Probability and Random Process (SWE3026)

Joint Distributions

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Covariance and Correlation

The covariance between X and Y is defined as

$$\mathrm{Cov}(X,Y) = E\big[(X - EX)(Y - EY)\big] = E[XY] - (EX)(EY).$$

Proof:

$$\begin{aligned} E\big[(X - EX)(Y - EY)\big] &= E\big[XY - X(EY) - (EX)Y + (EX)(EY)\big] \\ &= E[XY] - (EX)(EY) - (EX)(EY) + (EX)(EY) \\ &= E[XY] - (EX)(EY). \end{aligned}$$

Covariance and Correlation

Example. **Suppose $X \sim Uniform(1,2)$ and given $X = x$, Y is**

$$\boldsymbol{Y|X = x \sim Exponential(\lambda = x).}$$

Find $\text{Cov}(X, Y)$.

$$\boldsymbol{f_X(x) = \begin{cases} 1 & 1 \leq x \leq 2 \\ 0 & \text{else} \end{cases}, \quad f_{Y|X}(y|x) = \lambda e^{-\lambda y} u(y).}$$

Covariance and Correlation

Lemma. **The covariance has the following properties:**

1) $\mathrm{Cov}(X,X) = E[XX] - EXEX = E[X^2] - (EX)^2 = \mathrm{Var}(X).$

2) $X\&Y$ **independent:**

$$\mathrm{Cov}(X,X) = E[XY] - EXEY = E[X]E[Y] - EXEY = 0.$$

3) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

4) $\mathrm{Cov}(aX,Y) = a\mathrm{Cov}(X,Y) \quad a \in \mathbb{R}$

Covariance and Correlation

5) $\mathrm{Cov}(X + c, Y) = \mathrm{Cov}(X, Y)$

6) $\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$

7) $\mathrm{Cov}(X+Y, Z+W) = \mathrm{Cov}(X, Z)+\mathrm{Cov}(X, W)+\mathrm{Cov}(Y, Z)+\mathrm{Cov}(Y, W)$

$$\begin{aligned}\mathrm{Cov}(2X + Y, 3Z + W) =& 6\mathrm{Cov}(X, Z) + 2\mathrm{Cov}(X, W)+\\ & 3\mathrm{Cov}(Y, Z) + \mathrm{Cov}(Y, W)\end{aligned}$$

Covariance and Correlation

More generally

$$\operatorname{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \operatorname{Cov}(X_i, Y_j).$$

Covariance and Correlation

Variance of a sum:

If $Z = X + Y$, then

$$\begin{aligned} \mathrm{Var}(Z) &= \mathrm{Cov}(Z, Z) \\ &= \mathrm{Cov}(X+Y, X+Y) \\ &= \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) + \mathrm{Cov}(Y, Y) \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y). \end{aligned}$$

Covariance and Correlation

More generally, for $a, b \in \mathbb{R}$, we conclude:

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X, Y).$$

Covariance and Correlation

Correlation Coefficient:

$$\rho_{XY} = \rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var(X)\ Var(Y)}}} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

Covariance and Correlation

Properties of the correlation coefficient:

1) $-1 \leq \rho(X, Y) \leq 1;$

2) $\rho(aX + b, cY + d) = \rho(X, Y) \text{ for } a, c > 0;$

3) $\rho(X, Y) = 1 \quad \text{if } Y = aX + b \;\; a > 0;$
   $\rho(X, Y) = -1 \;\; \text{if } Y = aX + b \;\; a < 0.$

Covariance and Correlation

Definition. Consider two random variables X and Y :

1) If $\rho(X,Y) = 0$, we say that X and Y are uncorrelated.

2) If $\rho(X,Y) > 0$, we say that X and Y are positively correlated.

3) If $\rho(X,Y) < 0$, we say that X and Y are negatively correlated.

Bivariate Normal Distribution

Definition. Two random variables X and Y are said to be bivariate normal, or jointly normal, if $aX + bY$ has a normal distribution for all $a, b \in \mathbb{R}$.

$$b = 0,\ a = 1 \ \longrightarrow X : \text{Normal}$$

$$a = 0,\ b = 1 \ \longrightarrow Y : \text{Normal}$$

Bivariate Normal Distribution

Definition. Two random variables X and Y are said to have a **bivariate normal distribution** with parameters $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$ and ρ, if their joint PDF is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

Where $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and $\rho \in (-1,1)$ are all constants.

Bivariate Normal Distribution

If $\rho = 0$ ($X \& Y$ are uncorrelated) :

$$f_{XY}(x,y) = c \cdot \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}$$

$$= c' \underbrace{\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}}_{\text{Function of } X} \cdot d \underbrace{\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}}_{\text{Function of } Y}$$

Bivariate Normal Distribution

Theorem. **If X and Y are bivariate normal and uncorrelated, then they are independent.**

$$f_{XY}(x, y) = f_X(x) f_Y(y).$$

Bivariate Normal Distribution

Theorem. **Let X and Y be two bivariate normal random variables. Then, there exist independent standard normal random variables Z_1 and Z_2 such that**

$$\begin{cases} X = \sigma_X Z_1 + \mu_X \\ Y = \sigma_Y \left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2 \right) + \mu_Y \end{cases}$$

Bivariate Normal Distribution

Theorem. Suppose X and Y are jointly normal random variables with parameters $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$ and ρ . Then, given $X = x, Y$ is normally distributed with

$$E[Y|X = x] = \mu_Y + \rho\sigma_Y \frac{x - \mu_X}{\sigma_X},$$
$$\mathrm{Var}(Y|X = x) = (1 - \rho^2)\sigma_Y^2.$$

Bivariate Normal Distribution

Theorem. **If X and Y are bivariate normal & uncorrelated, then they are independent.**

$$E[Y|X=x] = \mu_Y + \rho\sigma_Y \frac{x-\mu_X}{\sigma_X},$$
$$\mathrm{Var}(Y|X=x) = (1-\rho^2)\sigma_Y^2.$$