Example 1 The positive unit normal to the plane $x - 2y + 2z - 6 = 0$ is

$$\mathbf{n} = \tfrac{1}{3}\mathbf{i} + (-\tfrac{2}{3})\mathbf{j} + \tfrac{2}{3}\mathbf{k}.$$

The distance from the origin to the plane is $p = |-6|/3 = 2$.

2 Find an equation of a plane that has a positive unit normal

$$\mathbf{n} = -\tfrac{6}{7}\mathbf{i} + \tfrac{2}{7}\mathbf{j} + (-\tfrac{3}{7})\mathbf{k}$$

and is 4 units from the origin.

Solution. From Theorem 15–9(d) we have for an equation of the plane

$$-\tfrac{6}{7}x + \tfrac{2}{7}y + (-\tfrac{3}{7})z - 4 = 0$$

or, equivalently, $6x - 2y + 3z + 28 = 0$.

Using the ideas developed in Theorem 15–7, we can now derive a formula for the distance from a point to a plane which will be quite analogous to the corresponding formula developed for the distance from a point to a line.

Theorem 15–10 *The distance from a point $P_1 = (x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is*

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

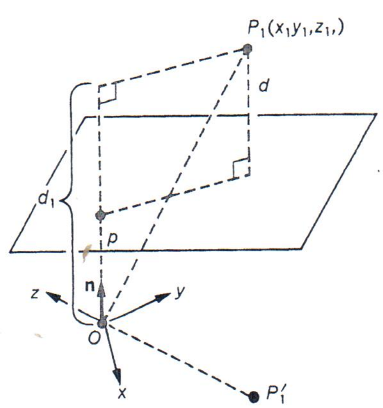

FIGURE 18

Proof Let $\mathbf{n}$ be the positive unit normal to the plane and

$$d_1 = \mathbf{n} \cdot \overrightarrow{OP_1}$$

be the length of the projection of $\overrightarrow{OP_1}$ on $\mathbf{n}$. Then d_1 can be positive, as for the point P_1 in Fig. 18, or zero, or negative, as for the point P_1' in the same figure. If d is the distance from P_1 to the plane, then in every case we have $d = |d_1 - p|$, where p is the distance from the origin to the plane. Now,

$$d_1 = \frac{Ax_1 + By_1 + Cz_1}{\pm\sqrt{A^2 + B^2 + C^2}}$$

and

$$p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

where the sign of the radical is chosen opposite to that of D. Hence

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

We observe that for $P_1 = (0, 0, 0)$, we have the formula for the distance from the origin to the plane,

$$d = p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}.$$

If $P_2 = (x_2, y_2, z_2)$ is any other point and $Ax_2 + By_2 + Cz_2 + D$ has the same sign as D, then P_2 is on the same side of the plane as the origin, whereas if they are opposite in sign, P_2 is on the opposite side of the plane from the origin.

Example 3 The distance from the point whose coordinates are (2, 3, 4) to the plane $3x + 3y + 4z - 12 = 0$ is

$$d = \frac{|3 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 - 12|}{\sqrt{34}}$$

$$= \frac{|19|}{\sqrt{34}} = \frac{19}{\sqrt{34}}.$$

The distance from the origin to the plane is

$$p = \frac{|-12|}{\sqrt{34}} = \frac{12}{\sqrt{34}}.$$

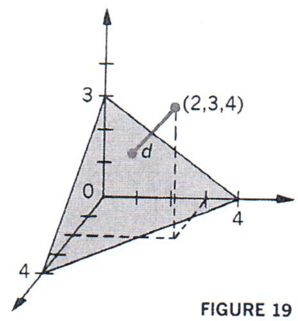

FIGURE 19

Since 19 and -12 are opposite in sign, we conclude that the point (2, 3, 4) is on the opposite side of the plane from the origin. A sketch of a portion of the plane and the points in Fig. 19 substantiates, approximately, these results.

Problems

Set A

Find the distance from the origin to each of the following planes. Sketch a portion of the planes.

1. $x + y - z + 3 = 0$
2. $20x - 5y - 4z + 63 = 0$
3. $\sqrt{3}\,x + 9y - \sqrt{21} = 0$
4. $13x + 16 = 0$
5. $-6x - \frac{1}{\sqrt{2}}y - 2z + 18 = 0$
6. $\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z - 7 = 0$
7. $x + y + z = 6$
8. $5x - 2y - z = 20.$

Find the distance from each of the following points to the plane $x - 4y - 8z + 16 = 0$. Sketch.

9. (2, 3, 4)
10. (0, 0, 1)
11. $(-4, 1, 1)$
12. (1, 5, 0)
13. $(-10, \frac{1}{2}, \frac{1}{2})$
14. $(0, 0, 10^m)$

Set B

Find the distance from each of the given points to the planes and determine whether the point is on the same side of the plane as the origin or on the opposite side.

15. (1, 1, 2) and $x + y + 2z - 4 = 0$
16. $(3, 0, -2)$ and $2x - 3y + z - 3 = 0$
17. $(-1, 2, 1)$ and $4x + 4y + 2z + 7 = 0$

18. $(-2, 3, 0)$ and $5x + 2y - 10 = 0$

19. $(2, 2, 2)$ and $3x + 4z - 12 = 0$

20. $(-5, -3, 6)$ and $x - 3y + 4z - 2 = 0$

21. What are the direction cosines of the positive unit normal to the plane $2x - y - z + a = 0$ if $a > 0$? if $a < 0$?

22. A plane has a positive unit normal

$$\mathbf{n} = \tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}$$

and is 3 units from the origin. What is a rectangular equation of the plane?

23. A line has direction cosines

$$-\frac{1}{\sqrt{2}}, \quad -\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{6}}$$

and lies in a plane. If $P = (2, 3, 4)$, is $\overrightarrow{OP}$ perpendicular to that plane?

24. A rectangular equation of a plane is $3x - 2y - 4z - 12 = 0$. Find the coordinates of the point P_0 in the plane that is the foot of the perpendicular from the origin to the plane.

25. Find equations and sketch a picture of the two planes that are parallel to the plane $x - 2y + 2z - 6 = 0$ and are 4 units from the given plane.

26. A plane has a positive unit normal

$$\mathbf{n} = \frac{1}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k},$$

and the point $P = (-2, 4, 5)$ lies in the plane. What is a rectangular equation of the plane?

27. Show that the planes $4x - 3y + z - 7 = 0$, and $4x - 3y + z + 6 = 0$ are parallel planes and find the distance between them.

28. Show that the planes $3x - 2y + z + 5 = 0$, and $-9x + 6y - 3z + 14 = 0$ are parallel planes and find the distance between them.

29. Show that a plane parallel to the plane

$$Ax + By + Cz + D = 0$$

and containing the point (x_0, y_0, z_0) has an equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Set C

30. Angles between two planes are defined as the angles between normals to the planes. What are the cosines of the angles between the planes whose equations are

$$2x + 3y - z + 5 = 0$$

and

$$-x + y + z + 1 = 0?$$

31. What are the measures of the angles formed by the plane $x + y - 2z = 0$ and the xy-coordinate plane?

32. The point $P = (x, y, z)$ is equidistant from the planes $2x - y - z + 6 = 0$ and $x + 2y - 7z + 12 = 0$. Write an equation satisfied by the coordinates (x, y, z). Prove that the set of all such points lies on two planes. What are these planes geometrically?

33. A plane is parallel to the plane $x - y + 3z = 1$ and is at a distance of 3 units from the point $P = (2, -1, 1)$. Find an equation of the plane. (There are two correct answers.)

34. Show that the planes $-2x + 5y - 3z + 8 = 0$ and $7x + 4y + 2z + 1 = 0$ are perpendicular planes.

Calculator Problem

Find the distance from the point $P = (-1.5, 3.8, -4.6)$ to the plane

$$\sqrt{3}x + 3.7y - \sqrt[3]{7}z + 0.8 = 0.$$

15-7 Vector Multiplication

In Section 15-3 the dot product of two vectors was defined to be a real number. In this section we define a new operation on vectors which will combine two 3-space vectors to produce another 3-space vector. We call this new operation *vector multiplication* and describe it in an intuitive manner.

Suppose $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ are vectors, and θ is the measure of the angle between them, $0 < \theta < \pi$. Consider the parallelogram $OAPB$ determined by these vectors. (See Fig. 20.) We define the *vector product* of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, to be a vector $\mathbf{c}$ that is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. The magnitude of $\mathbf{c}$ is the real number that is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. Of course, there are two vectors perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$, and so we must decide (if we wish $\mathbf{c}$ to be unique) which of these two vectors shall be the vector product of $\mathbf{a}$ and $\mathbf{b}$.

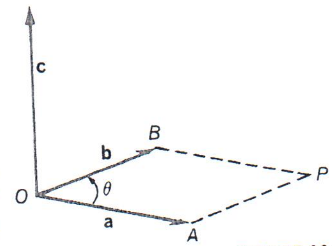

FIGURE 20

It is convenient to choose vector $\mathbf{c}$ as pointing in the same direction as is an advancing right-threaded screw when it is turned, or rotated, from $\mathbf{a}$ to $\mathbf{b}$ through the angle having measure θ. (See Fig. 21.)

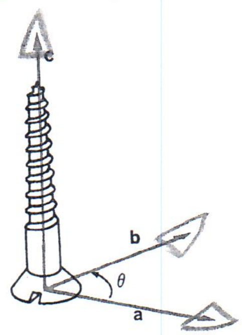

FIGURE 21

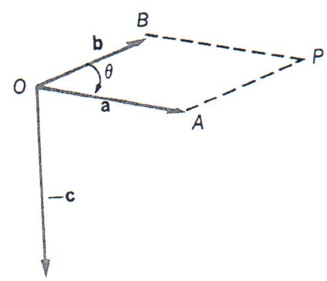

FIGURE 22

If we consider $\mathbf{b} \times \mathbf{a}$, then $\mathbf{b} \times \mathbf{a}$ would be a vector that is opposite in direction to $\mathbf{c}$. (See Fig. 22.) Thus

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \qquad \text{and} \qquad \mathbf{b} \times \mathbf{a} = -\mathbf{c}.$$

Example 1 Find $\mathbf{i} \times \mathbf{j}$.

Solution. Vectors $\mathbf{i}$ and $\mathbf{j}$ determine a parallelogram whose area is 1. (See Fig. 23.) Using the "right-hand" rule, we find that

$$\mathbf{i} \times \mathbf{j} = 1 \cdot \mathbf{k} = \mathbf{k}.$$

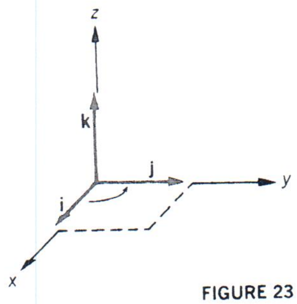

FIGURE 23

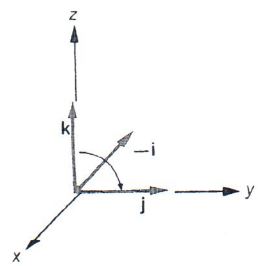

FIGURE 24

Example 2 From Fig. 24, we see that

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}.$$

3 If $\mathbf{a}$ and $\mathbf{b}$ are vectors and θ is the measure of the angle between them, then the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is

$$|\mathbf{a}|(|\mathbf{b}| \sin \theta).$$

(See Fig. 25.) Hence

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \theta.$$

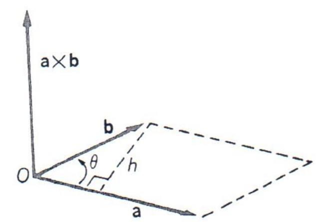

FIGURE 25

Definition 15–2 If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then the *vector product* of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is the vector $\mathbf{c}$ that is perpendicular to the plane determined by $\mathbf{a}$ and $\mathbf{b}$ and whose direction is chosen as described above. The magnitude of $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}|\,|\mathbf{b}| \sin \theta$, where θ is the measure of the angle between $\mathbf{a}$ and $\mathbf{b}$.

The vector product of two vectors is also called the *cross product,* or *outer product,* of the vectors, to distinguish it from the inner, or dot, product. (See the historical note at the end of this chapter.)

Remarks

1. Note that $\mathbf{c} = (|\mathbf{a}|\,|\mathbf{b}| \sin \theta)\mathbf{n}$, where $\mathbf{n}$ is the unit vector in the direction of $\mathbf{c}$.

2. If two vectors have the same direction or are opposite in direction, the parallelogram determined by the vectors will be a degenerate parallelogram and will have area zero. Therefore the vector product is the null vector.

Problems

Set A

1. Find the vector products for the table.

$\times$	i	j	k
i			
j			
k			

Find the vector that is described as the vector product indicated. Sketch a picture of the vector in each case.

2. $2\mathbf{i} \times 3\mathbf{j}$ **3.** $-2\mathbf{i} \times -3\mathbf{j}$ **4.** $2\mathbf{i} \times -3\mathbf{j}$

5. $\mathbf{j} \times 4\mathbf{k}$ **6.** $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ **7.** $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$

Set B

8. Find $(3\mathbf{i} - 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j})$. Sketch the vectors.
9. Find $3\mathbf{i} \times (3\mathbf{i} + 3\mathbf{j})$. [*Hint:* What is the angle between the two vectors?]
10. Find $2\mathbf{j} \times (\sqrt{3}\,\mathbf{j} + \mathbf{k})$. What is the angle between the vectors?
11. Is $(\mathbf{i} \times 2\mathbf{j}) \times 3\mathbf{i} = \mathbf{i} \times (2\mathbf{j} \times 3\mathbf{i})$?
12. Is $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{i} \times (\mathbf{i} \times \mathbf{j})$?
13. Is vector multiplication associative?
14. Is vector multiplication commutative?
15. If $|\mathbf{a}| = 4$ and $|\mathbf{b}| = 6$ and the angle between the vectors is $\pi/6$, what is the magnitude of $\mathbf{a} \times \mathbf{b}$?
16. Show that if $|\mathbf{r}| = \sqrt{2}$ and $|\mathbf{s}| = \sqrt{2}$ and the angle between the vectors is $5\pi/6$ then $\mathbf{r} \times \mathbf{s}$ is a vector of unit length.

Set C

17. Show that if **a** and **b** have position vectors $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, respectively, then an equation of the plane of **a** and **b** is

$$(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z = 0.$$

18. Show that the plane of Problem 17 has a unit normal

$$\mathbf{u} = \frac{(a_2b_3 - a_3b_2)}{d}\mathbf{i} + \frac{(a_3b_1 - a_1b_3)}{d}\mathbf{j} + \frac{(a_1b_2 - a_2b_1)}{d}\mathbf{k},$$

where
$$d = [(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2.]^{1/2}$$

Calculator Problem

Find the angle between the vectors

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}.$$

Then compute $|\mathbf{a} \times \mathbf{b}|$.

Use the results of Problems 17 and 18 to find the components of the vector product of **a** and **b**.

15-8 The Computation of Vector Products

It is desirable to be able to compute the vector product of two vectors in terms of the components of the two vectors. The following theorem gives a formula for doing this.

Theorem 15-11 *If*

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \textit{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then

$$\begin{aligned}\mathbf{v} &= \mathbf{a} \times \mathbf{b} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.\end{aligned} \tag{1}$$

Proof Although it is possible to derive formula (1) without knowing it in advance, it is much easier to simply *verify* that (1) is, indeed, correct. This is what we shall do.

If either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ and formula (1) also gives $\mathbf{0}$. Suppose now that neither $\mathbf{a}$ nor $\mathbf{b}$ is $\mathbf{0}$ but that $\mathbf{b}$ has the same direction as, or the opposite direction from, $\mathbf{a}$. Then $\mathbf{b} = \lambda\mathbf{a}$ for some real number λ, and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. But if $\mathbf{b} = \lambda\mathbf{a}$, then $b_1 = \lambda a_1$, $b_2 = \lambda a_2$, $b_3 = \lambda a_3$, and the right-hand member of (1) is also $\mathbf{0}$. Hence in this case also, $\mathbf{a} \times \mathbf{b}$ is given by formula (1).

Suppose now that neither $\mathbf{a}$ nor $\mathbf{b}$ is $\mathbf{0}$ and $\mathbf{b} \neq \lambda\mathbf{a}$ for any real number λ. Then not all the numbers

$$a_2b_3 - a_3b_2, \qquad a_3b_1 - a_1b_3, \qquad a_1b_2 - a_2b_1 \tag{2}$$

are 0.* Therefore the vector $\mathbf{v}$ defined by

$$\mathbf{v} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \tag{3}$$

is not the zero vector.

We shall now verify by Theorem 15-3 that $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (Fig. 26), and therefore to the plane of $\mathbf{a}$ and $\mathbf{b}$. This will show that $\mathbf{v}$ has either the correct direction or the opposite direction.

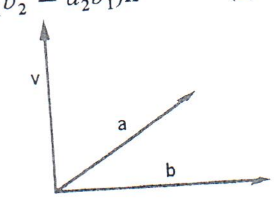

FIGURE 26

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*This may be seen as follows. Since $\mathbf{a} \neq \mathbf{0}$, at least one of its components is not 0. Suppose that $a_1 \neq 0$ and that all the numbers in (2) are 0. Then

$$b_2 = \frac{a_2b_1}{a_1}, \qquad b_3 = \frac{a_3b_1}{a_1},$$

and

$$\mathbf{b} = b_1\mathbf{i} + \frac{a_2b_1}{a_1}\mathbf{j} + \frac{a_3b_1}{a_1}\mathbf{k} = \frac{b_1}{a_1}(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = \frac{b_1}{a_1}\mathbf{a}.$$

But this equation asserts that $\mathbf{b} = \lambda\mathbf{a}$, with $\lambda = b_1/a_1$, which is contrary to assumption. If $a_1 = 0$, then either a_2 or a_3 does not equal 0, and a similar argument leads to a contradiction. Therefore if $b \neq \lambda\mathbf{a}$, not all of the numbers (2) are 0.

We have

$$\mathbf{v}\cdot\mathbf{a} = (a_2b_3 - a_3b_2)a_1 + (a_3b_1 - a_1b_3)a_2 + (a_1b_2 - a_2b_1)a_3 = 0,$$
$$\mathbf{v}\cdot\mathbf{b} = (a_2b_3 - a_3b_2)b_1 + (a_3b_1 - a_1b_3)b_2 + (a_1b_2 - a_2b_1)b_3 = 0, \tag{4}$$

where it is easily checked that the right-hand members of (4) are 0.

We now verify that $\mathbf{v}$ has the correct magnitude. We have

$$\begin{aligned}|\mathbf{v}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2\\ &= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b^2{}_1\\ &\quad - 2a_2a_3b_2b_3 - 2a_1a_3b_1b_3 - 2a_1a_2b_1b_2,\end{aligned} \tag{5}$$

and, if θ is the measure of the angle between $\mathbf{a}$ and $\mathbf{b}$,

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta)\\ &= |\mathbf{a}|^2|\mathbf{b}|^2\left(1 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{a}|^2|\mathbf{b}|^2}\right)\\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2\\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2\\ &= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2\\ &\quad - 2a_2a_3b_2b_3 - 2a_1a_3b_1b_3 - 2a_1a_2b_1b_2.\end{aligned} \tag{6}$$

Comparison of (5) and (6) shows that $|\mathbf{v}|^2 = |\mathbf{a}\times\mathbf{b}|^2$.

We have now proved that $\mathbf{v}$ is the correct magnitude and is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. To verify that $\mathbf{v}$ has the correct one of the two possible directions requires an argument based on continuity which we omit here. For our purpose it suffices to observe that the formula for $\mathbf{v}$ gives $\mathbf{a}\times\mathbf{b}$ when $\mathbf{a}$ and $\mathbf{b}$ are pairs of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. This last is left as an exercise for the student.

Remark

The components of $\mathbf{a}\times\mathbf{b}$ may seem difficult to remember, but after determinants are studied in Chapter 16 it will be easy to show that $\mathbf{a}\times\mathbf{b}$ can be expressed as a third-order determinant, namely,

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

Furthermore, we may show that vector multiplication is distributive with respect to addition of vectors so that memorization of the components of $\mathbf{a}\times\mathbf{b}$ is not really necessary.

Theorem 15–12 *If* **a**, **b** *and* **c** *are any vectors,*

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}).$$

Proof Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times [(b_1 + c_1)\mathbf{i} + (b_2 + c_2)\mathbf{j} + (b_3 + c_3)\mathbf{k}] \\
&= [a_2(b_3 + c_3) - a_3(b_2 + c_2)]\mathbf{i} + [a_3(b_1 + c_1) - a_1(b_3 + c_3)]\mathbf{j} \\
&\quad + [a_1(b_2 + c_2) - a_2(b_1 + c_1)]\mathbf{k} \qquad \text{Theorem 15–11} \\
&= [(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}] \\
&\quad + [a_2c_3 - a_3c_2)\mathbf{i} + (a_3c_1 - a_1c_3)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}]
\end{aligned}$$

Associativity and commutativity for addition of vectors and distributivity

$$= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}). \qquad \text{Theorem 15–11}$$

Example Find $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} = 2\mathbf{i}$ and $\mathbf{b} = (4\mathbf{j} + 3\mathbf{k})$.

Solution.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= 2\mathbf{i} \times (4\mathbf{j} + 3\mathbf{k}) \\
&= (2\mathbf{i} \times 4\mathbf{j}) + (2\mathbf{i} \times 3\mathbf{k}) \\
&= 8\mathbf{k} + (-6)\mathbf{j} = -6\mathbf{j} + 8\mathbf{k}
\end{aligned}$$

The sketch in Fig. 27 substantiates the results.

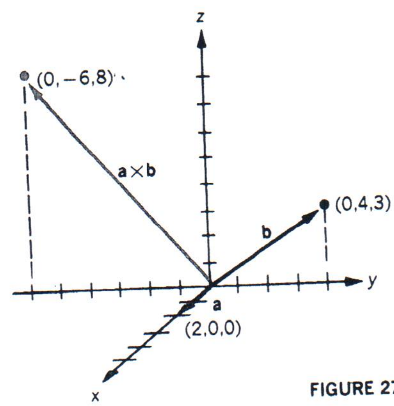


FIGURE 27

Problems

Set A

Use Theorem 15–11 to find the vector products below.

1. $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$
2. $(2\mathbf{i} + \mathbf{j}) \times (3\mathbf{j} + 2\mathbf{k})$
3. $(4\mathbf{i} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j})$
4. $(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$
5. $(-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
6. $(\mathbf{i} + \mathbf{j} + \mathbf{k})(2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$
7. Give the area of the parallelogram determined by the position vectors in Problems 1 through 6.

Set B

Use the distributive law for vector multiplication to compute the following cross products.

8. $2\mathbf{i} \times (\mathbf{j} + 3\mathbf{k})$
9. $3\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$
10. $(3\mathbf{i} + \mathbf{j}) \times (4\mathbf{j} + \mathbf{k})$
11. $(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (3\mathbf{i})$
12. $(3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
13. $(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \times (-3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Set C

14. Is $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{b} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a})$?

15. Let $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{s} = -3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. Find $\mathbf{r} \times \mathbf{s}$ and $\mathbf{s} \times \mathbf{r}$. What do you notice about the two cross products?

16. Prove that the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is *anticommutative,* that is, show that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

17. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = -5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Verify for these vectors that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

18. Show that if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the vectors of Problem 16, then
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

19. Verify that the formula for $\mathbf{v}$ in Theorem 15–11 gives the correct vector when $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors.

Calculator Problem

Find $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ where

$$\begin{aligned} \mathbf{a} &= 4.2\mathbf{i} + 3.5\mathbf{j} + 2.3\mathbf{k}, \\ \mathbf{b} &= 0.5\mathbf{i} - 1.2\mathbf{j} - 0.8\mathbf{k}, \\ \mathbf{c} &= -2.6\mathbf{i} + 7.3\mathbf{j} - 3.2\mathbf{k}. \end{aligned}$$

15–9 Applications of Vector Products

We wish to consider three special applications of vector products. The first of these concerns *areas of parallelograms and triangles in space.* Let us suppose that B, A, and C are successive vertices of a parallelogram, as in Fig. 28. Then by the definition of vector multiplication,

$$\text{area } \square BACD = |\overrightarrow{AB} \times \overrightarrow{AC}|$$

and

$$\text{area } \triangle BAC = \tfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|.$$

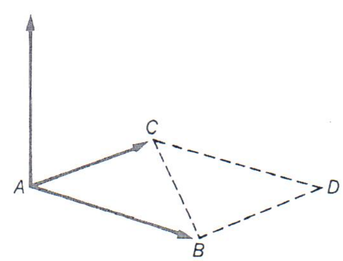


FIGURE 28

Example 1 Find the area of $\triangle ABC$, where $A = (2, 1, 3)$, $B = (-1, 0, 4)$ and $C = (4, 2, 2)$.

Solution. We proceed as follows:

$$\begin{aligned} \overrightarrow{AB} &= -3\mathbf{i} - \mathbf{j} + \mathbf{k}, \qquad \overrightarrow{AC} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} \\ \overrightarrow{AB} \times \overrightarrow{AC} &= [(-1)(-1) - 1 \cdot 1]\mathbf{i} + [1 \cdot 2 - (-3)(-1)]\mathbf{j} \\ &\quad + [(-3)1 - (-1)2]\mathbf{k} = -\mathbf{j} - \mathbf{k}; \\ \text{area } \triangle ABC &= \tfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \tfrac{1}{2}\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}/2. \end{aligned}$$

Our second application deals with finding equations for *planes perpendicular to two intersecting planes*. If a plane, π_3, is perpendicular to each of two intersecting planes, π_1 and π_2, then the normals of π_1 and π_2 must be perpendicular to the normals of π_3. (See Fig. 29.) Therefore the cross product of vectors normal to the intersecting planes π_1 and π_2 will produce a vector that will be normal to the set of planes perpendicular to π_1 and π_2.

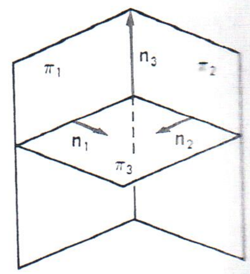


FIGURE 29

Example 2 Find an equation of the plane π that contains the point $P_1 = (3, 2, 4)$ and is perpendicular to each of the intersecting planes

$$\pi_1: x + y - z - 4 = 0 \qquad \text{and} \qquad \pi_2: x - 2y + 3z - 6 = 0.$$

Solution. The given planes have normals

$$\mathbf{n}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k} \qquad \text{and} \qquad \mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$

respectively. The required plane will have a normal $\mathbf{n}$ given by

$$\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} - 3\mathbf{k}.$$

The plane π must contain $P_1 = (3, 2, 4)$; so if $P = (x, y, z)$ is any point in π, then $\mathbf{n} \cdot \overrightarrow{P_1P} = 0$. Hence $(x - 3) \cdot 1 + (y - 2)(-4) + (z - 4)(-3) = 0$, so $x - 4y - 3z + 17 = 0$ is an equation of the plane π.

A final application of vector products is that dealing with *moment of force*. A force F acting on a lever of length d from a fixed point O tends to rotate the lever about O. (See Fig. 30.) The vector product, $\mathbf{d} \times \mathbf{F}$, is called the *moment of force* about O. Numerous applications of moments of force occur in physics.

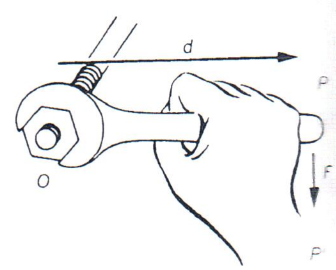


FIGURE 30

Example 3 The vector $\overrightarrow{PP'}$ acts on $\overrightarrow{OP}$ to produce a moment of force about O. What is this moment if $P = (2, 3, 1)$ and $P' = (-2, 6, 4)$?

Solution.

$$\overrightarrow{OP} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k},$$
$$\overrightarrow{PP'} = -4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k},$$
$$\text{moment of force about } O = \overrightarrow{OP} \times \overrightarrow{PP'} = 6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}.$$

Problems

Set A

Using cross products, find the area of each triangle whose vertices have the following coordinates.

1. (0, 0, 0), (1, 1, 1), (0, 0, 3)
2. (2, 0, 0), (0, 2, 0), (0, 0, 2)
3. (2, 0, 0), (0, 3, 0), (0, 0, 4)
4. (1, −1, 1), (2, 2, 2), (4, −2, 1)
5. (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)
6. Three of the vertices of a parallelogram are (1, 2, −1), (4, 0, −4) and (0, 6, 6). What is the area of the parallelogram?
7. The area of an equilateral triangle is $s^2\sqrt{3}/4$ where s is the length of one side of the triangle. Verify your result for the equilateral triangle in Problem 2.

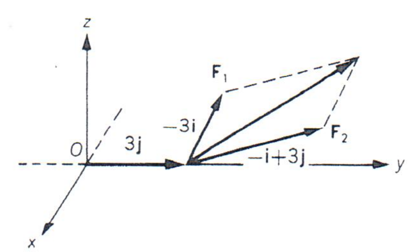
Set B

8. Find an equation of a plane through (0, 2, 0) that will be perpendicular to the pair of intersecting planes $x + 2z = 6$ and $x + y = 6$. Sketch a picture of all three planes.
9. Find an equation of a plane that contains the origin and is perpendicular to each of the planes $x + y + z = 4$ and $x - y + z = 4$. Sketch a picture.
10. Find the point of intersection of the three planes of Problem 9.
11. The volume of a tetrahedron is given by the formula $V = \frac{1}{3}Bh$, where B is the area of a base and h is the length of the altitude to this base. Find the volume of a tetrahedron whose triangular base has vertices at (3, 1, 0), (1, 4, 0), and (5, 5, 0). The fourth vertex is at (3, 1, 6).

Set C

In each problem, find an equation of the plane through the given noncollinear points. [*Hint:* In Problem 12, for example, the arrows ((1, 2, 2), (3, 3, −1)) and ((1, 2, 2), (−1, 5, 1)) lie in the plane. Therefore, the vector product of the vectors represented by these arrows, $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$, will be a normal to the plane.]

12. (1, 2, 2), (3, 3, −1), (−1, 5, 1)
13. (0, −1, 0), (2, 1, −2), (1, 0, 1)
14. (1, 1, 0), (3, 0, 1), (−1, 2, 1)
15. (1, 1, 1), (2, 4, −2), (5, 1, 0)
16. (2, 0, 0), (0, 3, 0), (0, 0, 4)
17. (2, 2, 1), (4, 2, −1), (6, −3, 2)
18. A force $\mathbf{F} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ acts at P, where $\overrightarrow{OP} = 5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. What is the moment of force about the point O? What is the magnitude of the moment of force?
19. In mechanics, a useful theorem due to Varignon states that the sum of the moments of force about a point O is equal to the moment of the sum of the forces about O. Use the diagram to verify that the sum of the moments of $\mathbf{F}_1$ and $\mathbf{F}_2$ about O is the same as the moment of force of $\mathbf{F}_1 + \mathbf{F}_2$ about O. What property of vectors does this illustrate?



Calculator Problem

Use the cross product to calculate the area of $\triangle ABC$ if $A = (3.4, -1.1, 0.8)$, $B = (1.5, -3.1, 1.2)$ and $C = (0.4, 1.2, 2.8)$.

Summary

Some of the more important concepts and formulas developed in this chapter are listed below. You should make certain that you understand the concepts and can apply the formulas.

1. The angle between two vectors or two rays.
2. The cosine of the angle between the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$:

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}|\,|\mathbf{b}|}.$$

3. The dot or scalar product of $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, where θ is the measure of the angle between $\mathbf{a}$ and $\mathbf{b}$.
4. If the components of $\mathbf{a}$ and $\mathbf{b}$ are (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively, then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
5. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.
6. The line $ax + by + c = 0$ has a positive unit normal

$$\mathbf{n} = \frac{a}{\pm\sqrt{a^2 + b^2}}\mathbf{i} + \frac{b}{\pm\sqrt{a^2 + b^2}}\mathbf{j},$$

where the sign of the radical is chosen opposite to that of c.

The distance p from the origin to the line is

$$p = \frac{|c|}{\sqrt{a^2 + b^2}}.$$

The distance d of the point $P_1 = (x_1, y_1)$ from the line is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

7. The plane $Ax + By + Cz + D = 0$ has a positive unit normal

$$\mathbf{n} = \frac{A\mathbf{i}}{\pm\sigma} + \frac{B\mathbf{j}}{\pm\sigma} + \frac{C\mathbf{k}}{\pm\sigma},$$

where $\sigma = \sqrt{A^2 + B^2 + C^2}$, and the sign of the radical is chosen opposite to that of D.

The distance p from the origin to the plane is

$$p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}.$$

The distance d from a point $P_1 = (x_1, y_1, z_1)$ to the plane is

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example 1 $\begin{vmatrix} -3 & 4 \\ -5 & 8 \end{vmatrix} = (-3)8 - (-5)4 = -24 + 20 = -4$

2 If

$$\begin{vmatrix} x & 2 \\ 3 & (x-1) \end{vmatrix} = 0$$

then by Definition 16–2 we have the quadratic equation

$$x(x-1) - 6 = 0 \qquad \text{or} \qquad x^2 - x - 6 = 0.$$

Hence $(x-3)(x+2) = 0$ and $x = 3$ or $x = -2$. Substituting either of these numbers for x in the determinant equation, we find that both numbers satisfy the equation.

Problems

Set A

Evaluate the determinants.

1. $\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 7 & -3 \\ -1 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 0.5 & 1.2 \\ -0.6 & 2.1 \end{vmatrix}$

4. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

5. $\begin{vmatrix} a + kc & b + kd \\ c & d \end{vmatrix}$

6. $\begin{vmatrix} x & 1 \\ 2 & x-1 \end{vmatrix}$

7. $\begin{vmatrix} x & x \\ x-1 & x+1 \end{vmatrix}$

8. $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Solve the equations.

9. $\begin{vmatrix} 3 & 1 \\ x & 2 \end{vmatrix} = 0$

10. $\begin{vmatrix} x-1 & 1 \\ -1 & x-2 \end{vmatrix} = 0$

11. $\begin{vmatrix} x-1 & x-2 \\ x-3 & x \end{vmatrix} = 0$

Solve by using Cramer's Rule.

12. $\begin{cases} 2x + y = 5, \\ x + 3y = 5 \end{cases}$

13. $\begin{cases} 3x + 2y = -11, \\ 2x - 3y = 10 \end{cases}$

14. $\begin{cases} 6x - 9y = 11, \\ 2x - 3y = 7 \end{cases}$

15. $\begin{cases} 2x + 5 = 3y, \\ 1 + y = -2x \end{cases}$

Set B

Solve by using Cramer's Rule, if possible.

16. $\begin{cases} 4x + 12y = 20, \\ 2x + 6y = 10 \end{cases}$

17. $\begin{cases} 3/x - 2/y = 2, \\ 9/x + 4/y = 1 \end{cases}$

18. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{2y} = \dfrac{1}{2}, \\ \dfrac{3}{x} + \dfrac{3}{2y} = 1 \end{cases}$

19. $\begin{cases} 0.4x - 0.1y = 0.5, \\ 1.1x + 2.2y = -1.1 \end{cases}$

20. $\begin{cases} x\tan\theta + y\sec\theta = \sec\theta + \tan\theta, \\ x\sec\theta + y\tan\theta = \tan\theta + \sec\theta \end{cases}$

Set C

21. Solve the inequality

$$\begin{vmatrix} x & x-2 \\ 5 & 10 \end{vmatrix} < 0.$$

22. Show that:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = -\begin{vmatrix} c & a \\ d & b \end{vmatrix}.$$

23. State the results of Problem 22 in words. Properties concerning rows and columns of determinants are valid for determinants of any order.

24. Show that

$$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix}.$$

State the results in words.

Problems

Set A

Evaluate the determinants.

1. $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}$

2. $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 6 \\ 1 & 2 & 13 \end{vmatrix}$

3. $\begin{vmatrix} -2 & 3 & 7 \\ 4 & -2 & 6 \\ -1 & 3 & 5 \end{vmatrix}$

4. Prove that

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a & a & d \\ b & b & e \\ c & c & f \end{vmatrix} = 0.$$

Solve the following systems, using Cramer's Rule, and also using elimination.

5. $\begin{cases} x + y + z = 1, \\ 3x + 3y - 3z = 2, \\ x - y - z = 0 \end{cases}$

6. $\begin{cases} x - y + 1 = 0, \\ y - z + 2 = 0, \\ x + z + 3 = 0 \end{cases}$

7. $\begin{cases} x - y + 1 = 0, \\ x + z - 6 = 0, \\ y + z - 7 = 0 \end{cases}$

Set B

8. Evaluate the determinant

$$\begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

by cofactors of
(a) the elements of the second row, and
(b) the elements of the third column.

9. Verify Theorem 16-4 for the determinant of Problem 8, using the elements of the third row and the cofactors of the first row.

Solve the following systems of equations using Cramer's rule. Then solve by reducing each to triangular form.

10. $\begin{cases} x + y + z = 1, \\ 3x + 3y - 3z = 2, \\ x - y - z = 0 \end{cases}$

11. $\begin{cases} x - y + 1 = 0, \\ x + z - 6 = 0, \\ y + z - 6 = 0 \end{cases}$

12. $\begin{cases} 2x - 5y - 7z = 18, \\ x + y + 8z = -35, \\ 4x + 6y + z = 13 \end{cases}$

13. $\begin{cases} x + y - z = 2, \\ -x + 2y + 4z = 5, \\ 2x + 5y + z = 9 \end{cases}$

14. $\begin{cases} x + 2y - z = 3, \\ 3x + y - 2z = 4 \end{cases}$

15. $\begin{cases} x - 2y + z = 3, \\ -x + 4y - 3z = 1, \\ 2x - 3y + z = 8 \end{cases}$

Set C

16. Show in two ways that the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is an equation of the line through the distinct points (x_1, y_1) and (x_2, y_2). [*Hints:* (1) There is such a line: $Ax + By + C = 0$. Therefore $Ax_1 + By_1 + C = 0$ and $Ax_2 + By_2 + C = 0$. These three homogeneous equations have a nontrivial solution. Apply the Corollary to Theorem 16-5. (2) The determinant equation is a linear equation in x and y and therefore represents a line. Show that (x_1, y_1) and (x_2, y_2) are on it.]

17. Use the procedure of Problem 16 to find the line through $(1, -2)$ and $(-2, -4)$.

18. Prove that if k times one row (or column) is added to another row (or column), then the determinant is unchanged.