Probability and Random Process (SWE3026)

Continuous and Mixed Random Variables

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Rationale

- **Consider that a continuous random variable *X* has a range in the form of an interval or a union of non-overlapping interview on the real line. This leads to the need for new tools to help you focus on continuous random variables.**
- **The theory of continuous random variables is analogous to the theory of discrete random variables. So, you may take any formula about discrete random variables and replace *sums* with *integrals* to come up with the corresponding formula for continuous random variables.**
- **Chapter 4 focuses on these relationships.**

Continuous Random Variables

So far, we have discussed discrete random variables:

For a discrete random variable X, $R_X = \{x_1, x_2, ...\}$, is a countable set, e.g.,

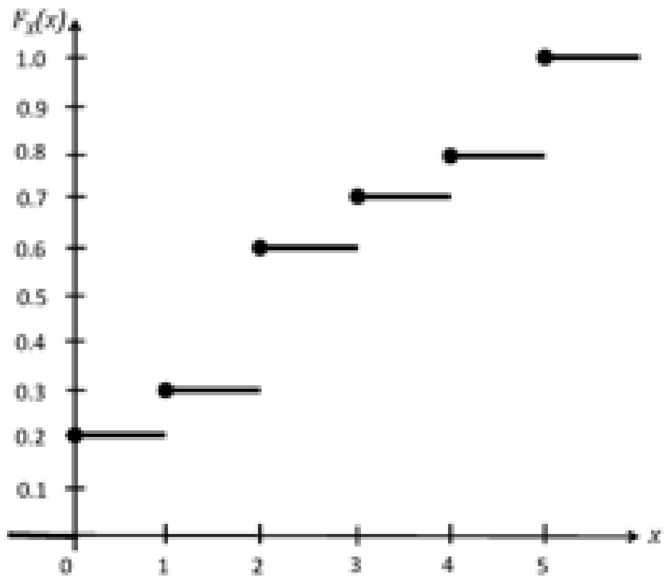
$\text{Range}(X) = \{1, 2, 3, 4, 5, 6\}$ or $\text{Range}(X) = \{1, 2, 3, \cdots\}$, etc.

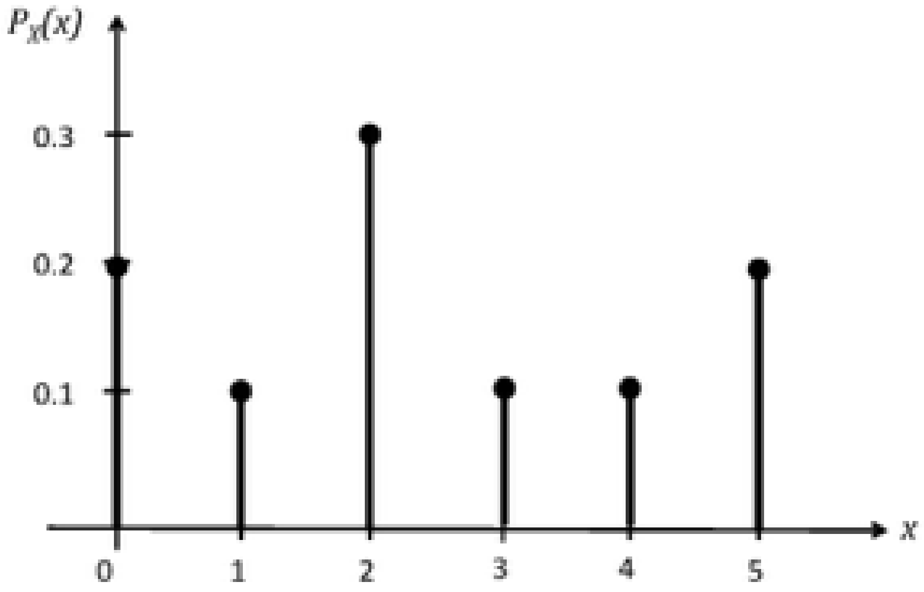
the CDF $F_X(x) = P(X \leq x)$, looks like a series of steps with jumps at

$x_1, x_2, x_3 \cdots$.

Discrete Random Variables: PMF & CDF

The jump at $x = x_k$ is given by the PMF $P_X(x_k)$





Cumulative Distributive Function (CDF)

Recall the general properties of a CDF:

1) $F_X(-\infty) = 0, \quad F_X(+\infty) = 1$

2) $y \geq x \;\Rightarrow\; F_X(y) \geq F_X(x)$

Continuous Random Variables

If $\mathrm{Range}(X)$ is not countable then X is not a discrete random variable.

Example. $\mathrm{Range}(X) = [a, b]$,

Remember: $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}, (a, b] = \{x \in \mathbb{R}, a < x \leq b\}$, **etc.**

Example.

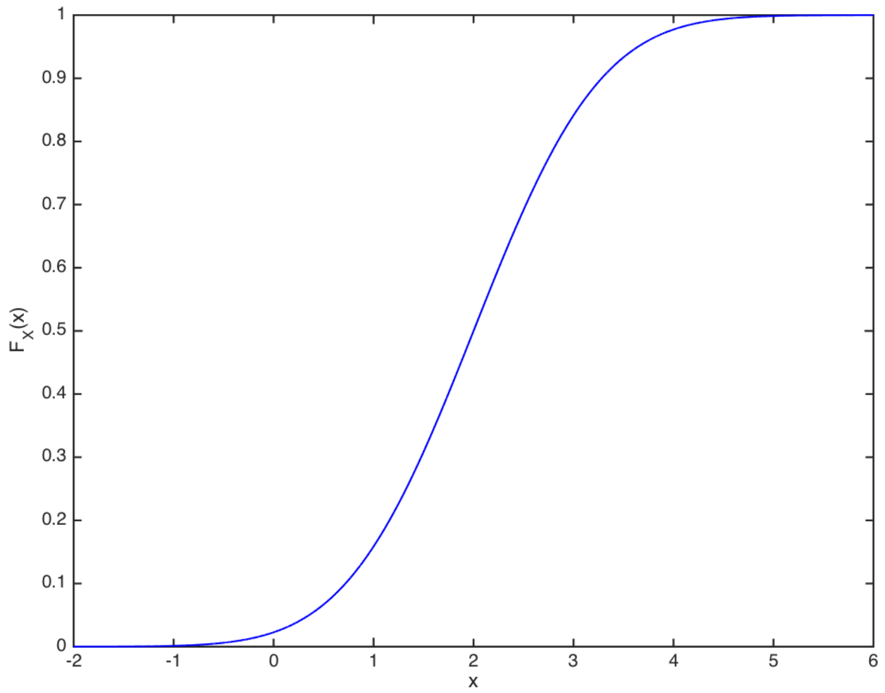
- T: **Lifetime of a light bulb,** $\mathrm{Range}(T) = [0, \infty)$.
- V: **Voltage across a resistor,** $\mathrm{Range}(V) = [0, V_{max}]$.

Continuous Random Variables

Now suppose we have a **continuous** **function having those properties – for example:**

- **A function like this is also a valid CDF** **.**$F_X(x)$ **– that is, it can also represent** **.**$P(X \leq x)$ **for some random variable** X**.**

$$F_X(x) = Prob\{X \leq x\}.$$



Continuous Random Variables

Definition. A random variable X having a CDF $F_X(x)$ that is a continuous function for all x in $\mathbb{R}$ is said to be a continuous random variable.

Example. Let $[a, b]$ be an interval in the real line (where a and b are real numbers with $a < b$). Let X be a number chosen at random from that interval.

"Chosen at random" means: If $a \leq x_1 \leq x_2 \leq b$, then

$$P(X \in [x_1, x_2]) = \frac{x_2 - x_1}{b - a},$$

Continuous Random Variables

Let's find the CDF $F_X(x) = P(X \leq x)$ **:**

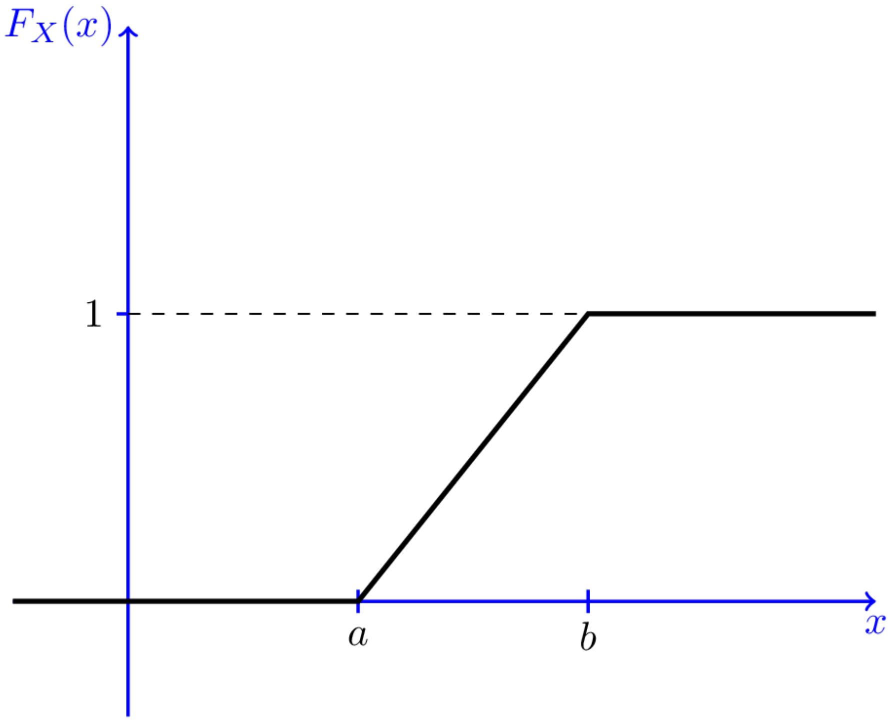
a) $F_X(x) = 0, \qquad \text{for } x < a,$

b) $F_X(x) = \dfrac{x-a}{b-a}, \qquad \text{for } a \leq x \leq b,$

c) $F_X(x) = 1, \qquad \text{for } x \geq b.$

Continuous Random Variables

In this case, X is called a $Uniform(a, b)$ random variable.



Continuous Random Variables

Example. Consider a line of length 1, i.e., $[0, 1]$. Place a dot at random on the line.

$X =$ Location of the dot, $\mathrm{Range}(X) = [0, 1]$.

a) What is $\mathrm{Prob}\{0 \leq X \leq 0.5\}$? $\mathbf{0.5}$

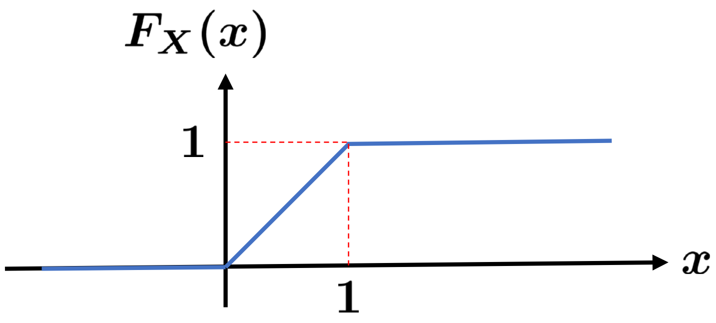
b) What is $\mathrm{Prob}\{0.2 \leq X \leq 0.8\}$? $\mathbf{0.6}$

c) For $0 \leq a \leq b \leq 1$, what is $\mathrm{Prob}\{a \leq X \leq b\}$? $\boldsymbol{a - b}$

d) Obtain the CDF of X.

Continuous Random Variables

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 0 \end{cases}$$



e) What is $\mathrm{Prob}\{X = 0.5\}$? why?

$$P(X = 0.5) = 0,$$

$$P(X = 0.5) \leq P(0.5 - \epsilon \leq X \leq 0.5 + \epsilon) = 2\epsilon,$$

$$\text{For all } \epsilon > 0, \Rightarrow P(X = 0.5) = 0.$$

Continuous Random Variables

- For a discrete random variable: the rate of increase in the CDF $F_X(x)$ is characterized by the PMF $P_X(x)$ - that is, by the locations and sizes of the jumps in the CDF.

- What characterizes the rate of increase for a continuous function?

The derivative of the function

Continuous Random Variables

- If the CDF $F_X(x)$ is a continuous function, then X is said to be a continuous random variable.
- PMF is Not well-defined for continuous random variables. Instead, we define probability density function (pdf).

Probability Density Function (PDF)

Definition. **pdf:**

$$f_X(x_1) = \lim_{\Delta \to 0} \frac{P(x_1 < X \leq x_1 + \Delta)}{\Delta}.$$

Remember that,

$$P(x_1 < X \leq x_1 + \Delta) = F_X(x_1 + \Delta) - F_X(x_1).$$

$$\Rightarrow \quad f_X(x_1) = \lim_{\Delta \to 0} \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} = \frac{dF_X(x_1)}{dx_1} = F_X'(x_1).$$

Probability Density Function (PDF)

Definition. Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. Then we have

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

is called the **probability density function (PDF)** of X.

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha) d\alpha.$$

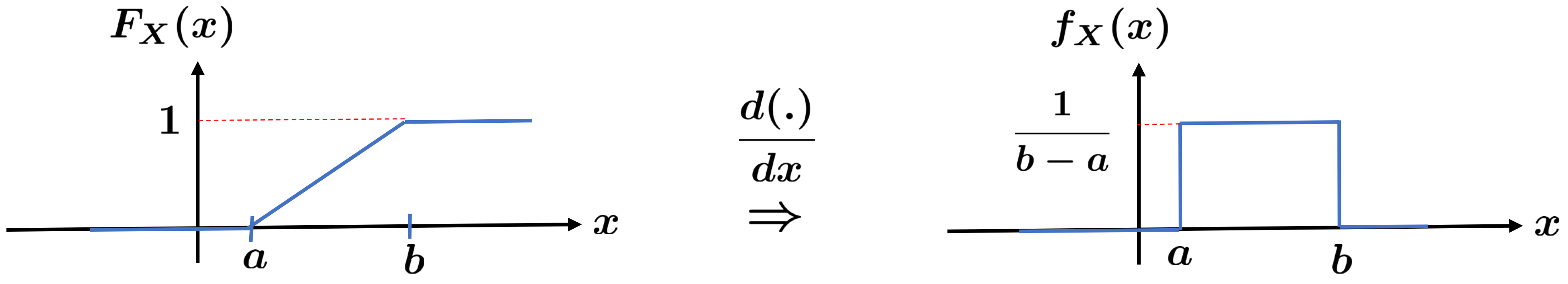
Probability Density Function (PDF)

Theorem.

$$
\begin{aligned}
P(a < X < b) &= F_X(b) - F_X(a) \\
&= \int_{-\infty}^{b} f_X(\alpha)d\alpha - \int_{-\infty}^{a} f_X(\alpha)d\alpha \\
&= \int_{a}^{b} f_X(\alpha)d\alpha.
\end{aligned}
$$

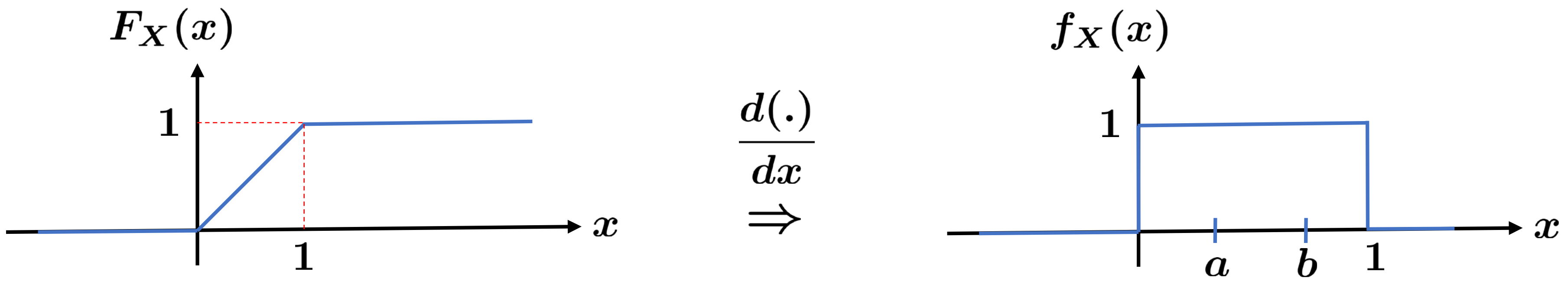
Probability Density Function (PDF)

Example. Say that $X \sim Uniform(a, b)$.



Probability Density Function (PDF)

Back to our dot and line example:



$$P(a \leq X \leq b) = \int_a^b f_X(x)dx = \int_a^b 1 \; dx = b - a, \quad 0 \leq a \leq b \leq 1$$

Probability Density Function (PDF)

Comparison with discrete random variable:

$$\text{Prob}\{a \leq X < b\} = \sum_{a \leq x_k < b} P_X(x_k) \quad \longrightarrow \quad \int_a^b f_X(x)dx$$

discrete

continuous

If we integrate over the entire real line, we must get 1, i.e.,

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

Probability Density Function (PDF)

In math: **A density function for a quantity is a non-negative function that is integrated to give that quantity.**

Let's look at the properties of a PDF:

1) **Since $F_X(x)$ is monotone non-decreasing, its derivative must satisfy**

$$f_X(x) \geq 0 \text{ for all } x \in \mathbb{R}.$$

2) $$\int_{-\infty}^{\infty} f_X(u)du = F_X(\infty) - F_X(-\infty) = 1 - 0 = 1.$$

Probability Density Function (PDF)

3) **In general:**

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du.$$

4) **We can extend the property 3 to:**

$$P(X \in A) = \int_A f_X(u)du, \quad \text{for any set } A.$$

Probability Density Function (PDF)

- So, for a discrete random variable having range $R_X = \{x_1, x_2, x_3, ...\}$, to find $P(X \in A)$, we sum the PMF over the points $x_k \in A$.

- For a continuous random variable, to find $P(X \in A)$, we integrate the PDF over the set A .

Probability Density Function (PDF)

Example. **Let X be a continuous random variable with the following PDF**

$$f_X(x) = \begin{cases} Ae^{-x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

a) **Find A.**

b) **Find $F_X(x)$.**

c) **Find $P(1 < X < 3)$.**

Probability Density Function (PDF)

Important note: $f_X(x)$ is not equal to $P(X = x)$. In fact, for a continuous random variable X we have $P(X = x) = 0$ for every point x. Also, it is possible in general to have $f_X(x) > 1$ for some values of x.

Expected Value and Variance

$$\sum_{k=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}; \quad P_X(x_k) \quad \longrightarrow \quad f_X(x),$$

So, we get:

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Expected Value and Variance

Law of the unconscious statistician (LOTUS) for continuous random variables:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Expected Value and Variance

Remember that the variance of any random variable is defined as

$$\mathrm{Var}(X) = E\big[(X - \mu_X)^2\big] = EX^2 - (EX)^2.$$

So for a continuous random variable, we can write

$$\begin{aligned}\mathrm{Var}(X) = E\big[(X - \mu_X)^2\big] &= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx \\ = EX^2 - (EX)^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X^2.\end{aligned}$$

Functions of Continuous Random Variables

Suppose that X is a continuous random variable having CDF $F_X(x)$ and PDF $f_X(x)$. Let $g: R \to R$ be some function, and let $Y = g(X)$.

Since X is a random variable, so is Y. We already know that we can find $E[Y]$ by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Functions of Continuous Random Variables

Question: How do we find the CDF and PDF for Y ?

Usually: It is easier to first find the CDF for Y , and then take the derivative to find the PDF.

Functions of Continuous Random Variables

There's another approach called the Method of Transformations that sometimes gives a quicker way of finding $f_Y(y)$ directly from $f_X(x)$.

- $g(x)$ **is differentiable**
- $g(x)$ **is a strictly increasing function, that is if** $x_1 < x_2$**, then** $g(x_1) < g(x_2)$

$$f_Y(y) = \begin{cases} \dfrac{f_X(x_1)}{g'(x_1)} = f_X(x_1)\dfrac{dx_1}{dy} \quad where\ g(x_1) = y \\ 0 \end{cases}$$

- $g(x)$ **is differentiable**
- $g(x)$ **is a strictly increasing function, that is if** $x_1 < x_2$**, then** $g(x_1) < g(x_2)$

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{g'(x_1)} = f_X(x_1)\frac{dx_1}{dy} & where\ g(x_1) = y \\ 0 & \end{cases}$$

Functions of Continuous Random Variables

Example. Now suppose that $f_X(x) = 4x^3, 0 < x \leq 1;\ 0\ otherwise$ and $Y = \frac{1}{X}$. Find PDF of Y

Functions of Continuous Random Variables

If X is a continuous random variable and $Y = g(X)$ is a function of X, then Y itself is a random variable.

Steps:

1) **Find R_Y.**

2) **Find CDF of Y : $F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$.**

3) **Find $f_Y(y) = \frac{d}{dy} F_Y(y)$.**

Summary of Continuous Random Variable

- **PDF:** $f_X(x) = \dfrac{dF_X(x)}{dx}$
- **Expected Value:** $EX = \int_{-\infty}^{\infty} x f_X(x)dx$
- **LOTUS:** $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$
- $P(a < X < b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$

Summary of Continuous Random Variable

Definition. Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. The function $f_X(x)$ defined by

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)d\alpha.$$

Summary of Continuous Random Variable

Consider a continuous random variable $\boldsymbol{X}$ with PDF $\boldsymbol{f_X(x)}$. We have

1) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.
2) $\displaystyle\int_{-\infty}^{\infty} f_X(u)du = 1.$
3) $\displaystyle P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du.$
4) More generally, for a set A, $\displaystyle P(X \in A) = \int_A f_X(u)du.$

Summary of Continuous Random Variable

Discrete RVs → **Continuous RVs**

PMF → **PDF**

$$\sum \longrightarrow \int$$

$$EX = \sum_{x_k \in R_X} x_k P_X(x_k) \longrightarrow EX = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k) \longrightarrow E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$\mathrm{Var}(X) = EX^2 - \big(EX\big)^2$$

Summary of Continuous Random Variable

Functions of Continuous RVs:

$$X \longrightarrow f_X(x); \;\; Y = g(X)$$

Steps:

1) $R_Y = \{g(x);\; x \in R_X\}.$

2) $F_Y(y) = P(Y \leq y) = P(g(X) \leq y).$

Read method of Transformations (4.1.3).