Example The coordinates of P_0 and P_1 are $(3, -2, 4)$ and $(-1, 3, 7)$, respectively. Find a vector equation of the line through P_0 and P_1. (Fig. 10.)

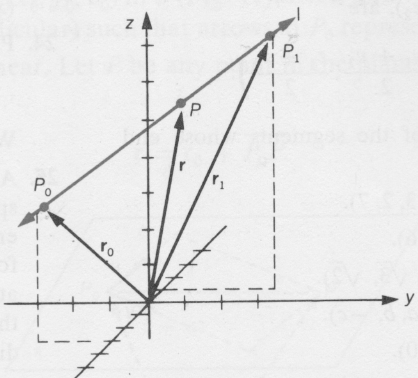


FIGURE 10

Solution. We have

$$\overrightarrow{OP_0} = \mathbf{r}_0 = 3\mathbf{i} + (-2)\mathbf{j} + 4\mathbf{k}$$

and

$$\begin{aligned}\overrightarrow{P_0P_1} &= (-1 - 3)\mathbf{i} + (3 + 2)\mathbf{j} + (7 - 4)\mathbf{k}.\\ &= -4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}\end{aligned}$$

Therefore the vector $\mathbf{r}$ to any point P on the line P_0P_1 is

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_0 + t\overrightarrow{P_0P_1}\\ &= 3\mathbf{i} + (-2)\mathbf{j} + 4\mathbf{k} + t(-4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})\\ &= (3 - 4t)\mathbf{i} + (-2 + 5t)\mathbf{j} + (4 + 3t)\mathbf{k}.\end{aligned}$$

Problems

Set A

Determine the vector equation of the line through each of the following pairs of points. (Let P_0 be the first point listed.)

1. $(0, 1, 2)$, $(3, 0, 2)$

2. $(3, -1, 5)$, $(2, 0, -5)$

3. $(4, 4, -2)$, $(-3, 2, -1)$

4. $(-3, -1, -3)$, $(3, 1, 3)$

5. (a_1, b_1, c_1), (a_2, b_2, c_2)

6. A line contains the point $(3, 2, 1)$ and is perpendicular to the xy-plane. Find a vector equation of the line.

A vector equation of a line is given by

$$\mathbf{r} = (4 + 2t)\mathbf{i} + (-1 + t)\mathbf{j} + (-3 + 4t)\mathbf{k}.$$

Find the coordinates of the terminal point of $\mathbf{r}$ for the following values of the parameter t.

7. 0 **8.** 1 **9.** -1

10. 2 **11.** $\frac{1}{2}$ **12.** $\frac{2}{3}$

Set B

13. Prove that the coordinates of the midpoint of a segment $\overline{P_1P_2}$, where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Find the midpoints of the segments whose end points are

14. $(-1, 5, 2)$ and $(-3, 2, 7)$.
15. $(1, 2, 3)$ and $(4, 5, 6)$.
16. $(\pi, \pi/2, 2)$ and $(3, \sqrt{3}, \sqrt{2})$.
17. $(a, -b, c)$ and $(-a, b, -c)$.
18. $(a, 0, 0)$ and $(0, b, 0)$.
19. $(0, 0, 0)$ and (a, b, c).
20. A point P is between $P_0 = (-1, 3, 2)$ and $P_1 = (5, 2, 5)$ and is twice as far from P_0 as P_1. Find the coordinates of P.
21. A point P is on the line containing $P_0 = (-2, -4, 1)$ and $P_1 = (3, 2, 2)$. If P_1 is the midpoint of the segment $\overline{P_0P}$, what are the coordinates of P?
22. A line contains the points $P_0 = (5, 5, 3)$ and $P_1 = (1, -3, 8)$. Find all possible coordinates of a point P that is twice as far from P_1 as it is from P_0.

Set C

23. Obtain the coordinates of the points that trisect the segment whose end points are (x_1, y_1, z_1) and (x_2, y_2, z_2).
24. Parametric equations of a line are

$$x = -2 + \tfrac{2}{3}d,\ y = 5 + \tfrac{1}{3}d,\ z = 1 - \tfrac{2}{3}d.$$

Write a vector equation of the line.

25. A point "moves" on a line with constant speed. At time $t = 0$, it is at $(0, 0, 0)$. At the end of one second it is at $(2, 4, \sqrt{5})$. Give a formula for the position vector to the point at the end of t seconds. What is the speed in the x-direction? in the y-direction? in the z-direction?
26. The midpoints of the sides of a triangle are (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3). Using vectors, find the coordinates of the vertices of the triangle.

Calculator Problem

Find, as decimals, the midpoint of a segment whose end points are

$$(\sqrt{11}, \ln 29, \sin 4.7)$$

and

$$\left(\frac{\pi}{5}, e^{3.2}, \cos\frac{15\pi}{17}\right).$$

Problems

Set A

1. If $\mathbf{v}$ has direction cosines

$$c_1 = \frac{1}{2}, \qquad c_2 = -\frac{1}{2} \qquad \text{and} \qquad c_3 = \frac{1}{\sqrt{2}}$$

and $\mathbf{v}'$ has direction cosines

$$c_1' = \frac{1}{3}, \qquad c_2' = \frac{5}{6} \qquad \text{and} \qquad c_3' = -\frac{\sqrt{6}}{6},$$

find the cosine of the angle between $\mathbf{v}$ and $\mathbf{v}'$.

Find the cosine of the angle between the following pairs of vectors.

2. $(\mathbf{i} + \mathbf{k})$ and $(\mathbf{j} + \mathbf{k})$
3. $(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
4. $(-\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ and $(3\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
5. $(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k})$ and $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Find the cosine of the angle between two position vectors whose end points are given below.

6. $(3, 1, 1), (-2, 4, -2)$
7. $(-4, 0, 3), (0, 6, -3)$
8. $(2, 4, 3), (2, -4, 3)$
9. $(x, y, z), (-x, -y, -z)$

Set B

10. Show that the angle between the unit basis vectors $\mathbf{i}$ and $\mathbf{j}$ is $\pi/2$.
11. Show that the angle between $\mathbf{i}$ and $-\mathbf{i}$ is π.

Find the cosine of the angle between the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ if P_1, P_2, P_3, and P_4 are, in order, the following points.

12. $(0, -1, 3), (-2, 1, 2), (-10, -2, -1), (10, 3, 3)$
13. $(-2, -3, 1), (0, 1, 2), (-5, -2, 1), (-2, 2, 3)$
14. $(7, -3, 1), (4, 2, -2), (6, 0, -3), (-4, -4, 4)$
15. $(5, 3, 4), (0, 0, 0), (-3, -2, -1), (1, 2, 3)$
16. Find the measure of the angles between $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and each of the positive coordinate axes.
17. Find the measure of angle XYZ if the coordinates of X, Y, and Z are $(5, 3, -1)$, $(4, -2, 6)$ and $(10, 10, 10)$ respectively.
18. Find the measure of the angle between $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $-4\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.
19. Find the measures of the angles between $\mathbf{v} = c\mathbf{i} + c\mathbf{j} + c\mathbf{k}$ and each of the positive coordinate axes.
20. Prove that $\mathbf{v}_1 = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ are perpendicular.

Set C

21. Show that the line through $(1, 3, 5)$ and $(3, 0, 1)$ is perpendicular to the line through $(1, 11, -1)$ and $(-5, 3, 2)$.
22. A triangle has vertices $(-1, -1, 2)$, $(3, 2, -1)$, and $(0, 4, 3)$. Find the measure of the angle whose vertex is $(-1, -1, 2)$.
23. Prove that the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, neither of which is the null vector, are perpendicular if and only if $a_1b_1 + a_2b_2 + a_3b_3 = 0$.
24. A directed line has direction cosines $1/2$, $-\sqrt{3}/2$, 0. Find direction cosines of a directed line in the xy-plane perpendicular to the given line.
25. A directed line parallel to the xy-plane has direction cosines l, m, and 0. Show that a directed line perpendicular to the given line and also parallel to the xy-plane has direction cosines $\pm m$, $\mp l$, 0.
26. Show that the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are vertices of a right triangle with the right angle at (x_1, y_1, z_1) if and only if

$$(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1) + (z_2 - z_1)(z_3 - z_1) = 0.$$

-4, -3, 4 $\quad \frac{4}{6} \quad \frac{4}{9} + \frac{4}{9} + \frac{1}{9}$ $\quad$ -1

Theorem 15–4 ***For all vectors* u, v, *and* w,**

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \qquad \textit{(commutative law)},$$

and

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w}) \qquad \textit{(distributive law)}.$$

The theorem is an easy consequence of Theorem 15–2, and its proof is left to the student. Observe that there is no vector that acts as a multiplicative identity for dot products as the number 1 does for ordinary multiplication. Furthermore, there is no associative law for dot products.

Problems

Set A

Compute the dot product of the following pairs of vectors.

1. $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} - \mathbf{j} - \mathbf{k}$

2. $\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

3. $3\mathbf{i} + \mathbf{j} - \mathbf{k}$, $-3\mathbf{i} - \mathbf{j} + \mathbf{k}$

4. $6\mathbf{i} - 9\mathbf{j} + 3\mathbf{k}$, $\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

5. $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

6. $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$

Let $\mathbf{n}$ be a unit vector and $\mathbf{u} = -6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. What is the length of the projection of $\mathbf{u}$ on $\mathbf{n}$ if the measure of the angle between the two vectors is

7. 0. **8.** $\pi/6$. **9.** $\pi/4$. **10.** $\pi/3$.

11. $\pi/2$. **12.** $2\pi/3$. **13.** π. **14.** $5\pi/6$.

Set B

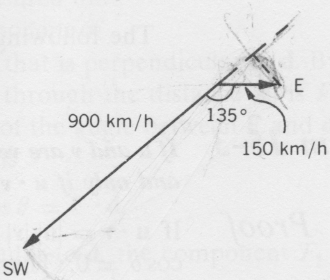
15. If $\mathbf{u} = \mathbf{i} + (-2)\mathbf{j} + 2\mathbf{k}$, what is $\mathbf{u} \cdot \mathbf{i}$? $\mathbf{u} \cdot \mathbf{j}$? $\mathbf{u} \cdot \mathbf{k}$?

16. Compute $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ where $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

17. Compute $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ using vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of Problem 16.

18. A force of 20 newtons moves an object along a line which forms an angle of 30° with the direction of the force. What is the component of the force along the line? What is the component of the force perpendicular to the line? How much work is done if the object is moved 8 m along the line?

19. A jet airplane flies southwest at 900 km/h in still air. The airplane encounters a jetstream of air moving due east at 150 km/h. What is the component of the jetstream along the line of flight of the plane? What will be the speed of the plane due to the wind?



20. The unit vector $\mathbf{n} = \frac{3}{7}\mathbf{i} + (-\frac{6}{7}\mathbf{j}) + \frac{2}{7}\mathbf{k}$ is perpendicular to a plane, π. The point $P = (3, 1, 4)$ is in the plane. Find $\overrightarrow{OP} \cdot \mathbf{n}$. What does the dot product represent geometrically? What is the distance from the origin O to the plane at Q?