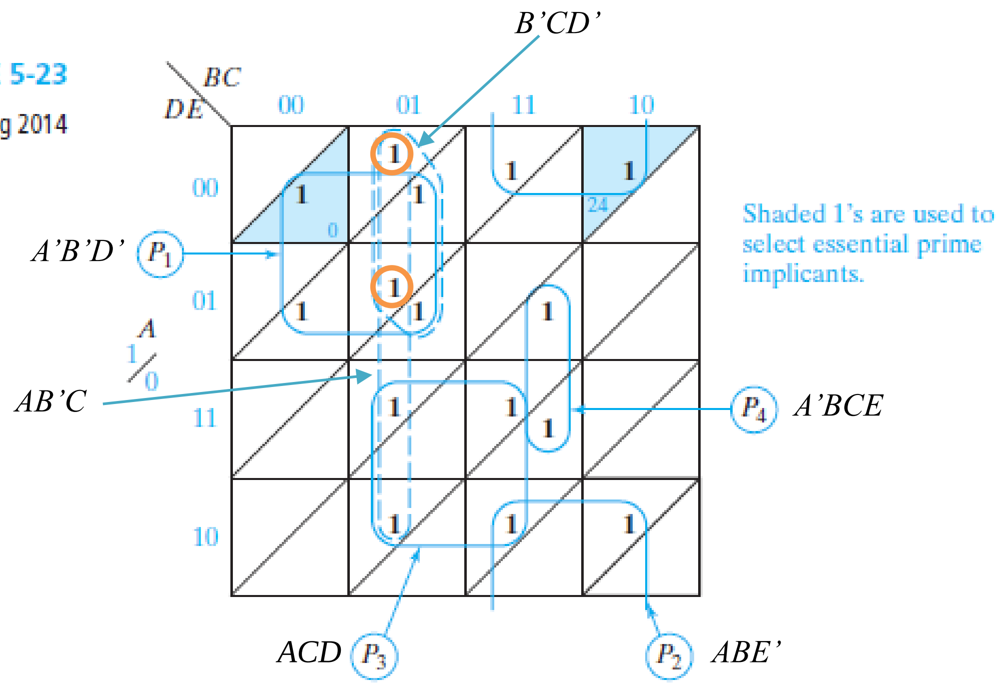
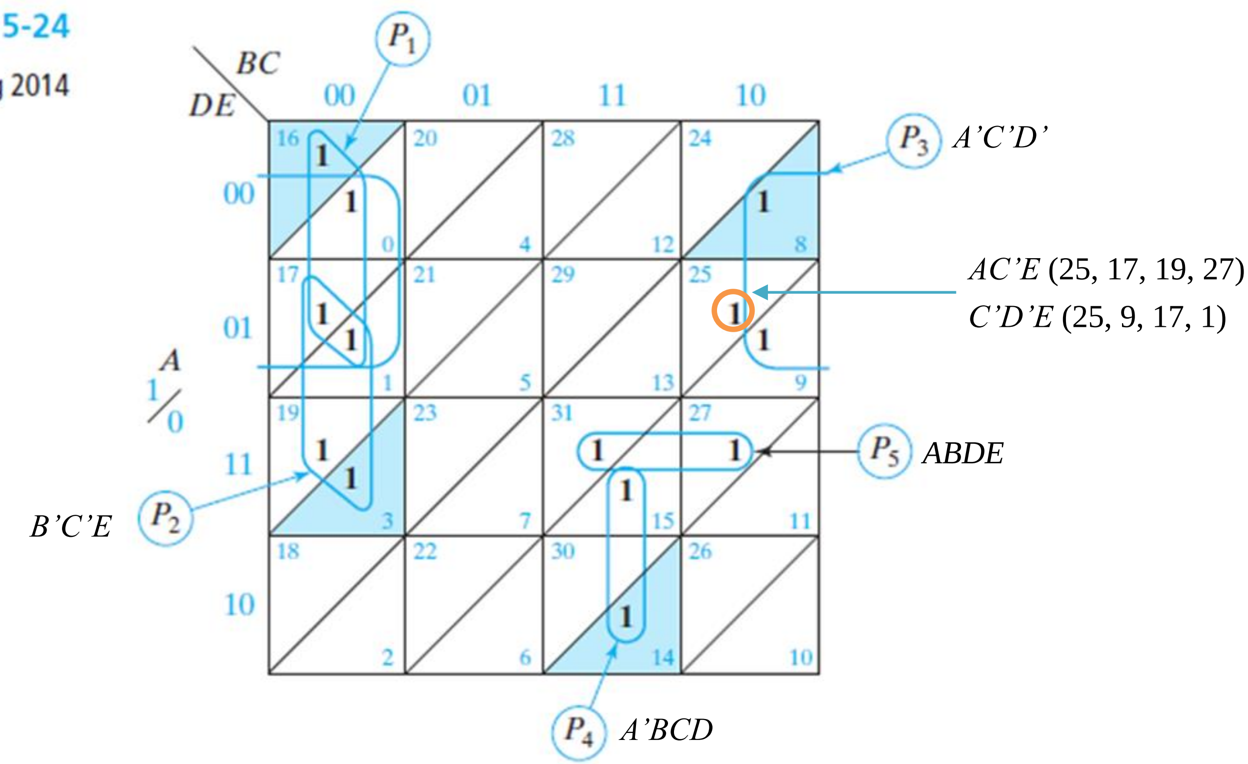


Example of Five-Variable Karnaugh Map:

- $F(A, B, C, D, E) = \sum m(0, 1, 3, 8, 9, 14, 15, 16, 17, 19, 25, 27, 31)$
- All 1's adjacent to m_{16} is covered by P_1. All 1's adjacent to m_3 is covered by P_2.
- All 1's adjacent to m_8 is covered by P_3. All 1's adjacent to m_{14} is covered by P_4.
- P_1, P_2, P_3, P_4 are essential prime implicants.
- The final solutions is

$$F = P_1 + P_2 + P_3 + P_4 + P_5 + AC'E \text{ , or } F = P_1 + P_2 + P_3 + P_4 + P_5 + C'D'E$$

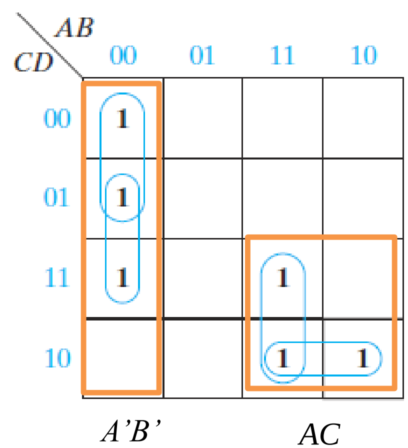
FIGURE 5-24



Other Uses of Karnaugh Maps:

- We can prove that two functions are equal by plotting them on maps and showing that they have the same Karnaugh map, i.e., minterm expansions for F (F = 1) and maxterm expansions for F' (F = 0)
- We can perform the AND operation (or the OR operation) on two functions by ANDing (or ORing) the 1's and 0's which appear in corresponding positions on their maps.
- A Karnaugh map can facilitate factoring an expression. Inspection of the map reveals terms which have one or more variables in common.

FIGURE 5-25



$F = A'B'(C' + D) + AC(B + D')$

Other Uses of Karnaugh Maps:

- When simplifying a function algebraically, the Karnaugh map can be used as a guide in determining what steps to take.
- Consider the function

$$F = ABCD + B'CDE + A'B' + BCE'$$

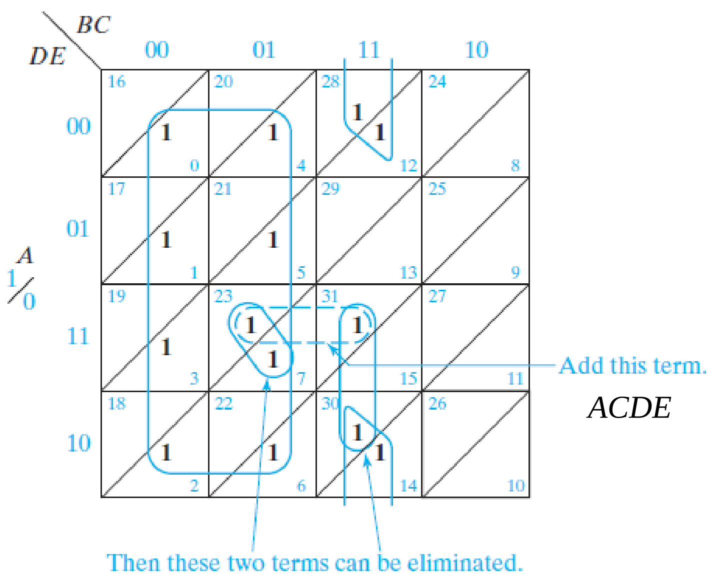
Add *ACDE* term using consensus theorem:

$$F = ABCD + B'CDE + A'B' + BCE' + ACDE$$

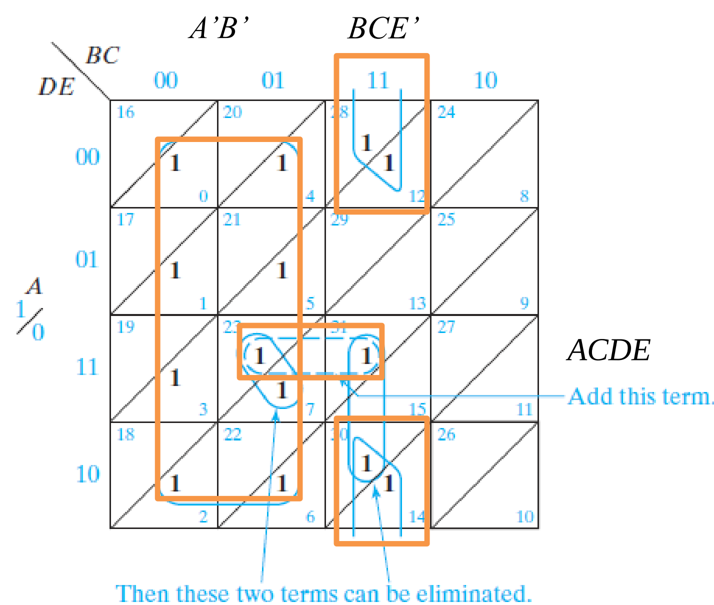
The minimum solution is

$$F = A'B' + BCE' + ACDE$$

FIGURE 5-26



(B'CDE, ABCD)

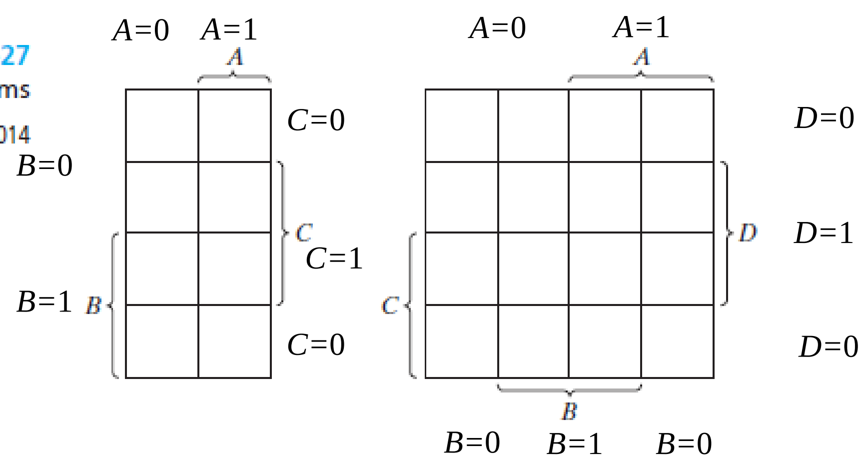


Other Forms of Karnaugh Maps

Veitch Diagrams:

- Instead of labeling the sides of a Karnaugh map with 0's and 1's, some people prefer to use the labeling shown below.
- For the half of the map labeled *A*, *A*=1; and for the other half, *A*=0.
- However, Karnaugh maps are more convenient to solve sequential circuit problems.

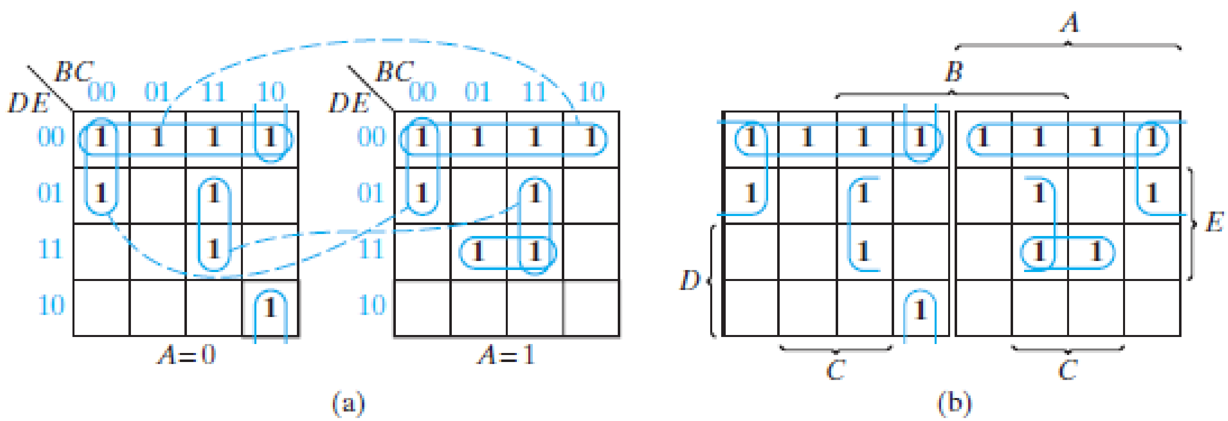
FIGURE 5-27
Veitch Diagrams



Other forms of Five-Variable Karnaugh Maps:

- One form simply consists of two four-variable maps side-by-side as in Figure 5-28(a).
- Figure 5-28(b) shows *mirror image map*, in which the first and eighth columns are "adjacent" as are second and seventh columns, third and sixth columns, and fourth and fifth columns.

FIGURE 5-28
Other Forms of Five-Variable Karnaugh Maps



$$F = D'E' + B'C'D' + BCE + A'BC'E' + ACDE$$