Probability and Random Process (SWE3026)

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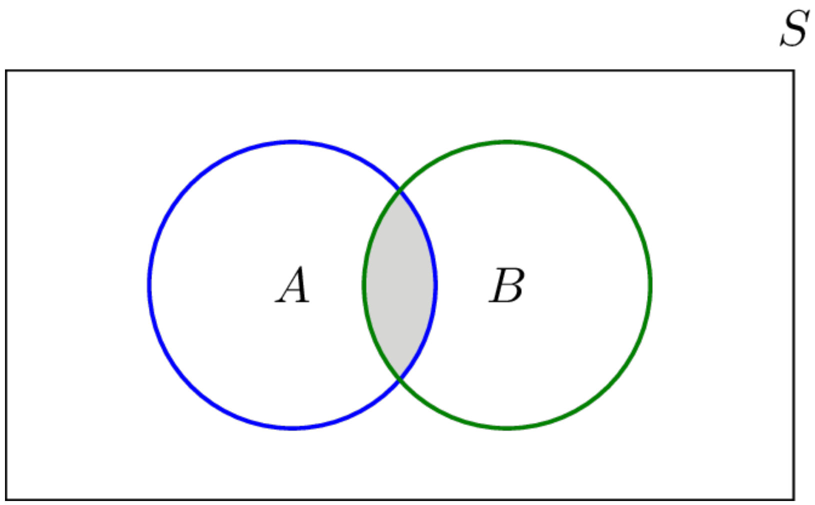
H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Conditional Probability

If A and B are two events in a sample space S, then

$$\boldsymbol{P(A \text{ given } B) = P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.}$$



Conditional Probability

Conditional probability satisfies the probability axioms :

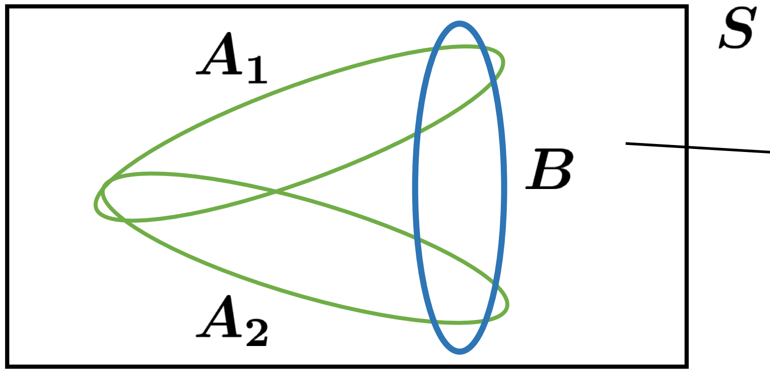
a) For any event A, $P(A|B) \geq 0.$

b) Conditional probability of B given B is $P(B|B) = 1.$

Conditional Probability

c) If $A_1, A_2, A_3, \cdots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cdots |B) = P(A_1|B) + P(A_2|B) + P(A_3|B) + \cdots .$$



$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B).$$

Conditional Probability

Example. **Roll two dice** $X_1, \; X_2$.

A **: 3 dots are shown at least on one die**

$$X_1 = 3 \quad \text{or} \quad X_2 = 3,$$

$$B: \; X_1 + X_2 = 6,$$

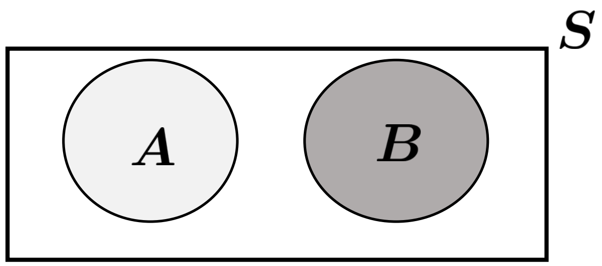
Find $P(A|B) = \dfrac{P(A \cap B)}{P(B)}.$

Conditional Probability

Special cases:

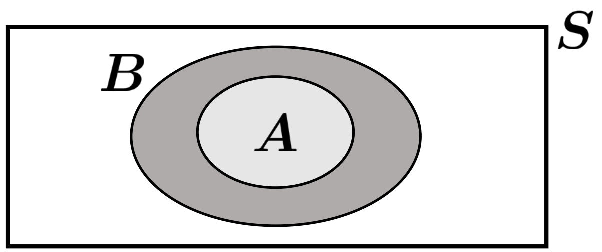
1) A and B are disjoint:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0.$$



2) $A \subset B$

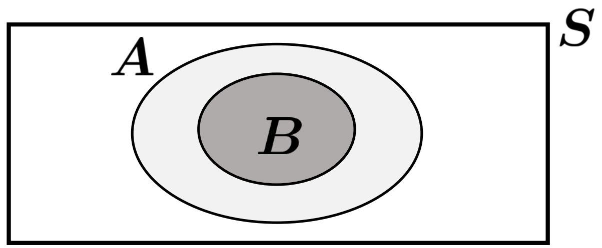
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$



Conditional Probability

3) $B \subset A$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$



Conditional Probability

Example. **Consider a family that has two children. We are interested in the children's genders. Our sample space is $\boldsymbol{S = \{(G, G), (G, B), (B, G), (B, B)\}}$. Also assume that all four possible outcomes are equally likely.**

Conditional Probability

Example. **Consider a family that has two children. We are interested in the children's genders. Our sample space is $S = \{(G, G), (G, B), (B, G), (B, B)\}$. Also assume that all four possible outcomes are equally likely.**

a. **What is the probability that both children are girls given that the first child is a girl?**

b. **We ask the parent: "Do you have at least one daughter?" He said "Yes". With this information, What is the probability that both children are girls?**

Conditional Probability

Chain Rule for Conditional Probability

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \;\Rightarrow\; P(B \cap A) = P(A)P(B|A).$$

We can extend this to 3 or more events:

$$P(B \cap A \cap C) = P(A)P(B|A)P(C|A \cap B).$$

Conditional Probability

Definition: **Two events A and B are independent if and only if**

$$P(A|B) = P(A), \text{ equivalently } \quad P(A \cap B) = P(A)P(B).$$

$$\Rightarrow P(A|B)P(B) = P(A)P(B) \Rightarrow P(A \cap B) = P(A)P(B).$$

Conditional Probability

Warning!

Disjoint (mutually exclusive) $\neq$ **Independent**

Disjoint: $A \cap B = \emptyset, \quad P(A \cup B) = P(A) + P(B)$

Independent: $P(A \cap B) = P(A)P(B), \quad P(A|B) = P(A), P(B|A) = P(B)$

Conditional Probability

Suppose A **and** B **are disjoint:**

If $P(B) \neq 0$, $\quad P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0}{P(B)} = 0.$

If $P(A) \neq 0, P(B) \neq 0$ $\&$ **disjoint** $\Rightarrow$ **Not independent.**

Conditional Probability

Example:

I pick a random number from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and call it N. Suppose that all outcomes are equally likely. Let A be the event that N is less than 7, and let B be the event that N is an even number. Are A and B independent? disjoint?

Conditional Probability

Remark:

1) $P(A|B) = P(A) \;\Rightarrow\; P(B|A) = P(B), \;\; (P(A), P(B) \neq 0)$

2) **If** $A \;\&\; B$ **are independent, then**

   a) $A^c \;\&\; B$ are independent.

   b) $A \;\&\; B^c$ are independent.

   c) $A^c \;\&\; B^c$ are independent.

Independence

Summary:

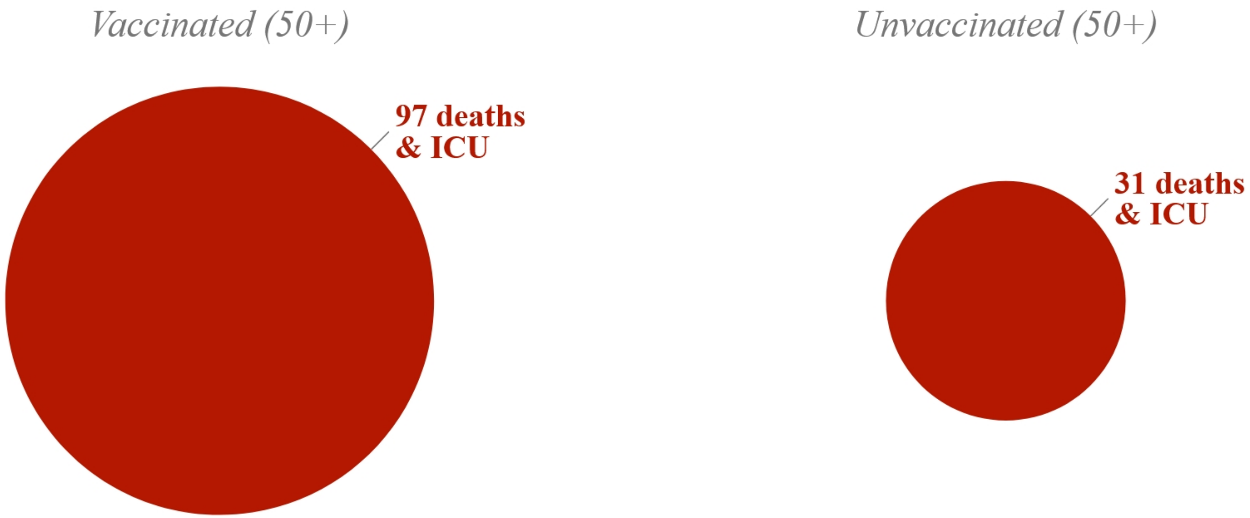
Concept	Meaning	Formulas
Disjoint	A and B cannot occur at the same time	$A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
Independent	A does not give any information about B	$P(A\|B) = P(A), P(B\|A) = P(B)$ $P(A \cap B) = P(A)P(B)$

Careful Explanation

http://news.sbs.co.kr/news/endPage.do?news_id=N1002027621

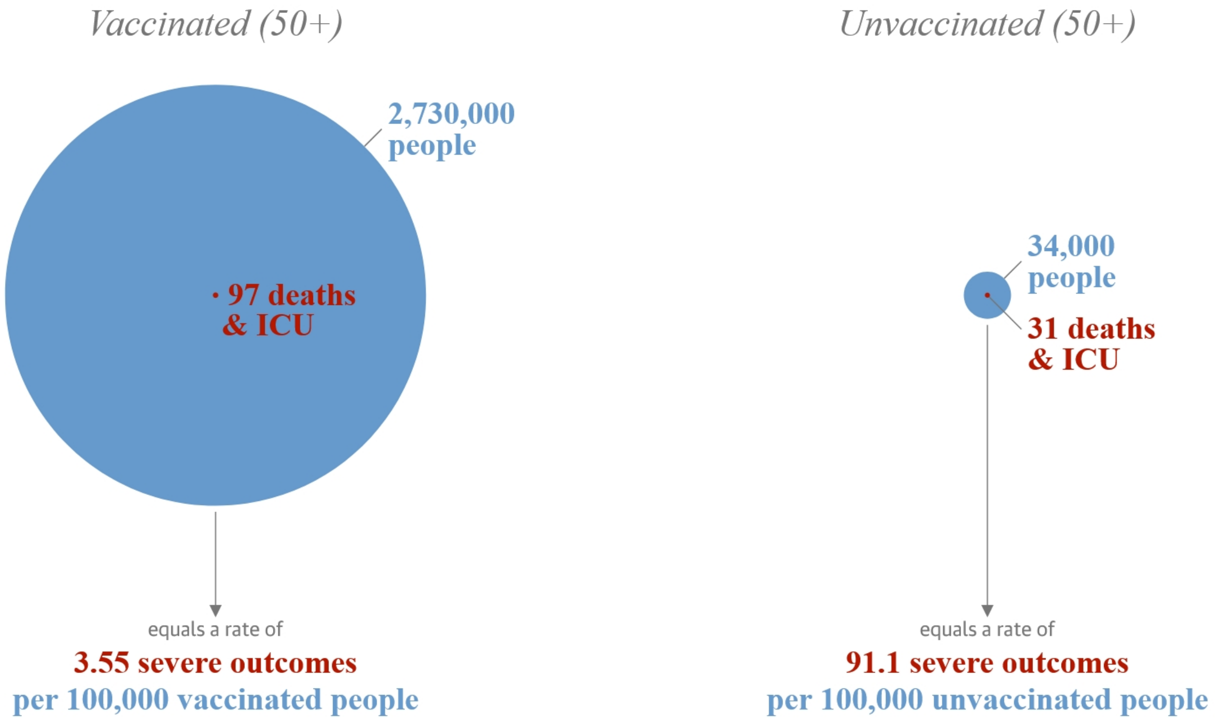
Careful Explanation

These **circles** compare the number of vaccinated and unvaccinated people aged 50 and over who have **died or ended up in ICU** due to Covid in NSW between 26 November 2021 and 1 January 2022. This looks bad without any other context - these **severe outcomes** seem higher for vaccinated than unvaccinated people



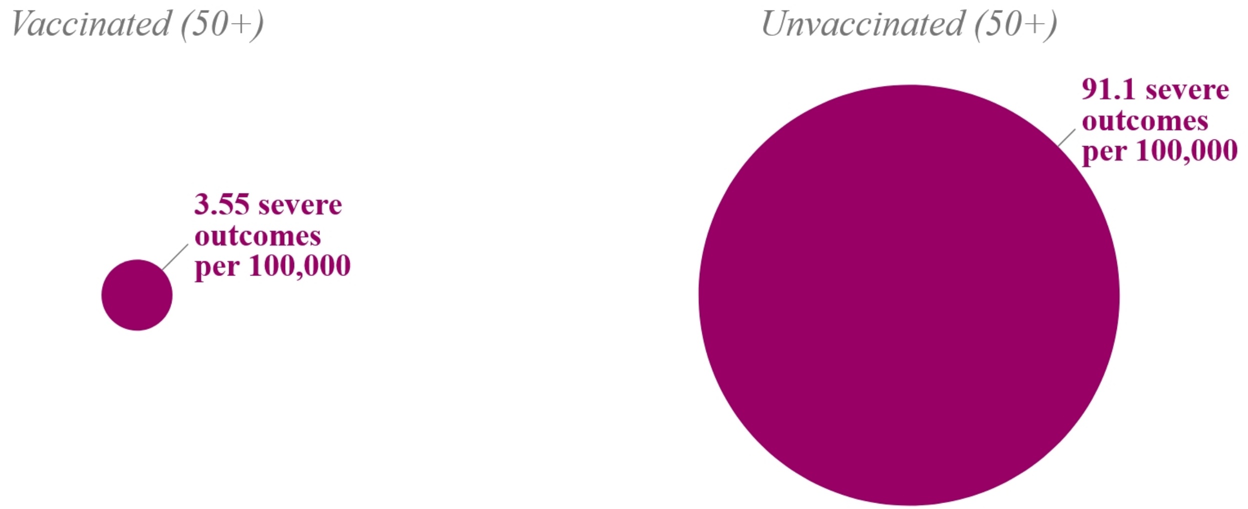
Careful Explanation

However, if you show the **severe outcomes** as a proportion of **all vaccinated and unvaccinated people** in the state, you can see that the number of vaccinated people is far larger than the number of unvaccinated people. This means that vaccinated people are far less likely to have a severe outcome from Covid



Careful Explanation

So now, resizing the circles by the rate of **severe outcomes per 100,000 people** we can see the rate is far lower in vaccinated people. This is because vaccines greatly reduce the chance of severe illness from Covid



https://www.theguardian.com/news/datablog/ng-interactive/2022/jan/28/the-simple-numbers-every-government-should-use-to-fight-anti-vaccine-misinformation