Problems

1, 2, 5–7, 12, 16, 18, 21, 23, 25, 26

Set A

Give the magnitude and direction cosines of the vectors given by the ordered pairs of points.

1. $((0, 0, 0), (4, 4, 2))$

2. $((5, 4, 1), (1, -3, 3))$

3. $((-3, -3, 4), (0, 0, 0))$

4. $((0, a, 0), (b, 0, 0))$

5. What is the position vector for the ordered pair of points $((3, 3, 4), (5, -2, 1))$?

The terminal points of two position vectors are given. Find the coordinates of the terminal point of the vector that is the sum of the given vectors. Sketch the vectors.

6. $(0, 2, 3)$ and $(3, 1, 0)$

7. $(-1, 5, 1)$ and $(4, 1, 1)$

8. $(-2, -2, 4)$ and $(5, 6, -2)$

9. $(-2, 4, 4)$ and $(1, -2, -1)$

If $\mathbf{r}$ has a position vector with terminal point at $(3, -2, 1)$, what is the terminal point of the following vectors?

10. $3\mathbf{r}$

11. $-\mathbf{r}$

12. $-5\mathbf{r}$

13. $4\mathbf{r} + (-4)\mathbf{r}$

14. $2\mathbf{r}$

15. $6\mathbf{r} + (-5)\mathbf{r}$

Set B

16. An arrow has initial point $(-1, -2, 1)$ and direction cosines $c_1 = -\frac{1}{3}$, $c_2 = \frac{2}{3}$, $c_3 = \frac{2}{3}$. If the arrow has magnitude 6, what are the coordinates of the end point of the arrow?

17. If $P = (6, -6, -2)$ and $\overrightarrow{OP'}$ has the same magnitude as $\overrightarrow{OP}$ but is opposite in direction, what are the coordinates of P'? Sketch the arrows.

18. $|\overrightarrow{OP}| = 1$, and $\overrightarrow{OP}$ has direction cosines

$$\left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right).$$

What are the coordinates of P? 1

Set C

19. Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ have position vectors to points $(3, -4, 2)$ and $(-3, 4, -2)$, respectively. What are the coordinates of the point P such that

$$\overrightarrow{OP} = \mathbf{v}_1 + \mathbf{v}_2?$$

20. Does each vector in space have an additive inverse? That is, if $\mathbf{v}$ is any vector, is there a vector $\mathbf{r}$ such that $\mathbf{v} + \mathbf{r} = \mathbf{0}$?

21. If $\mathbf{r}$ and $\mathbf{t}$ are any vectors, is there a vector $\mathbf{s}$ such that $\mathbf{r} + \mathbf{s} = \mathbf{t}$?

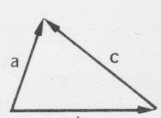
22. Define subtraction for vectors.

23. Show that $\mathbf{v} + \mathbf{v} = 2\mathbf{v}$ for every vector $\mathbf{v}$.

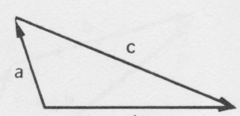
24. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, where $A = (3, 1, 2)$, $B = (-2, 2, -1)$, and $C = (4, 0, 3)$, find the coordinates of the point P such that $\overrightarrow{OP} = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$. Find the coordinates of the point Q such that $\overrightarrow{OQ} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$. Is addition of vectors associative?

In each figure, $\mathbf{c}$ is the difference of two vectors. Write the proper subtraction statement for each picture.

25.



26.



Calculator Problem

Find the coordinates of the terminal point of the position vector for $\overrightarrow{P_1P_2}$ where $P_1 = (-2.6, 1.8, -3.7)$ and $P_2 = (4.5, -7.2, -5.1)$. Then find the magnitude, direction cosines and direction angles of the position vector.

14-3 Unit Basis Vectors

In Chapter 12 we saw that each vector in the plane can be represented as a linear combination of two unit vectors directed along the positive x- and positive y-axes. In a similar manner, we may show that every vector in space can be represented as a linear combination of three unit vectors. A most convenient choice of the unit vectors is the three unit vectors directed along the positive x-, y-, and z-axes, as shown in Fig. 6. We shall use **i**, **j**, and **k** to denote these unit vectors.

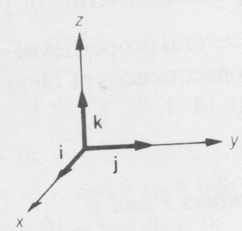


FIGURE 6

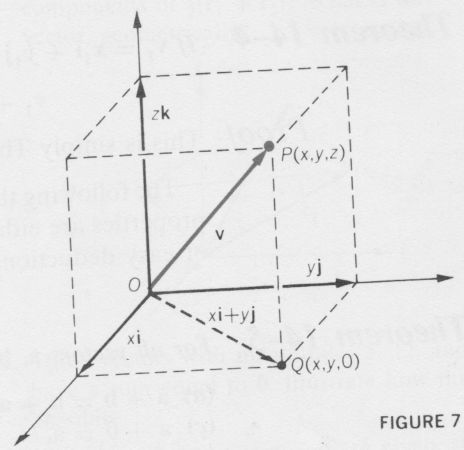


FIGURE 7

Let us consider a vector $\mathbf{v} = \overrightarrow{OP}$ as in Fig. 7. Then $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$ are vectors directed along the axes, and

$$x\mathbf{i} + y\mathbf{j} = \overrightarrow{OQ}.$$

Because $\overrightarrow{QP} = z\mathbf{k}$,

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$$

and

$$\mathbf{v} = \overrightarrow{OP} = (x\mathbf{i} + y\mathbf{j}) + z\mathbf{k}. \tag{1}$$

In the same way,

$$\mathbf{v} = x\mathbf{i} + (y\mathbf{j} + z\mathbf{k}). \tag{2}$$

Thus, if $P' \neq P$, then $\overrightarrow{OP'} \neq \overrightarrow{OP}$, and it follows that every vector is a sum of unique vectors directed along the axes. From equations (1) and (2), we see that the way in which the vectors are grouped, or associated, is immaterial. We therefore write

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Definition 14–3 The *components* of $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ are the real numbers x, y, and z. The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the *unit basis vectors* for this coordinate system.

Theorem 14–3 ***Two vectors are equal if and only if the corresponding components of these vectors are equal relative to the same coordinate system.***

Proof The coordinates of P, which are the components of $\overrightarrow{OP}$, uniquely determine the vector $\overrightarrow{OP}$.

Theorem 14–4 ***If*** $\mathbf{v}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ ***and*** $\mathbf{v}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, ***then***

$$\mathbf{v}_1 + \mathbf{v}_2 = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}.$$

Proof This is simply Theorem 14–2 expressed in terms of the unit basis vectors.

The following theorem exhibits several properties of vector addition. These properties are either immediate consequences of Definitions 14–1 and 14–2 or easy deductions from Theorem 14–4.

Theorem 14–5 ***For all vectors a, b, c and real numbers r and s,***

(*a*) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$,	(*b*) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$,
(*c*) $\mathbf{a} + \mathbf{0} = \mathbf{a}$,	(*d*) $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$,
(*e*) $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$,	(*f*) $(rs)\mathbf{a} = r(s\mathbf{a})$,
(*g*) $0 \cdot \mathbf{a} = \mathbf{0}$,	(*h*) $1 \cdot \mathbf{a} = \mathbf{a}$.

Proof (a) Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$. By Theorem 14–4, commutativity of addition of real numbers, and Theorem 14–3 we have

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k} \\ &= (x_2 + x_1)\mathbf{i} + (y_2 + y_1)\mathbf{j} + (z_2 + z_1)\mathbf{k} \\ &= \mathbf{b} + \mathbf{a}.\end{aligned}$$

Proofs for some of the other parts of Theorem 14–5 are left for the student.

Problems

2, 4, 9, 12, 15, 16, 19, 24, 28, 29

Set A

1. Sketch a picture as in Fig. 7 to show that a position vector $\overrightarrow{OP}$ may be represented as $\overrightarrow{OP} = (x\mathbf{i} + z\mathbf{k}) + y\mathbf{j}$.

2. If $P_1 = (7, 4, -1)$ and $P_2 = (3, -5, 4)$, what are the components of $\overrightarrow{P_1P_2}$? Express $\overrightarrow{P_1P_2}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Sketch the following vectors, add them, and sketch their sum.

3. $2\mathbf{i} + 3\mathbf{j}$, $\mathbf{j} + \mathbf{k}$
4. $2\mathbf{i} + \mathbf{j} + (-1)\mathbf{k}$, $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
5. $3\mathbf{i}$, $-2\mathbf{k}$
6. $4\mathbf{i}$, $-2\mathbf{j} + 2\mathbf{k}$

Given that

$$\mathbf{r} = \frac{\sqrt{3}}{2}\mathbf{i} + 3\mathbf{j} - \sqrt{2}\mathbf{k}, \quad \mathbf{s} = \sqrt{3}\,\mathbf{i} - 6\mathbf{j} + 2\sqrt{2}\mathbf{k},$$

$$\mathbf{t} = 5\mathbf{j} - 3\mathbf{k}, \quad \mathbf{u} = \mathbf{i} + 3\mathbf{j},$$

find the following.

7. $\mathbf{r} + \mathbf{s}$ **8.** $\mathbf{s} + \mathbf{t}$ **9.** $\mathbf{u} + \mathbf{s} + \mathbf{t}$

10. $\frac{3}{4}\mathbf{r} + \frac{3}{4}\mathbf{t}$ **11.** $\mathbf{r} + 7\mathbf{u}$ **12.** $\frac{1}{2}(\mathbf{s} + 3\mathbf{u})$

Set B

Find the magnitude of the following vectors.

13. $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

14. $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

15. $\mathbf{r} = 3\mathbf{i} + (-4)\mathbf{j} + \mathbf{k}$

16. $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{(-2)}{5}\mathbf{j} + \frac{2\sqrt{3}}{5}\mathbf{k}$

17. $\mathbf{t} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Find real numbers x, y, and z such that

18. $x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k} + 3\mathbf{i} - \mathbf{j} = 4\mathbf{i} + 3\mathbf{k}$.

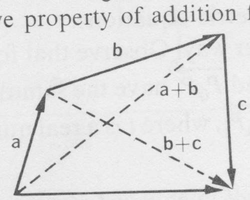
19. $7x\mathbf{i} + (y - 3)\mathbf{j} + 6\mathbf{k} = 10\mathbf{i} + 8\mathbf{j} - 3z\mathbf{k}$.

20. $(x + 4)\mathbf{i} + (y - 5)\mathbf{j} + (z - 1)\mathbf{k} = 0$.

21. Verify that $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ for the vectors

$$\mathbf{a} = 2\mathbf{i} - \tfrac{1}{2}\mathbf{j} + 5\mathbf{k},$$
$$\mathbf{b} = \mathbf{i} - \mathbf{j} - 2\mathbf{k},$$
$$\mathbf{c} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

22. Explain how the figure below illustrates the associative property of addition for vectors.



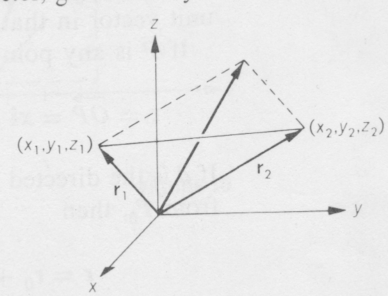
Set C

23. Prove part (b) of Theorem 14–5.

24. Prove part (d) of Theorem 14–5.

25. Let $\mathbf{v}$ be a position vector such that $|\mathbf{v}| = L$. Given that the direction cosines of $\mathbf{v}$ are c_1, c_2, and c_3, show that $\mathbf{v} = L(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$.

26. In the figure, $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$. What are the components of $\frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$? What is this vector, geometrically?



27. Three vectors with magnitudes 5, 12, and 13 have a sum equal to **0**. Illustrate how this is possible.

28. If $\mathbf{u} = \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$ are given, show how to find $\mathbf{a}$ and $\mathbf{b}$ geometrically. [*Hint:* What is $\mathbf{u} + \mathbf{v}$? What is $\mathbf{u} - \mathbf{v}$?]

29. If $P_1 = (3, -2, 2)$ and $P_2 = (4, 5, -1)$, show that $\overrightarrow{OP_1} \perp \overrightarrow{OP_2}$. [*Hint:* Use the distance formula and the converse of the Pythagorean Theorem.]

Calculator Problem

If $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and

$$\mathbf{a} = 1.2\mathbf{i} + 3.3\mathbf{j} - 6.5\mathbf{k},$$
$$\mathbf{b} = -0.9\mathbf{i} - 1.7\mathbf{j} + 2.1\mathbf{k},$$

find the components of $\mathbf{c}$, the magnitude of $\mathbf{c}$, and the direction cosines of $\mathbf{c}$.