Probability and Random Process (SWE3026)

Joint Distributions

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Rationale

- **If you were to examine the population of your community, you might notice that each household has a different number of people. Each of those household members has a different age, a different income, a different number of hobbies, etc.**
- **Each of these results is a random variable. In this Lesson, you explore the concept of comparing two or more random variables, because you grasp comparing two, the extension to *n* random variables is straightforward.**

Joint Probability Mass Function (PMF)

PMF:

$$P_X(x_k) = P(X = x_k), \quad R_X = \text{Range}(X).$$

Joint Probability Mass Function (PMF) for X and Y:

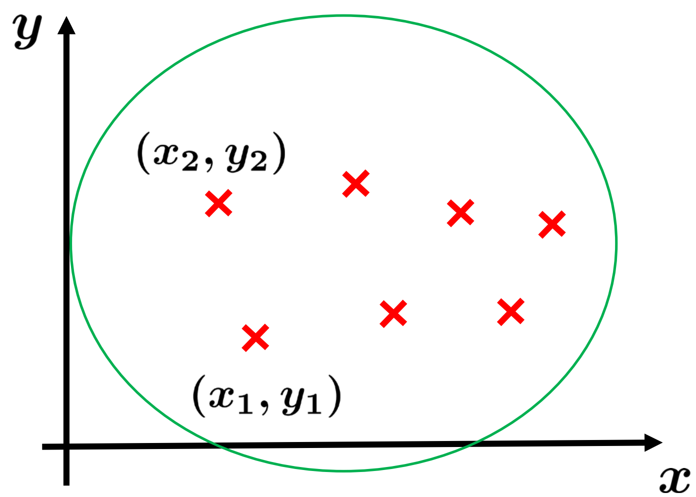
$$P_{XY}(x_j, y_j) = P(X = x_k, Y = y_j).$$

Joint Probability Mass Function (PMF)

R_{XY} = all possible value for (X, Y).

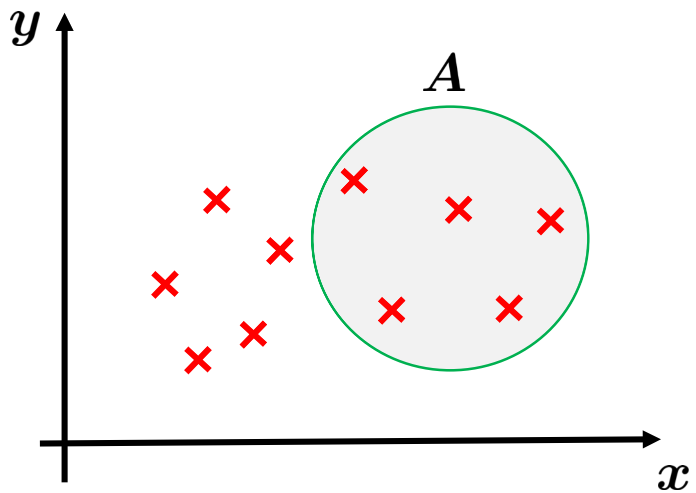
$$= \{(x_i, y_j) | x_i \in R_X, y_j \in R_Y\}.$$

$$\sum_{(x_i, y_j) \in R_{XY}} P_{XY}(x_i, y_j) = 1$$



Joint Probability Mass Function (PMF)

$$P\big((X,Y) \in A\big) = \sum_{(x_i,y_j)\in A} P_{XY}(x_i, y_j)$$



Joint Probability Mass Function (PMF)

Marginal PMFs

I have $P_{XY}(x_i y_j)$, how do I find PMF of X, $P_X(x_i)$?

$$
\begin{aligned}
P_X(x_i) &= P(X = x_i) \\
&= \sum_{y_j \in R_Y} P(X = x_i, Y = y_j) \qquad \text{law of total probablity} \\
&= \sum_{y_j \in R_Y} P_{XY}(x_i, y_j).
\end{aligned}
$$

Joint Probability Mass Function (PMF)

Marginal PMFs

$$P_X(x_i) = \sum_{y_j \in R_Y} P_{XY}(x_i, y_j), \qquad \text{for any } x_i \in R_X$$

$$P_Y(y_j) = \sum_{x_i \in R_X} P_{XY}(x_i, y_j), \qquad \text{for any } y_j \in R_Y$$

Joint Probability Mass Function (PMF)

Example. **Consider two random variables X and Y with joint PMF given in Table.**

a) **Find the marginal PMFs of X and Y.**

b) **Find $P(Y=0|X=0)$.**

c) **Are X and Y independent?**

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{2}$	$\frac{1}{3}$
$X=1$	$\frac{1}{6}$	0

Joint Cumulative Distributive Function (CDF)

Remember that, for a random variable X , we define the CDF as

$$F_X(x) = P(X \leq x).$$

The joint cumulative distribution function of two random variables

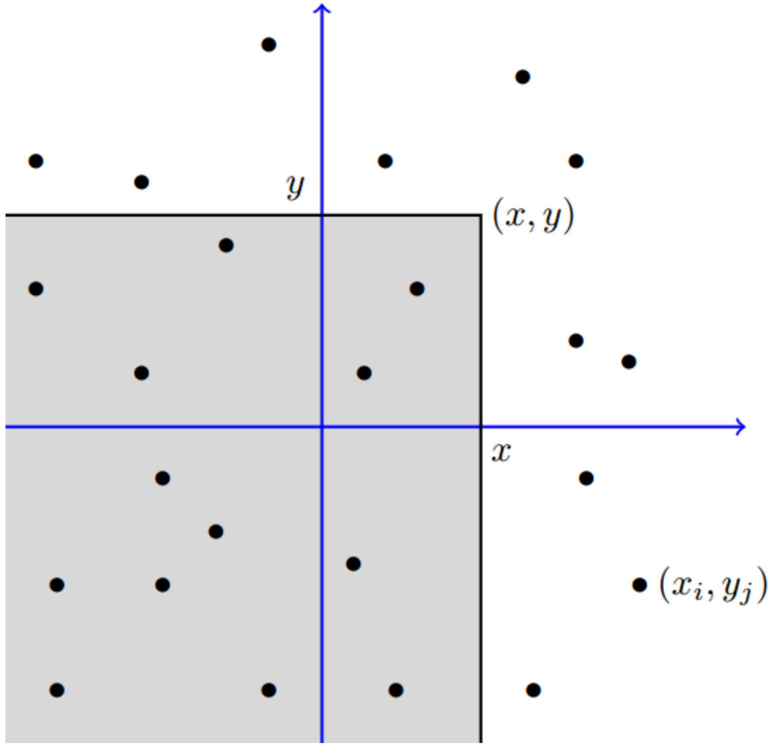
$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

and

Joint Cumulative Distributive Function (CDF)

$$F_{XY}(x, y) = P(\square).$$

$$F_{XY}(1, 2) = P(X \leq 1, Y \leq 2).$$



Joint Cumulative Distributive Function (CDF)

Marginal CDFs of X and Y:

$$F_{XY}(x, \infty) = P(X \leq x, Y < \infty) = P(X \leq x) = F_X(x),$$

$$F_{XY}(\infty, y) = P(X < \infty, Y < y) = P(Y \leq y) = F_Y(y),$$

$$F_{XY}(\infty, \infty) = 1,$$

$$F_{XY}(-\infty, y) = 0, \qquad \text{for any } y,$$

$$F_{XY}(x, -\infty) = 0, \qquad \text{for any } x.$$

$$0 \leq F_{XY}(x, y) \leq 1$$

Joint Cumulative Distributive Function (CDF)

Example. **Toss a fair coin twice,**

$$\textbf{First:}\ \begin{cases} X = 1 & H \\ X = 0 & T \end{cases} \qquad \textbf{Second:}\ \begin{cases} Y = 1 & H \\ Y = 0 & T \end{cases}$$

X **and** Y **are independent. Find the joint PMF and joint CDF for** X **and** Y**.**

Joint Cumulative Distributive Function (CDF)

Two discrete random variables X and Y are independent if

$$P_{XY}(x, y) = P_X(x)P_Y(y), \qquad \text{for all } x, y.$$

Equivalently, X and Y are independent if

$$F_{XY}(x, y) = F_X(x)F_Y(y), \qquad \text{for all } x, y.$$

Joint Cumulative Distributive Function (CDF)

So far:

Joint PMF

- $P_{XY}(x_i, y_j) = P(X = x_i, Y = y_j)$.
- $R_{XY} =$ all possible value for (X, Y).
- **Marginal PMFs**

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \qquad \text{LOTP}$$

$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y), \qquad \text{LOTP}$$

Joint Cumulative Distributive Function (CDF)

Joint CDF:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

$$F_{XY}(3, 2) = P(X \leq 3, Y \leq 2).$$

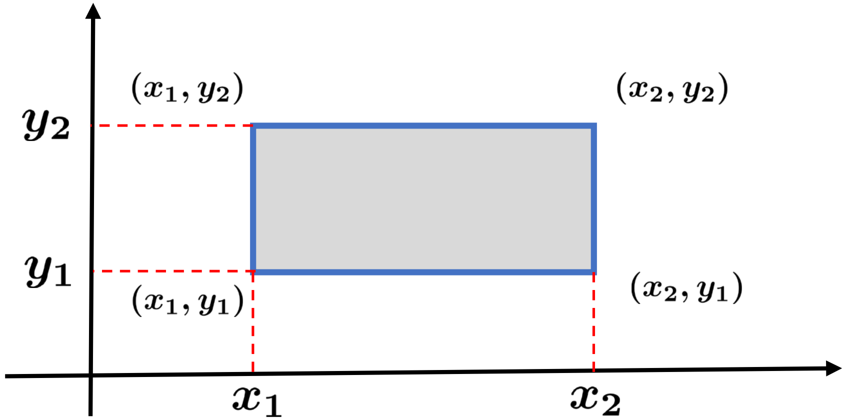
Remember:

$$P(a < X \leq b) = F_X(b) - F_X(a),$$

Joint Cumulative Distributive Function (CDF)

Lemma. For two random variables X and Y, and real numbers $x_1 \leq x_2, \;\; y_1 \leq y_2$, we have

$$P(x_1 < X \leq x_2, \;\; y_1 < Y \leq y_2) = \\ F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1).$$



Conditioning and Independence

Conditioning:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.$$

$$P(X = x_i|A) = \frac{P(X = x_i \text{ and } A)}{P(A)},$$

For example $A: \quad Y = y_j.$

Conditioning and Independence

Conditional PMF and CDF:

For a discrete random variable X and event A, the conditional PMF of X given A is defined as

$$\begin{aligned} P_{X|A}(x_i) &= P(X = x_i|A) \\ &= \frac{P(X = x_i \text{ and } A)}{P(A)}, \quad \text{for any } x_i \in R_X. \end{aligned}$$

Similarly, we define the conditional CDF of X given A as

$$F_{X|A}(x) = P(X \leq x|A).$$

Conditioning and Independence

PMF: $P_X(x_i) = P(X \leq x_i)$

Conditional PMF: $P_{X|A}(x_i) = P(X = x_i | A),$

Conditional CDF: $F_{X|A}(x) = P(X \leq x | A),$

Conditioning and Independence

Let $A: \quad Y = y_j.$

Conditional PMF of X given $Y = y_j$:

$$\begin{aligned} P_{X|Y}(x_i|y_j) &= P(X = x_i|Y = y_j) \\ &= \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} \\ &= \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}. \end{aligned}$$

Conditioning and Independence

Similarly, we can define the conditional probability of Y given X :

$$P_{Y|X}(y_j|x_i) = P(Y = y_j|X = x_i)$$
$$= \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}.$$

Conditioning and Independence

For discrete random variables X and Y, the conditional PMFs of X given Y and vice versa are defined as

$$P_{X|Y}(x_i|y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)},$$

$$P_{Y|X}(y_j|x_i) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}$$

for any $x_i \in R_X$ and $y_j \in R_Y$.

Conditioning and Independence

Example. Consider two random variables X and Y with joint PMF given in the following Table.

Find $P_{X|Y}(x|2)$, conditional PMF of X given $Y = 2$.

	$Y = 1$	$Y = 2$
$X = 1$	$\frac{1}{3}$	$\frac{1}{12}$
$X = 2$	$\frac{1}{6}$	0
$X = 4$	$\frac{1}{12}$	$\frac{1}{3}$

Conditioning and Independence

Independent Random Variables:

Two discrete random variables X and Y are independent if

$$P_{XY}(x_i, y_j) = P_X(x_i)P_Y(y_j), \quad \text{for all } x_i, y_j.$$

Equivalently

$$P_{X|Y}(x_i|y_j) = P_X(x_i), \quad P_{Y|X}(y_j|x_i) = P_Y(y_j).$$

Equivalently

$$F_{XY}(x, y) = F_X(x)F_Y(y), \quad \text{for all } x, y.$$

Conditioning and Independence

Conditional Expectation:

$$E[X] = \sum_{x_i \in R_X} x_i P_X(x_i),$$

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i),$$

$$E[X|Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y_j)$$

Conditioning and Independence

Example. **Consider two random variables X and Y with joint PMF given in Table.**

Find $E[X|Y=2]$ and $\mathrm{Var}(X|Y=2)$.

	$Y=1$	$Y=2$
$X=1$	$\frac{1}{3}$	$\frac{1}{12}$
$X=2$	$\frac{1}{6}$	0
$X=4$	$\frac{1}{12}$	$\frac{1}{3}$

$$\frac{1}{12} + 0 + \frac{4}{3}$$

$$\frac{1}{12} + 0 + \frac{16}{3}$$

Conditioning and Independence

Law of Total Probability:

$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A | Y = y_j) P_Y(y_j), \quad \textbf{for any set } A.$$

Conditioning and Independence

Law of Total Probability:

If $B_1, B_2, B_3, \ldots$ is a partition of the sample space S, then we have

$$P(A) = \sum_j P(A \cap B_j) = \sum_j P(A|B_j)P(B_i).$$

$B_j : Y = y_j,$

$$P(A) = \sum_j P(A|Y = y_j)P(Y = y_j).$$

Conditioning and Independence

Law of Total Expectation:

If $B_1, B_2, B_3, \ldots$ is a partition of the sample space S, then we have

$$EX = \sum_j E[X|B_j]P(B_j).$$

$$B_j : \; Y = y_j,$$

$$EX = \sum_{y_j \in R_Y} E[X|Y = y_j]P_Y(y_j).$$

Conditioning and Independence

Example. Suppose that the number of customers visiting a fast food restaurant in a given day is $N \sim Poisson(\lambda)$. Assume that each customer purchases a drink with probability p, independently from other customers and independently from the value of N. Let X be the number of customers who purchase drinks. Find EX.