Probability and Random Process (SWE3026)

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Random experiment

Random experiment: A phenomenon whose outcome cannot be predicted with certainty, such as

Random experiment:

- Roll a die
- Roll a die three times
- Flip a coin

Random experiment

Outcome:

An outcome is the result of a random experiment.

- **Roll a die $\longrightarrow$ 3**
- **Roll a die 3 times $\longrightarrow$ (2, 3, 6)**

Random experiment

Events:

An event is collection of possible outcomes.

- **Roll a die (Event=E)**

$$\boldsymbol{E_1 = \{1, 3, 5\}, \quad E_2 = \{2, 4\}, \quad E_3 = \{6\}}$$

Random experiment

Sample Space:

The sample space is the set of all possible outcomes.

- **Roll a die:** ← random experiment

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- **Roll a die three times**

$$\Omega = \{(1,1,1), (1,1,2), \cdots, (1,1,6), (2,1,1), \cdots, (6,6,6)\}$$

an outcome

Random experiment

- **Event** $\longleftrightarrow$ **Set**
- **Sample space** $\longleftrightarrow$ **Universal Set**
- **Outcome** $\longleftrightarrow$ **Element**

We say that an event A occurs if the outcome of the experiment is an element of A.

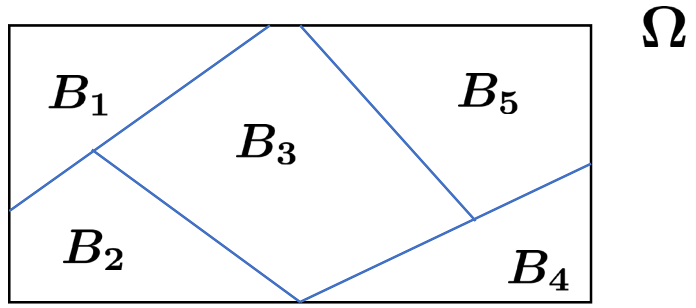
Random experiment

Partition:

A partition is a collectively exhaustive, and mutually exclusive set of events, i.e.,

$B_1, B_2, \cdots, B_n$ **is a Partition if**

- $B_1 \cup B_2 \cup B_3 \cup \cdots B_n = \Omega$
- $B_i \cap B_j = \emptyset, \ i \neq j.$



Summary of Random experiment

a) Review of set theory

b) Random experiments: **Roll a die, etc.**

- **Outcome:** An outcome is a result of random experiment.
  - Roll a die $\longrightarrow$ 3
  - Roll a die three times $\longrightarrow$ (3,6,2)

Summary of Random experiment

- **Sample Space:** The set of all possible outcomes (S).
  - Roll a die $\longrightarrow$ $S = \{1, 2, 3, 4, 5, 6\}$
- **Event:** An event is a collection of possible outcomes.

  $\Rightarrow$ An event is subset of S.
  - Roll a die : $E_1 = \{1, 2\}, \quad E_2 = \{4\}$

Summary of Random experiment

➢ We also say that an event A has occurred if the outcome of the experiment is an element of A.

- Roll a die $\longrightarrow$ 2 ,

E_1 : has occurred

E_2 : has not occurred

- Roll a die 3 times

$$S = \{(1,1,1),(1,1,2),\cdots,(1,1,6),(2,1,1),\cdots,(6,6,6)\}$$

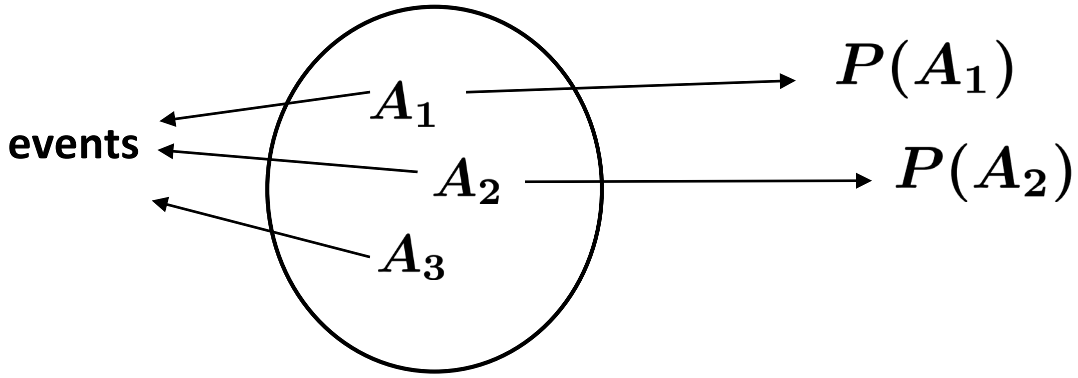
$\Rightarrow$ 6^3 elements.

Probability

Event $A \longrightarrow P(A)$ **: Probability of** A**.**

We assign a probability $P(A)$ **to every event** A**.**

$P(A)$ **: The portion of times event** A **is observed in a large number of runs of the experiment.**



Probability

Axioms of Probability

Definition. A probability measure $P(.)$ is a function that maps events in the sample space S to real numbers. Such that:

1) For any event A, $P(A) \geq 0$.

2) Probability of the sample space S is $P(S) = 1$.

3) For any countable collection $A_1, A_2, A_3, \cdots$ of disjoint events

$$P(A_1 \cup A_2 \cup A_3 \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Probability

Roll a fair die (fair: outcomes are equally likely).

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{6\})$$

disjoint

3rd axiom:

$$P(\{1\} \cup \{2\} \cup \{3\} \cup \cdots \{6\}) = P(\{1\}) + P(\{2\}) + \cdots + P(\{6\}) = 6P(\{1\})$$

$$1 = P(S) = P(\{1, 2, \cdots, 6\})$$

Probability

$$\Rightarrow \;\; P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$$

$$P(\{1,2\}) = P(\{1\} \cup \{2\}) = P(1) + P(2) = \frac{2}{6} = \frac{1}{3}.$$

disjoint

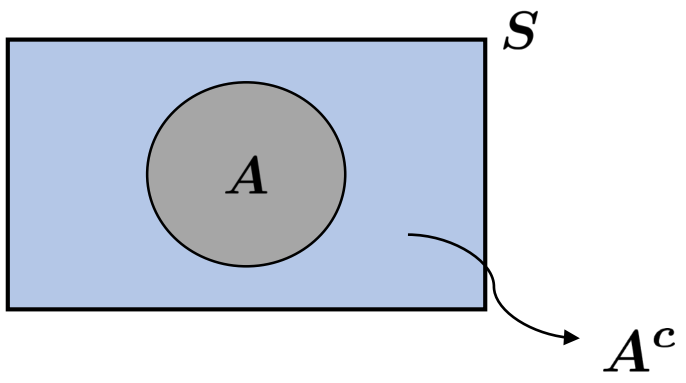
➢ **Equally likely outcomes:**

$$P(A) = \frac{|A|}{|S|}$$

Probability

Using the axioms:

1) $P(A)$, what is $P(A^c)$.



$$A \cup A^c = S$$

$$\Rightarrow \; P(A \cup A^c) = P(s) = 1$$

disjoint

$$\Rightarrow \; P(A) + P(A^c) = 1 \; \Rightarrow \; P(A^c) = 1 - P(A)$$

Probability

2) $P(\emptyset) = 0,$

$\emptyset$: empty

$$P(\emptyset) = P(S^c) = 1 - P(S) = 1 - 1 = 0.$$

3) $P(A) \leq 1,$

$$P(A) = 1 - \underbrace{P(A^c)}_{\geq 0} \Rightarrow P(A) \leq 1.$$

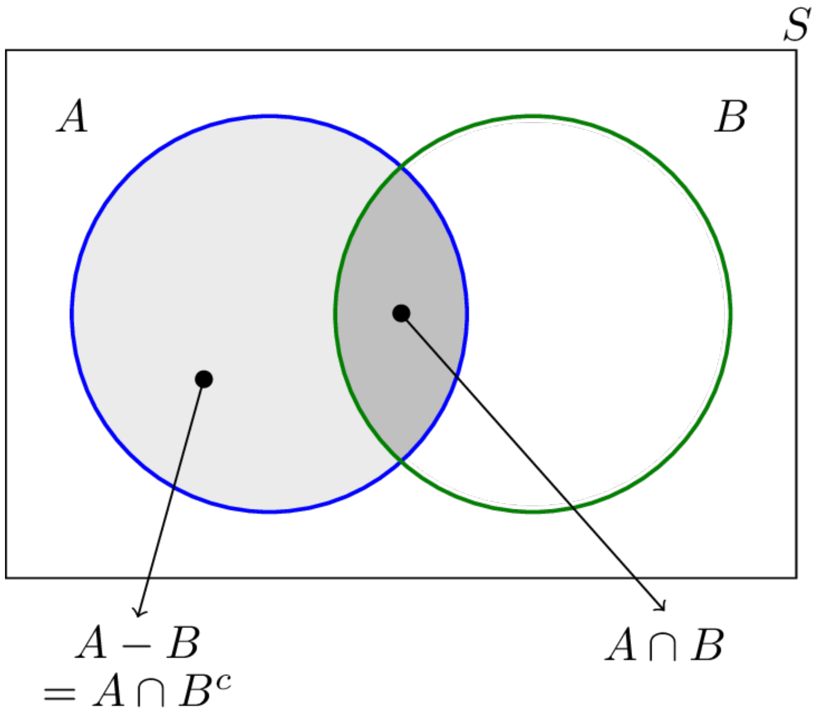
Probability

4) $P(A - B) = P(A) - P(A \cap B)$,

$$A = (A \cap B) \cup (A - B)$$

$(A \cap B)$ **and** $(A - B)$ **are disjoint.**

$$\begin{aligned} P(A) &= P\big((A \cap B) \cup (A - B)\big) \\ &= P(A \cap B) + P(A - B). \end{aligned}$$



Probability

5) $P(A \cup B) = P(A) + P(B) - P(A \cap B),$

$\cup$: or , $\cap$: and

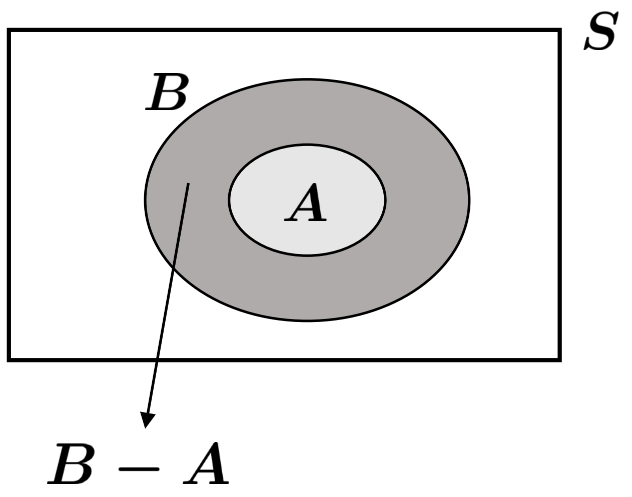
Use Venn diagram.

6) $A \subset B \;\Rightarrow\; P(A) \leq P(B),$

$P(B) = P(A) + P(B - A)$

$P(B - A) \geq 0,\;$ B and $(A - B)$ are disjoint.

$\Rightarrow\; P(B) \geq P(A)$



Probability

Example:

Roll a die twice and observed X_1 and X_2.

(a) Find S.

(b) $A : X_1 + X_2 = 4,$ Find the elements in A, and $P(A)$.

(c) $B : X_1 + X_2 = 6$ or 7, Find $P(B)$.

Summary

Probability:

$A \subset S, \; P(A):$

$(1) \; P(A) \geq 0$

$(2) \; P(S) = 1$

$(3) \; A_1, A_2, \cdots \text{ disjoint } P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

- **Finite Sample Space with equally likely outcomes:** $P(A) = \dfrac{|A|}{|S|}$

Sample Space

Sample Space:

a) Countable: $\boldsymbol{S = \{s_1, s_2, s_3, \cdots\}.}$

S : **Discrete Probability Space**

$\boldsymbol{A = \{a_1, a_2, a_3, \cdots\},}$

$$\Rightarrow \ \boldsymbol{P(A) = P(a_1) + P(a_2) + P(a_3) + \cdots}$$

Sample Space

b) **Uncountable**

S : **Continuous Probability Space**

$$\boldsymbol{S} = \mathbb{R}^{+} = \{\boldsymbol{x} \in \mathbb{R},\ \boldsymbol{x} \geq \mathbf{0}\}$$

Continuous Probability Space

Example: I choose a point completely at random in $[0, 1]$.

a) $P([0, 0.5]) = 0.5$

b) $P([0, 0.25]) = 0.25$

c) $P([a, b]) = b - a, \quad 0 \leq a \leq b \leq 1$

d) $P(\{0.5\}) = 0 = P([0.5, 0.5]) = 0.5 - 0.5 = 0$

Continuous Probability Space

Key point: **Axioms of Probability applies to continuous probability spaces.**

Example: **Suppose we know that the probability that a certain machine lasts more than or equal to x years is :**

$$P(T \geq x) = \frac{1}{2^x}, \quad T: \text{ Lifetime}$$

Find the following sets:

a) $P(T \geq 1)$

b) $P(T \geq 2)$

c) $P(1 \leq T \leq 2)$