

Question:	1	2	3	4	Total
Points:	20	40	20	20	100
Score:					

**SKKU SWE3026\_41 Probability and Random Process**  
**2021 Spring Final Exam**

**Student ID:** \_\_\_\_\_

**Name:** \_\_\_\_\_

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- **Write your answer in Python 3.** We will grade your answers based only on Python 3.  
**Python 3를 사용해 문제를 해결하세요.** 채점은 Python 3 기준으로만 진행됩니다.

1. (20 points) (No partial points) Answer the following questions.

1-1. (15 points) Answer the questions with "T" if the answer is true or "F" otherwise. Other characters are not accepted.

1-1-1. (1 point) The instructor name of this class is JinYeong Bak.

**[Answer box]**

1-1-2. (1 point) Negative rational number is countable infinite set.

**[Answer box]**

1-1-3. (1 point) Disjoint and independent of events are the same.

**[Answer box]**

1-1-4. (1 point) Law of the unconscious statistician (LOTUS) for discrete random variables is  $E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(g(x_k))$ .

**[Answer box]**

1-1-5. (1 point) If  $X$  and  $Y$  are random variables, then  $E[E[X|Y]] = E[Y]$ .

**[Answer box]**

- 1-1-6. (1 point) When we find upper bounds on  $P(X \geq \alpha n)$  for  $X \sim \text{Binomial}(n, p)$ , the upper bound from Chebyshev's inequalities is tighter than the upper bound from Chernoff bounds.

[Answer box]

- 1-1-7. (1 point) The delta function of zero  $\delta(0)$  is 0.

[Answer box]

- 1-1-8. (1 point) If  $X$  and  $Y$  are independent random variables, then  $E[g(X)h(Y)] = g(E[X])h(E[Y])$ .

[Answer box]

- 1-1-9. (1 point) If  $X$  and  $Y$  are independent random variables and then the moment generating function of a random variables  $Z = X + Y$  is  $M_Z(s) = M_X(s)M_Y(s)$ .

[Answer box]

- 1-1-10. (1 point) Bayesian statistician think that the unknown quantity is assumed to be a fixed one.

[Answer box]

- 1-1-11. (1 point) The normalized random variable is as follows:  $Z = \frac{X-\mu}{\sigma}$  when  $E[X] = \mu$ ,  $\text{Var}[X] = \sigma^2$ .

[Answer box]

- 1-1-12. (1 point) A systematic way of parameter estimation that finds the parameter value that maximizes the posterior distribution is MLE.

**[Answer box]**

- 1-1-13. (1 point) We should minimize the cost function to find the best parameters of linear regression.

**[Answer box]**

- 1-1-14. (1 point) A random sequence is a random process.

**[Answer box]**

- 1-1-15. (1 point) In a Markov chain,  $X_m$  random variable depends on  $X_{m-1}$  and  $X_{m-2}$  that are previous values.

**[Answer box]**

1-2. (5 points) Choose the best answer to fill in the blank from the options given.

1-2-1. (1 point) We want to model the arrival of customers at a service facility. The most useful random variable is .

- ① Pascal
- ② Hypergeometric
- ③ Geometric
- ④ Poisson

1-2-2. (1 point) Bayes' rule is

$$P(B|A) = \frac{P(A|B)P(B)}{\text{$$

- ①  $P(A)$
- ②  $P(A, B)$
- ③  $P(B)$
- ④  $P(A|B)$
- ⑤  $P(B|A)$
- ⑥  $P(B, A)$

1-2-3. (1 point) We have a set with  $n$  elements, and we want to draw  $k$  samples from the set such that ordering does not matter and repetition is not allowed. The number of  $k$ -element subsets of the set is .

- ①  $P_k^n$
- ②  $n^k$
- ③  $k^n$
- ④  $\binom{n}{k}$
- ⑤  $\binom{k}{n}$
- ⑥  $\binom{n+k-1}{k}$

1-2-4. (1 point) If the correlation coefficient of two random variables  $X$  and  $Y$  is zero, we say that  $X$  and  $Y$  are .

- ① correlated
- ② uncorrelated
- ③ positively correlated
- ④ negatively correlated
- ⑤ independent
- ⑥ dependent

1-2-5. (1 point) For a discrete random variable  $X$  and event  $A$ , the  of  $X$  given  $A$  is defined as

$$\frac{P(X = x_i \text{ and } A)}{P(A)}$$

- ① marginal PDF
- ② marginal PMF
- ③ marginal CDF
- ④ conditional PDF
- ⑤ conditional PMF
- ⑥ conditional CDF

2. (40 points) Choose the best answer for the following questions. (Caution) Select the last option 'I do not know' when you do not solve the problem. We will give zero scores if you select the option. If not, we will check your answer and give **negative points if it is incorrect**. For example, when you choose the wrong option except 'I do not know', you will get -6 points for each wrong answer. Please select your answer carefully.

- 2-1. (6 points) There are 15 people in a party, including Hannah and Sarah. We divide the 15 people into 3 groups, where each group has 5 people. What is the probability that Hannah and Sarah are in the same group?

- ① 0
- ②  $\frac{1}{3}$
- ③  $\frac{1}{7}$
- ④  $\frac{2}{7}$
- ⑤  $\frac{2}{9}$
- ⑥  $\frac{5}{9}$
- ⑦ I do not know

- 2-2. (6 points) Let  $N$  be the number of phone calls made by the customers of a phone company in a given hour. Suppose that  $N \sim \text{Poisson}(\beta)$ , where  $\beta > 0$  is known. Let  $X_i$  be the length of the  $i$ 'th phone call, for  $i = 1, 2, \dots, N$ . We assume  $X_i$ 's are independent of each other and also independent of  $N$ . We further assume  $X_i \sim \text{Exponential}(\lambda)$ , where  $\lambda > 0$  is known. Let  $Y$  be the sum of the lengths of the phone calls, i.e.,

$$Y = \sum_{i=1}^N X_i$$

What is  $E[Y]$ ?

Hint:  $E[N] = \beta$ ,  $E[X] = \frac{1}{\lambda}$

- ①  $\beta + \frac{1}{\lambda}$
- ②  $\beta^2 + \frac{1}{\lambda}$
- ③  $\beta \frac{1}{\lambda}$
- ④  $\beta \lambda$
- ⑤  $N \beta \lambda$
- ⑥  $N \beta \lambda^2$
- ⑦ I do not know

- 2-3. (6 points) In a communication system, each codeword consists of 1000 bits. Due to the noise, each bit may be received in error with probability 0.1. It is assumed bit errors occur independently. Since error correcting codes are used in this system, each codeword can be decoded reliably if there are less than or equal to 125 errors in the received codeword, otherwise the decoding fails. Using the central limit theorem, find the probability of decoding failure.

Hint:  $E[X] = p$ ,  $\text{Var}[X] = p(1 - p)$  where  $X \sim \text{Bernoulli}(p)$

- ①  $\Phi(\frac{10}{9})$
- ②  $\Phi(\frac{25}{90})$

- ③  $\Phi(\frac{25}{\sqrt{90}})$
- ④  $1 - \Phi(\frac{10}{9})$
- ⑤  $1 - \Phi(\frac{25}{90})$
- ⑥  $1 - \Phi(\frac{25}{\sqrt{90}})$
- ⑦ I do not know

2-4. (6 points) Suppose we would like to test the hypothesis that at least 10% of students suffer from allergies. We collect a random sample of 225 students and 21 of them suffer from allergies. Compute the P-value of the hypothesis.

- ①  $\Phi(-\frac{1}{2})$
- ②  $\Phi(-\frac{1}{3})$
- ③  $\Phi(-\frac{1}{4})$
- ④  $\Phi(\frac{1}{4})$
- ⑤  $\Phi(\frac{1}{3})$
- ⑥  $\Phi(\frac{1}{2})$
- ⑦ I do not know

2-5. (6 points) Consider the Markov chain with three states  $S = \{1, 2, 3\}$ , that has the state transition matrix as below:

$$P = \begin{bmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

Suppose  $P(X_1 = 1) = 1/2$  and  $P(X_1 = 2) = 1/4$ . Find  $P(X_1 = 3, X_3 = 1)$

- ①  $1/32$
- ②  $1/16$
- ③  $3/32$
- ④  $3/16$
- ⑤  $5/32$
- ⑥  $5/16$
- ⑦ I do not know

2-6. (10 points) There are 1000 households in a town. Specifically, there are 100 households with one member, 200 households with 2 members, 300 households with 3 members, 200 households with 4 members, 100 households with 5 members, and 100 households with 6 members.

2-6-1. (5 points) We pick a household at random, and define the random variable  $X$  as the number of people in the chosen household. Find the expected value of  $X$ .

- ① 2.2
- ② 3.3
- ③ 4.4
- ④ 5.5
- ⑤ 6.6
- ⑥ 7.7
- ⑦ I do not know



2-6-2. (5 points) We pick a person in the town at random, and define the random variable  $Y$  as the number of people in the household where the chosen person lives. Find the expected value of  $Y$ . Please raise decimals to the next whole number.  
Ex)  $9.63 \rightarrow 10, 7.21 \rightarrow 7$

- ① 1
- ② 2
- ③ 3
- ④ 4
- ⑤ 5
- ⑥ 6
- ⑦ I do not know

3. (20 points) Jun wants to find a lower bound of  $\log P(X)$  where  $X$  is a random variable. Jin suggests that one of the lower bound is  $\sum_Z P(Z|X) \log \frac{P(Z,X)}{P(Z|X)}$  where  $Z$  is another random variable. However, Jun is hard to know the reason. He asks Jin the reason and Jin writes the steps of finding the lower bound of  $\log P(X)$  as below. Write the in-depth reason for each step for Jun.

$$\log P(X) = \log \sum_Z P(Z, X) \quad (1)$$

$$= \log \sum_Z P(Z, X) \frac{P(Z|X)}{P(Z|X)} \quad (2)$$

$$= \log \sum_Z P(Z|X) \frac{P(Z, X)}{P(Z|X)} \quad (3)$$

$$\geq \sum_Z P(Z|X) \log \frac{P(Z, X)}{P(Z|X)} \quad (4)$$

[Answer box for (1)]

[Answer box for (2)]

[Answer box for (3)]

[Answer box for (4)]

4. (20 points) This question is related to your homework assignments - Naive Bayes Classifier. Read the instructions carefully and answer each question according to the instruction.

**[Program]**

```

1 import re
2 import math
3
4 def main():
5     training1_sentence = "John likes to watch movies. Mary likes movies
6     too."
7     training2_sentence = "In the machine learning, " \
8     "naive Bayes classifiers are a family of
9     simple probabilistic classifiers."
10    testing_sentence = "John also likes to watch football games."
11
12    alpha = 0.1
13    prob1 = 0.4
14    prob2 = 0.6
15
16    print(naive_bayes(training1_sentence, training2_sentence,
17    testing_sentence, alpha, prob1, prob2))
18
19 def naive_bayes(training1_sentence, training2_sentence, testing_sentence
20 , alpha, prob1, prob2):
21     bow_train1 = create_BOW(training1_sentence)
22     bow_train2 = create_BOW(training2_sentence)
23     bow_test = create_BOW(testing_sentence)
24
25     classify1 = math.log( $\frac{(4-1-1)}{(4-1-3)}$ ) +  $\frac{(4-1-2)}{(4-1-4)}$ 
26     classify2 = math.log( $\frac{(4-1-3)}{(4-1-4)}$ ) +  $\frac{(4-1-2)}{(4-1-4)}$ 
27
28     return normalize_log_prob(classify1, classify2)
29
30 def calculate_doc_prob(bow_train, bow_test, alpha):
31     total_dict = list(bow_train[0].keys())
32     for token in bow_test[0].keys():
33         if token not in total_dict:
34             total_dict.append(token)
35
36     N = 0
37     for n in bow_train[1]:
38         N = N + n
39
40     prob_dic = {}
41     for word in bow_test[0].keys():
42         if word not in bow_train[0].keys():
43              $\frac{(4-1-5)}{(4-1-5)}$  = alpha / (N + alpha * len(total_dict))
44         else:
45             n = bow_train[0][word]
46              $\frac{(4-1-5)}{(4-1-5)}$  = (bow_train[1][n] + alpha) / (N + alpha * len(
47 total_dict))
48     logprob = 0
49     for word in bow_test[0].keys():
50         index = bow_test[0][word]
51         logprob += math.log( $\frac{(4-1-5)}{(4-1-5)}$ ) * bow_test[1][index]
52
53     return logprob

```

```
49 def normalize_log_prob(prob1, prob2):
50     maxprob = max(prob1, prob2)
51     prob1 -= maxprob
52     prob2 -= maxprob
53     prob1 = math.exp(prob1)
54     prob2 = math.exp(prob2)
55     normalize_constant = 1.0 / float(prob1 + prob2)
56     prob1 *= normalize_constant
57     prob2 *= normalize_constant
58     return prob1, prob2
59
60 def replace_non_alphabetic_chars_to_space(sentence):
61     return re.sub(r'[^a-z]+', ' ', sentence)
62
63 def create_BOW(sentences):
64     bow_dict = {}
65     bow = []
66     sentences = sentences.lower()
67     sentences = replace_non_alphabetic_chars_to_space(sentences)
68     words = sentences.split()
69     for token in words:
70         if len(token) < 1:
71             continue
72         if token not in bow_dict:
73             new_idx = len(bow)
74             bow.append(0)
75             bow_dict[token] = new_idx
76         bow[bow_dict[token]] += 1
77     return bow_dict, bow
78
79 if __name__ == "__main__":
80     main()
81
```

4-1. (15 points) Fill in the blanks (4-1-1), (4-1-2), (4-1-3), and (4-1-4) to complete the program code of Naive Bayes Classifier.

**[Answer box for (4-1-1)]**

**[Answer box for (4-1-2)]**

**[Answer box for (4-1-3)]**

**[Answer box for (4-1-4)]**

**[Answer box for (4-1-5)]**

- 4-2. (5 points) Explain the role of the 'alpha' variable in line 10. Hint: What if the value of 'alpha' is zero?

**[Answer box]**