Probability and Random Process (SWE3026)

Joint Distributions

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Two Continuous Random Variables

PDF:

$$P(X \in A) = \int_A f_X(x)dx,$$

Joint PDF:

$$P\big((X,Y) \in A\big) = \iint\limits_A f_{XY}(x,y)dxdy.$$

Two Continuous Random Variables

If we choose $A = \mathbb{R}^2$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1.$$

Two Continuous Random Variables

Definition. Two random variables X and Y are jointly continuous if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)dxdy,$$

The function $f_{XY}(x,y)$ is called the joint probability density function (PDF) of X and Y.

Two Continuous Random Variables

CDF:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du,$$

Joint CDF:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)dudv,$$

and

Two Continuous Random Variables

$$f_X(x) = \frac{d}{dx} F_X(x),$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

Two Continuous Random Variables

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)\,du\,dv,$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Two Continuous Random Variables

The joint cumulative function of two random variables X and Y is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

The joint CDF satisfies the following properties:

1) $F_X(x) = F_{XY}(x, \infty)$, for any x (marginal CDF of X);

2) $F_Y(y) = F_{XY}(\infty, y)$, for any y (marginal CDF of Y);

3) $F_{XY}(\infty, \infty) = 1$;

Two Continuous Random Variables

4) $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0;$

5) $P(x_1 < X \leq x_2,\ \ y_1 < Y \leq y_2) =$

$$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

6) if X and Y are independent, then $F_{XY}(x, y) = F_X(x)F_Y(y)$.

Two Continuous Random Variables

Marginal PDFs:

For discrete random variables:

$$P_X(x) = \sum_{y \in R_Y} P_{XY}(x, y).$$

For continuous random variables:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy, \quad \text{for all } x,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx, \quad \text{for all } y.$$

Conditioning and Independence

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

conditional CDF:

$$F_{X|A}(x) = P(X \leq x|A).$$

Conditioning and Independence

Example. Let $A : a \le X \le b, \;\; X$: continuous

$$F_{X|A}(x) = P(X \le x|A) = P(X \le x|a \le X \le b) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > b \end{cases}$$

If $a \le x \le b$:

$$F_{X|A}(x) = \frac{P(X \le x, a \le X \le b)}{P(a \le X \le b)} = \frac{P(a \le X \le x)}{P(a \le X \le b)}$$

$$= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}.$$

Conditioning and Independence

Finally, if $x > b$, then $F_{X|A}(x) = 1$. Thus, we obtain

$$F_{X|A}(x) = \begin{cases} 1 & x > b \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \le x < b \\ 0 & \text{otherwise} \end{cases} \qquad A: \{a \le X \le b\}$$

Then the conditional PDF of X given A is given by

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{F_X(b) - F_X(a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Conditioning and Independence

In general, for a random variable X and an event A, we have the following:

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx,$$

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx,$$

$$\mathrm{Var}(X|A) = E[X^2|A] - (E[X|A])^2$$

Conditioning and Independence

Conditioning by Another Random Variable:

For discrete random variables: the conditional PMF of X given $Y = y$ is given by

$$P_{X|Y}(x|y) = \frac{P_{XY}(x, y)}{P_Y(y)}.$$

Conditioning and Independence

For continuous random variables:

The conditional PDF of X given $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

The conditional PDF of Y given $X = x$ is given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

Conditioning and Independence

For two jointly continuous random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, we have

$$\boldsymbol{E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx,}$$

$$\boldsymbol{E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)dx,}$$

$$\boldsymbol{\mathrm{Var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2}$$

Summary of Conditioning

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

conditional CDF and PDF given A : (e.g., $A = \{a \leq X \leq b\}$)

$$F_{X|A}(x) = P(X \leq x|A),$$

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x).$$

Summary of Conditioning

In general, for a random variable X and an event A, we have the following:

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx,$$

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx,$$

$$\mathrm{Var}(X|A) = E[X^2|A] - (E[X|A])^2$$

Summary of Conditioning

Conditioning by Another Random Variable:

For discrete random variables: the conditional PMF of X given $Y = y$ is given by

$$P_{X|Y}(x|y) = \frac{P_{XY}(x, y)}{P_Y(y)}.$$

Summary of Conditioning

For continuous random variables:

The conditional PMF of X given $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}.$$

The conditional PMF of Y given $X = x$ is given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}.$$

Summary of Conditioning

For two jointly continuous random variables X and Y, we have

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx,$$

$$E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx,$$

$$\mathrm{Var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dx\,dy.$$

Conditioning and Independence

Law of Total Probability:

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x) f_X(x)\, dx,$$

Law of Total Expectation:

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x)\, dx = E[E[Y|X]].$$

Conditioning and Independence

Law of Total Variance:

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X]).$$

Conditioning and Independence

Independent Random Variables:

Discrete:

$$P_{XY}(x_i, y_j) = P_X(x_i) P_Y(y_j) \quad \text{for all } x_i, y_j.$$

Continuous:

$$f_{XY}(x, y) = f_X(x) f_Y(y), \quad \text{for all } x, y.$$

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$$F_{XY}(x, y) = F_X(x) F_Y(y), \quad \text{for all } x, y.$$

Conditioning and Independence

Example. **Determine whether X and Y are independent.**

$$f_{XY}(x,y)=\begin{cases}2e^{-x-2y}, & x,y>0\\ 0, & \text{otherwise}\end{cases}$$

Functions of Two Continuous Random Variables

LOTUS for two continuous random variables:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y).$$

Functions of Two Continuous Random Variables

Theorem. Let X and Y be two jointly continuous random variables. Let $(Z, W) = g(X, Y) = (g_1(X, Y), g_2(X, Y))$, where $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a continuous one-to-one (invertible) function with continuous partial derivatives. Let $h = g^{-1}$, i.e., $(X, Y) = h(Z, W) = (h_1(Z, W), h_2(Z, W))$. Then Z and W are jointly continuous and their joint PDF, $f_{ZW}(z, w)$, for $(z, w) \in R_{ZW}$ is given by

$$f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w))|J|,$$

Functions of Two Continuous Random Variables

Where J is the Jacobian of h defined by

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \frac{\partial h_1}{\partial z} . \frac{\partial h_2}{\partial w} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial w} .$$

Functions of Two Continuous Random Variables

Example.

Let X and Y be two independent standard normal random variables. Let also

$$\begin{cases} Z = 2X - Y \\ W = -X + Y \end{cases}$$

Find $f_{ZW}(z, w)$

Functions of Two Continuous Random Variables

If X and Y are two jointly continuous random variables and $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z-w)dw = \int_{-\infty}^{\infty} f_{XY}(z-w, w)dw.$$

If X and Y are also independent, then

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z-w)dw. \end{aligned}$$

Summary of Independence

Two continuous random variables X and Y are independent if

$$f_{XY}(x, y) = f_X(x) f_Y(y), \qquad \text{for all } x, y.$$

Conditioning and Independence

Two continuous random variables X and Y are independent, then we have

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \overbrace{f_{XY}(x,y)}^{f_X(x)f_Y(y)} dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y) dxdy$$
$$= \int_{-\infty}^{\infty} y f_Y(y) \left[\int_{-\infty}^{\infty} x f_X(x) dx\right] dy = \underbrace{\int_{-\infty}^{\infty} x f_X(x) dx}_{EX} \underbrace{\int_{-\infty}^{\infty} y f_Y(y) dy}_{EY}$$

Conditioning and Independence

More generally: X **and** Y **independent**

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$