Probability and Random Process (SWE3026)

Random Processes

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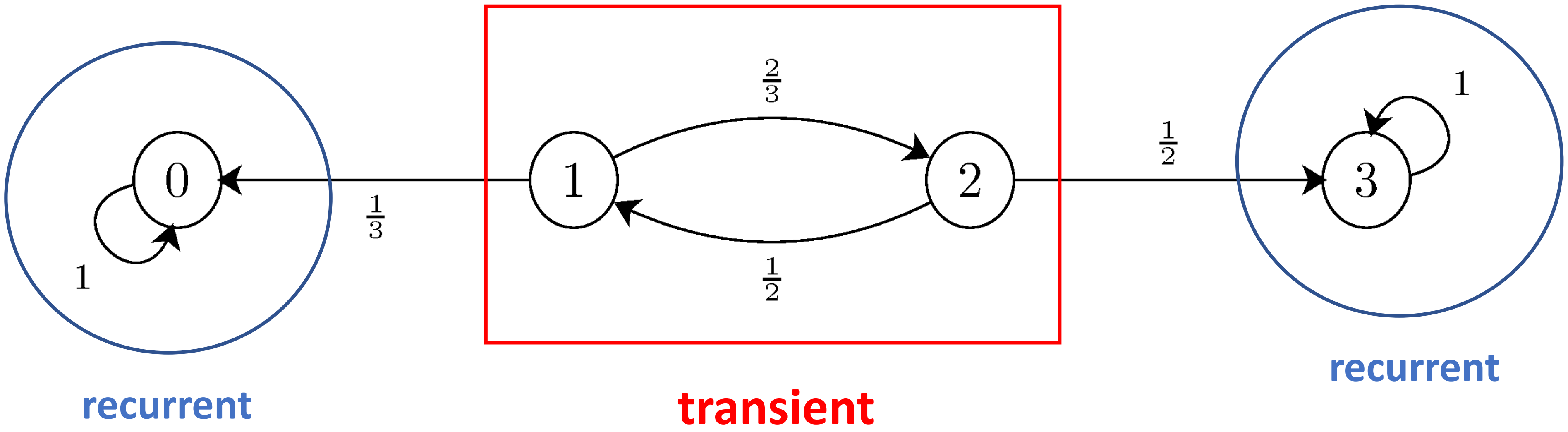
H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Using the Law of Total Probability with Recursion

Absorption Probabilities:

Example.



Using the Law of Total Probability with Recursion

$$
\begin{aligned}
a_0 &= P(\text{absorption in } 0 | X_0 = 0), \\
a_1 &= P(\text{absorption in } 0 | X_0 = 1), \\
a_2 &= P(\text{absorption in } 0 | X_0 = 2), \\
a_3 &= P(\text{absorption in } 0 | X_0 = 3).
\end{aligned}
$$

Using the Law of Total Probability with Recursion

How do we find a_i ?

Main Idea: Apply the law of total probability

$$a_i = \sum_k a_k p_{ik}, \qquad \text{for } i = 0, 1, 2, 3$$

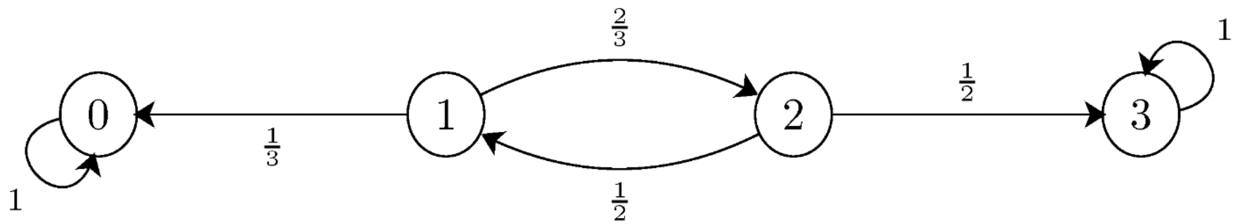
Using the Law of Total Probability with Recursion

Thus,

$$\begin{aligned} a_0 &= a_0, \\ a_1 &= \frac{1}{3}a_0 + \frac{2}{3}a_2, \\ a_2 &= \frac{1}{2}a_1 + \frac{1}{2}a_3, \\ a_3 &= a_3. \end{aligned}$$

We also know $a_0 = 1$ and $a_3 = 0$. Then we have

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{4}.$$

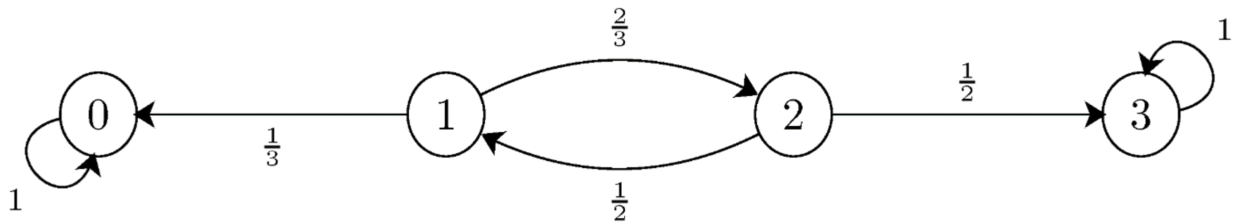


Using the Law of Total Probability with Recursion

$$b_i = P(\text{absorption in } 1 | X_0 = i)$$

Since $a_i + b_i = 1$, we conclude

$$b_0 = 0, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{3}{4}, \quad b_3 = 1.$$



Using the Law of Total Probability with Recursion

Similar idea (using LOTP) can be used to find.

- **Mean Hitting Times:**

  The expected time until the process hits a certain set of state for the first time.

- **Mean Return Times:**

  The expected time until returning to state i .

Using the Law of Total Probability with Recursion

Mean Hitting Times

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $A \subset S$ be a set of states. Let T be the first time the chain visits a state in A. For all $i \in S$, define

$$t_i = E[T | X_0 = i].$$

Using the Law of Total Probability with Recursion

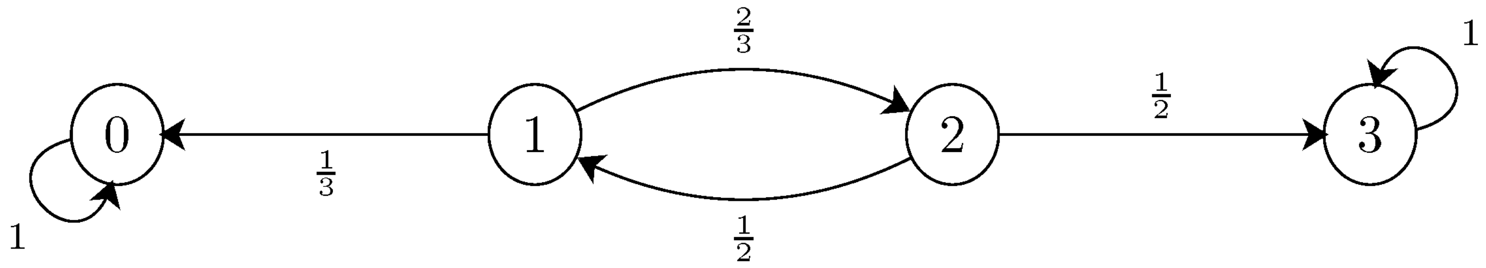
Mean Hitting Times

By the above definition, we have $t_j = 0$, for all $j \in A$. To find the unknown values of t_i's, we can use the following equations

$$\boldsymbol{t_i = 1 + \sum_k t_k p_{ik}, \qquad \text{for } i \in S - A.}$$

Using the Law of Total Probability with Recursion

Example.



t_i : **The number of steps needed until the chain hits the state 0 or 3 given $X_0 = i$.**

$t_0 = t_3 = 0,$

Using the Law of Total Probability with Recursion

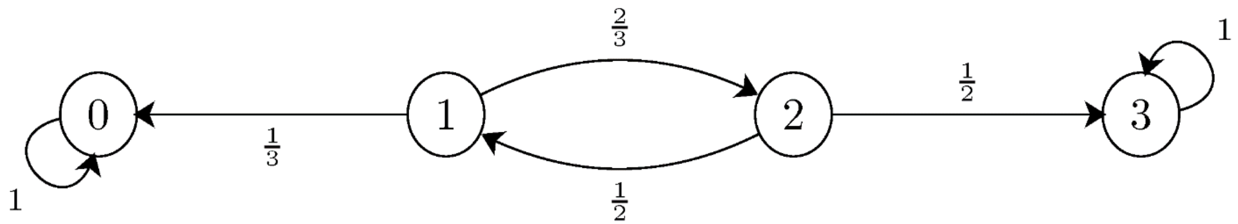
$$t_1 = 1 + \frac{1}{3}t_0 + \frac{2}{3}t_2 = 1 + \frac{2}{3}t_2.$$

Similarly, we can write

$$t_2 = 1 + \frac{1}{2}t_1 + \frac{1}{2}t_3 = 1 + \frac{1}{2}t_1.$$

Solving the above equations, we obtain

$$t_1 = \frac{5}{2}, \quad t_2 = \frac{9}{4}.$$



Using the Law of Total Probability with Recursion

Mean Return Times:

$$r_l = E[R_l | X_0 = l].$$

R_l : return to state l.

Using the Law of Total Probability with Recursion

Mean Return Times

Consider a finite irreducible Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $l \in S$ be a state. Let r_l be the mean return time to state l. Then

$$r_l = 1 + \sum_k t_k p_{lk},$$

Using the Law of Total Probability with Recursion

Mean Return Times

where t_k is the expected time until the chain hits state l given $X_0 = k$. Specifically,

$$t_l = 0,$$

$$t_k = 1 + \sum_j t_j p_{kj}, \qquad \text{for } k \neq l.$$

Stationary and Limiting Distributions

Long-term behavior of Markov chains

The fraction of time that the Markov chain spends at state i as time $n \to \infty$.

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 0) & P(X_n = 1) & \cdots & \end{bmatrix}$$

Stationary and Limiting Distributions

The initial state (X_0) does not matter as n becomes large. Thus, we can write

$$\lim_{n\to\infty} P(X_n = 0|X_0 = i) = \frac{b}{a+b},$$
$$\lim_{n\to\infty} P(X_n = 1|X_0 = i) = \frac{a}{a+b}.$$

Stationary and Limiting Distributions

Limiting Distributions

The probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$ is called the limiting distribution of the Markov chain X_n if

$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i)$$

for all $i, j \in S$, and we have

$$\sum_{j \in S} \pi_j = 1.$$

Stationary and Limiting Distributions

How to find the limiting distribution?

Finite Markov chains:

$$\pi^{(n+1)} = \pi^{(n)} P = \pi^{(n)}, \quad n \to \infty$$

Steady-state

So solve $\pi = \pi P \;\rightarrow\; \pi$ **Steady-state (limiting) distribution**

Stationary and Limiting Distributions

Theorem. Consider a finite Markov chain $\{X_n, n = 0, 1, 2, ...\}$ where $X_n \in S = \{0, 1, 2, \cdots, r\}$. Assume that the chain is **irreducible** and **aperiodic**. Then,

1. The set of equations

$$\pi = \pi P,$$
$$\sum_{j \in S} \pi_j = 1$$

has a unique solution.

Stationary and Limiting Distributions

2. **The unique solution to the above equations is the limiting distribution of the Markov chain, i.e.,**

$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i),$$

   for all $i, j \in S$ **.**

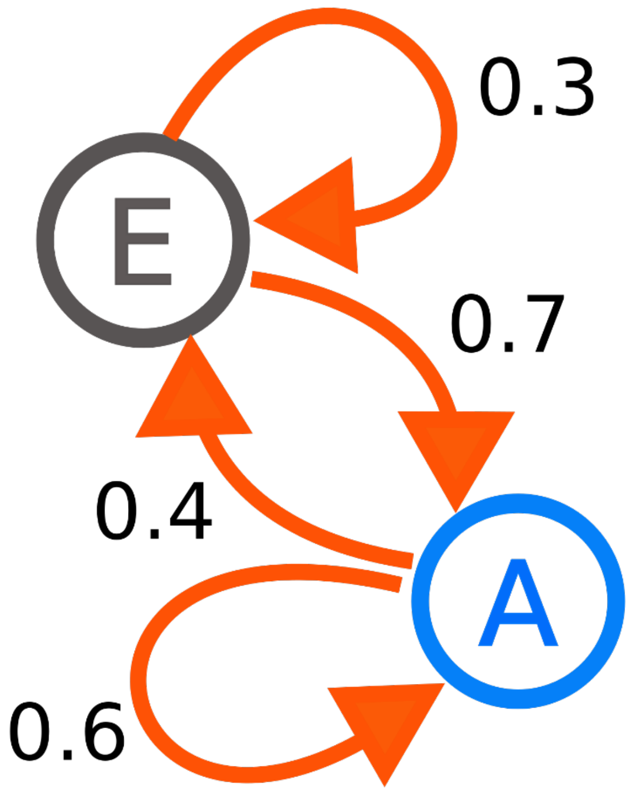
3. **We have**

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j \in S,$$

   where r_j **is the mean return time to state** j **.**

Markov Decision Process

Markov Process



Markov Decision Process