Dynamics of a Mono-Propelled VTOL Drone

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Introduction

- Skipper is a mono-propelled vertical-takeoff vertical-landing drone (VTOL). What does this mean in practice?
  - Skipper has one propeller located at the bottom of the drone which is used for lift and steering.
  - This motor is connected to a gimbal with two DOF.
  - By default this configuration does not have roll control so either a reaction wheel or gyroscope must be used.
- The goal of Skipper is to model the dynamics of Hopper and design a flight computer for the former which can be adapted to the latter.

Figure : US Navy's Aerovel Flexrotor.

Prerequisites

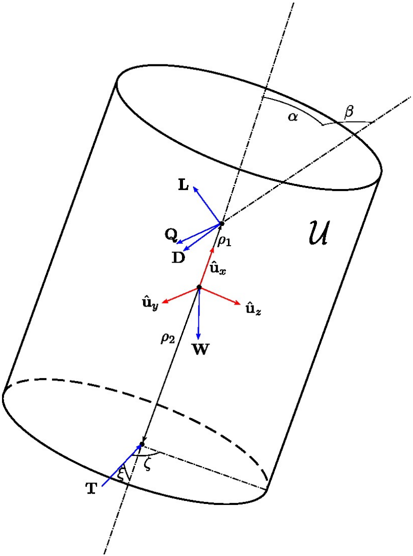
- At first glance dynamic equations may seen overwhelming; however, in reality most of the terms are simply algebra left over from the derivations.
- There isn't time to cover all the necessary background information for the design of a flight computer so below is a list of useful resources. This knowledge will be assumed for the rest of the presentation.
    - Reference frames and basis vectors.
    - The transport theorem.
    - Rotating and translating frames and an example.
    - Newton's laws of motion.
    - Angular momentum.
    - Rigid body kinematics - angular momentum and inertia.
    - Euler's laws of motion and their generalization along with an example.
    - First order form of differential equations.
    - Euler's angles and coordinate transformation matricies.
- In addition for those interested both UF's Dr. John Conklin and Dr. Anil Rao have uploaded their dynamics courses online (although Dr. Rao's is incomplete).

Constructing a Dynamic Model

Purpose and Assumptions

- The goal of the presentation is to derive the equations of motion which govern the movement of Skipper.
- By default Skipper may not move how we want it to, so we want to design a controller which will input values into these equations to achieve our desired result.
- For simplicity we will make a few assumptions about Skipper:
  - Skipper is a rigid cylinder.
  - The center of mass (CM) of skipper is fixed to the center of this cylinder.
  - All aerodynamic forces act at the aerodynamic center (AC) of the cylinder.

- We define a basis fixed to Skipper, $\{\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z\} \in \mathcal{U}$. This is rotated from the inertial frame, $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} \in \mathcal{I}$, through a 3-2-1 Euler angle sequence, also known as yaw, ψ, pitch, θ, and roll, ϕ.
  - $\hat{\mathbf{u}}_x$ points along Skipper's axis of symmetry.
  - $\hat{\mathbf{u}}_z$ points at 90° CW from $\hat{\mathbf{u}}_x$ in an arbitrarily fixed direction.
  - $\hat{\mathbf{u}}_y = \hat{\mathbf{u}}_z \times \hat{\mathbf{u}}_x$.
- We are going to want to quantity the motion of $\mathcal{U}$ in $\mathcal{I}$ because our perspective is the later. In addition, $\mathcal{U}$ is acceleration and hence not an inertial frame so Newton's laws would not be valid.



- To transform from $\mathcal{I}$ to $\mathcal{U}$ we must use the aforementioned 3-2-1 or yaw-pitch-roll Euler angle sequence.
- We can break apart the transformation into three separate transformations of the form: $\mathcal{I} \rightarrow \mathcal{J} \rightarrow \mathcal{K} \rightarrow \mathcal{U}$.
  - $\mathcal{I} \rightarrow \mathcal{J}$: Rotation through ψ about Z/3-axis.
    - These axes are simply lines which extend infinite from the bases unit vectors.
  - $\mathcal{J} \rightarrow \mathcal{K}$: Rotation through θ about Z'/2'-axis, ie. the new Y/2-axis.
  - $\mathcal{K} \rightarrow \mathcal{U}$: Rotation through ϕ about X"/1'-axis, ie. the new X/1-axis.

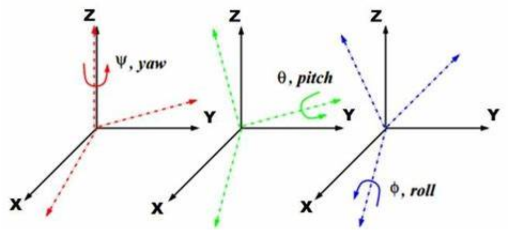


Figure : 3-2-1 Euler Angle sequence.

- Each stage of our transformation has an associated coordinate transformation matrix which allows the conversion of any vector from the initial frame to the later.
    - For example, to convert from $\mathcal{I} \rightarrow \mathcal{J}$ the transformation matrix, $\mathbf{T}_{\mathcal{I}}^{\mathcal{J}}$, is given by,

$$\mathbf{T}_{\mathcal{I}}^{\mathcal{J}} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

- To go from $\mathcal{I} \rightarrow \mathcal{U}$ we simply chain these transformation matrices to get

$$\begin{aligned} \mathbf{T}_{\mathcal{I}}^{\mathcal{U}} &= \mathbf{T}_{\mathcal{I}}^{\mathcal{J}} \mathbf{T}_{\mathcal{J}}^{\mathcal{K}} \mathbf{T}_{\mathcal{K}}^{\mathcal{U}} \\ &= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix} \end{aligned} \tag{2}$$

- If we have a vector $\mathbf{b}$ with components written in the $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$-basis and we want these components in the $\{\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z\}$-basis we would write

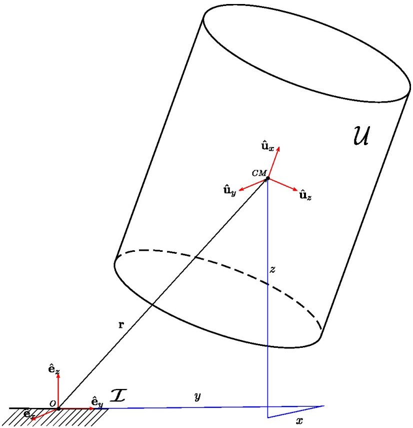
$$\begin{bmatrix} b_{u_x} \\ b_{u_y} \\ b_{u_z} \end{bmatrix} = \mathbf{T}_{\mathcal{I}}^{\mathcal{U}} \begin{bmatrix} b_{e_x} \\ b_{e_y} \\ b_{e_z} \end{bmatrix} \tag{3}$$

- Note that via a property of transformation matrices if we instead wanted to go from $\mathcal{U} \rightarrow \mathcal{I}$ it can be shown that

$$\mathbf{T}_{\mathcal{U}}^{\mathcal{I}} = (\mathbf{T}_{\mathcal{I}}^{\mathcal{U}})^{\mathsf{T}} \tag{4}$$

- We are now ready to derive Skipper's equations of motion (EOM).
- To begin, we need to find the position, **r** of Skipper's CM with respect to a point O fixed in $\mathcal{I}$.
  - For our purposes let O be where Skipper takes off.
  - Also, define three distances (x, y, z) along $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$ respectively.
  - **r** can then be defined as

$$\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z \qquad (5)$$



Kinematics

Angular Velocity

- Next, we need to define the angular velocity of $\mathcal{U}$ relative to $\mathcal{I}$, ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{U}}$, so that we may use the transport theorem.
  - For later convenience we define $\mathcal{U}$ as

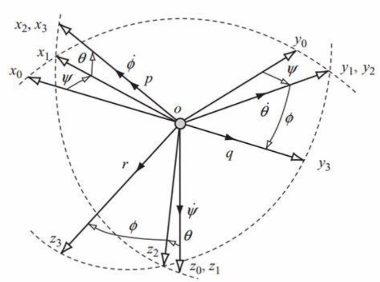
$$ {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{U}} = p\hat{\mathbf{u}}_x + q\hat{\mathbf{u}}_y + r\hat{\mathbf{u}}_z \tag{6} $$

  where (p, q, r) are the roll rate, pitch rate, and yaw rate respectively found by finding the components of $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ along each of $\{\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z\}$.
- It can be seen that

$$ p = \dot{\phi} - \dot{\psi}\sin\theta \tag{7} $$

$$ q = \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \tag{8} $$

$$ r = \dot{\psi}\cos\phi\cos\theta - \dot{\theta}\sin\phi \tag{9} $$



- To find the inertial velocity, ${}^{\mathcal{I}}\mathbf{v}$, and acceleration, ${}^{\mathcal{I}}\mathbf{a}$, of Skipper we simply take the time derivative of $\mathbf{r}$ with respect to $\mathcal{I}$ twice.
- Since $\mathbf{r}$ was defined in $\mathcal{I}$ we don't need the transport theorem and can simply find that

$$ {}^{\mathcal{I}}\mathbf{v} = \dot{x}\hat{\mathbf{e}}_x + \dot{y}\hat{\mathbf{e}}_y + \dot{z}\hat{\mathbf{e}}_z \tag{10} $$

and likewise,

$$ {}^{\mathcal{I}}\mathbf{a} = \ddot{x}\hat{\mathbf{e}}_x + \ddot{y}\hat{\mathbf{e}}_y + \ddot{z}\hat{\mathbf{e}}_z \tag{11} $$

  - In some contexts (usually boats) the components of velocity, $(\dot{x}, \dot{y}, \dot{z})$, are referred to as the surge, sway, and heave; (u, v, w).

- The moment of inertial tensor, **I**, depends on the choice of reference frame and coordinate system. We choose $\mathcal{U}$ as it yields a value of **I** corresponds to the moment of inertia of a cylinder with at least one coordinate axis along the axis of cylinder and centered at the CM.
  - Let Skipper have a height of H, a radius of R, and a mass of M; $\mathbf{I}^{\mathcal{U}}_{CM}$ is given by

$$\mathbf{I}^{\mathcal{U}}_{CM} = \begin{bmatrix} \frac{M(3R^2+H^2)}{12} & 0 & 0 \\ 0 & \frac{M(3R^2+H^2)}{12} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{bmatrix} \tag{12}$$

- The inertial angular of momentum of the CM of Skipper is given by

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{H}_{CM} &= \mathbf{I}^{\mathcal{U}}_{CM}\, {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{U}} \\ &= \begin{bmatrix} \frac{MR^2}{2} & 0 & 0 \\ 0 & \frac{M(3R^2+H^2)}{12} & 0 \\ 0 & 0 & \frac{M(3R^2+H^2)}{12} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \\ &= \frac{MR^2}{2} p\hat{\mathbf{u}}_x + \frac{M(3R^2+H^2)}{12} q\hat{\mathbf{u}}_z + \frac{M(3R^2+H^2)}{12} r\hat{\mathbf{u}}_y \\ &= I_{xx} p\hat{\mathbf{u}}_x + I_{yy} q\hat{\mathbf{u}}_y + I_{zz} r\hat{\mathbf{u}}_z \end{aligned} \tag{13}$$

- We begin by our kinetics analysis by addressing the thrust, $\mathbf{T}$ generated by Skipper's rotors.
- The gimbal used to vector our thrust gives us 5 degrees of freedom (DOF) and is comprised of two joints.
    - The first joint, J_1, allows rotation about $\hat{\mathbf{u}}_z$ by some angle, ξ.
    - The second joint is attached to a lever arm stretching from the first and allows rotation about the rotated $\hat{\mathbf{u}}_y$ axis by some angle, ζ.
- The rotations can be modeled as a 3-2 incomplete Euler angle sequence from $\mathcal{U} \to \mathcal{T}$, the frame of the fully vectored gimbal in which $\mathbf{T}$ acts along $\hat{\mathbf{t}}_x$.
- To convert from $\mathcal{U} \to \mathcal{T}$ we can use the transformation matrix, $\mathbf{T}_{\mathcal{U}}^{\mathcal{T}}$, given by,

$$\mathbf{T}_{\mathcal{U}}^{\mathcal{T}} = \begin{bmatrix} \cos\xi\cos\zeta & \sin\xi\cos\zeta & -\sin\zeta \\ -\sin\xi & \cos\xi & 0 \\ \cos\xi\sin\zeta & \sin\xi\sin\zeta & \cos\zeta \end{bmatrix} \tag{14}$$

- Skipper experiences an aerodynamic force, $\mathbf{F}_a$, which acts in accordance with the relative wind (ie. Skipper's velocity).
    - The angle of attack, α, measures the angle between the relative wind and Skipper's axis of symmetry in the $\hat{\mathbf{u}}_x\hat{\mathbf{u}}_z$-plane.
    - The sideslip angle, β, measures the angle between the relative wind and axis of symmetry in the $\hat{\mathbf{u}}_x\hat{\mathbf{u}}_y$-plane.
- α and β can be used to define a frame $\{\hat{\mathbf{w}}_x, \hat{\mathbf{w}}_y, \hat{\mathbf{w}}_z\} \in \mathcal{W}$ aligned with the relative wind via a 2-3 incomplete Euler angle sequence.
- $\mathbf{F}_a$ can be decomposed in $\mathcal{W}$ into: lift, $\mathbf{L}$, along $-\hat{\mathbf{w}}_z$; drag, $\mathbf{D}$, along $-\hat{\mathbf{w}}_x$; and a side force, $\mathbf{Q}$, along $\hat{\mathbf{w}}_y$. In vector form,

$$\mathbf{F}_a = -D\hat{\mathbf{w}}_x + Q\hat{\mathbf{w}}_y - L\hat{\mathbf{w}}_z \tag{15}$$

- To convert from $\mathcal{U} \to \mathcal{W}$ we can use the transformation matrix

$$\mathbf{T}_{\mathcal{U}}^{\mathcal{W}} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{16}$$

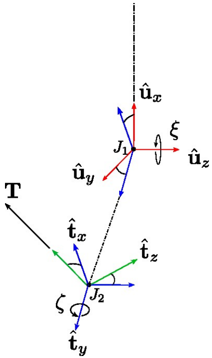


Figure : Thrust vectoring.

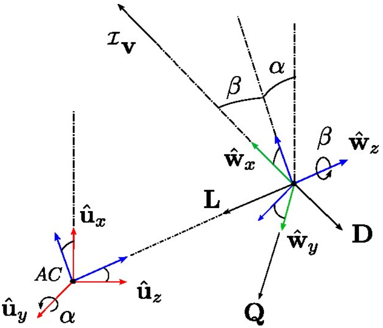


Figure : Orientation of aerodynamic forces.

- Skipper's weight, $\mathbf{F}_g$, is given by

$$\mathbf{F}_g = -Mg\hat{\mathbf{e}}_z \tag{17}$$

- Perturbation forces, $\mathbf{F}_p$ represent primarily wind gusts which hit Skipper as well as an other unexpected forces acting at a random point on the body of Skipper.
  - We can model these forces as having components along all basis directions in whichever frame is most convenient.
  - In practice these forces are used to test the stability of our controller and can be generated through a random method such as a Monte-Carlo simulation.

- With the kinematics derived and forces defined we can now move into the kinetics.
  - Because Skipper is modeled as a rigid body we turn to Euler's laws instead of Netwon's laws for deriving our EOM.
- We know that Euler's 1st law is given by

$$\mathbf{F} = m\,^{\mathcal{I}}\mathbf{a} \tag{18}$$

and we may expand this out, writing $\mathbf{F}_P$ in terms of $\mathcal{U}$, to

$$\mathbf{T} + \mathbf{F}_a + \mathbf{F}_g + \mathbf{F}_p = M\,^{\mathcal{I}}\mathbf{a} \tag{19}$$

$$\begin{split} T\hat{\mathbf{t}}_x - D\hat{\mathbf{w}}_x + Q\hat{\mathbf{w}}_y - L\hat{\mathbf{w}}_z - Mg\hat{\mathbf{e}}_z \\ + F_{p,x}\hat{\mathbf{u}}_x + F_{p,y}\hat{\mathbf{u}}_y + F_{p,z}\hat{\mathbf{u}}_z = M(\ddot{x}\hat{\mathbf{e}}_x + \ddot{y}\hat{\mathbf{e}}_y + \ddot{z}\hat{\mathbf{e}}_z) \end{split} \tag{20}$$

- We know that Euler's (generalized) 2nd law is given by

$$\mathbf{M}_Q - (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\,{}^{\mathcal{I}}\mathbf{a}_Q + \boldsymbol{\tau} = {}^{\mathcal{I}}\frac{d}{dt}\,{}^{\mathcal{I}}\mathbf{H}_Q \tag{21}$$

and if simplified by choosing $CM = Q$ reduces to

$$\mathbf{M}_{CM} + \boldsymbol{\tau} = {}^{\mathcal{I}}\frac{d}{dt}\,{}^{\mathcal{I}}\mathbf{H}_{CM} \tag{22}$$

- Only forces acting at a distance from the CM generate a moment, and it this case these are $\mathbf{T}, \mathbf{F}_F$, and $\mathbf{F}_P$.
  - F_A acts at the AC a distance ρ_1 from the CM along $\hat{\mathbf{u}}_x$.
  - $\mathbf{T}$ acts at the center of thrust, CT, a distance ρ_2 from the CM along $-\hat{\mathbf{u}}_x$.
  - F_p acts at an arbitrary point P at a distance ρ_3 with components along all of $\{\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z\}$
- We can expand (19) using our knowledge of the forces to

$$\begin{aligned}\rho_1\hat{\mathbf{u}}_x \times (-D\hat{\mathbf{w}}_x + Q\hat{\mathbf{w}}_y - L\hat{\mathbf{w}}_z) - \rho_2\hat{\mathbf{u}}_x \times \mathbf{T} + (\rho_{3,x}\hat{\mathbf{u}}_x + \rho_{3,y}\hat{\mathbf{u}}_y + \rho_{3,z}\hat{\mathbf{u}}_z)\\ \times (F_{p,x}\hat{\mathbf{u}}_x + F_{p,y}\hat{\mathbf{u}}_y + F_{p,z}\hat{\mathbf{u}}_z) + \tau_r\hat{\mathbf{u}}_x = {}^{\mathcal{I}}\frac{d}{dt}(I_{xx}p\hat{\mathbf{u}}_x + I_{yy}q\hat{\mathbf{u}}_y + I_{zz}r\hat{\mathbf{u}}_z)\end{aligned} \tag{23}$$

where τ_r is the roll control torque due to either a reaction wheel or gyroscope.

- To compute the equations of motion we first convert every vector to the $\mathcal{U}$ basis and then equate components.
- We have a maximum of six equations for (20) and (23) and thus can have six unknown: x, y, z, ϕ, θ and ψ. α and β are not true unknowns as they can be solved for from the former variables. Finally, $\xi, \zeta, T, L, D, Q, \tau_r, M, H$ and R are all controlled variable or known functions.
- The couple equations of motion are as follows:

$$T\cos\xi\cos\zeta - D\cos\alpha\cos\beta + Q\sin\beta - L\sin\alpha\cos\beta + Mg\sin\theta = \ddot{x}\cos\theta\cos\psi + \ddot{y}\cos\theta\sin\psi - \ddot{z}\sin\theta \tag{24}$$

$$\begin{aligned}&-T\sin\xi + D\cos\alpha\sin\beta + Q\cos\beta + L\sin\alpha\sin\beta - Mg\sin\phi\sin\theta\\ &\quad= \ddot{x}(\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi) + \ddot{y}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) + \ddot{z}\sin\phi\cos\theta\end{aligned} \tag{25}$$

$$\begin{aligned}&T\cos\xi\sin\zeta + D\sin\alpha + L\cos\alpha - Mg\cos\phi\cos\theta\\ &\quad= \ddot{x}(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi) + \ddot{y}(-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi) + \ddot{z}\cos\phi\cos\theta\end{aligned} \tag{26}$$

$$\rho_{3,x}(\rho_{3,y}F_{p,z} - \rho_{3,z}F_{p,y}) + \tau_r = I_{xx}\dot{p} + pqr(I_{zz} - I_{yy}) \tag{27}$$

$$\rho_2 T\cos\xi\sin\zeta - \rho_1(D\sin\alpha + L\cos\alpha) - \rho_{3,y}(\rho_{3,x}F_{p,z} - \rho_{3,z}F_{p,x}) = I_{yy}\dot{q} + pqr(I_{zz} - I_{xx}) \tag{28}$$

$$\rho_2 T\sin\xi + \rho_1(D\cos\alpha\sin\beta + L\sin\alpha\sin\beta) + \rho_{3,z}(\rho_{3,x}F_{p,y} - \rho_{3,y}F_{p,x}) = I_{zz}\dot{r} + pqr(I_{yy} - I_{xx}) \tag{29}$$

Conclusion

- The equations of motion, (24-29), must be modified before they are ready for use in a controls system.
  - They need to be decoupled, ie. each independent variable much be solved for. This can be done using a computer algebra system (CAS) or MATLAB.
  - They must be converted to first order form.
  - There are an abundance of nonlinear (trigonometric) terms which must be linearized.
- In addition, all of the aerodynamic parameters must be calculated - this can be performed using computational fluid dynamics (CFD) simulations or wind tunnel experiments.
- Also keep in mind that we made a variety of simplifying assumptions which likely can't be kept for Hopper's control system.
  - $\mathbf{F}_a$ must be taken to act at the changing center of pressure, CP, instead of the AC.
  - In reality M and hence $\mathbf{I}^{\mathcal{U}}_{CM}$ and the CM change with time.
  - Hopper won't really be a cylinder and their will likely be non-negligible aerodynamic forces inside its frame.
- Ultimately, this presentation serves as a good starting point and will be the foundation of all future GNC work relating to Skipper and later Hopper.