Problems

Set A

Find the rectangular coordinates of the points with the given polar coordinates.

1. $(3, \pi/6)$ **2.** $(4, -5\pi/6)$

3. $(-4, -5\pi/4)$ **4.** $(-5, \pi/2)$

5. $(\sqrt{3}, -\pi/3)$ **6.** $(2, 2)$

Find polar coordinates of the points with rectangular coordinates.

7. $(5, 5)$ **8.** $(-3, 4)$

9. $(-2, -2)$ **10.** $(2, -2\sqrt{3})$

11. $(\frac{3}{5}, \frac{4}{5})$ **12.** $(3 \sin 4, 3 \cos 4)$

Find polar equations of the following curves and sketch them.

13. $x^2 + y^2 = a^2$ **14.** $x + 5 = 0$

15. $x + y = 2$ **16.** $x^2 + y^2 = 4y$

17. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

18. $xy = 1$

Find rectangular equations of the curves with the following polar equations and sketch them.

19. $r = a \sin \theta$ **20.** $r = 4$

21. $r = 1/(1 - \cos \theta)$ **22.** $r^2 = a^2 \sin 2\theta$

23. $r = \tan \theta$ **24.** $r^2 = \tan \theta \sec^2 \theta$

Set B

Find polar equations of the following curves and sketch them.

25. $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$

26. $y = b$

27. $x \cos \gamma + y \sin \gamma = p$

28. $4x^2 + 3y^2 - 2y - 1 = 0$

29. $y^2 = x^3$

30. $x^2 + 4y^2 = 4$

Find rectangular equations of each curve and sketch them.

31. $r = a \sin 2\theta$ **32.** $r = \cos \theta - \sin \theta$

33. $r = \theta$ **34.** $r = \tan \frac{1}{2}\theta$

Set C

35. Show that the distance between the points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}.$$

36. Prove that a straight line not passing through the origin has an equation of the form $r(A \cos \theta + B \sin \theta) = 1$. If the line passes through the origin, what is the form of a polar equation for it?

37. Find a rectangular equation of the line through the points P_1 and P_2 if the polar coordinates of these points are $P_1 = (4, \pi/2)$ and $P_2 = (4\sqrt{2}, -\pi/4)$.

Calculator Problem

Calculate the distance between the points P_1 and P_2 of Problem 37 in two ways.

(a) Use the formula given in Problem 35.

(b) Find the rectangular coordinates of the points and then use the distance formula for the rectangular coordinate plane.

19–7 Intersections

To find the intersections of curves given in rectangular coordinates, one simply solves the two simultaneous equations, because we know that a point is on both curves if and only if its (unique) coordinates satisfy both equations.

To find the intersection of curves with polar equations, more care must be taken because a point has many sets of polar coordinates. Thus on one curve, the point may be given by one pair of coordinates while on the other curve, it is given by a different pair. Naturally, solution of the two simultaneous polar equations (if any solution exists) will yield points on both curves, but it may not give all of them. The procedure we shall follow is to

(a) solve the two polar equations, and

(b) graph the two equations to see whether all points of intersection have been obtained.

Example Find the points of intersection of the circle $r = \sin\theta$ and the cardioid $r = 1 - \sin\theta$.

Solution. Eliminating θ between the two equations gives $r = 1 - r$ and $r = \frac{1}{2}$, so $\sin\theta = \frac{1}{2}$ and $\theta = \pi/6$ or $5\pi/6$. The two points of intersection so obtained are shown in Fig. 20. Furthermore, we see from the figure that the curves also intersect at the pole. As a point of the circle, the pole has coordinates $(0, 0)$, $(0, \pi)$, $(0, 2\pi)$, and as a point of the cardioid, it has coordinates $(0, \pi/2)$. We may visualize this situation by thinking of two ships, both starting at P and sailing along the two curves to O. They pass through O at different times because their paths from their starting point to O are different. One would go from P to O directly; the other would go from P to R to S and then to O.

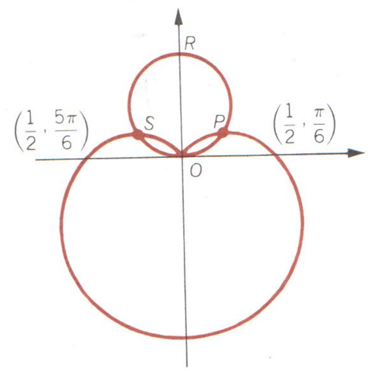

FIGURE 20

Problems

Set A

Find the points of intersection of the pairs of curves with the following polar equations. Sketch each pair.

1. $r = 3$, $r = 3 + 2\cos\theta$ **2.** $r = 2$, $r = 2\sin 2\theta$

3. $r = \sin\theta$, $r = \cos 2\theta$ **4.** $r = \cos\theta$, $r = \sin 2\theta$

5. $r = \sin 2\theta$, $r = \cos 2\theta$ **6.** $r = 2$, $r^2 = 8\cos 2\theta$

Set B

Find the points of intersections of the pairs of curves. Sketch the curves.

7. $r = 2$, $r = \sec^2 \frac{1}{2}\theta$

8. $r^2 = a^2\cos 2\theta$, $r^2 = a^2\sin 2\theta$

9. $r^2 = \sin 2\theta$, $r = \sin\theta$

10. $r = \cos\theta$, $r = 1 + \cos\theta$

11. $r = 1 - \sin\theta$, $r = \cos 2\theta$

12. $r^2 = \cos 2\theta$, $r^2 = \sin 2\theta$

Set C

13. Find the point of intersection of the pair of lines whose polar equations are

$$r = \frac{-1}{\sin\theta + \cos\theta}$$

and

$$r = \frac{2}{3\sin\theta - 2\cos\theta}.$$

14. Find rectangular equations of the lines in Problem 13. Solve the simultaneous equations and obtain the point of intersection in rectangular coordinates.

15. Show, by an example, that even though (r_0, θ_0) may satisfy the equation of a polar curve, other coordinates for the same point, $(r_0, \theta_0 + 2k\pi)$ or $(-r_0, \theta_0 + k\pi)$, may not satisfy the equation.

Calculator Problem

Sketch the curve $r^2 = 16 \sin \frac{1}{2}\theta$. Use increments of $\pi/12$ (15°) for θ.

19-8 Conics

Equations of conics in polar coordinates are quite simple if the focus is at the pole and the directrix is perpendicular to the polar axis. The polar form of an equation of a conic is often used in calculation of orbits of planets or satellites.

Let us suppose that the focus and directrix of a conic are as in Fig. 21. Then P is on the conic of eccentricity e if and only if $|PF| = |PM|e$. Since $|PF| = r$ and $|PM| = p + r\cos\theta$, we have $r = (p + r\cos\theta)e$ or

$$r = \frac{pe}{1 - e\cos\theta}. \qquad (1)$$

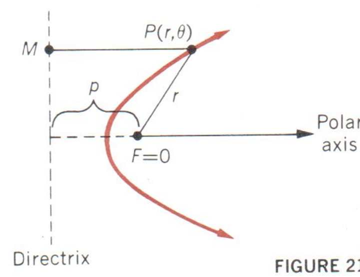

FIGURE 21

Equation (1) can be used to quickly sketch the conic, as the following example shows.

Example Sketch the curve with the equation

$$r = \frac{4}{2 - \cos\theta}.$$