Probability and Random Process (SWE3026)

Multiple Random Variables

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Probability Bounds

1) Union Bound and its Extensions
2) Markov and Chebyshev's Inequalities
3) Chernoff Bounds
4) Cauchy-Schwarz Inequality
5) Jenson's Inequality

Probability Bounds

Usefulness:

1) Generality
2) When exact computation is not possible.

Probability Bounds

Remember:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &\leq P(A) + P(B). \end{aligned}$$

More generally:

$$P(A \cup B \cup C \cup \cdots) \leq P(A) + P(B) + P(C) + \cdots$$

Probability Bounds

The Union Bound

For any events $A_1, A_2, \ldots, A_n$, we have

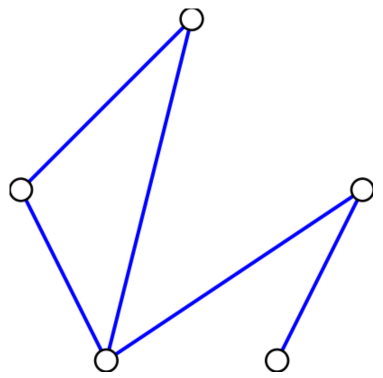
$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

Probability Bounds

Example. **Random Graphs:** $G(n, p)$.

n : The number of nodes

p : Probability of connection between two nodes (independent from other edges).



$n = 5$

Probability Bounds

Example. **Let B_n be the event that a graph randomly generated according to $G(n, p)$ model has at least one isolated node (a node that is not connected to any other nodes). Show that**

$$P(B_n) \leq n(1-p)^{n-1}.$$

b) And conclude that for any $\epsilon > 0$, if $p = p_n = (1+\epsilon)\dfrac{\ln(n)}{n}$ then

$$\lim_{n \to \infty} P(B_n) = 0.$$

Probability Bounds

It is an interesting exercise to calculate $P(B_n)$ exactly using the inclusion-exclusion principle:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right).$$

Probability Bounds

Generalization of the Union Bound: Bonferroni Inequalities

For any events $A_1, A_2, \ldots, A_n$, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i);$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j);$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k).$$

$$\vdots$$

Probability Bounds

Markov's Inequality

If X is any nonnegative random variable, then

$$P(X \geq a) \leq \frac{EX}{a}, \quad \text{for any } a > 0.$$

Probability Bounds

Let X be any positive continuous random variable, we can write

$$\begin{aligned} EX &= \int_0^\infty x f_X(x)dx \geq \int_a^\infty x f_X(x)dx \qquad \text{(for any } a > 0 \text{ and } x > a) \\ &\geq \int_a^\infty a f_X(x)dx = a \underbrace{\int_a^\infty f_X(x)dx}_{P(X \geq a)} \end{aligned}$$

$$\Rightarrow \; P(X \geq a) \leq \frac{EX}{a}, \qquad \text{for any } a > 0.$$

Probability Bounds

Chebyshev's Inequality

If X is any random variable, then for any $b > 0$, we have

$$P(|X - EX| \geq b) \leq \frac{Var(X)}{b^2}.$$

Probability Bounds

Let X be any random variable. If $Y = (X - EX)^2$, then Y is a nonnegative random variable.

$$P(Y \geq b^2) \leq \frac{EY}{b^2} = \frac{\mathrm{Var}(X)}{b^2},$$

$$P\big((X - EX)^2 \geq b^2\big) = P\big(|X - EX| \geq b\big).$$

Probability Bounds

Chernoff Bounds

$$P(X \geq a) \leq e^{-sa} M_X(s), \quad \text{for all } s > 0$$

$$P(X \leq a) \leq e^{-sa} M_X(s), \quad \text{for all } s < 0$$

Probability Bounds

For $s > 0$, we can write

$$
\begin{aligned}
P(X \geq a) &= P(e^{sX} \geq e^{sa}) \\
&\leq \frac{E[e^{sX}]}{e^{sa}}, \quad \text{by Markov's inequality.} \\
&= \frac{M_X(s)}{e^{sa}}
\end{aligned}
$$

Comparison between Bounds

Let's find upper bounds on $\boldsymbol{P(X \geq \alpha n)}$ for $\boldsymbol{X} \sim Binomial(\boldsymbol{n}, \boldsymbol{p})$ where $\boldsymbol{p} = \frac{\mathbf{1}}{\mathbf{2}}, \alpha = \frac{\mathbf{3}}{\mathbf{4}}$.

1) **Markov Inequalities** $$\boldsymbol{P(X \geq a) \leq \frac{EX}{a}, \quad \text{for any } a > 0.}$$

2) **Chebyshev's Inequalities** $$\boldsymbol{P\big(|X - EX| \geq b\big) \leq \frac{Var(X)}{b^2}.}$$

3) **Chernoff Bounds** $$\boldsymbol{P(X \geq a) \leq e^{-sa} M_X(s), \quad \text{for all } s > 0}$$

Probability Bounds

Cauchy-Schwarz Inequality

For any two random variables X and Y, we have

$$|EXY| \leq \sqrt{E[X^2]E[Y^2]},$$

where equality holds if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$.

Probability Bounds

$|\rho(X,Y)| \le 1:$

assuming $EX = EY = 0$

$$|\rho(X,Y)| = \left|\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}\right| = \left|\frac{EXY}{\sqrt{E[X^2]E[Y^2]}}\right| \le 1$$

$$\Rightarrow \;\; |EXY| \le \sqrt{E[X^2]E[Y^2]}.$$

Probability Bounds

Jensen's Inequality

Remember

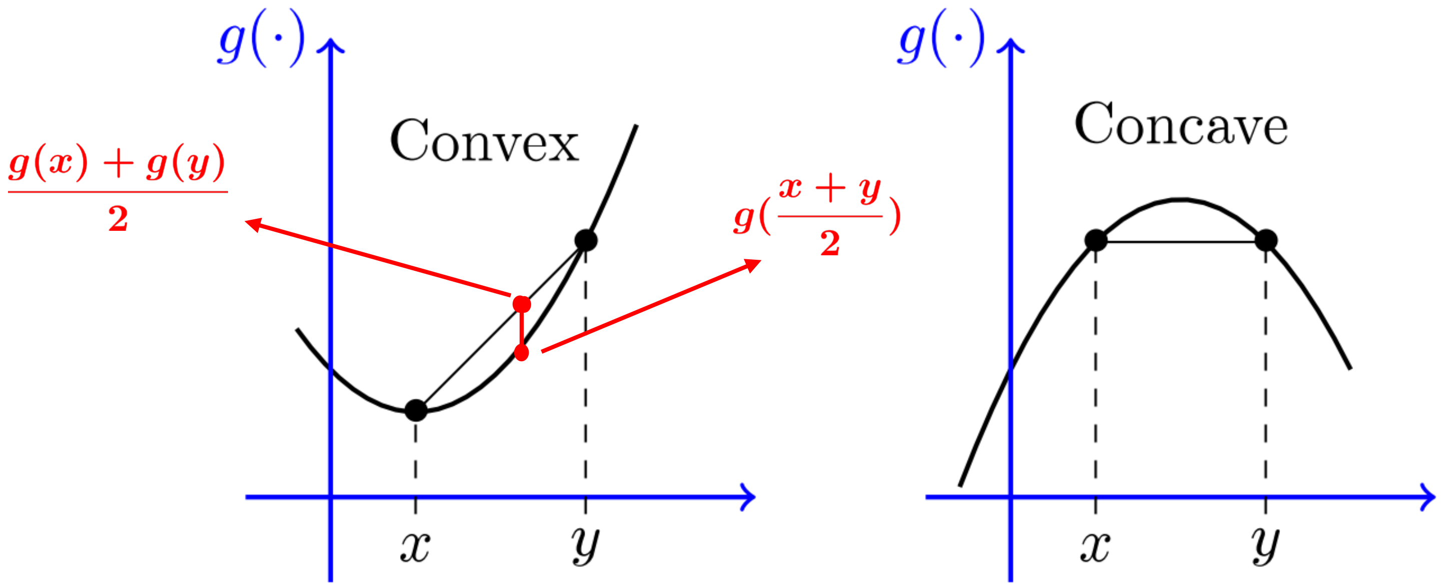
$$Var(X) = EX^2 - (EX)^2 \geq 0 \;\Rightarrow\; EX^2 \geq (EX)^2.$$

If $g(x) = x^2$,

$$E[g(X)] \geq g(E[X]).$$

Probability Bounds

Jensen's Inequality



$$\Rightarrow \quad \textbf{Convex}: \frac{g(x)+g(y)}{2} \geq g\left(\frac{x+y}{2}\right)$$

Probability Bounds

Definition. Consider a function $g : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$. We say that g is a **convex** function if, for any two points x and y in I and any $\alpha \in [0, 1]$, we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y).$$

We say that g is **concave** if

$$g(\alpha x + (1-\alpha)y) \geq \alpha g(x) + (1-\alpha)g(y).$$

Probability Bounds

Jensen's Inequality

If $g(x)$ is a convex function on R_X, and $E[g(X)]$ and $g(E[X])$ are finite, then

$$E[g(X)] \geq g(E[X]).$$

Probability Bounds

A twice-differentiable function $g : I \to \mathbb{R}$ is convex if and only if $g''(x) \geq 0$ for all $x \in I$.