Cheat Sheet

Metric Spaces

Definition 1 (Metric Space)**.** *A metric space is a set X equipped with a function d such that:*

1. $d(x, y) = 0 \iff x = y$ *(identity of indiscernibles),*
2. $d(x, y) = d(y, x)$ *(symmetry),*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

Topology

Definition 2. *A subset $U \subseteq X$ of a metric space (X, d) is **open** if for every $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$, where $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε.*

Definition 3. *A subset $F \subseteq X$ of a metric space (X, d) is **closed** if its complement $X \setminus F$ is open.*

Proposition 1. *The intersection of any collection of closed sets is closed. The union of any collection of open sets is open.*

Proposition 2. *A subset $F \subseteq X$ of a metric space (X, d) is closed if and only if it contains all its limit points.*

Definition 4 (Limit Point)**.** *A point $x \in X$ is a **limit point** of a subset $A \subseteq X$ if every open ball $B(x, \varepsilon)$ centered at x with radius $\varepsilon > 0$ intersects A in some point other than x itself.*

Connectedness

Definition 5 (Connected/Disconnected Metric Space)**.** *We say that a metric space (X, d) is connected if the only subsets $E \subseteq X$ that are simultaneously open and closed are $E = X$ and $E = \emptyset$. A metric space that is not connected is called disconnected.*

Lemma 1. *A metric space (X, d) is disconnected if and only if there are two non-empty, disjoint, open subsets $A, B \subseteq X$ such that $X = A \cup B$.*

Definition 6 (Connected Subset)**.** *A subset $E \subseteq X$ of a metric space (X, d) is connected if (E, d_E) is a connected metric space, where d_E is the restriction of d to $E \times E$.*

Proposition 3. *A subset $E \subseteq X$ of a metric space (X, d) is disconnected if and only if $E \subseteq U \cup V$ for two open subsets $U, V \subseteq X$ such that $U \cap E \neq \emptyset$, $V \cap E \neq \emptyset$, and $(U \cap V) \cap E = \emptyset$.*

Proposition 4. *An open subset $E \subseteq X$ of a metric space (X, d) is disconnected if and only if there are two non-empty, disjoint, open subsets $A, B \subseteq X$ such that $E = A \cup B$.*

Theorem 1. *The interval $(0, 1) \subseteq \mathbb{R}$ is connected.*

Theorem 2. *Suppose that $f : X \to Y$ is a continuous map between metric spaces. If $E \subseteq X$ is connected, then $f(E) \subseteq Y$ is connected.*

Definition 7 (Property P)**.** *We say that a set $I \subseteq \mathbb{R}$ has property P if whenever $a \leq z \leq b$ with $a, b \in I$, then $z \in I$.*

Theorem 3. *A non-empty subset $I \subseteq \mathbb{R}$ is connected if and only if it is an interval.*

Corollary 1. *Let (X, d) be a connected metric space and $f : X \to \mathbb{R}$ be a continuous function. Then the image of f is an interval $I \subseteq \mathbb{R}$.*

Theorem 4 (Bolzano's Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. If $f(a) < c$ and $f(b) > c$, then there is some $x \in (a, b)$ such that $f(x) = c$.*

Definition 8 (Path Connectedness)**.** *Let (X, d) be a metric space. A path connecting two points x and $y \in X$ is a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We say that (X, d) is path connected if every pair of points of X is connected by a path.*

Theorem 5. *A path connected metric space is connected.*

Compactness

Definition 9 (Compact Metric Space)**.** *A metric space (X, d) is called compact if given any collection of open sets $\{U_\alpha \subseteq X : \alpha \in I\}$ such that:*

$$X = \bigcup_{\alpha \in I} U_\alpha,$$

one can find finitely many indices $\alpha_1, \ldots, \alpha_n \in I$ such that:

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Proof. The metric space $X = \mathbb{R}$ is not compact. To see this, consider the open cover $\{U_\alpha : \alpha \in [1, \infty)\}$ where $U_\alpha = (-\alpha, \alpha)$. For any finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, the union is $(-M, M)$ with $M = \max\{\alpha_1, \ldots, \alpha_n\}$, which does not cover $\mathbb{R}$ entirely. □

Definition 10 (Compact Subset)**.** *A subset $E \subseteq X$ of a metric space (X, d) is compact if for every open cover $\{U_\alpha \subseteq X : \alpha \in I\}$ such that:*

$$E \subseteq \bigcup_{\alpha \in I} U_\alpha,$$

there exists a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ *such that:*

$$E \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Lemma 2. *A subset* $E \subseteq X$ *of a metric space* (X, d) *is compact if and only if the metric space* (E, d_E) *is compact, where* d_E *is the restriction of* d *to* $E \times E$.

Theorem 6. *The closed interval* $[0, 1] \subseteq \mathbb{R}$ *is compact.*

Proof. Let $\{U_\alpha \subseteq \mathbb{R} : \alpha \in I\}$ be an open cover of $[0, 1]$. Assume, by contradiction, that no finite subcover exists. Construct a nested sequence of intervals $[0, 1] = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$, where each $I_n = [a_n, b_n]$ cannot be covered by finitely many sets in the open cover. The length of I_n decreases geometrically, and the intersection of all I_n contains exactly one point x. This x must belong to some U_α, contradicting the assumption that no finite subcover exists. □

Theorem 7 (Compactness and Continuity)**.** *If* $f : X \to Y$ *is a continuous function and* $K \subseteq X$ *is compact, then* $f(K) \subseteq Y$ *is compact.*

Corollary 2. *Any finite, closed interval* $[a, b] \subseteq \mathbb{R}$ *is compact.*

Lemma 3. *A compact subset* K *of a metric space* X *is closed.*

Theorem 8 (Weierstrass Theorem)**.** *A continuous function* $f : [a, b] \to \mathbb{R}$ *attains its maximum and minimum values. That is, there exist* $x_m, x_M \in [a, b]$ *such that:*

$$f(x_m) = \min_{x \in [a,b]} f(x), \quad f(x_M) = \max_{x \in [a,b]} f(x).$$

Theorem 9 (Heine-Borel Theorem)**.** *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

Definition 11 (Sequential Compactness)**.** *A metric space (X, d) is sequentially compact if every sequence of points in X has a convergent subsequence converging to a point in X.*

Corollary 3 (Sequential Compactness)**.** *A metric space (X, d) is compact if and only if it is sequentially compact.*

Completeness

Definition 12 (Cauchy Sequence)**.** *A sequence* $\{x_n\}$ *in a metric space* (X, d) *is a* ***Cauchy sequence*** *if for every* $\varepsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that:*

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

Definition 13 (Complete Metric Space)**.** *A metric space* (X, d) *is called* ***complete*** *if every Cauchy sequence in* X *converges to a point in* X.

Lemma 4. *Any converging sequence in a metric space is a Cauchy sequence.*

Lemma 5. *A Cauchy sequence $\{x_n\}$ in a metric space has bounded diameter. Specifically, there exists a constant C such that:*

$$\sup_{n,m} d(x_n, x_m) \leq C.$$

Lemma 6. *If a Cauchy sequence $\{x_n\}$ in a metric space has a converging subsequence $\{x_{n_k}\}$, then $\{x_n\}$ converges to the same limit.*

Theorem 10. *A compact metric space is complete.*

Theorem 11. *The space $\mathbb{R}^n$ with the standard Euclidean metric is a complete metric space.*

Theorem 12. *The space ℓ^2 (the space of square-summable sequences) is a complete metric space.*

Sequences of Functions

Definition 14 (Pointwise Convergence)**.** *A sequence of functions $\{f_n : I \to \mathbb{R}\}$ is said to converge **pointwise** to a function $f : I \to \mathbb{R}$ on a set $E \subseteq I$ if for every $x \in E$:*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Definition 15 (Uniform Convergence)**.** *A sequence of functions $\{f_n : I \to \mathbb{R}\}$ is said to converge **uniformly** to a function $f : I \to \mathbb{R}$ on a set $E \subseteq I$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that:*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and } x \in E.$$

Proposition 5. *A sequence $\{f_n\}$ of functions in $B(E, \mathbb{R})$ converges in the metric:*

$$d(f, g) = \sup_{x\in E} |f(x) - g(x)|$$

if and only if it converges uniformly on E.

Theorem 13. *The metric space $B(E, \mathbb{R})$ (the space of bounded functions on E) is complete under the uniform metric.*

Continuity

Theorem 14 (Intermediate Value Theorem (IVT))**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) \neq f(b)$ and k is any value between $f(a)$ and $f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = k$.*

Theorem 15 (Extreme Value Theorem (Weierstrass))**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains both a maximum and a minimum value on $[a, b]$. That is, there exist $x_m, x_M \in [a, b]$ such that*

$$f(x_m) \leq f(x) \leq f(x_M) \quad \text{for all } x \in [a, b].$$

Definition 16 (Continuity at a Point). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is* ***continuous*** *at a point $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:*

$$|f(x) - f(c)| < \varepsilon \quad \textit{whenever} \quad |x - c| < \delta.$$

In other words, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if x is within δ of c, then $f(x)$ is within ε of $f(c)$.

Definition 17 (Continuity on an Interval). *A function $f : [a, b] \to \mathbb{R}$ is said to be continuous on the interval $[a, b]$ if it is continuous at every point in the interval.*

Uniform Continuity

Definition 18 (Uniform Continuity). *A function $f : [a, b] \to \mathbb{R}$ is said to be* ***uniformly continuous*** *on $[a, b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:*

$$|f(x) - f(y)| < \varepsilon \quad \textit{whenever} \quad |x - y| < \delta \quad \textit{for all} \quad x, y \in [a, b].$$

In other words, the choice of δ does not depend on the point x, but is valid for all $x, y \in [a, b]$.

Definition 19 (Lipschitz Continuity). *A function $f : X \to Y$, where X and Y are metric spaces, is said to be* ***Lipschitz continuous*** *if there exists a constant $L \geq 0$ such that for all $x, y \in X$, we have:*

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y),$$

where d_X and d_Y are the metrics on X and Y, respectively. The smallest such constant L is called the ***Lipschitz constant*** *of f.*

Differentiation

Definition 20 (Existence of Derivative). *Let f be a real-valued function defined on an open interval containing a point a. We say f is differentiable at a if*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite.

$$f'(y) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Theorem 16. *If f is differentiable at a point a, then f is continuous at a.*

Theorem 17. *Let $f : I \to \mathbb{R}$, where I is an open interval. If $x_0 \in I$ is a point of local minimum or maximum of $f(x)$, and f is differentiable at x_0, then*

$$f'(x_0) = 0.$$

Theorem 18 (Rolle's Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$, differentiable on the open interval (a, b), and suppose $f(a) = f(b)$. Then there exists at least one point $c \in (a, b)$ such that*

$$f'(c) = 0.$$

Theorem 19 (Mean Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 4. *Let f be a differentiable function on an interval (a, b). Then*

(i) f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$;

(ii) f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$;

(iii) f is increasing if $f'(x) \geq 0$ for all $x \in (a, b)$;

(iv) f is decreasing if $f'(x) \leq 0$ for all $x \in (a, b)$.

Definition 21 (IVT for Derivatives). *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] $x \in (x_1, x_2)$ such that $f'(x) = c$.*

Definition 22 (Differentiability at a Point). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is differentiable at a point $c \in \mathbb{R}$ if the following limit exists:*

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

Power Series

Radius of Convergence

For a power series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

the radius of convergence describes the distance within which the series converges absolutely and uniformly, and outside of which it diverges.

Definition 23 (Ratio Test).

$$\frac{1}{R} = \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Definition 24 (Root Test).

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}.$$

Definition 25 (Taylor Series)**.** *The Taylor series of a function $f(x)$ around a point a is given by*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Definition 26 (Partial Sum)**.** *The N-th partial sum $b_N(x)$ of a power series is given by*

$$b_N(x) = \sum_{n=0}^{N} a_n x^n.$$

Definition 27 (Remainder)**.** *Given a function $f(x)$ and its Taylor series expansion at $x = a$:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

the remainder $R_N(x)$ is defined as the difference between the function $f(x)$ and the partial sum $b_N(x)$:

$$R_N(x) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Integration

Definition 28. *Let f be a function. If f is bounded and has compact support, then f is Riemann integrable.*

Theorem 20. *A function f is Riemann integrable if and only if for all $\epsilon > 0$, there exist step functions ϕ and ψ such that*

$$\phi \leq f \leq \psi$$

and

$$\int_{-\infty}^{\infty} \psi - \int_{-\infty}^{\infty} \phi < \epsilon.$$

Theorem 21. *A function f is integrable if and only if there exist step functions ϕ_n and ψ_n such that*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} \psi_n.$$

Theorem 22. *If f and g are Riemann integrable, then*

$$\int_{-\infty}^{\infty} (f(x) + g(x))\, dx = \int_{-\infty}^{\infty} f(x)\, dx + \int_{-\infty}^{\infty} g(x)\, dx.$$

Lemma 7. *If f is continuous, then f is Riemann integrable on any compact set.*

Lemma 8. *If f is increasing, then f is Riemann integrable on any compact set.*

Theorem 23. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing or decreasing function. If f is Riemann integrable, then f is Riemann integrable on any compact interval $[a, b]$.*

Theorem 24 (Integral Mean Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists a point $c \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Theorem 25. *Let f be an integrable function on the interval $[a, b]$. Define the function $F(x)$ as*

$$F(x) = \int_a^x f(t)\,dt.$$

Then, F is differentiable, and

$$F'(x) = f(x) \text{ for all } x \in [a, b].$$

Theorem 26. *Let $G : [a, b] \to \mathbb{R}$ be such that $G'(x) = f(x)$ for all $x \in [a, b]$. Then,*

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

Definition 29. *Let $X \subset B[a, b]$ denote the subset of integrable functions.*

Theorem 27. *Let $X \subseteq B[a, b]$ be a subset of the space of functions $B[a, b]$. If X is closed in $B[a, b]$, then the following is equivalent:*

- *If $f_n \to f$ in $B[a, b]$ and $f_n \in X$ for all n, then f is the limit of f_n and is integrable.*

Theorem 28. *A function f is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exist step functions φ and ψ such that:*

$$\varphi(x) \leq f(x) \leq \psi(x) \quad \text{for all } x \in [a, b],$$

and

$$\int_a^b \psi(x)\,dx - \int_a^b \varphi(x)\,dx < \epsilon.$$

Corollary 5. *If $X \subset B[a, b]$, then X is complete, because it is closed in a complete space.*

Corollary 6. *Let $f : (-R, R) \to \mathbb{R}$ be a power series of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then $f(x)$ is integrable on all compact intervals that are subsets of $(-R, R)$.

Corollary 7. *Let $f_n(x)$ be a sequence of functions. If $f_n(x)$ is converging uniformly to $f(x)$ on any compact interval $[a, b[$ that is a subset of $(-R, R)$, then the convergence is uniform on that interval.*

1. Sum of 1

$$\sum_{k=1}^{n} 1 = n$$

2. Sum of k

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

3. Sum of k^2

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

4. Sum of k^3

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$