Midterm II Prep

MAT 310

Spring 2025

4 Polynomials

4.1 Recall complex properties

$$z = \mathrm{Re}(z) + \mathrm{Im}(z) \cdot i.$$
$$\overline{z} = \mathrm{Re}(z) - \mathrm{Im}(z) \cdot i.$$
$$|z| = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}.$$

4.2 Zeros of Polynomials

Definition 1. A number $\lambda \in \mathbb{F}$ is called a zero (or root) of a polynomial $p \in \mathcal{P}(\mathbb{F})$ if $p(\lambda) = 0$.

Theorem 1. Let $p \in \mathcal{P}(\mathbb{F})$ be a polynomial of degree $m \geq 1$, and let $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ of degree $m - 1$ such that:

$$p(z) = (z - \lambda)q(z)$$

for all $z \in \mathbb{F}$.

From this theorem, we notice that each zero of a polynomial corresponds to a degree-one factor.

Rule 1. A polynomial $p \in \mathcal{P}(\mathbb{F})$ of degree $m \geq 1$ has **at most** m zeros in $\mathbb{F}$.

4.3 Division algorithm for polynomials

Theorem 2. Let $p, s \in \mathcal{P}(\mathbb{F})$ be polynomials, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that:

$$p(z) = s(z)q(z) + r(z),$$

where $\deg r < \deg s$.

4.4 Factorization of Polynomials over $\mathbb{C}$

Theorem 3 (Fundamental Theorem of Algebra)**.** Every nonconstant polynomial $p \in \mathcal{P}(\mathbb{C})$ with complex coefficients has at least one zero in $\mathbb{C}$.

Note 1. A constant polynomaial is a polynomial with degree = 0.

Theorem 4. If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization of the form

$$p(z) = c(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m),$$

where $c \in \mathbb{C}$, $c \neq 0$, and $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$.

4.5 Factorization of Polynomials over $\mathbb{R}$

Theorem 5. Let $p \in \mathcal{P}(\mathbb{C})$ be a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is its conjugate $\overline{\lambda}$.

For example, if $p(z) = z^2 + 1$, then $\lambda = i$ is a zero since $p(i) = i^2 + 1 = -1 + 1 = 0$. The conjugate $\overline{\lambda} = -i$ is also a zero, as $p(-i) = (-i)^2 + 1 = -1 + 1 = 0$.

Theorem 6. Let $b, c \in \mathbb{R}$. The quadratic polynomial $x^2 + bx + c$ factors as $(x - \lambda_1)(x - \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $b^2 \geq 4c$.

Theorem 7. Let $p \in \mathcal{P}(\mathbb{R})$ be a nonconstant polynomial. Then p has a unique factorization (except for the order of the factors) of the form:

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M),$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_M, c_1, \ldots, c_M \in \mathbb{R}$, and each quadratic factor $x^2 + b_k x + c_k$ is irreducible over $\mathbb{R}$, satisfying $b_k^2 < 4c_k$ for each $k = 1, \ldots, M$.

5 Eigenvalues and Eigenvectors

5.1 Invariant Subspaces

Definition 2. A linear map from a vector space to itself is called an operator.

Definition 3. Suppose $T \in \mathcal{L}(V)$. A subspace $U \subset V$ is called **invariant** under T if $Tu \in U$ for every $u \in U$.

Example 1. Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is defined by $Tp = p'$. The subspace $\mathcal{P}_4(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$, consisting of polynomials of degree at most 4, is invariant under T. For any $p \in \mathcal{P}_4(\mathbb{R})$, the derivative $p' = Tp$ has degree at most 3, so $Tp \in \mathcal{P}_4(\mathbb{R})$, satisfying the definition of an invariant subspace.

For instance, if $p(x) = x^4 + 2x^3 \in \mathcal{P}_4(\mathbb{R})$, then $Tp = p' = 4x^3 + 6x^2$, which has degree 3 and is in $\mathcal{P}_4(\mathbb{R})$.

5.2 Eigenvalues

Definition 4. Suppose $T \in \mathcal{L}(V)$, where V is a vector space over a field $\mathbb{F}$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists a vector $v \in V$, $v \neq 0$, such that $Tv = \lambda v$.

Example 2. Define an operator $T \in \mathcal{L}(\mathbb{F}^3)$ by:

$$T(x, y, z) = (7x + 3z, 3x + 6y + 9z, -6y),$$

for all $(x, y, z) \in \mathbb{F}^3$. Consider the vector $v = (3, 1, -1) \in \mathbb{F}^3$. Then:

$$T(3, 1, -1) = (7 \cdot 3 + 3 \cdot (-1), 3 \cdot 3 + 6 \cdot 1 + 9 \cdot (-1), -6 \cdot 1) = (18, 6, -6) = 6(3, 1, -1).$$

Since $v \neq 0$ and $Tv = 6v$, the number 6 is an eigenvalue of T.

Theorem 8. Suppose V is a finite-dimensional vector space over a field $\mathbb{F}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Definition 5 (Definition 8)**.** Suppose $T \in \mathcal{L}(V)$, where V is a vector space over a field $\mathbb{F}$, and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Theorem 9. Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Theorem 10. Suppose V is a finite-dimensional vector space. Then each operator $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.

5.3 More Invariant Subspaces

Note 2. Suppose $T \in \mathcal{L}(V)$ and m is a positive integer. The powers of T are defined as follows:

- $T^m \in \mathcal{L}(V)$ is the operator given by $T^m = T \circ \cdots \circ T$ (m times).
- T^0 is the identity operator I on V.
- If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by $T^{-m} = (T^{-1})^m$.

Definition 6. Suppose $T \in \mathcal{L}(V)$, where V is a vector space over a field $\mathbb{F}$, and let $p \in \mathcal{P}(\mathbb{F})$ be a polynomial given by:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m,$$

for all $z \in \mathbb{F}$, with $a_0, a_1, \ldots, a_m \in \mathbb{F}$. Then the operator $p(T) \in \mathcal{L}(V)$ is defined by:

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m,$$

where I is the identity operator on V.

Definition 7. Let $p, q \in \mathcal{P}(\mathbb{F})$ be polynomials over a field $\mathbb{F}$. The product polynomial $pq \in \mathcal{P}(\mathbb{F})$ is defined by:

$$(pq)(z) = p(z)q(z),$$

for all $z \in \mathbb{F}$.

Note 3. Polynomial multiplication is commutative.

Theorem 11. Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then the null space null $p(T)$ and the range range $p(T)$ are invariant under T.

5.4 The Minimal Polynomial

Theorem 12. Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Definition 8. A **monic polynomial** is a polynomial whose highest-degree coefficient equals 1.

Theorem 13. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

Theorem 14. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1) \cdots (z - \lambda_m),$$

where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

Theorem 15. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

5.5 Upper-Triangular Matrices

6 Inner Product Spaces

6.1 Inner Products and Norms

Definition 9 (dot product)**.** For $x, y \in \mathbb{R}^n$, the dot product of x and y, denoted by $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Note 4. An inner product is a generalization of the dot product to any vector space.

Definition 10. An inner product on a vector space V over a field $\mathbb{F}$ assigns to each pair of elements $u, v \in V$ a number $\langle u, v\rangle \in \mathbb{F}$ and has the following properties:

- **Positivity**: $\langle v, v\rangle \geq 0$ for all $v \in V$.
- **Definiteness**: $\langle v, v\rangle = 0$ if and only if $v = 0$.
- **Additivity in first slot**: $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.
- **Homogeneity in first slot**: $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$.
- **Conjugate symmetry**: $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

Definition 11. An **inner product space** is a vector space V along with an inner product on V.

Below are some important properties of the inner product.

Definition 12.
- For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v\rangle$ is a linear map from V to $\mathbf{F}$.
- $\langle 0, v\rangle = 0$ for every $v \in V$.
- $\langle v, 0\rangle = 0$ for every $v \in V$.
- $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.
- $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

6.2 Norms

Definition 13. For $v \in V$, the norm of v, denoted by $\|v\|$, is defined by $\|v\| = \sqrt{\langle v, v\rangle}$. The norm has the following properties:

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbf{F}$.

Definition 14 (Orthogonality)**.** Two vectors $u, v \in V$ are called **orthogonal** if $\langle u, v\rangle = 0$. Note that 0 is the orthogonal to every vector and is the only vector that is orthogonal to itself.

Theorem 16 (Pythagorean Theorem)**.** Suppose $u, v \in V$. If u and v are orthogonal, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Definition 15 (Cauchy-Schwarz Inequality)**.** Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Definition 16 (Triangle Inequality)**.** Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

Definition 17 (Parallelogram Inequality)**.** Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$