Probability and Random Process (SWE3026)

Joint Distributions

JinYeong Bak
jy.bak@skku.edu
College of Computing, SKKU

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Functions of Two Random Variables

Let X and Y are two random variables, and suppose that $Z = g(X, Y)$,

$\Rightarrow$ Z is random variable.

Law of the unconscious statistician (LOTUS) for two discrete random variables:

$$E[g(X)] = \sum_{x_i \in R_X} g(x_i) P_X(x_i)$$

$$E[g(X,Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j) P_{XY}(x_i, y_j)$$

Functions of Two Random Variables

Example.

Linearity of Expectation: For two discrete random variables X and Y, show that $E[X+Y] = EX + EY$.

$$
\begin{aligned}
E[X+Y] &= \sum_{(x_i,y_j)\in R_{XY}} (x_i + y_j) P_{XY}(x_i, y_j) \\
&= \sum_{(x_i,y_j)\in R_{XY}} x_i P_{XY}(x_i, y_j) + \sum_{(x_i,y_j)\in R_{XY}} y_j P_{XY}(x_i, y_j) \\
&= \sum_{x_i\in R_X} \sum_{y_j\in R_Y} x_i P_{XY}(x_i, y_j) + \sum_{x_i\in R_X} \sum_{y_j\in R_Y} y_j P_{XY}(x_i, y_j) \\
&= \sum_{x_i\in R_X} x_i \sum_{y_j\in R_Y} P_{XY}(x_i, y_j) + \sum_{y_j\in R_Y} y_j \sum_{x_i\in R_X} P_{XY}(x_i, y_j) \\
&= \sum_{x_i\in R_X} x_i P_X(x_i) + \sum_{y_j\in R_Y} y_j P_Y(y_j) \qquad \text{(marginal PMF (Equation 5.1))} \\
&= EX + EY.
\end{aligned}
$$

Functions of Two Random Variables

General scenario:

$$Z = g(X, Y),$$

Find PMF of Z.

1) $R_Z = \{g(x_i, y_j);\ (x_i, y_j) \in R_{XY}\},$

2) **For** $n \in R_Z;\ P_Z(n) = P(Z = n) = P(g(X, Y) = n)$

$$= \sum_{\substack{(x_i, y_j) \in R_{XY} \\ g(x_i, y_j) = n}} P_{XY}(x_i, y_j).$$

Conditional Expectation and Variance

Conditional Expectation as a Function of a Random Variable:

$$E[X|Y=y] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y).$$

Note that $E[X|Y=y]$ depends on the value of y, so we can write

$$g(y) = E[X|Y=y].$$

Conditional Expectation and Variance

Thus, we can think of $E[X|Y=y]$ as a function of the value of the random variable Y. We then write

$$g(Y) = E[X|Y].$$

If Y is a random variable with range $R_Y = \{y_1, y_2, \cdots\}$, then

$$E[X|Y] = \begin{cases} E[X|Y=y_1] & \text{with probability } P(Y=y_1) \\ E[X|Y=y_2] & \text{with probability } P(Y=y_2) \\ \vdots & \vdots \end{cases}$$

Conditional Expectation and Variance

Example. Consider two random variables X and Y with joint PMF given in the following Table $Z = E[X|Y]$.

$E[X|Y=0] = \frac{2}{3}$

a) Find the Marginal PMFs of X and Y.

b) Find the conditional PMF of X given $Y = 0$ and $Y = 1$.

$P_{X|Y}(1,0) = \frac{2}{3}$

$(0|0) = \frac{1}{3}$

c) Find the PMF of Z. $\rightarrow$ $\frac{2}{3}$ $p = \frac{3}{5}$

0 $p = \frac{2}{5}$

d) Find EZ, and check that $EZ = EX$.

$2/5$

e) Find $\mathrm{Var}(Z)$.

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
$X=1$	$\frac{2}{5}$	0

Conditional Expectation and Variance

Rule: **Taking out what is known.**

$$E[g(X)h(Y)|X] = g(X)E[h(Y)|X].$$

Proof: **Note that $E[g(X)h(Y)|X]$ is a random variable that is a function of X.**
If $X = x$ then

$$\begin{aligned} E[g(X)h(Y)|X = x] &= E[g(x)h(Y)|X = x] \\ &= g(x)E[h(Y)|X = x] \qquad (g(x) \text{ is a constant}). \end{aligned}$$

$$\Rightarrow \; E[g(X)h(Y)|X] = g(X)E[h(Y)|X].$$

Conditional Expectation and Variance

Iterated Expectations:

Let $g(Y) = E[X|Y]$,

Then,

$$\begin{aligned} E[X] &= \sum_{y_j \in R_Y} E[X|Y = y_j] P_Y(y_j) \\ &= \sum_{y_j \in R_Y} g(y_j) P_Y(y_j) \\ &= E[g(Y)] \qquad\qquad \text{by LOTUS} \\ &= E[E[X|Y]]. \end{aligned}$$

Conditional Expectation and Variance

Iterated Expectations:

$$\text{Law of Iterated Expectations: } \boldsymbol{E[X] = E[E[X|Y]]}$$

This is equal to the low of total Expectation.

Conditioning and Independence

Example. **Suppose that the number of customers visiting a fast food restaurant in a given day is $N \sim Poisson(\lambda)$. Assume that each customer purchases a drink with probability $p,$ independently from other customers and independently from the value of N. Let X be the number of customers who purchase drinks. Find EX.**

Conditional Expectation and Variance

If X and Y are **independent** random variables, then

1. $E[X|Y] = EX$;
2. $E[g(X)|Y] = E[g(X)]$;
3. $E[XY] = EXEY$;
4. $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

Conditional Expectation and Variance

Conditional Variance:

Let $\mu_{X|Y}(y) = E[X|Y=y]$, then

$$\begin{aligned}\mathrm{Var}(X|Y=y) &= E\left[(X-\mu_{X|Y}(y))^2|Y=y\right] \\ &= E\left[X^2|Y=y\right]-\mu_{X|Y}(y)^2.\end{aligned}$$

Note that $\mathrm{Var}(X|Y=y)$ is a function of y. We define $\mathrm{Var}(X|Y)$ is a function of Y. That is, $\mathrm{Var}(X|Y)$ is a random variable whose value equals $\mathrm{Var}(X|Y=y)$.

Conditional Expectation and Variance

Law of Total Variance:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

Conditional Expectation and Variance

Example. **Let N be the number of customers that visit a certain store in a given day. Suppose that we know $E[N]$ and $\mathrm{Var}(N)$. Let X_i be the amount that the $i\mathrm{th}$ customer spends on average. We assume X_i's are independent of each other and also independent of N. We further assume they have the same mean and variance**

$$\begin{aligned} EX_i &= EX, \\ \mathrm{Var}(X_i) &= \mathrm{Var}(X). \end{aligned}$$

Let Y be the store's total sales, i.e.,

$$Y = \sum_{i=1}^{N} X_i.$$

Find EY and $\mathrm{Var}(Y)$.