Probability and Random Process (SWE3026)

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H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Classification of Natural Phenomena

- **Deterministic phenomenon**

  Observed the same result when the conditions are the same

- **Random phenomenon**

  Observed different results even the conditions are the same

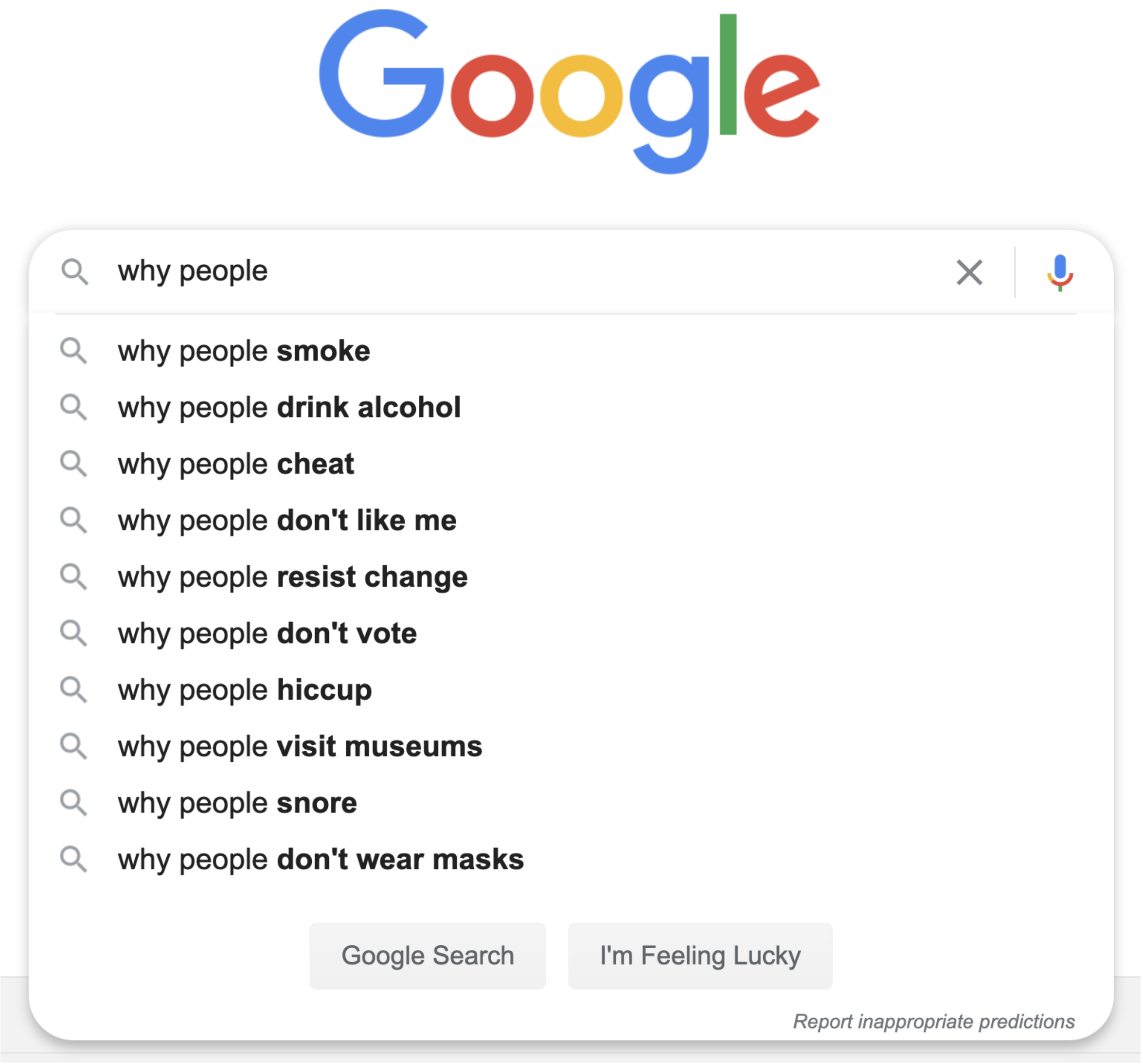
Mathematical Model

- **Procedure**
  - Identification of model
  - Solution for certain quantities of interest (mathematics)
  - Verification of model (physical experiment)
  - Modification of model (based on experimental results)
- **Usage of mathematical model**
  - Applying to similar other situations & predicting the outcome (Analysis)
  - Suggesting alternative solution for a given problem (Design)

Probability

- **Coin toss**
  - The probability that a coin toss will come up heads is 50% (if the coin is fair)
  - The coin will come up heads about ½ of the time if we flip the coin a lot

Probability - Language Model



Review of Set Theory

A set is an unordered collection of things (elements).

- $A = \{\clubsuit, \diamondsuit\}$ $\quad\quad \diamondsuit \in A; \quad \heartsuit \notin A$
- $B = \{1, 2, 3\};$
- $C = \{x^2 : x = 1, 2, 3\} = \{1, 4, 9\};$
- $D = \{H, T\};$

Review of Set Theory

- **The set of natural numbers,** $\mathbb{N} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots\}$**.**

- **The set of integers,** $\mathbb{Z} = \{\cdots, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots\}$**.**

- **The set of real number** $\mathbb{R}$**.**

Review of Set Theory

Set A is a subset of set B if every element of A is also an element of B.

We write $A \subset B$, where " $\subset$ " indicates "subset ".

$$\boldsymbol{A \subset B \;\equiv\; (x \in A) \;\Rightarrow\; (x \in B)}$$

Example:

- $\boldsymbol{E = \{1, 4\}; \quad C = \{1, 4, 9\} \;\Rightarrow\; E \subset C}$.
- $\mathbb{N} \subset \mathbb{Z}$.

Review of Set Theory

$A = B$ if and only if $A \subset B$ and $B \subset A$.

Example:

- $\{1, 2, 3\} = \{3, 2, 1\}$
- $\{a, a, b\} = \{a, b\}$

Universal set: The set of all things that we could possibly consider in a given context.

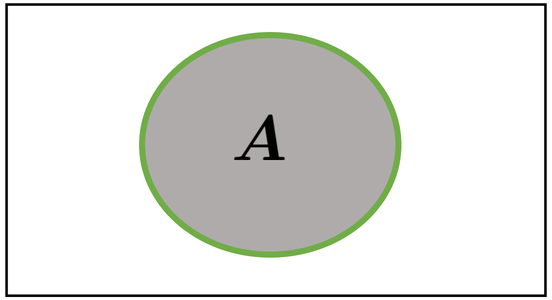
$S =$ **Universal set** $= \Omega;$

$\emptyset =$ **Null set ;** $\emptyset = \{\};$

For any set $A;\ \emptyset \subset A.$

Venn Diagrams

In a Venn diagram any set is depicted by a closed surface.



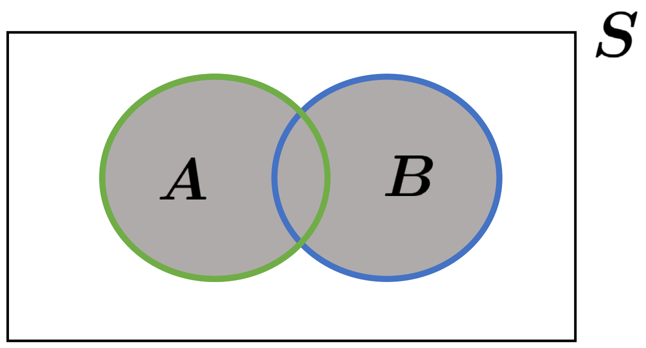
$$\Omega = S$$

Set Operations

Union: The union of two sets A and B is denoted by $A \cup B$ and consist of all objects in A or B.

$x \in (A \cup B)$ if and only if $(x \in A)$ or $(x \in B)$.

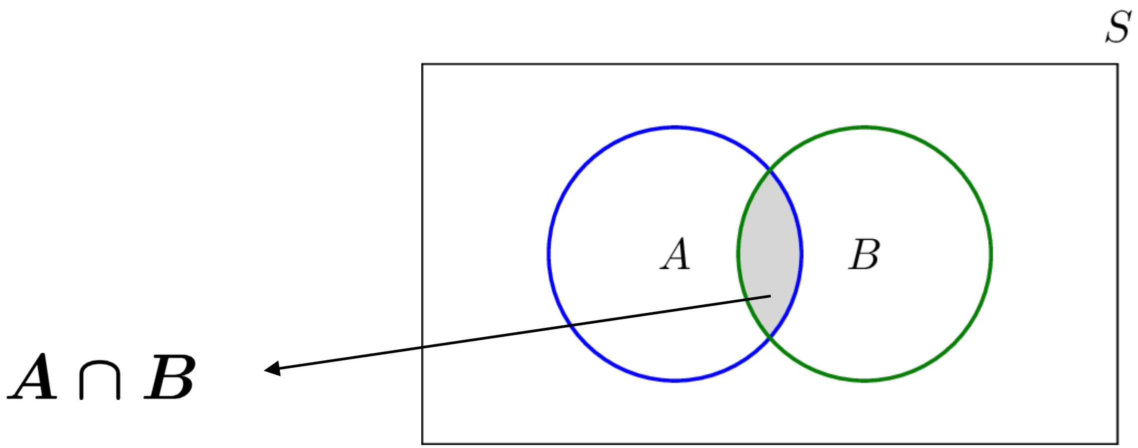
$\{1, 2\} \cup \{3\} = \{1, 2, 3\}$



Set Operations

Intersection:

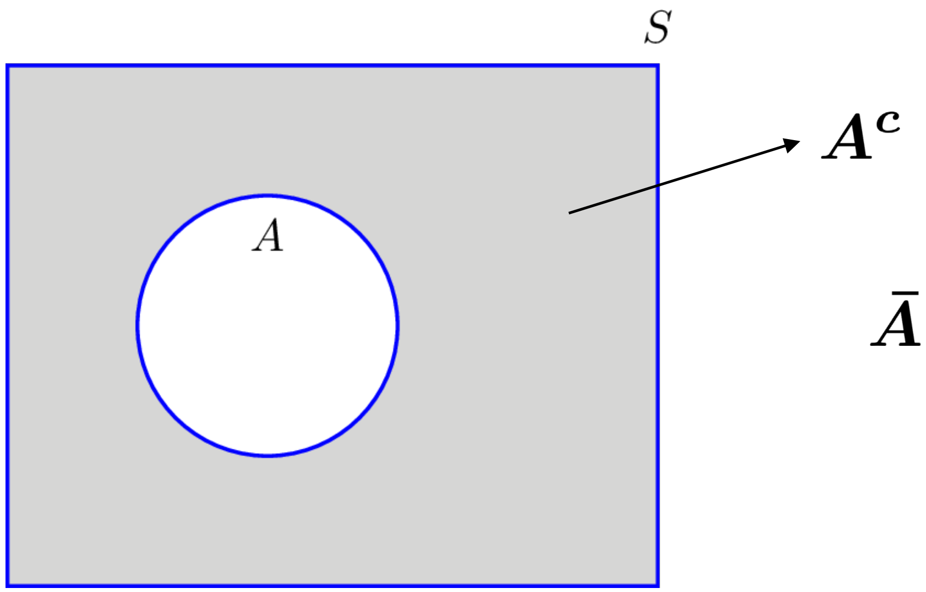
The intersection of two sets A and B is denoted by $A \cap B$ and consist of all objects in both A and B.



Set Operations

Complement:

The complement of a set A, denoted by A^c, is the set of all elements in S (Ω) that are Not in A.



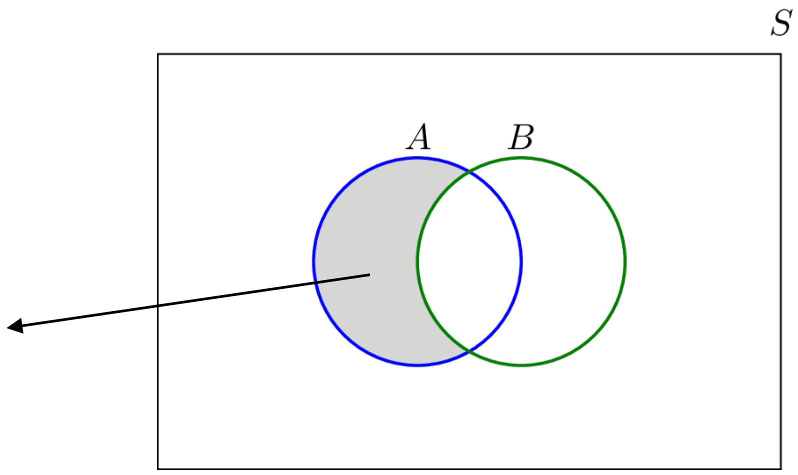
$$\bar{\boldsymbol{A}} = \boldsymbol{A^c}$$

Set Operations

Difference (subtraction):

The subtraction of set B from A $(A - B)$ is all elements in A that are not in B.

$$A - B = A \cap B^c; \ (x \in A) \text{ and } (x \notin B).$$



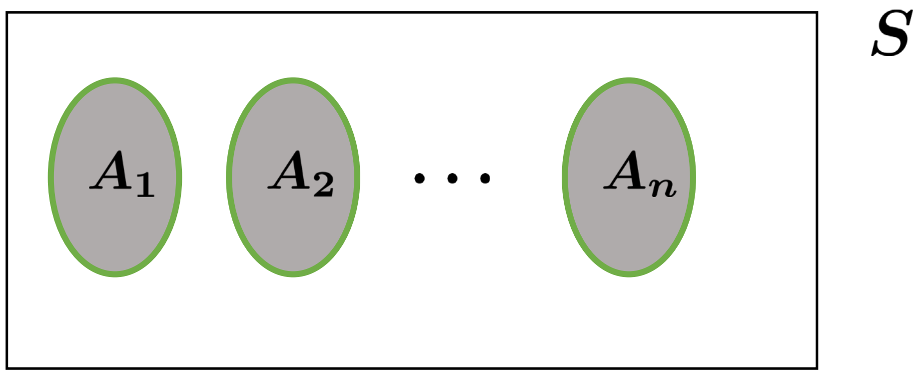
$A - B$

Set Operations

Mutually exclusive set (disjoint):

Two sets A and B are mutually exclusive (or disjoint) if $A \cap B = \emptyset$.

- **$A_1, A_2, \cdots, A_n$ are m.e. if $A_i \cap A_j = \emptyset, \ i \neq j$.**



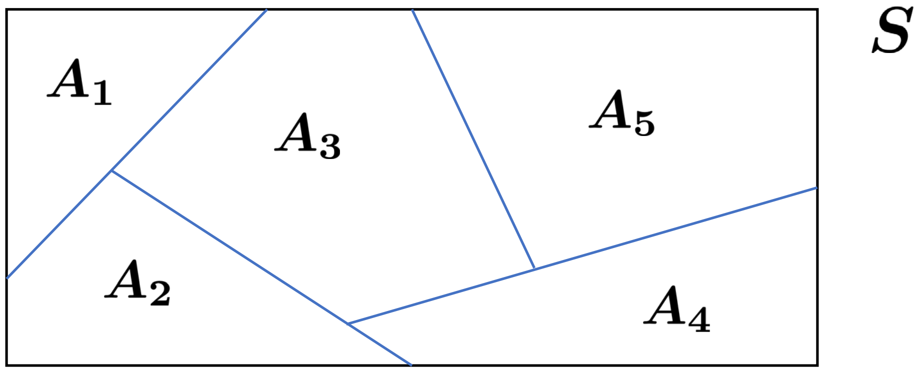
Set Operations

Partition:

A collection of sets $A_1, A_2, \cdots , A_n$ is a Partition of S if

a) They are disjoint .

b) $A_1 \cup A_2 \cup A_3 \cup \cdots A_n = S$



Set Operations

Theorem : De Morgan's law

$$(A \cup B)^c = \overline{A \cup B} = A^c \cap B^c$$

Set Operations

Example:

Let $S = \{1, 2, 3, 4, 5, 6\},$ **and** $A = \{1, 2\},\ B = \{2, 4, 6\}.$

a) $A \cup B$ **b)** $A \cap B$ **c)** A^c

d) B^c **f)** $(A \cup B)^c$ **g)** $A^c \cap B^c$

- **The sets** $\{1, 2\},\ \{3, 4, 5\},\ \{6\}$ **form a partition of** S.

Set Operations

Theorem : Distributive law

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

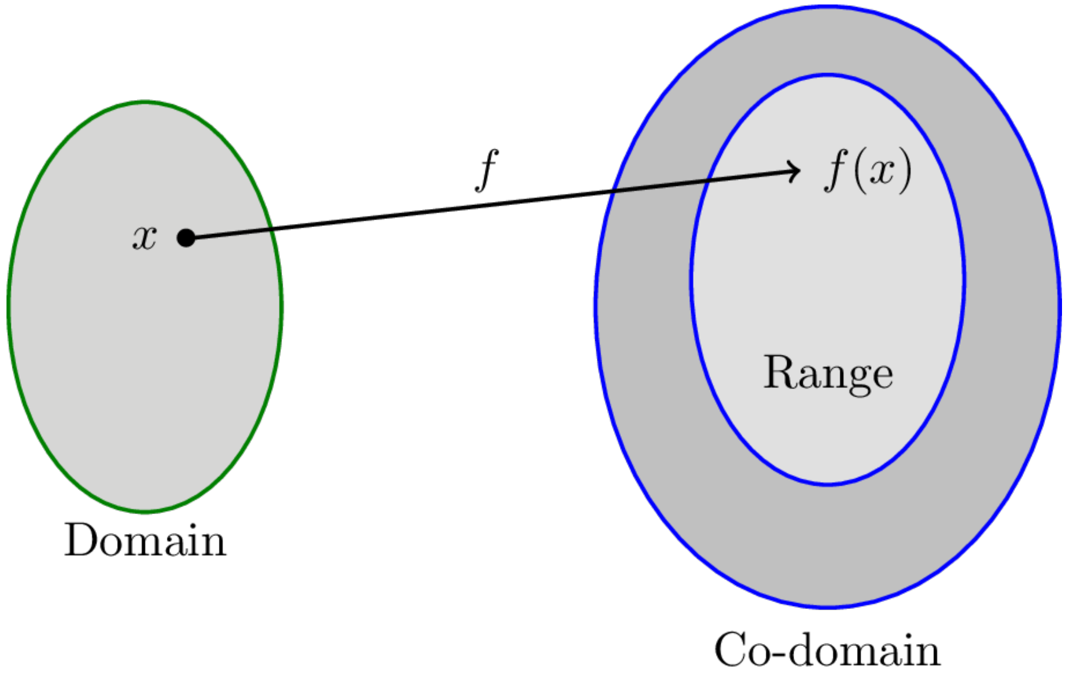
Functions

$\boldsymbol{f : X \to Y}.$

$\boldsymbol{X}$: **Domain**

$\boldsymbol{Y}$: **Co-domain**

$\forall\ \boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{f(x)} \in \boldsymbol{Y}$



Range: the set of all the possible values of $f(x)$. (Range $\subset \boldsymbol{Y}$)

Functions

Example:

Consider the function $f : \mathbb{R} \to \mathbb{R}$, **defined as** $f(x) = x^2$.

$X = Y = \mathbb{R}$;

$\mathrm{Range}(f) = \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$.

- **one-to-one (invertible):** $f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$

Countable and Uncountable Sets

Cardinality of a set A is the number of elements in A; $|A|$.

- A set is finite if $|A| < \infty$.
- A set if countable if it is finite Or the elements of A can be enumerated or listed in a sequence $a_1, a_2, a_3, \cdots$, that is,

$$A = \bigcup_{k=1}^{\infty} \{a_k\}, \qquad A = \{a_1, a_2, a_3, \cdots\}$$

Ex: $\mathbb{N} = \{1, 2, 3, \cdots\}$ is countable.

Countable and Uncountable Sets

Uncountable: **Not countable.**

e.g., $\mathbb{R}$; $[\,0, 1]$

Equivalently: A **set is countably infinite if it is in one-to-one correspondence with**

$$\boldsymbol{N = \{1, 2, 3, \cdots\} = \bigcup_{k=1}^{\infty} \{k\}}$$

Countable and Uncountable Sets

Example:

$\mathbb{Z}$ (set of integers) is countable (countably infinite).

Because $$\mathbb{Z} = \{\underset{a_1}{\underset{\uparrow}{0}}, \underset{a_2}{\underset{\uparrow}{1}}, \underset{a_3}{\underset{\uparrow}{-1}}, 2, -2, 3, -3, \cdots\}$$

Countable and Uncountable Sets

Example:

Show that a set of the form $B = \bigcup_{i,j=1}^{\infty} \{b_{ij}\} = \bigcup_{i}^{\infty} \bigcup_{j}^{\infty} \{b_{ij}\}$ **is countable.**

Example:

Show that the positive rational number form a countable set: $\mathbb{Q}^{+} = \bigcup_{i,j=1}^{\infty} \{\frac{i}{j}\}$.

Countable and Uncountable Sets

Example:

Show that the positive rational number form a countable set:

$$\mathbb{Q}^+ = \bigcup_{i,j=1}^{\infty} \{\frac{i}{j}\}.$$

Countable and Uncountable Sets

But $\mathbb{R}$ is not countable.

In fact, any interval $[a, b]$ where $b > a$ is not countable.

$$[a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R},\ a \leq x < b\}$$

Countable and Uncountable Sets

But $\mathbb{R}$ is not countable.

Proof) Proof by contradiction

We assume that $f(n) = r, n \in N;\ r \in R, 0 < r < 1$

$$f(1) = 0.11233\ldots$$
$$f(2) = 0.23458\ldots$$
$$f(3) = 0.84635\ldots$$
$$f(4) = 0.25494\ldots$$
$$f(5) = 0.69473\ldots$$
$$\ldots$$

Countable and Uncountable Sets

But $\mathbb{R}$ is not countable.

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$$\ldots$$

We can make a new number $x = 0.24704 \ldots$ by following this rule

Extract nth number and Add 1 (if the number is 9, change it to 0)