Algebra Through Problem Solving

by

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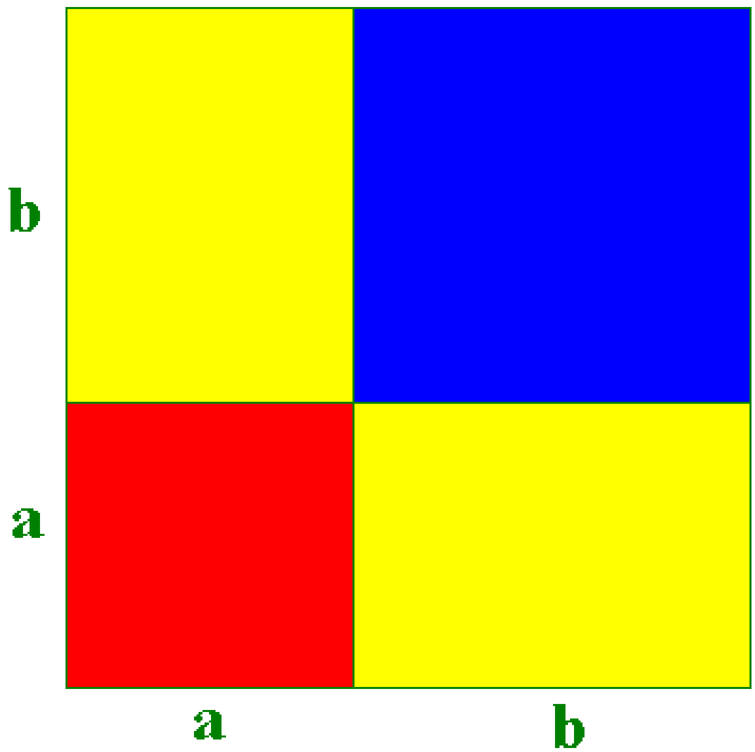

$$(a + b)^2 = ???$$

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To

Josephine Hillman

PREFACE

This book is an outgrowth of the authors' work in conducting problem solving seminars for undergraduates and high school teachers, in directing mathematics contests for undergraduates and high school students, and in the supervision of an undergraduate research participation program. Their experience has shown that interest in and knowledge of mathematics can be greatly strengthened by an opportunity to acquire some basic problem solving techniques and to apply these techniques to challenging problems for which the prerequisite knowledge is available.

Many students who have not had this opportunity lose confidence in themselves when they try unsuccessfully to solve non-routine problems such as those in the Mathematics Magazine or in the Putnam Intercollegiate Mathematics Competitions, conducted by the Mathematical Association of America. Those who gain self-confidence by work on challenging material at the proper level also generally have increased motivation for mastering significant mathematical concepts and for making original contributions to mathematical knowledge.

The topics chosen for this book are particularly appropriate since they are at a fairly elementary level and exhibit the interdependence of mathematical concepts. Many generalizations are suggested in the problems; the perceptive reader will be able to discover more.

The authors express their debt to all who have influenced this effort. We are especially grateful to Leonard Klosinski, Roseanna Torretto, and Josephine Hillman for their invaluable assistance.

Albuquerque, New Mexico — Abraham P. Hillman
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INTRODUCTION

Knowledge of mathematics together with the ability to apply this knowledge to non-routine problems will be very valuable in the more and more automated world we face. Routine problems will tend to be solved mechanically, while new and challenging problems arise at a rapid rate for human minds to solve.

We hope that this book will be helpful in the process of mastering some aspects of mathematics and becoming proficient in using this knowledge. The topics selected contain important ideas that are often lost in the regular curriculum. An effort has been made to develop most of the theoretical material through sequences of related and progressively more sophisticated problems that follow the necessary text material and illustrative examples. In later chapters some proofs of a considerably more involved nature are omitted or dealt with only in special cases. The sets of problems provide opportunities for recognizing mathematical patterns and for conjecturing generalizations of specific results.

And now some words of advice to prospective problem solvers: some of the problems of this book are easy, some may take longer to solve than any previously encountered, and some may prove to be too difficult. If a problem is difficult, it may be helpful to look at the surrounding problems. One does not have to do all of the problems in a chapter before going on. In fact, some of the hard problems will appear to be easier if one returns to them after progressing through later chapters. The statements in problems preceded by the symbol "R" are required for subsequent work and should be specially noted.

Answers or hints are provided for the odd-numbered problems except starred problems and those that contain the answer in the problem. However, the greatest benefit comes from trying the problems; one should postpone looking at the answers as long as possible.

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Chapter 1

THE PASCAL TRIANGLE

In modern mathematics, more and more stress is placed on the context in which statements are true. In elementary mathematics this generally means an emphasis on a clear understanding of which number systems possess certain properties. We begin, then, by describing briefly the number systems with which we will be concerned. These number systems have developed through successive enlargements of previous systems.

At one time a "number" meant one of the **natural numbers**: 1,2,3,4,5,6,... . The next numbers to be introduced were the fractions:1/2, 1/3, 2/3, 1/4, 3/4, 1/5,..., and later the set of numbers was expanded to include zero and the negative integers and fractions. The number system consisting of zero and the positive and negative integers and fractions is called the system of **rational numbers**, the word "rational" being used to indicate that the numbers are ratios of integers. The integers themselves can be thought of as ratios of integers since 1 = 1/1, -1 = -1/1, 2 = 2/1, -2 = -2/1, 3 = 3/1, etc.

The need to enlarge the rational number system became evident when mathematicians proved that certain constructible lengths, such as the length $\sqrt{2}$ of a diagonal of a unit square, *cannot* be expressed as rational numbers. The system of **real numbers** then came into use. The real numbers include all the natural numbers; all the fractions; numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{(2+\sqrt{6})/3}$ which represent constructible lengths; numbers such as $\sqrt[3]{2}$ and π which do not represent lengths constructible from a given unit with compass and straightedge; and the negatives of all these numbers. Modern technology and science make great use of a still larger number system, called the **complex numbers**, consisting of the numbers of the form $a + bi$ with a and b real numbers and $i^2 = -1$.

Our first topic is the Pascal Triangle, an infinite array of natural numbers. We begin by considering expansions of the powers $(a + b)^n$ of a sum of two terms. Clearly, $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$. Then $(a + b)^3 = (a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b)$. We expand this last expression as the sum of all products of a term of $a^2 + 2ab + b^2$ by a term of $a + b$ in the following manner:

$$
\begin{array}{llll}
a^2 & + 2ab & + b^2 & \\
a & + b & & \\
\hline
a^3 & + 2a^2b & + ab^2 & \\
 & a^2b & + 2ab^2 & + b^3 \\
\hline
a^3 & + 3a^2b & + 3ab^2 & + b^3
\end{array}
$$

Hence $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. If $a \neq 0$ and $b \neq 0$ (a is not equal to zero and b is not equal to zero), this may be written

$$(a + b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

The terms of the expanded form are such that the exponent for a starts as 3 and decreases by one each time, while the exponent of b starts as 0 and increases by one each time. Thus the sum of the exponents is 3 in each term.

One might guess that by analogy the expansion of $(a + b)^4$ involves a^4, a^3b, a^2b^2, ab^3, and b^4. This is verified by expanding $(a + b)^4 = (a + b)^3(a + b) = (a^3 + 3a^2b + 3ab^2 + b^3)(a + b)$ as follows:

(1)	a^3	$+ 3a^2b$	$+ 3ab^2$	$+ b^3$	
(2)	a	$+ b$			
(3)	a^4	$+ 3a^3b$	$+ 3a^2b^2$	$+ ab^3$	
(4)		a^3b	$+ 3a^2b^2$	$+ 3ab^3$	$+ b^4$
(5)	a^4	$+ 4a^3b$	$+ 6a^2b^2$	$+ 4ab^3$	$+ b^4$

Thus we see that a^4, a^3b, a^2b^2, ab^3, and b^4 are multiplied by 1, 4, 6, 4, 1 to form the terms of the expansion. The numbers 1, 4, 6, 4, 1 are the coefficients of the expansion. Examination of expressions (1) to (5), above, shows that these coefficients are obtainable from the coefficients 1, 3, 3, 1 of $(a + b)^3$ by means of the following condensed versions of (3), (4), and (5):

(3*)	1	3	3	1	
(4*)		1	3	3	1
(5*)	1	4	6	4	1

We now tabulate the coefficients of $(a + b)^n$ for $n = 0, 1, 2, 3, 4, \ldots$ in a triangular array:

n	Coefficients of $(a + b)^n$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
...	. . . . . .

One may observe that the array is bordered with 1's and that each number inside the border is the sum of the two closest numbers on the previous line. This observaion simplifies the generation of additional lines of the array. For example, the coefficients for $n = 5$ are 1, 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, 4 + 1 = 5, and 1.

The above triangular array is called the **Pascal Triangle** in honor of the mathematician Blaise Pascal (1623-1662). A notation for the coefficients of $(a + b)^n$ is

$$\binom{n}{0}, \binom{n}{1}, \ \ldots \ , \binom{n}{n}. \tag{6}$$

For example, one writes $(a + b)^4$ in this notation as

$$\binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

where $\binom{4}{0} = 1 = \binom{4}{4}$, $\binom{4}{1} = 4 = \binom{4}{3}$, and $\binom{4}{2} = 6$.

A two-term expression is called a **binomial**, and an expansion for an expression such as $(a + b)^n$ is called a **binomial expansion**. The coefficients listed in (6) above are called **binomial coefficients**.

Note that the symbol $\binom{n}{k}$ denotes the coefficient of $a^{n-k}b^k$, or of a^kb^{n-k}, in the expansion of $(a + b)^n$. Thus $\binom{3}{1}$ is the coefficient 3 of a^2b or of ab^2 in the expansion of $(a + b)^3$,and $\binom{4}{2}$ is the coefficient 6 of x^2y^2 in $(x + y)^4$. One reads $\binom{n}{k}$ as "binomial coefficient n choose k" or simply as "n choose k." The reason for this terminology is given in Chapter 7.

In Figure 1, (on page 4) we see how n and k give us the location of $\binom{n}{k}$ in the Pascal Triangle. The number n in $\binom{n}{k}$ is the row number and k is the diagonal number if one adopts the convention of labeling the rows or diagonals as 0, 1, 2, ...

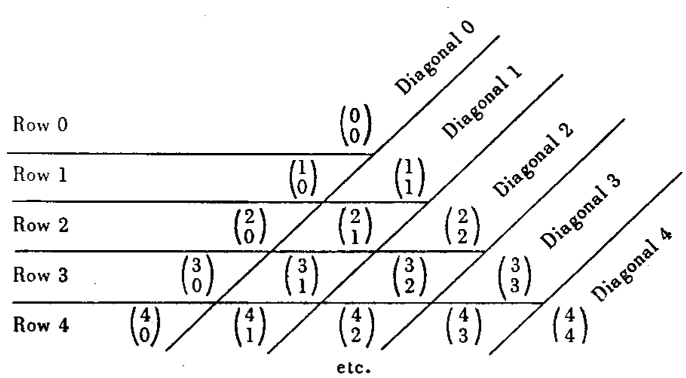

FIGURE 1

The formulas $\binom{n}{0} = 1 = \binom{n}{n}$ recall the fact that the Pascal Triangle is bordered with 1's. The rule that each number inside the border of 1's in the Pascal Triangle is the sum of the two closest entries on the previous line may be written as

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Example 1. Expand $(2x + 3y^2)^3$.

Solution: The expansion $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is an identity which remains true when one substitutes $a = 2x$ and $b = 3y^2$ and thus obtains

$$\begin{aligned}(2x + 3y^2)^3 &= (2x)^3 + 3(2x)^2(3y^2) + 3(2x)(3y^2)^2 + (3y^2)^3 \\ &= 8x^3 + 3(4x^2)(3y^2) + 3(2x)(9y^4) + 27\, y^6 \\ &= 8x^3 + 36x^2y^2 + 54xy^4 + 27y^6.\end{aligned}$$

Example 2. Show that $\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \binom{7}{3} + \binom{8}{4} = \binom{9}{4}$.

Solution: Using the fact that $\binom{4}{0} = 1 = \binom{5}{0}$ and the formula

$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ we see that

$$\binom{8}{4} + \binom{7}{3} + \binom{6}{2} + \binom{5}{1} + \binom{4}{0} = \binom{8}{4} + \binom{7}{3} + \binom{6}{2} + \binom{5}{1} + \binom{5}{0}$$

$$= \binom{8}{4} + \binom{7}{3} + \binom{6}{2} + \binom{6}{1}$$

$$= \binom{8}{4} + \binom{7}{3} + \binom{7}{2}$$

$$= \binom{8}{4} + \binom{8}{3}$$

$$= \binom{9}{4}.$$

Problems for Chapter 1

1. Give the value of $\binom{5}{2}$, that is, of the coefficient of a^3b^2 in $(a + b)^5$.

2. Give the value of $\binom{5}{4}$.

3. Find s if $\binom{5}{4} = \binom{5}{s}$ and s is not 4.

4. Find t if $\binom{5}{t} = \binom{5}{0}$ and t is not 0.

5. Obtain the binomial coefficients for $(a + b)^3$ from those for $(a + b)^2$ in the style of lines (3*), (4*), (5*) on page 2.

6. Obtain the binomial coefficients for $(a + b)^6$ from those for $(a + b)^5$ in the style of lines (3*), (4*), (5*) on page 2.

7. Generate the lines of the Pascal Triangle for $n = 6$ and $n = 7$, using the technique described at the top of page 3.

8. Find $\binom{8}{0}$, $\binom{8}{1}$, $\binom{8}{2}$, $\binom{8}{3}$, and $\binom{8}{4}$.

9. Use $\binom{9}{1} = 9$ and $\binom{9}{2} = 36$ to find $\binom{9}{7}$ and $\binom{9}{8}$.

10. Use $\binom{9}{1} = 9$ and $\binom{9}{2} = 36$ to find $\binom{10}{2}$ and $\binom{10}{8}$.

11. Expand $(5x + 2y)^3$.

12. Expand $(x^2 - 4y^2)^3$ by letting $a = x^2$ and $b = -4y^2$ in the expansion of $(a + b)^3$.

13. Show that:

(a) $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

(b) $(x - y)^5 = \binom{5}{0}x^5 - \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 - \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 - \binom{5}{5}y^5$.

14. Show that:

(a) $(x + 1)^6 + (x - 1)^6 = 2\left[\binom{6}{0}x^6 + \binom{6}{2}x^4 + \binom{6}{4}x^2 + \binom{6}{6}\right]$.

(b) $(x + y)^6 - (x - y)^6 = 2\left[\binom{6}{1}x^5y + \binom{6}{3}x^3y^3 + \binom{6}{5}xy^5\right]$.

15. Show that $(x + h)^3 - x^3 = h(3x^2 + 3xh + h^2)$.

16. Show that $(x + h)^{100} - x^{100} = h\left[\binom{100}{1}x^{99} + \binom{100}{2}x^{98}h + \binom{100}{3}x^{97}h^2 + \ldots + \binom{100}{100}h^{99}\right]$.

17. Find numerical values of c and m such that cx^3y^m is a term of the expansion of $(x + y)^8$.

18. Find d and n such that dx^5y^4 is a term of $(x + y)^n$.

19. Find each of the following:

(a) $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$.

(b) $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$.

(c) $2\binom{4}{0} + 2\binom{4}{1} + \binom{4}{2}$.

(d) $2\left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2}\right]$.

(e) $2\left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2}\right] + \binom{6}{3}$.

(f) $2\left[\binom{7}{7} + \binom{7}{6} + \binom{7}{5} + \binom{7}{4}\right]$.

20. Find the sum of the 101 binomial coefficients for $n = 100$ by assigning specific values to a and b in the identity

$$(a+b)^{100} = \binom{100}{0}a^{100} + \binom{100}{1}a^{99}b + \binom{100}{2}a^{98}b^2 + \ldots$$
$$+ \binom{100}{98}a^2b^{98} + \binom{100}{99}ab^{99} + \binom{100}{100}b^{100}.$$

21. Find each of the following:

(a) $\binom{2}{0} - \binom{2}{1} + \binom{2}{2}$.

(b) $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$.

22. Find each of the following:

(a) $\binom{100}{0} - \binom{100}{1} + \binom{100}{2} - \binom{100}{3} + \ldots - \binom{100}{99} + \binom{100}{100}$.

(b) $\binom{101}{0} - \binom{101}{1} + \binom{101}{2} - \binom{101}{3} + \ldots + \binom{101}{100} - \binom{101}{101}$.

23. Find each of the following:

(a) $\binom{4}{0} + \binom{4}{2} + \binom{4}{4}$.

(b) $\binom{5}{0} + \binom{5}{2} + \binom{5}{4}$.

(c) $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}$.

(d) $\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7}$.

24. Find each of the following:

(a) $\binom{1000}{0} + \binom{1000}{2} + \binom{1000}{4} + \ldots + \binom{1000}{1000}$.

(b) $\binom{1000}{1} + \binom{1000}{3} + \binom{1000}{5} + \ldots + \binom{1000}{999}$.

25. Find r, s, t, and u, given the following:

(a) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{r}{3}$.

(b) $\binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \binom{6}{4} = \binom{s}{4}$.

(c) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} = \binom{t}{2}$.

(d) $\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} = \binom{u}{3}$.

26. Express $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \cdots + \binom{100}{4}$ as a binomial coefficient.

27. Express $\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \cdots + \binom{995}{990}$ as a binomial coefficient.

28. Show that

(a) $n = \binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \binom{3}{0} + \cdots + \binom{n-1}{0} = \binom{n}{1}$.

(b) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$.

(c) $\binom{n}{k} + 2\binom{n}{k+1} + \binom{n}{k+2} = \binom{n+2}{k+2}$ for $0 \le k \le n - 2$.

29. Show that $\binom{9}{4}\binom{5}{3} = \binom{9}{3}\binom{6}{4} = \binom{9}{2}\binom{7}{3}$.

30. Show that $\binom{10}{1}\binom{9}{2}\binom{7}{3} = \binom{10}{4}\binom{6}{3}\binom{3}{2} = \binom{10}{2}\binom{8}{4}\binom{4}{1}$.

31. Expand $(x + y + z)^4$ by expanding $(w + z)^4$, then replacing w by $x + y$, and expanding further.

32. Expand $(x + y - z)^4$.

33. The sum of squares $\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$ is expressible in the form $\binom{2m}{m}$.

Find m.

34. Express each of the following in the form $\binom{2m}{m}$:

(a) $\binom{4}{0}\binom{4}{4} + \binom{4}{1}\binom{4}{3} + \binom{4}{2}\binom{4}{2} + \binom{4}{3}\binom{4}{1} + \binom{4}{4}\binom{4}{0}$.

(b) $2\left[\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2\right]$.

(c) $\binom{6}{3}^2 + 2\left[\binom{6}{2}^2 + \binom{6}{1}^2 + \binom{6}{0}^2\right]$.

35. Use $(x^3 + 3x^2 + 3x + 1)^2 = [(x + 1)^3]^2 = (x + 1)^6$ to show the following:

(a) $\binom{3}{0}^2 = \binom{6}{0}$.

(b) $\binom{3}{0}\binom{3}{1} + \binom{3}{1}\binom{3}{0} = \binom{6}{1}$.

(c) $\binom{3}{0}\binom{3}{2} + \binom{3}{1}\binom{3}{1} + \binom{3}{2}\binom{3}{0} = \binom{6}{2}$.

(d) $\binom{3}{0}\binom{3}{3} + \binom{3}{1}\binom{3}{2} + \binom{3}{2}\binom{3}{1} + \binom{3}{3}\binom{3}{0} = \binom{6}{3}$.

36. Express $\binom{100}{0}^2 + \binom{100}{1}^2 + \binom{100}{2}^2 + \cdots + \binom{100}{100}^2$ in the form $\binom{2m}{m}$.

37. How many of the 3 binomial coefficients $\binom{n}{k}$ with n = 0 or 1 are odd?

38. How many of the 10 binomial coefficients $\binom{n}{k}$ with n = 0, 1, 2, or 3 are odd?

39. How many binomial coefficients $\binom{n}{k}$ are there with n = 0, 1, 2, 3, 4, 5, 6, or 7, and how many of these are odd?

40. Show that all eight of the coefficients $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \ldots, \binom{7}{7}$ are odd.

41. For what values of *n* among 0, 1, 2,..., 20 are all the coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ on row *n* of the Pascal Triangle odd?

42. If *n* is an answer to the previous problem, how many of the binomial coefficients on row *n* + 1 of the Pascal Triangle are odd?

*43. How many binomial coefficients $\binom{n}{k}$ are there with n = 0, 1, 2,..., 1022, or 1023, and how many of these are even?

Chapter 2

THE FIBONACCI AND LUCAS NUMBERS

The great Italian mathematician, Leonardo of Pisa (c. 1170-1250), who is known today as Fibonacci (an abbreviation of filius Bonacci), expanded on the Arabic algebra of North Africa and introduced algebra into Europe. The solution of a problem in his book *Liber Abacci* uses the sequence

(F) $\qquad 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,$ ÿ

One of the many applications of this ***Fibonacci sequence*** is a theorem about the number of steps in an algorithm for finding the greatest common divisor of a pair of large integers. We study the sequence here because it provides a wonderful opportunity for discovering mathematical patterns.

The numbers shown in (F) are just the beginning of the unending Fibonacci sequence. The rule for obtaining more terms is as follows:

RECURSIVE PROPERTY. The sum of two consecutive terms in (F) is the term immediately after them.

For example, the term after 55 in (F) is 34 + 55 = 89 and the term after that is 55 + 89 = 144.

To aid in stating properties of the Fibonacci sequence, we use the customary notation F_0, F_1, F_2, F_3, ÿ for the integers of the Fibonacci sequence. That is, $F_0 = 0$, $F_1 = 1$, $F_2 = F_0 + F_1 = 1$, $F_3 = F_1 + F_2 = 2$, $F_4 = F_2 + F_3 = 3$, $F_5 = F_3 + F_4 = 5$, $F_6 = F_4 + F_5 = 8$, and so on. When F_n stands for some term of the sequence, the term just after F_n is represented by F_{n+1}, the term after F_{n+1} is F_{n+2}, and so on. Also, the term just before F_n is F_{n-1}, the term just before F_{n-1} is F_{n-2} and so on.

We now can define the Fibonacci numbers formally as the sequence F_0, F_1, ÿ having the two following properties.

INITIAL CONDITIONS $F_0 = 0$ and $F_1 = 1$.

RECURSION RULE $F_n + F_{n+1} = F_{n+2}$ for $n = 0, 1, 2,$ ÿ.

Next let S_n stand for the sum of the Fibonacci numbers from F_0 through F_n. That is,

$$\begin{array}{lll}
S_0 = F_0 = 0 & & \\
S_1 = F_0 + F_1 & = 0 + 1 & = 1 \\
S_2 = F_0 + F_1 + F_2 & = 0 + 1 + 1 = 2 & \\
S_3 = F_0 + F_1 + F_2 + F_3 & = S_2 + F_3 & = 2 + 2 = 4
\end{array}$$

and in general, $S_n = F_0 + F_1 + F_2 +$ ÿ $+ F_n = S_{n-1} + F_n$.

We tabulate some of the values and look for a pattern.

n	0	1	2	3	4	5	6	7	$\cdots$
F_n	0	1	1	2	3	5	8	13	$\cdots$
S_n	0	1	2	4	7	12	20	33	$\cdots$

Is there a relationship between the numbers on the third line of this table and the Fibonacci numbers? One pattern is that each of the terms of the sequence $S_0, S_1, S_2, \ldots$ is 1 less than a Fibonacci number. Specifically, we have

$$\begin{aligned} S_0 &= F_2 - 1 = 0 \\ S_1 &= F_3 - 1 = 1 \\ S_2 &= F_4 - 1 = 2 \\ S_3 &= F_5 - 1 = 4 \\ S_4 &= F_6 - 1 = 7 \\ S_5 &= F_7 - 1 = 12 \end{aligned}$$

and might conjecture that $S_n = F_{n+2} - 1$ for $n = 0, 1, 2, \ldots$. Does this formula hold for $n = 6$? Yes, since

$$\begin{aligned} S_6 &= F_0 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6 \\ &= S_5 + F_6 \\ &= (F_7 - 1) + F_6 \\ &= (F_6 + F_7) - 1 \\ &= F_8 - 1. \end{aligned}$$

The first steps in proving our conjecture correct for all the terms in the unending sequence $S_0, S_1, S_2, \ldots$ are rewriting the recursion formula $F_n + F_{n+1} = F_{n+2}$ as $F_n = F_{n+2} - F_{n+1}$ and then using this to replace each Fibonacci number in the sum S_n by a difference as follows:

$$\begin{aligned} S_n &= F_0 + F_1 + F_2 + \cdots + F_{n-1} + F_n \\ S_n &= (F_2 - F_1) + (F_3 - F_2) + (F_4 - F_3) + \cdots + (F_{n+1} - F_n) + (F_{n+2} - F_{n+1}). \end{aligned}$$

Next we rearrange the terms and get

$$\begin{aligned} S_n &= -F_1 + (F_2 - F_2) + (F_3 - F_3) + \cdots + (F_{n+1} - F_{n+1}) + F_{n+2} \\ S_n &= -F_1 + 0 + 0 + \cdots + 0 + F_{n+2} \\ S_n &= F_{n+2} - F_1 = F_{n+2} - 1. \end{aligned}$$

Thus we have made our conjecture (that is, educated guess) into a theorem.

The fundamental relation $F_n = F_{n+2} - F_{n+1}$ can also be used to define F_n when n is a negative integer. Letting $n = -1$ in this formula gives us $F_{-1} = F_1 - F_0 = 1 - 0 = 1$. Similarly, one finds that $F_{-2} = F_0 - F_{-1} = 0 - 1 = -1$ and $F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$. In this way one can obtain F_n for any negative integer n.

Some of the values of F_n for negative integers n are shown in the following table:

n	...	-6	-5	-4	-3	-2	-1
F_n	...	-8	5	-3	2	-1	1

Perhaps the greatest investigator of properties of the Fibonacci and related sequences was François Edouard Anatole Lucas (1842-1891). A sequence related to the F_n bears his name. The ***Lucas sequence***, 2, 1, 3, 4, 7, 11, 18, 29, 47, ..., is defined by

$$L_0 = 2,\ L_1 = 1,\ L_2 = L_1 + L_0,\ L_3 = L_2 + L_1,\ \ldots,\ L_{n+2} = L_{n+1} + L_n, \ldots\ .$$

Some of the many relations involving the F_n and the L_n are suggested in the problems below. These are only a very small fraction of the large number of known properties of the Fibonacci and Lucas numbers. In fact, there is a mathematical journal, *The Fibonacci Quarterly*, devoted to them and to related material.

Problems for Chapter 2

1. For the Fibonacci numbers F_n show that:

   (a) $F_3 = 2F_1 + F_0$. (b) $F_4 = 2F_2 + F_1$. (c) $F_5 = 2F_3 + F_2$.

2. The relation $F_{n+2} = F_{n+1} + F_n$ holds for all integers n and hence so does $F_{n+3} = F_{n+2} + F_{n+1}$. Combine these two formulas to find an expression for F_{n+3} in terms of F_{n+1} and F_n.

3. Find r, given that $F_r = 2F_{101} + F_{100}$.

4. Express $F_{157} + 2F_{158}$ in the form F_s.

5. Show the following:

   (a) $F_4 = 3F_1 + 2F_0$. (b) $F_5 = 3F_2 + 2F_1$.

6. Add corresponding sides of the formulas of the previous problem and use this to show that $F_6 = 3F_3 + 2F_2$.

7. Express F_{n+4} in terms of F_{n+1} and F_n.

8. Find s, given that $F_s = 3F_{200} + 2F_{199}$.

9. Find t, given that $F_t = 5F_{317} + 3F_{316}$.

10. Find numbers a and b such that $F_{n+6} = aF_{n+1} + bF_n$ for all integers n.

11. Show the following:

(a) $F_0 + F_2 + F_4 + F_6 = F_7 - 1$. (b) $F_0 + F_2 + F_4 + F_6 + F_8 = F_9 - 1$.
(c) $F_1 + F_3 + F_5 + F_7 = F_8$. (d) $F_1 + F_3 + F_5 + F_7 + F_9 = F_{10}$.

12. The relation $F_{n+2} = F_{n+1} + F_n$ can be rewritten as $F_{n+1} = F_{n+2} - F_n$. Use this form to find a compact expression for $F_a + F_{a+2} + F_{a+4} + F_{a+6} + ... + F_{a+2m}$.

13. Find p, given that $F_p = F_1 + F_3 + F_5 + F_7 + ... + F_{701}$.

14. Find u and v, given that $F_u - F_v = F_{200} + F_{202} + F_{204} + ... + F_{800}$.

15. Show the following:

(a) $F_4 = 3F_2 - F_0$. (b) $F_5 = 3F_3 - F_1$. (c) $F_6 = 3F_4 - F_2$.

16. Use the formulas $F_{n+4} = 3F_{n+1} + 2F_n$ and $F_{n+1} = F_{n+2} - F_n$ to express F_{n+4} in terms of F_{n+2} and F_n.

17. Show the following:

(a) $2(F_0 + F_3 + F_6 + F_9 + F_{12}) = F_{14} - 1$. (b) $2(F_0 + F_3 + F_6 + F_9 + F_{12} + F_{15}) = F_{17} - 1$.

18. Show the following:

(a) $2(F_1 + F_4 + F_7 + F_{10} + F_{13}) = F_{15}$. (b) $2(F_1 + F_4 + F_7 + F_{10} + F_{13} + F_{16}) = F_{18}$.

19. By addition of corresponding sides of formulas of the two previous problems, find expressions for:

(a) $2(F_2 + F_5 + F_8 + F_{11} + F_{14})$. (b) $2(F_2 + F_5 + F_8 + F_{11} + F_{14} + F_{17})$.

20. (i) Prove that $F_{n+3} - 4F_n - F_{n-3} = 0$ for $n \$ 3$.
(ii) Prove that $F_{n+4} - 7F_n + F_{n-4} = 0$ for $n \$ 4$.
*(iii) For $m \$ 4$, find a compact expression for $F_a + F_{a+4} + F_{a+8} + ... + F_{a+4m}$.

21. Evaluate each of the following sums:

(a) $\binom{2}{0} \% \binom{1}{1}$. (b) $\binom{3}{0} \% \binom{2}{1}$.

(c) $\binom{4}{0} \% \binom{3}{1} \% \binom{2}{2}$. (d) $\binom{5}{0} \% \binom{4}{1} \% \binom{3}{2}$.

(e) $\binom{6}{0} \% \binom{5}{1} \% \binom{4}{2} \% \binom{3}{3}$. (f) $\binom{7}{0} \% \binom{6}{1} \% \binom{5}{2} \% \binom{4}{3}$.

22. Find m, given that $\binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4} = F_m$.

23. Find r, s, and t given that:

(a) $\binom{2}{0}F_0 + \binom{2}{1}F_1 + \binom{2}{2}F_2 = F_r$. ($b$) $\binom{2}{0}F_1 + \binom{2}{1}F_2 + \binom{2}{2}F_3 = F_s$.

(c) $\binom{2}{0}F_7 + \binom{2}{1}F_8 + \binom{2}{2}F_9 = F_t$.

24. Find u, v, and w, given that:

(a) $\binom{3}{0}F_0 + \binom{3}{1}F_1 + \binom{3}{2}F_2 + \binom{3}{3}F_3 = F_u$.

(b) $\binom{3}{0}F_1 + \binom{3}{1}F_2 + \binom{3}{2}F_3 + \binom{3}{3}F_4 = F_v$.

(c) $\binom{3}{0}F_7 + \binom{3}{1}F_8 + \binom{3}{2}F_9 + \binom{3}{3}F_{10} = F_w$.

25. Find r,s, and t, given that:

(a) $(F_7)^2 + (F_8)^2 = F_r$. (b) $(F_8)^2 + (F_9)^2 = F_s$. (c) $(F_9)^2 + (F_{10})^2 = F_t$.

26. Find u, v, *and* w, given that:

(a) $(F_3)^2 - (F_2)^2 = F_u\,F_{u+3}$. ($b$) $(F_4)^2 - (F_3)^2 = F_v\,F_{v+3}$. ($c$) $(F_9)^2 - (F_8)^2 = F_w\,F_{w+3}$.

27. Let L_0, L_1, L_2 ,... be the Lucas sequence. Prove that

$$L_0 + L_1 + L_2 + L_3 + \ldots + L_n = L_{n+2} - 1.$$

28. Find r, given that $L_r = 2L_{100} + L_{99}$.

29. Find s, given that $L_s = 3L_{201} + 2L_{200}$.

30. Find t, given that $L_t = 8L_{999} + 5L_{998}$.

31. Show that $L_0 + L_2 + L_4 + L_6 + L_8 + L_{10} = L_{11} + 1$.

32. Find m, given that $L_0 + L_2 + L_4 + L_6 + \ldots + L_{400} = L_m + 1$.

33. Derive a formula for $L_1 + L_3 + L_5 + L_7 + \ldots + L_{2m+1}$.

34. Conjecture, and test in several cases, formulas for:

(a) $L_0 + L_3 + L_6 + L_9 + \ldots + L_{3m}$. (b) $L_1 + L_4 + L_7 + L_{10} + \ldots + L_{3m+1}$.

(c) $L_2 + L_5 + L_8 + L_{11} + \ldots + L_{3m+2}$. (d) $\binom{n}{0}L_k + \binom{n}{1}L_{k+1} + \binom{n}{2}L_{k+2} + \ldots + \binom{n}{n}L_{k+n}$.

35. Find r, s, and t, given that $F_2L_2 = F_r$, $F_3L_3 = F_s$, and $F_4L_4 = F_t$.

36. Find u, v, and w, given that $F_{10}/F_5 = L_u$, $F_{12}/F_6 = L_v$, and $F_{14}/F_7 = L_w$.

37. Evaluate the following:

(a) $(F_1)^2 - F_0F_2$. (b) $(F_2)^2 - F_1F_3$. (c) $(F_3)^2 - F_2F_4$. (d) $(F_4)^2 - F_3F_5$.

38. Evaluate the expressions of the previous problem with each Fibonacci number replaced by the corresponding Lucas number.

39. Which of the Fibonacci numbers F_{800}, F_{801}, F_{802}, F_{803}, F_{804}, and F_{805} are even?

40. Which of the Fibonacci numbers of the previous problem are exactly divisible by 3?

Chapter 3

FACTORIALS

A sequence which occurs frequently in mathematics is

$$1,\ 1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ \ldots .$$

We tabulate this in the form

n	0	1	2	3	4	5	6	...
$n!$	1	1	2	6	24	120	720	...

where the notation $n!$ (read as **n factorial**) is used for the number on the second line that is thus associated with n. Clearly $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc. The definition of $n!$ can be given as follows:

$$0! = 1,\ 1! = 1(0!),\ 2! = 2(1!),$$
$$3! = 3(2!),\ \ldots,\ (n+1)! = (n+1)(n!),\ \ldots .$$

The expression $n!$ is not defined for negative integers n. One reason is that the relation $(n + 1)! = (n + 1)(n!)$ becomes $1 = 0\cdot(-1)!$ when $n = -1$, and hence there is no way to define $(-1)!$ so that this relation is preserved.

Problems for Chapter 3

1. Find the following:

   (a) $7!$.
   (b) $(3!)^2$.
   (c) $(3^2)!$.
   (d) $(3!)!$.

2. Find the following:

   (a) $8!$.
   (b) $(2!)(3!)$.
   (c) $(2\cdot 3)!$.

3. Show that $\binom{5}{2}(2!)(3!) = 5!$ and $\binom{7}{3}(3!)(4!) = 7!$.

4. Find c and d, given that $\binom{6}{2}(2!)(4!) = c!$ and $\binom{8}{3}(3!)(5!) = d!$.

5. Write as a single factorial:

   (a) $3! \cdot 4 \cdot 5$.
   (b) $4! \cdot 210$.
   (c) $n!(n + 1)$.

6. Express $a!(a^2 + 3a + 2)$ as a single factorial.

7. Find a and b such that $11 \cdot 12 \cdot 13 \cdot 14 = a!/b!$.

8. Find e, given that $(n + e)!/n! = n^3 + 6n^2 + 11n + 6$.

9. Express $(n + 4)!/n!$ as a polynomial in n.

10. Find numbers $a, b, c, d,$ and e such that $(n + 5)!/n! = n^5 + an^4 + bn^3 + cn^2 + dn + e$.

11. Calculate the following sums:

    (a) $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3$.
    (b) $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + 4! \cdot 4$.
    (c) $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + 4! \cdot 4 + 5! \cdot 5$.

12. Conjecture a compact expression for the sum $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n$ and test it for several values of n.

13. Show that $(n + 1)! - n! = n! \cdot n$.

14. Show that $(n + 2)! - n! = n!(n^2 + 3n + 1)$.

15. Find numbers $a, b,$ and c such that $(n + 3)! - n! = n!(n^3 + an^2 + bn + c)$ holds for $n = 0, 1, 2, \ldots$.

16. Use the formula in Problem 13 to derive a compact expression for the sum in Problem 12.

17. Use the formula in Problem 14 to derive a compact expression for

$$0! + 11(2!) + 29(4!) + ... + (4m^2 + 6m + 1)[(2m)!].$$

18. Derive a compact expression for

$$5(1!) + 19(3!) + 41(5!) + ... + (4m^2 + 2m - 1)[(2m - 1)!].$$

19. Derive compact expressions for:

(a) $0! + 5(1!) + 11(2!) + ... + (n^2 + 3n + 1)(n!)$.

*(b) $0! + 2(1!) + 5(2!) + ... + (n^2 + 1)(n!)$.

20. Derive a compact expression for $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + ... + \dfrac{n}{(n+1)!}$

21. Show that:

(a) $6! = 3! \cdot 2^3 \cdot 3 \cdot 5$.

(b) $8! = 4! \cdot 2^4 \cdot 3 \cdot 5 \cdot 7$.

(c) $10! = 5! \cdot 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 9$.

22. Express r, s, and t in terms of m so that

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2m - 1) = r!/(s! \cdot 2^t).$$

Chapter 4

ARITHMETIC AND GEOMETRIC PROGRESSIONS

A finite sequence such as

$$2, 5, 8, 11, 14, \ldots , 101$$

in which each succeeding term is obtained by adding a fixed number to the preceding term is called an **arithmetic progression**. The general form of an arithmetic progression with n terms is therefore

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots ,\ a + (n - 1)d$$

where a is the first term and d is the fixed **difference** between successive terms.

In the arithmetic progression above, the first term is 2 and the common difference is 3. The second term is $2 + 3\cdot 1$. the third term is $2 + 3\cdot 2$. the fourth is $2 + 3\cdot 3$. and the nth is $2 + 3(n - 1)$. Since $101 - 2 = 99 = 3\cdot 33$ or $101 = 2 + 3(34 - 1)$, one has to add 3 thirty-three times to obtain the nth term. This shows that there are thirty-four terms here. The sum S of these thirty-four terms may be found by the following technique. We write the sum with the terms in the above order and also in reverse order, and add:

$$\begin{array}{rcccccccccc}
S & = & 2 & + & 5 & + & 8 & + \cdots + & 98 & + & 101 \\
S & = & 101 & + & 98 & + & 95 & + \cdots + & 5 & + & 2 \\
\hline
2S & = & (2 + 101) & + & (5 + 98) & + & (8 + 95) & + \cdots + & (98 + 5) & + & (101 + 2) \\
 & = & 103 & + & 103 & + & 103 & + \cdots + & 103 & + & 103 \\
2S & = & 34\cdot 103 & = & 3502. & & & & & &
\end{array}$$

Hence $S = 3502/2 = 1751$.

Using this method, one can show that the sum

$$T_n = 1 + 2 + 3 + 4 + \ldots + n$$

of the first n positive integers is $n(n + 1)/2$. Some values of T_n are given in the table which follows.

n	1	2	3	4	5	6	...
T_n	1	3	6	10	15	21	...

The sequence T_n may be defined for all positive integers n by

$$T_1 = 1,\ T_2 = T_1 + 2,\ T_3 = T_2 + 3,$$

$$T_4 = T_3 + 4,\ \ldots,\ T_{n+1} = T_n + (n + 1),\ \ldots\ .$$

The values 1, 3, 6, 10, 15, ... of T_n are called **triangular numbers** because they give the number of objects in triangular arrays of the type shown in Figure 2.

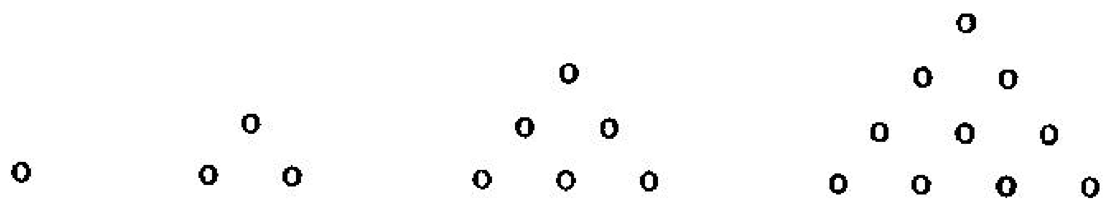

Figure 2

An arithmetic progression may have a negative common difference d. One with $a = 7/3$, $d = -5/3$, and $n = 8$ is:

$$7/3,\ 2/3,\ -1,\ -8/3,\ -13/3,\ -6,\ -23/3,\ -28/3.$$

The **average** (or **arithmetic mean**) of n numbers is their sum divided by n. For example, the average of 1, 3, and 7 is 11/3. If each of the terms of a sum is replaced by the average of the terms, the sum is not altered. We note that the average of all the terms of an arithmetic progression is the average of the first and last terms, and that the average is the middle term when the number of terms is odd, that is, whenever there is a middle term.

If a is the average of r and s, then it can easily be seen that r, a, s are consecutive terms of an arithmetic progression. (The proof is left to the reader.) This is why the average is also called the arithmetic mean.

A finite sequence such as

$$3,\ 6,\ 12,\ 24,\ 48,\ 96,\ 192,\ 384$$

in which each term after the first is obtained by multiplying the preceding term by a fixed number, is called a **geometric progression**. The general form of a geometric progression with n terms is therefore

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1}.$$

Here a is the first term and r is the fixed multiplier. The number r is called the **ratio** of the progression, since it is the ratio (i.e., quotient) of a term to the preceding term.

We now illustrate a useful technique for summing the terms of a geometric progression.

Example. Sum $5 + 5{\cdot}2^2 + 5{\cdot}2^4 + 5{\cdot}2^6 + 5{\cdot}2^8 + \ldots + 5{\cdot}2^{100}$.

Solution: Here the ratio r is $2^2 = 4$. We let S designate the desired sum and write S and rS as follows:

$$\begin{aligned} S &= 5 + 5{\cdot}2^2 + 5{\cdot}2^4 + 5{\cdot}2^6 + \ldots + 5{\cdot}2^{100} \\ 4S &= \phantom{5 + {}} 5{\cdot}2^2 + 5{\cdot}2^4 + 5{\cdot}2^6 + \ldots + 5{\cdot}2^{100} + 5{\cdot}2^{102}. \end{aligned}$$

Subtracting, we note that all but two terms on the right cancel out and we obtain

$$3S = 5{\cdot}2^{102} - 5$$

or

$$3S = 5(2^{102} - 1).$$

Hence we have the compact expression for the sum:

$$S = \frac{5(2^{102} - 1)}{3}.$$

If the ratio r is negative, the terms of the geometric progression alternate in signs. Such a progression with $a = 125$, $r = -1/5$, and $n = 8$ is

$$125, -25, 5, -1, 1/5, -1/25, 1/125, -1/625.$$

The geometric mean of two positive real numbers a and b is $\sqrt{ab}$, the positive square root of their product; the geometric mean of three positive numbers a, b, and c is $\sqrt[3]{abc}$. In general, the **<u>geometric mean</u>** of n positive numbers is the nth root of their product. For example, the geometric mean of 2, 3, and 4 is $\sqrt[3]{2{\cdot}3{\cdot}4} = \sqrt[3]{8{\cdot}3} = 2\sqrt[3]{3}$.

Problems for Chapter 4

1. (a) Find the second, third, and fourth terms of the arithmetic progression with the first term -11 and difference 7.
   (b) Find the next three terms of the arithmetic progression -3, -7, -11, -15, ...

2. (a) Find the second, third, and fourth terms of the arithmetic progression with the first term 8 and difference -3.
   (b) Find the next three terms of the arithmetic progression 7/4, 1, 1/4, -1/2, -5/4, -2, ...

3. Find the 90th term of each of the following arithmetic progressions:

   (a) 11, 22, 33, 44, ...
   (b) 14, 25, 36, 47, ...
   (c) 9, 20, 31, 42, ...

4. For each of the following geometric progressions, find e so that 3^e is the 80th term.

   (a) 3, 9, 27, ...
   (b) 1, 3, 9, ...
   (c) 81, 243, 729, ...

5. Find x, given that 15, x, 18 are consecutive terms of an arithmetic progression.

6. Find x and y so that 14, x, y, 9 are consecutive terms of an arithmetic progression.

7. Sum the following:

   (a) 7/3 + 2/3 + (-1) + (-8/3) + (-13/3) + ... + (-1003/3).
   (b) (-6) + (-2) + 2 + 6 + 10 + ... + 2002.
   (c) The first ninety terms of 7/4, 1, 1/4, -1/2, -5/4, -2, ...
   (d) The first n odd positive integers, that is, $1 + 3 + 5 + ... + (2n - 1)$.

8. Sum the following:

   (a) 12 + 5 + (-2) + (-9) + ... + (-1073).
   (b) (-9/5) + (-1) + (-1/5) + 3/5 + 7/5 + 11/5 + ... + 2407.
   (c) The first eighty terms of -3, -7, -11, -15, ...
   (d) The first n terms of the arithmetic progression a, $a + d$, $a + 2d$, ...

9. Find the fourth, seventh, and ninth terms of the geometric progression with first term 2 and ratio 3.

10. Find the fourth and sixth terms of the geometric progression with first term 2 and ratio -3.

11. Find the next three terms of the geometric progression 2, 14, 98, ...

12. Find the next three terms of the geometric progression 6, -2, 2/3, -2/9, ...

13. Find both possible values of x if 7, x, 252 are three consecutive terms of a geometric progression.

14. Find all possible values of y if 400, y, 16 are three consecutive terms of a geometric progression.

15. Find a compact expression for each of the following:

(a) $1 + 7 + 7^2 + 7^3 + ... + 7^{999}$.
(b) $1 - 7 + 7^2 - 7^3 + ... - 7^{999}$.
(c) $1 + 7 + 7^2 + 7^3 + ... + 7^{n-1}$.

16. Find a compact expression for each of the following:

(a) $1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + ... + \frac{1}{3^{88}}$.

(b) $8 + \frac{8}{3^2} + \frac{8}{3^4} + \frac{8}{3^6} + ... + \frac{8}{3^{188}}$.

(c) $8 + 8 \cdot 3^{-2} + 8 \cdot 3^{-4} + 8 \cdot 3^{-6} + ... + 8 \cdot 3^{-2m}$.

17. Find n, given that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \binom{n}{2}$.

*18. Find m, given that $1 + 2 + 3 + 4 + ... + 1000 = \binom{m}{m-2}$.

19. Given that a is the average of the numbers r and s, show that r, a, s are three consecutive terms of an arithmetic progression and that their sum is $3a$.

20. Show that r^3, r^2s, rs^2, s^3 are four consecutive terms of a geometric progression and that their sum is $(r^4 - s^4)/(r - s)$.

21. Find the geometric mean of each of the following sets of positive numbers:

(a) 6, 18.
(b) 2, 6, 18, 54.
(c) 2, 4, 8.
(d) 1, 2, 4, 8, 16.

22. Find the geometric mean of each of the following sets of numbers:

(a) 3, 4, 5.
(b) 3, 4, 5, 6.
(c) $1, 7, 7^2, 7^3$.
(d) a, ar, ar^2, ar^3, ar^4.

23. Find the geometric mean of 8, 27, and 125.

24. Find the geometric mean of a^4, b^4, c^4, and d^4.

25. Let b be the middle term of a geometric progression with $2m + 1$ positive terms and let r be the common ratio. Show that:

(a) The terms are $br^{-m}, br^{-m+1}, \dots, br^{-1}, b, br, \dots, br^m$.
(b) The geometric mean of the $2m + 1$ numbers is the middle term.

26. Show that the geometric mean of the terms in a geometric progression of positive numbers is equal to the geometric mean of any two terms equally spaced from the two ends of the progression.

27. Find a compact expression for the sum $x^n + x^{n-1}y + x^{n-2}y^2 + \dots + xy^{n-1} + y^n$.

28. Find a compact expression for the arithmetic mean of $x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n$.

29. A 60-mile trip was made at 30 miles per hour and the return at 20 miles per hour.

(a) How many hours did it take to travel the 120 mile round trip?
(b) What was the average speed for the round trip?

30. Find x, given that 1/30, $1/x$, and 1/20 are in arithmetic progression. What is the relation between x and the answer to Part (b) of problem 29?

31. Verify the factorization $1 - x^7 = (1 - x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$ and use it with $x = 1/2$ to find a compact expression for

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6.$$

32. Use the factorization $1 + x^{99} = (1 + x)(1 - x + x^2 - x^3 + x^4 - \dots + x^{98})$ to find compact expressions for the following sums:

(a) $1 - 5^{-1} + 5^{-2} - 5^{-3} + \dots - 5^{-97} + 5^{-98}$.
(b) $a - ar + ar^2 - ar^3 + \dots - ar^{97} + ar^{98}$.

33. Let $a_1, a_2, a_3, \dots, a_{3m}$ be an arithmetic progression, and for $n = 1, 2, \dots, 3m$ let A_n be the arithmetic mean of its first n terms. Show that A_{2m} is the arithmetic mean of the two numbers A_m and A_{3m}.

34. Let $g_1, g_2, \dots, g_{3m}$ be a geometric progression of positive terms. Let A, B, and C be the geometric means of the first m terms, the first $2m$ terms, and all $3m$ terms, respectively. Show that $B^2 = AC$.

35. Let S be the set consisting of those of the integers 0, 1, 2, ..., 30 which are divisible exactly by 3 or 5 (or both), and let T consist of those divisible by neither 3 nor 5.

(a) Write out the sequence of numbers in S in their natural order.
(b) In the sequence of Part (a), what is the arithmetic mean of terms equally spaced from the two ends of the sequence?
(c) What is the arithmetic mean of all the numbers in T?
(d) Find the sum of the numbers in T.

36. Find the sum 4 + 5 + 6 + 8 + 10 + 12 + 15 + ... + 60,000 of all the positive integers not exceeding 60,000 which are integral multiples of at least one of 4, 5, and 6.

37. Let $u_1, u_2, ..., u_t$ satisfy $u_{n+2} = 2u_{n+1} - u_n$ for $n = 1, 2, ..., t - 2$. Show that the t terms are in arithmetic progression.

38. Find a compact expression for the sum $v_1 + v_2 + ... + v_t$ in terms of v_1 and v_2, given that $v_{n+2} = (v_{n+1})^2/v_n$ for $n = 1, 2, ..., t - 2$.

39. Let $a_n = 2^n$ be the nth term of the geometric progression $2, 2^2, 2^3, ..., 2^t$. Show that $a_{n+2} - 5a_{n+1} + 6a_n = 0$ for $n = 1, 2, ..., t - 2$.

40. For what values of r does the sequence $b_n = r^n$ satisfy $b_{n+2} - 5b_{n+1} + 6b_n = 0$ for all n?

41. Let a be one of the roots of $x^2 - x - 1 = 0$. Let the sequence $c_0, c_1, c_2, ...$ be the geometric progression $1, a, a^2, ...$. Show that:

(a) $c_{n+2} = c_{n+1} + c_n$.
(b) $c_2 = a^2 = a + 1$.
(c) $c_3 = a^3 = 2a + 1$.
(d) $c_4 = 3a + 2$.
(e) $c_5 = 5a + 3$.
(f) $c_6 = 8a + 5$.

42. For the sequence $c_0, c_1, ...$ of the previous problem, express c_{12} in the form $aF_u + F_v$, where F_u and F_v are Fibonacci numbers, and conjecture a similar expression for c_m.

*43. In the sequence 1/5, 3/5, 4/5, 9/10, 19/20, 39/40, ... each succeeding term is the average of the previous term and 1. Thus:

$$\frac{3}{5} = \frac{1}{2}\left(\frac{1}{5} + 1\right),\ \frac{4}{5} = \frac{1}{2}\left(\frac{3}{5} + 1\right),\ \frac{9}{10} = \frac{1}{2}\left(\frac{4}{5} + 1\right),\ \ldots .$$

(a) Show that the twenty-first term is $1 - \dfrac{1}{5 \cdot 2^{18}}$.

(b) Express the nth term similarly.

(c) Sum the first five hundred terms.

*44. In the sequence 1, 2, 3, 6, 7, 14, 15, 30, 31, ... a term in an even numbered position is double the previous term, and a term in an odd numbered position (after the first term) is one more than the previous term.

(a) What is the millionth term of this sequence?

(b) Express the sum of the first million terms compactly.

Chapter 5

MATHEMATICAL INDUCTION

In mathematics, as in science, there are two general methods by which we can arrive at new results. One, deduction, involves the assumption of a set of axioms from which we deduce other statements, called theorems, according to prescribed rules of logic. This method is essentially that used in standard courses in Euclidean geometry.

The second method, induction, involves the guessing or discovery of general patterns from observed data. While in most branches of science and mathematics the guesses based on induction may remain merely conjectures, with varying degrees of probability of correctness, certain conjectures in mathematics which involve the integers frequently can be proved by a technique of Pascal called mathematical induction. Actually, this technique in not induction, but is rather an aid in proving conjectures arrived at by induction.

THE PRINCIPLE OF MATHEMATICAL INDUCTION: *A statement concerning positive integers is true for all positive integers if (a) it is true for 1, and (b) its being true for any integer k implies that it is true for the next integer k + 1.*

If one replaces (a) by (a'),"it is true for some integer *s*," then (a') and (b) prove the statement true for all integers greater than or equal to *s*. Part (a) gives only a starting point; this starting point may be any integer - positive, negative, or zero.

Let us see if mathematical induction is a reasonable method of proof of a statement involving integers n. Part (a) tells us that the statement is true for $n = 1$. Using (b) and the fact that the statement is true for 1, we obtain the fact that it is true for the next integer 2. Then (b) implies that it is true for $2 + 1 = 3$. Continuing in this way, we would ultimately reach any fixed positive integer.

Let us use this approach on the problem of determining a formula which will give us the number of diagonals of a convex polygon in terms of the number of sides. The three-sided polygon, the triangle, has no diagonals; the four-sided polygon has two. An examination of other cases yields the data included in the following table:

n = number of sides	3	4	5	6	7	8	9	...	n	...
D_n = number of diagonals	0	2	5	9	14	20	27	...	D_n	...

The task of guessing the formula, if a formula exists, is not necessarily an easy one, and there is no sure approach to this part of the over-all problem. However, if one is perspicacious, one observes the following pattern:

$$\begin{aligned} 2D_3 &= 0 = 3\cdot 0 \\ 2D_4 &= 4 = 4\cdot 1 \\ 2D_5 &= 10 = 5\cdot 2 \\ 2D_6 &= 18 = 6\cdot 3 \\ 2D_7 &= 28 = 7\cdot 4. \end{aligned}$$

This leads us to conjecture that

$$2D_n = n(n - 3)$$

or

$$D_n = \frac{n(n-3)}{2}$$

Now we shall use mathematical induction to prove this formula. We shall use as a starting point $n = 3$, since for n less than 3 no polygon exists. It is clear from the data that the formula holds for the case $n = 3$. Now we assume that a k-sided polygon has $k(k - 3)/2$ diagonals. If we can conclude from this that a $(k + 1)$-sided polygon has $(k + 1)[(k + 1) - 3]/2 = (k + 1)(k - 2)/2$ diagonals, we will have proved that the formula holds for all positive integers greater than or equal to 3.

Consider a k-sided polygon. By assumption it has $k(k - 3)/2$ diagonals. If we place a triangle on a side AB of the polygon, we make it into a $(k + 1)$-sided polygon. It has all the diagonals of the k-sided polygon plus the diagonals drawn from the new vertex N to all the

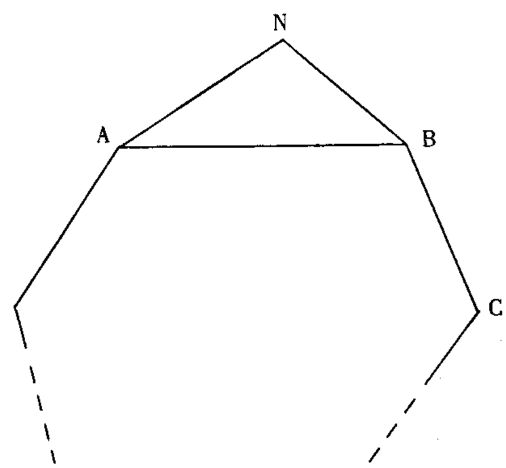

Figure 3

vertices of the previous k-sided polygon except 2, namely A and B. In addition, the former side AB has become a diagonal of the new $(k + 1)$-sided polygon. Thus a $(k + 1)$-sided polygon has a total of $\frac{k(k-3)}{2} + (k - 2) + 1$ diagonals. But:

$$\frac{k(k-3)}{2} + (k - 2) + 1$$
$$= \frac{k^2 - 3k + 2k - 2}{2}$$

$$= \frac{k^2 - k - 2}{2}$$
$$= \frac{(k + 1)(k - 2)}{2}$$
$$= \frac{(k + 1)[(k + 1) - 3]}{2}$$

This is the desired formula for $n = k + 1$.

So, by assuming that the formula $D_n = n(n - 3)/2$ is true for $n = k$, we have been able to show it true for $n = k + 1$. This, in addition to the fact that it is true for $n = 3$, proves that it is true for all integers greater than or equal to 3. (The reader may have discovered a more direct method of obtaining the above formula.)

The method of mathematical induction is based on something that may be considered one of the axioms for the positive integers: If a set S contains 1, and if, whenever S contains an integer k, S contains the next integer $k + 1$, then S contains all the positive integers. It can be shown that this is equivalent to the principle that in every non-empty set of positive integers there is a least positive integer.

Example 1. Find and prove by mathematical induction a formula for the sum of the first n cubes, that is, $1^3 + 2^3 + 3^3 + ... + n^3$.

Solution: We consider the first few cases:

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 100. \end{aligned}$$

We observe that $1 = 1^2$, $9 = 3^2$, $36 = 6^2$, and $100 = 10^2$. Thus it appears that the sums are the squares of triangular numbers 1, 3, 6, 10, ... In Chapter 4 we saw that the triangular numbers are of the form $n(n + 1)/2$. This suggests that

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n + 1)}{2}\right]^2.$$

It is clearly true for $n = 1$. Now we assume that it is true for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k + 1)}{2}\right]^2.$$

Can we conclude from this that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2?$$

We can add $(k + 1)^3$ to both sides of the known expression, obtaining:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= (k+1)^2 \frac{k^2}{4} + (k+1)^3 \\
&= \frac{(k+1)^2 (k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2 (k+2)^2}{4} \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2 \\
&= \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2.
\end{aligned}$$

Hence the sum when $n = k + 1$ is $[n(n + 1)/2]^2$, with n replaced by $k + 1$, and the formula is proved for all positive integers n.

Our guessed expression for the sum was a fortunate one!

Example2. Prove that $a - b$ is a factor of $a^n - b^n$ for all positive integers n.

Proof: Clearly, $a - b$ is a factor of $a^1 - b^1$; hence the first part of the induction is verified, that is, the statement is true for $n = 1$. Now we assume that $a^k - b^k$ has $a - b$ as a factor:

$$a^k - b^k = (a - b)M.$$

Next we must show that $a - b$ is a factor of $a^{k+1} - b^{k+1}$. But

$$\begin{aligned}
a^{k+1} - b^{k+1} &= a{\cdot}a^k - b{\cdot}b^k \\
&= a{\cdot}a^k - b{\cdot}a^k + b{\cdot}a^k - b{\cdot}b^k \\
&= (a - b)a^k + b(a^k - b^k).
\end{aligned}$$

Now, using the assumption that $a^k - b^k = (a - b)M$ and substituting, we obtain:

$$a^{k+1} - b^{k+1} = (a - b)a^k + b(a - b)M$$
$$= (a - b)\ [a^k + bM].$$

We see from this that $a - b$ is a factor of $a^{k+1} - b^{k+1}$ and hence $a - b$ is a factor of $a^n - b^n$ for n equal to any positive integer. It is easily seen that the explicit factorization is

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}).$$

Example 3. Prove that $n(n^2 + 5)$ is an integral multiple of 6 for all integers n, that is, there is an integer u such that $n(n^2 + 5) = 6u$.

Proof: We begin by proving the desired result for all the integers greater than or equal to 0 by mathematical induction.

When $n = 0$, $n(n^2 + 5)$ is 0. Since $0 = 6 \cdot 0$ is a multiple of 6, the result holds for $n = 0$.

We now assume it true for $n = k$, and seek to derive from this its truth for $n = k + 1$. Hence we assume that

$$k(k^2 + 5) = 6r \tag{1}$$

with r an integer. We then wish to show that

$$(k + 1)[(k + 1)^2 + 5] = 6s \tag{2}$$

with s an integer. Simplifying the difference between the left-hand sides of (2) and (1), we obtain

$$(k + 1)[(k + 1)^2 + 5] - k(k^2 + 5) = 3k(k + 1) + 6. \tag{3}$$

Since k and $k + 1$ are consecutive integers, one of them is even. Then their product $k(k + 1)$ is even, and may be written as $2t$, with t an integer. Now (3) becomes

$$(k + 1)[(k + 1)^2 + 5] - k(k^2 + 5) = 6t + 6 = 6(t + 1). \tag{4}$$

Transposing, we have

$$(k + 1)[(k + 1)^2 + 5] = k(k^2 + 5) + 6(t + 1).$$

Using (1), we can substitute $6r$ for $k(k^2 + 5)$. Hence

$$(k + 1)[(k + 1)^2 + 5] = 6r + 6(t + 1) = 6(r + t + 1).$$

Letting s be the integer $r + t + l$, we establish (2), which is the desired result when $n = k + 1$. This completes the induction and proves the statement for $n \geq 0$.

Now let n be a negative integer, that is, let $n = -m$, with m a positive integer. The previous part of the proof shows that $m(m^2 + 5)$ is of the form $6q$, with q an integer. Then

$$n(n^2 + 5) = (-m)[(-m)^2 + 5] = -m(m^2 + 5) = -6q = 6(-q),$$

a multiple of 6. The proof is now complete.

We have seen that binomial coefficients, Fibonacci and Lucas numbers, and factorials may be defined inductively, that is, by giving their initial values and describing how to get new values from previous values. Similarly, one may define an arithmetic progression $a_1, a_2, \ldots, a_t$ as one for which there is a fixed number d such that $a_{n+1} = a_n + d$ for $n = 1, 2, \ldots, t - 1$. Then the values of a_1 and d would determine the values of all the terms. A geometric progression $b_1, \ldots, b_t$ is one for which there is a fixed number r such that $b_{n+1} = b_n r$ for $n = 1, 2, \ldots, t - 1$; its terms are determined by b_1 and r.

It is not surprising that mathematical induction is very useful in proving results concerning quantities that are defined inductively, however, it is sometimes necessary or convenient to use an alternate principle, called **strong mathematical induction**.

STRONG MATHEMATICAL INDUCTION: *A statement concerning positive integers is true for all the positive integers if there is an integer q such that (a) the statement is true for 1, 2, ... , q, and (b) when* $k \geq q$, *the statement being true for 1, 2, ... , k implies that it is true for k + 1.*

As in the case of the previous principle, this can be modified to apply to statements in which the starting value is an integer different from 1.

We illustrate strong induction in the following:

Example 4. Let $a, b, c, r, s,$ and t be fixed integers. Let $L_0, L_1, \ldots$ be the Lucas sequence. Prove that

$$\text{(A)} \qquad rL_{n+a} = sL_{n+b} + tL_{n+c}$$

is true for $n = 0, 1, 2, \ldots$ if it is true for $n = 0$ and $n = 1$.

Proof: We use strong induction. It is given that (A) is true for $n = 0$ and $n = 1$. Hence, it remains to assume that $k \geq 1$ and that (A) is true for $n = 0, 1, 2, \ldots, k$, and to use these assumptions to prove that (A) holds for $n = k + 1$.

We therefore assume that

$$\begin{aligned} rL_a &= sL_b + tL_c \\ rL_{1+a} &= sL_{1+b} + tL_{1+c} \\ rL_{2+a} &= sL_{2+b} + tL_{2+c} \\ &\cdots \\ rL_{k-1+a} &= sL_{k-1+b} + tL_{k-1+c} \\ rL_{k+a} &= sL_{k+b} + L_{k+c} \end{aligned}$$

and that there are at least two equations in this list. Adding corresponding sides of the last two of these equations and combining like terms, we obtain

$$r(L_{k+a} + L_{k-1+a}) = s(L_{k+b} + L_{k-1+b}) + t(L_{k+c} + L_{k-1+c}).$$

Using the relation $L_{n+1} + L_n = L_{n+2}$ for the Lucas numbers, this becomes

$$rL_{k+1+a} = sL_{k+1+b} + tL_{k+1+c}$$

which is (A) when $n = k + 1$. This completes the proof.

Problems for Chapter 5

In Problems 1 to 10 below, use mathematical induction to prove each statement true for all positive integers n.

1. The sum of the interior angles of a convex $(n + 2)$-sided polygon is $180n$ degrees.

2. $1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$.

3. (a) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = n(4n^2 - 1)/3$.

   (b) $1{\cdot}3 + 3{\cdot}5 + 5{\cdot}7 + \ldots + (2n - 1)(2n + 1) = n(4n^2 + 6n - 1)/3$.

   (c) $\dfrac{1}{1{\cdot}3} + \dfrac{1}{2{\cdot}4} + \dfrac{1}{3{\cdot}5} + \ldots + \dfrac{1}{n(n + 2)} = \dfrac{n(3n + 5)}{4(n + 1)(n + 2)}$.

   (d) $1 + 2a + 3a^2 + \ldots + na^{n-1} = [1 - (n + 1)a^n + na^{n+1}]/(1 - a)^2$.

4. (a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$.

   (b) $1{\cdot}3 + 2{\cdot}4 + 3{\cdot}5 + \ldots + n(n + 2) = n(n + 1)(2n + 7)/6$.

   (c) $\dfrac{5}{1{\cdot}2}{\cdot}\dfrac{1}{3} + \dfrac{7}{2{\cdot}3}{\cdot}\dfrac{1}{3^2} + \dfrac{9}{3{\cdot}4}{\cdot}\dfrac{1}{3^3} + \ldots + \dfrac{2n + 3}{n(n + 1)}{\cdot}\dfrac{1}{3^n} = 1 - \dfrac{1}{3^n(n + 1)}$.

5. $(1^3 + 2^3 + 3^3 + \ldots + n^3) + 3(1^5 + 2^5 + 3^5 + \ldots + n^5) = 4(1 + 2 + 3 + \ldots + n)^3$.

6. $(1^5 + 2^5 + 3^5 + \ldots + n^5) + (1^7 + 2^7 + 3^7 + \ldots + n^7) = 2(1 + 2 + 3 + \ldots + n)^4$

*7. $3^n + 7^n - 2$ is an integral multiple of 8.

*8. $2 \cdot 7^n + 3 \cdot 5^n - 5$ is an integral multiple of 24.

9. $x^{2n} - y^{2n}$ has $x + y$ as a factor.

10. $x^{2n+1} + y^{2n+1}$ has $x + y$ as a factor.

11. For all integers n, prove the following:

    (a) $2n^3 + 3n^2 + n$ is an integral multiple of 6.
    (b) $n^5 - 5n^3 + 4n$ is an integral multiple of 120.

12. Prove that $n(n^2 - 1)(3n + 2)$ is an integral multiple of 24 for all integers n.

13. Guess a formula for each of the following and prove it by mathematical induction:

    (a) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dots + \dfrac{1}{n(n+1)}$.

    (b) $(x + y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)\dots(x^{2^n} + y^{2^n})$.

14. Guess a formula for each of the following and prove it by mathematical induction:

    (a) $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)$.

    (b) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \dots + \dfrac{1}{(2n-1)(2n+1)}$.

15. Guess a simple expression for the following and prove it by mathematical induction:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right).$$

16. Find a simple expression for the product in Problem 15, using the factorization

$$x^2 - y^2 = (x - y)(x + y).$$

17. Prove the following properties of the Fibonacci numbers F_n for all integers n greater than or equal to 0:

(a) $2(F_s + F_{s+3} + F_{s+6} + \ldots + F_{s+3n}) = F_{s+3n+2} - F_{s-1}$.

(b) $F_{-n} = (-1)^{n+1}F_n$.

(c) $\binom{n}{0}F_s + \binom{n}{1}F_{s+1} + \binom{n}{2}F_{s+2} + \ldots + \binom{n}{n}F_{s+n} = F_{s+2n}$.

18. Discover and prove formulas similar to those of Problem 17 for the Lucas numbers L_n.

19. Use Example 4, in the text above, to prove the following properties of the Lucas numbers for $n = 0, 1, 2, \ldots$, and then prove them for all negative integers n.

(a) $L_{n+4} = 3L_{n+2} - L_n$.

(b) $L_{n+6} = 4L_{n+3} + L_n$.

(c) $L_{n+8} = 7L_{n+4} - L_n$.

(d) $L_{n+10} = 11L_{n+5} + L_n$.

20. State an analogue of Example 4 for the Fibonacci numbers instead of the Lucas numbers and use it to prove analogues of the formulas of Problem 19.

21. In each of the following parts, evaluate the expression for some small values of n, use this data to make a conjecture, and then prove the conjecture true for all integers n.

(a) $F_{n+1}^2 - F_n F_{n+2}$.

(b) $\dfrac{F_{n+2}^2 - F_{n+1}^2}{F_n}$.

(c) $F_{n-1} + F_{n+1}$.

22. Discover and prove formulas similar to the first two parts of the previous problem for the Lucas numbers.

23. Prove the following for all integers m and n:

(a) $L_{m+n+1} = F_{m+1}L_{n+1} + F_mL_n$.

(b) $F_{m+n+1} = F_{m+1}F_{n+1} - F_mF_n$.

24. Prove that $(F_{n+1})^2 + (F_n)^2 = F_{2n+1}$ for all integers n.

25. Let a and b be the roots of the quadratic equation $x^2 - x - 1 = 0$. Prove that:

(a) $F_n = \dfrac{a^n - b^n}{a - b}$.

(b) $L_n = a^n + b^n$.

(c) $F_n L_n = F_{2n}$.

(d) $a^n = aF_n + F_{n-1}$ and $b^n = bF_n + F_{n-1}$.

26. The sequence 0, 1, ½, 3/4, 5/8, 11/16, ... is defined by

$$u_0 = 0,\ u_1 = 1,\ u_2 = \frac{u_1 + u_0}{2},\ \ldots,\ u_{n+2} = \frac{u_{n+1} + u_n}{2},\ \ldots .$$

Discover and prove a compact formula for u_n as a function of n.

27. The Pell sequence 0, 1, 2, 5, 12, 29, ... is defined by

$$P_0 = 0, P_1 = 1, P_2 = 2P_1 + P_0, \ldots, P_{n+2} = 2P_{n+1} + P_n, \ldots .$$

Let $x_n = P_{n+1}^2 - P_n^2$, $y_n = 2P_{n+1}P_n$, and $z_n = P_{n+1}^2 + P_n^2$. Prove that for every positive integer n the numbers x_n, y_n, and z_n are the lengths of the sides of a right triangle and that x_n and y_n are consecutive integers.

28. Discover and prove properties of the Pell sequence that are analogous to those of the Fibonacci sequence.

29. Let the sequence 1, 5, 85, 21845, ... be defined by

$$c_1 = 1, c_2 = c_1(3c_1 + 2), \ldots, c_{n+1} = c_n(3c_n + 2), \ldots .$$

Prove that $c_n = \dfrac{4^{2^{n-1}} - 1}{3}$ for all positive integers n.

30. Let a sequence be defined by $d_1 = 4$, $d_2 = (d_1)^2, \ldots, d_{n+1} = (d_n)^2, \ldots$.

Show that $d_n = 3c_n + 1$, where c_n is as defined in the previous problem.

31. Prove that $1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + n(n + 1)(n + 2) = \dfrac{n(n + 1)(n + 2)(n + 3)}{4}$.

*32. Certain of the above formulas suggest the following:

$$1\cdot 2\cdots m + 2\cdot 3\cdots(m+1) + \cdots + n(n+1)\cdots(n+m-1) = \frac{n(n+1)...(n+m)}{m+1}.$$

Prove it for general m.

*33. Prove that $n^5 - n$ is an integral multiple of 30 for all integers n.

*34. Prove that $n^7 - n$ is an integral multiple of 42 for all integers n.

*35. Show that every integer from 1 to $2^{n+1} - 1$ is expressible uniquely as a sum of distinct powers of 2 chosen from 1, 2, 2^2, ... , 2^n.

*36. Show that every integer s from $-\dfrac{3^{n+1}-1}{2}$ to $\dfrac{3^{n+1}-1}{2}$ has a unique expression of the form

$$s = c_0 + 3c_1 + 3^2c_2 + ... + 3^nc_n$$

where each of c_0, c_1, ..., c_n is 0, 1, or -1.

Chapter 6

THE BINOMIAL THEOREM

In Chapter 1 we defined $\binom{n}{r}$ as the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$, and tabulated these coefficients in the arrangement of the Pascal Triangle:

n	Coefficients of $(a + b)^n$													
0								1						
1							1		1					
2						1		2		1				
3					1		3		3		1			
4				1		4		6		4		1		
5			1		5		10		10		5		1	
6		1		6		15		20		15		6		1
...	.		.		.		.		.		.		.	.

We then observed that this array is bordered with 1's; that is, $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for $n =$ 0, 1, 2, ... We also noted that each number inside the border of 1's is the sum of the two closest numbers on the previous line. This property may be expressed in the form

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}. \tag{1}$$

This formula provides an efficient method of generating successive lines of the Pascal Triangle, but the method is not the best one if we want only the value of a single binomial coefficient for a large n, such as $\binom{100}{3}$. We therefore seek a more direct approach.

It is clear that the binomial coefficients in a diagonal adjacent to a diagonal of 1's are the numbers 1, 2, 3, ... ; that is, $\binom{n}{1} = n$. Now let us consider the ratios of binomial coefficients to the previous ones on the same row. For $n = 4$, these ratios are:

(2) $$4/1,\ 6/4 = 3/2,\ 4/6 = 2/3,\ 1/4.$$

For $n = 5$, they are

(3) $$5/1,\ 10/5 = 2,\ 10/10 = 1,\ 5/10 = 1/2,\ 1/5.$$

The ratios in (3) have the same pattern as those in (2) if they are rewritten as

$$5/1,\ 4/2,\ 3/3,\ 2/4,\ 1/5.$$

It is easily seen that this pattern also holds on the line for $n = 8$, and that the coefficients on that line are therefore:

(4) $$1,\ \frac{8}{1},\ \frac{8}{1}\cdot\frac{7}{2},\ \frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3},\ \frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3}\cdot\frac{5}{4},\ \ldots .$$

The binomial coefficient $\binom{8}{3} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}$ can be rewritten as

$$\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{8!}{3!5!}.$$

Similarly, $\binom{8}{2} = \frac{8!}{2!6!}$ and $\binom{8}{4} = \frac{8!}{4!4!}$.

This leads us to conjecture that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ holds in all cases. We prove this by mathematical induction in the following theorem.

THEOREM: If n and r are integers with $0 \le r \le n$, then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof: If $n = 0$, the only allowable value of r is 0 and $\binom{0}{0} = 1$. Since

$$\frac{n!}{r!(n-r)!} = \frac{0!}{0!0!} = 1$$

the formula holds for $n = 0$.

Now let us assume that it holds for $n = k$. Then

$$\binom{k}{r-1} = \frac{k!}{(r-1)!(k-r+1)!}, \binom{k}{r} = \frac{k!}{r!(k-r)!}.$$

Using (1), above, we now have

$$\begin{aligned}\binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r} &= \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!}\\ &= \frac{k!r}{(r-1)!r(k-r+1)!} + \frac{k!(k-r+1)}{r!(k-r)!(k-r+1)}\\ &= \frac{k!(r+k-r+1)}{r!(k-r+1)!}\\ &= \frac{k!(k+1)}{r!(k-r+1)!}\\ &= \frac{(k+1)!}{r!(k-r+1)!}.\end{aligned}$$

Since the formula

$$\binom{k+1}{r} = \frac{(k+1)!}{r!(k-r+1)!}$$

is the theorem for $n = k + 1$, the formula is proved for all integers $n \geq 0$, with the exception that our proof tacitly assumes that r is neither 0 nor $k + 1$; that is, it deals only with the coefficients inside the border of 1's. But the formula

$$\binom{k+1}{r} = \frac{(k+1)!}{r!(k-r+1)!}$$

shows that each of $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ is $\frac{(k+1)!}{0!(k+1)!} = 1$.

Hence the theorem holds in all cases.

The above theorem tells us that the coefficient of $x^r y^s$ in $(x+y)^n$ is

$$\frac{n!}{r!s!}.$$

Since this expression has the same value when r and s are interchanged, we again see that the binomial coefficients have the symmetry relation

$$\binom{n}{r} = \binom{n}{n-r}.$$

By writing out the factorials more explicitly, we see that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots2\cdot1}{1\cdot2\cdot3\ldots r(n-r)(n-r-1)\ldots2\cdot1}.$$

Cancelling common factors, we now have

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{1\cdot2\cdot3\ \ldots\ r}.$$

This is the alternate form of the theorem illustrated for $n = 8$ in (4), above.

We can now rewrite the expansion of $(a+b)^n$ in the form

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot2}a^{n-2}b^2 + \ldots$$

$$+ \frac{n(n-1)\ldots(n-r+1)}{1\cdot2\ \ldots\ r}a^{n-r}b^r + \cdots + b^n.$$

This last formula it generally called the **<u>Binomial</u> <u>Theorem.</u>**

The formulas

$$\binom{n}{0} = 1, \binom{n}{r} = \frac{n(n-1)...(n-r+1)}{1\cdot 2 \ ... \ r} \text{ for } r > 0$$

enable us to extend the definition of $\binom{n}{r}$, previously defined only for integers n and r with

$0 \le r \le n,$ to allow n to be any integer. We then have, for example, $\binom{2}{5} = 0$, $\binom{-2}{7} = -8$,

and $\binom{-3}{8} = 45.$

It can easily be shown that the formula

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

holds with the extended definition as it did with the original definition.

Now the identity

$$2\binom{m}{2} + \binom{m}{1} = 2\frac{m(m-1)}{2} + m = m^2 - m + m = m^2$$

holds for all integers m, and we can use the formulas

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + ... + \binom{n}{1} = \binom{n+1}{2}$$

$$\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + ... + \binom{n}{2} = \binom{n+1}{3}$$

to show that

$$
\begin{aligned}
1^2 + 2^2 + \dots + n^2 &\\
&= \left[2\binom{1}{2} + \binom{1}{1}\right] + \left[2\binom{2}{2} + \binom{2}{1}\right] + \dots + \left[2\binom{n}{2} + \binom{n}{1}\right] \\
&= 2\left[\binom{1}{2} + \binom{2}{2} + \dots + \binom{n}{2}\right] + \left[\binom{1}{1} + \binom{2}{1} + \dots + \binom{n}{1}\right] \\
&= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\
&= \frac{2n^3 - 2n}{6} + \frac{3n^2 + 3n}{6} \\
&= \frac{2n^3 + 3n^2 + n}{6} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}
$$

Frequently in mathematical literature a short notation for sums is used which involves the Greek capital letter sigma, written $\sum$. In this notation,

$$a_1 + a_2 + \dots + a_n$$

is written as

$$\sum_{i=1}^{n} a_i$$

and the auxiliary variable i is called the **index of summation**. Thus, for example,

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{6} 1 = 1 + 1 + 1 + 1 + 1 + 1 = 6$$

$$\sum_{j=1}^{n-1} j^2 = 1^2 + 2^2 + 3^2 + \dots + (n-1)^2.$$

Under the capital sigma, one indicates the symbol that is used as the index of summation and the initial value of this index. Above the sigma, one indicates the final value. The general polynomial $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ of degree n can be written as

$$\sum_{k=0}^{n} a_k x^{n-k}.$$

One easily sees that

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$$

since

$$\begin{aligned}\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i &= (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 + \ldots + b_n) \\ &= (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) \\ &= \sum_{i=1}^{n} (a_i + b_i).\end{aligned}$$

Also, $\sum_{i=1}^{n} (ca_i) = c \sum_{i=1}^{n} a_i$, the proof of which is left to the reader. However,

$$\left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right) \neq \sum_{i=1}^{n} (a_i b_i)$$

as can easily be shown by counterexample. (See Problem 19 of this chapter.)

A corresponding notation for products uses the Greek letter pi:

$$\prod_{i=1}^{n} a_i = a_1 a_2 \ldots a_n.$$

In this notation, $n!$ for $n \geq 1$ can be expressed as $\prod_{k=1}^{n} k$.

In solving problems stated in terms of the sigma or pi notation, it is sometimes helpful to rewrite the expression in the original notation.

Problems for Chapter 6

1. Find each of the following:

   (a) The coefficient of x^4y^{16} in $(x + y)^{20}$.
   (b) The coefficient of x^5 in $(1 + x)^{15}$.
   (c) The coefficient of x^3y^{11} in $(2x - y)^{14}$.

2. Find each of the following:

   (a) The coefficient of $a^{13}b^4$ in $(a + b)^{17}$.
   (b) The coefficient of a^{11} in $(a - 1)^{16}$.
   (c) The coefficient of a^6b^6 in $(a - 3b)^{12}$.

3. Find integers a, b, and c such that $6\binom{n}{3} = n^3 + an^2 + bn + c$ for all integers n.

4. Find integers p, q, r, and s such that $4!\binom{n}{4} = n^4 + pn^3 + qn^2 + rn + s$ for all integers n.

5. Prove that $\binom{n}{3} = 0$ for $n = 0, 1, 2$.

6. Given that k is a positive integer, prove that $\binom{n}{k} = 0$ for $n = 0, 1, \ldots, k - 1$.

7. Find $\binom{-1}{r}$ for $r = 0, 1, 2, 3, 4,$ and 5. .

8. Find $\binom{-2}{r}$ for $r = 0, 1, 2, 3, 4,$ and 5. .

9. Prove that $\binom{-3}{r} = (-1)^r\binom{r+2}{2}$ for $r = 0, 1, 2, \ldots$.

10. Prove that $\binom{-4}{r} = (-1)^r \binom{r+3}{3}$ for $r = 0, 1, 2, \ldots$.

11. Let m be a positive integer and r a non-negative integer. Express $\binom{-m}{r}$ in terms of a binomial coefficient $\binom{n}{k}$ with $0 \le k \le n$.

12. In the original definition of $\binom{n}{r}$ as a binomial coefficient, it was clear that it was always an integer. Explain why this is still true in the extended definition.

13. Show that $\binom{n}{a}\binom{n-a}{b} = \dfrac{n!}{a!b!(n-a-b)!}$ for integers a, b, and n, with $a \ge 0$, $b \ge 0$, and $n \ge a + b$.

14. Given that $n = a + b + c + d$ and that a, b, c, and d are non-negative integers, show that

$$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}\binom{n-a-b-c}{d} = \frac{n!}{a!b!c!d!}.$$

15. Express $\sum_{k=1}^{n} [a + (k-1)d]$ as a polynomial in n.

16. Express $\prod_{k=1}^{n} (2k)$ compactly without using the Π notation.

17. Show that $\prod_{k=1}^{n} a_k = \prod_{j=0}^{n-1} a_{j+1}$.

18. Show that $\sum_{k=1}^{n-2} b_k = \sum_{i=3}^{n} b_{i-2}$.

19. Evaluate $\left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} b_i\right)$ and $\sum_{i=1}^{2} (a_i b_i)$ and show that they are not always equal.

20. Show that $\left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right) = \prod_{i=1}^{n} (a_i b_i)$.

21. Prove by mathematical induction that $\sum_{i=0}^{n} \binom{s+i}{s} = \binom{s+1+n}{s+1}$.

22. Prove that $\sum_{j=0}^{n} \binom{s+j}{j} = \binom{s+1+n}{n}$.

23. Express $\sum_{k=1}^{n-2} \frac{k(k+1)}{2}$ as a polynomial in n.

24. Express $\sum_{k=1}^{n-2} \binom{k+1}{k-1}$ as a polynomial in n.

25. Write $6\left[\binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right]$ as a polynomial in n, and then use the fact that $\binom{n}{r}$ is always an integer to give a new proof that $n(n^2 + 5)$ is an integral multiple of 6 for all integers n.

26. (a) Write $4!\left[\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right]$ as a polynomial in n.

(b) Show that $n^4 - 2n^3 + 11n^2 + 14n$ is an integral multiple of 24 for all integers n.

27. Find numbers s and t such that $n^3 = n(n - 1)(n - 2) + sn(n - 1) + tn$ holds for $n = 1$ and $n = 2$.

28. Find numbers a and b such that $n^3 = 6\binom{n}{3} + a\binom{n}{2} + b\binom{n}{1}$ for all integers n.

29. Find numbers r, s, and t such that $n^4 = n(n - 1)(n - 2)(n - 3) + rn(n - 1)(n - 2) + sn(n - 1) + tn$ for $n = 1$, 2, and 3. Using these values of r, s, and t, show that

$$n^4 = 24\binom{n}{4} + 6r\binom{n}{3} + 2s\binom{n}{2} + t\binom{n}{1}$$

for all integers n.

30. Find numbers a, b, c, and d such that

$$n^5 = 5!\binom{n}{5} + a\binom{n}{4} + b\binom{n}{3} + c\binom{n}{2} + d\binom{n}{1}.$$

31. Express $\sum_{k=1}^{n} k^4$ as a polynomial in n.

32. Express $\sum_{k=1}^{n} k^5$ as a polynomial in n.

33. We define a sequence S_0, S_1, S_2, ... as follows: When n is an even integer $2t$, let

$S_n = S_{2t} = \sum_{j=0}^{t}\binom{t+j}{t-j}$. When n is an odd integer $2t + 1$, let $S_n = S_{2t+1} = \sum_{j=0}^{t}\binom{t+1+j}{t-j}$.

Prove that $S_{2t} + S_{2t+1} = S_{2t+2}$ and $S_{2t+1} + S_{2t+2} = S_{2t+3}$ for $t = 0, 1, 2, \ldots$.

34. For the sequence defined in Problem 33, prove that S_n is the Fibonacci number F_{n+1}.

*35. Prove the following property of the Fibonacci numbers:

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j F_{s+2n-2j} = F_{s+n}.$$

*36 Prove an analogue of the formula of Problem 35 for the Lucas numbers.

37 Find a compact expression, without using the sigma notation, for

$$1 \cdot n + 2(n-1) + 3(n-2) + \cdots + (n-1)\cdot 2 + n \cdot 1,$$

that is, for $\sum_{k=0}^{n-1}(k+1)(n-k)$.

Chapter 7

COMBINATIONS AND PERMUTATIONS

We have seen in the previous chapter that $(a + b)^n$ can be written as

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

where we have the specific formula for the binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1 \cdot 2 \cdot 3 \ldots k}.$$

We now look at a different interpretation of these numbers and will see why $\binom{n}{k}$ is called "n choose k." Let

$$M = (1 + x_1)(1 + x_2)(1 + x_3).$$

Expanding this product, we get

$$M = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3.$$

The terms of this expansion correspond to the subsets of $S = \{x_1, x_2, x_3\}$. That is, we can associate the term 1 with the empty subset of S; the terms x_1, x_2, and x_3 with the singleton subsets of S; the terms x_1x_2, x_1x_3, and x_2x_3 with the doubleton subsets; and $x_1x_2x_3$ with S itself. (S is the only subset with 3 elements.)

Next we replace each of x_1, x_2, and x_3 by x in our two expressions for M. This results in

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

Thus we see that $\binom{3}{k}$ for $k = 0, 1, 2, 3$ is the number of ways of choosing a subset of k elements from a set S of 3 elements. Similarly, one can see that the number of ways of choosing k elements from a set of n elements is $\binom{n}{k}$.

For example, the set {1, 2, 3, 4, 5} with 5 elements has $\binom{5}{3}$ subsets having 3 elements. Since

$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10,$$

it is not too difficult to write out all ten of these subsets as

{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5},
{1, 4, 5}, {2, 3, 4}, {2, 3, 5}, {2, 4, 5}, {3, 4, 5}.

If we drop the braces enclosing the elements of each subset, the resulting sequence is said to be a **combination** of 3 things chosen from the set {1, 2, 3, 4, 5}. Thus the ten combinations of 3 things chosen from this set of 5 objects are

1, 2, 3;	1, 2, 4;	1, 2, 5;	1, 3, 4;	1, 3, 5;
1, 4, 5;	2, 3, 4;	2, 3, 5;	2, 4, 5;	3, 4, 5.

Note that changing the order in which the objects of a combination are written does not change the combination. For example, 1, 2, 4 is the same combination as 1, 4, 2.

The formula $\binom{n}{k} = \binom{n}{n-k}$ tells us that the entries on row n of the Pascal Triangle read the same left to right as they do right to left. The combinatorial significance of this formula is that the number of ways of choosing k elements from a set of n elements is equal to the number of ways of omitting n - k of the elements.

A problem analogous to that of finding the coefficients of a binomial expansion is that of finding the coefficents in

$$(x + y + z)^n.$$

These coefficients, called the **trinomial coefficients**, are naturally more complicated but, fortunately, can be expressed in terms of the binomial coefficients in the following way. Let us look for the coefficient of $x^6y^3z^1$ in $(x + y + z)^{10}$. From this product of ten factors we must choose six x's, three y's, and one z. We can choose six x's from a set of ten in $\binom{10}{6}$ ways and three

y's from the remaining four factors in $\binom{4}{3}$ ways, and the z from the remaining factor in

$\binom{1}{1}$ way. Therefore the trinomial coefficient of $x^6y^3z^1$ in $(x + y + z)^{10}$ can be written as

$\binom{10}{6}\binom{4}{3}\binom{1}{1}$, or, since $\binom{1}{1} = 1$, as $\binom{10}{6}\binom{4}{3}$. We can obtain an alternate

representation of this number, however, by choosing the z first in $\binom{10}{1}$ ways, then the six x's

from the remaining nine, and finally the three y's from the remaining three. Thus the coefficient

would appear as $\binom{10}{1}\binom{9}{6}\binom{3}{3}$ or $\binom{10}{1}\binom{9}{6}$. Hence $\binom{10}{6}\binom{4}{3} = \binom{10}{1}\binom{9}{6}$. By

choosing the y's first, one can see that this coefficient could also be expressed as

$\binom{10}{3}\binom{7}{6}$ or $\binom{10}{3}\binom{7}{1}$. The reader may find other forms of the coefficient.

This problem can be generalized similarly to find the coefficients of $(x + y + z + \cdots + w)^n$; they are called **multinomial** **coefficients**. It can readily be shown that the coefficient of $x^a y^b z^c \ldots w^d$ in the expansion of $(x + y + z + \cdots + w)^n$ is

$$\frac{n!}{a!b!c!\ldots d!}$$

where, of course, the sum $a + b + c + \cdots + d$ of the exponents must be n.

Another interesting problem, and one with frequent applications, is that of finding the number of ways in which one can arrange a set of objects in a row, that is, the number of **permutations** of the set. Let us consider the set of four objects a, b, c, d. They can be arranged in the following ways:

a b c d	*b a c d*	*c a b d*	*d a b c*
a b d c	*b a d c*	*c a d b*	*d a c b*
a c b d	*b c a d*	*c b a d*	*d b a c*
a c d b	*b c d a*	*c b d a*	*d b c a*
a d b c	*b d a c*	*c d a b*	*d c a b*
a d c b	*b d c a*	*c d b a*	*d c b a*

Rather than write them all out, if we are only interested in the number of arrangements, we may think of the problem thus: We have four spaces to fill. If we put, for example, the *b* in the first, we have only the *a*, *c*, and *d* to choose from in filling the remaining three. And if we put the *d* in the second, we have only *a* and *c* for the remaining; and so forth. So we have four choices for the first space, three for the second, two for the third, and one for the fourth. This gives us $4 \cdot 3 \cdot 2 \cdot 1$, or 4! arrangements of four objects. This argument can be used to show that there are $n!$ arrangements of n objects.

We may also consider the possibility of arranging, in a row, r objects chosen from a set of n. We have n choices for the first space, $n - 1$ for the second, $n - 2$ for the third, and so on. Finally we have $n - r + 1$ choices for the rth space, giving a total of $n(n - 1)(n - 2)\ldots(n - r + 1)$ possible arrangements (or permutations). This can be written in terms of factorials as follows:

$$\begin{aligned} & n(n-1)(n-2)\ldots(n-r+1) \\ & \quad = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1} \\ & \quad = \frac{n!}{(n-r)!}. \end{aligned}$$

It should be noted that this is not the number of combinations of r objects taken from a set of n, since in permutations order is important; in combinations it is not. For example, if we consider the three objects *a*, *b*, and *c*, the number of permutations of two objects chosen from them is $3 \cdot 2$, the arrangements *ab, ba, bc, cb, ca, ac*. However, the number of combinations is $\dfrac{3!}{2!1!} = 3$, and the combinations are *a* and *b*, *b* and *c*, and *a* and *c*.

Next we define even and odd permutations of 1, 2, ..., n; this topic is used in Chapter 9 and in higher algebra.

We begin with the case $n = 3$, that is the numbers 1, 2, 3. With each permutation

$$i, j, k$$

of these three numbers, we associate the product of differences

$$p = (j - i)(k - i)(k - j).$$

If p is positive, the associated permutation is called **<u>even</u>**; if p is negative, the associated permutation is **<u>odd</u>**. Three of the 3! permutations of 1, 2, 3 are even and three are odd. The even ones are listed in the first column, and the odd ones in the second column:

1, 2, 3	1, 3, 2
2, 3, 1	2, 1, 3
3, 1, 2	3, 2, 1

For general n, a permutation $i, j, h, k, \ldots, r, s$ of 1, 2, 3, ... , n is associated with the product

$$p = [(j - i)][(h - i)(h - j)][(k - i)(k - j)(k - h)] \ldots$$
$$[(s - i)(s - j)(s - h)(s - k)\ldots(s - r)]$$

of all the differences of two of $i, j, h, k, \ldots, r, s$ in which the number that appears first is subtracted from the other. If the permutation $i, j, h, k, \ldots, r, s$ is written in the notation $a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n$, then the product p takes the form

$$p = [(a_2 - a_1)][(a_3 - a_1)(a_3 - a_2)][(a_4 - a_1)(a_4 - a_2)(a_4 - a_3)]\ldots$$
$$[(a_n - a_1)(a_n - a_2)(a_n - a_3)(a_n - a_4)\ldots(a_n - a_{n-1})].$$

If the product p is positive, the permutation is **<u>even</u>**; if p is negative, the permutation is **<u>odd</u>**.

Problems for Chapter 7

1. Write out all the combinations of two letters chosen from *a, b, c, d,* and *e*.

2. Write out all the combinations of three letters chosen from *a, b, c, d,* and *e*.

3. Write out all the permutations of two letters chosen from *a, b, c, d,* and *e*.

4. Write out all the permutations of three letters chosen from *a, b, c, d,* and *e*.

5. Find the positive integer that is the coefficient of $x^3y^7z^2$ in $(x + y + z)^{12}$.

6. Express the trinomial coefficient of the previous problem in six ways as a product of two binomial coefficients.

7. How many combinations are there of 1, 2, 3, or 4 elements from a set of 5 elements?

8. How many non-empty proper subsets are there of a set of n elements? That is, how many combinations are there of 1, 2, ... , or n - 1 elements?

9. Express the coefficient of $x^3y^7w^2$ in $(x + y + z + w)^{12}$ in six different ways as a product of two binomial coefficients.

10. Express the coefficient of $x^2y^3z^4w^2$ in $(x + y + z + w)^{11}$ in six different ways as a product of three binomial coefficients.

11. Find the coefficient of $x^2y^9z^3w$ in $(2x + y - z + w)^{15}$.

12. Find the coefficient of x^ryzw in $(x + y + z + w)^{r+3}$.

13. Show that $x^5y^2z^9$ has the same coefficient in $(x + y + z + w)^{16}$ as in $(x + y + z)^{16}$.

14. What is the relation between the coefficient of xy^7z^2 in $(x + y - z)^{10}$ and its coefficient in $(x - y + z + w)^{10}$?

15. Let a, b, and n be positive integers, with $n > a + b$. Show that

$$\binom{n}{a}\binom{n-a}{b}+\binom{n}{b}\binom{n-b}{a-1}+\binom{n}{a}\binom{n-a}{b-1}=\binom{n+1}{a}\binom{n-a+1}{b}.$$

16. Express the coefficient of $x^2y^4z^6$ in $(x + y + z)^{12}$ as the sum of three of the trinomial coefficients in the expansion of $(x + y + z)^{11}$.

17. What is the sum of all the trinomial coefficients in $(x + y + z)^{100}$?

18. What is the sum of the coefficients in each of the following:

    (a) $(x + y - z)^{100}$?

    (b) $(x - y + z - w)^{100}$?

19. List the even permutations of 1, 2, 3, 4.

20. List the odd permutations of 1, 2, 3, 4.

R 21. Let P be a permutation $i, j, h, \ldots k$ of $1, 2, 3, \ldots, n$.

(a) Show that if i and j are interchanged, P changes from odd to even or from even to odd.

(b) Show that if any two adjacent terms in P are interchanged, P changes from odd to even or from even to odd.

(c) Show that the interchange of any two terms in P can be considered to be the result of an odd number of interchanges of adjacent terms.

(d) Show that if any two terms in the permutation P are interchanged, P changes from odd to even or from even to odd.

(e) Given that $n \geq 2$, show that half of the permutations of 1, 2, ... , n are even and half are odd, that is, that there are $\frac{n!}{2}$ even permutations and the same number of odd ones.

R 22. (a) Let P be a permutation a, b, c, d of the numbers 1, 2, 3, 4. Let $d = 4$ and let Q be the associated permutation a, b, c of 1, 2, 3. Show that P and Q are either both even or both odd.

(b) Let R be a permutation $i, j, ..., h, n$ of the numbers 1, 2, ..., n-1, n in which the last term of R is n. Let S be the associated permutation $i, j, ..., h$ of 1, 2, ..., n-1 obtained by dropping the last term of R. Show that R and S are either both even or both odd.

23. How many triples of positive integers r, s, and t are there with $r < s < t$ and:

(a) $r + s + t = 52$?

(b) $r + s + t = 352$?

24. The arrangement $\begin{bmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & 8 \end{bmatrix}$ has the property that the numbers increase as one goes down or to the right.

(a) How many other arrangements are there of the numbers 1, 2, ..., 8 in 2 rows and 4 columns with this property?

(b) How many arrangements are there of the numbers 1, 2, ..., 14 in 2 rows and 7 columns with this property?

(c) How many arrangements are there of the numbers 1, 2, ..., 12 in 3 rows and 4 colunms with this property?

Chapter 8

POLYNOMIAL EQUATIONS

An nth degree polynomial with complex coefficients is of the form

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

where the a_i are complex numbers and $a_0 \neq 0$. (Of course, the coefficients $a_0, \ldots, a_n$ may be real numbers, since a real number is a special case of a complex number.)

Thus a first degree polynomial is of the form $ax + b$ and a second degree polynomial of the form $ax^2 + bx + c$, with $a \neq 0$ in both cases. A non-zero constant a is a polynomial of degree zero; the constant zero is also a polynomial, but it is not assigned a degree.

The polynomial equation $y = x^2 - 6x + 1$ defines y to be a **function of x** on the domain of all complex numbers; that is, it provides a rule for assigning a unique complex number y to each complex number x. The table

x	4	3	2	1	0	i	$2i$
$y = x^2 - 6x + 1$	-7	-8	-7	-4	1	$-6i$	$-3 - 12i$

shows that this functional rule assigns -4 to 1, 1 to 0 , $-6i$ to i, etc. A rule that makes y a function of x assigns precisely one value y to a fixed x; however, the same number y may be assigned to more than one x, as is seen here with -7 assigned to 2 and to 4.

It is sometimes convenient to represent the rule that defines y to be a function of x by the symbol $f(x)$. This notation enables one to express in a simple way the number assigned to a given x by the function. For example, $f(1)$, $f(2)$, and $f(3)$ stand for the numbers assigned to 1, 2, and 3, respectively. If $f(x) = x^2 - 6x + 1$, then $f(1) = -4$, $f(2) = -7$, $f(3) = -8$,

$$f(\sqrt{2}) = (\sqrt{2})^2 - 6(\sqrt{2}) + 1,$$

$$f(a + b) = (a + b)^2 - 6(a + b) + 1,$$

and

$$f(x + 1) = (x + 1)^2 - 6(x + 1) + 1.$$

Notice that $f(a + b)$ is not necessarily the same as $f(a) + f(b)$, since $f(a + b)$ is the result of replacing x in $x^2 - 6x + 1$ by $a + b$ and is *not* f times $a + b$.

If several functions are involved in a given discussion, one may use $g(x)$, $F(x)$, $p(x)$, $q(x)$, and so on, as alternates for $f(x)$.

8.1 THE FACTOR AND REMAINDER THEOREMS

If an nth degree polynomial $p(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ can be factored in the form

$$p(x) = a_0(x - r_1)(x - r_2)...(x - r_n), \quad a_0 \neq 0,$$

then the roots of the polynomial equation $p(x) = 0$ are found by setting each of the factors equal to zero, since a product of complex numbers is zero if and only if at least one of the factors is zero. Therefore, the roots are $r_1, ..., r_n$. We wish to establish a form of converse to this result: we wish to show that if r is a root of a polynomial equation $p(x) = 0$ then it follows that $x - r$ is a factor of $p(x)$; that is $p(x)$ can be expressed in the form

$$p(x) = (x - r)q(x)$$

where $q(x)$ is a polynomial in x.

THE FACTOR THEOREM: Let

$$p(x) = a_0x^n + a_1x^{n-1} + ... + a_n$$

be a polynomial in x. If r is a root of $p(x)$, that is, if $p(r) = 0$, then $x - r$ is a factor of $p(x)$.

Proof: Using the hypothesis that $p(r) = 0$, we have

$$\begin{aligned} p(x) &= p(x) - 0 \\ p(x) &= p(x) - p(r) \\ p(x) &= (a_0x^n + a_1x^{n-1} + \cdots + a_n) - (a_0r^n + a_1r^{n-1} + \cdots + a_n) \end{aligned}$$

(1) $$p(x) = a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots + (a_n - a_n).$$

Since $x - r$ is a factor of $x^n - r^n$, $x^{n-1} - r^{n-1}$, and so on (see Example 2, Chapter 5), it follows that $x - r$ is a factor of the entire right side of equation (1), and so is a factor of $p(x)$.

We next use this theorem to obtain information concerning the case in which r is not a root of $p(x)$.

THE REMAINDER THEOREM: Let $p(x)$ be a polynomial. Then for every complex number r there is a polynomial $q(x)$ such that

(2) $$p(x) = (x - r)q(x) + p(r)$$

Proof: Let us define a new polynomial $f(x)$ by

$$f(x) = p(x) - p(r)$$

Then $f(r) = p(r) - p(r) = 0$. Hence r is a root of $f(x)$ and, by the Factor Theorem, above, $x - r$ is a factor of $f(x)$, and so there is a polynomial $q(x)$ such that

$$f(x) = (x - r)q(x)$$

Now $p(x) - p(r) = (x - r)q(x)$, since both sides are equal to $f(x)$; equation (2) is then obtained by transposing $p(r)$.

The polynomial $p(r)$ is the **remainder** in the division of $p(x)$ by $x - r$. In specific cases, the **quotient** polynomial $q(x)$ of (2), above, may be found by long division or by a more compact form of division called synthetic division. We first illustrate these techniques on the example in which

$$p(x) = x^3 - 7x^2 + 4x + 9, \qquad r = 2.$$

Dividing $p(x)$ by $x - 2$, we have

$$\begin{array}{r|llll}
 & x^2 & -5x & -6 & \\ \cline{2-5}
x - 2 & x^3 & -7x^2 & +4x & +9 \\
 & x^3 & -2x^2 & & \\ \cline{2-3}
 & & -5x^2 & +4x & \\
 & & -5x^2 & +10x & \\ \cline{3-5}
 & & & -6x & +9 \\
 & & & -6x & +12 \\ \cline{4-5}
 & & & & -3
\end{array}$$

This shows that

$$x^3 - 7x^2 + 4x + 9 = (x - 2)(x^2 - 5x - 6) - 3.$$

That is, $p(x) = (x - 2)q(x) + p(2)$, with $q(x) = x^2 - 5x - 6$ and $p(2) = -3$.

The synthetic form of the division is as follows:

$$\begin{array}{r|rrrr}
2 & 1 & -7 & 4 & 9 \\
 & & 2 & -10 & -12 \\ \hline
 & 1 & -5 & -6 & \boxed{-3}
\end{array}$$

The steps in this synthetic form of the division are explained in the treatment of the general case which follows.

The synthetic division of

$$p(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

by $x - h$ is in the form

$$\begin{array}{r|rrrrrr} h & a_0 & a_1 & a_2 & ... & a_{n-1} & a_n \\ \hline & & b_1 & b_2 & ... & b_{n-1} & b_n \\ \hline & c_0 & c_1 & c_2 & ... & c_{n-1} & \boxed{c_n} \end{array}$$

where $c_0 = a_0$, $b_1 = hc_0$, $c_1 = a_1 + b_1$, $b_2 = hc_1$, $c_2 = a_2 + b_2$, ..., $b_n = hc_{n-1}$, $c_n = a_n + b_n$. In general, each b is h times the previous c, $c_0 = a_0$, and each succeeding c is the sum of the a and b above it. The last c, c_n, is the value of $p(h)$, and the other c's are the coefficients of $q(x)$ in the formula $p(x) = (x - h)q(x) + p(h)$; they give us the expression

$$p(x) = (x - h)(c_0x^{n-1} + c_1x^{n-2} + ... + c_{n-2}x + c_{n-1}) + c_n .$$

Example 1. Express $p(x) = x^5 + 25x^2 + 7$ in the form $(x + 3)q(x) + p(-3)$.

Solution: We note that $h = -3$ and that $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 25$, $a_4 = 0$, and $a_5 = 7$ in this problem. The synthetic division is therefore written

$$\begin{array}{r|rrrrrr} -3 & 1 & 0 & 0 & 25 & 0 & 7 \\ & & -3 & 9 & -27 & 6 & -18 \\ \hline & 1 & -3 & 9 & -2 & 6 & \boxed{-11} \end{array}$$

Hence:

$$x^5 + 25x^2 + 7 = (x + 3)(x^4 - 3x^3 + 9x^2 - 2x + 6) - 11.$$

Example 2. Use synthetic division to show that 5 is a root of $p(x) = 2x^3 - 40x - 50 = 0$, and use this fact to solve the equation.

Solution: We divide $p(x)$ by $x - 5$ with the object of showing that the remainder $p(5)$ is zero. Thus:

$$
\begin{array}{r|rrrr}
5 & 2 & 0 & -40 & -50 \\
  &   & 10 & 50 & 50 \\
\hline
  & 2 & 10 & 10 & \boxed{0}
\end{array}
$$

This shows us that $p(x) = (x - 5)(2x^2 + 10x + 10)$. The roots of $p(x) = 0$ are therefore obtained from

$$x - 5 = 0, \quad 2(x^2 + 5x + 5) = 0$$

as 5 and $(-5 \pm \sqrt{25-20})/2$; we have, then

$$5, \quad (-5 + \sqrt{5})/2, \quad \text{and} \quad (-5 - \sqrt{5})/2.$$

Example 3. Let $f(x) = 9x^3 + x^2 - 7x + 4$. Find numbers a, b, c, and d such that

$$f(x) = a + b(x+1) + c(x+1)^2 + d(x+1)^3. \tag{3}$$

We give two solutions.

First solution: Letting $x = -1$ in (3), we see that $a = f(-1)$. We therefore use synthetic division to express $f(x)$ in the form $(x + 1)g(x) + f(-1)$ and find that $g(x) = 9x^2 - 8x + 1$ and $a = f(-1) = 3$. Now (3) becomes

$$(x + 1)(9x^2 - 8x + 1) + 3 = 3 + b(x + 1) + c(x + 1)^2 + d(x + 1)^3.$$

On each side we subtract 3 and then divide by $x + 1$, thus obtaining

$$g(x) = 9x^2 - 8x + 1 = b + c(x+1) + d(x+1)^2. \tag{4}$$

Letting $x = -1$, we see that b = $g(-1)$. We therefore treat $g(x)$ as $f(x)$ was treated above, and find that $g(x) = (x + 1)(9x - 17) + 18$. Hence $b = 18$. Then (4) becomes

$$(x + 1)(9x - 17) + 18 = b + c(x + 1) + d(x + 1)^2.$$

This leads to

$$9x - 17 = c + d(x + 1)$$

or

$$9(x + 1) - 26 = c + d(x + 1).$$

Hence $c = -26$ and then $d = 9$.

Alternate solution: Let $x + 1 = y$. Then $x = y - 1$ and

$$f(x) = f(y - 1) = 9(y - 1)^3 + (y - 1)^2 - 7(y - 1) + 4.$$

Expanding and collecting like terms, we obtain

$$\begin{aligned} f(x) &= 3 + 18y - 26y^2 + 9y^3 \\ &= 3 + 18(x + 1) - 26(x + 1)^2 + 9(x + 1)^3. \end{aligned}$$

8.2 INTEGRAL ROOTS

Let the coefficients a_i of the polynomial equation

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$$

be integers. Then it can be shown that the only possibilities for integral roots are the integral divisors of the last coefficient a_n. For example, an integer that is a root of

$$x^4 + x^3 + x^2 + 3x - 6 = 0$$

would have to be one of the eight integral divisors ± 1, ± 2, ± 3, ± 6 of -6. Trial of each of these eight integers, as in Example 2 in Section 8.1, would show that 1 and -2 are the only integral roots. The work can be reduced, when one root is found, by substituting the quotient polynomial for the original polynomial in further work. Thus

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & 1 & 3 & -6 \\ & & 1 & 2 & 3 & 6 \\ \hline & 1 & 2 & 3 & 6 & \boxed{0} \end{array}$$

shows that $x^4 + x^3 + x^2 + 3x - 6 = (x - 1)(x^3 + 2x^2 + 3x + 6)$. Hence, 1 is a root and the other roots are the roots of the equation $x^3 + 2x^2 + 3x + 6 = 0$.

8.3 RATIONAL ROOTS

We now consider a polynomial equation

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0, \quad a_0 \neq 0$$

of degree n with integer coefficients a_i. It can be shown that if there is a rational root p/q, with p and q integers having no common integral divisor greater that 1, then p must be an integral divisor of a_n and q must be an integral divisor of a_0. For example, if the rational number p/q in lowest terms is a root of

$$6x^4 - x^3 - 6x^2 - x - 12 = 0$$

then p must be one of the twelve integral divisors $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$ of -12 and q one of the integral divisors of 6. Without losing any of the possibilities, we may restrict q to be positive, that is, to be one of the integers 1, 2, 3, 6. The possible rational roots, therefore, are

$$\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12,\ \pm1/2,\ \pm3/2,\ \pm1/3,\ \pm2/3,\ \pm4/3,\ \pm1/6.$$

Trials would show that 3/2 and -4/3 are the only rational roots.

Example. Prove that $\sqrt{2} + \sqrt{3}$ is not a rational number.

Solution: Let $a = \sqrt{2} + \sqrt{3}$. Then

$$\begin{aligned} a - \sqrt{2} &= \sqrt{3} \\ a^2 - 2\sqrt{2}a + 2 &= 3 \\ a^2 - 1 &= 2\sqrt{2}a \\ a^4 - 2a^2 + 1 &= 8a^2 \\ a^4 - 10a^2 + 1 &= 0. \end{aligned}$$

Hence a is a root of $x^4 - 10x^2 + 1 = 0$. This fourth degree polynomial equation has integer coefficients. The rule on rational roots tells us that the only possible rational roots are 1 and -1. Substituting, we see that neither 1 nor -1 is a root. Hence there are no rational roots. Since a is a root, it follows that a is not rational.

Problems for Sections 8.1, 8.2, and 8.3

1. Express $p(x) = x^4 + 5x^3 - 10x - 12$ in the form $(x + 2)q(x) + p(-2)$.

2. Express $f(x) = 5x^5 - x^4 - x^3 - x^2 - x - 2$ in the form $(x - 1)g(x) + f(1)$.

3. Show that -1 is a root of $x^3 + 3x^2 - 2 = 0$, and find the other roots.

4. Show that 2 is a root of $x^3 - 6x + 4 = 0$, and find the other roots.

5. Find a, given that -4 is a root of $5x^6 - 7x^5 + 11x + a = 0$.

6. Find b, given that 3 is a root of $x^7 - 10x^5 + 8x^3 + 4x^2 - 3x + b = 0$.

7. Find all the integral roots of $x^4 - 2x^3 - x^2 - 4x - 6 = 0$, and then find the other roots.

8. Find all the integral roots of $x^5 - 8x^4 + 15x^3 + 8x^2 - 64x + 120 = 0$, and then find the other roots.

9. Let $f(x) = (x - a)^3 - x^3 + a^3$. Find $f(0)$ and $f(a)$, and use this information to find two factors of $f(x)$.

10. Let $g(x) = (x - a)^5 - x^5 + a^5$. Show that $f(x)$ is divisible by x and by $x - a$, and find the other factors.

11. Find all the integral roots of $3x^4 + 20x^3 + 36x^2 + 16x = 0$, and then find the other roots.

12. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x$; that is, let $a_n = 0$. Also, let the a_i be integers. Show that any non-zero integral root of $f(x) = 0$ is an integral divisor of a_{n-1}.

13. Find a rational root of $3x^3 + 4x^2 - 21x + 10 = 0$, and then find the other roots.

14. Find all the roots of $6x^4 + 31x^3 + 25x^2 - 33x + 7 = 0$.

15. Find all the roots of $81x^5 - 54x^4 + 3x^2 - 2x = 0$.

16. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x$ with the a_i integers. State a necessary condition for a non-zero rational number to be a root of $f(x) = 0$.

17. Given that a and b are integers, what are possibilities for rational roots of $x^3 + ax^2 + bx + 30 = 0$?

18. Let $f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ with the a_i integers. Note that $a_0 = 1$. Show that any rational root of $f(x) = 0$ must be an integer.

19. Let $f(x)$ be a polynomial. Let r and s be roots of $f(x) = 0$ and let $r \neq s$. Show that there exist polynomials $g(x)$ and $h(x)$ such that all of the following are true:

    (a) $f(x) = (x - r)g(x)$.
    (b) $g(s) = 0$.
    (c) $g(x) = (x - s)h(x)$.
    (d) $f(x) = (x - r)(x - s)h(x)$.

20. Let $f(x) = 0$ be a polynomial equation with distinct roots r, s, and t. Show that $f(x) = (x - r)(x - s)(x - t)p(x)$, with $p(x)$ a polynomial.

21. Prove that if $r_1, r_2 \ldots, r_n$ are distinct roots of a polynomial equation $f(x) = 0$, then $f(x)$ is a multiple of $(x - r_1)(x - r_2)\ldots(x - r_n)$.

22. Prove that $\sqrt{3} - \sqrt{2}$, $\sqrt{2} - \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$ are all irrational.

23. Prove that $\sqrt{5} + \sqrt{3}$, $\sqrt{5} - \sqrt{3}$, $-\sqrt{5} + \sqrt{3}$, and $-\sqrt{5} - \sqrt{3}$ are all irrational.

24. Prove that $\sqrt[3]{14}$ is irrational.

25. Find an eighth-degree polynomial equation with integer coefficients that has

$\sqrt{2} + \sqrt{3} + \sqrt{7}$ as a root.

26. If $f(x)$ is a function of x, the notation $\Delta f(x)$ represents $f(x + 1) - f(x)$. Show that

$\Delta x^2 = 2x + 1$ and $\Delta x^3 = 3x^2 + 3x + 1$.

27. Let $\Delta f(x) = f(x + 1) - f(x)$. Find $\Delta f(x)$ for each of the following:

(a) $f(x) = a + bx$.
(b) $f(x) = a + bx + cx^2$.
(c) $f(x) = a + bx + cx^2 + dx^3$.
(d) $f(x) = x^n$, with n a positive integer.

28. Find $f(x + 2) - 2f(x + 1) + f(x)$ for:

(a) $f(x) = a + bx$.
(b) $f(x) = a + bx + cx^2$.

29. Find $f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x)$ for:

(a) $f(x) = a + bx$.
(b) $f(x) = a + bx + cx^2$.
(c) $f(x) = a + bx + cx^2 + dx^3$.

30. Let $\Delta^n f(x)$, with n a positive integer, be defined inductively by

$$
\begin{aligned}
\Delta^1 f(x) &= \Delta f(x) = f(x+1) - f(x),\\
\Delta^2 f(x) &= \Delta[\Delta f(x)] = \Delta[f(x+1) - f(x)]\\
&= [f(x+2) - f(x+1)] - [f(x+1) - f(x)],\\
\Delta^3 f(x) &= \Delta[\Delta^2 f(x)],\\
&\ldots,\\
\Delta^{m+1} f(x) &= \Delta[\Delta^m f(x)],\\
&\ldots .
\end{aligned}
$$

[The function $\Delta^n f(x)$ is called the nth difference of $f(x)$.] Show that

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+n-k).$$

31. Let $\Delta^n f(x)$ be defined as in Problem 30 above and show:

(a) $\Delta^n f(x) = 0$, if $f(x)$ is a polynomial of degree less than n.

(b) $\Delta^n f(x) = n!a_0$, if $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$.

32. Let $f(x) = a + bx + cx^2$. Let $r = f(0)$, $s = f(1) - f(0)$, and $t = f(2) - 2f(1) + f(0)$.

(a) Show that $f(x) = r + sx + tx(x-1)/2$.
(b) Generalize this problem.

33. Let $f(x) = 5x^4 - 6x^3 - 3x^2 + 8x + 2$. Use repeated synthetic division to find numbers a, b, c, d, and e such that

$$f(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 + e(x-2)^4.$$

34. Use the method of the alternate solution for Example 3 in Section 8.1 to do Problem 33.

35. Let $f(x) = x^3 + ax^2 + bx + c$, and let a, b, c, and r be complex numbers. Show that

$$f(x) = f(r) + (3r^2 + 2ar + b)(x-r) + s(x-r)^2 + (x-r)^3,$$

and express s in terms of a and r.

36. Let $f(x) = x^3 + ax^2 + bx + c$, and let r be a root of $f(x) = 0$. Show that $f(x)$ is divisible by $(x-r)^2$ if and only if $3r^2 + 2ar + b = 0$.

37. Let $f(x) = x^3 + ax^2 + bx + c$, $g(x) = 3x^2 + 2ax + b$, and $h(x) = 6x + 2a$. Show that $f(x) = (x - r)^3$ if and only if $f(r) = g(r) = h(r) = 0$.

38. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$. Find s, t, and u in terms of a, b, c, and r such that

$$f(x) = f(r) + s(x - r) + t(x - r)^2 + u(x - r)^3 + (x - r)^4.$$

39. Do the methods of this chapter enable you to solve $x^3 - 3x + 1 = 0$?

8.4 SYMMETRIC FUNCTIONS

If we multiply out $(x - a)(x - b)(x - c)(x - d)$, we obtain an expression of the form $x^4 - s_1x^3 + s_2x^2 - s_3x + s_4$, where

$$\begin{aligned}
s_1 &= a + b + c + d,\\
s_2 &= ab + ac + ad + bc + bd + cd,\\
s_3 &= abc + abd + acd + bcd,\\
s_4 &= abcd.
\end{aligned}$$

We note that s_k is the sum of all products of a, b, c, and d taken k at a time. It is also clear that the s_k are **symmetric functions** of a, b, c, and d; that is, they do not change value when any two of a, b, c, and d are interchanged.

The **Fundamental Theorem of Algebra**, the proof of which is too advanced for this book, states that the general nth degree polynomial with complex coefficients

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n, \quad a_0 \neq 0$$

has a factorization into linear factors

$$a_0(x - r_1)(x - r_2) \dots (x - r_n)$$

where the r_i are complex numbers. One can then see that $(-1)^k a_k/a_0$ is the sum of all the products of k factors chosen from $r_1, r_2, \dots, r_n$.

The **absolute value** of a real number x is written as $|x|$ and is defined as follows: If $x \geq 0$, then $|x| = x$; if $x < 0$, then $|x| = -x$.

Problems for Section 8.4

1. Let $3(x - r)(x - s) = 3x^2 - 12x + 8$. Find the following:

(a) $r + s$.
(b) rs.
(c) $(r + s)^2$.
(d) $r^2 + s^2$.
(e) $r^2 - 2rs + s^2$.
(f) $|r - s|$.

2. Find the sum, product, and absolute value of the difference of the roots of $5x^2 + 7x - 4 = 0$.

3. Let $(x - r)(x - s) = x^2 + x + 1$. Show the following:

   (a) $r = -(s + 1)$, $s = -(r + 1)$.
   (b) $r^3 + r^2 + r = 0 = s^3 + s^2 + s$.
   (c) $r = s^2$, $s = r^2$.
   (d) $r^{-1} + s^2 = -1$, $s^{-1} + r^2 = -1$.
   (e) $r^4 + r^{-1}s^{-1} + s^4 = 0$.
   (f) $r^9 - r^6 + r^3 - 1 = 0 = s^9 - s^6 + s^3 - 1$.
   (g) $r^{10} + s^7 + r^4 + s = -2 = s^{10} + r^7 + s^4 + r$.
   (h) $(r^2 - r + 1)(s^2 - s + 1)(r^4 - r^2 + 1)(s^4 - s^2 + 1) = 16$.

4. Let r be a root of $x^2 + x + 1 = 0$. Show the following:

   (a) $x^3 - a^3 = (x - a)(x - ar)(x - ar^2)$.
   (b) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + ry + r^2z)(x + r^2y + \mathrm{r}\mathrm{z})$.

5. Let a, b, and c be the roots of $x^3 + 3x + 3 = 0$. Find $(a + 1)(b + 1)(c + 1)$.

6. Given that $(x - a)(x - b) = x^2 - px + q$, express each of the following in terms of p and q:

   (a) $a + b$.
   (b) ab.
   (c) $a^2 + 2ab + b^2$.
   (d) $a^2 + ab + b^2$.
   (e) $ab(a^2 + ab + b^2)$.
   (f) a^3b^3.
   (g) The coefficients of the expansion of $(x - a^2)(x - ab)(x - b^2)$.

7. Let $(x - r)(x - s) = x^2 - px + q$, and let $(x - r^3)(x - r^2s)(x - rs^2)(x - s^3) = x^4 - ax^3 + bx^2 - cx + d$. Express a, b, c, and d in terms of p and q.

8. Let $(x - a)(x - b) = x^2 - ex + f$, $(x - c)(x - d) = x^2 - gx + h$, and $(x - ac)(x - ad)(x - bc)(x - bd) = x^4 - px^3 + qx^2 - rx + s$. Find p, q, r, and s in terms of e, f, g, and h.

9. Let $(x - a)(x - b)(x - c) = x^3 - 3x + 1$. Find each of the following:

   (a) $2(a + b + c)$.
   (b) $(a + b)(a + c) + (a + b)(b + c) + (a + c)(b + c)$.
   (c) $(a + b)(a + c)(b + c)$.
   (d) the equation $\mathrm{y}^3 - py^2 + qy - r = 0$ whose roots are $a + b$, $a + c$, and $b + c$.

10. Do Problem 9 with $(x - a)(x - b)(x - c) = x^3 + 3x - 1$.

11. Let $s_1 = a + b + c$, and $s_2 = ab + ac + bc$, and $s_3 = abc$. Find numbers x, y, z, t, u, v, and w such that for all a, b, and c:

(a) $a^3 + b^3 + c^3 = xs_3 + ys_1s_2 + zs_1^3$.
(b) $a^4 + b^4 + c^4 = ts_1s_3 + us_2^2 + vs_1^2s_2 + ws_1^4$.

12. Let $(x - r)(x - s)(x - t) = x^3 - ax^2 + bx - c$. Express $r^5 + s^5 + t^5$ in terms of a, b, and c.

13. Let $(x -1)(x -2)(x - 3)...(x - n) = x^n - s_1x^{n-1} + s_2x^{n-2} - ... + (-1)^ns_n$. Show the following:

(a) $s_1 = \binom{n+1}{2}$.

(b) $s_n = n!$.

(c) $2s_2 = (1^3 + 2^3 + ... + n^3) - (1^2 + 2^2 + ... + n^2)$.

Chapter 9

DETERMINANTS

Let us consider a pair of simultaneous equations:

(A) $2x - 3y = 9$
(B) $8x + 5y = 7$

We seek numbers x and y which satisfy (A) and (B) simultaneously, that is, a pair x,y for which both (A) and (B) become true statements. Such a pair also satisfies the equations

(A') $10x - 15y = 45$
(B') $24x + 15y = 21$

obtained by respectively multiplying both sides of (A) by 5 and both sides of (B) by 3. A pair x, y satisfying both (A') and (B') has to satisfy

(C) $34x = 66$

which is obtained by adding corresponding sides of (A') and (B'). The only root of (C) is $x = 66/34 = 33/17$.

If one replaces x by 33/17 in (A), one finds that

$$\begin{aligned} 2(33/17) - 3y &= 9 \\ 66/17 - 3y &= 9 \\ (66/17) - 9 &= 3y \\ -87/17 &= 3y \\ -29/17 &= y \end{aligned}$$

Hence the only pair of numbers x,y which might satisfy (A) and (B) simultaneously is $x = 33/17$, $y = -29/17$. Our method of obtaining these numbers shows that they do satisfy (A). We check that they also satisfy (B) by substituting in the left side as follows:

$$8\cdot\frac{33}{17} + 5\cdot\frac{-29}{17} = \frac{264}{17} + \frac{-145}{17} = \frac{119}{17} = 7$$

This shows that $x = 33/17$, $y = -29/17$ is the unique pair that satisfies (A) and (B) simultaneously.

An equation of the form $ax + by = c$ in which a, b and c are known numbers is called a **first-degree equation** in x and y, or a **linear equation** in x and y. Thus (A) and (B) are an example of a pair of **simultaneous linear equations**. The method illustrated above for solving such a pair is called **elimination**. More specifically, we eliminated y to obtain equation (C).

Eliminating y (or x) from the simultaneous equations

(D) $6x - 15y = -10$
(E) $4x - 10y = -7$

leads, upon multiplying both sides of (D) by 2 and both sides of (E) by -3, to

(D') $12x - 30y = -20$
(E') $-12x + 30y = 21$

and, upon adding corresponding sides of (D') and (E'), to

(F) $$0 = 1$$

Assuming that there is a pair x, y satisfying (D) and (E) simultaneously has led us to the false conclusion that 0 = 1. Hence there is no pair x, y which satisfies (D) and (E) simultaneously.

9.1 DETERMINANTS OF ORDER 2

Let us now examine the general pair of simultaneous first-degree equations:

(G) $$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

We are going to apply the elimination technique discussed above to (G) and then introduce the related concept of a determinant, which has important applications in the theory of systems of equations and in other fields.

If we multiply both sides of the first equation in (G) by b_2 and both sides of the second equation by b_1, we obtain

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 \\ a_2b_1x + b_1b_2y &= c_2b_1 \end{aligned}$$

Whence, by subtraction, we obtain:

(1) $$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$$

If we use the equations in (G) to eliminate x rather than y, the result is

(2) $$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1.$$

If $a_1b_2 - a_2b_1$ is not zero, we see that the solution of the system (G) is found from (1) and (2) in the form

(3)
$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}, \qquad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

We note that the denominators are the same, that the numerator in the expression for x is obtained from the denominator by replacing the coefficients a_1 and a_2 of x in (G) with c_1 and c_2, respectively, and that the numerator in the expression for y is obtained from the denominator by replacing the coefficients b_1 and b_2 of y with c_1 and c_2.

This motivates us to introduce the notation

(4)
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The square array bordered by vertical lines in (4) is called a two-by-two (2x2) **determinant**. With this notation, the equations in (3) can be rewritten as

(5)
$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided that the common denominator is not zero.

Thus we see that the solution of a system of simultaneous first-degree equations can be written as ratios of determinants in which the denominator is the determinant made up of coefficients of x and y in the order in which they appear in the equations, while the numerator for x is the same determinant with the coefficients of x replaced by the constants, and the numerator for y is the determinant of the denominator with the coefficients of y replaced by the constants. This technique is called **Cramer's Rule**. The common denominator is called the **determinant of the system**.

Example 1. Solve by determinants:

$$3x + 2y = 5$$
$$x - 7y = 2$$

Solution: We first evaluate the determinant in the denominator (the determinant of the system), as follows:

$$\begin{vmatrix} 3 & 2 \\ 1 & -7 \end{vmatrix} = 3(-7) - 1\cdot 2 = -21 - 2 = -23$$

Since this determinant is not zero, the system has the unique solution:

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 2 & -7 \end{vmatrix}}{-23} = \frac{5(-7) - 2\cdot 2}{-23} = \frac{-35 - 4}{-23} = \frac{-39}{-23} = \frac{39}{23}$$

$$y = \frac{\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}}{-23} = \frac{3\cdot 2 - 1\cdot 5}{-23} = \frac{6 - 5}{-23} = \frac{1}{-23} = -\frac{1}{23}.$$

Example 2. Use determinants to investigate the simultaneous equations:

$$\text{(H)} \qquad \begin{aligned} 10x - 14y &= 5 \\ 15x - 21y &= 8 \end{aligned}$$

Solution: The determinant of the system is

$$D = \begin{vmatrix} 10 & -14 \\ 15 & -21 \end{vmatrix} = 10(-21) - 15(-14) = -210 + 210 = 0.$$

Since $D = 0$, we cannot solve for x and y in the form of (3) above. However, we can consider forms (1) and (2), since they do not involve division by zero. Evaluating the other determinants, we have

$$\begin{vmatrix} 5 & -14 \\ 8 & -21 \end{vmatrix} = 5(-21) - 8(-14) = -105 + 112 = 7$$

$$\begin{vmatrix} 10 & 5 \\ 15 & 8 \end{vmatrix} = 10\cdot 8 - 15\cdot 5 = 80 - 75 = 5.$$

Thus (1) and (2) lead to

$$\text{(I)} \qquad \begin{aligned} 0 &= 0\cdot x = 7 \\ 0 &= 0\cdot y = 5. \end{aligned}$$

Because the contradictory equations (I) are implied by the system (H), this latter system has no simultaneous solution.

It can be shown that if the determinant of system (G) is zero, then (G) has no solution or an infinite number of solutions.

9.2 DETERMINANTS OF ORDER 3

When the elimination technique is applied to the general system

(1) $$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

of first-degree equations in the three unknowns x, y, and z, one obtains equations of the form

(2) $$Dx = E, \quad Dy = F, \quad Dz = G$$

where D is the three-by-three determinant (or determinant of order 3) defined by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

In the equations above, E, F, and G are obtained by substituting the column of d's for the column of a's, b's, or c's, respectively, in D. For example,

$$F = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1d_2c_3 - a_1d_3c_2 + a_2d_3c_1 - a_2d_1c_3 + a_3d_1c_2 - a_3d_2c_1.$$

If $D \neq 0$, it follows from the equations (2) above that the system of simultaneous equations (1) has the unique solution

(3) $$x = \frac{E}{D}, \; y = \frac{F}{D}, \; z = \frac{G}{D}.$$

As in the case of 2 by 2 determinants, it can be shown that if $D = 0$, then the simultaneous equations (1) either have no common solution or an infinite number of common solutions. The determinant D is called the **determinant of the system**. The technique of expressing the solution of a system of simultaneous linear equations in terms of ratios of determinants when the determinant of the system is not zero, as in (3) above, is called **Cramer's Rule**.

We note that, according to the definition

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

given above, the three-by-three determinant D consists of a sum of products of the form $\pm\ a_ib_jc_k$ where i, j, k is a permutation of 1, 2, 3, and the plus sign is chosen when the permutation is even and the minus sign when it is odd. (For a definition of even and odd permutations, see Chapter 7.) Since there is a term corresponding to each permutation, the number of terms is 3! = 6, half preceded by a plus sign and half by a minus sign. (See Problem 21, Chapter 7.) Is should be noted that these observations also apply to two-by-two determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

in that here the permutation 1, 2 is even and the permutation 2, 1 is odd, so that the 2! = 2 terms are preceded by the appropriate signs.

Example. Evaluate the determinant of the following system and thus show that the system has a unique solution:

$$\begin{aligned} 2x - 3z &= 10 \\ 5x + 4y &= 11 \\ y - 6z &= -3 \end{aligned}$$

Solution:

$$D = \begin{vmatrix} 2 & 0 & -3 \\ 5 & 4 & 0 \\ 0 & 1 & -6 \end{vmatrix}$$

$$\begin{aligned} &= 2\cdot4\cdot(-6) - 2\cdot1\cdot0 + 5\cdot1\cdot(-3) - 5\cdot0\cdot(-6) + 0\cdot0\cdot0 - 0\cdot4\cdot(-3) \\ &= -48 - 15 = -63 \end{aligned}$$

Since $D \neq 0$, there is a unique solution.

We now introduce a double subscript notation for the entries in a determinant, with the first subscript giving the row and the second giving the column. In this notation, a determinant of order 3 can be written as

(4) $$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and the expanded value is

(5) $$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Each term in (5) is of the form $\pm a_{1i}a_{2j}a_{3k}$ with the plus sign used if *i*, *j*, *k* is an even permutation of 1, 2, 3 and the minus sign if *i*, *j*, *k* is an odd permutation. The terms in (5) can be grouped in several ways, one of which is

(6) $$D = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

The coefficient $a_{21}a_{32}$ - $a_{22}a_{31}$ of a_{13} in (6) is the value of the 2 by 2 determinant that results when the entire first row and third column of D are removed; the coefficients of a_{11} and $-a_{12}$ in (6) can be characterized similarly.

This motivates the following definitions: Let a_{ij} be a given entry in the determinant D of (4) and let M_{ij} be the 2 by 2 determinant obtained by deleting the *i*th row and the *j*th column of D. This determinant M_{ij} is called the **minor** of the entry a_{ij}. The **cofactor** C_{ij} of the entry a_{ij} is defined by the formula

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

For example, the minor of a_{23} is

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{12}a_{31}$$

and the cofactor of a_{23} is

$$C_{23} = (-1)^{2+3}M_{23} = -(a_{11}a_{32} - a_{12}a_{31}).$$

It now can easily be seen that equation (6) may be rewritten as

$$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

or as

$$D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

Problems for Sections 9.1 and 9.2

1. Solve the following system by determinants, that is, by using Cramer's rule.

$$\begin{aligned} 9x - 7y &= 11 \\ 5x + 2y &= -4 \end{aligned}$$

2. Show by determinants that the following system has no solution.

$$\begin{aligned} 15x + 20y &= -13 \\ 21x + 28y &= 5 \end{aligned}$$

3. Solve the system given in the example of Section 9.2.

4. Use Cramer's rule to solve the following system:

$$\begin{aligned} 7x - 3y - z &= 0 \\ 2x - 2y + z &= 5 \\ -x + y + 2z &= 25 \end{aligned}$$

5. Use Cramer's rule to solve the following system:

$$\begin{aligned} x - y + 3z &= 15 \\ 2x - 4y + z &= 6 \\ 3x + 3y - 6z &= -3 \end{aligned}$$

6. Solve the following system for *x, y,* and *z*:

$$\begin{aligned} 2x + y &= 0 \\ y + 2z &= 0 \\ x + z &= 0 \end{aligned}$$

7. Solve the following system for *x, y,* and *z* in terms of *a, b,* and *c*:

$$\begin{aligned} x + y &= c \\ y + z &= a \\ x + z &= b \end{aligned}$$

8. Show the following for the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(a) $D = a_1M_1 - a_2M_2 + a_3M_3$, where M_1, M_2, and M_3 are the minors of a_1, a_2, and a_3 respectively.

(b) $D = a_2A_2 + b_2B_2 + c_2C_2$, where A_2, B_2, and C_2 are the cofactors of a_2, b_2, and c_2, respectively.

9. Show the following for the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

(a) $a_{31}M_{21} - a_{32}M_{22} + a_{33}M_{23} = 0.$

(b) $a_{11}C_{13} + a_{21}C_{23} + a_{31}C_{33} = 0.$

10. Show the following:

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$

(b) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

11. (a) Show that $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$

(b) Show that if each element of a fixed row of a three by three determinant D is multiplied by a factor k, the new determinant equals kD.

12. (a) Evaluate $\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$

(b) Show that if a three by three determinant D has a row of zeros, then $D = 0$.

(c) Show that if a three by three determinant D has a column of zeros, then $D = 0$.

13. Show the following:

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$.

(b) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$.

(c) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$.

(d) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$.

(e) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

14. Evaluate $\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix}$.

15. Show that in a three by three determinant if the elements of one row are a constant k times the elements of another row, then the determinant equals zero.

16. (a) Use the definition of a determinant to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + a_2' & b_2 + b_2' & c_2 + c_2' \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2' & b_2' & c_2' \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(b) Show that if the elements of a given row of a three by three determinant D are $f_1 + g_1$, $f_2 + g_2$, $f_3 + g_3$, then $D = D_1 + D_2$ where D_1 results from D by replacing the given row by f_1, f_2, f_3 and D_2 by replacing the given row by g_1, g_2, g_3.

17. Show that if two rows of a three by three determinant D are u_1, u_2, u_3 and v_1, v_2, v_3, respectively, and if D^* is the same as the determinant D except that the row u_1, u_2, u_3 is replaced by $u_1 + kv_1$, $u_2 + kv_2$, $u_3 + kv_3$ where k is a constant, then $D^* = D$.

9.3 DETERMINANTS OF ORDER *n*

We have defined the 2 by 2 determinant

(1) $$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

to be the number $ad - bc$ obtained from the square array

(2) $$\begin{matrix} a & b \\ c & d \end{matrix}$$

of 4 numbers in two rows and two columns. Thus, bordering the array (2) with vertical lines converts the array into a symbol for the number D. Similarly, a 3 by 3 determinant is a number obtained in a certain manner from a square array of 9 numbers.

Our next objective is to define an n by n determinant. More precisely, we seek an unambiguous rule for obtaining a number from an n by n square array of numbers and want this rule to agree with the previous definitions when $n = 2$ or 3. Let

(3) $$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix}$$

be an array of n^2 numbers a_{ij}, where the first subscript designates the row and the second designates the column.

There are $n!$ products

$$a_{1i}a_{2j}a_{3h}\cdots a_{nk}$$

with exactly one factor from each row and exactly one factor from each column. The determinant D associated with the array (3) is the sum of the $n!$ terms

$$\pm a_{1i}a_{2j}a_{3h}\cdots a_{nk}$$

where the plus sign is used when the permutation

$$i, j, h, \ldots , k$$

is even, and the minus sign is used when the permutation is odd. As before, the array (3) is bordered with vertical lines in writing the symbol

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{4}$$

for a determinant D of order n.

The ***minor*** M_{ij} of the entry a_{ij} in D is the determinant of order n - 1 obtained by deleting the ith row and jth column of D. The ***cofactor*** of a_{ij} is defined by

$$C_{ij} = (-1)^{i+j}M_{ij}\,.$$

Associated with the determinant D of (4) is the determinant

$$D' = \begin{vmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \cdots & & & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix}$$

in which the entries in the ith column are the corresponding entries in the ith row of D. The determinant D' is called the **transpose** of D; it is easily seen that D is then the transpose of D'. For example

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \text{ and } \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

are transposes of each other.

The following theorem enables one to prove results involving columns of a determinant from the corresponding results on the rows and vice versa.

THE TRANSPOSE THEOREM: Let D' be the transpose of a determinant D. Then $D' = D$.

Proof: Let a_{rs} and b_{rs} be the entries of D and D', respectively, in the rth row and sth column. Since D' is the transpose of D, $b_{rs} = a_{sr}$. By definition, D is the sum of $n!$ terms

(5) $$\pm\, a_{1i}a_{2j}\cdots a_{nh}$$

where the plus sign is used if the permutation

$$i, j, ..., h$$

is even and the minus sign if it is odd. Also, D' is the sum of $n!$ terms

(6) $$\pm\, b_{1x}b_{2y}\cdots b_{nz}$$

where the sign is plus if

(7) $$x,\ y,\ \dots,\ z$$

is an even permutation, and minus otherwise. Since $b_{rs} = a_{sr}$, each term (6) can be rewritten as

(8) $$\pm\, a_{x1}a_{y2}\cdots a_{zn}$$

The terms (8) are all the products, with signs attached, in which there is exactly one factor from each row and from each column of D; hence the terms (8) are the terms of the expansion of D, except that the signs may not agree. We will therefore prove that $D = D'$ by showing that (8) has the same sign as a term of the expansion of D that it has in the expansion of D'. We do this by describing a method of determining the sign whether or not the row numbers of the entries are in the order 1, 2, ..., n. Let

(9) $$\pm\, a_{pu}a_{qv}\cdots a_{rw}$$

be a term of the expansion of D with its factors in any order. Then the row subscript numbers and the column subscript numbers give us the two permutations:

$$(10) \qquad \begin{matrix} p, & q, & \ldots, & r \\ u, & v, & \ldots, & w \end{matrix}$$

We wish to prove that the plus sign should be used in (9) if both of the permutations of (10) are even or both are odd, and that the minus sign should be used when one is even and the other odd. When the entries in (9) have their row numbers in the normal order, the permutations (10) are of the form

$$\begin{matrix} 1, 2, \ldots, n \\ i, j, \ldots, h \end{matrix}$$

with the top one even, and hence the new 2-permutation rule indicates that the sign should be plus when the bottom permutation is even and minus otherwise. This agrees with the definition of a determinant and shows that the new rule is correct in this case.

We can go from the order $a_{1i}a_{2j}...a_{nh}$ of the factors to any order $a_{pu}a_{qv}...a_{rw}$ by means of a number of interchanges of adjacent factors. Whenever one such interchange is made, the row subscript permutation and column subscript permutation will each change from even to odd or from odd to even. This means that the new rule will continue to indicate the correct sign as these interchanges are made. We are especially interested in the case in which the column numbers are in order, that is, the case in which (9) is (8)

$$\pm a_{x1}a_{y2}\cdots a_{zn}.$$

The permutations (10) then become

$$(11) \qquad \begin{matrix} x, & y, & \ldots, & z \\ 1, & 2, & \ldots, & n \end{matrix}$$

with the bottom one even. Hence the 2-permutation rule indicates that the plus sign is used if and only if the permutation $x, y, ..., z$ of (11) is even. This is exactly the rule for determining the sign of (8)

$$\pm a_{x1}a_{y2}\cdots a_{zn}$$

as a term of D'. Hence the sign for (8) is the same either as a term of D' or of D, since the sign in both cases agrees with the 2-permutation rule. This shows that $D' = D$ and completes the proof.

Let

$$(12) \qquad \begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{matrix}$$

be a system of n simultaneous first-degree equations in n unknowns, $x_1, x_2, ..., x_n$. The determinant D of (4) is the **determinant of the system** for (12).

Let $D_1, D_2, ..., D_n$ be the determinants that result when the column of b's in (12) is substituted for the first, second, ..., nth column, respectively, of the D of (4). The general **Cramer's Rule** states that if $D \neq 0$, then the system (12) has the unique solution

(13) $$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ \ldots,\ x_n = \frac{D_n}{D}$$

and that if $D = 0$, then the system either has no solution or an infinite number of solutions. We do not give the proof of this rule for general n.

When the number of equations in (12) is large, it becomes very difficult to evaluate the $n + 1$ determinants in (13) by the methods discussed in this book. More advanced texts describe variations of the elimination techniques that are practical for the numerical approximation of determinants of large order or in the solution of systems (12) with n large. (For example, see the description of Crout's Method in the appendix of F. B. Hildebrand's *Methods of Applied Mathematics*, Prentice-Hall, 1952.)

Problems for Section 9.3

In Problems 1-7, below, D represents a determinant of order n. Prove each statement either from the definition of an n by n determinant, by using the Transpose Theorem, or by using previous results.

R 1. If all the entries on a given row (or column) of D are multiplied by a fixed number k, the value of D is multiplied by k.

R 2. If each entry in a given row (or column) of D is zero, then $D = 0$.

R 3. (a) If any two columns of a determinant D are interchanged, the resulting determinant D_1 equals $-D$. (See Problem 21, Chapter 7.)

(b) If any two rows of a determinant D are interchanged, the resulting determinant D_2 equals $-D$.

R 4. If the entries of a row (or column) of D are a constant k times the corresponding entries of another row (or column), then $D = 0$.

R 5. If the entries of a given row (or column) of D are $f_1 + g_1$, $f_2 + g_2$, ..., $f_n + g_n$, then $D = D_1 + D_2$, where D_1 results from D by replacing the given row (or column) by $f_1, f_2, \ldots, f_n$ and D_2 by replacing the given row (or column) by $g_1, g_2, \ldots, g_n$.

R 6. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the entries of two rows (or columns) of D, and let D^* result from replacing $v_1, v_2, \ldots, v_n$ in D by $v_1 + ku_1, v_2 + ku_2, \ldots, v_n + ku_n$, respectively. Then $D^* = D$.

R 7. Let a_{ij} be the entry in the ith row and jth column of D.

(a) If S is the sum of all the terms of the expansion of D that involve a_{nn}, then $S = a_{nn}M_{nn} = a_{nn}C_{nn}$, where M_{nn} and C_{nn} are the minor and cofactor of a_{nn}. (See Problem 22, Chapter 7.)

(b) Let T be the sum of all the terms of the expansion of D that involve a fixed entry a_{hk}. Then $T = (-1)^{h+k}a_{hk}M_{hk} = a_{hk}C_{hk}$. (Use Problem 3, above, and Part (a) of this problem.)

(c) If h is one of the numbers 1, 2, ..., n, then each term of the expansion of D has one and only one of the entries $a_{h1}, a_{h2}, ..., a_{hn}$ as a factor.

(d) $D = (-1)^{h+1}a_{h1}M_{h1} + (-1)^{h+2}a_{h2}M_{h2} + \cdots + (-1)^{h+n}a_{hn}M_{hn}.$

(e) $D = a_{h1}C_{h1} + a_{h2}C_{h2} + \cdots + a_{hn}C_{hn}.$

(f) If k is any one of the numbers 1, 2, ..., n then $D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}$.

8. (a) Find specific numbers a, b, c, and d such that the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has the values listed in the table below. Check, using $f(5) = 165$.

x	1	2	3	4
$f(x)$	1	10	35	84

(b) Follow the instructions of the previous part for the table below. Check, using $f(4) = 30$.

x	0	1	2	3
$f(x)$	0	1	5	14

9. (a) Show that

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}$$

and that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}.$$

(b) Let D be an n by n determinant, with the element a_{ij} in the ith row and jth column 0 if $i > j$. Show that $D = a_{11}a_{22}a_{33}...a_{nn}$.

10. (a) Evaluate $\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ a & 1 & a \\ a^2 & a & 1 \end{vmatrix}$.

(b) Show that $\begin{vmatrix} 1 & a & a^2 & a^3 \\ a & 1 & a & a^2 \\ a^2 & a & 1 & a \\ a^3 & a^2 & a & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1-a^2 & a-a^3 & a^2-a^4 \\ 0 & 0 & 1-a^2 & a-a^3 \\ 0 & 0 & 0 & 1-a^2 \end{vmatrix}$.

(c) Evaluate $\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ a & 1 & a & a^2 & a^3 \\ a^2 & a & 1 & a & a^2 \\ a^3 & a^2 & a & 1 & a \\ a^4 & a^3 & a^2 & a & 1 \end{vmatrix}$.

(d) In the determinants of Parts (a), (b), and (c) of this problem, the element c_{ij} in the ith row and jth column is $a^{|i-j|}$. Establish a compact formula for the value of the n by n determinant with $c_{ij} = a^{|i-j|}$. (See Section 8.4 for a definition of |x|.)

11. Show that $\begin{vmatrix} r-s & s-t & t-r \\ s-t & t-r & r-s \\ t-r & r-s & s-t \end{vmatrix} = 0$.

12. Show that $\begin{vmatrix} x_2+x_3 & x_1+x_3 & x_1+x_2 \\ y_2+y_3 & y_1+y_3 & y_1+y_2 \\ z_2+z_3 & z_1+z_3 & z_1+z_2 \end{vmatrix} = 2\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$.

13. Show that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$.

14. Show that $\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = (a+b+c+d)(a-b+c-d)[(a-c)^2 + (b-d)^2]$.

15. Evaluate $\begin{vmatrix} x+y & z & z \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$.

16. Evaluate

(a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$.

(b) $\begin{vmatrix} 1+a & 1+a & 1+a & 1+a \\ 1+a & a & a & a \\ 1+a & a & 1+a & a \\ 1+a & a & a & 1+a \end{vmatrix}$.

17. Let F_0, F_1, F_2, ... be the Fibonacci sequence.

(a) Show that $\begin{vmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{vmatrix} = -\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix}$.

(b) Find numbers x, y, and z such that $F_{2n} = xF_{\mathrm{n}}^2 + yF_nF_{n+1} + zF_{n+1}^2$.

(c) Find x, y, z, and w such that $F_{3n} = xF_n^3 + yF_n^2F_{n+1} + zF_nF_{n+1}^2 + wF_{n+1}^3$.

(d) Find analogues of the formulas above for the Lucas numbers.

18. Evaluate:

(a) $\begin{vmatrix} 1 & x & y & z+w \\ 1 & y & z & w+x \\ 1 & z & w & x+y \\ 1 & w & x & y+z \end{vmatrix}$.

(b) $\begin{vmatrix} x & y & z & w \\ y & x & z & w \\ y & x & w & z \\ x & y & w & z \end{vmatrix}$.

19. Let D be an n by n determinant with c_{ij} the entry in the ith row and jth column. Show that $D = 0$ if n is odd and $c_{ij} + c_{ji} = 0$ for all i and j.

20. Evaluate the n by n determinant with the entry c_{ij} in the ith row and jth column satisfying each of the following conditions: (It may be helpful to begin with small values of n and to try to find a pattern which suggests a proof.)

(a) $c_{ij} = \binom{i+j-2}{j-1}$.
(b) $c_{ij} = c_{1j}$ if $i > j$.
(c) $c_{ij} = a + |i - j|d$. (See Section 8.4 for a definition of |x|.)
(d) $c_{ij} = 1$ if $j - i$ is -1, 0, or a positive even integer, and $c_{ij} = 0$ for other values of $j - i$.
(e) $c_{ij} = a + x$ if $j > i$, $c_{ij} = b + x$ if $j < i$, and $c_{ii} = r_i + x$.
(f) $c_{ij} = \dfrac{1}{(j+2-i)!}$ for $i < j + 2$ and $c_{ij} = 0$ for $i \geq j + 2$.

9.4 VANDERMONDE AND RELATED DETERMINANTS

Determinants in which the elements of each column (or row) are the terms 1, r, r^2, ..., r^{n-1} of a geometric progression are called **Vandermonde determinants**, named for Alexandre-Théophile Vandermonde (1735-1796), who was the first to give a systematic treatment of the theory of determinants.

Let us evaluate the 4 by 4 Vandermonde determinant

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & b & c \\ x^2 & a^2 & b^2 & c^2 \\ x^3 & a^3 & b^3 & c^3 \end{vmatrix}.$$

The expansion by minors of first column entries as outlined in Problem 7, Section 9.3, yields the following:

$$D = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} - x\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + x^2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} - x^3\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Letting r, s, t, and u stand for the 3 by 3 determinants in this expression, we may write

$$D = f(x) = r - sx + tx^2 - ux^3.$$

If we let $x = a$ in D, two columns become identical and, by Problem 4, Section 9.3, D becomes zero. This means that $f(a) = 0$, and it follows from the Factor Theorem that $x - a$ is a factor of $f(x)$. Similarly, $x - b$ and $x - c$ are factors of $f(x)$.

If two of the numbers a, b, and c are equal, D has identical columns and thus is zero. We therefore assume that a, b, and c are distinct numbers. Then $f(x)$ is a multiple of the product of $x - a$, $x - b$, and $x - c$. Since $f(x)$ is a polynomial of degree 3 or less and has $-u$ as the coefficient of x^3, this means that $f(x)$ must be $-u(x - a)(x - b)(x - c)$. This can be written as

$$u(a - x)(b - x)(c - x).$$

We leave it as an exercise for the reader (in Problem 1 below) to show that the 3 by 3 determinant u is expressible as $(b - a)(c - a)(c - b)$. Substituting this for u in the above gives the result:

$$D = (a - x)(b - x)(c - x)(b - a)(c - a)(c - b).$$

Problems for Section 9.4

R 1. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - a)(c - a)(c - b)$.

2. Show that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (b - a)(c - a)(c - b)(a + b + c)$.

3. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (b - a)(c - a)(c - b)(bc + ca + ab)$.

4. Express $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ as a product of 6 first-degree factors.

5. Express $\begin{vmatrix} 1 & -x & x^2 & -x^3 \\ 1 & -y & y^2 & -y^3 \\ 1 & -z & z^2 & -z^3 \\ 1 & -w & w^2 & -w^3 \end{vmatrix}$ as a product of 6 first-degree factors.

6. Evaluate $\begin{vmatrix} 1 & a+x \\ 1 & a+y \end{vmatrix}$ and $\begin{vmatrix} 1 & a+x & b+cx+x^2 \\ 1 & a+y & b+cy+y^2 \\ 1 & a+z & b+cz+z^2 \end{vmatrix}$ in factored form.

7. Evaluate $\begin{vmatrix} 1 & 2-x & 3+4x+x^2 & 5-x^3 \\ 1 & 2-y & 3+4y+y^2 & 5-y^3 \\ 1 & 2-z & 3+4z+z^2 & 5-z^3 \\ 1 & 2-w & 3+4w+w^2 & 5-w^3 \end{vmatrix}$ in factored form.

8. Show the following:

(a) $\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 1 & y & y^2 \end{vmatrix} = (y-x)^2.$

(b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ x & 1 & 0 & y \\ x^2 & 2x & 2 & y^2 \\ x^3 & 3x^2 & 6x & y^3 \end{vmatrix} = 2(y-x)^3.$

9. Find integers r, s, and t such that

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ x & 1 & y & z \\ x^2 & 2x & y^2 & z^2 \\ x^3 & 3x^2 & y^3 & z^3 \end{vmatrix} = (y-x)^r(z-x)^s(z-y)^t.$$

10. Show that $\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 0 & 1 & 2x & 3x^2 & 4x^3 \\ 0 & 0 & 2 & 6x & 12x^2 \\ 1 & y & y^2 & y^3 & y^4 \\ 0 & 1 & 2y & 3y^2 & 4y^3 \end{vmatrix} = 2(y-x)^6.$

11. Find integers r, s, and t such that $\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ x & 1 & y & 1 & z \\ x^2 & 2x & y^2 & 2y & z^2 \\ x^3 & 3x^2 & y^3 & 3y^2 & z^3 \\ x^4 & 4x^3 & y^4 & 4y^3 & z^4 \end{vmatrix} = (y-x)^r(z-x)^s(z-y)^t$.

12. Solve the following system of equations if $a < b < c$:

$$\begin{aligned} x + y + z &= 3 \\ ax + by + cz &= a + b + c \\ a^2x + b^2y + c^2z &= a^2 + b^2 + c^2. \end{aligned}$$

13. (a) Show that the system

$$\begin{aligned} x + y + z &= 3 \\ a^2x + b^2y + c^2z &= a^2 + b^2 + c^2 \\ a^3x + b^3y + c^3z &= a^3 + b^3 + c^3 \end{aligned}$$

has a solution if the a, b, c are distinct and $ab + bc + ca \neq 0$.

(b) Calculate the solution.

14. Evaluate:

(a) $\begin{vmatrix} 1 & a & a^2 + 2bc \\ 1 & b & b^2 + 2ca \\ 1 & c & c^2 + 2ab \end{vmatrix}$.

(b) $\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}$.

(c) $\begin{vmatrix} 1 & a & a^2 & a^3 + 2bcd \\ 1 & b & b^2 & b^3 + 2cda \\ 1 & c & c^2 & c^3 + 2dab \\ 1 & d & d^2 & d^3 + 2abc \end{vmatrix}$.

15. Let D be the general n by n Vandermonde determinant with the entry c_{ij} in the ith row and jth column given by $c_{ij} = a_j^{i-1}$. Show that D is the product

$$(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)...(a_n - a_{n-1})$$

of all the different $a_r - a_s$ with $1 \leq s < r \leq n$.

Chapter 10

INEQUALITIES

We deal only with real numbers in this chapter.

10.1 ELEMENTARY PROPERTIES

If a and b are real numbers, $a < b$ (read "a is less that b") is defined to mean that $b - a$ is positive. This definition and the following three properties can be used in proving elementary properties of inequalities:

(1) **Closure of the positive numbers**. If a and b are positive numbers, then $a + b$ and ab are positive numbers.

(2) **Trichotomy**. One and only one of the following is true for a given real number a: (a) a is zero; (b) a is positive; (c) $-a$ is positive.

(3) **Roots**. If p is a positive number and n is a positive integer, then there is exactly one positive number r such that $r^n = p$. (This number r is called the positive nth root of p or the principal nth root of p.)

We can write the statement $a < b$ in the form $b > a$ (read "b is greater than a"). The notation $a \leq b$ (read "a is less than or equal to b") means that either $a < b$ or $a = b$, and $b \geq a$ is defined analogously. The notation $x < y < z$ or $z > y > x$ means that $x < y$ and $y < z$ are true simultaneously.

As mentioned is Section 8.4, the **absolute value** of a real number x is written as $|x|$ and is defined as follows: If $x \geq 0$, then $|x| = x$; if $x < 0$, then $|x| = -x$.

Example 1. Show that if $x < y$ and $y < z$, that is $x < y < z$, then $x < z$.

Solution: If $x < y$ then $y - x = p$, a positive number, and similarly $y < z$ implies that $z - y = q$ where q is positive. Hence $(z - y) + (y - x) = q + p$, or

$$z - x = q + p.$$

Since $z - x$ is the sum of the positive numbers q and p, it is positive by the closure property and thus $x < z$ by the definition.

Example 2. Show that if $x < y$ and $p > 0$, then $px < py$.

Solution: If $x < y$, then $y - x$ is positive and, by closure, the product $p(y - x)$ of positive numbers is positive; that is, $py - px$ is positive. Now we have $px < py$ by definition.

Example 3. Show that if m and n are integers and $m < n$, then $n - m \geq 1$.

Solution: Since $m < n$, it follows that $n - m$ is positive. Since the difference of integers is an integer and the least positive integer is 1, $n - m$ is at least 1; that is $n - m \geq 1$.

Problems for Section 10.1

R 1. (a) Show that if $x < y$, then $x + z < y + z$.
(b) Show that if $x < y$, then $x - w < y - w$.

R 2. Show that if $x < y$ and $q < 0$, then $qx > qy$.

R 3. (a) Show that if $x > 0$ or $x < 0$, then $x^2 > 0$.
(b) Show that for all real x, $x^2 \geq 0$.
(c) Show that $1 > 0$.

R 4. (a) Show that if $x > 0$, then $\frac{1}{x} > 0$, and if $x < 0$, then $\frac{1}{x} < 0$.

(b) Show that if $0 < x < y$ or $x < y < 0$, then $\frac{1}{x} > \frac{1}{y}$.

R 5. Show the following:

(a) If $0 < x < y$ and n is a positive integer, then $x^{2n-1} < y^{2n-1}$.
(b) If $x < 0 < y$ and n is a positive integer, then $x^{2n-1} < y^{2n-1}$.
(c) If $x < y < 0$ and n is a positive integer, then $x^{2n-1} < y^{2n-1}$.

R 6. (a) Show that if $0 < x < y$ and n is a positive integer, then $x^{2n} < y^{2n}$.
(b) Show that if $y < x < 0$ and n is a positive integer, then $x^{2n} < y^{2n}$.

R 7. Show that if n is a positive integer and $x^{2n-1} < y^{2n-1}$, then $x < y$. (See Problem 5.)

R 8. (a) Show that if $x^{2n} < y^{2n}$ and $y > 0$, then $-y < \pm x < y$. (See Problem 6.)
(b) Show that if $x^{2n} < y^{2n}$ and $y < 0$, then $y < \pm x < -y$.
(c) Use Parts (a) and (b) to show that if $x^{2n} < y^{2n}$, then $-|y| < \pm x < |y|$.

R 9. Prove the following by mathematical induction:

(a) If $a_1, a_2, ..., a_n$ are positive, so is $a_1 + a_2 + ... + a_n$.
(b) If $a_1, a_2, ..., a_n$ are positive, so is $a_1a_2...a_n$.
(c) If $a_1 < b_1, a_2 < b_2, ..., a_n < b_n$, then $a_1 + a_2 + ... + a_n < b_1 + b_2 + ... + b_n$.
(d) If $0 < a_1 < b_1, 0 < a_2 < b_2, ..., 0 < a_n < b_n$, then $a_1a_2...a_n < b_1b_2...b_n$.

R 10. Show that:

(a) $x^2 - 2xy + y^2 \geq 0$.
(b) $x^2 + y^2 \geq 2xy$.

(See Problem 3.)

11. Given that $x \neq y$, show that:

(a) $x - y \neq 0$.
(b) $x^2 - 2xy + y^2 > 0$.
(c) $x^2 + y^2 > 2xy$.

12. Find all the integers n such that $2n^2 - 3 < 8n$.

13. (a) Given that $0 < a < b$, show that $a^2 < ab < b^2$.
(b) Given that $0 < a < b$, show that $3a^2 < a^2 + ab + b^2 < 3b^2$.

14. (a) Given that $a < b$, show that $a < \frac{a + b}{2} < b$.

(b) Given that $a < b$, show that $a < \frac{2a + b}{3} < \frac{a + 2b}{3} < b$.

15. Given that $0 < x < y$, show the following:

(a) $\frac{x - 1}{x} < \frac{y - 1}{y}$.

(b) $\frac{x}{x + 1} < \frac{y}{y + 1}$.

16. Given that $0 < x \leq y \leq z$ and that $x + y > z$, show that $\frac{x}{x + 1} + \frac{y}{y + 1} > \frac{z}{z + 1}$.

17. (a) Given that $0 < a < b \leq c$, show that $c > \frac{a + b}{2}$.

(b) Given that $0 < a \leq b < c$, show that $c > \frac{a + b}{2}$.

(c) Given that $0 < a \leq b \leq c$ and a < c, show that $c > \frac{a + b}{2}$.

R 18. Given that $0 < a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{n-1} \leq a_n$ and $a_1 < a_n$, show that $a_n > (a_1 + a_2 + \dots + a_{n-1})/(n - 1)$.

19. Find integers a, b, and c such that $0 < a < b < c$, $a + b > c$, and c is as small as possible.

20. Let m and n be positive integers, and let 1, m, and n be the lengths of the sides of a triangle. Show that $m = n$.

21. Given that $x > 0$ and $y > 0$, show that $(x + y)^n > x^{n-1}(x + ny)$ for all integers $n \geq 2$.

22. Given that $1 + x \geq 0$, prove by mathematical induction that $(1 + x)^n \geq 1 + nx$ for all positive integers n.

23. Prove that $2\sqrt{x} < \dfrac{1}{\sqrt{x+1} - \sqrt{x}}$ for all positive real numbers x.

24. Prove that $\dfrac{2}{3}n\sqrt{n} < \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$ for all positive integers n.

25. Prove that $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} < \dfrac{(4n+3)\sqrt{n}}{6}$ for all positive integers n.

26. Use the fact that $1<b$ and $x<y$ imply $b^x<b^y$ to prove the inequalities $\sqrt{2} \leq a_n < a_{n+1} < 2$ for the sequence $a_1, a_2, \ldots$ defined by

$$a_1 = \sqrt{2},\ a_2 = \left(\sqrt{2}\right)^{a_1},\ \ldots,\ a_{n+1} = \left(\sqrt{2}\right)^{a_n},\ \ldots .$$

27. Use the fact that $0 < b < 1$ and $x < y$ imply $b^x > b^y$ to prove the inequalities

$$u_1 < u_2,\ u_2 > u_3,\ \ldots,\ u_{2k-1} < u_{2k},\ u_{2k} > u_{2k+1}, \ldots$$

for the sequence $u_1, u_2, \ldots$ defined by

$$u_1 = \frac{1}{\sqrt{2}},\ u_2 = \left(\frac{1}{\sqrt{2}}\right)^{u_1},\ \ldots,\ u_{n+1} = \left(\frac{1}{\sqrt{2}}\right)^{u_n},\ \ldots .$$

10.2 FURTHER INEQUALITIES

In this section we develop a technique for investigating the range of values assumed by a quadratic function. In subsequent work we shall assume as known the results of the examples in Section 10.1 and of Problems 1 to 8 in Section 10.1.

Example: Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. Let D be the ***discriminant*** $b^2 - 4ac$. Show that if $D > 0$, then $f(x)$ takes on both positive and negative values.

Solution: Completing squares, we obtain

$$\begin{aligned} f(x) = ax^2 + bx + c &= \frac{4a^2x^2 + 4abx + 4ac}{4a} \\ &= \frac{4a^2x^2 + 4abx + b^2 - (b^2 - 4ac)}{4a} \\ &= \frac{(2ax + b)^2 - D}{4a}. \end{aligned}$$

If $x = -b/2a$, $2ax + b = 0$, and so $f(-b/2a) = -D/4a$. We first consider the case in which $a > 0$. This and $D > 0$ imply that $f(x) = -D/4a < 0$. We wish to show that $f(x)$ also takes on positive values.

We consider values of x greater that $(-b + \sqrt{D})/2a$. Then

$$\begin{aligned} x &> \frac{-b + \sqrt{D}}{2a} \\ 2ax &> -b + \sqrt{D} \\ 2ax + b &> \sqrt{D} > 0 \\ (2ax + b)^2 &> D \\ (2ax + b)^2 - D &> 0 \\ f(x) = \frac{(2ax + b)^2 - D}{4a} &> 0. \end{aligned}$$

Thus we have proved the desired result for the case $a > 0$. If $a < 0$, let $g(x) = -ax^2 - bx - ac$. Since the coefficient of x^2 in $g(x)$ is positive, $g(x)$ takes on both positive and negative values by the previous case. Then so does $f(x) = -g(x)$.

Problems for Section 10.2

1. Let a and b be real numbers. Prove that $a^2 + b^2 \geq 0$ and that $a^2 + b^2 = 0$ if and only if $a = b = 0$.

2. Let $c_1, c_2, \ldots, c_n$ be real numbers. Prove that $c_1^2 + c_2^2 + \cdots + c_n^2 \geq 0$ and $c_1^2 + c_2^2 + \cdots + c_n^2 = 0$ if and only if each $c_i = 0$.

3. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a > 0$. Let D be the discriminant $b^2 - 4ac$. Show the following:

(a) If $D < 0$, then $f(x) > 0$ for all x, and $f(x) = 0$ has no real roots.
(b) If $D = 0$, then $f(x) \geq 0$ for all x, and $f(x) = 0$ has one real root.
(c) If $D > 0$, then $f(x) < 0$ for $\frac{-b - \sqrt{D}}{2a} < x < \frac{-b + \sqrt{D}}{2a}$, $f(x) > 0$ for $x < \frac{-b - \sqrt{D}}{2a}$ or $x > \frac{-b + \sqrt{D}}{2a}$, and $f(x) = 0$ has two real roots.

4. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a < 0$. Let $D = b^2 - 4ac$. Show the following:

(a) If $D < 0$, then $f(x) < 0$ for all x, and $f(x) = 0$ has no real roots.

(b) If $D = 0$, then $f(x) \leq 0$ for all x, and $f(x) = 0$ has one real root.

(c) If $D > 0$, then $f(x) > 0$ for $\frac{-b - \sqrt{D}}{2a} > x > \frac{-b + \sqrt{D}}{2a}$, $f(x) < 0$ for $x > \frac{-b - \sqrt{D}}{2a}$ or $x < \frac{-b + \sqrt{D}}{2a}$, and $f(x) = 0$ has two real roots.

*5. Let $F_1, F_2, F_3, \ldots$ be the sequence of Fibonacci numbers 1, 1, 2, 3, 5, ... and let $R_n = F_{n+1}/F_n$ for $n = 1, 2, 3, \ldots$. Do the following:

(a) Show that $R_{n+1} = 1 + \frac{1}{R_n}$.
(b) Prove that $R_{2n-1} < R_{2n+1} < R_{2n}$ and $R_{2n+1} < R_{2n+2} < R_{2n}$ for all positive integers n, that is, that $R_1 < R_3 < R_5 < R_7 < \ldots < R_8 < R_6 < R_4 < R_2$.

10.3 INEQUALITIES AND MEANS

We recall that the arithmetic mean of $a_1, a_2, \ldots, a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

and the geometric mean is

$$\sqrt[n]{a_1 a_2 \cdots a_n}.$$

We restrict $a_1, \ldots, a_n$ to be positive in discussing geometric means, since otherwise the definition might express the mean as an even root of a negative number.

We shall use A_n for the arithmetic mean of $a_1, a_2, \ldots, a_n$ and G_n for the geometric mean.

THEOREM: Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Then

$$A_n \geq G_n$$

that is,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Proof: We proceed by mathematical induction. When $n = 1$, it is clear that $A_1 = a_1$ and $G_1 = a_1$. Hence $A_1 = G_1$, and the theorem holds for $n = 1$.

We next prove it for $n = 2$. Since a_1 and a_2 are positive, $\sqrt{a_1}$ and $\sqrt{a_2}$ exist in the real number system by the roots property of Section 10.1, and so

$$\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geq 0$$

by Problem 3(b) of section 10.1. It follows that

$$\begin{aligned} a_1 - 2\sqrt{a_1}\sqrt{a_2} + a_2 &\geq 0 \\ a_1 + a_2 &\geq 2\sqrt{a_1}\sqrt{a_2} \\ \frac{a_1 + a_2}{2} &\geq \sqrt{a_1 a_2}. \end{aligned}$$

This is precisely the statement $A_2 \geq G_2$ of the theorem for $n = 2$.

We now assume the inequality true for $n = k$, that is, we assume that $A_k \geq G_k$, and with this as a basis shall prove that $A_{k+1} \geq G_{k+1}$.

If $a_1 = a_2 = \ldots = a_{k+1}$ then $A_{k+1} = a_1, G_{k+1} = a_1$, and so $A_{k+1} \geq G_{k+1}$. It remains to investigate the case in which the a_i are not all equal. Without loss of generality, we may assume that the a_i are numbered so that

$$0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{k+1}.$$

The fact that the a's are not all equal implies that $a_1 < a_{k+1}$. It now follows from Problem 18 of Section 10.1 that $a_{k+1} > A_k$. Since

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}$$

we have $kA_k = a_1 + a_2 + \ldots + a_k$, and hence

$$\begin{aligned}A_{k+1} &= \frac{a_1 + a_2 + \cdots + a_k + a_{k+1}}{k+1} \\ &= \frac{kA_k + a_{k+1}}{k+1} \\ &= \frac{(k+1)A_k + (a_{k+1} - A_k)}{k+1} \\ &= A_k + \frac{a_{k+1} - A_k}{k+1}.\end{aligned}$$

Let $(a_{k+1} - A_k)/(k + 1) = p$. We have seen above that $a_{k+1} > A_k$; this implies that $p > 0$. Now $A_{k+1} = A_k + p$. We raise both sides of this equality to the $(k + 1)$th power, obtaining

$$\begin{aligned}(A_{k+1})^{k+1} &= (A_k + p)^{k+1} \\ &= (A_k)^{k+1} + (k+1)(A_k)^k p + \binom{k+1}{2}(A_k)^{k-1}p^2 + \cdots + p^{k+1}.\end{aligned}$$

Since $p > 0$ and $A_k > 0$, all the terms in the binomial expansion on the right side are positive. There are $k + 2$ terms in this expansion; hence there are at least 4 terms. Now $(A_{k+1})^{k+1}$ is greater than the sum of the first two terms:

$$(A_{k+1})^{k+1} > (A_k)^{k+1} + (k + 1)p(A_k)^k.$$

Since $(k + 1)p = a_{k+1} - A_k$, this becomes

$$\begin{aligned}(A_{k+1})^{k+1} &> (A_k)^{k+1} + a_{k+1}(A_k)^k - (A_k)^{k+1} \\ (A_{k+1})^{k+1} &> a_{k+1}(A_k)^k.\end{aligned}$$

Having assumed above that $A_k \geq G_k$, we now have

$$\begin{aligned}&(A_{k+1})^{k+1} > a_{k+1}(A_k)^k \geq a_{k+1}(G_k)^k = a_{k+1}(a_1a_2\cdots a_k) \\ &(A_{k+1})^{k+1} > (G_{k+1})^{k+1}.\end{aligned}$$

From Problems 7 and 8, Section 10.1, it follows that $A_{k+1} > G_{k+1}$, and the theorem is proved.

We have actually proved more than is stated in the theorem; we have shown that $A_n > G_n$ unless all the a_i are equal.

There is a third type of mean that is used quite often: the **harmonic** **mean**. The harmonic mean of numbers a_1, a_2, ..., a_n, none of which is zero, is given by

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

In a **harmonic progression**, defined as a progression of the form

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots,\ \frac{1}{a+(n-1)d}$$

each term, except the end ones, is the harmonic mean of its two adjacent terms; that is, if a_1, a_2, ..., a_n are in harmonic progression, then

$$\frac{2}{\dfrac{1}{a_{k-1}}+\dfrac{1}{a_{k+1}}} = a_k.$$

The proof is left to the reader.

10.4 THE CAUCHY-SCHWARZ INEQUALITY

Another famous inequality is given various names in different texts, although in the United States it is usually referred to as the **Cauchy - Schwarz Inequality** (named for Augustin Cauchy, 1789-1857; and Hermann Amandus Schwarz, 1843-1921). Some call it the Schwarz Inequality, while others, including the Russians, call it the Cauchy-Buniakowski Inequality.

THEOREM: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any real numbers. Then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

that is,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

Proof: We define a polynomial function $f(x)$ by

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \ldots + (a_nx + b_n)^2.$$

Clearly $f(x)$ is positive or zero for all real numbers x, since it is a sum of squares. Now

$$\begin{aligned} f(x) &= (a_1^2x^2 + 2a_1b_1x + b_1^2) + (a_2^2x^2 + 2a_2b_2x + b_2^2) + \cdots + (a_n^2x^2 + 2a_nb_nx + b_n^2) \\ &= (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2). \end{aligned}$$

Let $a_1^2 + a_2^2 + \cdots + a_n^2 = A$, $a_1b_1 + a_2b_2 + \cdots + a_nb_n = B$, and $b_1^2 + b_2^2 + \cdots + b_n^2 = C$ so that

$f(x) = Ax^2 + 2Bx + C$. Since $f(x) \geq 0$ for all x, the discriminant $D \leq 0$, since $D > 0$ implies that $f(x)$ is sometimes positive and sometimes negative. (See Section 10.2.) Now

$$D = (2B)^2 - 4AC = 4B^2 - 4AC \leq 0.$$

Hence $B^2 - AC \leq 0$, and so $AC \geq B^2$. Translating this back into our original notation, we have the Cauchy-Schwarz Inequality:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2.$$

Examination of the above proof shows that

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} a_ib_i\right)^2$$

if and only if there is a fixed number x such that $a_ix + b_i = 0$ for all i, that is, the a_i and b_i are proportional.

The hypothesis for the inequality on the arithmetic and geometric means is that the numbers are all positive. The numbers in the Chauchy-Schwarz Inequality need not be positive. In fact,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

is unaltered by changes in the signs of the a_i and b_i, while

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2$$

is largest when all the signs are positive.

Problems for Sections 10.3 and 10.4

1. Given that $a, b, c, d, x, y, z,$ and w are positive real numbers, prove the following [from (a) to (z)]:

   (a) If $x + y = 2$, then $xy \leq 1$.
   (b) If $xyz = 1$, then $x + y + z \geq 3$.
   (c) If $xyz = 1$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$.

(d) If $x + y + z = 1$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$.

(e) $\left(\frac{a + b + c}{3}\right)^3 \geq abc$.

(f) $(a + b + c + d)^4 \geq 256abcd$.

(g) $(x + y)(x - y)^2 \geq 0$.

(h) $x^3 + y^3 \geq x^2y + xy^2$.

(i) $x^4 + y^4 \geq x^3y + xy^3 \geq 2x^2y^2$.

(j) $x^5 + y^5 \geq x^4y + xy^4 \geq x^3y^2 + x^2y^3$.

(k) $x + \frac{1}{x} \geq 2$.

(l) $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$.

(m) $\frac{x}{y} + \frac{x}{z} + \frac{y}{z} + \frac{y}{x} + \frac{z}{x} + \frac{z}{y} \geq 6$.

(n) $xy(x + y) + yz(y + z) + zx(z + x) \geq 6xyz$.

(o) $\frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x} \geq 4$.

(p) $a + b + c \geq \sqrt{bc} + \sqrt{ca} + \sqrt{ab}$.

(q) $3(a + b + c + d) \geq 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd} + \sqrt{cd}\right)$.

(r) $(x + y)(y + z)(z + x) \geq 8xyz$.

(s) $[(x + y)(x + z)(x + w)(y + z)(y + w)(z + w)]^2 \geq 4096(xyzw)^3$.

(t) If $x + y + z = 1$, then $(1 - z)(1 - x)(1 - y) \geq 8xyz$.

(u) If $x + y + z = 1$, then $\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)\left(\frac{1}{z} - 1\right) \geq 8$.

(v) $(y + z + w)(x + z + w)(x + y + w)(x + y + z) \geq 81xyzw$.

(w) If $x + y + z + w = 1$, then $(1 - x)(1 - y)(1 - z)(1 - w) \geq 81xyzw$.

(x) $(ab + xy)(ax + by) \geq 4abxy$.

(y) $[(ab + cd)(ac + bd)(ad + bc)]^2 \geq 64(abcd)^3$.

(z) If $x + y = 1$, then $x^2 + y^2 \geq \frac{1}{2}$.

2. Given that a, b, and c are positive real numbers, show that

$$(a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \geq 9a^2b^2c^2.$$

Is this true for all real numbers a, b, and c?

3. Show that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n.$$

4. Let $a_1, a_2, \ldots, a_{n-1}, a_n$ be positive, and let $a_i, a_j, \ldots, a_h, a_k$ be a permutation of these n numbers. Show that $\frac{a_1}{a_i} + \frac{a_2}{a_j} + \cdots + \frac{a_{n-1}}{a_h} + \frac{a_n}{a_k} \geq n$.

5. Let a, b, x and y be real numbers, with $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$. Show that:

   (a) $(ax + by)^2 \leq 1$.
   (b) $(ax - by)^2 \leq 1$.
   (c) $-1 \leq ax + by \leq 1$.
   (d) $-1 \leq ax - by \leq 1$.

6. Let a, b, c, x, y, and z be real numbers with $a^2 + b^2 + c^2 = 1 = x^2 + y^2 + z^2$. Show that:

   (a) $(ax + by + cz)^2 \leq 1$.
   (b) $-1 \leq ax + by + cz \leq 1$.

7. Let a, b, c, d, and e be real numbers. Show the following:

   (a) $a^2 + b^2 \geq 2ab$.
   (b) $a^2 + b^2 + c^2 \geq bc + ac + ab$.
   (c) $3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$.
   (d) $2(a^2 + b^2 + c^2 + d^2 + e^2) \geq a(b + c + d + e) + b(c + d + e) + c(d + e) + de$.

8. Show that $a^2 + b^2 + 1 \geq b + a + ab$ for all real numbers a and b.

9. Show that if x and y are positive real numbers with $x + y = 1$, then

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{25}{2}.$$

10. Show that if x, y and z are positive real numbers with $x + y + z = 1$, then

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(z + \frac{1}{z}\right)^2 \geq \frac{100}{3}.$$

11. Let a, b, and c be positive real numbers. Show that $\sqrt{3(bc + ca + ab)} \leq a + b + c$.

12. Let a, b, c, and d be positive. Show that

$$2\sqrt{6(ab + ac + ad + bc + bd + cd)} \ \le\ 3(a + b + c + d).$$

13. Let A_n, G_n, and H_n be the arithmetic, geometric, and harmonic mean, respectively, of positive numbers $a_1, a_2, \ldots, a_n$. Assuming $A_n \ge G_n$, show that $A_n \ge G_n \ge H_n$.

14. Show that $n^n \ge 1\cdot 3\cdot 5\cdots(2n - 1)$.

15. Show that $(1^k + 2^k + \cdots + n^k)^n \ \ge\ n^n(n!)^k$ for all positive integers n and k.

16. Show the following:

(a) $(n + 1)^n \ge 2\cdot 4\cdot 6\cdots(2n)$.

(b) $n^n\left(\dfrac{n + 1}{2}\right)^{2n} \ge (n!)^3$ for all positive integers n.

17. Show the following:

(a) $n\cdot 1 + (n - 1)\cdot 2 + (n - 2)\cdot 3 + \cdots + 2(n - 1) + 1\cdot n = \dbinom{n + 2}{3}$.

(b) $\left[\dfrac{(n + 1)(n + 2)}{6}\right]^{n/2} \ge n!$.

18. Show the following:

(a) $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$.

(b) $2^n \ge 1 + n\left(\sqrt{2}\right)^{n-1}$ for all positive integers n.

19. Do the following:

(a) Show that $\left(1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}\right)^n \ge \dfrac{n^n}{\sqrt{n!}}$.

(b) Show that $\sqrt{n + 1} - \sqrt{n} > \dfrac{1}{2\sqrt{n + 1}}$ for all positive integers n.

(c) Show by mathematical induction that $2\sqrt{n} > 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$.

(d) Show that $n! > \left(\frac{n}{4}\right)^n$.

20 Show that $\prod_{k=0}^{n} \binom{n}{k} \leq \left(\frac{2^n - 2}{n - 1}\right)^{n-1}$ for $n \geq 2$.

21. Do the following:

(a) Find the arithmetic mean of $a_1, a_2, ..., a_{100}$, given that $a_1 = 1$ and $a_2 = a_3 = ... = a_{100} = 100/99$.

(b) Prove that $\left(\frac{100}{99}\right)^{99/100} < \frac{101}{100}$.

(c) Prove that $\left(1 + \frac{1}{99}\right)^{99} < \left(1 + \frac{1}{100}\right)^{100}$.

(d) Prove that $\left(1 + \frac{1}{n}\right)^{n} < \left(1 + \frac{1}{n+1}\right)^{n+1}$ for all positive integers n.

22. Do the following:

(a) Find the arithmetic mean of $a_0, a_1, a_2, ..., a_{100}$, given that $a_0 = 1$ and $a_1 = a_2 = ... = a_{100} = 99/100$.

(b) Prove that $100^{201} > 99^{100} \cdot 101^{101}$.

(c) Prove that $\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}$ for all positive integers n.

23. Do the following:

(a) Find the arithmetic and geometric means of the roots of $x^4 - 8x^3 + 18x^2 - 11x + 2 = 0$, given that all the roots are positive.

(b) Given that all the roots of $x^6 - 6x^5 + ax^4 + bx^3 + cx^2 + dx + 1 = 0$ are positive, find $a, b, c,$ and d.

(c) Find all the roots of $x^{11} - 11x^{10} + ... - 1 = 0$, given that each root is positive.

24. Given tht a, b, and c are the lengths of the sides of a triangle, show that

$$3(bc + ac + ab) \le (a + b + c)^2 < 4(bc + ac + ab).$$

25. For all real numbers $a, b, c, x, y,$ and z show that

$$\sqrt{a^2 + b^2 + c^2} + \sqrt{x^2 + y^2 + z^2} \ge \sqrt{(a + x)^2 + (b + y)^2 + (c + z)^2}.$$

26. For all real numbers $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$, show that

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \ge \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2}.$$

27. Show that if a and b are positive real numbers and m and n are positive integers, then

$$\frac{m^m n^n}{(m + n)^{m+n}} \ge \frac{a^m b^n}{(a + b)^{m+n}}.$$

28. Let F_n and L_n be the nth Fibonacci and nth Lucas number, respectively. Prove that

$$\left(\frac{F_{4n}}{n}\right)^n > L_2 L_6 L_{10} \ldots L_{4n-2}$$

for all integers $n \ge 2$.

ANSWERS TO THE
ODD-NUMBERED PROBLEMS

Chapter1, page 5

1. 10.

3. 1.

5. $$\begin{array}{llll} 1 & 2 & 1 & \\ \underline{} & \underline{1} & \underline{2} & \underline{1} \\ 1 & 3 & 3 & 1. \end{array}$$

7. 1, 1 + 5 = 6, 5 + 10 = 15, 10 + 10 = 20, 10 + 5 = 15, 5 + 1 = 6, and 1;
1, 1 + 6 = 7, 6 + 15 = 21, 15 + 20 = 35, 20 + 15 = 35, 15 + 6 = 21, 6 + 1 = 7, and 1.

9. $\binom{9}{7}$ ' $\binom{9}{2}$ ' 36, $\binom{9}{8}$ ' $\binom{9}{1}$ ' 9.

11. $125x^3 + 150x^2y + 60xy^2 + 8y^3$.

17. $c = 56$, $m = 5$.

19. (a) 4. (b) 8. (c) 16. (d) 32. (e) 64. (f) 128.

21. (a) 0. (b) 0.

23. (a) 8. (b) 16. (c) 32. (d) 64.

25. $r = 6$, $s = 7$, $t = 6$, $u = 7$.

27. $\binom{996}{990}$.

31. $x^4 \ \% \ 4x^3y \ \% \ 6x^2y^2 \ \% \ 4xy^3 \ \% \ y^4 \ \% \ 4x^3z \ \% \ 12x^2yz \ \% \ 12xy^2z \ \%$
$4y^3z \ \% \ 6x^2z^2 \ \% \ 12xyz^2 \ \% \ 6y^2z^2 \ \% \ 4xz^3 \ \% \ 4yz^3 \ \% \ z^4$.

33. 3.

37. 3.

39. 36, of which 27 are odd.

41. 0, 1, 3, 7, and 15.

Chapter 2, page 14

3. 103.

7. $F_{n+4} = 3F_{n+1} + 2F_n$.

9. 321.

13. 702.

19. (a) F_{16} - 1. (b) F_{19} - 1.

21. (a) 2. (b) 3. (c) 5. (d) 8. (e) 13. (f) 21.

23. $r = 4$, $s = 5$, and $t = 11$.

25. $r = 15$, $s = 17$, and $t = 19$.

29. 204

33. $L_{2m+2} - 2$.

35. $r = 4$, $s = 6$, and $t = 8$.

37. (a) 1. (b) -1. (c) 1. (d) -1.

Chapter 3, page 18

1. (a) 5040. (b) 36. (c) 362,880. (d) 720.

5. (a) 5!. (b) 7!. (c) $(n + 1)!$.

7. $a = 14$ and $b = 10$ or $a = 24024$ and $b = 24023$.

9. $n^4 + 10n^3 + 35n^2 + 50n + 24$.

11. (a) 23. (b) 119. (c) 719.

15. $a = 6$, $b = 11$, and $c = 5$.

17. $(2m + 2)! - 1$.

19. (a) $(n + 2)! + (n + 1)! - 2$.

Chapter 4, page 23

1. (a) -4, 3, 10. (b) -19, -23, -27.

3. (a) 990. (b) 993. (c) 988.

5. 33/2.

7. (a) -33,698. (b) 501,994. (c) -11,385/4. (d) n^2.

9. 54, 1458, 13122.

11. 686, 4802, 33614.

13. ± 42.

15. (a) $(7^{1000} - 1)/6$. (b) $(1 - 7^{1000})/8$. (c) $(7^n - 1)/6$.

17. 9.

21. (a) $6\sqrt{3}$. (b) $6\sqrt{3}$. (c) 4. (d) 4.

23. 30.

27. $(x^{n+1} - y^{n+1})/(x - y)$.

29. (a) 5. (b) 24 miles per hour.

31. $2[1 - (1/2)^7]$.

35. (a) 0, 3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, 27, 30.
(b) 15. (c) 15. (d) $16 \bullet 15 = 240$.

Chapter 5, page 35

13. (a) $1 \& \frac{1}{n \% 1}$. (b) $\frac{x^{2^{n\%1}} \& y^{2^{n\%1}}}{x \& y}$.

15. $(n + 1)/2n$ for $n > 1$.

21. Using mathematical induction, one can show that:

(a) $F_{n+1}^2 - F_nF_{n+2} = (-1)^n$.

(b) $\dfrac{F_{n+2}^2 - F_{n+1}^2}{F_n} = F_{n+3}$.

(c) $F_{n-1} + F_{n+1} = L_n$.

Chapter 6, page 47

1. (a) 4,845. (b) 3,003. (c) -2,912.

3. $a = -3$, $b = 2$, $c = 0$.

7. 1, -1, 1, -1, 1, -1.

11. $\binom{-m}{r} = (-1)^r\binom{r+m-1}{m-1}$.

15. $(d/2)n^2 + [a - (d/2)]n$.

19. $\left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} b_i\right) = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$ and $\sum_{i=1}^{2}(a_ib_i) = a_1b_1 + a_2b_2$. These are not always equal, since, for example, they are unequal for $a_1 = a_2 = b_1 = b_2 = 1$.

23. $(1/6)n^3 - (1/2)n^2 + (1/3)n$.

25. $n^3 + 5n$.

27. $s = 3$, $t = 1$.

29. $r = 6$, $s = 7$, $t = 1$.

31. $(1/5)n^5 + (1/2)n^4 + (1/3)n^3 - (1/30)n$.

37. $(n^3 + 3n^2 + 2n)/6$.

Chapter 7, page 55

1. *a, b; a, c; a, d; a, e; b, c; b, d; b, e; c, d; c, e; d, e.*

3. *a, b; b, a; a, c; c, a; a, d; d, a; a, e; e, a; b, c; c, b; b, d; d, b; b, e; e, b; c, d; d, c; c, e; e, c; d, e; e, d.*

5. 7,920.

7. 30.

9. $\binom{12}{3}\binom{9}{7}$, $\binom{12}{3}\binom{9}{2}$, $\binom{12}{7}\binom{5}{3}$, $\binom{12}{7}\binom{5}{2}$, $\binom{12}{2}\binom{10}{7}$, $\binom{12}{2}\binom{10}{3}$.

11. -1,201,200.

17. 3^{100}.

19. 1,2,3,4; 2,1,4,3; 3,1,2,4; 4,1,3,2;
1,3,4,2; 2,4,3,1; 3,2,4,1; 4,3,2,1;
1,4,2,3; 2,3,1,4; 3,4,1,2; 4,2,1,3.

23. (a) 200. (b) 10,150.

Chapter 8

Sections 8.1, 8.2 and 8.3, page 64

1. $(x + 2)(x^3 + 3x^2 - 6x + 2) - 16$.

3. $\&1 \pm \sqrt{3}$.

5. $a = -27{,}604$.

7. $\&1$, 3, $\sqrt{2}\,i$, $\&\sqrt{2}\,i$.

9. $f(0) = 0 = f(a)$. Two factors are x and $x - a$.

11. 0, -2, -4, -2/3.

13. $\frac{5}{3}, \ \frac{-3+\sqrt{17}}{2}, \ \frac{-3-\sqrt{17}}{2}$.

15. $0, \ -\frac{1}{3}, \ \frac{2}{3}, \ \frac{1+\sqrt{3}i}{6}, \ \frac{1-\sqrt{3}i}{6}$.

17. $\pm1, \ \pm2, \ \pm3, \ \pm5, \ \pm6, \ \pm10, \ \pm15, \ \pm30$.

25. $x^8 - 48x^6 + 536x^4 - 1728x^2 + 400 = 0$.

27. (a) b.
(b) $c(2x + 1) + b$.
(c) $d(3x^2 + 3x + 1) + c(2x + 1) + b$.
(d) $\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} + \cdots + \binom{n}{n-1}x + \binom{n}{n}$.

29. (a) 0. (b) 0. (c) $6d$.

33. $a = 38$, $b = 84$, $c = 81$, $d = 34$, $e = 5$.

39. No.

Section 8.4, page 68

1. (a) 4. (b) 8/3. (c) 16. (d) 32/3. (e) 16/3. (f) $\frac{4\sqrt{3}}{3}$.

5. 1.

7. $a = p^3 - 2pq$, $b = p^4q - 3p^2q^2 + 2q^3$, $c = p^3q^3 - 2pq^4$, $d = q^6$.

9. (a) 0. (b) -3. (c) 1. (d) $y^3 - 3y - 1 = 0$.

11. (a) $x = 3$, $y = -3$, $z = 1$.
(b) $t = 4$, $u = 2$, $v = -4$, $w = 1$.

Chapter 9

Sections 9.1 and 9.2, page 78

1. $x = -6/53$, $y = -91/53$.

3. $x = 103/21$, $y = -71/21$, $z = -4/63$.

5. $x = 5$, $y = 2$, $z = 4$.

7. $x = (b + c - a)/2$, $y = (c + a - b)/2$, $z = (a + b - c)/2$.

Sections 9.3, page 85

15. $4xyz$.

17. (b) $x = -1$, $y = 2$, $z = 0$.

    (c) $x = 2$, $y = -3$, $z = 3$, $w = 0$.

    (d) $\begin{vmatrix} L_{n+2} & L_{n+1} \\ L_{n+1} & L_n \end{vmatrix} = -\begin{vmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{vmatrix}$,

    $$L_{2n} = \frac{3L_n^2 - 2L_nL_{n+1} + 2L_{n+1}^2}{5},$$

    $$L_{3n} = \frac{2L_n^3 - 3L_n^2L_{n+1} + 3L_nL_{n+1}^2}{5}.$$

Section 9.4, page 90

5. $(w - x)(w - y)(w - z)(z - x)(z - y)(y - x)$.

7. $(w - x)(w - y)(w - z)(z - x)(z - y)(y - x)$.

9. $r = 2$, $s = 2$, $t = 1$.

11. $r = 4$, $s = 2$, $t = 2$.

13. (b) $x = 1$, $y = 1$, $z = 1$.

Chapter 10

Section 10.1, page 95

19. $a = 2$, $b = 3$, $c = 4$.

Sections 10.3 and 10.4, page 103

21. (a) 101/100.

23. (a) The arithmetic mean is 2; the geometric mean is $\sqrt[4]{2}$.
(b) $a = 15$, $b = -20$, $c = 15$, $d = -6$.
(c) The roots are equal, and so all equal to 1, since their arithmetic mean is the same as their geometric mean.

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