Probability and Random Process (SWE3026)

Introduction to Random Processes

JinYeong Bak
jy.bak@skku.edu
College of Computing, SKKU

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Stationary Processes

A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is strict-sense stationary or simply stationary if, for all $t_1, t_2, \cdots, t_r \in \mathbb{R}$ and all $\Delta \in \mathbb{R}$, the joint CDF of

$$X(t_1), X(t_2), \cdots, X(t_r)$$

is the same as the joint CDF of

$$X(t_1 + \Delta), X(t_2 + \Delta), \cdots, X(t_r + \Delta).$$

Stationary Processes

That is, for all real numbers $x_1, x_2, \cdots, x_r$, we have

$$\begin{aligned} F_{X(t_1)X(t_2)\cdots X(t_r)}(x_1, x_2, \cdots, x_r) \\ = F_{X(t_1+\Delta)X(t_2+\Delta)\cdots X(t_r+\Delta)}(x_1, x_2, \cdots, x_r). \end{aligned}$$

Stationary Processes

A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ is strict-sense stationary or simply stationary, if for all $n_1, n_2, \cdots, n_r \in \mathbb{Z}$ and all $D \in \mathbb{Z}$, the joint CDF of

$$X(n_1), X(n_2), \cdots, X(n_r)$$

is the same as the joint CDF of

$$X(n_1 + D), X(n_2 + D), \cdots, X(n_r + D).$$

Stationary Processes

That is, for all real numbers $x_1, x_2, \cdots, x_r$**, we have**

$$F_{X(n_1)X(n_2)\cdots X(n_r)}(x_1, x_2, \cdots, x_n) = F_{X(n_1+D)X(n_2+D)\cdots X(n_r+D)}(x_1, x_2, \cdots, x_r).$$

Stationary Processes

Weak-Sense Stationary Processes:

A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is weak-sense stationary or wide-stationary (WSS) if

1) $\mu_X(t) = \mu_X$, for all $t \in \mathbb{R}$.

2) $R_X(t_1, t_2) = R_X(t_1 - t_2)$, for all $t_1, t_2 \in \mathbb{R}$.

Stationary Processes

Weak-Sense Stationary Processes:

A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ is weak-sense stationary or wide-stationary (WSS) if

1) $\mu_X(n) = \mu_X, \text{ for all } n \in \mathbb{Z}.$

2) $R_X(n_1, n_2) = R_X(n_1 - n_2), \text{ for all } n_1, n_2 \in \mathbb{Z}.$

Stationary Processes

Properties of $R_X(\tau)$:

$\tau = t_1 - t_2$

If $X(t)$ WSS, $R_X(t_1 - t_2) = R_X(\tau)$

1) $t_1 = t_2 \;\rightarrow\; E[X(t_1)^2] \geq 0 \quad \Rightarrow\; R_X(0) = E[X(t)^2] \geq 0,$

2) $R_X(-\tau) = E[X(t-\tau)X(t)] = E[X(t)X(t-\tau)] = R_X(\tau)$

$$\Rightarrow\; R_X(\tau) = R_X(-\tau), \quad \text{for all } \tau \in \mathbb{R}.$$

Stationary Processes

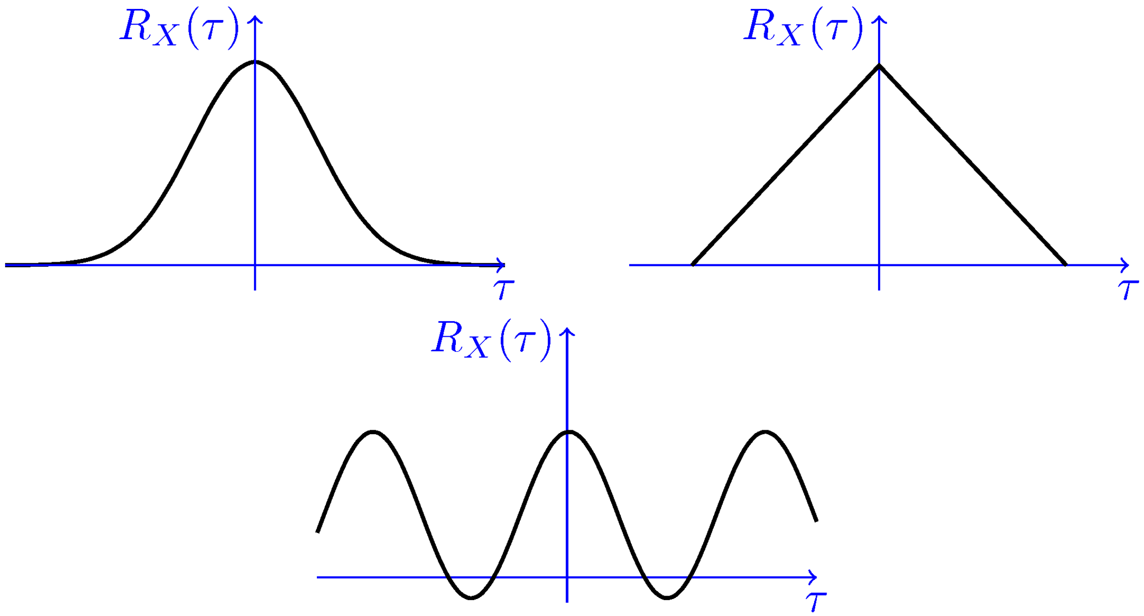
3) $|R_X(\tau)| \leq R_X(0), \quad \text{for all } \tau \in \mathbb{R}.$

Proof:

$$X = X(t)$$

$$Y = X(t-\tau)$$

$$\underbrace{|EXY|}_{R_X(\tau)} \leq \sqrt{\underbrace{E[X^2]}_{R_X(0)}\,\underbrace{E[Y^2]}_{R_X(0)}}$$



Stationary Processes

Jointly Wide-Sense Stationary Processes:

Two random processes $\{X(t), t \in \mathbb{R}\}$ and $\{Y(t), t \in \mathbb{R}\}$ are said to be jointly wide-sense stationary if

1) $X(t)$ **and** $Y(t)$ **are each wide-sense stationary.**

2) $R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$.

- **For WSS $X(t)$ & $Y(t)$, $\rightarrow$ $R_{XY}(\tau) = R_{XY}(-\tau)$.**

Stationary Processes

Cross Covariance:

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1) \cdot \mu_Y(t_2).$$

Summary of Random Process

$\boldsymbol{X(t), \quad t \in (0, \infty) \; or \; (-\infty, \infty)}$ **Continuous-time**

$\boldsymbol{X(n), \quad n \in \mathbb{Z}}$ **Discrete-time**

$$\boldsymbol{\mu_X(t) = E[X(t)] = \mu_X,}$$

$\boldsymbol{R_X(t_1, t_2) = E[X(t_1)X(t_2)] = R_X(t_1 - t_2)}$ **for WSS**

Summary of Random Process

$$R_X(0) = E[X(t)^2] \geq 0$$

$$R_X(\tau) = R_X(-\tau), \quad \text{for all } \tau \in \mathbb{R}.$$

$$|R_X(\tau)| \leq R_X(0), \quad \text{for all } \tau \in \mathbb{R}.$$

Gaussian Random Processes

A random process $\{X(t), t \in J\}$ is said to be a **Gaussian (normal) random process** if, for all

$$t_1, t_2, \ldots, t_n \in J,$$

the random variables $X(t_1), X(t_2), \cdots, X(t_n)$ are jointly normal.