Probability and Random Process (SWE3026)

Discrete Random Variables

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Expectation

Expected value (= mean=average):

$$a_1, a_2, \ldots, a_n \;\Rightarrow\; \bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Definition. Let X be a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$. The expected value of X, denoted by EX is defined as

$$EX = \mu_X = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Expectation

Repeat the experiment N times (N large).

$$P(X = x_k) = P_X(x_k) = \frac{\text{(The number of times } X = x_k)}{N} = \frac{N_k}{N},$$

$$\Rightarrow \ N_k \approx N P_X(x_k),$$

$$\begin{aligned}
\text{Average } &= \frac{N_1 x_1 + N_2 x_2 + N_3 x_3 + \ldots}{N} \\
&\approx \frac{x_1 N P_X(x_1) + x_2 N P_X(x_2) + x_3 N P_X(x_3) + \ldots}{N} \\
&= x_1 P_X(x_1) + x_2 P_X(x_2) + x_3 P_X(x_3) + \ldots \\
&= EX.
\end{aligned}$$

Expectation

Example. **Let** $X \sim Bernoulli(p),$ **find** EX**.**

Expectation

Example. **Let** $X \sim Geometric(p)$**, find** EX**.**

Expectation

Example. **Let** $X \sim Poisson(\lambda)$, **find** EX.

Summary

Discrete RVs:

- **Range:** $R_X = \{x_1, x_2, x_3, \ldots\}$.
- **PMF:** $P_X(x_k) = P(X = x_k)$.
- **CDF:** $F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}$.
- **Excepted value:** $\mu_X = E[X] = \sum_{x_k \in R_X} x_k P_X(x_k)$.

Summary

- $X \sim Bernoulli(p), \quad EX = p.$
- $X \sim Geometric(p), \quad EX = \dfrac{1}{p}.$
- $X \sim Poisson(\lambda), \quad EX = \lambda.$

Functions of Random Variables

If X is a random variable and $Y = g(X)$, then Y itself is a random variable.

For example: $Y = X^2$.

$$R_X = \{x_1, x_2, x_3, ...\} \;\rightarrow\; R_Y = \{g(x) | x \in R_X\} = \{g(x_1), g(x_2), \cdots\}.$$

Then,

$$P_Y(y_k) = P(Y = y_k) = P(g(X) = y_k).$$

Functions of Random Variables

Example. **Let X be a discrete random variable uniformly distributed with $P_X(k) = \frac{1}{5}$, for $k = \{-2, -1, 0, 1, 2\}$. Let $Y = |X|$.**

a) Find PMF of Y, $P_Y(y)$.

b) Find EY.

Functions of Random Variables

Law of the unconscious statistician (LOTUS) for discrete random variables:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k).$$

In the previous example $g(X) = |X|$,

$$E[|X|] = |-2| . \frac{1}{5} + |-1| . \frac{1}{5} + |0| . \frac{1}{5} + |1| . \frac{1}{5} + |2| . \frac{1}{5}$$
$$= \frac{6}{5}.$$

Functions of Random Variables

Linearity of expectation:

$$Y = aX + b \Rightarrow E[Y] = aEX + b, \quad a, b \in \mathbb{R}$$

Proof: Here $g(X) = aX + b$, so using LOTUS we have

$$E[Y] = E[g(X)] = \sum g(x_k) P_X(x_k) = \sum_{x_k \in R_X} (ax_k + b) P_X(x_k)$$

$$= a \underbrace{\sum_{x_k \in R_X} x_k P_X(x_k)}_{EX} + b \underbrace{\sum_{x_k \in R_X} P_X(x_k)}_{1}$$

$$= aEX + b.$$

Functions of Random Variables

More generally (Linearity of expectation):

$$Y = X_1, X_2, \cdots, X_n \;\Rightarrow\; E[Y] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

Example. $X \sim Binomial(n, p),\; EX?$

Functions of Random Variables

Example. Let $R_X = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$ **, such that**

$$P_X(0) = P_X(\frac{\pi}{4}) = P_X(\frac{\pi}{2}) = P_X(\frac{3\pi}{4}) = P_X(\pi) = \frac{1}{5}$$

Find $E[\sin(X)]$**.**

Variance

The variance is a measure of how spread out the distribution of a random variable is.

$$EX = \mu_X \;\rightarrow\; E[X - \mu_X] = E[X] - E[\mu_X] = \mu_X - \mu_X = 0.$$

The variance of a random variable X, with mean $EX = \mu_X$, is defined as

$$\mathrm{Var}(X) = E\left[(X - \mu_X)^2\right], \quad \mu_X = EX.$$

Variance

The standard deviation of a random variable X is defined as

$$\mathrm{SD}(X) = \sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Variance

Theorem. **Computational formula for the variance:**

$$\mathrm{Var}(X) = E\left[X^2\right] - \left(EX\right)^2.$$

Proof:

$$\begin{aligned}
\mathrm{Var}(X) &= E\left[(X - \mu_X)^2\right] \\
&= E\left[X^2 - 2\mu_X X + \mu_X^2\right] \\
&= E\left[X^2\right] - 2E\left[\mu_X X\right] + E\left[\mu_X^2\right] \qquad \text{by linearity of expectation.} \\
&= E\left[X^2\right] - 2\mu_X^2 + \mu_X^2 \\
&= E\left[X^2\right] - \mu_X^2.
\end{aligned}$$

Variance

Example. $X \sim Bernoulli(p),\ \mathrm{Var}(X)$?

Variance

Theorem. For a random variable X and real numbers a and b,

$$\mathrm{Var}(aX + b) = a^2\ \mathrm{Var}(X).$$

Proof:

If $Y = aX + b,\ EY = aEX + b$. Thus,

$$\begin{aligned}\mathrm{Var}(Y) &= E[(Y - EY)^2] \\ &= E\left[\left((aX + b) - (aEX + b)\right)^2\right] \\ &= E[a^2(X - \mu_X)^2] \\ &= a^2 E[(X - \mu_X)^2] = a^2\ \mathrm{Var}(X)\end{aligned}$$

Variance

Theorem. If $X_1, X_2, \cdots, X_n$ are independent random variables and $X = X_1 + X_2 + \cdots + X_n$, then

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n).$$

Variance

Example. $X \sim Binomial(n, p),\ \mathrm{Var}(X)?$