Probability and Random Process (SWE3026)

Multiple Random Variables

JinYeong Bak
jy.bak@skku.edu
College of Computing, SKKU

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Rationale

Until now in this course, you have been working with one and two random variables and how they might be extended to more.

You will now begin considering three or more variables.

As the number of random variables increases, you will notice how the functions become computationally intractable. This leads to an exploration of other techniques.

Joint Distributions and Independence

Let $X_1, X_2, X_3, \cdots, X_n$ be n discrete random variables.

Joint PMF:

$$P_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n).$$

$$P_{XYZ}(1, 2, \sqrt{3}) = P(X = 1, Y = 2, Z = \sqrt{3})$$

Joint Distributions and Independence

Let $X_1, X_2, X_3, \cdots, X_n$ be n continuous random variables.

Joint PDF: $f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$

$$P\Big((X_1, X_2, \cdots, X_n) \in A\Big)$$
$$= \int \cdots \int_A \cdots \int f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n.$$

Joint Distributions and Independence

The joint CDF of n random variables $X_1, X_2, X_3, \cdots, X_n$ (both discrete and continuous) is defined as

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n).$$

Joint Distributions and Independence

Random variables $X_1, X_2, X_3, \cdots, X_n$ are **independent**, if

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = F_{X_1}(x_1)F_{X_2}(x_2)\cdots F_{X_n}(x_n).$$

Discrete:

$$P_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P_{X_1}(x_1)P_{X_2}(x_2)\cdots P_{X_n}(x_n).$$

Continuous:

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n).$$

Joint Distributions and Independence

If random variables $X_1, X_2, X_3, \cdots, X_n$ are independent, then

$$E[X_1 X_2 \cdots X_n] = E[X_1]E[X_2] \cdots E[X_n].$$

Joint Distributions and Independence

Definition. **Random variables $X_1, X_2, X_3, \cdots, X_n$ are said to be independent and identically distributed (i.i.d.) if they are *independent*, and they have the *same marginal distributions* :**

$$F_{X_1}(x) = F_{X_2}(x) = \ldots = F_{X_n}(x), \text{ for all } x \in \mathbb{R}.$$

Joint Distributions and Independence

Example. if random variables $X_1, X_2, X_3, \cdots, X_n$ are i.i.d., they will have the same means and variances, so we can write

$$\begin{aligned} E[X_1 X_2 \cdots X_n] &= E[X_1]E[X_2]\cdots E[X_n] && (\ X_i\text{'s are indepenednt}) \\ &= E[X_1]E[X_1]\cdots E[X_1] && (X_i\text{'s are identically distributed}) \\ &= E[X_1]^n. \end{aligned}$$

Sums of Random Variables

$$Y = X_1 + X_2 + \cdots + X_n$$

With regards to the linearity of expectation:

$$EY = EX_1 + EX_2 + \cdots + EX_n.$$

Sums of Random Variables

Variance of a sum of two and three random variables is

$$\begin{aligned}\mathrm{Var}(X_1 + X_2) &= \mathrm{Cov}(X_1 + X_2, X_1 + X_2) \\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X_1, X_2).\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X_1 + X_2 + X_3) &= \mathrm{Cov}(X_1 + X_2 + X_3, X_1 + X_2 + X_3) \\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) + 2\mathrm{Cov}(X_1, X_2) \\ &+ 2\mathrm{Cov}(X_1, X_3) + 2\mathrm{Cov}(X_2, X_3).\end{aligned}$$

Sums of Random Variables

Generally,

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum_{i<j} \operatorname{Cov}(X_i, X_j).$$

Sums of Random Variables

If $X_1, X_2, X_3, \cdots, X_n$ are uncorrelated (i.e. $\text{Cov}(X_i, X_j) = 0, \ \text{for } i \neq j$), then

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

If $X_1, X_2, X_3, \cdots, X_n$ are independent then they are uncorrelated, thus

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

Moment Generating Functions

Definition. The nth moment of a random variable X is defined to be $E[X^n]$.

The nth central moment of X is defined to be $E[(X - EX)^n]$.

Moment Generating Functions

The moment generating function (MGF) of a random variable X is a function $M_X(s)$ defined as

$$M_X(s) = E\left[e^{sX}\right].$$

We say that MGF of X exists, if there exists a positive constant a such that $M_X(s)$ is finite for all $s \in [-a, a]$.

Moment Generating Functions

Example. **For each of the following random variables, find the MGF.**

a) X **is a discrete random variable, with PMF**

$$P_X(k) = \begin{cases} \frac{1}{3} & k = 1 \\ \frac{2}{3} & k = 2 \end{cases}$$

$e^{s} P_X(1) + e^{s \cdot 2} P_X(2)$

$= e^{s} \cdot \frac{1}{3} + e^{2s} \cdot \frac{2}{3}$

b) Y **is a** $Uniform(0, 1)$ **random variable.**

$E[e^{sx}] = \int_0^1 e^{sx}\, dx = \frac{e^{s}-1}{s}$

Moment Generating Functions

Finding Moments from MGF:

Remember

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$e^{sX} = \sum_{k=0}^{\infty} \frac{(sX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k s^k}{k!}.$$

Moment Generating Functions

Thus, we have

$$M_X(s) = E[e^{sX}] = \sum_{k=0}^{\infty} E[X^k]\frac{s^k}{k!}.$$

Moment Generating Functions

We can obtain all moments of X^k from its MGF:

$$M_X(s) = \sum_{k=0}^{\infty} E[X^k] \frac{s^k}{k!},$$

$$E[X^k] = \frac{d^k}{ds^k} M_X(s)|_{s=0}.$$

Moment Generating Functions

Example. **Let $X \sim Uniform(0, 1)$. Find all of its moments, $E[X^k]$.**

Moment Generating Functions

Theorem. Consider two random variables X and Y. Suppose that there exists a positive constant c such that MGFs of X and Y are finite and identical for all values of s in $[-c, c]$. Then,

$$F_{\mathbf{X}}(t) = F_{\mathbf{Y}}(t), \text{ for all } t \in \mathbb{R}.$$

MGF determines the distribution.

Moment Generating Functions

Sum of Independent Random Variables:

If X and Y are independent RVs and $Z = X + Y$ then,

$$\begin{aligned} M_Z(s) &= E[e^{sZ}] = E[e^{s(X+Y)}] \\ &= E[e^{sX}e^{sY}] = E[e^{sX}]E[e^{sY}] \qquad \text{(Since } X \& Y \text{ independent)} \\ &= M_X(s)M_Y(s). \end{aligned}$$

Moment Generating Functions

If $X_1, X_2, \cdots, X_n$ are n independent random variables, then

$$M_{X_1+X_2+\cdots+X_n}(s) = M_{X_1}(s) M_{X_2}(s) \cdots M_{X_n}(s).$$

$M_X(s) = (1-p+pe^s)^n$

Moment Generating Functions

Example. If $X \sim Binomial(n, p)$ find the MGF of X .

$X = X_1 + X_2 + \cdots X_n$

$\therefore M_X(s) = M_{X_1}(s) \cdots M_{X_n}(s)$

$= e^0 P_{X_1}(0) + e^s P_X(1)$

$=$

Moment Generating Functions

Example. **Using MGFs prove that if $X \sim Binomial(m, p)$ and $Y \sim Binomial(n, p)$ are independent, then**

$$X + Y \sim Binomial(m + n, p).$$

Characteristic Functions

If a random variable does not have a well-defined MGF, we can use the characteristic function defined as

$$\phi_X(\omega) = E[e^{j\omega X}],$$

where $j = \sqrt{-1}$ and ω is a real number.

Characteristic Functions

$$|\phi_X(\omega)| = |E[e^{j\omega X}]| \leq E[|e^{j\omega X}|] \leq 1.$$

If X and Y are independent, and $Z = X + Y$, then

$$\phi_Z(\omega) = \phi_X(\omega)\phi_Y(\omega).$$