Probability and Random Process (SWE3026)

Statistical Inference

JinYeong Bak
jy.bak@skku.edu
College of Computing, SKKU

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Objectives

Test whether a hypothesis is true or false

Hypothesis

Definition

H_0: null hypothesis, initially assumed to be true

H_1: alternative hypothesis, contradictory to H_0

Example

Let’s consider a radar system that uses radio waves to detect aircraft.

H_0: No aircraft is present

H_1: An aircraft is present

Example

You have a coin and you would like to check whether it is fair or not.

let θ be the probability of heads, $\theta = P(Head)$. You have two hypotheses:

H_0: The coin is fair, $\theta = \frac{1}{2}$

H_1: The coin is not fair, $\theta \neq \frac{1}{2}$

Which hypothesis is true?

Example

Experiment)

We toss the coin 100 times and record the number of heads

X**: the number of heads that we observe**

$$X \sim Binomial(100, \theta)$$

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With a threshold t:

if $|X - 50| \leq t$, accept H_0 & reject H_1

if $|X - 50| > t$, accept H_1 & reject H_0

Example

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X: the number of heads that we observe

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With a threshold t:

if $|X - 50| \leq t$, accept H_0 & reject H_1

if $|X - 50| > t$, accept H_1 & reject H_0

How do we choose t?

$$P(type\ 1\ error) = P(|X - 50| > t\ when\ H_0\ is\ true)$$

$$P(type\ 1\ error) \leq \alpha$$

Example

$$X \sim Binomial(100, \theta_0 = \frac{1}{2})$$

if $|X - 50| \leq t$, accept H_0 & reject H_1

$$P(type\ 1\ error) = P(|X - 50| > t\ when\ H_0\ is\ true) \leq \alpha = 0.05$$

By the central limit theorem:

$$Y = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{X - 50}{5} \sim N(0, 1)$$

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$$P(type\ 1\ error) = P(|X - 50| > t \mid H_0) = P\left(\left|\frac{X - 50}{5}\right| > \frac{t}{5}\middle| H_0\right) = P(|Y| > \frac{t}{5} | H_0)$$

With a threshold $c = \frac{t}{5}$:

if $|Y| \le c$, accept H_0 & reject H_1

if $|Y| > c$, accept H_1 & reject H_0

Example

$$X \sim Binomial(100, \theta_0 = \frac{1}{2})$$

if $|X - 50| \leq t$, accept H_0 & reject H_1

$$P(type\ 1\ error) = P(|X - 50| > t\ when\ H_0\ is\ true) = P(|Y| > c|H_0) \leq \alpha = 0.05$$

$$Y = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{X - 50}{5} \sim N(0,1)$$

With a threshold $c = \frac{t}{5}$:

if $|Y| \leq c$, accept H_0 & reject H_1

$$P(|Y| > c) = 1 - P(-c \leq Y \leq c) \cong 2 - 2\Phi(\mathrm{c}) = \alpha = 0.05$$
$$c = \Phi^{-1}(0.975) = 1.96$$

Example

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$$c = \Phi^{-1}(0.975) = 1.96$$

if $|Y| \leq 1.96$, accept H_0 & reject H_1

if $|Y| > 1.96$, accept H_1 & reject H_0

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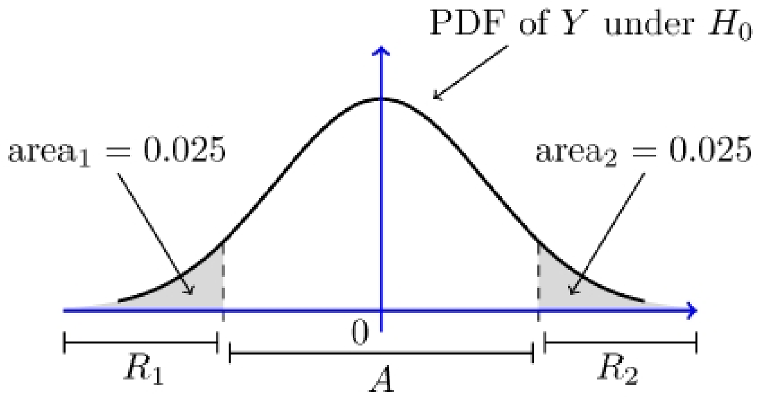

A = Acceptance Region

$R = R_1 \cup R_2$ = Rejection Region

$\alpha = \mathrm{P}(\text{type I error}) = \text{area}_1 + \text{area}_2 = 0.05$

Example

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if $|X - 50| \le t$, accept H_0 & reject H_1

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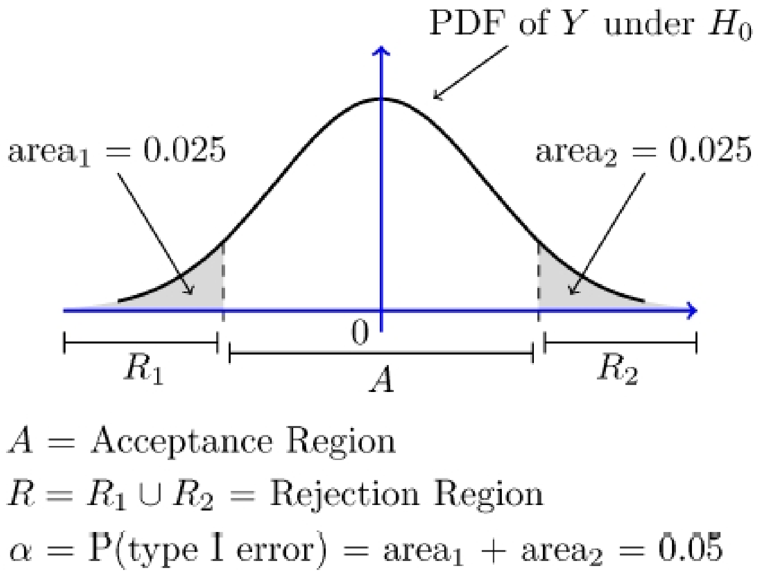

if $|X - 50| \le 9.8$, accept H_0 & reject H_1

if $|X - 50| > 9.8$, accept H_1 & reject H_0

Hypothesis Testing for the Mean

Definition

H_0: null hypothesis, initially assumed to be true

H_1: alternative hypothesis, contradictory to H_0

Example

We have n random samples from a distribution and let's make inference about the mean of the distribution μ

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

Example – Known Variance

Let $X_1, X_2, \dots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ is unknown but σ^2 is known. Design a level α test to choose between

$\boldsymbol{H_0: \mu = \mu_0, H_1: \mu \neq \mu_0}$

Example – Unknown Variance

Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ and σ^2 are unknown. Design a level α test to choose between

$H_0\!: \mu = \mu_0, H_1\!: \mu \neq \mu_0$

Example

The average adult male height in a certain country is 170 cm. We suspect that the men in a certain city in that country might have a different average height due to some environmental factors. We pick a random sample of size 9 from the adult males in the city and obtain the following values for their heights (in cm):

176.2 157.9 160.1 180.9 165.1 167.2 162.9 155.7 166.2

Assume that the height distribution in this population is normally distributed. Here, we need to decide between $H_0: \mu = 170$, $H_1: \mu \neq 170$.

Based on the observed data, is there enough evidence to reject H_0 at significance level $\alpha = 0.05$?

The average adult male height in a certain country is 170 cm. We suspect that the men in a certain city in that country might have a different average height due to some environmental factors. We pick a random sample of size 9 from the adult males in the city and obtain the following values for their heights (in cm):

176.2 157.9 160.1 180.9 165.1 167.2 162.9 155.7 166.2

Assume that the height distribution in this population is normally distributed. Here, we need to decide between $H_0: \mu = 170, H_1: \mu \neq 170$.

Based on the observed data, is there enough evidence to reject H_0 at significance level $\alpha = 0.05$?

Hypothesis Testing for the Mean

We have n random samples from a distribution and let's make inference about the mean of the distribution μ

Two-sided test

$\boldsymbol{H_0 : \mu = \mu_0}$

$\boldsymbol{H_1 : \mu \neq \mu_0}$

One-sided test

$\boldsymbol{H_0 : \mu \leq \mu_0}$

$\boldsymbol{H_1 : \mu > \mu_0}$

or

$\boldsymbol{H_0 : \mu \geq \mu_0}$

$\boldsymbol{H_1 : \mu < \mu_0}$

P-value

Definition

The lowest significance level α that results in rejecting the null hypothesis

Intuitively)

If the P-value is small, it means that the observed data is very unlikely to have occurred under H_0,

so we are more confident in rejecting the null hypothesis.

Example

You have a coin and you would like to check whether it is fair or not.

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H_0: The coin is fair, $\theta = \frac{1}{2}$

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We toss the coin 100 times and observe 60 heads.

Can we reject H_0 at significance level $\alpha = 0.05$?

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We toss the coin 100 times and observe 60 heads.

Can we reject $\boldsymbol{H_0}$ at significance level $\boldsymbol{\alpha = 0.01}$?

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We toss the coin 100 times and observe 60 heads.

What is the P-value?