13-1 Introduction

In this chapter we shall see that we can coordinatize space in much the same way we did a plane. We shall also see further similarities: Parametric equations of a line in space are quite similar to those of a line in the plane. The distance formula is a simple extension of the formula in the plane. And finally, planes in space can be described simply by means of equations.

In this discussion we shall need certain facts about lines and planes in space. The required properties will be stated as postulates for space.

Postulates for Space

P1. There is a set of points called euclidean space. Certain subsets of space are called lines. Certain other subsets of space are called planes. Space contains at least four points not in any one plane.

P2. There is exactly one plane containing any three noncollinear points (Fig. 1).

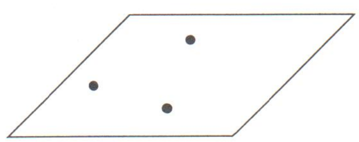


FIGURE 1

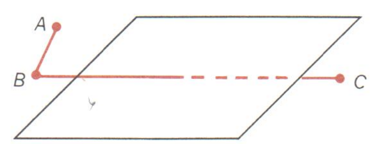


FIGURE 2

P3. Each plane satisfies all the postulates for euclidean plane geometry. In particular, the line joining two points of a plane lies in that plane, and the postulates for congruence of segments and angles apply whether or not the segments and angles are in the same plane.

P4. Each plane separates space (Fig. 2). This means that the set of points not in the plane consists of two subsets on either side of the plane, called half-spaces, with the following properties:

(a) If two points are in the same subset, the segment joining them does not intersect the plane.

(b) If two points are in different subsets, the segment joining them does intersect the plane.

From these few postulates all the theorems of space, or *solid,* geometry can be derived. However, we will not prove any of these theorems in the text. All that is required is an understanding of the elementary aspects of space geometry and the ability to visualize and draw some space figures. In this regard, you should observe that the drawing problem is one of representing three-dimensional relations in a plane, in fact, in only a portion of a plane, namely a sheet of paper. Planes will be indicated by a drawing of a parallelogram, suggesting a portion of a plane. Lines that are behind planes as viewed by the observer (you) are dashed, as in Fig. 3.

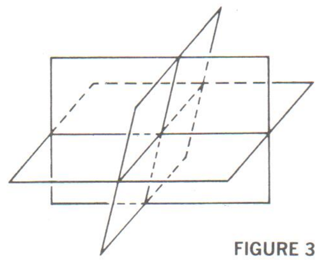


FIGURE 3

13-2 Space Coordinatization

Just as in the plane we defined a one-to-one mapping of ordered pairs of real numbers onto the points of the plane, so for space we analogously define a one-to-one mapping of ordered *triples* of real numbers onto space.

As was the case in the plane, the assigning of coordinates to space also depends on certain choices. These choices can be made in different ways.

Let us choose a unit of length and a point O, called the *origin of coordinates* (Fig. 4a). Next, let us choose three mutually perpendicular lines through O. We call any one of these lines the *first axis,* either of the other two the *second axis,* and the remaining line the *third axis.* The planes containing pairs of coordinate axes are called the *coordinate planes.*

Next we coordinatize each axis (Fig. 4b). This step involves choosing a positive direction on each axis. We can now define a mapping from the points of space to ordered triples of real numbers (Fig. 4c). For each point P, we consider the plane through P parallel to the coordinate plane of the second and third axes. This plane meets the first axis in a point with coordinate a_1, called the *first coordinate* of P. In the same manner, planes parallel to the other coordinate planes determine the second and third coordinates, a_2 and a_3, of P. We indicate the coordinates of point P by (a_1, a_2, a_3). Frequently we simply refer to the ordered triple (a_1, a_2, a_3) as a point.

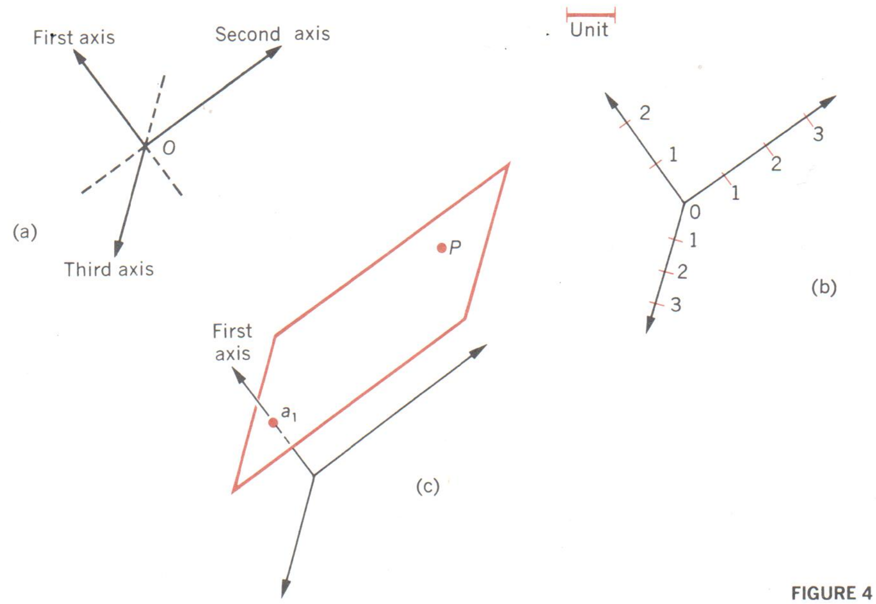


FIGURE 4

Because the planes through P parallel to the coordinate planes are unique, the definition of coordinates given above defines a function, or mapping, of points of space into ordered triples of real numbers. This mapping is one-to-one. Since we know that for each point there is a unique triple (a_1, a_2, a_3), it remains to show that each triple corresponds to exactly one point. This converse is left to the student.

A coordinate system provides a one-to-one mapping between the points of space and the set of all ordered triples of real numbers. The coordinates of a point are called *rectangular cartesian coordinates.* The word *rectangular* comes from our choice of axes which are mutually perpendicular. This choice is not necessary but is highly convenient. (See Problem 26 of the problem set.)

There are several conventions which are observed in coordinate space geometry. Usually we give names to the three axes, which are related to the variable used to denote the coordinate. Thus, if we use x to denote any first coordinate, and y and z to denote the second and third coordinates, respectively, then it is natural to speak of the x-, y-, and z-coordinates of a point, as well as the x-, y-, and z-axes. Of course, there is nothing compulsory about the use of the letters x, y, and z. Any letters would do as well. We could also have a-, b-, and c-axes, or u-, v-, and w-axes, etc.

Furthermore, there is no definite way to orient the axes, but it is customary in textbooks to picture the axes as shown in Fig. 5. The y- and z-axes are in the plane of the page, and the positive x-axis extends out from the page toward the reader. In the drawing, the x-axis is at an angle which is intended to convey a feeling of perspective, and the unit of length is shortened along the x-axis to enhance this effect. But these drawings are *not* perspective drawings, since parallel lines are not drawn to converge. Lines parallel in space are parallel in these drawings.

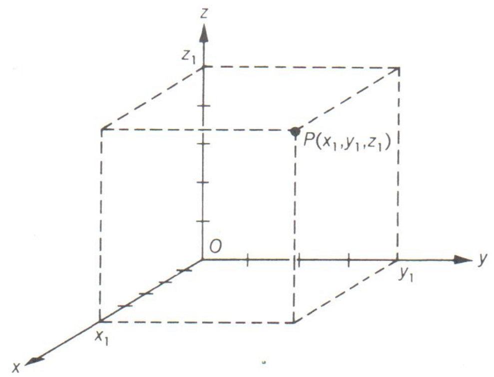


FIGURE 5

The plane containing the x- and y-axes is the *xy-coordinate plane,* or simply, the xy-plane. Likewise there are yz- and xz-coordinate planes. The coordinate planes separate space into eight *octants.* Usually only one of them is given a name; this is the *first octant,* which consists of the set of all points having positive coordinates only. The two sides of a plane are called *half-spaces.*

Remark

We have started with a geometric object, namely euclidean space, and have given it coordinates. But often, in the applications of mathematics to problems of physics, chemistry, engineering and other fields it is the converse situation that prevails. We are given ordered triples of real numbers and use space to illustrate graphically the relations between the ordered triples. In some cases there is no need to choose the same unit for all the coordinate axes. The following example illustrates this.

Example

A rectangular plate is heated, but not uniformly. Consider the temperature at each point of the plate. Discuss the problem of plotting temperature as a function of position. What is the relevant portion of space? (Fig. 6.)

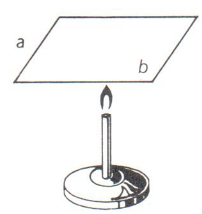


FIGURE 6

Solution. Let x and y denote the distances from one corner of the rectangular plate and t the temperature. Fig. 7 shows the temperature as a function of position. The graph would be a kind of surface, but not necessarily a plane, because the plate was not heated uniformly. Only the first octant would be needed to show the graph.

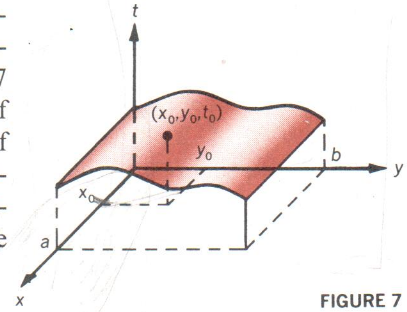
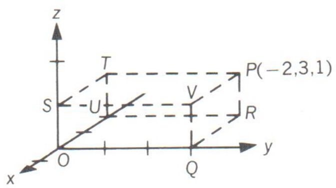


FIGURE 7

Problems

Set A

In the figure, the coordinates of point P are $(-2, 3, 1)$. Give the coordinates of each of these points.



1. Q **2.** R **3.** S

4. T **5.** U **6.** V

7. Plot the following points on the same coordinate axis. $(0, 0, 0)$, $(0, 1, 2)$, $(0, -1, 2)$, $(1, -1, 2)$, $(-2, 2, -3)$, $(4, 0, 3)$, $(4, 0, -3)$, $(\pi, \pi/2, 0)$, $(-\sqrt{2}, \sqrt{2}, 0)$.

8. A point is on the y-axis. What can you say about its x- and z-coordinates? What equations in x, y, and z are satisfied if and only if the point (x, y, z) is on the y-axis?

Set B

9. Describe the eight octants in terms of the coordinates of points in them. Also describe the octants in terms of the half-spaces formed by the coordinate planes.

10. A plane is parallel to the xz-coordinate plane and passes through $(-1, 3, -4)$. What can you say about the coordinates of any point (x, y, z) in the plane?

11. What is the set of all points (x, y, z) for which $x = -2$? the set of all points (x, y, z) for which $x = 1$ and $y = -2$?

12. A point lies in the half-space formed by the yz-plane that contains the point $(-3, 4, 5)$. What can you say about the coordinates of any point (x, y, z) in this half-space?

13. What is the set of all points (x, y, z) for which $y < 0$?

14. What is the set of all points (x, y, z) for which $x + 2 \leq 0$?

15. What is the set of all points (x, y, z) for which $x^2 > 0$?

16. Draw a picture of the set of all points (x, y, z) for which $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$.

17. A line is parallel to the z-axis and passes through $(-1, -2, -3)$. Find several other points on the line.

Set C

Plot the following pairs of points. Then draw the line segment connecting them. Draw a plane containing the segment that is parallel to the y-axis. Draw a second plane containing the segment parallel to the z-axis.

18. $(2, -2, 0)$, $(-1, 2, 3)$

19. $(3, 0, 2)$, $(0, 3, 3)$

20. $(3, 0, 3)$, $(0, 2, -1)$

21. $(2, 0, -2)$, $(0, 3, 1)$

22. A line passes through $(-1, -2, 3)$ and $(3, -2, 1)$. Find a pair of equations, one in x and z and the other in y, that are satisfied if and only if the point (x, y, z) is on the line.

23. What is the set of all points (x, y, z) such that $2x + 3y = 6$?

24. A particle moves along a straight line subject to a force that depends on its position and the time t. If the force is always in one direction and no greater than 100 lbs, discuss the portion of space relevant to plotting force as a function of distance and time.

25. The internal energy E of a given quantity of a gas depends on the pressure p and temperature T. What is the relevant portion of space?

26. In assigning coordinates for space, it is not necessary to choose mutually perpendicular axes. Describe how to set up a coordinate system for space using any three concurrent noncoplanar lines as axes.

27. Show that to each ordered triple of real numbers (a_1, a_2, a_3), there corresponds a unique point P in space.

Calculator Problem

A point P has space coordinates of $(\sqrt[3]{15}, -\sqrt{33}, \sqrt[4]{147})$. What are the coordinates of the point with *integer* coordinates that is closest to point P?

13-3 The Distance Formula

Let us consider any two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. We choose $P_3 = (x_2, y_2, z_1)$ (Fig. 8). Then the points P_1 and P_3 are in the plane parallel to the xy-coordinate plane where $z = z_1$. The distance $|P_1P_3|$ is given by the distance formula in the plane,

$$|P_1P_3| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The points P_2 and P_3 are on a line parallel to the z-axis and

$$|P_2P_3| = \sqrt{(z_1 - z_2)^2} = |z_1 - z_2|.$$

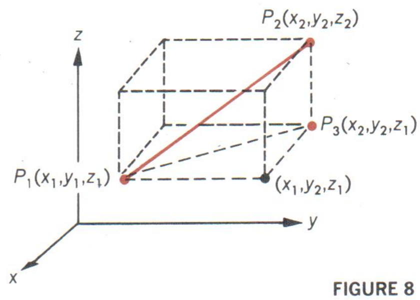


FIGURE 8

The triangle $P_1P_2P_3$ is a right triangle; hence by the Pythagorean Theorem,

$$|P_1P_2|^2 = |P_1P_3|^2 + |P_2P_3|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

and we obtain the *distance formula*

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The distance formula in three dimensions is an extension of the distance formula in two dimensions. There is one extra term. If both P_1 and P_2 lie in the xy-plane, then $z_1 = z_2 = 0$ and we get the two-dimensional formula.

Example Find . the distance between $P_1 = (-1, 1, 1)$ and $P_2 = (1, 2, 3)$.

Solution. Using the distance formula above, we have

$$|P_1P_2| = \sqrt{[1-(-1)]^2 + (2-1)^2 + (3-1)^2}$$
$$= \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3.$$

Problems

Set A

Plot the following pairs of points and find the distance between them.

1. $(2, 0, 1), (10, 1, 5)$
2. $(2, -1, -1), (-1, 3, 3)$
3. $(3, 2, 0), (0, 0, 6)$
4. $(0, 0, 0), (1, 1, 1)$
5. $(0, 0, 0), (2, -9, 6)$
6. $(1 - \sqrt{2}, 1 + \sqrt{3}, 5), (1 + \sqrt{2}, 1 - \sqrt{3}, 1)$
7. The eight vertices of a box have coordinates $(0, 0, 0)$, $(4, 0, 0)$, $(4, 5, 0)$, $(0, 5, 0)$, $(0, 5, 3)$, $(0, 0, 3)$, $(4, 0, 3)$, and $(4, 5, 3)$. Draw the box. Then find the length of the longest diagonal of the box.
8. The dimensions of a rectangular room are 4 m x 6 m x 3 m. What is the distance between a point on the floor in one corner and a point on the ceiling in the far opposite corner?

Set B

9. What is the distance between the pair of points whose coordinates are (a, b, c), $(a + d, b + e, c + f)$?
10. The vertices of $\triangle ABC$ are $A = (3, -2, 4)$, $B = (5, 5, 5)$ and $6 = (-4, 4, -2)$. Find the length of each side of the triangle.
11. The vertices of $\triangle XYZ$ are $X = (2, -1, 1)$, $Y = (4, 2, 1)$ and $Z = (2, -1, 4)$. Sketch the triangle. Show that the triangle is a right triangle by showing that the sum of the squares of two of the sides is equal to the square of the third side.
12. Are the points $(1, -1, 1)$, $(2, 2, 2)$, and $(4, -2, 1)$ vertices of a right triangle?
13. Show that $(2, 6, -3)$, $(-4, 3, -3)$, and $(-2, 7, 2)$ are vertices of an isosceles triangle.

Set C

14. Is the point $(1, 0, 4)$ inside or outside the sphere of radius π with center at $(2, 3, 3)$?
15. Determine x such that $(x, 0, 0)$ will be on the sphere of radius $3\sqrt{3}$ with center at $(2, 3, 3)$.
16. Give an equation in x, y, z that is satisfied if and only if the point (x, y, z) is on the sphere of radius r with center at the origin. Explain your reasoning.
17. Show that if a sphere of radius r has its center at (a, b, c) then any point $P(x, y, z)$ on the sphere has coordinates such that
$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$
18. Find an equation in x, y, and z satisfied by the coordinates of all points (x, y, z) that are equidistant from $(2, 3, -2)$ and $(6, 11, 8)$.
19. Find the points on the y-axis at a distance 6 from the point $(2, 1, 4)$.
20. Show that if the line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is perpendicular to the line through (x_1, y_1, z_1) and (x_3, y_3, z_3), then
$$(x_2 - x_1)(x_1 - x_3) + (y_2 - y_1)(y_1 - y_3) + (z_2 - z_1)(z_1 - z_3) = 0.$$
21. Using the distance formula, show that $|P_1P_2| = |P_2P_1|$.

Calculator Problem

Heron of Alexandria (c. 50 A.D.) discovered a formula for the area of a triangle in terms of the lengths of the three sides of the triangle. Heron's formula for the area A of any triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the lengths of the three sides and s is the semiperimeter of the triangle. That is, $s = \frac{1}{2}(a + b + c)$.

Use the distance formula to find the lengths of the three sides of a triangle whose vertices are $(6, 1, -4)$, $(4, 5, 3)$ and $(-2, -4, 7)$. Then use Heron's formula to find the area of the triangle.

13-4 Direction Cosines of Lines in Space

In the next section we will obtain parametric equations of lines in space in a manner analogous to that for lines in the plane. For this purpose we need to define direction angles and direction cosines of a line in space.

Let l be any line and (x_0, y_0, z_0) a point on it. Next we coordinatize the line with point (x_0, y_0, z_0) as the origin (Fig. 9). Then each point of l has associated with it a unique real number, d, the directed distance from (x_0, y_0, z_0). In

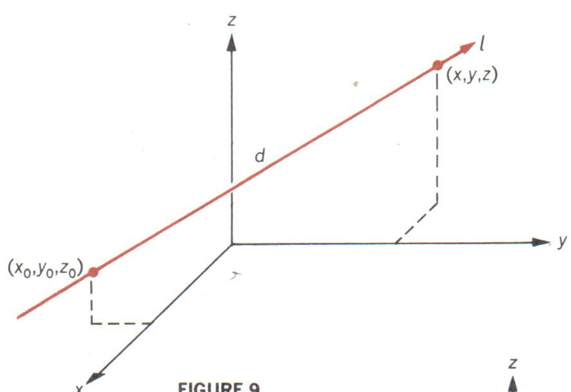


FIGURE 9

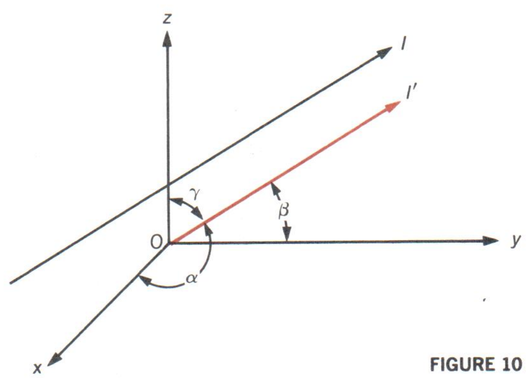


FIGURE 10

this way l becomes a directed line. To define direction angles we consider a directed line l' parallel to l and in the same direction (as in Fig. 10) and through the origin.

Definition 13-1 With the notation already described, the *direction angles* of the directed line l are the angles made by the positive ray from O on l' and the positive x-, y-, and z-axes. The angles have measure α, β, and γ, respectively.

Remark The oppositely directed line has direction angles with measures

$$\pi - \alpha, \qquad \pi - \beta, \qquad \pi - \gamma.$$

Definition 13-2 The *direction cosines* of the directed line l are the numbers c_1, c_2, c_3:

$$\begin{aligned} c_1 &= \cos\alpha = \text{the } x\text{-direction cosine,} \\ c_2 &= \cos\beta = \text{the } y\text{-direction cosine} \\ c_3 &= \cos\gamma = \text{the } z\text{-direction cosine.} \end{aligned}$$

Remark The oppositely directed line has these direction cosines:

$$\begin{aligned} \cos(\pi - \alpha) &= -\cos\alpha, \\ \cos(\pi - \beta) &= -\cos\beta, \\ \cos(\pi - \gamma) &= -\cos\gamma. \end{aligned}$$

What we now desire are equations that give the direction cosines of a line in terms of rectangular coordinates of points on the line. These are supplied by the following theorem, analogous to Theorem 11-1 for the plane.

Theorem 13-1 ***Suppose l is a directed line in the plane. Let (x_0, y_0, z_0) and (x, y, z) be on l, and let d be the directed distance from (x_0, y_0, z_0) to (x, y, z). Then***

$$\begin{aligned} c_1 &= \cos\alpha = \frac{x - x_0}{d}, \\ c_2 &= \cos\beta = \frac{y - y_0}{d}, \qquad (1) \\ c_3 &= \cos\gamma = \frac{z - z_0}{d}. \end{aligned}$$

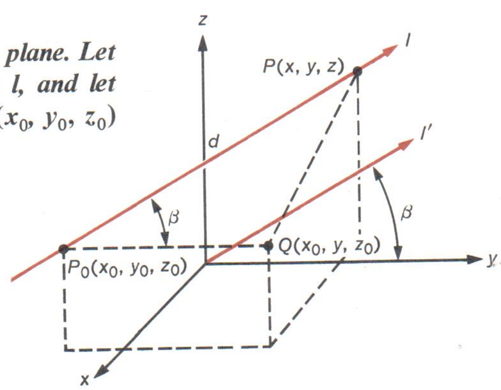


FIGURE 11

Proof We shall prove only the second equality. The others are proved similarly. Referring to Fig. 11, it is an easy exercise in geometry to see that $\angle PP_0Q$ is congruent to the y-direction angle if (x, y, z) is on the positive side of P_0. Because $\triangle PP_0Q$ is a right triangle with right angle at Q, we have

$$\cos\beta = \frac{y - y_0}{d}. \tag{2}$$

If (x, y, z) is on the other side of (x_0, y_0, z_0), then both numerator and denominator change sign, so equation (2) remains valid.

Example What are the direction cosines of the line through $P(1, -1, 2)$ and $Q(1, 3, 6)$ if it is directed from P to Q? (See Fig. 12.)

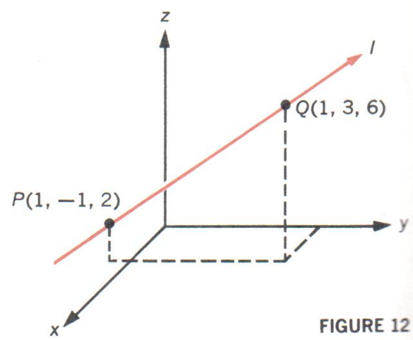


FIGURE 12

Solution. The directed distance from point P to Q is
Q is

$$d = \sqrt{(1-1)^2 + (3+1)^2 + (6-2)^2} = \sqrt{32} = 4\sqrt{2}.$$

From equations (1) we have

$$\cos\alpha = \frac{1-1}{4\sqrt{2}} = 0 \quad \text{and so} \quad \alpha = \frac{\pi}{2},$$

$$\cos\beta = \frac{3+1}{4\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad \text{and so} \quad \beta = \frac{\pi}{4},$$

$$\cos\gamma = \frac{6-2}{4\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad \text{and so} \quad \gamma = \frac{\pi}{4}.$$

Observe that even without equations (1) it is clear from Fig. 12 that $\alpha = \pi/2$ and $\beta = \gamma = \pi/4$.

There is an important relation connecting the direction cosines of any line in space, which we state as a theorem.

Theorem 13-2 ***If c_1, c_2, c_3 are the direction cosines of a line, then***

$$c_1^2 + c_2^2 + c_3^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The proof is an easy application of the distance formula and equations (1), and is left to the student.

The result of Theorem 13–2 raises the question of whether there is a line with given direction cosines a, b, and c such that $a^2 + b^2 + c^2 = 1$. The answer to this question is given by Theorem 13–3, which is the converse of Theorem 13–2.

Theorem 13–3 ***If a, b, and c are real numbers and $a^2 + b^2 + c^2 = 1$, then there is at least one line whose x-, y-, and z-direction cosines are a, b, and c, respectively.***

The proof of the theorem is left as an exercise and can easily be made by considering the line through $(0, 0, 0)$ and (a, b, c).

Problems

Set A

Find direction cosines of the lines through the following pairs of points, and directed from the first point to the second.

1. $(0, 0, 0)$, $(-3, 2, 6)$
2. $(0, 0, 0)$, $(5, 0, 0)$
3. $(1, 2, 0)$, $(4, 0, 4)$
4. $(3, 0, 2)$, $(0, 3, -1)$
5. $(0, 0, 0)$, (a, b, c)
6. (x, y, z), $(x + a, y + b, z + c)$
7. Verify that $c_1^2 + c_2^2 + c_3^2 = 1$ for Problems 1 through 6.
8. Give the direction cosines for Problems 1, and 3 if the lines are directed from the second point to the first.

Set B

9. The line through $(2, -3, 2)$ and $(-4, -3, 4)$ lies in a plane parallel to the xz-plane. Sketch the line. Find the direction cosines of the line.
10. A line is parallel to the xy-plane. Show that $c_3 = 0$.
11. A line is parallel to the z-axis. When directed, what direction cosines can it have? What direction angles?
12. Show that the three numbers $\frac{1}{2}$, $\frac{2}{3}$ and $\sqrt{11}/6$ are the direction cosines of some line.
13. Can a line have direction angles of $\pi/3$, $\pi/4$ and $\pi/3$?
14. Can one have direction angles

$$\alpha = \beta = \gamma = \pi/3?$$

Interpret geometrically. [*Hint:* Try holding a pencil so that it forms an angle of $\pi/3$ with each of the coordinate axes in space.]

Set C

15. What are the direction cosines of the directed line from $(0, 0, 0)$ to (a, b, c)? Suppose further that $a^2 + b^2 + c^2 = 1$. What theorem has been proved?
16. Prove Theorem 13–2.
17. Prove Theorem 13–3.
18. Suppose p, q, and r are real numbers such that not all three are zero. Show that if $k = \sqrt{p^2 + q^2 + r^2}$ then p/k, q/k, and r/k are direction cosines of some directed line.
19. If α, β, and γ are direction angles of a line, prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.

Calculator Problem

A line has direction cosines $c_1 = c_2 = c_3$. Find these direction cosines and determine the direction angles of the line.

13-5 Parametric Equations of Lines

We shall obtain parametric equations of a line in space quite analogously to parametric equations of a line in the plane.

As in the development of parametric equations of a line in a plane, there are two coordinate systems associated with the parametric equations of a line in space. One is the rectangular coordinate system represented by the ordered triple (x, y, z) and the other is the coordinate system represented by the directed distance d.

Theorem 13-4 ***Given a directed line l with direction cosines c_1, c_2, and c_3, let (x_0, y_0, z_0) be a point on l. Then a point (x, y, z) on l at a directed distance d from (x_0, y_0, z_0) has coordinates***

$$\begin{aligned} x &= x_0 + c_1 d, \\ y &= y_0 + c_2 d, \\ z &= z_0 + c_3 d. \end{aligned} \tag{1}$$

Conversely, if x, y, and z are given by equations (1), then the point (x, y, z) is on l. (See Fig. 13.)

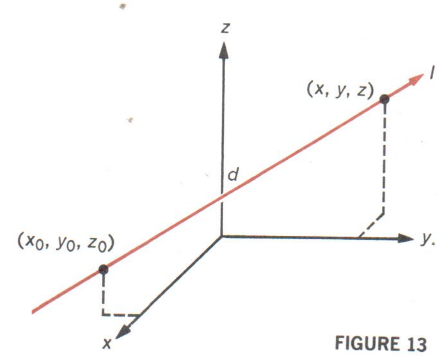


FIGURE 13

Proof If (x, y, z) is on l and $(x, y, z) \neq (x_0, y_0, z_0)$, then $d \neq 0$ and one has, by Theorem 13-1,

$$c_1 = \frac{x - x_0}{d}, \qquad c_2 = \frac{y - y_0}{d}, \qquad c_3 = \frac{z - z_0}{d}. \tag{2}$$

These equations imply equations (1). If $(x, y, z) = (x_0, y_0, z_0)$, then $d = 0$ and equations (1) are valid.

Conversely, if equations (1) are satisfied and $d \neq 0$, then equations (2) are satisfied. But these imply that (x, y, z) is on a line through (x_0, y_0, z_0) with direction cosines c_1, c_2, c_3. If $d = 0$, then $(x, y, z) = (x_0, y_0, z_0)$.

Equations (1) are *parametric equations* of the line with *parameter* d.

Example Find parametric equations of the line through (1, 3, 1) and (3, 0, 5).

Solution. We must first decide which point to use for (x_0, y_0, z_0). Let us suppose that (1, 3, 1) is selected (Fig. 14). Then the positive direction on l must be chosen. Suppose that we select (3, 0, 5) to be on the positive side. Then the distance between (3, 0, 5) and (1, 3, 1) is $\sqrt{29}$, and the direction cosines of the directed line are

$$c_1 = \frac{3-1}{\sqrt{29}} = \frac{2}{\sqrt{29}}, \qquad c_2 = \frac{0-3}{\sqrt{29}} = \frac{-3}{\sqrt{29}}, \qquad c_3 = \frac{5-1}{\sqrt{29}} = \frac{4}{\sqrt{29}}.$$

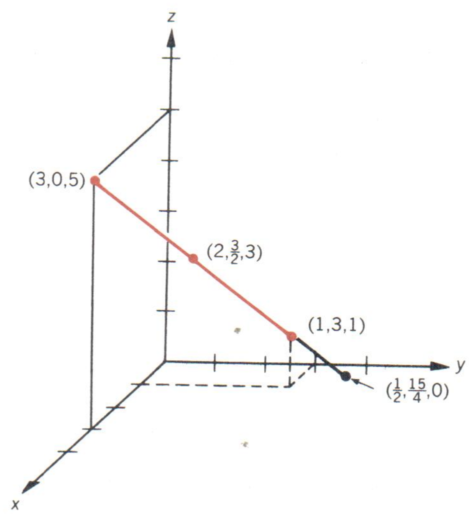


FIGURE 14

Had we selected the other ray from (1, 3, 1) to be the positive side of l, the direction cosines would all have been opposite in sign.

Parametric equations of the line are therefore

$$\begin{cases} x = 1 + \dfrac{2}{\sqrt{29}}\,d, \\[2ex] y = 3 - \dfrac{3}{\sqrt{29}}\,d, \\[2ex] z = 1 + \dfrac{4}{\sqrt{29}}\,d. \end{cases}$$

Other points of the line are easily found by using different values for d. For example, the midpoint of the segment between (1, 3, 1) and (3, 0, 5) can be found by setting d equal to one-half the distance between the points or $\sqrt{29}/2$. The coordinates of the midpoint are

$$x = 1 + \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{2} = 1 + 1 = 2$$

$$y = 3 - \frac{3}{\sqrt{29}} \cdot \frac{\sqrt{29}}{2} = 3 - \frac{3}{2} = \frac{3}{2}$$

$$z = 1 + \frac{4}{\sqrt{29}} \cdot \frac{\sqrt{29}}{2} = 1 + 2 = 3$$

The point at which the line pierces the xy-plane must have a z coordinate of 0. Setting

$$z = 1 + \frac{4}{\sqrt{29}} d = 0$$

we conclude that $d = -\sqrt{29}/4$, and the x- and y-coordinates are

$$x = 1 + \left(\frac{2}{\sqrt{29}} \cdot \frac{-\sqrt{29}}{4}\right) = \frac{1}{2},$$

$$y = 3 - \left(\frac{3}{\sqrt{29}} \cdot \frac{-\sqrt{29}}{4}\right) = \frac{15}{4}.$$

The point where the line pierces the plane is $(\frac{1}{2}, \frac{15}{4}, 0)$.

Problems

Set A

Find parametric equations of the lines through the following pairs of points. Use the first point as (x_0, y_0, z_0). Plot the pairs of points and find one other point on each line.

1. $(0, 0, 0)$, $(3, 2, 6)$

2. $(2, 2, -2)$, $(2, 5, 3)$

3. $(2, 4, 0)$, $(4, 0, 5)$

4. $(0, 0, 0)$, (e, f, g)

5. $(2, -1, 2)$, $(-1, 2, -1)$

6. (a, b, c), $(a + e, b + f, c + g)$

For Problems 7–11 use the line whose parametric equations are

$$x = 3 + \frac{2}{\sqrt{17}} d, \qquad y = -1 - \frac{3}{\sqrt{17}} d,$$

$$\text{and } z = 3 + \frac{2}{\sqrt{17}} d.$$

7. Give the coordinates for (x, y, z) when $d = 0$.

8. Give the coordinates for (x, y, z) when $d = \sqrt{17}$.

9. Find the midpoint of the segment whose end points are those given in Problems 7 and 8.

10. Verify that $c_1^2 + c_2^2 + c_3^2 = 1$.

11. Find the point where the line pierces each of the coordinate planes.

Set B

12. A line contains the point $(4, 3, 1)$ and is perpendicular to the xy-plane. Draw the line. Find parametric equations of the line.

13. Draw the line that has parametric equations

$$x = 5, \qquad y = 2 + \frac{1}{\sqrt{2}} d,$$

$$\text{and } z = -2 - \frac{1}{\sqrt{2}} d.$$

14. A line pierces the xy-plane at $A = (-2, 1, 0)$ and the xz-plane at $B = (-\frac{3}{2}, 0, 1)$. Find the point P on the line, directed from A to B, that is at a directed distance 3 from A.

15. Show that the segment with end points $A = (6, 4, -3)$ and $B = (2, 0, 5)$ and the segment with end points $C = (8, -5, -1)$ and $D = (0, 9, 3)$ have the same midpoint. What are the coordinates of the common midpoint?

Set C

16. A line has parametric equations $x = x_0$, $y = y_0$, and $z = z_0 + d$. Sketch the line. Show that the distance from (x, y, z) to (x_0, y_0, z_0) is $|d|$.

17. Given that (x_1, y_1, z_1) is on the line with parametric equations $x = x_0 + c_1 d$, $y = y_0 + c_2 d$, $z = z_0 + c_3 d$, show that the line also has parametric equations

$$x = x_1 + c_1 d',$$
$$y = y_1 + c_2 d',$$
$$z = z_1 + c_3 d',$$

where, of course, d' is not necessarily equal to d.

18. Show that if (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points on a line, then the midpoint of the line segment with these endpoints will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Calculator Problem

A line has direction angles $\alpha = \pi/5$, $\beta = 2\pi/5$ and $\gamma = \pi/3$. Show that

$$c_1^2 + c_2^2 + c_3^2 = 1.$$

Suppose the line contains the point (4, 3, 2). Find parametric equations of the line. Determine the coordinates of the points where the line pierces each of the three coordinate planes.

13-6 Planes

We shall return to the study of planes again in Chapter 15, where we shall prove that if A, B, C are real numbers, not all 0, then the set of points (x, y, z) such that

$$Ax + By + Cz + D = 0 \qquad (1)$$

is a plane. Equation (1) is called an equation of the plane, and sometimes we simply say "the plane $Ax + By + Cz + D = 0$." Conversely, every plane has an equation of the form (1).

For now, we shall take this basic theorem for granted. We shall draw some planes and find equations of some.

Example 1 To sketch the plane $2x + 3y + 6z = 12$, it will suffice to find three points in the plane. Then we can "see" where it goes. If the plane is not parallel to any of the axes, it will cut each axis. These three *intercepts* are easy to obtain. For the given plane (Fig. 15) they are $(6, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 2)$. The plane, or rather a triangular portion of the plane, is then easy to draw.

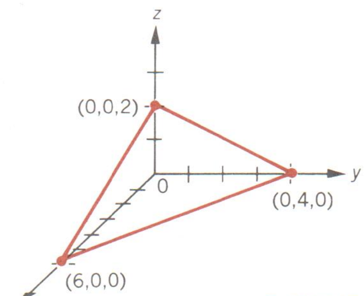


FIGURE 15

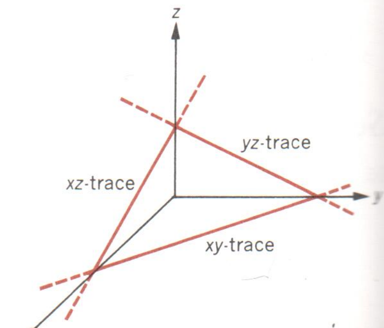


FIGURE 16

Another way of looking at the same problem is to examine the *traces* of the given plane in each of the coordinate planes (Fig. 16). The trace in any coordinate plane is the line of intersection of the given plane and that coordinate plane. Thus from Fig. 16 we see that the trace in the xy-plane is the intersection of the planes

$$z = 0, \qquad 2x + 3y = 12;$$

the trace in the yz-plane is the intersection of the planes

$$x = 0, \qquad 3y + 6z = 12;$$

the trace in the zx-plane is the intersection of the planes

$$y = 0, \qquad 2x + 6z = 12.$$

Example 2 Sketch the plane $3x + 2z = 6$.

Solution. What is striking here is that the variable y is missing from the equation. The x- and z-intercepts are at $(2, 0, 0)$ and $(0, 0, 3)$. There is no y-intercept because x and z cannot both be 0. Since the plane does not cut the y-axis, it is parallel to the y-axis, and the plane appears as in Fig. 17.

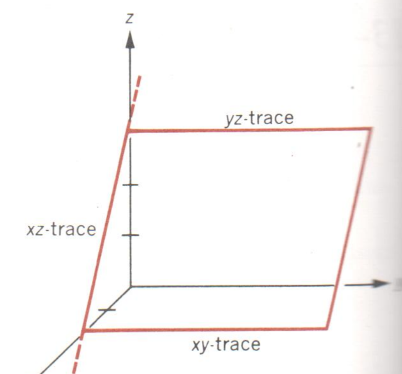


FIGURE 17

Example 3 Find an equation of the plane through the points $(0, -1, 1)$, $(6, 2, 1)$, and $(2, 1, 2)$.

Solution. Our first remark is that if the three points are collinear, there will be many planes through them. However, by finding direction cosines, it is easy to see that the three given points are *not* collinear, and so there is a unique plane. Therefore we wish to determine constants A, B, C, D such that the equation $Ax + By + Cz + D = 0$ is satisfied by the coordinates of all three points:

$$A(0) + B(-1) + C(1) + D = 0, \tag{2}$$

$$A(6) + B(2) + C(1) + D = 0, \tag{3}$$

$$A(2) + B(1) + C(2) + D = 0. \tag{4}$$

Although it may turn out that any one of A, B, C, or D is 0, we know that not all of A, B, C are 0 because the unique plane through the given points has an equation of the type we seek.

Let us see what we can discover about the relations of A, B, C, and D from equations (2), (3), and (4). From equations (2) and (3) we obtain by subtraction

$$6A + 3B = 0,$$

where $A = -\frac{1}{2}B$. If we use $-\frac{1}{2}B$ for A in equation (4), we obtain

$$-B + B + 2C + D = 0,$$

where $C = -\frac{1}{2}D$. Then equation (2) gives

$$B = C + D = -\tfrac{1}{2}D + D = \tfrac{1}{2}D.$$

Therefore

$$\begin{aligned} A &= -\tfrac{1}{2}B = -\tfrac{1}{4}D, \\ B &= \tfrac{1}{2}D, \\ C &= -\tfrac{1}{2}D, \end{aligned} \tag{5}$$

and the plane has an equation

$$-\tfrac{1}{4}Dx + \tfrac{1}{2}Dy + (-\tfrac{1}{2}Dz) + D = 0. \tag{6}$$

Certainly $D \neq 0$, for otherwise equation (6) is the triviality, $0 = 0$, and does not represent a plane. Hence equation (6) is equivalent to

$$x - 2y + 2z - 4 = 0,$$

and this is an equation of the plane we sought.

Remarks

1. What we have done in the above solution is to solve for three of the constants in terms of the fourth. We cannot tell in advance which one to select for the fourth; however, in practice this causes no trouble. We simply successively eliminate A, B, C, or D until we obtain a solution analogous to equations (5).

2. Equations (2), (3), and (4) are three equations in four unknowns. We are not actually interested in these numbers A, B, C, and D, but only in their ratios. In equation (6), D can be any non-zero number. In other words, the plane

$$Ax + By + Cz + D = 0 \tag{1}$$

is also represented by

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0$$

if $D \neq 0$. Then the three ratios, A/D, B/D, and C/D can be found from equations (2), (3), and (4), and the plane is determined.

But what happens if $D = 0$? Then we cannot divide equation (1) by D. Yet, there is a plane, $Ax + By + Cz = 0$, through the three given points, for which at least one of A, B, or C is not 0. Thus we may divide in equation (1) by any one of A, B, or C which is not 0. More generally, we can multiply equation (1) by any number $k \neq 0$ to obtain the equivalent equation

$$kAx + kBy + kCz + kD = 0. \tag{7}$$

If $k = 1/D$, we have the situation of the example. If $k = 1/A$ ($A \neq 0$), then we would solve for the three ratios B/A, C/A, D/A.

If the intercepts of a plane are known, it is particularly easy to write an equation of the plane, as the following theorem shows.

Theorem 13-5 ***If a plane has non-zero x-, y-, and z-intercepts of (a, 0, 0), (0, b, 0) and (0, 0, c) respectively, then an equation of the plane is***

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{8}$$

Conversely, equation (8) is an equation of a plane with the given intercepts.

Proof Substituting the intercepts in (1) we have

$$Aa + D = 0, \qquad Bb + D = 0 \qquad \text{and} \qquad Cc + D = 0.$$

Therefore

$$A = \frac{-D}{a}, \qquad B = \frac{-D}{b} \qquad \text{and} \qquad C = \frac{-D}{c}.$$

Substituting these values in (1) gives

$$\frac{-D}{a}x + \frac{-D}{b}y + \frac{-D}{c}z = -D.$$

Dividing both sides of the equation by $-D$ gives the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Conversely, equation (8) is equivalent to equation (1) because $A = 1/a$, $B = 1/b$, $C = 1/c$ and $D = -1$. It is easy to show that equation (8) yields the desired intercepts.

Example A plane has x-, y-, and z-intercepts of 3, -2, and 2 respectively. Find an equation of the plane.

Solution. From Theorem 13–5 an equation of the plane is

$$\frac{x}{3} + \frac{y}{-2} + \frac{z}{2} = 1$$

or equivalently

$$2x - 3y + 3z = 6.$$

A portion of the plane is shown in Fig. 18.

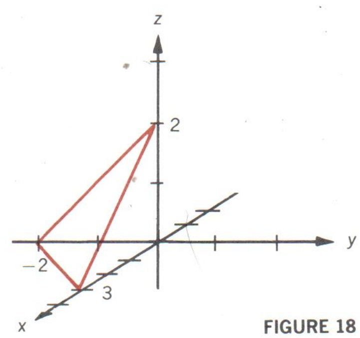


FIGURE 18

Problems

Set A

Sketch the following planes. In each case find the intercepts and the traces in the coordinate planes.

1. $x - y + 2z = 3$

2. $x + 2y = 5$

3. $x + y + z = a,\ a > 0$

4. $2y + 3z = 10$

5. $x - 2y - z + 4 = 0$

6. $2x - 2z + 5 = 0$

Find an equation of the plane passing through the following points.

7. $(1, 2, 2), (3, 3, -1), (-1, 5, 1)$

8. $(0, -1, 0), (2, 1, -2), (1, 0, -1)$

9. $(1, 1, 0), (3, 0, 1), (-1, 2, -1)$

10. $(2, 2, 1), (4, 2, -1), (6, -3, 2)$

Set B

Find an equation of the plane whose x-, y-, and z-intercepts are as follows. Sketch a portion of the plane.

11. $x = 4$, $y = -1$, $z = 3$

12. $x = -5$, $y = 6$, $z = 2$

13. $x = 3$, $y = 4$, $z = -4$

14. $x = -3$, $y = -3$, $z = 3$

15. Traces of a plane in each of the coordinate planes are given by

$$3x + 5z = 15,$$
$$2x - 5y = 10,$$

and

$$2z - 3y = 6.$$

Sketch a portion of the plane and find an equation of the plane with these traces.

Set C

Find the point or points of intersection, if any, of the following sets of three planes.

16. $x + y + z = 5$, $2x - y + z = 0$, $3x - 4y + 5z + 4 = 0$

17. $x + 2y = 6$, $2x = y$, $z = 3$ (Draw the planes.)

18. $x = a$, $y = b$, $z = c$ (Draw the planes.)

19. $x + 2y - 3z = 5$, $2x + y + z = 6$, $3x - y + 4z = -1$

20. $x + y + z = 5$, $x - y - z + 5 = 0$, $3x - y - z + 5 = 0$. (Draw the planes.)

21. $x - y + 2z = 6$, $2x - 2y + 4z = -4$, $5x - 5y + 10z = 15$ (Draw the planes.)

22. Show that if not all of A, B, and C are 0, then

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

is an equation of a plane through (x_0, y_0, z_0).

23. Given distinct planes

$$A_1x + B_1y + C_1z + D_1 = 0$$

and

$$A_2x + B_2y + C_2z + D_2 = 0$$

that are not parallel, and given a real number k, show that the set of all points (x, y, z) for which

$$A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) = 0$$

is a plane through the line of intersection of the given planes.

24. Use Problem 23 to obtain an equation of the plane through the line of intersection of the planes $x - 3y + z = 5$ and $2x - y - z = 5$, and containing the point $(1, 1, 1)$.

25. The line through $(4, 4, 1)$ is perpendicular to a plane at $(3, 2, -1)$. Find an equation of the plane.

26. Show that if a line contains a point (a, b, c) and is perpendicular to a plane at (d, e, f), then

$$(a - d)x + (b - e)y + (c - f)z + (d^2 + e^2 + f^2 - ad - be - cf) = 0$$

is an equation of the plane.

Calculator Problem

Find the intercepts of the plane

$$3.66x - 8.34y + 6.83z = 2.18.$$

Round the coordinates of the intercepts to the nearest hundredth.

Summary

We have assumed in this chapter that you are familiar with the simpler aspects of space geometry. We observed that these space properties are consequences of only four postulates other than those of plane geometry. Then using the fact that a line can be assigned coordinates, we were able to coordinatize space, that is, to show the existence of a one-to-one correspondence between the points of space and the set of ordered triples (x, y, z) of real numbers. A distance formula between pairs of points in space was developed as a result of the establishment of a coordinate system.

Direction angles and direction cosines gave us parametric equations of a line in space in a manner analogous to that used for a line in the plane. In the plane, each line can be represented by two parametric equations. In space, a set of three parametric equations is needed to represent a line.

Finally, we considered rectangular equations of planes and observed that by considering the intercepts of a plane or the traces of the plane in the coordinate planes, it is quite easy to draw pictures of planes.

Some of the more important relations developed in this chapter are listed below.

1. The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2):
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
2. Direction cosines of a line: If d is the directed distance from (x_0, y_0, z_0) to (x_1, y_1, z_1), then the line containing these points has direction cosines
$$c_1 = \frac{x_1 - x_0}{d}, \qquad c_2 = \frac{y_1 - y_0}{d}, \qquad c_3 = \frac{z_1 - z_0}{d}.$$
3. Parametric equations of a line:
$$x = x_0 + c_1 d, \qquad y = y_0 + c_2 d, \qquad z = z_0 + c_3 d,$$
where d is the parameter and is the directed distance along the line.
4. Equation of a plane: Every plane has an equation of the form
$$Ax + By + Cz + D = 0,$$
where not all of the real numbers A, B, and C are 0. The set of all points (x, y, z) satisfying $Ax + By + Cz + D = 0$ is a plane.
5. Intercepts and traces of planes: The plane $Ax + By + Cz + D = 0$ has intercepts
$$\left(\frac{-D}{A}, 0, 0\right), \qquad \left(0, \frac{-D}{B}, 0\right), \qquad \text{and} \qquad \left(0, 0, \frac{-D}{C}\right)$$
If A, B, and C are not 0.
Traces of the plane are given by
$$Ax + By + D = 0, \quad By + Cz + D = 0, \quad \text{and} \quad Ax + Cz + D = 0.$$

Chapter 13 Test

13-2

1. In which coordinate plane is the point whose coordinates are $(6, 0, -3)$?
2. A plane is parallel to the xy-coordinate plane. What is true about all the coordinates (x, y, z) of points in this plane?

13-3

3. Find the distance from the origin to $(3, -4, 5)$.
4. Find the distance between $(2, 3, 1)$ and $(-2, 4, 5)$.

13-4

5. Can a line have direction angles of $\pi/3$, $\pi/4$ and $\pi/4$? Explain.
6. Find the direction cosines of the line from $(-2, -4, -1)$ directed toward $(0, -8, 3)$.
7. What are the direction angles of a line
   (a) parallel to the y-axis?
   (b) parallel to the xz-plane?

13-5

8. Find parametric equations of the line through $(-3, 5, 4)$ and $(-1, 2, 2)$. Use the first point as (x_0, y_0, z_0).
9. Parametric equations of a line are

$$x = 1 + \frac{3d}{5}, \qquad y = 4 - \frac{3d}{5}, \qquad z = -2 + \frac{\sqrt{7}\,d}{5}.$$

   Find the coordinates of the point where the line pierces the yz-plane.
10. Find the coordinates of the midpoints of the segment joining $(7, -2, 5)$ and $(-3, 6, -3)$.
11. Parametric equations of a line are

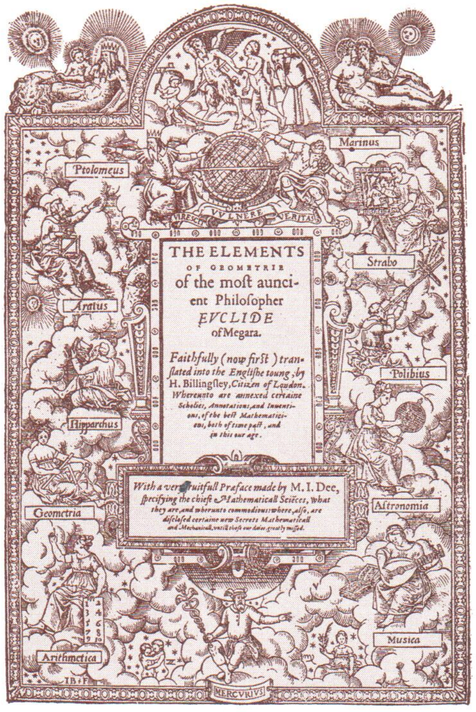
$$x = 3 + \frac{1}{\sqrt{6}}d, \qquad y = -4 + \frac{2}{\sqrt{6}}d, \qquad z = 7 - \frac{1}{\sqrt{6}}d.$$

   Find the coordinates of a point on the line when $d = 2\sqrt{6}$.

13-6

12. What are the intercepts of the plane $2x - 5y - 2z = 10$? Sketch a portion of the plane.
13. Give the equations of the traces of the plane $x - 3y + 2z = 6$ in each of the coordinate planes.
14. Find an equation of the plane that contains the three noncollinear points $(7, 0, 0)$, $(1, 1, 4)$, and $(7, 7, -2)$.
15. The traces of a plane are $2x + 5y = 10$, $3x + 5z = 15$, and $3y + 2z = 6$. What is a rectangular equation of the plane?
16. The three planes $2x + y + 2z = 7$, $x + y + z = 3$, and $x - y - z = 5$ have exactly one point in common. Find this point of intersection.

Historical Note



Title page of first translation of Euclid's Elements, printed in 1570.

Little is known of Euclid the man. The information we have is by inference from comments by other writers concerning his mathematics. What seems to be certain is that he lived around 300 B.C., that he learned his mathematics from pupils of Plato, and that he lived much of his life in Alexandria, where he taught and founded a school. His major work, *The Elements,* consists of thirteen "books," parts of which are familiar to high school students from either geometry or algebra. The word "elements" should not be construed in its common current sense to mean simple, or elementary. Rather, the word refers to the *foundations* or basic building blocks from which the entire edifice of mathematics (of his day) was to be constructed.

Of these books only XI and XIII are concerned with solid geometry. Most of the elementary theorems are to be found in Book XI among the first 19 propositions. The remaining propositions (20 to 39) of Book XI are concerned with solid polyhedra and their volumes. Euclid used no additional postulates for his geometry of space. More recently, however, mathematicians have determined the need for four additional postulates. Book XIII is devoted to propositions about the five regular polyhedra.