Probability and Random Process (SWE3026)

Introduction to Random Processes

JinYeong Bak
jy.bak@skku.edu
College of Computing, SKKU

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

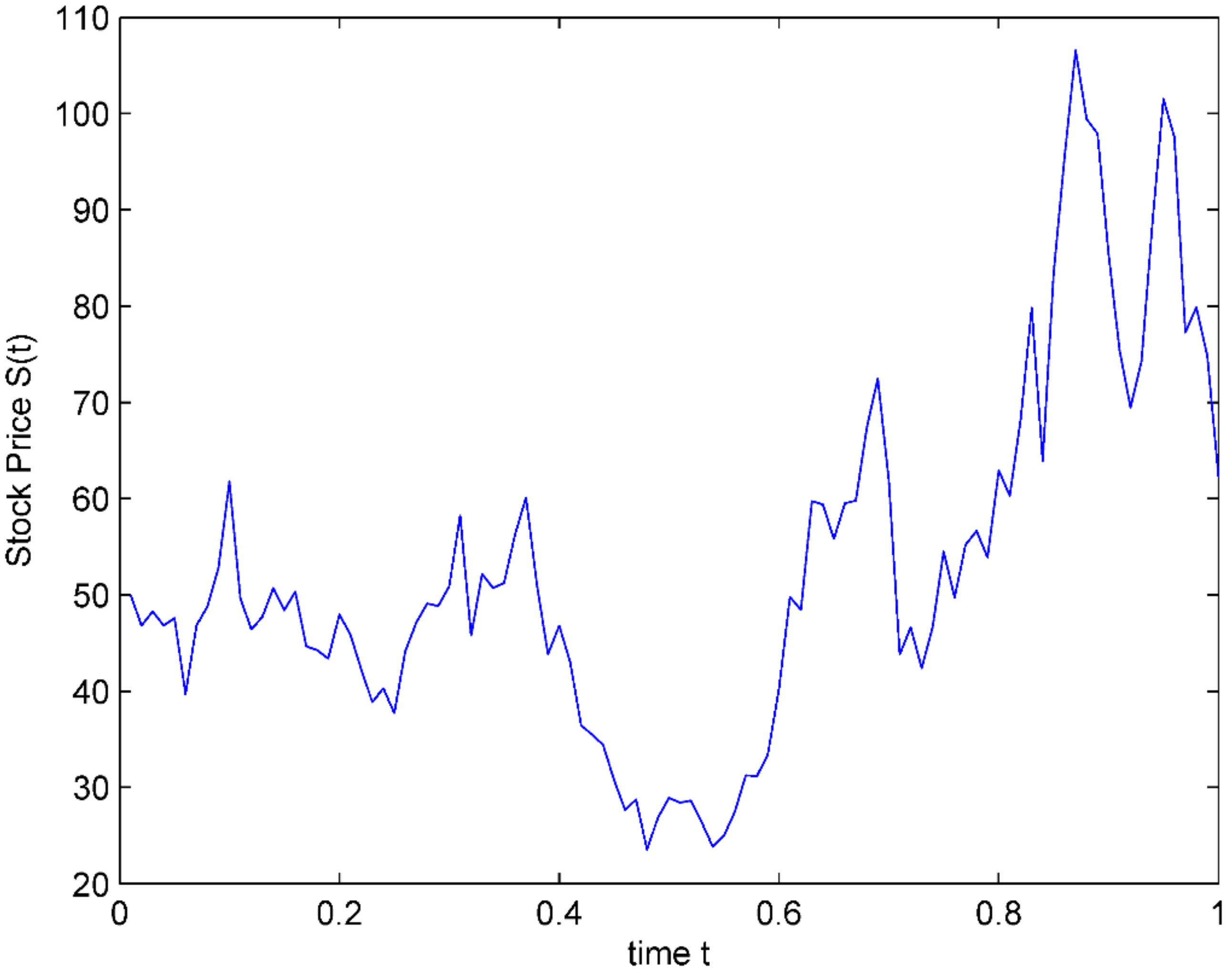
Rationale

In the real world, you may be interested in multiple observations of random values over a period of time.

One example might be watching a company's stock price fluctuate over time.

Knowing that random processes are collections of random variables, you possess the knowledge needed to analyze these random processes.

Random Process

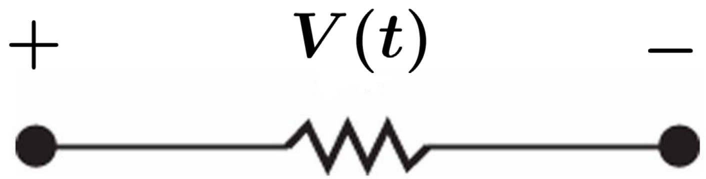


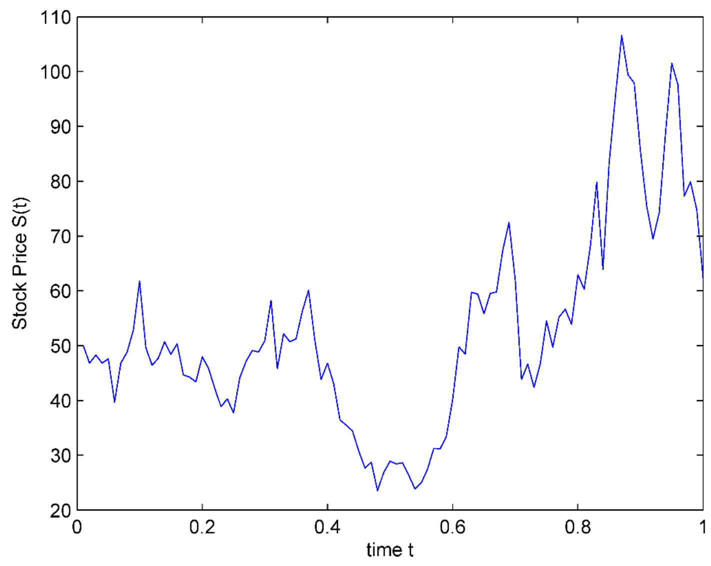
Random Process

Random process (stochastic process)

A family of random variable (usually infinite)

Example. Many random phenomena that occur in nature are function of time.





Random Process

At any time $t = t_1$, $X(t_1)$ is a random variable

Each of the possible outcomes (functions) is called a sample function.

These are examples of continuous-time random processes:

$$\{\boldsymbol{X}(\boldsymbol{t}), \boldsymbol{t} \in [\mathbf{0}, \mathbf{1}]\}$$

Random Process

Discrete-time random process:

$$\{X[n], n \in \mathbb{Z}\} \quad \text{or} \quad \{X_n, n \in \mathbb{Z}\}.$$

$$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

Random Process

A random process is a collection of random variables usually indexed by time.

$$\{X(t), t \in J\}.$$

Random Process

A continuous-time random process is a random process $\{X(t), t \in J\}$, where J is an interval on the real line such as $[-1, 1]$, $[0, \infty)$, $(-\infty, \infty)$, etc.

A discrete-time random process (or a random sequence) is a random process $\{X(n) = X_n, n \in J\}$, where J is a countable set such as $\mathbb{N}$ or $\mathbb{Z}$.

Example

You have 1000 dollars to put in an account with interest rate R, compounded annually. That is, if X_n is the value of the account at year n, then

$$X_n = 1000(1+R)^n, \quad \text{for } n = 0, 1, 2, \cdots .$$

The value of R is a random variable that is determined when you put the money in the bank, but it does not change after that. In particular, assume that

$$R \sim Uniform(0.04, 0.05).$$

a) **Find all possible sample functions for the random process**

$$\{X_n, n = 0, 1, 2, ...\}.$$

b) **Find the expected value of your account at year three. That is, find $E[X_3]$.**

You have 1000 dollars to put in an account with interest rate R, compounded annually. That is, if X_n is the value of the account at year n, then

$$X_n = 1000(1+R)^n, \qquad \text{for } n = 0, 1, 2, \cdots .$$

The value of R is a random variable that is determined when you put the money in the bank, but it does not change after that. In particular, assume that

$$R \sim Uniform(0.04, 0.05).$$

a) Find all possible sample functions for the random process

$$\{X_n, n = 0, 1, 2, ...\}.$$

b) Find the expected value of your account at year three. That is, find $E[X_3]$.

Example

Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \qquad \text{for all } t \in [0, \infty),$$

where A and B are independent normal $N(1, 1)$ random variables.

a) Find all possible sample functions for this random process.

b) Define the random variable $Y = X(1)$. Find the PDF of Y.

c) Let also $Z = X(2)$. Find $E[YZ]$.

Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \qquad \text{for all } t \in [0, \infty),$$

where A and B are independent normal $N(1, 1)$ random variables.

a) Find all possible sample functions for this random process.

b) Define the random variable $Y = X(1)$. Find the PDF of Y.

c) Let also $Z = X(2)$. Find $E[YZ]$.

Mean and Correlation Functions

Mean Function of a Random Process

For a random process $\{X(t), t \in J\}$ the mean function $\mu_X(t) : J \to \mathbb{R}$, is defined as

$$\mu_X(t) = E[X(t)].$$

Mean and Correlation Functions

For a random process $\{X(t), t \in J\}$, the autocorrelation function or, simply, the correlation function, $R_X(t_1, t_2)$, is defined by

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)], \quad \text{for } t_1, t_2 \in J.$$

Mean and Correlation Functions

For a random process $\{X(t), t \in J\}$, the autocovariance function or, simply, the covariance function, $C_X(t_1, t_2)$, is defined by

$$\begin{aligned} C_X(t_1, t_2) &= \mathrm{Cov}\big(X(t_1), X(t_2)\big) \\ &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2), \quad \text{for } t_1, t_2 \in J. \end{aligned}$$

Multiple Random Processes

For two random processes $\{X(t), t \in J\}$ and $\{Y(t), t \in J\}$:

- the cross-correlation function $R_{XY}(t_1, t_2)$, is defined by

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)], \quad \text{for } t_1, t_2 \in J;$$

- the cross-covariance function $C_{XY}(t_1, t_2)$, is defined by

$$\begin{aligned} C_{XY}(t_1, t_2) &= \mathrm{Cov}\big(X(t_1), Y(t_2)\big) \\ &= R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2), \quad \text{for } t_1, t_2 \in J. \end{aligned}$$

Multiple Random Processes

Two random processes $\{X(t), t \in J\}$ and $\{Y(t), t \in J'\}$ are said to be **independent** if, for all

$$t_1, t_2, \ldots, t_m \in J$$
$$\text{and}$$
$$t'_1, t'_2, \ldots, t'_n \in J',$$

the set of random variables

$$X(t_1), X(t_2), \cdots, X(t_m)$$

are independent of the set of random variables

$$Y(t'_1), Y(t'_2), \cdots, Y(t'_n).$$