**Algorithm**

# Mathematical Background

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# Floor and Ceiling Functions

- The *floor* function maps any real number $x$ onto the greatest integer less than or equal to $x$:

$$\lfloor 3.2 \rfloor = \lfloor 3 \rfloor = 3$$
$$\lfloor -5.2 \rfloor = \lfloor -6 \rfloor = -6$$

- The *ceiling* function maps $x$ onto the least integer greater than or equal to $x$:

$$\lceil 3.2 \rceil = \lceil 4 \rceil = 4$$
$$\lceil -5.2 \rceil = \lceil -5 \rceil = -5$$

- The <*cmath*> library implements these as
  - `double floor(double);`   `double ceil(double);`

# Logarithms

- If $n = e^m$, we define

  $$m = \ln(\, n \,)$$

  - It is always true that $e^{\ln(n)} = n$

- Exponentials grow faster than any non-constant polynomial

  $$\lim_{n \to \infty} \frac{e^n}{n^d} = \infty$$

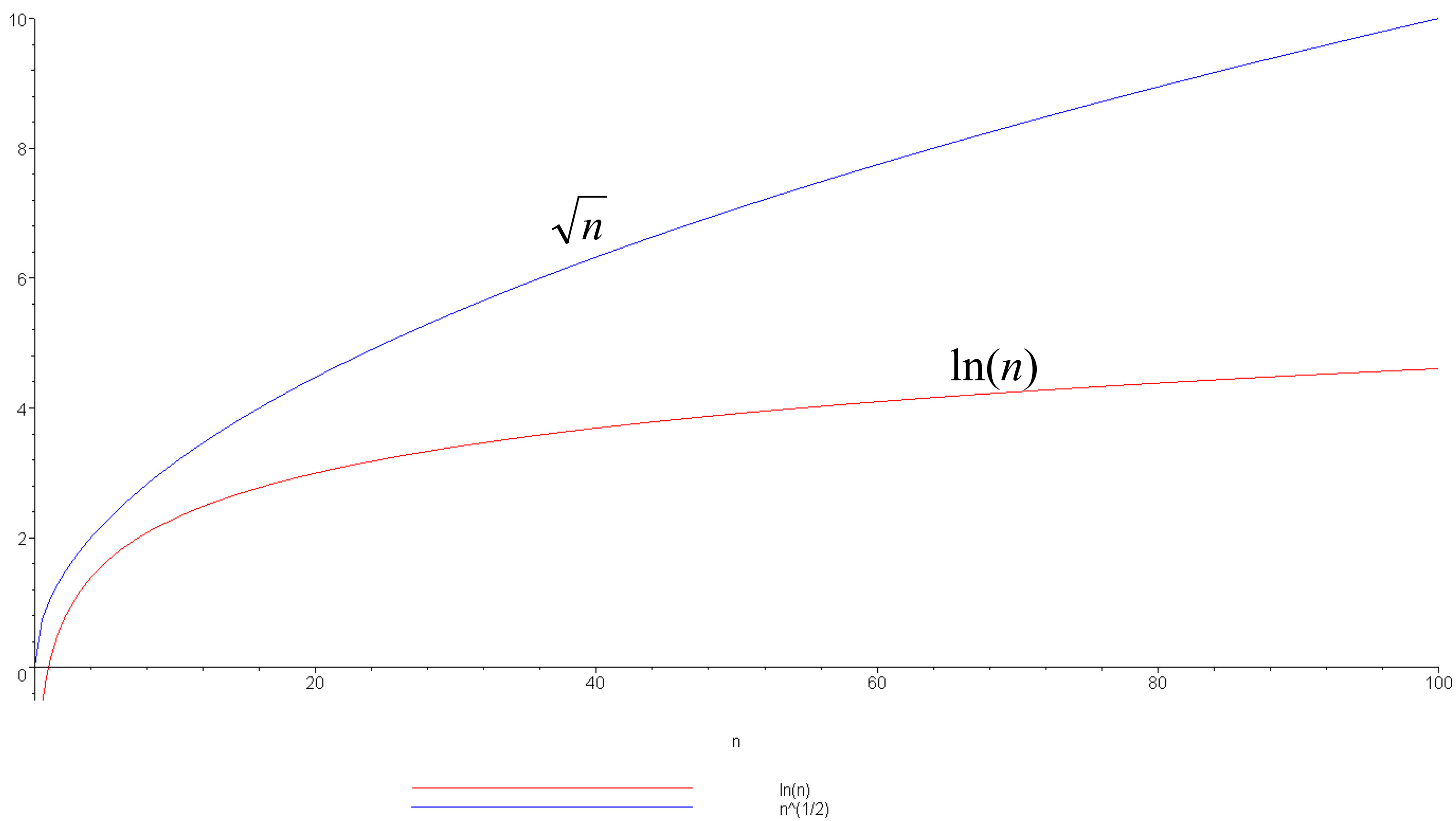
  - for any $d > 0$

- Thus, their inverses—logarithms—grow slower than any polynomial

  $$\lim_{n \to \infty} \frac{\ln(n)}{n^d} = 0$$

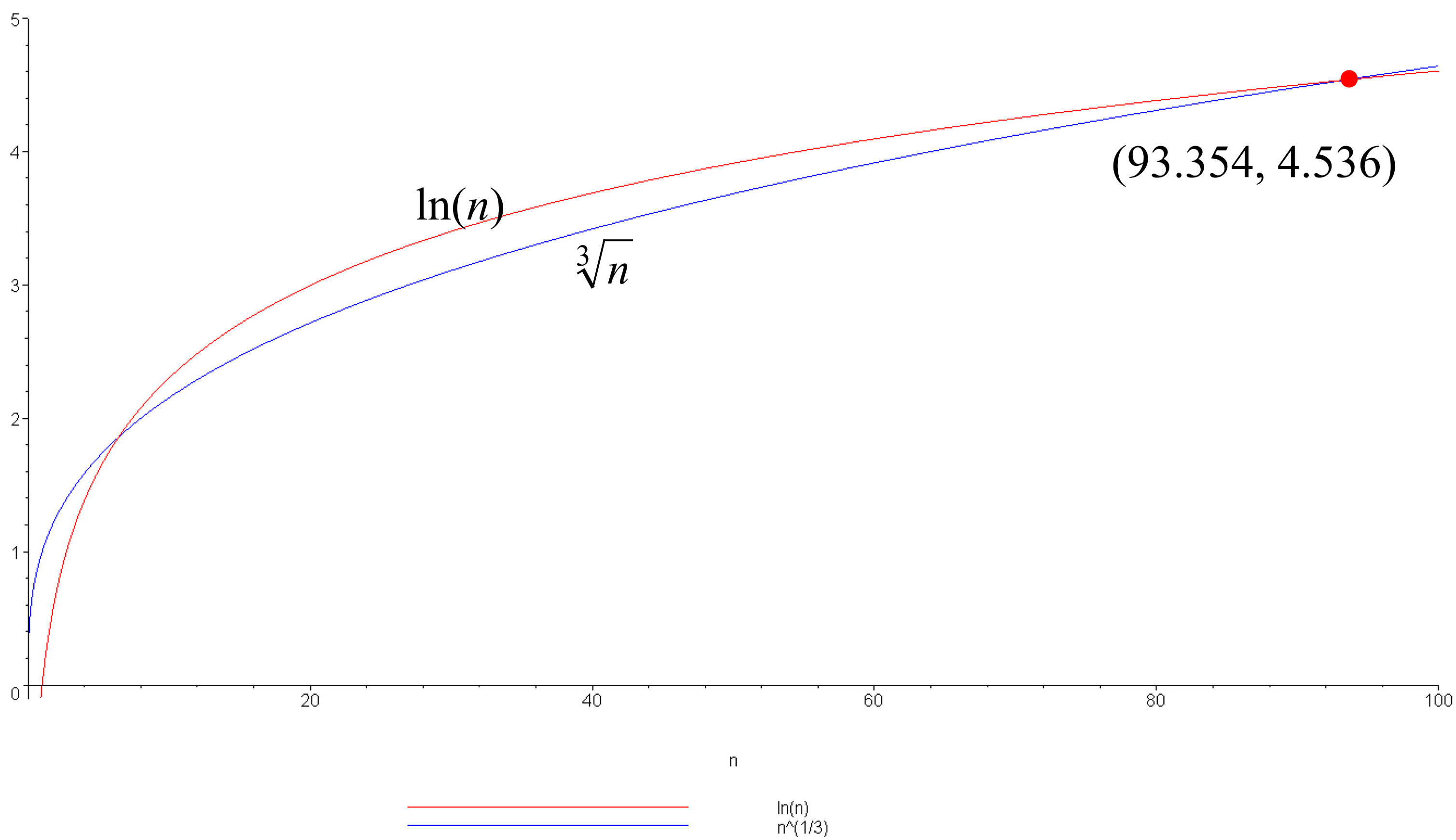
# Logarithms

- Example: $f(n) = n^{1/2} = \sqrt{n}$ is strictly greater than $\ln(n)$

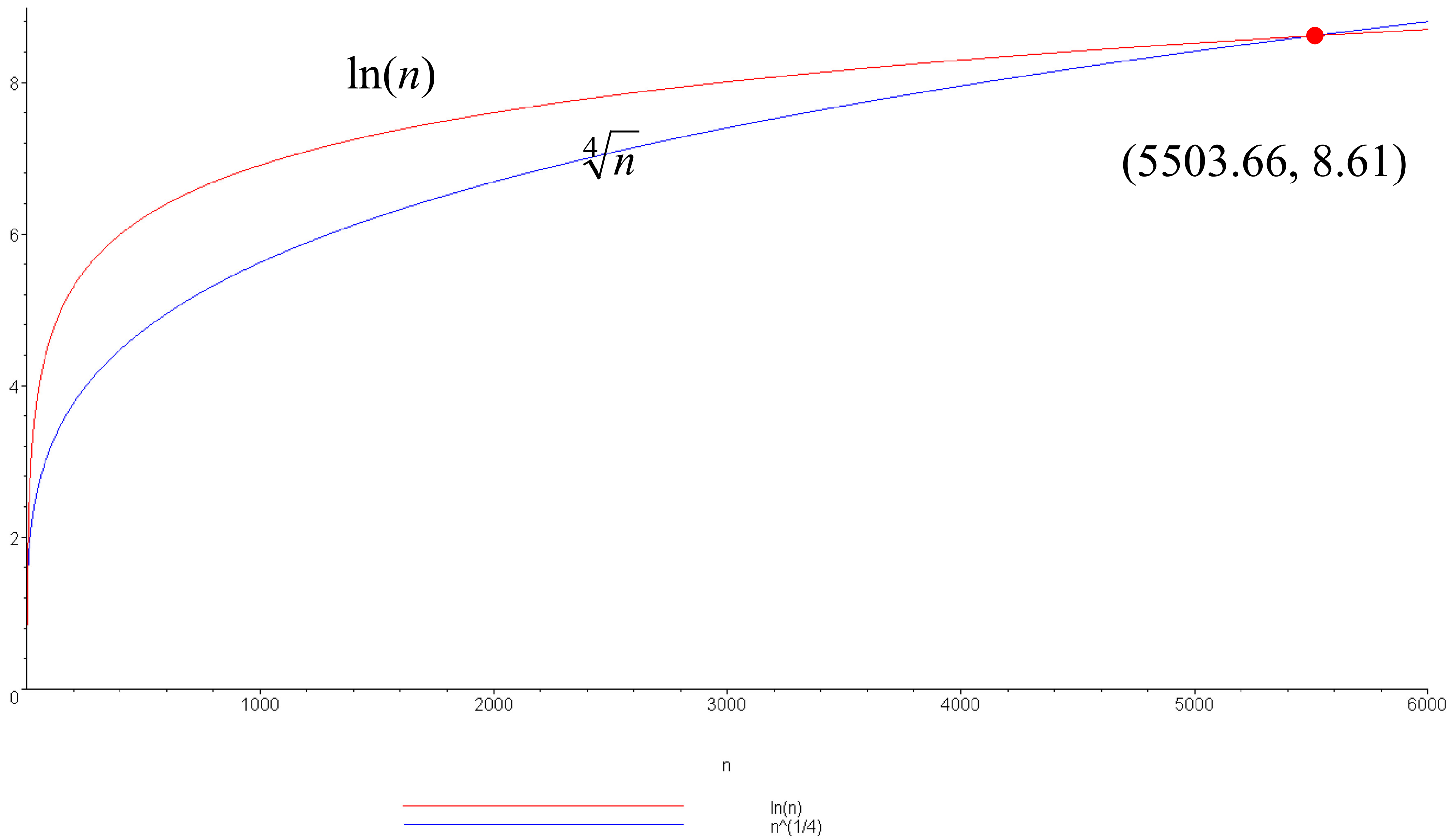


# Logarithms

- $f(n) = n^{1/3} = \sqrt[3]{n}$ grows slower but only up to $n = 93$



# Logarithms



ln(n)
$\sqrt[4]{n}$
(5503.66, 8.61)
8
6
4
2
0
1000
2000
3000
4000
5000
6000
n
ln(n)
n^(1/4)

# Logarithms

- We have compared logarithms and polynomials
  - How about $\log_2(n)$, $\ln(n)$, and $\log_{10}(n)$

- You have seen the formula:

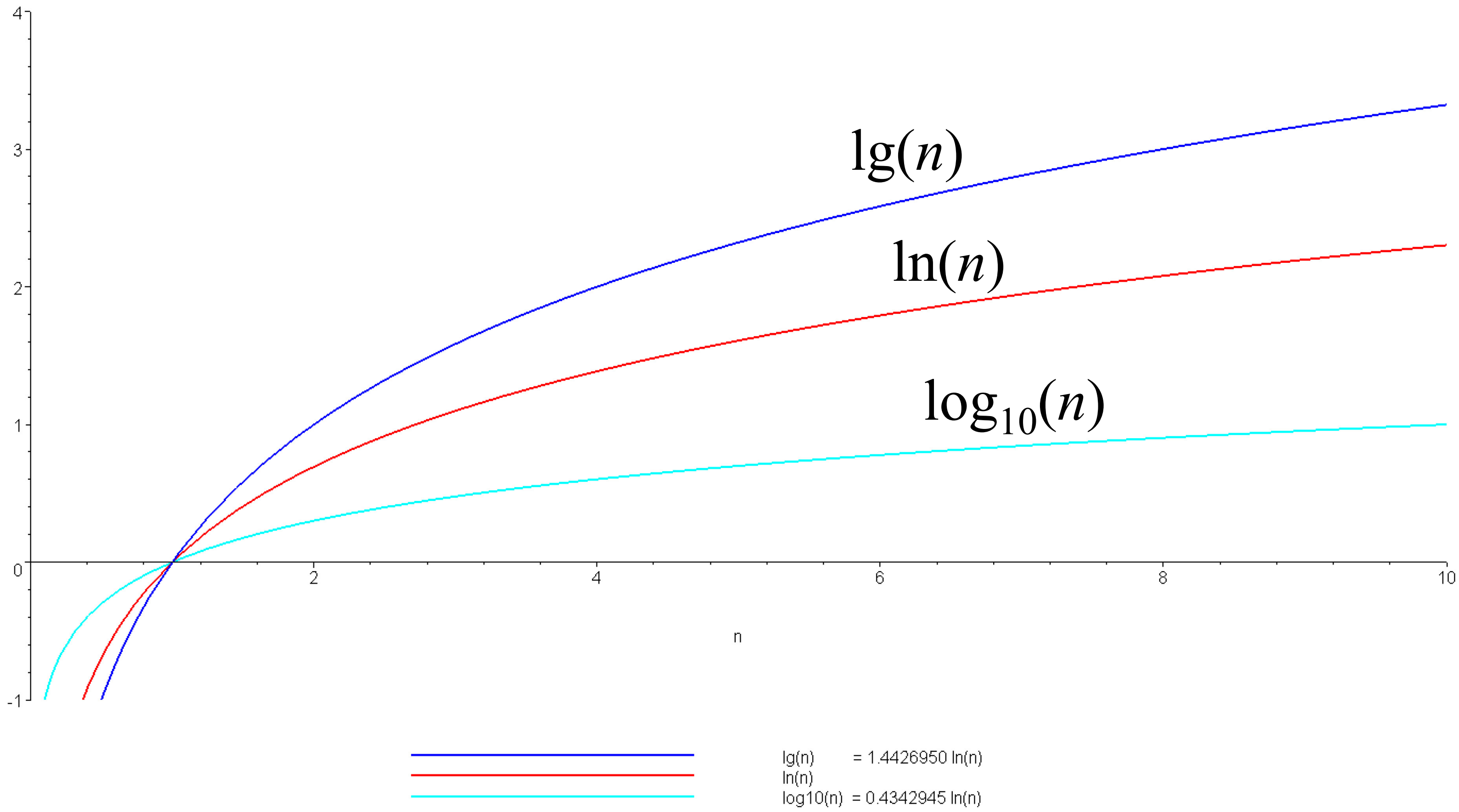
$$\log_b(n) = \frac{\ln(n)}{\ln(b)}$$

  - where, $\ln(b)$ is a constant

- All logarithms are scalar multiples of each others

# Logarithms

- A plot of $\log_2(n) = \lg(n), \ln(n)$, and $\log_{10}(n)$



# Logarithms

- Note: the base-2 logarithm $\log_2(n)$ is written as $\lg(n)$

- It is an industry standard to implement the natural logarithm $\ln(n)$ as

```
double log( double );
```

- The *common* logarithm $\log_{10}(n)$ is implemented as

```
double log10( double );
```

# Logarithms

- You should also be aware of the relationship:

$$\begin{aligned} \lg(2^{10}) &= \lg(1024) &&= 10 \\ \lg(2^{20}) &= \lg(1\,048\,576) &&= 20 \end{aligned}$$

$$\begin{aligned} \lg(10^{3}) &= \lg(1000) &&\approx 10 \quad \text{kilo} \\ \lg(10^{6}) &= \lg(1\,000\,000) &&\approx 20 \quad \text{mega} \\ \lg(10^{9}) & &&\approx 30 \quad \text{giga} \\ \lg(10^{12}) & &&\approx 40 \quad \text{tera} \end{aligned}$$

# L'Hôpital's Rule

- If you are attempting to determine

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

- But both $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ , it follows

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{(1)}(n)}{g^{(1)}(n)}$$

Note: $f^{(k)}(n)$ is the $k^{\text{th}}$ derivative

- Repeat as necessary

# Arithmetic Series

- Each term in an arithmetic series is increased by a constant value (usually 1) :

$$0+1+2+3+\cdots+n=\sum_{k=0}^{n} k=\frac{n(n+1)}{2}$$

  - Proof 1: write out the series twice and add each column

$$\begin{array}{ccccccccccccccc}
 & 1 & + & 2 & + & 3 & + \cdots + & n-2 & + & n-1 & + & n \\
+ & n & + & n-1 & + & n-2 & + \cdots + & 3 & + & 2 & + & 1 \\
\hline
 & (n+1) & + & (n+1) & + & (n+1) & + \cdots + & (n+1) & + & (n+1) & + & (n+1)
\end{array}$$

$$= n\,(n+1)$$

    - Since we added the series twice, we must divide the result by 2

# Arithmetic Series

- Each term in an arithmetic series is increased by a constant value (usually 1) :

$$0+1+2+3+\cdots+n=\sum_{k=0}^{n}k=\frac{n(n+1)}{2}$$

  - Proof 2: By mathematical induction
    - The statement is true for $n = 0$: $\sum_{i=0}^{0}k=0=\frac{0\cdot 1}{2}=\frac{0(0+1)}{2}$
    - Assume that the statement is true for an arbitrary $n$: $\sum_{k=0}^{n}k=\frac{n(n+1)}{2}$
    - For $n + 1$, we have: $\sum_{k=0}^{n+1}k=(n+1)+\sum_{i=0}^{n}k$

$$=(n+1)+\frac{n(n+1)}{2}$$

$$=\frac{(n+1)2+(n+1)n}{2}$$

$$=\frac{(n+1)(n+2)}{2}$$

# Polynomial Series

- $$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$
- …
- However, it is easier to see the pattern

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{n^3}{3}$$

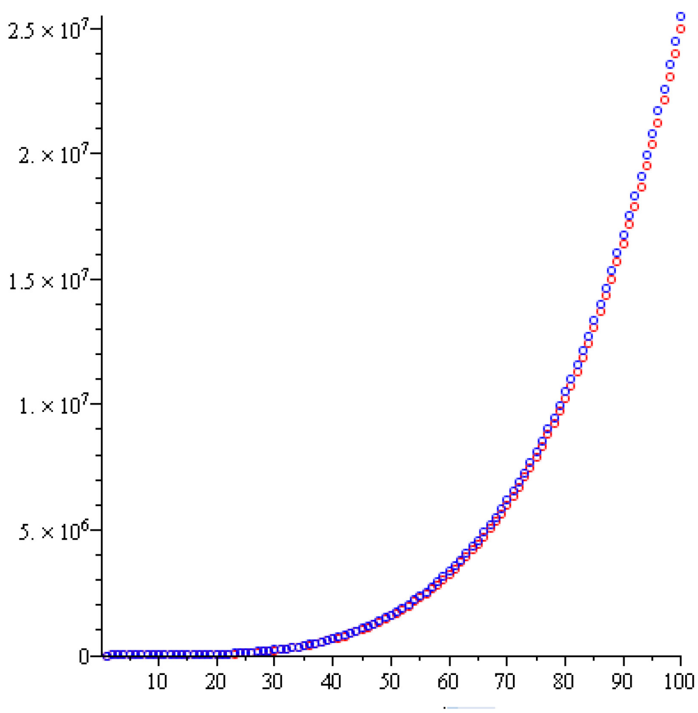
$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \approx \frac{n^4}{4}$$

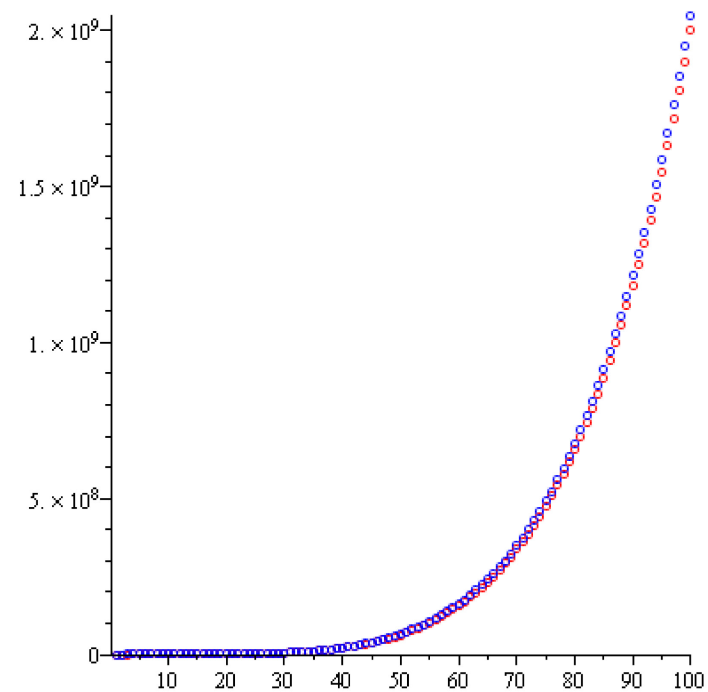
# Polynomial Series

- We can generalize this formula

$$\sum_{k=0}^{n} k^d \approx \frac{n^{d+1}}{d+1}$$

- Demonstrating with $d = 3$ and $d = 4$

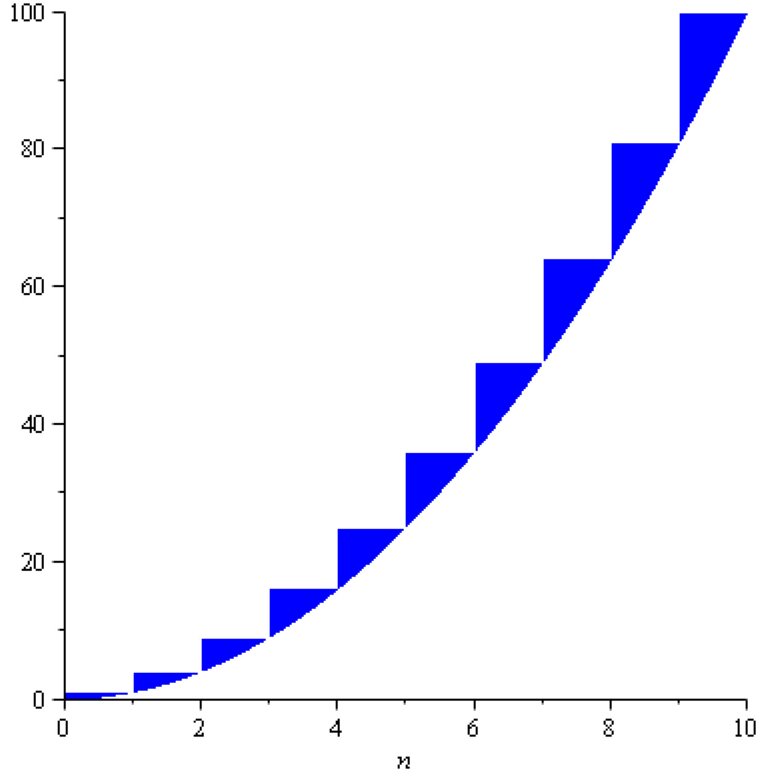


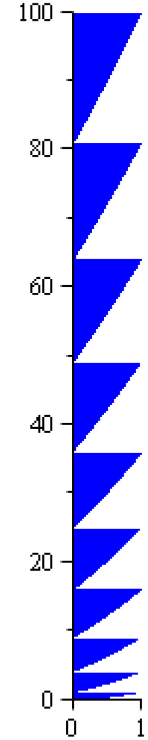



# Polynomial Series

- How large is the error?
  - Assuming $d > 1$, shifting the errors, we can see that they would be:

$$\frac{n^d}{2} \le \sum_{k=0}^{n} k^d - \frac{n^{d+1}}{d+1} < n^d \ll n^{d+1}$$



# Polynomial Series

- The ratio between the error and the actual value goes to 0:
  - In the limit, as $n \to \infty$, the ratio between the sum and the approximation goes to 1:

$$\lim_{n \to \infty} \frac{\dfrac{n^{d+1}}{d+1}}{\sum_{k=0}^{n} k^d} = 1$$

  - The relative error of the approximation goes to 0

# Geometric Series

- The geometric series with common ratio $r$:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

  - If $|r| < 1$, then it is also true that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

# Geometric Series

- Elegant proof: using $1 = \frac{1-r}{1-r}$

$$\begin{aligned}
\sum_{k=0}^{n} r^k &= \frac{(1-r)\sum_{k=0}^{n} r^k}{1-r} \\
&= \frac{\sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k}{1-r} \\
&= \frac{(1+r+r^2+\cdots+r^n)-(r+r^2+\cdots+r^n+r^{n+1})}{1-r} \\
&= \frac{1-r^{n+1}}{1-r}
\end{aligned}$$

# Geometric Series

- Proof by induction:
  - The formula is correct for $n = 0$: $\sum_{k=0}^{0} r^k = r^0 = 1 = \frac{1 - r^{0+1}}{1 - r}$
  - Assume that the formula $\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$ is true for an arbitrary $n$; then

$$\sum_{k=0}^{n+1} r^k = r^{n+1} + \sum_{k=0}^{n} r^k = r^{n+1} + \frac{1 - r^{n+1}}{1 - r} = \frac{(1 - r)r^{n+1} + 1 - r^{n+1}}{1 - r}$$

$$= \frac{r^{n+1} - r^{n+2} + 1 - r^{n+1}}{1 - r} = \frac{1 - r^{n+2}}{1 - r} = \frac{1 - r^{(n+1)+1}}{1 - r}$$

  - and therefore, by the process of mathematical induction, the statement is true for all $n \geq 0$

# Geometric Series

- A common geometric series with the ratios $r = \frac{1}{2}$ and $r = 2$

$$\sum_{i=0}^{n}\left(\frac{1}{2}\right)^{i} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2 - 2^{-n}$$

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i} = 2$$

$$\sum_{k=0}^{n} 2^{k} = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

# Combinations

- Given $n$ distinct items, in how many ways can you choose $k$ of these?
  - I.e., "In how many ways can you combine $k$ items from $n$?"
  - For example, given the set $\{1, 2, 3, 4, 5\}$, I can choose three of these in any of the following ways:

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\},$$
$$\{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}$$

- The number of ways such items can be chosen is written

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

  - is read as "$n$ choose $k$"

- A recursive definition: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

# Combinations

- You have also seen this in expanding polynomials:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

  - For example,

$$\begin{aligned}
(x+y)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k} \\
&= \binom{4}{0} y^4 + \binom{4}{1} xy^3 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^3 y + \binom{4}{4} x^4 \\
&= y^4 + 4xy^3 + 6x^2 y^2 + 4x^3 y + x^4
\end{aligned}$$

# Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Any Question?