Probability and Random Process (SWE3026)

Multiple Random Variables

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Random Vectors

Vectors

Column vector:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow a = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$a + b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}_{1\times 3}$$

Random Vectors

Matrix multiplication

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 2 \end{bmatrix}_{2\times 3}$$

$$\longrightarrow \qquad \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 2 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0+3+1 \\ 0+4-2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{2\times 1}$$

Random Vectors

Matrix multiplication

$$A_{m \times n} \cdot B_{n \times l} = C_{m \times l}$$

$$A = [a_{ij}] \;\Rightarrow\; A^T = [a_{ji}],$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Random Vectors

When we have n random variables $X_1, X_2, X_3, \cdots, X_n$ we can put them in a (column) vector $\mathbf{X}$.

$$\mathrm{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_{n \times 1}.$$

Random Vectors

CDF of the random vector $\mathbf{X}$

$$\begin{aligned} F_{\mathrm{X}}(\mathrm{x}) &= F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) \\ &= P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n). \end{aligned}$$

PDF of the random vector $\mathbf{X}$

$$f_{\mathrm{X}}(\mathrm{x}) = f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n).$$

Random Vectors

Expectation:

The expected value vector or the mean vector of the random vector X is defined as

$$\boldsymbol{E}\mathbf{X} = \begin{bmatrix} EX_1 \\ EX_2 \\ \vdots \\ EX_n \end{bmatrix}.$$

Random Vectors

random matrix

$$\mathrm{M} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1n} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \cdots & \boldsymbol{X}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{X}_{m1} & \boldsymbol{X}_{m2} & \cdots & \boldsymbol{X}_{mn} \end{bmatrix}.$$

Random Vectors

Mean matrix of M

$$
\boldsymbol{E}\mathrm{M} = \begin{bmatrix} \boldsymbol{EX_{11}} & \boldsymbol{EX_{12}} & \cdots & \boldsymbol{EX_{1n}} \\ \boldsymbol{EX_{21}} & \boldsymbol{EX_{22}} & \cdots & \boldsymbol{EX_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{EX_{m1}} & \boldsymbol{EX_{m2}} & \cdots & \boldsymbol{EX_{mn}} \end{bmatrix}.
$$

Random Vectors

Linearity of expectation

$$\mathbf{Y} = \mathbf{AX} + \mathbf{b},$$

$\mathbf{Y}$ & $\mathbf{X}$: **Random Vector**

$\mathbf{A}$ & $\mathbf{b}$: **fixed (non-random) matrices**

$$\mathbf{Y}_{m\times 1} = \underbrace{\mathbf{A}_{m\times n}\,\mathbf{X}_{n\times 1}}_{m\times 1} + \mathbf{b}_{m\times 1}$$

Random Vectors

Linearity of expectation

$$\underbrace{E\mathbf{Y}}_{m \times 1} = \underbrace{\mathbf{A}}_{m \times n}\underbrace{\mathbf{E}\mathbf{X}}_{n \times 1} + \underbrace{\mathbf{b}}_{m \times 1}.$$

Also, if $\mathbf{X_1}, \mathbf{X_2}, \cdots, \mathbf{X_k}$ are n-dimensional random vectors, then we have

$$\mathbf{E}[\mathbf{X_1} + \mathbf{X_2} + \cdots + \mathbf{X_k}] = \mathbf{E}\mathbf{X_1} + \mathbf{E}\mathbf{X_2} + \cdots + \mathbf{E}\mathbf{X_k}.$$

Random Vectors

Correlation and Covariance Matrix

$$\mathrm{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \qquad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \times \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

Random Vectors

Correlation and Covariance Matrix

For a random vector X, we define the correlation matrix, $\mathrm{R_X}$, as

$$\mathrm{R_X} = \mathrm{E}[\mathrm{X}\mathrm{X}^{\mathrm{T}}] = E\begin{bmatrix} X_1^2 & X_1X_2 & \dots & X_1X_n \\ X_2X_1 & X_2^2 & \dots & X_2X_n \\ \vdots & \vdots & \vdots & \vdots \\ X_nX_1 & X_nX_2 & \dots & X_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} EX_1^2 & E[X_1X_2] & \dots & E[X_1X_n] \\ EX_2X_1 & E[X_2^2] & \dots & E[X_2X_n] \\ \vdots & \vdots & \vdots & \vdots \\ E[X_nX_1] & E[X_nX_2] & \dots & E[X_n^2] \end{bmatrix}_{n\times n}$$

$$n = 1 \Rightarrow \ \mathrm{R_X} = EX^2$$

Random Vectors

Covariance

$$\mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{E}[(\boldsymbol{X} - \boldsymbol{E}\boldsymbol{X}).(\boldsymbol{Y} - \boldsymbol{E}\boldsymbol{Y})],$$

Random Vectors

The covariance matrix, $\mathbf{C_X}$, is defined as

$$\mathbf{C_X} = \mathbf{E}[(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}]$$

$$= E\begin{bmatrix} (X_1-EX_1)^2 & (X_1-EX_1)(X_2-EX_2) & \dots & (X_1-EX_1)(X_n-EX_n) \\ (X_2-EX_2)(X_1-EX_1) & (X_2-EX_2)^2 & \dots & (X_2-EX_2)(X_n-EX_n) \\ \vdots & \vdots & \vdots & \vdots \\ (X_n-EX_n)(X_1-EX_1) & (X_n-EX_n)(X_2-EX_2) & \dots & (X_n-EX_n)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \dots & \mathrm{Cov}(X_1,X_n) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2,X_n) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Cov}(X_n,X_1) & \mathrm{Cov}(X_nX_2) & \dots & \mathrm{Var}(X_n) \end{bmatrix}.$$

Random Vectors

$$\mathrm{Var}(X) = \mathrm{Cov}(X, X) = EX^2 - (EX)^2$$

$$\Rightarrow \ \mathrm{C_X} = \mathrm{R_X} - \mathbf{EXEX^T}.$$

$$E(XX^T)$$

Random Vectors

Correlation matrix of X:

$$\mathbf{R_X = E[XX^T]}$$

Covariance matrix of X:

$$\mathbf{C_X = E[(X - EX)(X - EX)^T] = R_X - EXEX^T}$$

Random Vectors

Note:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \qquad m, n = 1 \quad \mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$$

$$\Rightarrow \ \mathrm{Var}(Y) = a^2 \mathrm{Var}(X) = a \mathrm{Var}(X) a^T$$

$$(AB)^T = B^T A^T$$

Random Vectors

Proof: **Note that by linearity of expectation, we have**

$$\mathbf{EY} = \mathbf{AEX} + \mathbf{b}.$$

By definition, we have

$$\begin{aligned}
\mathbf{C_Y} &= \mathbf{E}[(\mathbf{Y} - \mathbf{EY})(\mathbf{Y} - \mathbf{EY})^{\mathbf{T}}] \\
&= \mathbf{E}[(\mathbf{AX} + \mathbf{b} - \mathbf{AEX} - \mathbf{b})(\mathbf{AX} + \mathbf{b} - \mathbf{AEX} - \mathbf{b})^{\mathbf{T}}] \\
&= E[\mathbf{A}(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}] \\
&= \mathbf{AE}[(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^{\mathbf{T}}]\mathbf{A}^{\mathbf{T}} \qquad \textbf{(by linearity of expectation)} \\
&= \mathbf{AC_X A^T}.
\end{aligned}$$

Random Vectors

Normal (Gaussian) Random Vectors:

Random variables $X_1, X_2, \cdots, X_n$ are said to be jointly normal if, for all $a_1, a_2, \cdots, a_n \in \mathbb{R}$, the random variable

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

is a normal random variable.

Random Vectors

A random vector

$$\mathrm{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \qquad (X_1, X_2, \cdots, X_n)$$

is said to be normal or Gaussian if the random variables $X_1, X_2, \cdots, X_n$ are jointly normal.

Random Vectors

Standard Normal Random Variable:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}, \qquad \text{for all } z \in \mathbb{R}.$$

$$X \sim N(\mu, \sigma^2),$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Random Vectors

Standard normal random vector:

$$\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}, \qquad Z_1, Z_2, \cdots, Z_n \longrightarrow \text{i.i.d.}$$

$$Z_i \sim N(0,1)$$

Random Vectors

Then,

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) &= f_{Z_1, Z_2, \ldots, Z_n}(z_1, z_2, \ldots, z_n) \\ &= \prod_{i=1}^{n} f_{Z_i}(z_i) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} z_i^2\right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z_1^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z_2^2}{2}\right\} \cdots \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right\}. \end{aligned}$$

Random Vectors

For a standard normal random vector $\mathbf{Z}$, where Z_i's are i.i.d. and $Z_i \sim N(0,1)$, the PDF is given by

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right\}.$$

Random Vectors

Generally,

For a normal random vector $\mathbf{X}$ with mean $\mathbf{m}$ and covariance matrix $\mathbf{C}$, the PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{C}}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right\}.$$

$$\mathbf{C} \to \mathrm{Var}(X) = \sigma^2 f_X(x) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T . \frac{1}{\sigma^2}(\mathbf{x}-\mu)\right\}$$

Random Vectors

If $\mathbf{X} = [X_1, X_2, ..., X_n]^T$ is a normal random vector, and we know $\mathrm{Cov}(X_i, X_j) = 0$ for all $i \neq j$, then $X_1, X_2, ..., X_n$ are independent.

Another important result is that if

$$X \sim N(\mu_X, \sigma_X^2) \rightarrow Y = aX + b \Rightarrow Y \sim N(a\mu_X + b, a^2\sigma_X^2)$$

$$\mathbf{C_Y = AC_XA^T}$$

Random Vectors

If $\mathbf{X} = [X_1, X_2, ..., X_n]^T$ is a normal random vector, $\mathbf{X} \sim N(\mathbf{m}, \mathbf{C})$, $\mathbf{A}$ is an m by n fixed matrix, and $\mathbf{b}$ is an m-dimensional fixed vector, then the random vector $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ is a normal random vector with mean $\mathbf{A}E\mathbf{X} + \mathbf{b}$ and covariance matrix $\mathbf{ACA}^T$.

$$\mathbf{Y} \sim N(\mathbf{A}E\mathbf{X} + \mathbf{b}, \mathbf{ACA}^T).$$