Probability and Random Process (SWE3026)

Discrete Random Variables

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Rationale

- In general, to analyze random experiments, we usually focus on some numerical aspects of the experiment.
- For example, in a soccer game we may be interested in the number of goals, shots, shots on goal, corners kicks, fouls, etc.
- In a nutshell, a random variable is a real-valued variable whose value is determined by an underlying random experiment.

Random Variables

Random experiments have sometimes numerical outputs, such as

- Lifetime of a certain product: $0 \leq T < \infty$
- Amount of money a gambler wins on a trip to the casino
- etc.

Even if the event is not numerical, it can often be considered in terms of numbers (for convenience and mathematical analysis).

Random Variables

Example. Toss a coin five times. Observe the number of heads:

$$S = \{TTTT, TTTH, \cdots, HHHH\}.$$

We define a random variable that gets its value from the outcome of the random experiment:

$$X = 1, 2, 3, 4, \text{or } 5.$$

Definition: A random variable is a real-valued variable that gets its value from a random experiment.

Random Variables

Formal Definition: **A random variable is a real-valued function on the sample space:**

$$X : S \to \mathbb{R}, X(\{HHTHT\}) = 3.$$

Definition: **Range of X is the set of possible values for X.**

In the above example, $\mathrm{Range}(X) = R = \{1, 2, 3, 4, 5\}.$

We show random variables with capital letters X, Y, Z.

Random Variables

Example. **Flip a coin twice, X = the number of heads**

$$\mathrm{Range}(X) = R = \{0, 1, 2\}.$$

Example. **T: Lifetime of a certain product:**

$$\mathrm{Range} = \{X : X \in \mathbb{R}; x \geq 0\} = \mathbb{R}^{+} = [0, \infty).$$

Random Variables

Countable set:

a) Finite set

b) One-to-one correspondence with Natural Numbers

$$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$$

i.e., $R = \{a_1, a_2, a_3, \cdots\}$

i.e., I can “list” the elements.

Random Variables

Countably infinite sets:

$$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$$

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

List: $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \cdots\}$

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$ **Countable**

Random Variables

However $\mathbb{R}$ is Not countable, in fact

$[\mathbf{0}, \mathbf{1}] = \{x \in \mathbb{R},\ \mathbf{0} \leq \boldsymbol{X} \leq \mathbf{1}\}$ **is Not countable.**

Discrete Random Variables

Definition: X is a discrete random variable, if its range is countable.

$$R_X = \{x_1, x_2, x_3, \ldots\}.$$

We show the values in the range by lower case letters.

Probability Mass Function

Definition: X is a discrete random variable,

$$\text{Range}(X) = R_X = \{x_1, x_2, x_3, \ldots\}.$$

The function:

$$P_X(x_k) = P(X = x_k), \ \text{for } k = 1, 2, 3, \ldots,$$

is called the probability mass function (PMF) of X.

Probability Mass Function

Example 1. **Toss a fair coin twice, $X =$ # of heads.**

Find the range of X, R_X, as well as its probability mass function P_X.

Probability Mass Function

Example 2: $X =$ **# of rolls of a die until the first 6 appears.**

Find the range of $X, R_X,$ **as well as its probability mass function** P_X**.**

Probability Mass Function

Thm. For a discrete random variable with PMF $P_X(x)$ and Range

$$R_X = S_X = \{x_1, x_2, x_3, \cdots\}$$

a) $0 \le P_X(x_k) \le 1 \text{ for all } x_k \in S_X.$

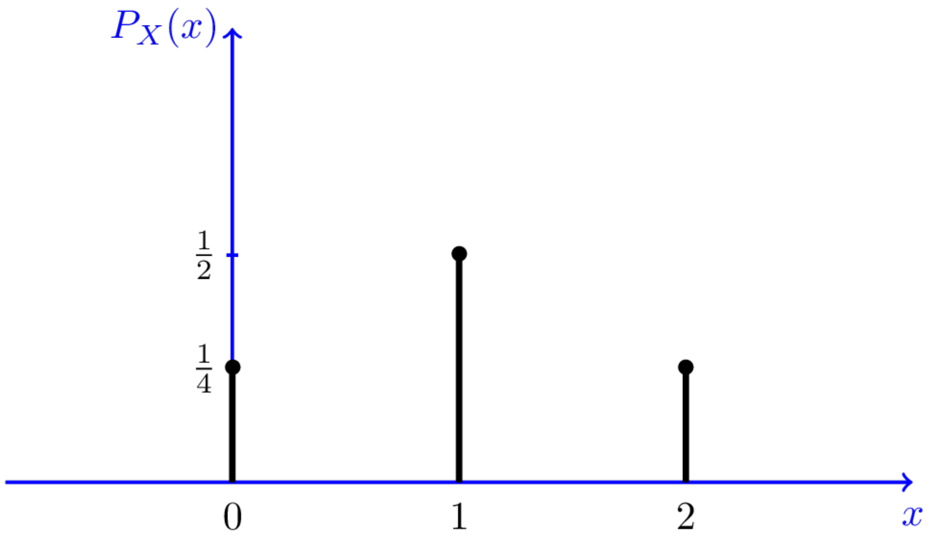
b) $\sum_{k=1}^{\infty} P_X(x_k) = 1.$

c) $A \subset S_X, P(X \in A) = P(A) = \sum_{x_k \in A} \text{Prob}\{X = x_k\} = \sum_{x_k \in A} P_X(x_k)$

Probability Mass Function

If we repeat the experiment over and over and plot the histogram, it will look like

The PMF in example 1



Independent Random Variables

Definition: Consider two discrete random variables X and Y. We say that X and Y are independent if

$$P\Big(X = x, Y = y\Big) = P(X = x)P(Y = y), \quad \text{for all } x, y.$$

In general, if two random variables are independent, then you can write

$$P\Big(X \in A, Y \in B\Big) = P(X \in A)P(Y \in B), \quad \text{for all sets } A \text{ and } B.$$

Independent Random Variables

Definition: Consider n discrete random variables $X_1, X_2, X_3, \dots, X_n$. We say that $X_1, X_2, X_3, \dots, X_n$ are independent if

$$P\left(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n\right)$$

$$= P(X_1 = x_1)P(X_2 = x_2)\dots P(X_n = x_n), \quad \text{for all } x_1, x_2, \dots, x_n.$$

Independent Random Variables

Example 1. I toss a fair coin twice, and define X to be the # of heads I observe. I toss the coin two more times and define Y to be the # of heads that I observe this time.

Find $P(\ (X < 2)\ and\ (Y > 1))$

Summary of Random Variables

- **Random Variables** $X : S \longrightarrow \mathbb{R}$

- **Discrete Random Variable** $R_X = \text{Range}(X)$ **is countable, i.e.,**

$$R_X = \{x_1, x_2, x_3, \ldots\}.$$

- **PMF:**

$$P_X(x_k) = P(X = x_k)$$

- **Independent Random Variable**