Probability and Random Process (SWE3026)

Continuous and Mixed Random Variables

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

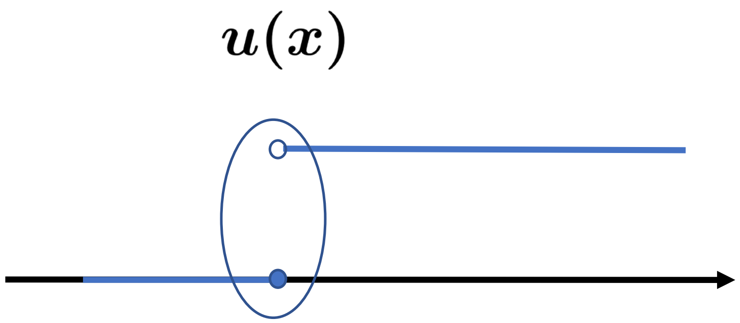
Mixed Random Variables

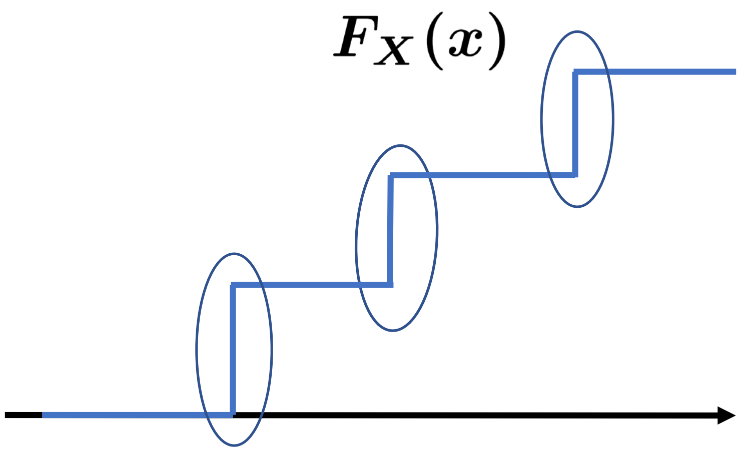
We will:

1) Define PDF (generalized PDF) for discrete random variables by using the delta function.
2) Introduce mixed random variables.

Mixed Random Variables

$$f_X(x) = \frac{d}{dx} F_X(x),$$





CDF for discrete RVs

Mixed Random Variables

Idea: Define the "derivative" of $u(x)$, $\dfrac{d}{dx}u(x)$ and that we can extend the PDF to the discrete RVs.

$\Rightarrow$ **delta function** $\delta(x)$

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{else} \end{cases}$$

Mixed Random Variables

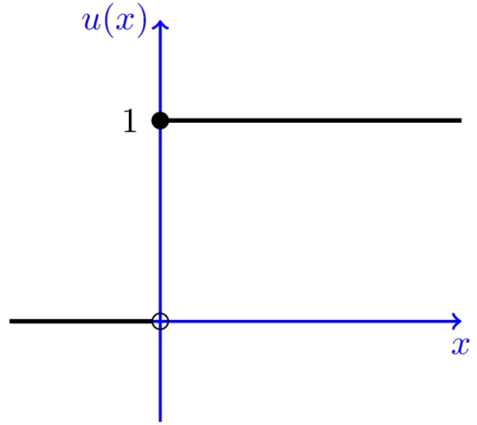
Delta function: $\delta(x)$

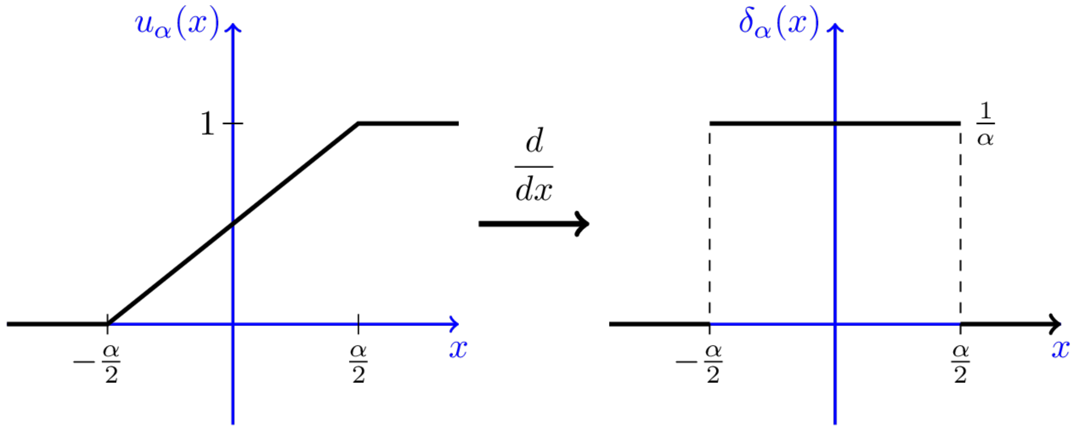
Where α is small.

$$\int \delta(x)dx = 1$$

$$\alpha = 0 \Rightarrow \delta(0) = \frac{1}{\alpha} = \infty$$

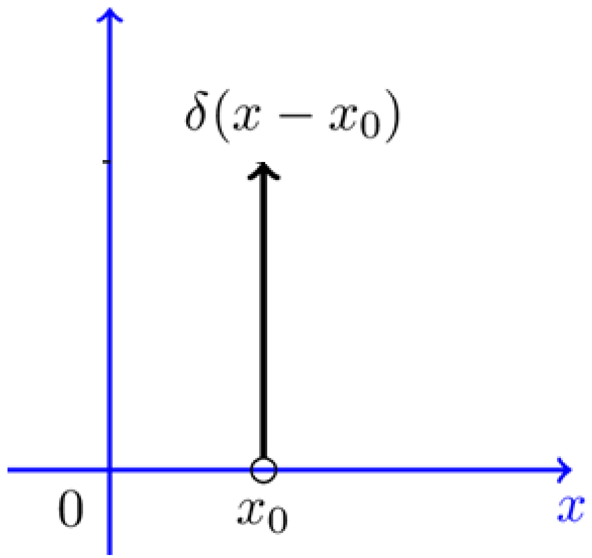
$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{else} \end{cases}$$





Mixed Random Variables

$$\delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & \text{else} \end{cases}$$

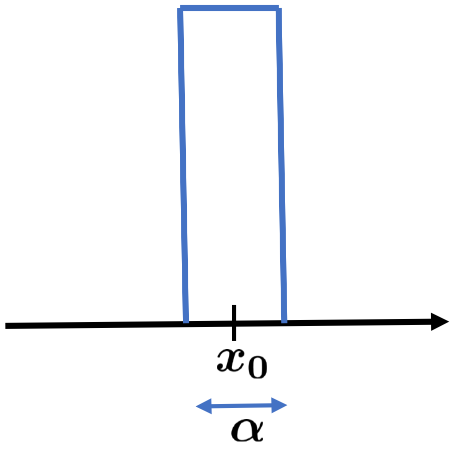


Mixed Random Variables

Lemma. **Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. We have**

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0).$$

$$g(x).\frac{1}{\alpha} \approx \frac{1}{\alpha} g(x_0)$$



Proof:

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx \approx \alpha \left(g(x_0).\frac{1}{\alpha} \right) = g(x_0).$$

Mixed Random Variables

Example. **Let** $g(x) = 2(x^2 + x),$

$$\int_{-\infty}^{\infty} g(x)\delta(x-1)dx = g(1) = 2(1^2 + 1) = 4.$$

Mixed Random Variables

Definition: **Properties of the delta function**

We define the delta function $\delta(x)$ as an object with the following properties:

1) $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{else} \end{cases}$

2) $\delta(x) = \dfrac{d}{dx}u(x)$**, where** $u(x)$ **is the unit step function.**

3) $\displaystyle\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1$ **for any** $\epsilon > 0.$

Mixed Random Variables

4) For any $\epsilon > 0$ and any function $g(x)$ that is continuous over $(x_0 - \epsilon, x_0 + \epsilon)$, we have

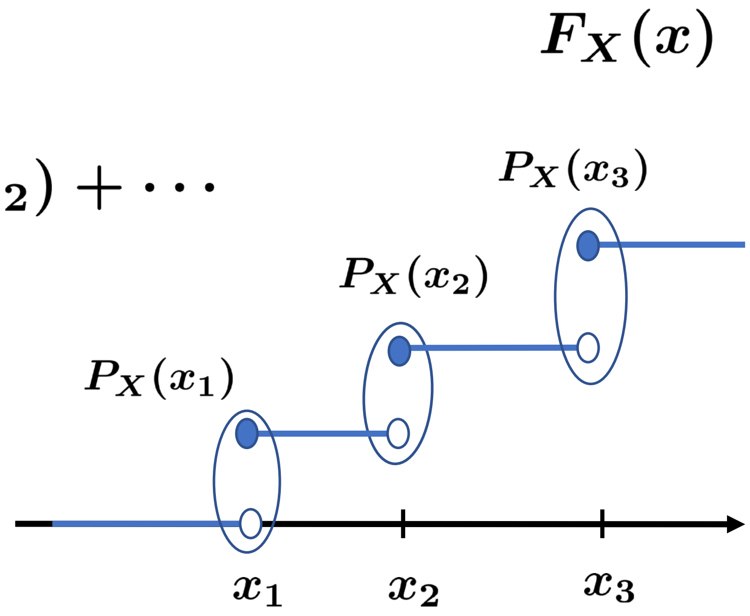
$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = \int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)dx = g(x_0).$$

Mixed Random Variables

Discrete Random Variable X

$$R_X = \{x_1, x_2, x_3, \cdots\}$$

$$F_X(x) = \sum_{x_k \in R_X} P_X(x_k) u(x - x_k)$$
$$= P_X(x_1)u(x - x_1) + P_X(x_2)u(x - x_2) + \cdots$$



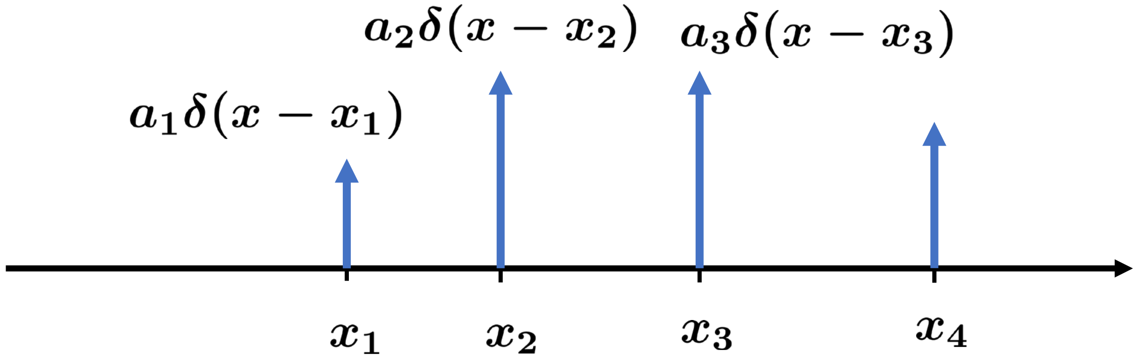
Mixed Random Variables

generalized PDF

$$a_k = P(X = x_k) = P_X(x_k),$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \sum_{x_k \in R_X} a_k u(x - x_k)$$

$$= \sum_{x_k \in R_X} a_k \delta(x - x_k).$$



Mixed Random Variables

For a discrete random variable X with range $R_X = \{x_1, x_2, x_3, ...\}$ and PMF $P_X(x_k)$, we define the (generalized) probability density function (PDF) as

$$f_X(x) = \sum_{x_k \in R_X} P_X(x_k)\delta(x - x_k).$$

Mixed Random Variables

Is it true $\int_{-\infty}^{+\infty} f_X(x)dx = 1$ **?**

$$\int_{-\infty}^{+\infty} f_X(x)dx = \int_{-\infty}^{+\infty} \sum_{x_k \in R_X} a_k \delta(x - x_k).dx$$

$$= \sum_{x_k \in R_X} a_k \int_{-\infty}^{+\infty} \delta(x - x_k).dx \qquad \left(\int_{-\infty}^{+\infty} \delta(x - x_k).dx = 1\right)$$

$$= \sum_{x_k \in R_X} a_k = \sum_{x_k \in R_X} P_X(x_k) = 1.$$

Mixed Random Variables

Example. $EX = \sum_{k} x_k P_X(x_k).$

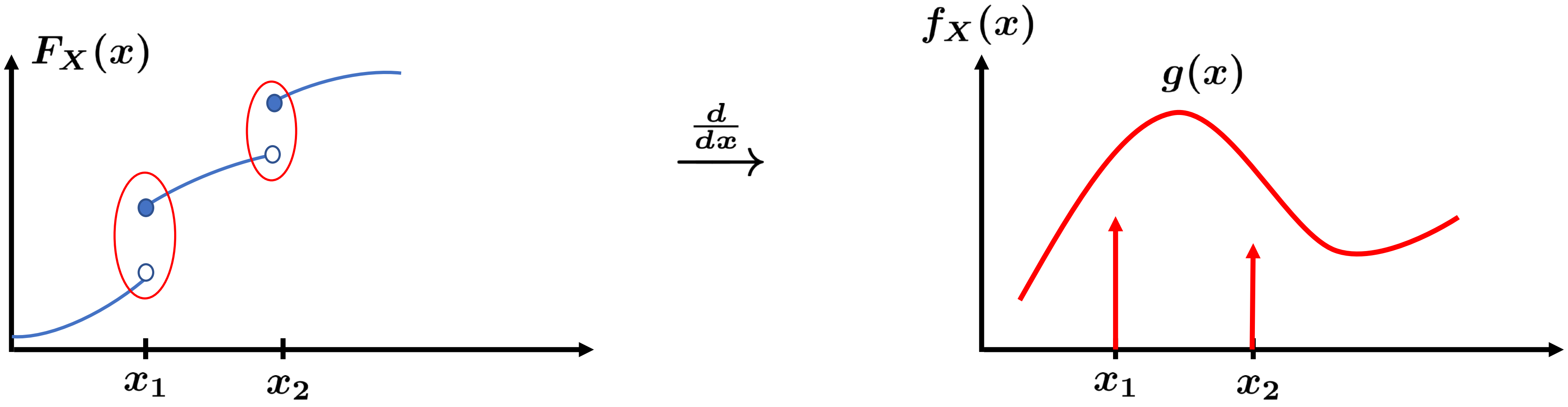
$$EX = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{x_k \in R_X} a_k \delta(x - x_k) dx$$

$$= \sum_{x_k \in R_X} a_k \underbrace{\int_{-\infty}^{\infty} x \delta(x - x_k) dx}_{x_k} = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Mixed Random Variables

Random variables:

- Discrete
- Continuous
- Mixed random variable



Mixed Random Variables

The (generalized) PDF of a mixed random variable can be written in the form

$$f_X(x) = \underbrace{\sum_k a_k \delta(x - x_k)}_{\text{Discrete}} + \underbrace{g(x)}_{\text{Continuous}},$$

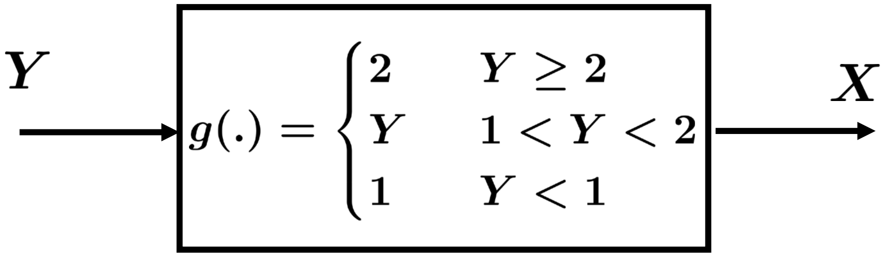
where $a_k = P(X = x_k)$**, and** $g(x) \geq 0$ **does not contain any delta functions. Furthermore, we have**

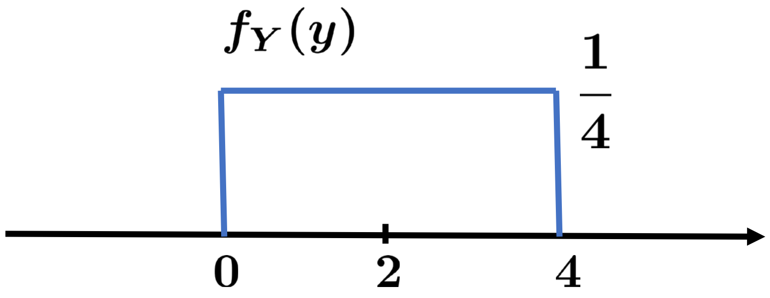
$$\int_{-\infty}^{\infty} f_X(x)dx = \sum_k a_k + \int_{-\infty}^{\infty} g(x)dx = 1.$$

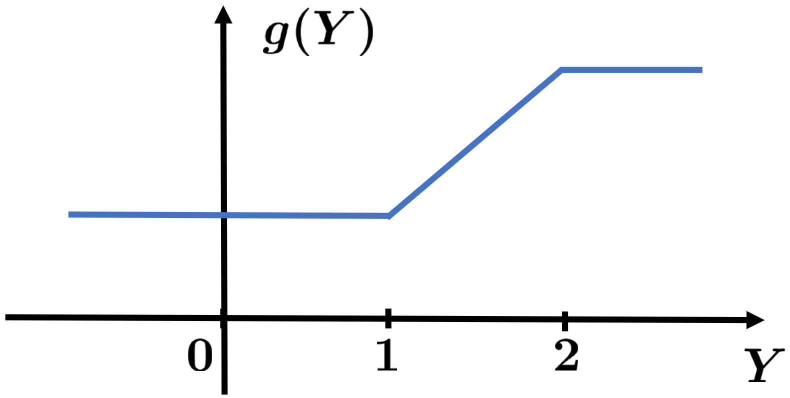
Mixed Random Variables

Example. Let $Y \sim Uniform(0,4)$,

$$X = \begin{cases} 2 & Y \geq 2 \\ Y & 1 < Y < 2 \\ 1 & Y < 1 \end{cases}$$







Mixed Random Variables

a) Find the CDF of X.

b) Find the generalized PDF of X .

c) Find EX and $\mathrm{Var}(X)$.

Mixed Random Variables

So mixed random variable:

$$f_X(x) = \underbrace{\sum_k P(X = x_k)\delta(x - x_k)}_{\text{Discrete part}} + \underbrace{g(x)}_{\text{Continuous part}}.$$

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx = \sum_k x_k P(X = x_k) + \int_{-\infty}^{\infty} x g(x) dx.$$