6 FUZZY BEHAVIOUR

FEB 2023

Dr. Liu Fan

isslf@nus.edu.sg

Total Slides: 107

Module objective

Knowledge and understanding

- Understand the fundamentals of fuzzy set, fuzzy logic.
- Understand the fundamentals of fuzzy inference system for robotic applications

Key skills

- Design and build fuzzy inference robotic system using Python.

Major reference

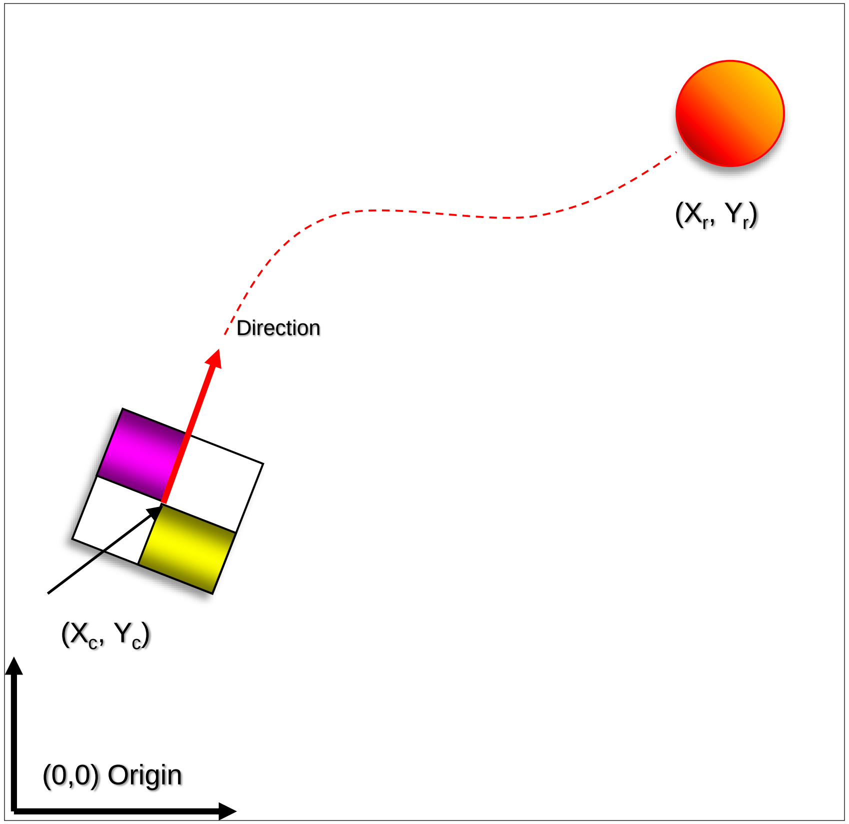
- [Tutorial] Michael Negnevitsky, ***Artificial Intelligence: A Guide to Intelligent Systems***, Pearson Education, 2011.
- [Comprehensive] Timothy J. Ross, ***Fuzzy Logic with Engineering Applications***, Wiley, 2010.
- ECE/CS/ME 539, Introduction to Artificial Neural Network and Fuzzy Systems (Year 2018), https://aefis.wisc.edu/index.cfm/page/AefisCourse.ABETSyllabusForm?courseid=866
- EP33FLO, Fuzzy Logic (Year 2020), available at website http://cmp.felk.cvut.cz/~navara/fl/

Topics

- Fundamentals of fuzzy set
- Fuzzy inference for robotic systems
- Design and build a fuzzy-based intelligent control system
- Demo on fuzzy-controlled car

Motivation: Robot navigation

- How does the robot follow the ball?

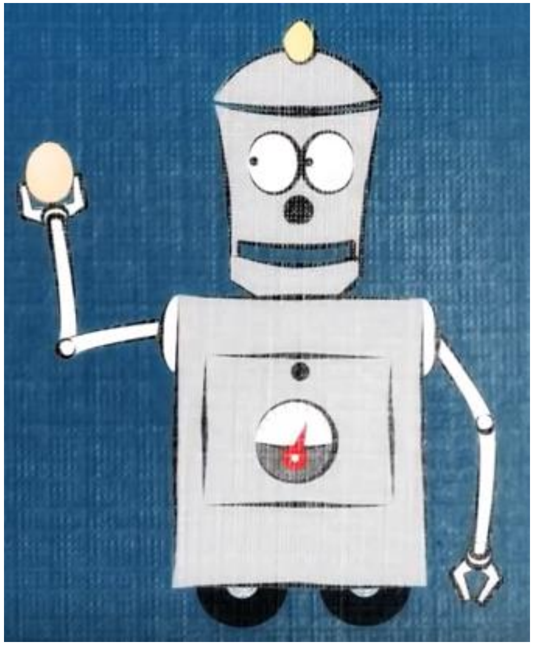


- Calculate angle from the ball
- Calculate distance from the ball
- Calculate optimum speed to reach the ball
- Calculate the steering angle to reach the ball
- Move the robot

Source: Napoleon H. Reyes, Intelligent Robotics 159.741, http://www.massey.ac.nz/~nhreyes/MASSEY/159741.htm

Fuzzy logic

- An egg-boiling fuzzy logic robot
- https://www.youtube.com/watch?v=J_Q5X0nTmrA
- Fuzzy logic: An introduction
- https://www.youtube.com/watch?v=P8wY6mi1vV8

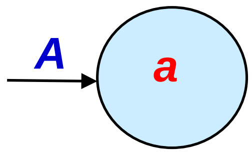


Topics

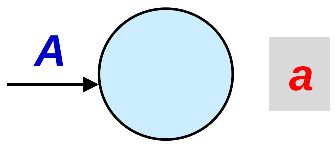
- Fundamentals of fuzzy set
- Fuzzy inference for robotic systems
- Design and build a fuzzy-based intelligent control system
- Demo on fuzzy-controlled car

Fuzzy set vs. crisp set (1)

- Crisp set
  - for an individual *a*, $a \in A$ means
    - *a* is a member of the set *A*



  - while $a \notin A$ means
    - *a* is *not* a member of the set *A*



  - There are *only* two possible relationships between the individual *a* and the set *A*:
    - $a \in A$ (membership = 1 → 100% belonging) or
    - $a \notin A$ (membership = 0 → 0% belonging)

Fuzzy set vs. crisp set (2)

- Fuzzy set
  - There is no full/zero membership in general
    - Is 45-yrs *old* or *young*?
    - Is 70/100 marks a *good* or *poor* exam result?
- Object dependent vs. subject dependent
  - Crisp: Ice, water, steam defined on temperature
  - Fuzzy: Cold, warm
- Sharp vs. Unsharp boundary
  - Crisp: Teenage (between 13 and 19 years old)
  - Fuzzy: Young, old

Fuzzy set

- The basic idea of the fuzzy set theory is that an element belongs to a fuzzy set with a certain degree of membership.
- The classical example in the fuzzy set theory is tall men. The elements of the fuzzy set “tall men” are all tall men, but the degrees of membership depend on their height.

Name	Height, cm	Crisp	Fuzzy
Chris	198	1	1.00
Mark	181	1	0.82
John	179	1	0.78
Mike	172	0	0.24
Steven	167	0	0.15
Peter	158	0	0.06

Michael Negnevitsky, *Artificial Intelligence: A Guide to Intelligent Systems*, Pearson Education, 2011

Fuzzy set

- Fuzzy set A of universe X is defined by function $\mu_A(x)$ call the membership function of A

$$\mu_A(x): X \rightarrow [0,1]$$

- where

$\mu_A(x) = 1$ if x is totally in A

$\mu_A(x) = 0$ if x is not in A

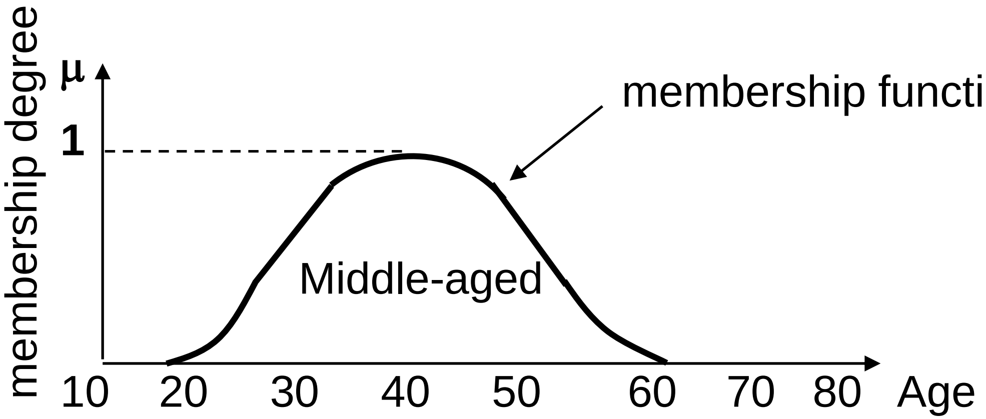
$0 < \mu_A(x) < 1$ if x is partly in A

- For any elements x of universe X, membership function $\mu_A(x)$ equals the degree to which x is an element of set A.

Michael Negnevitsky, *Artificial Intelligence: A Guide to Intelligent Systems*, Pearson Education, 2011

Notations of fuzzy set (1)

- A fuzzy set can be represented in many ways
  - Graphical representation



Notations of fuzzy set (2)

- List representation (for discrete universe)

  $A = \{<x_1, \mu_A(x_1)>, <x_2, \mu_A(x_2)>, \ \dots <x_n, \mu_A(x_n)>\}$

or $A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \dots + \mu_A(x_n)/x_n$

small-integer = {<0, 1>, <1,0.94>, <2,0.8>, …, <100,0>}

small-integer = 1/0 + 0.94/1 + 0.8/2 + 0.64/3 + …+ 0/100

Note: The symbol "/" stands for the correspondence between an element in the universal set and its membership grade in the fuzzy set. The symbol "+" merely connects the elements.

Notations of fuzzy set (3)

- Function representation (for continuous universe)

$$A = \int_X \mu_A(x) \,/\, x$$

(*) The symbol "$\int$" indicates the union of the elements in A.

- The generalized notation commonly used in the literature has the form:

  $A = \Sigma A(x)/x, \qquad x \in X$

- An important feature

  Probability: $\Sigma_X p(x_i) = 1$

  Fuzzy set: $\Sigma_X \mu(x_i) \neq 1$

Notations of fuzzy set (4)

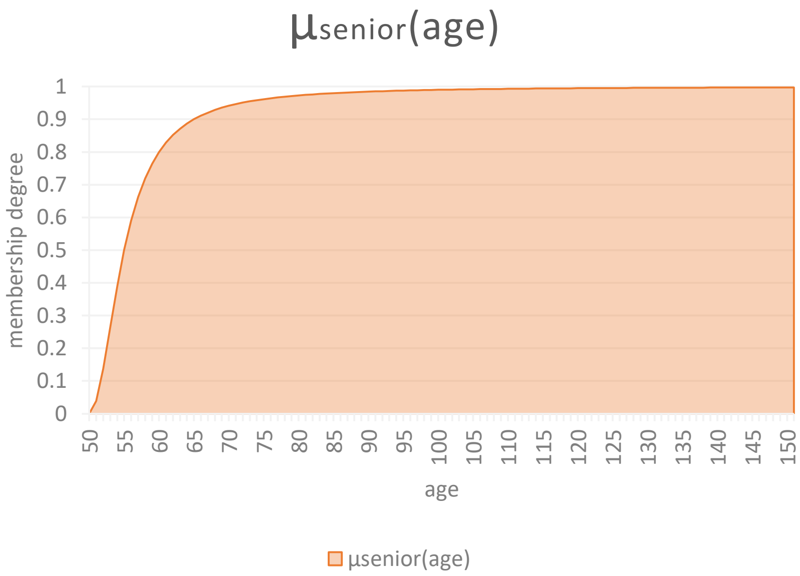
- Example:
- A senior membership function for $51 \leq age \leq 150$
  - Membership degrees of different ages

$\mu_{senior}(55) = 0.5$

$\mu_{senior}(60) = 0.8$

$\mu_{senior}(70) \approx 0.94$

$$\mu_{senior}(age) = \frac{1}{1+(\frac{5}{age-50})^2}$$

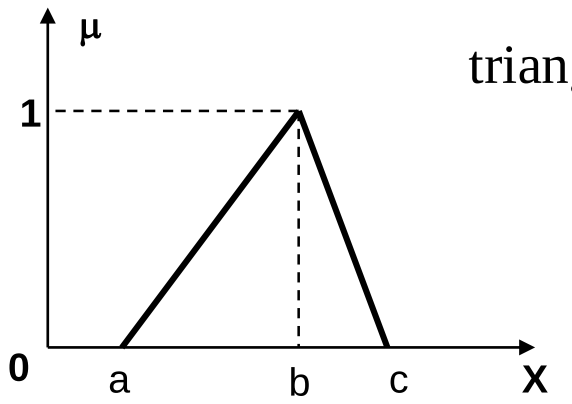


Membership function (1)

- In most cases of fuzzy sets, it would be impractical to list all the pairs defining a membership function. A more convenient and concise way to define a membership function is to express it as a mathematical formula.
- Parameterized functions commonly used to define membership functions
  - Triangular
  - Trapezoidal
  - Gaussian
  - Bell
  - Sigmoidal

Membership function (2)

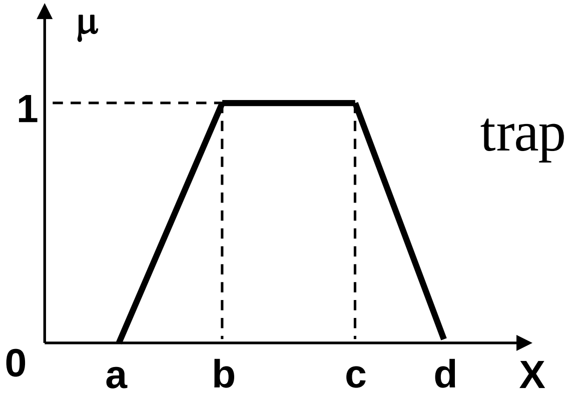
- Triangular membership function, specified by three parameters {*a*, *b*, *c*} (*a* < *b* < *c*)



$$\text{triangular}(x; a, b, c) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & c \le x \end{cases}$$

Membership function (3)

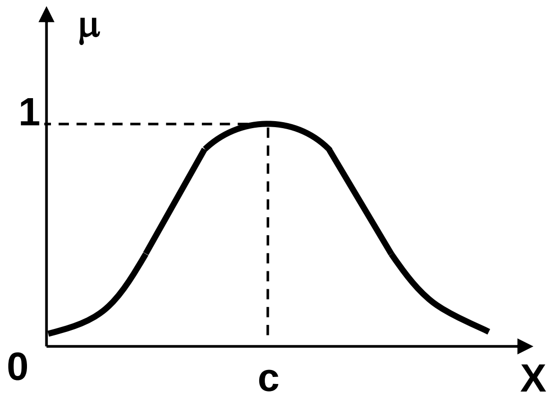
- Trapezoidal membership function, specified by four parameters {*a*, *b*, *c*, *d*} ($a < b \le c < d$)



$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & b \le x \le c \\ \dfrac{d-x}{d-c} & c \le x \le d \\ 0 & d \le x \end{cases}$$

Membership function (3)

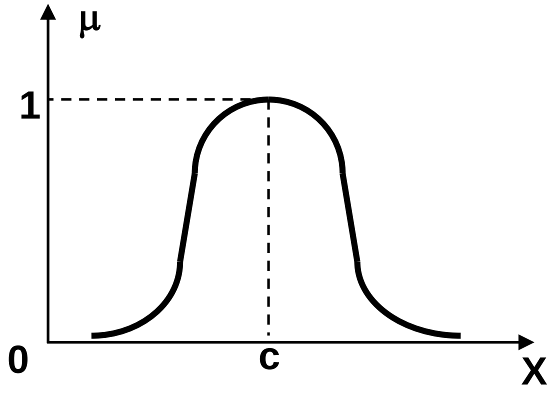
- Gaussian membership function, specified by two parameters $\{c, \sigma\}$



$$Gaussian(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Membership function (4)

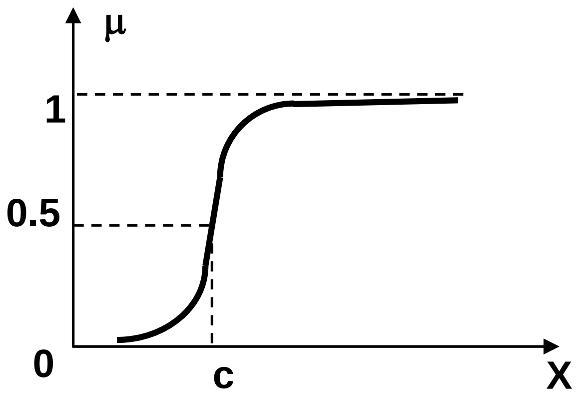
- Bell membership function, specified by three parameters {*a*, *b*, *c*}, where *b* is usually positive. (If *b* is negative, the shape of this membership function becomes an upside-down bell)



$$\text{bell}(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

Membership function (5)

- Sigmoidal membership function, specified by two parameters {*a*, *c*}, where *a* controls the slope at the crossover point *x* = *c*



$$Sigmoid(x; a, c) = \frac{1}{1 + \exp(-a(x - c))}$$

Exercise

Define fuzzy sets

- Driving speed: high, low
- Driving distance: short, long

Determine the universe of discourse

- Continuous or discrete
- Range

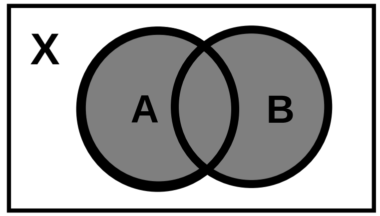
Determine membership function

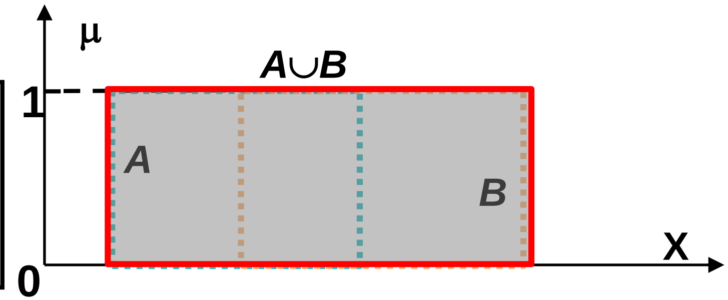
- mathematical formula (linear or non-linear)
- list representation

Basic set operations: Crisp set

- Union of A and B

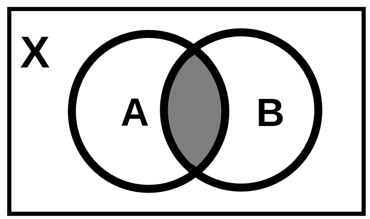
$A \cup B = \{x \mid x \in A \;\; or \;\; x \in B\}$

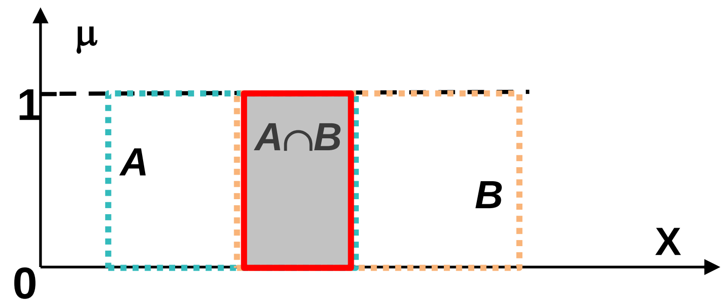




- Intersection of A and B

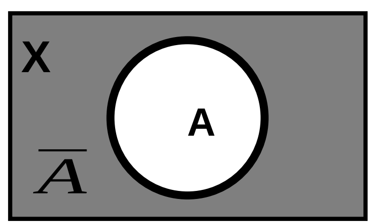
$A \cap B = \{x \mid x \in A \;\; and \;\; x \in B\}$

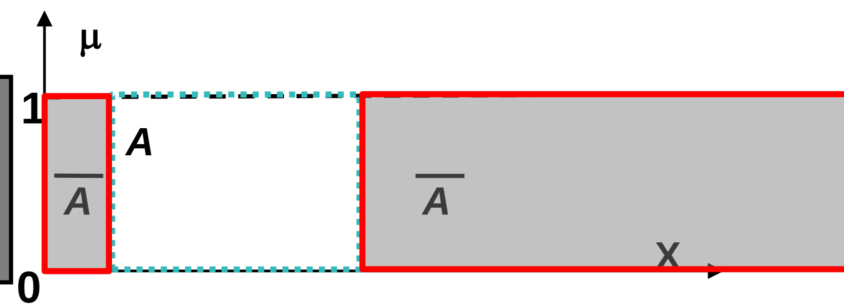




- Complement

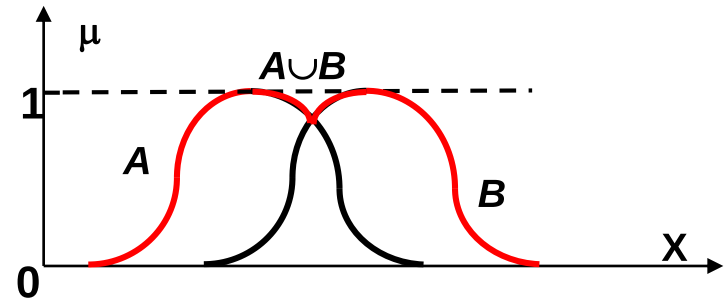
$\overline{A} = \{x \mid x \in X, x \notin A\}$





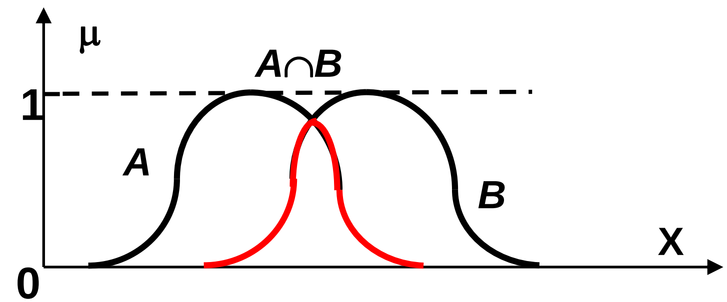
- Fuzzy Union (OR)

$$A \cup B = \int_X \boldsymbol{max}[\mu_A(x),\ \mu_B(x)] \,/\, x$$



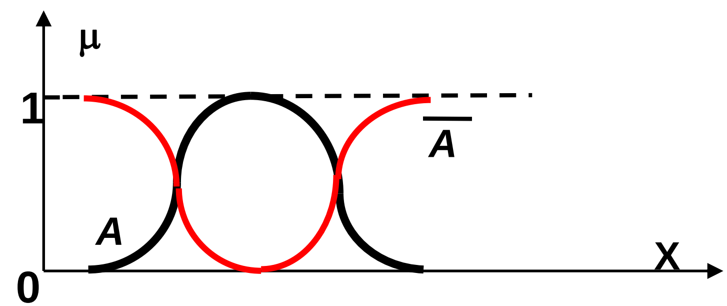
- Fuzzy Intersection(AND)

$$A \cap B = \int_X \boldsymbol{min}[\mu_A(x),\ \mu_B(x)] \,/\, x$$



- Fuzzy Complement (NOT)

$$\overline{A} = \int_X [1 - \mu_A(x)] \,/\, x$$



Example

- Example: Universal set of ages

  X = {5, 10, 20, 30, 40, 50, 60, 70, 80}.

  Given

  young = 1/5+1/10+.8/20+.5/30+.2/40+.1/50+0/60+0/70+0/80

  old = 0/5+0/10+.1/20+.2/30+.4/40+.6/50+.8/60+1/70+1/80

  adult = 0/5+0/10+.8/20+1/30+1/40+1/50+1/60+1/70+1/80

  We have

  Not young = 0/5+0/10+.2/20+.5/30+.8/40+.9/50+1/60+1/70+1/80

  young∩old = 0/5+0/10+.1/20+.2/30+.2/40+.1/50+0/60+0/70+0/80

  young∪old =1/5+1/10+.8/20+.5/30+.4/40+.6/50+.8/60+1/70+1/80

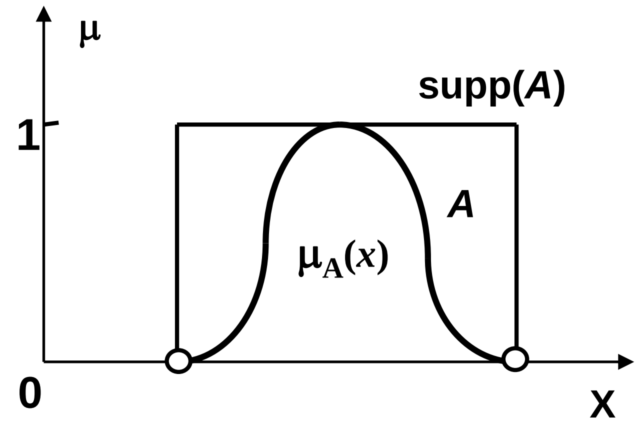
Concepts of fuzzy set: Support

- The support of a fuzzy set A, supp(A), in the universal set X is the crisp set that contains all the elements $x \in$ X that have a nonzero membership

  grade in A. That is supp(A) = $\{x \mid x \in X, \mu_A(x) > 0\}$
- Example:

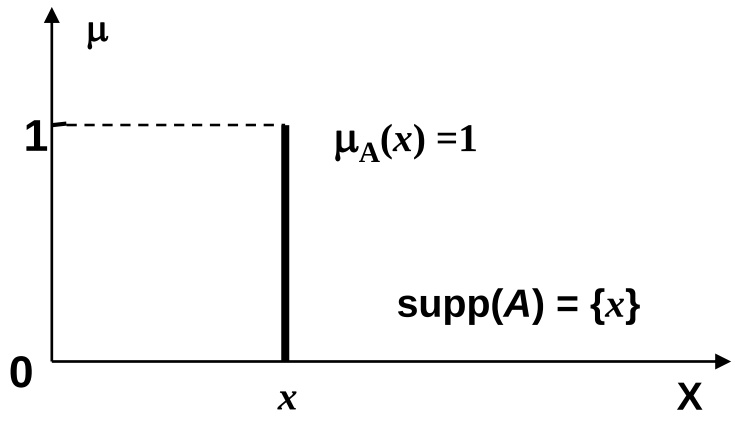
small-integer = 1/0 + 0.94/1 + …+ 0.02/99 + 0/100

supp(small-integer) = {0, 1, 2, …, 99}



Concepts of fuzzy set: Singleton

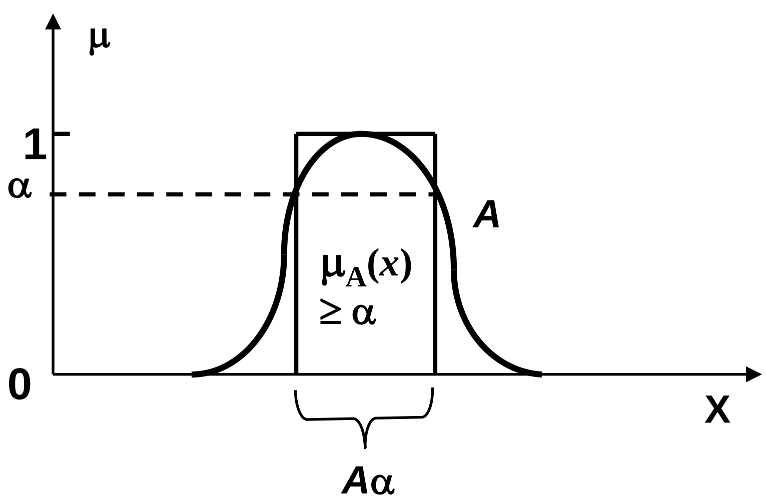
- Singleton: A fuzzy set *A* whose support is a single point in the universal set X with $\mu_A(x) = 1$.
  - $A = \{(x, \mu_A(x))\} = \{(x, 1)\}$
  - $\text{supp}(A) = \{x\}$



Concepts of fuzzy set: α—cut

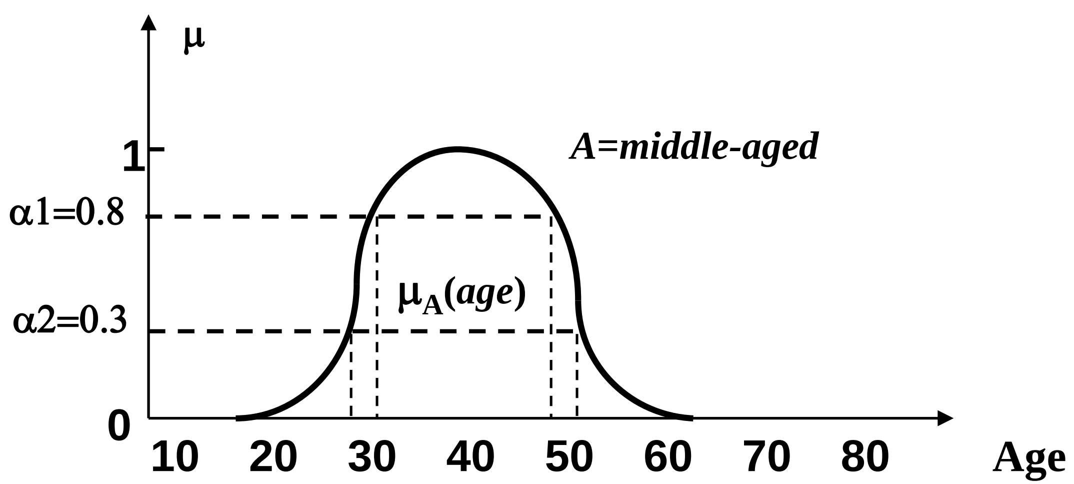
- α-cut set: The crisp set of elements that belong to the fuzzy set A at least to the degree α ($0 < \alpha < 1$) is called the α-level set or α-cut

$$A_\alpha = \{x \mid x \in X, \mu_A(x) \geq \alpha\}$$



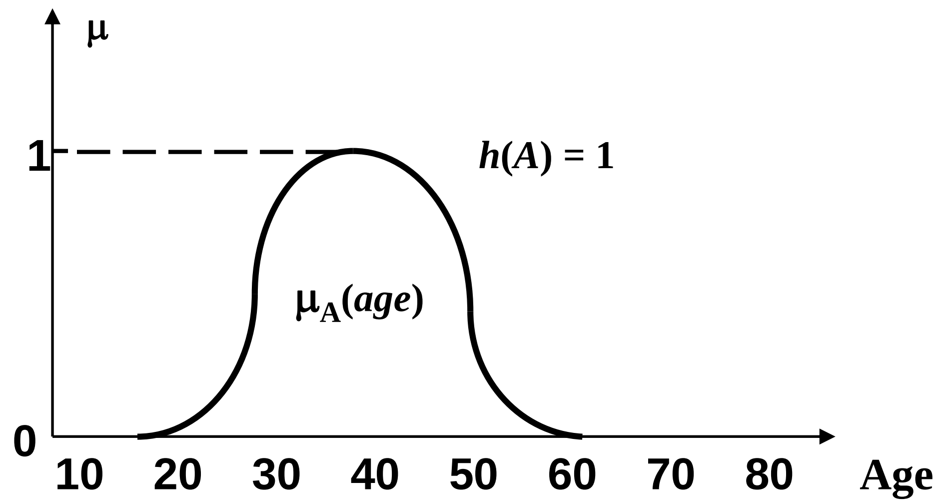
Concepts of fuzzy set: α—cut

- Example
  - fuzzy set A = middle-aged
  - supp(A) = {20, 21, …, 59,60} (suppose that only integers are used)
  - $A_{0.8}$ = {35, 36, …, 44,45}
  - $A_{0.3}$ = {30, 31, …, 49,50}



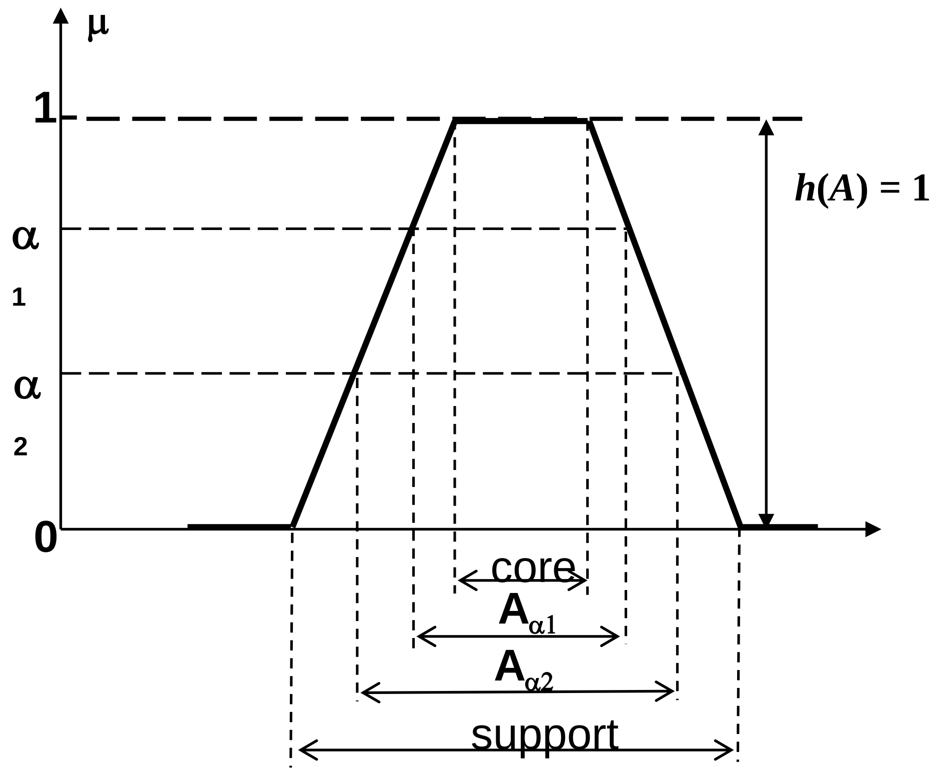
Concepts of fuzzy set: Height

- The height of a fuzzy set, $h(A)$, is the largest membership grade attained by any element in that fuzzy set, or the largest value of α for which the α-cut is not empty.



Concepts of fuzzy set: Core

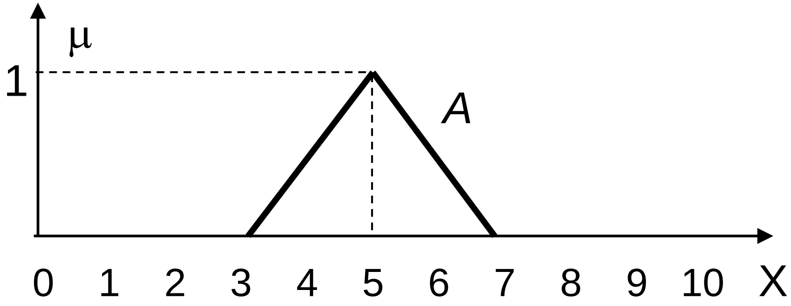
- The core of the fuzzy set A, core(A), is defined by the special α-cut for $\alpha = 1$
- $\text{core}(A) = A_1 = \{x \mid x \in X, \mu_A(x)=1\}$



Exercise

(1) Given fuzzy set A defined on the universal set X = [0, 10], find the following sets:

- Supp(A), Core(A)



(2) Given the universal set X = $\{x_1, x_2, x_3, x_4, x_5\}$ and two fuzzy sets defined on X:

- $A = 0/x_1 + 0.3/x_2 + 0.7/x_3 + 1/x_4 + 0.1/x_5$,
- $B = 0.1/x_1 + 0.6/x_2 + 0.8/x_3 + 0.5/x_4 + 0/x_5$.

Find the following sets

$$Supp(A), Core(B), (A \cup \bar{B})_{0.4}, (A \cap \bar{B})$$

Topics

- Fundamentals of fuzzy set
- Fuzzy Inference System
- Design and build a fuzzy-based intelligent control system
- Demo on fuzzy-controlled car

A fuzzy rule

Antecedent
(IF statement/premise)

IF	Speed is fast
THEN	Stopping_distance is long

Consequent
(THEN statement/premise)

Linguistic variable

Linguistic value

Other examples of fuzzy rule

Multiple antecedents (inputs)

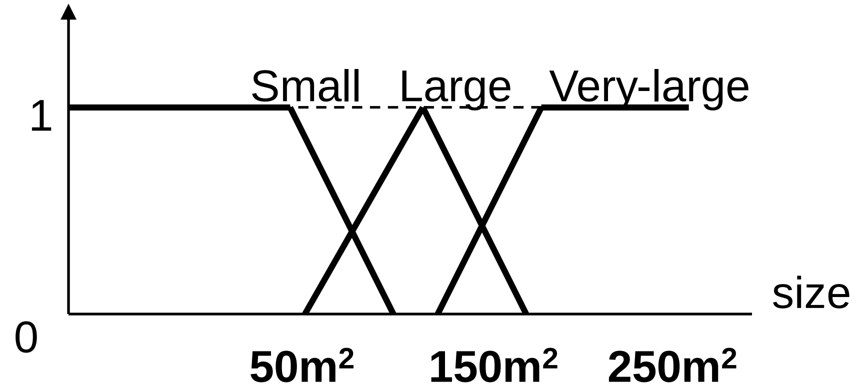
IF	Service is excellent
	OR
	Food is delicious
THEN	Tip is generous

Multiple consequents (outputs)

IF	Temperature is hot
THEN	Hot_water is reduced
	AND
	Cold_water is increased

Linguistic variable and value

- Linguistic variable
  - A variable that takes values which are not numbers but words or sentences in natural or artificial language.
- Example:
  - Speed is a linguistic variable if it takes values such as slow, fast, very fast, and so on
  - Size is a linguistic variable if it takes values such as small, large, very large, and so on.



Fuzzy IF-THEN rule

- A general form of a fuzzy if-then rule (also known as *fuzzy rule*, *fuzzy implication*, or *fuzzy conditional statement*) (multi-input-single-output):

  R^i: IF x is A_i, …, y is B_i, THEN $z = C_i$

  where,

  - x, …, y, and z are linguistic variables
  - A_i, …, B_i, and C_i are the linguistic values of x, …, y, and z, and defined by fuzzy subsets in the universe of discourses X, …, Y, and Z, respectively.

Fuzzy IF-THEN rule

- Example:

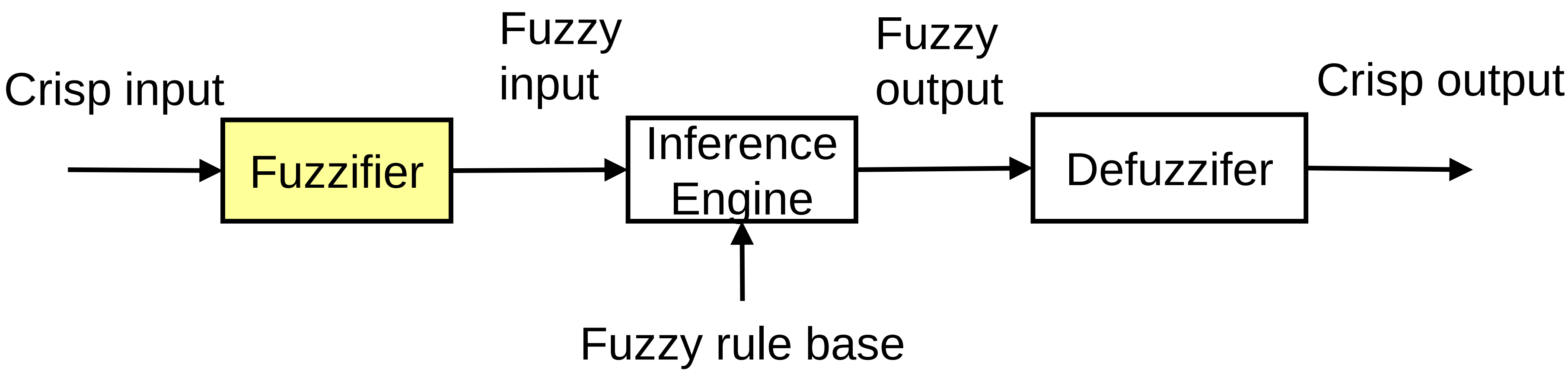
IF the speed is high, THEN apply the brake a little

"the speed is high"	—	antecedent
"apply the brake a little"	—	consequent
speed and apply-brake	—	linguistic variables
high and a little	—	linguistic values

Definition of Fuzzy Inference System

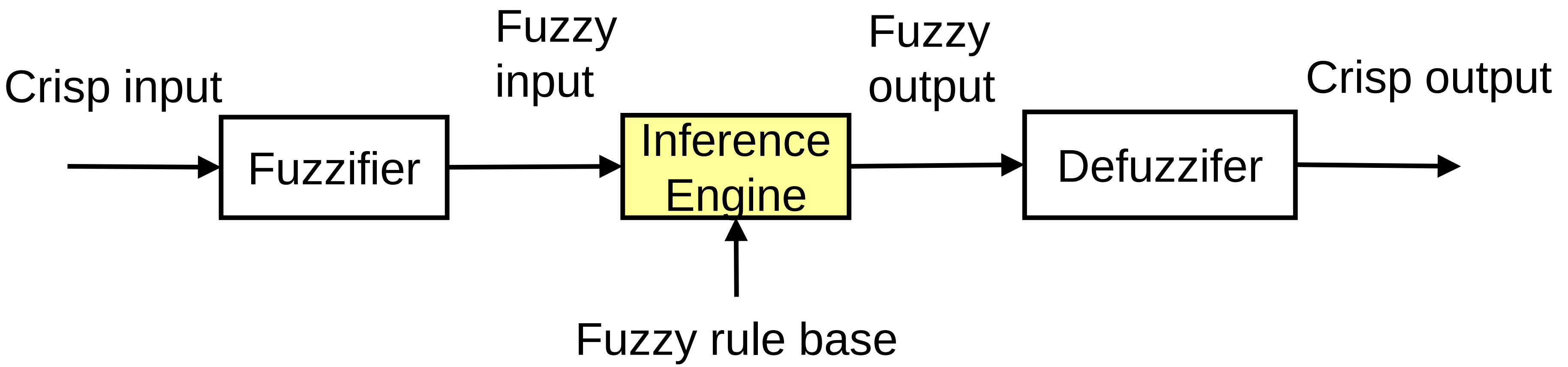
- Definition: Fuzzy inference System (FIS) is the process of formulating the mapping from a given input to an output using fuzzy logic.
- A nonlinear mapping that derives its output based on fuzzy reasoning and a set of fuzzy if-then rules.
- Also knows as
  - Fuzzy models
  - Fuzzy-rule-based system
  - Fuzzy Logic Controller

Fuzzy inference system (1)

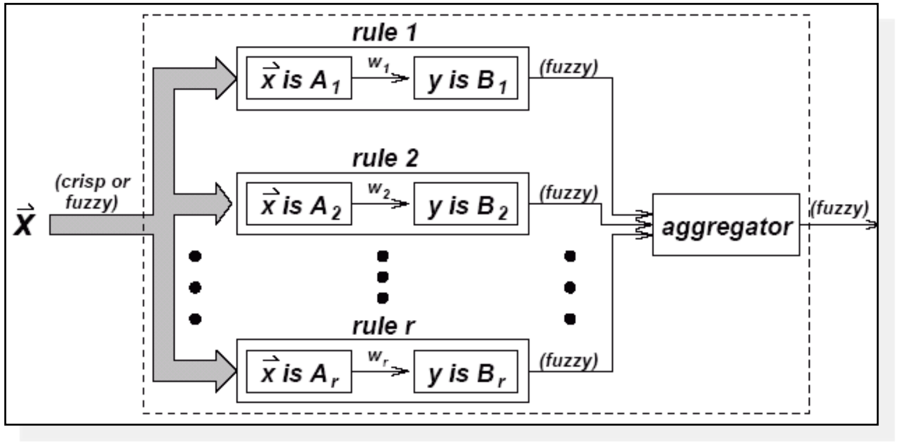


Converts the crisp input to a linguistic variable using the membership functions stored in the fuzzy knowledge base.

Fuzzy inference system (2)

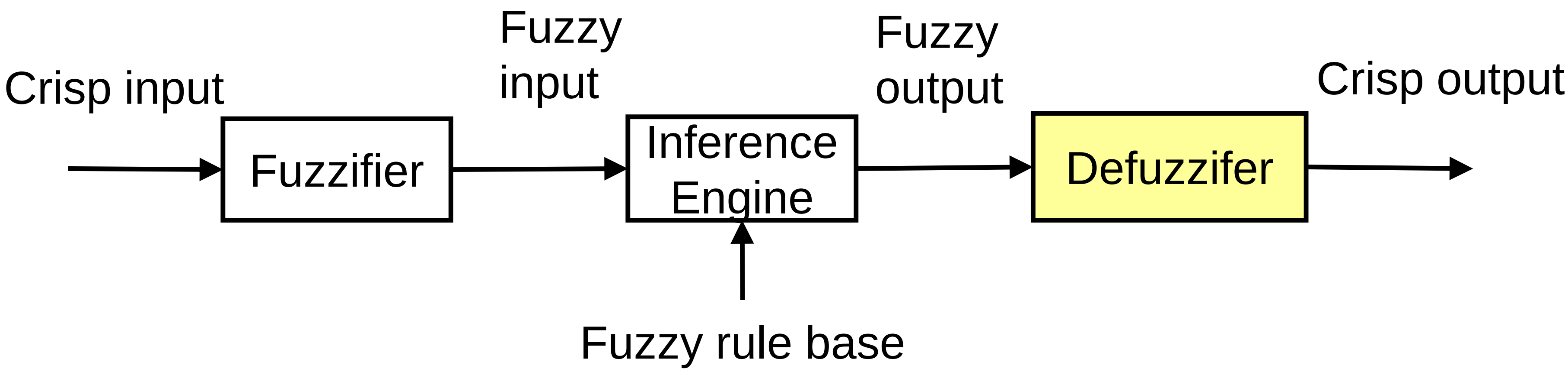


Use If-Then type fuzzy rules to convert the fuzzy input to the fuzzy output.



Source: Michael Negnevitsky, *Artificial Intelligence: A Guide to Intelligent Systems*, Pearson Education, 2011.

Fuzzy inference system (3)



Converts the fuzzy output of the inference engine to crisp output using membership functions analogous to the ones used by the fuzzifier.

Fuzzy inference engine

- Fuzzy inference engine realizes the mechanism of fuzzy reasoning/approximate reasoning
  - Given a fuzzy rule A $\rightarrow$ B and an input A'
    - the conclusion B' will be derived
- There are different models for fuzzy reasoning
  - Zadeh's compositional rule of inference,
  - Mamdani's inference,
  - Sugeno inference, … …

  using different
    - fuzzy implication, and
    - compositional operator (for calculating logical AND/OR)

Three popular FIS

- Mamdani FIS
- Sugeno FIS

- The differences between these FISs lie in the consequents of their fuzzy rules, and thus their aggregation and defuzzification procedures differ accordingly.

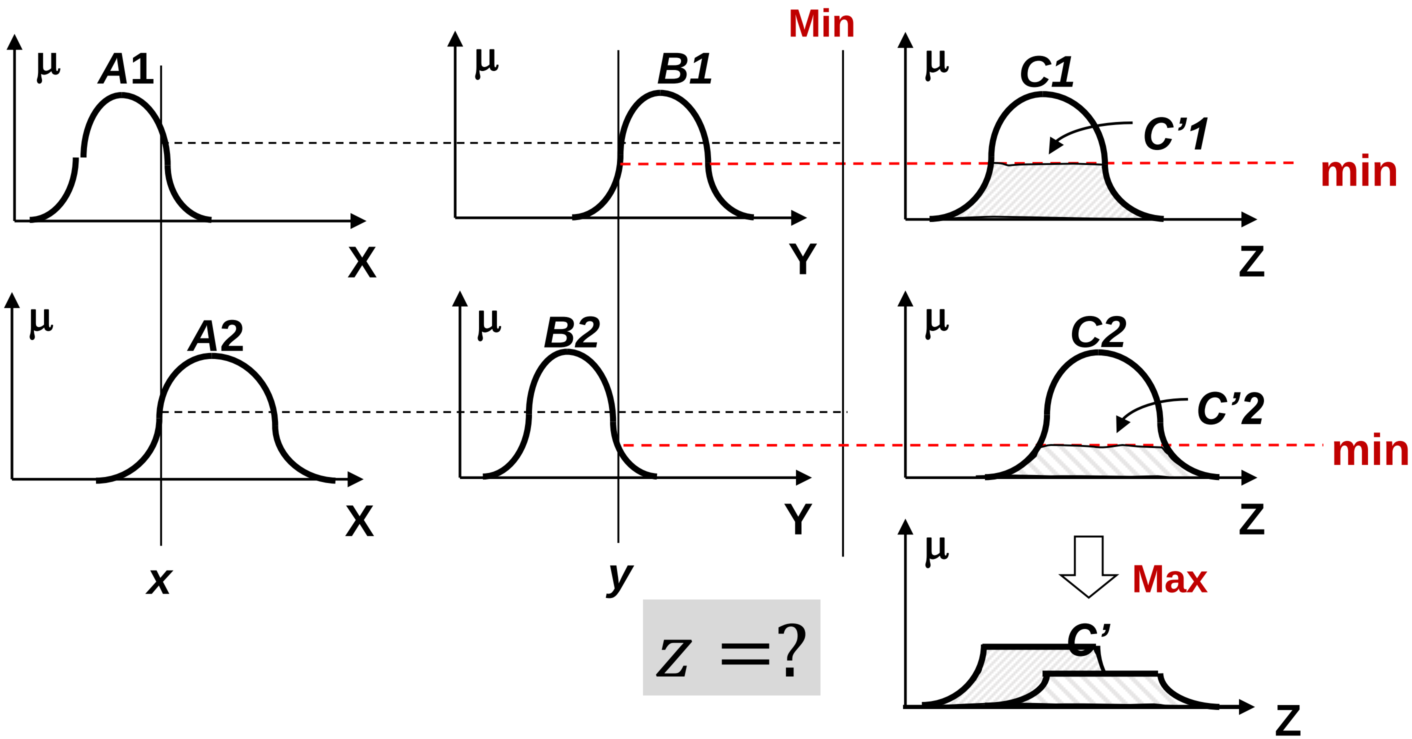
Mamdani FIS

- It is the most commonly used fuzzy inference technique.
- It was created by Professor Ebrahim Mamdani of London University who built one of the first fuzzy systems to control a steam engine and boiler combination in 1975.
- The Mamdani FIS is performed in four steps:
  1. Fuzzification of the input variables
  2. Rule evaluation (inference)
  3. Aggregation of the rule outputs (composition)
  4. Defuzzification

Image by Free-photo on Pixabay

Two-rule Mamdani with min and max operators

- Example:
  - IF *x* is A1 and *y* is B1 THEN z is C1
  - IF *x* is A2 and *y* is B2 THEN z is C2

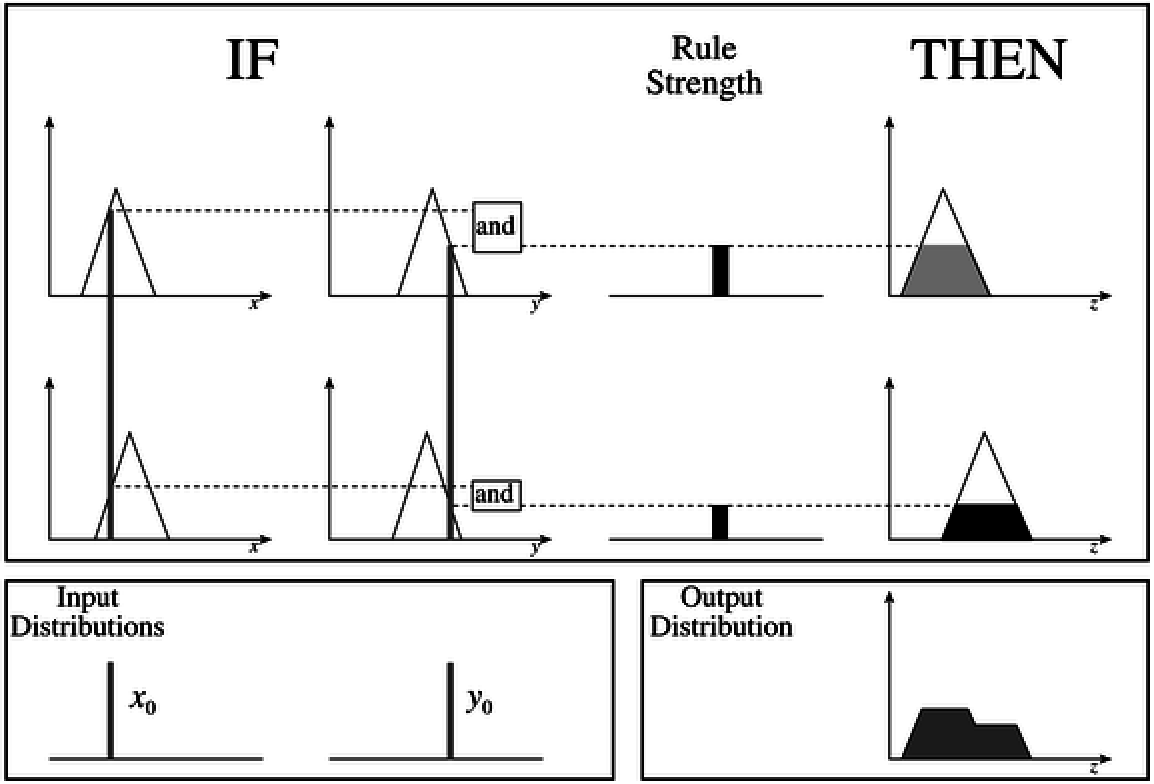


$$z = ?$$

Essential steps for Mamdani Inference

1. Fuzzification of the input variables
2. Rule evaluation (inference)
3. Aggregation of the rule output
4. Defuzzification

A two input, two rule Mamdani FIS with crisp inputs



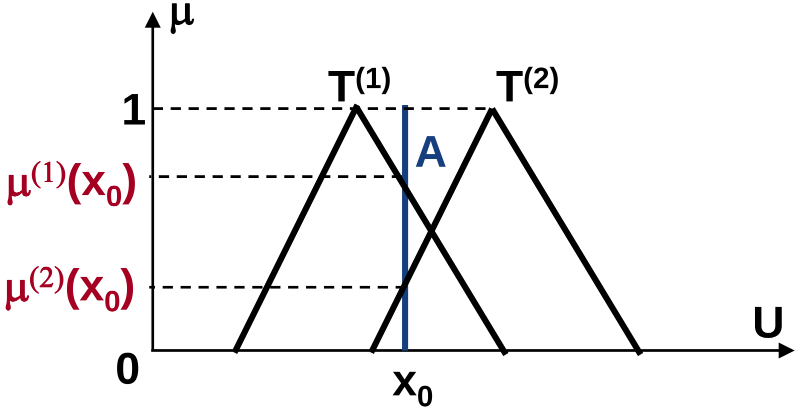
https://www.cs.princeton.edu/courses/archive/fall07/cos436/HIDDEN/Knapp/fuzzy004.htm

Fuzzy system: Fuzzification

- The function of fuzzification is to transform measurement data into valuation of a subjective value.
- It can be defined as a mapping from observed input space to labels of fuzzy sets (linguistic values) in a specified input universe of discourse.
- In fuzzy control applications, the observed data are usually crisp (though they may be corrupted by noise)

Fuzzy system: Fuzzification

- A simple fuzzification approach is
  - to first convert a crisp value x_0 into a fuzzy singleton A within the specified universe of discourse
  - the fuzzy singleton is then mapped to the fuzzy sets
    - $T^{(1)}$ with degree $\mu^{(1)}(x_0)$,
    - $T^{(2)}$ with degree $\mu^{(2)}(x_0)$,



Fuzzy system: Fuzzy rule base

- Fuzzy rule base is a collection of fuzzy IF-THEN rules (i.e.: the antecedents and/or consequents involve linguistic variables). It characterizes the simple input-output relation of the system.
  - E.g.: In a fuzzy control system, the fuzzy control rules evaluate the process state at time *t* and compute and decide the control actions:
    - Input = linguistic values
    - Output = linguistic/crisp

Fuzzy system: Rule evaluation

- Classical if-then rules are triggered if the antecedent can be matched (=true); firing (applying) the rule then allows for inferring the consequent.
- Fuzzy if-then rules are triggered only partially: antecedent is matched with fuzzified input (if rule has multiple antecedents, apply given fuzzy operator to obtain a single number).

Fuzzy system: Rule evaluation

- Fuzzy "AND"
  - ***min***$[\mu_A(x), \mu_B(x)]$
- Fuzzy "OR"
  - ***max***$[\mu_A(x), \mu_B(x)]$

Fuzzy system: Result aggregation

- Multiple fuzzy sets as different linguistic values are usually defined for same linguistic variable, so
  - Given input values, multiple fuzzy rules may be fired at the same time with different strengths
- Inference result is an aggregation of outputs from the multiple fired rules

Fuzzy system: Defuzzification

- Defuzzification is to perform a mapping from a space of linguistic values (decision, or actions) defined over an output universe of discourse into a space of non-fuzzy (crisp) decision action.
- Some typical methods of defuzzification:
  - *Center of Area* (COA) method
  - *Center of Maximum* (COM)
  - *Mean of Maximum method* (MOM)

Note: There is no systematic procedure for choosing a defuzzification method.

Defuzzification: COA

- Center of Area (COA) method
  - Generates the center of gravity of the possibility distribution of a control action

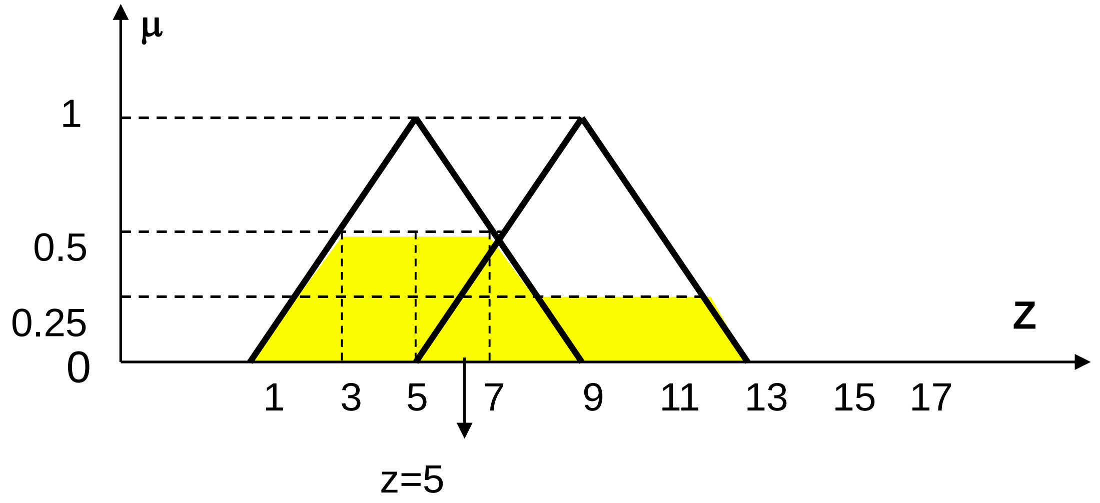
for discrete universe of discourse

$$z^*_{COA} = \frac{\sum_{j=1}^{n} \mu_C(z_j) z_j}{\sum_{j=1}^{n} \mu_C(z_j)}$$

for continuous universe of discourse

$$z^*_{COA} = \frac{\int_z \mu_C(z) z dz}{\int_z \mu_C(z) dz}$$

Defuzzification: COA



$z^*_{COA} =$

$= [1\times 0 + (3+5+7)\times 0.5 + (9+11)\times 0.25] + 13\times 0] / (0+0.5+0.5+0.5+0.25+0.25+0)$

$= 12.5 / 2 = 6.25$

Defuzzification: COM

- Center of Maximum (COM) method
  - Identical to COA that uses singleton membership functions.
  - Instead of area (all z_j as in COA method), using only the typical values of each related term and balancing the weights on those representative points

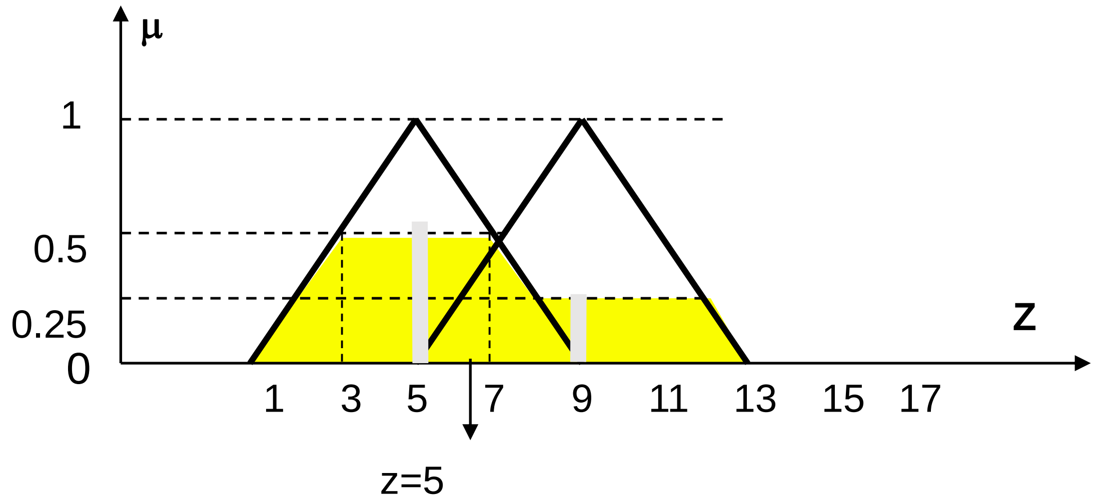
$$z^*_{COM} = \frac{\sum_{j=1}^{n} \mu_c(\bar{z}_j)\bar{z}_j}{\sum_{j=1}^{n} \mu_c(\bar{z}_j)}$$

  - Where $\bar{z}$ is the centroid of each membership function.

Defuzzification: COM

- Continue previous example, but using COM:

$$z^*_{COM} = (5 \times 0.5 + 9 \times 0.25) / (0.5 + 0.25) = 4.75/0.75 = 6.33$$



Defuzzification: MOM

- Mean of Maximum (MOM) method
  - Generates an action that represents the mean value of all actions whose membership functions reach the maximum.

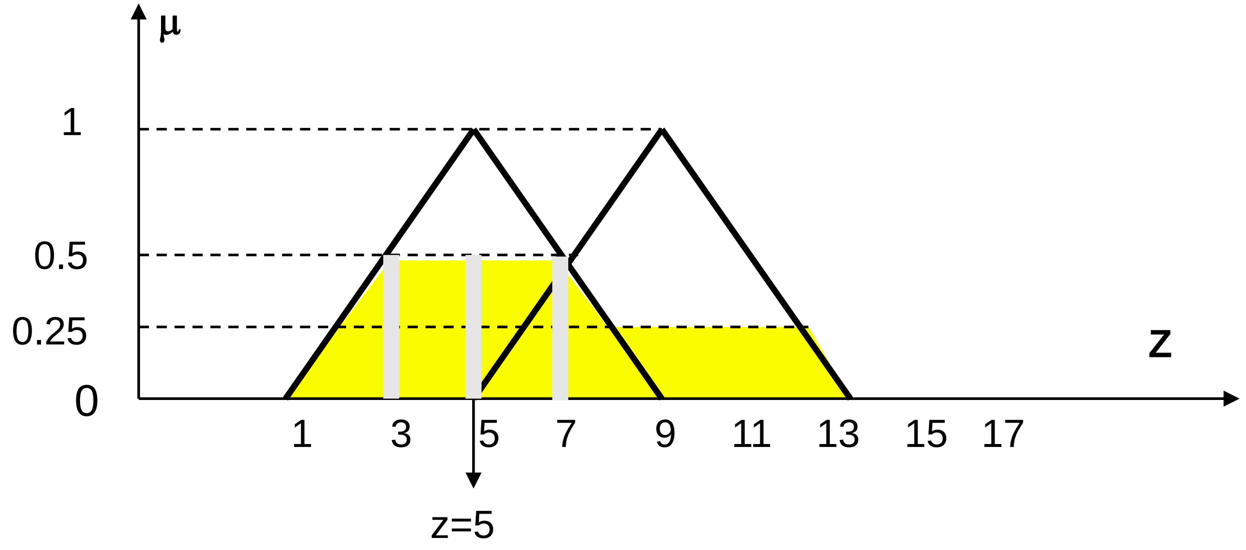
$$z^{*}_{MOM} = \sum_{j=1}^{m} \frac{z_j}{m}$$

where z_j is the support value at which the membership function reaches the maximum value μ^* and m is the number of such support values (for continuous universe of discourse, it is the average of the maximizing z at which the MF reach a maximum μ^*)

Defuzzification: MOM

- Applying MOM to the previous example:

z^*_{MOM} = AVERAGE(3:7)=5



Example

- Assume two rules:
  - Rule 1: IF x is A1 AND y is B1 THEN z is C1
  - Rule 2: IF x is A2 AND y is B2 THEN z is C2

Membership functions:

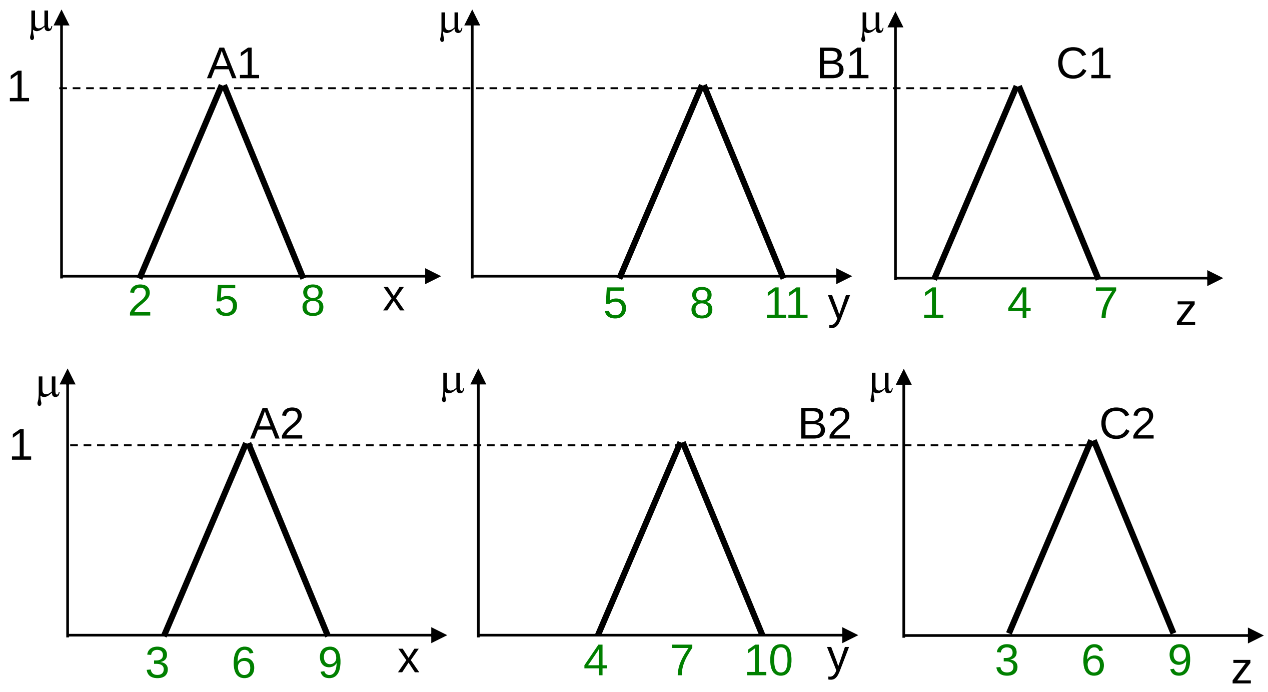
$$\mu_{A1}(x) = \begin{cases} \dfrac{x-2}{3} & 2 \le x \le 5 \\ \dfrac{8-x}{3} & 5 < x \le 8 \end{cases} \qquad \mu_{A2}(x) = \begin{cases} \dfrac{x-3}{3} & 3 \le x \le 6 \\ \dfrac{9-x}{3} & 6 < x \le 9 \end{cases}$$

$$\mu_{B1}(y) = \begin{cases} \dfrac{y-5}{3} & 5 \le y \le 8 \\ \dfrac{11-y}{3} & 8 < y \le 11 \end{cases} \qquad \mu_{B2}(y) = \begin{cases} \dfrac{y-4}{3} & 4 \le y \le 7 \\ \dfrac{10-y}{3} & 7 < y \le 10 \end{cases}$$

$$\mu_{C1}(z) = \begin{cases} \dfrac{z-1}{3} & 1 \le z \le 4 \\ \dfrac{7-z}{3} & 4 < z \le 7 \end{cases} \qquad \mu_{C2}(z) = \begin{cases} \dfrac{z-3}{3} & 3 \le z \le 6 \\ \dfrac{9-z}{3} & 6 < z \le 9 \end{cases}$$

Example: Membership function

Example (membership functions)



Example: Rule evaluation

Example

- Assume that we are reading **sensor values**
  - $x_0 = 4$ and $y_0 = 8$

    $\mu_{A1}(x_0) = 2/3$ $\mu_{B1}(y_0) = 1$

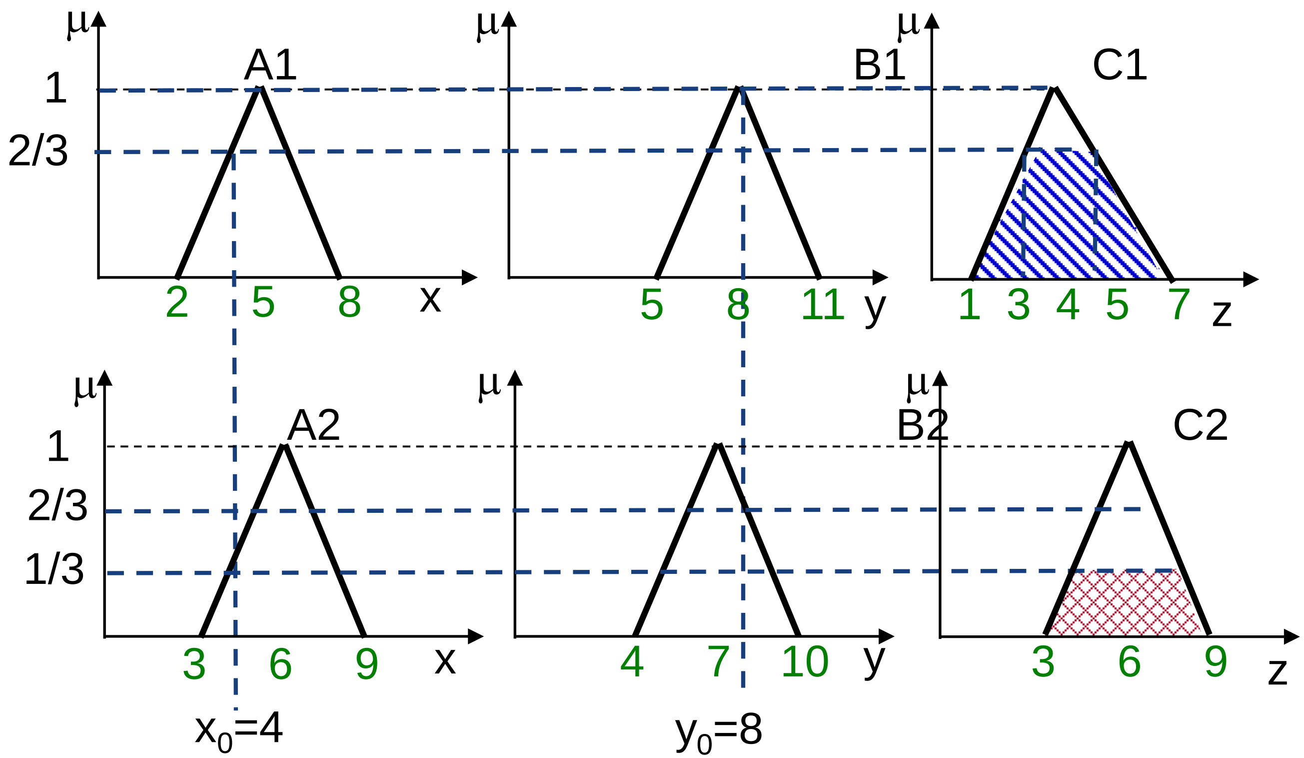
    $\mu_{A2}(x_0) = 1/3$ $\mu_{B2}(y_0) = 2/3$

- Finding the strength of rules:
  - for Rule 1: $\alpha_1 = Min(\mu_{A1}(x_0), \mu_{B1}(y_0)) = 2/3$
  - for Rule 2: $\alpha_2 = Min(\mu_{A2}(x_0), \mu_{B2}(y_0)) = 1/3$

Example: Rule evaluation

Example

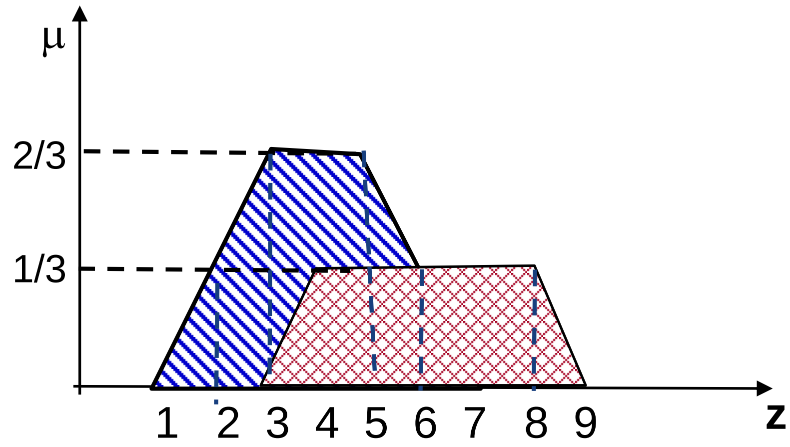
- Applying α_1 to the conclusion of Rule 1, and α_2 to the conclusion of Rule 2, respectively (implication).



Example: Result aggregation

Example

- Aggregating qualified consequents from Rule 1 and Rule 2



- Defuzzification is needed to get a single crisp value output.

Example: Defuzzification

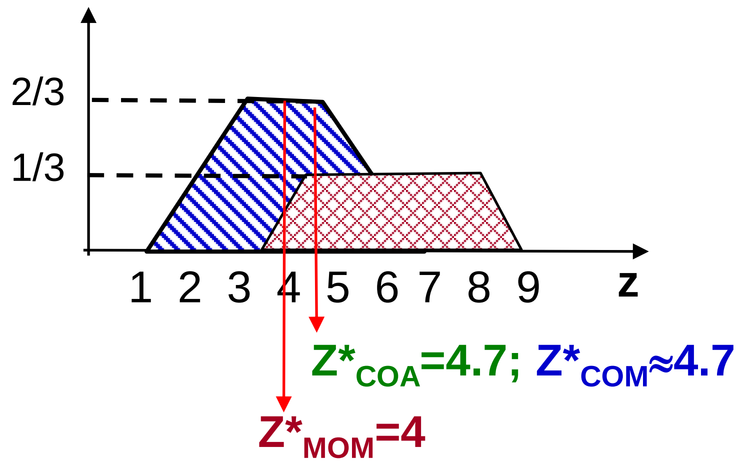
Example

- Defuzzification

$$z^*_{COA} = \frac{2\cdot\frac{1}{3}+3\cdot\frac{2}{3}+4\cdot\frac{2}{3}+5\cdot\frac{2}{3}+6\cdot\frac{1}{3}+7\cdot\frac{1}{3}+8\cdot\frac{1}{3}}{\frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 4.7$$

$$z^*_{COM} = \frac{4\cdot\frac{2}{3}+6\cdot\frac{1}{3}}{\frac{2}{3}+\frac{1}{3}} \approx 4.7$$

$$z^*_{MOM} = \frac{3+4+5}{3} = 4$$



Main steps: Fuzzy inference system (1)

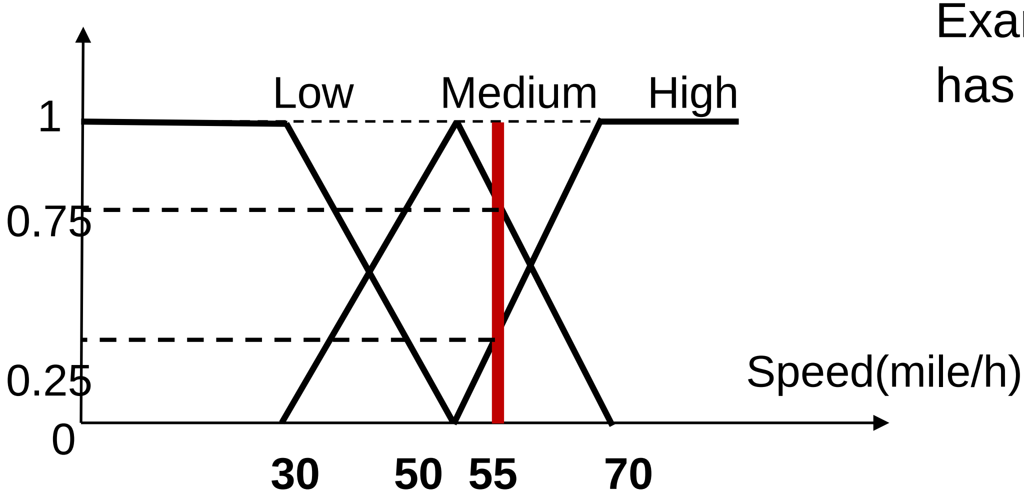
- Select relevant input and output variables
- Determine the number of linguistic terms associated with each input and output variables (fuzzy partition)
- Choosing an appropriate family of parameterised membership functions used in the rule base.
  - Piecewise linear: Triangular, Trapezoidal, …
  - Non-linear, differentiable : Gaussian, Bell, …
- Determining the parameters of membership functions used in the rule base
  - e.g.: through interviewing human experts
- Choose a specific type of fuzzy inference system

Main steps: Fuzzy inference system(2)

- Design a collection of fuzzy if-then rules (fuzzy rule base) in a symbolic style:
  - i.e. IF *x* is *A* THEN *y* is *B*
- Learning/Refining the parameters (fuzzy rules, membership functions) by other techniques:
  - e.g.: neural networks, genetic algorithms
- Defuzzification. Map from linguistic values (decision, or actions) defined over an output universe of discourse into a space of crisp decision action.

Fuzzy IF-THEN rule

- The speed 55mile/h can be represented by a fuzzy singleton and then matched with fuzzy subsets low, medium, and high.



Example: The fuzzy singleton (55mile/h) has matching degrees

- » 0 with fuzzy subset *low*
  - ✓ $\mu_{low}(55) = 0$
- » 0.75 with fuzzy subset *medium*
  - ✓ $\mu_{medium}(55) = 0.75$
- » 0.25 with fuzzy subset *high*
  - ✓ $\mu_{high}(55) = 0.25$

TSK FIS

- Proposed by Takagi, Sugeno, and Kang
- For developing a systematic approach to generating fuzzy rules from a given input-output data set.
- The format of the TSK rule is

IF *x* is A, …, *y* is B, THEN *z* = f(x,y)

- x, y, z are linguistic variables;
- A and B are fuzzy sets;
- f(x,y) is a mathematical function, usually polynomial in the input variables x and y.
- **Zero-order** TSK model: f is a constant
- **First-order** TSK model: f(x,y) is a first-order polynomial

TSK FIS

- The output is a weighted average:

$$z = \frac{\sum \mu_{A_i, B_k}(x, y) f_{m(i,k)}(x, y)}{\sum \mu_{A_i, B_k}(x, y)}$$

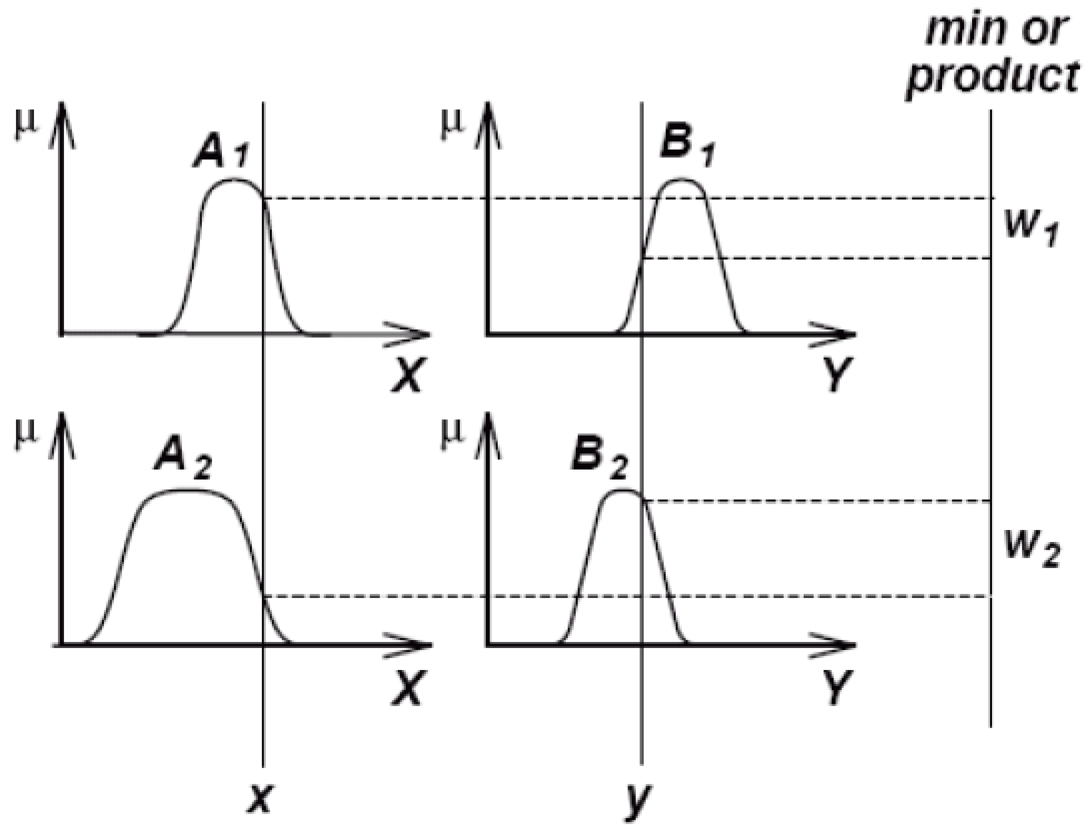
Double summation over all *i* (x MFs) and all *k* (y MFs)

$$= \frac{\sum w_i f_i(x, y)}{\sum w_i}$$

Summation over all *i* (fuzzy rules)

where w_i is the firing strength of the *i*-th output

Fuzzy Reasoning Procedure for a First order Sugeno Fuzzy Model



$$z_1 = p_1 x + q_1 y + r_1$$

$$z_2 = p_2 x + q_2 y + r_2$$

weighted average

$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

Example: Two-input Single-output Sugeno fuzzy model

- An example of a two-input single-output Sugeno fuzzy model with four rules:
- IF x is small AND y is small THEN z = -x+y+1
- IF x is small AND y is large THEN z = -y+3
- IF x is large AND y is small THEN z = -x+3
- IF x is large AND Y is large then z = x+y+2

Mamdani or TSK?

- Mamdani method is widely accepted for capturing expert knowledge. It allows us to describe the expertise in more intuitive, more human-like manner. However, Mamdani-type fuzzy inference entails a substantial computational burden.
- On the other hand, Sugeno method is computationally effective and works well with optimisation and adaptive techniques, which makes it very attractive in adaptive problems, particularly for dynamic nonlinear systems.

FIS Optimisation

- Why to optimise?
  - Trial and error tuning is laborious
  - It can be impossibly complicated if number of input parameters are large
  - Too many parameters to tune: number of rules, membership functions, rule consequents
  - Arbitrariness of FIS is eliminated

Optimisation methods

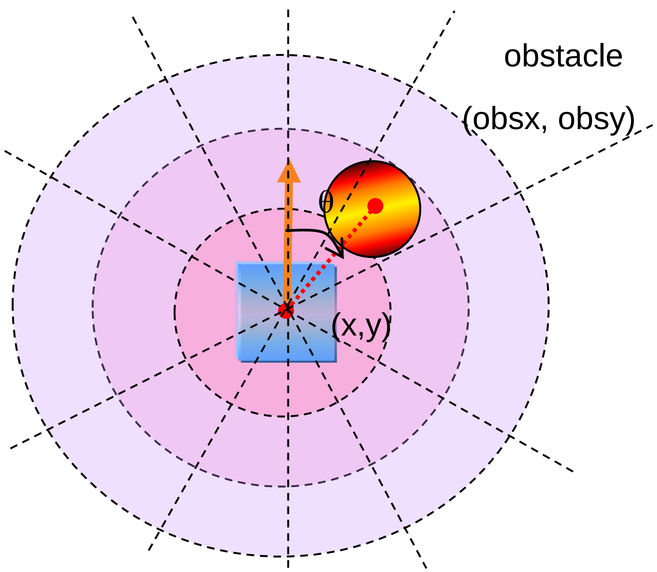
- Adaptive Neuro-Fuzzy systems
  - Multiplayer perceptron neural networks (ANFIS)
- Evolutionary techniques (genetic algorithms)
- Clustering methods (Fuzzy C-means)
- Etc.

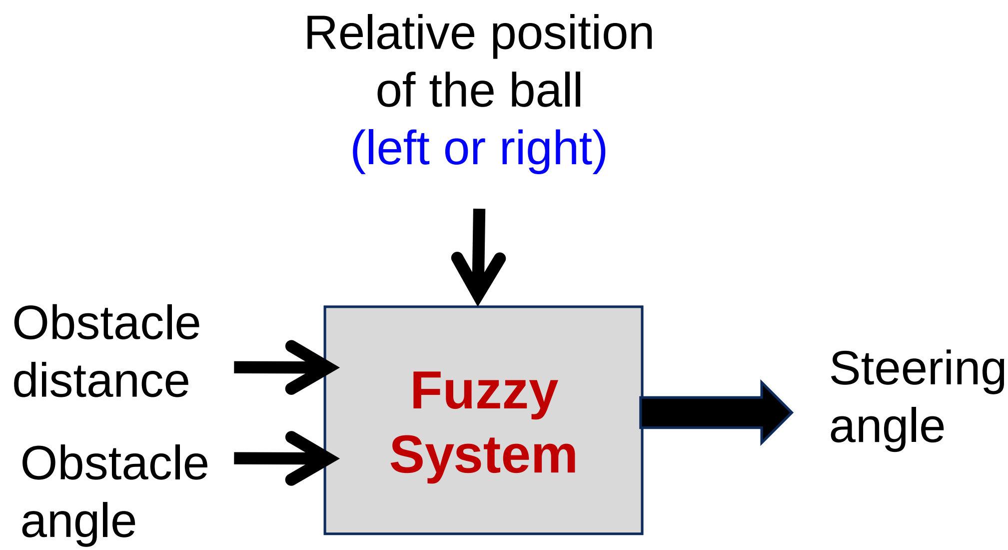
Summary

- Concept of fuzzy sets, main difference between crisp and fuzzy sets.
- Different types of fuzzy membership.
- Different representation methods of fuzzy sets.
- Linguistic variables and fuzzy rules.
- Operation of fuzzy sets.
- Fuzzy inference system: Mamdani and TSK.

Case study: Obstacle avoidance

IF	Distance is near
	AND
	Obstacle is at the right hand side
	AND
	Angle is small
THEN	Turn sharp left





Source: Michal Cap, Motion Planning for Autonomous Vehicles, https://cw.fel.cvut.cz/old/_media/courses/ae4m36pah/motion-planning.pdf

Topics

- Fundamentals of fuzzy set
- Fuzzy inference for robotic systems
- Design and build a fuzzy-based intelligent control system
- Demo on fuzzy-controlled car

Fuzzy PID controller

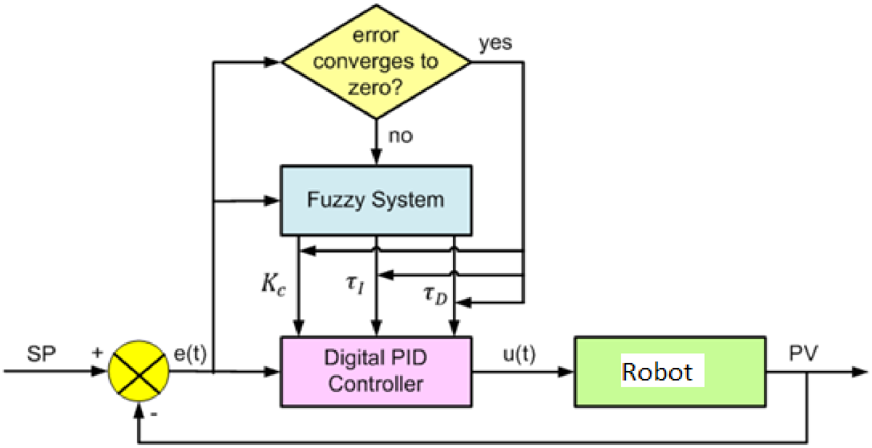


Figure 1 PID gain tuning using Fuzzy

- The PID controller automatically adjusts the control output based on the difference between the set point (SP) and the measured process variable (PV), as a control error e(t). The controller value u(t) is transferred as the system input.

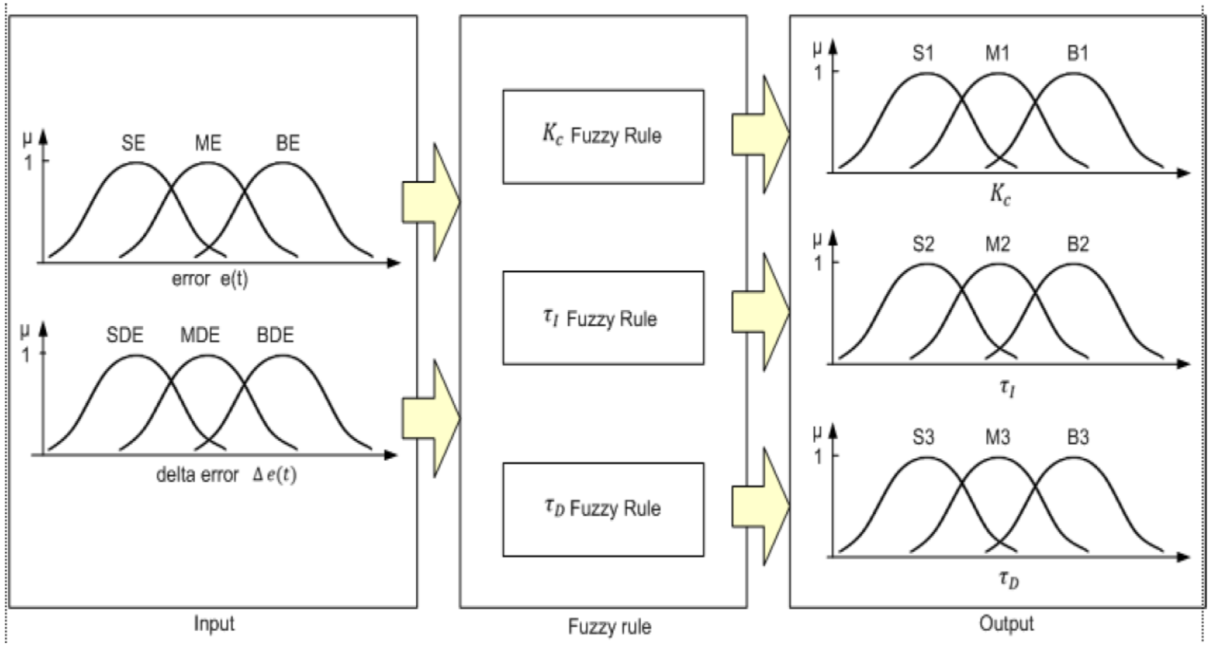
$$e(t) = SP - PV$$

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I} \sum_{i=1}^{n_t} e_i(t)\, \Delta t - K_c\, \tau_D \frac{PV_{n_t} - PV_{n_t - 1}}{\Delta t}$$

- u_{bias} is a constant that is usually set to the value u(t) when the first controller switches from manual to automatic mode.
- 3 determinants of the success of the control process, namely K_c, τ_I, τ_D. Here, we use fuzzy logic to tune the three parameters.

Source: Rahmat, B. and Nugroho, B., 2019. Fuzzy and artificial neural networks-based intelligent control systems using Python. *Nusantara Science and Technology Proceedings*, pp.152-170.

Fuzzy System Design



- The fuzzy system design for the PID gain tuning process requires a mechanism for how to make gain adjustments from PID, based on errors $e(t)$ and delta error $\Delta e(t)$.
- If the error has converged towards zero, then the existing gain is maintained. But if it is still far from converging towards zero, it is necessary to change the gain following the rule that has been designed.

Fuzzy Rule System

Rule	Input		Output		
	error	delta_error	K_c	τ_I	τ_D
1	SE	SDE	S1	S2	S3
2	SE	MDE	S1	S2	S3
3	SE	BDE	S1	S2	S3
4	ME	SDE	M1	M2	M3
5	ME	MDE	M1	M2	M3
6	ME	BDE	M1	M2	M3
7	BE	SDE	B1	B2	B3
8	BE	MDE	B1	B2	B3
9	BE	BDE	B1	B2	B3

- Two inputs: $e(t)$ and $\Delta e(t)$
- Three outputs: K_c, τ_I, τ_D
- 9 rules for K_c
- 9 rules for τ_I
- 9 rules for τ_D

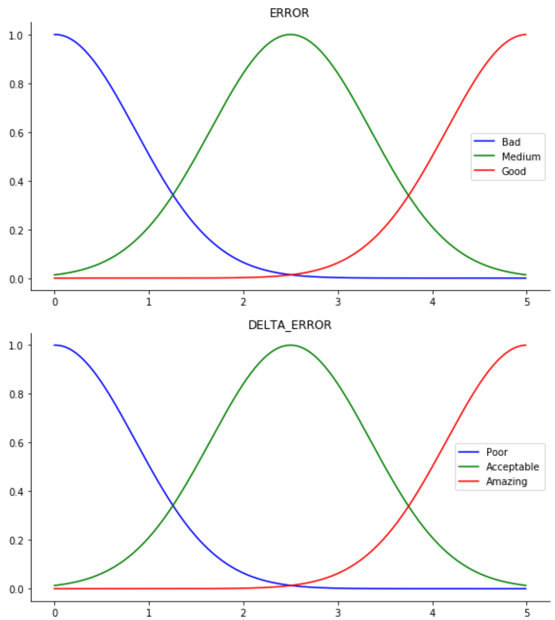
- Sample fuzzy rules:

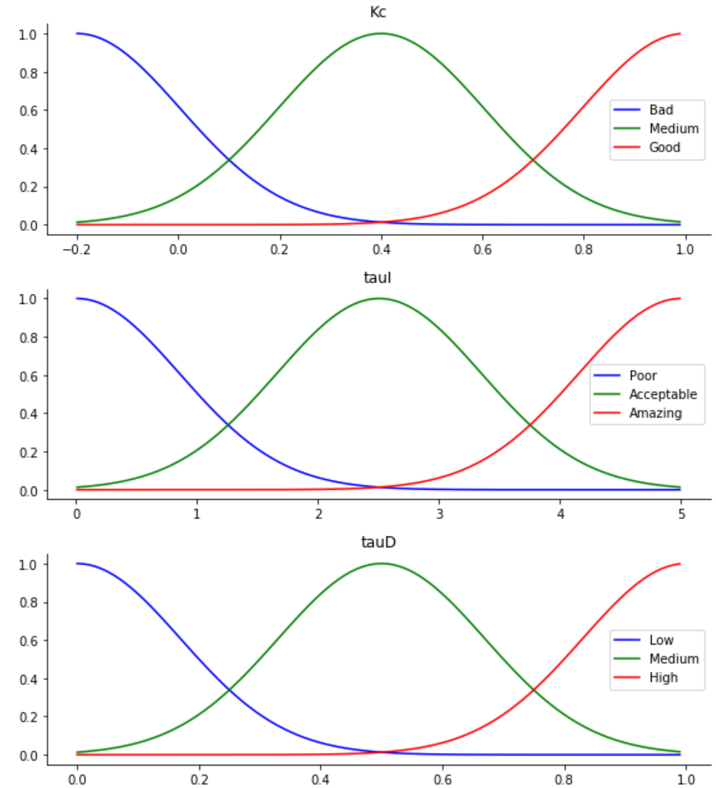
IF error is SE OR delta_error is SDE, THEN Kc is S1.

IF error is SE OR delta_error is SDE, THEN τ_I is S2.

IF error is SE OR delta_error is SDE, THEN $\tau_D$is S3.

Results





$$K_c = 0.3933$$
$$\tau_I = 2.4983$$
$$\tau_D = 0.4933$$

Exercises

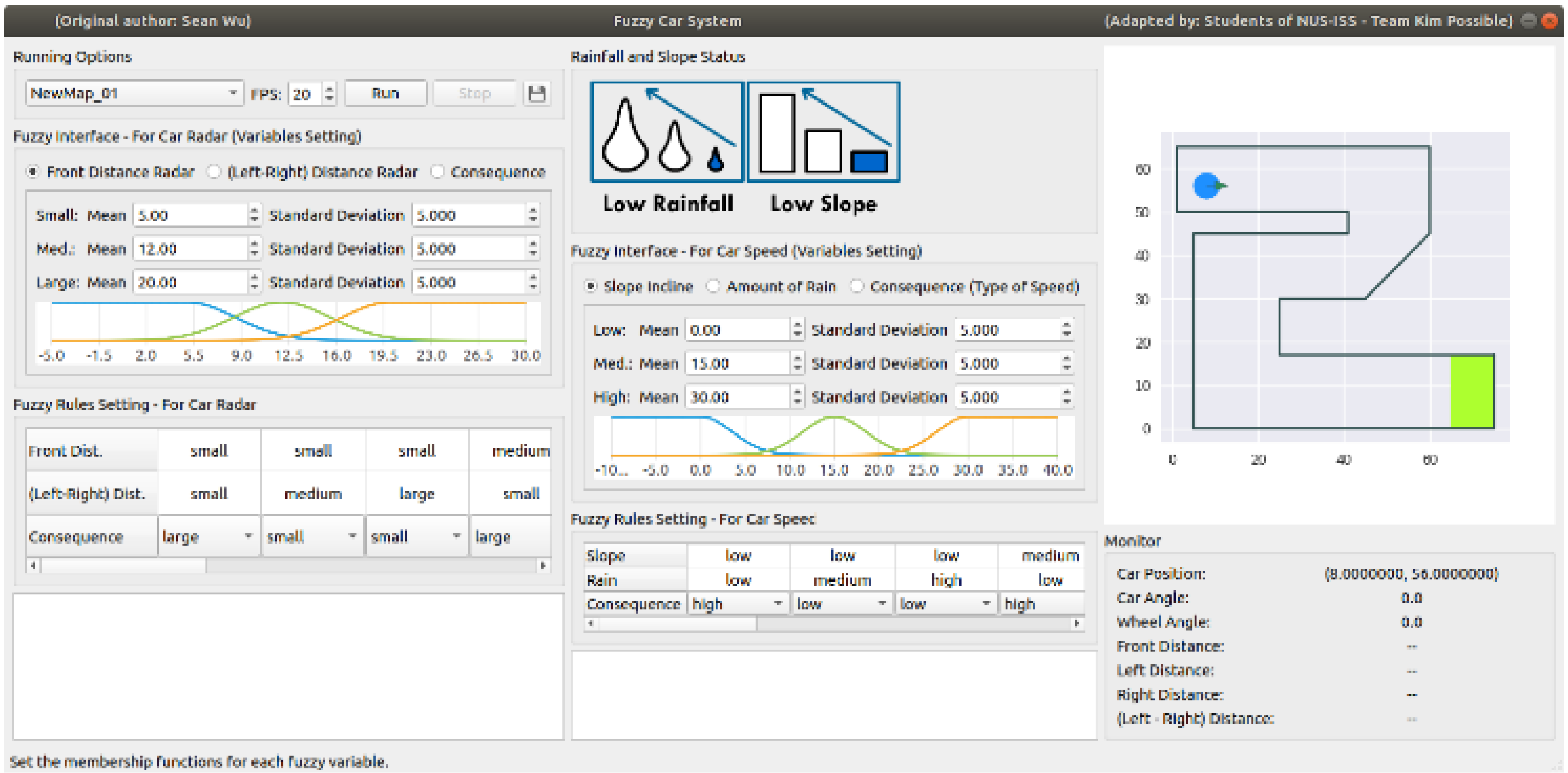
- Refer to the following scripts:
  - Fuzzy_logic.ipynb
  - Fuzzy_PID_Controller.ipynb

Topics

- Fundamentals of fuzzy set
- Fuzzy inference for robotic systems
- Design and build fuzzy inference robotic system in Python
- Demo on fuzzy-controlled car

Demo

- Fuzzy car system



Demo

Build a fuzzy car system

- Define input and output variables
- Identify what is the mapping function you need approximate
- Decide fuzzy partition of the input and output spaces
- Design fuzzy rules, describe the mapping relation
- Design the inference mechanism, choose a fuzzy implication, defuzzification, etc
- Implement/Evaluate/Fine tune the system

Demo

- Installation: Type following commands in Anaconda Prompt

```
conda create -n rbsvcar python=3.6
activate rbsvcar
pip install numpy==1.15.0 matplotlib==2.2.3 PySide2==5.11.1 scikit-fuzzy==0.4.0
```

- Activate your virtual environment in Anaconda Prompt

```
activate rbsvcar
```

- Browse to the folder that has demo Python codes, Run the fuzzy controlled car demo program

```
Python main.py
```

Demo

There are 5 input fuzzy sets:

- 3 radars (Front-Left-Right) within fuzzy range of Small, Medium, Large. The left and right radar range from –15 to 30 (map unit). The front radar ranges from –20 to 20 (map unit).
- Rainfall with fuzzy range of Low, Medium, High, from 0-30mm.
- Slope terrain with fuzzy range of Low, Medium, High from angle of 0 to 30.
- The input value of the three radars will be combined to calculate the wheel turning angle
- The input value of rainfall and slope will be used to calculate car velocity

There are 2 output consequence sets:

- Wheel turning angle: Small, Medium, Large.
- Velocity (speed): Low, Medium, High.

Thank you☺

EXTRA SLIDES

A COMPLETE FIS example

We examine a simple two-input one-output problem that includes three rules:

<u>Rule: 1</u>	IF x is A3	OR	y is B1	THEN	z is C1
<u>Rule: 2</u>	IF x is A2	AND	y is B2	THEN	z is C2
<u>Rule: 3</u>	IF x is A1			THEN	z is C3

Real-life example for these kinds of rules:

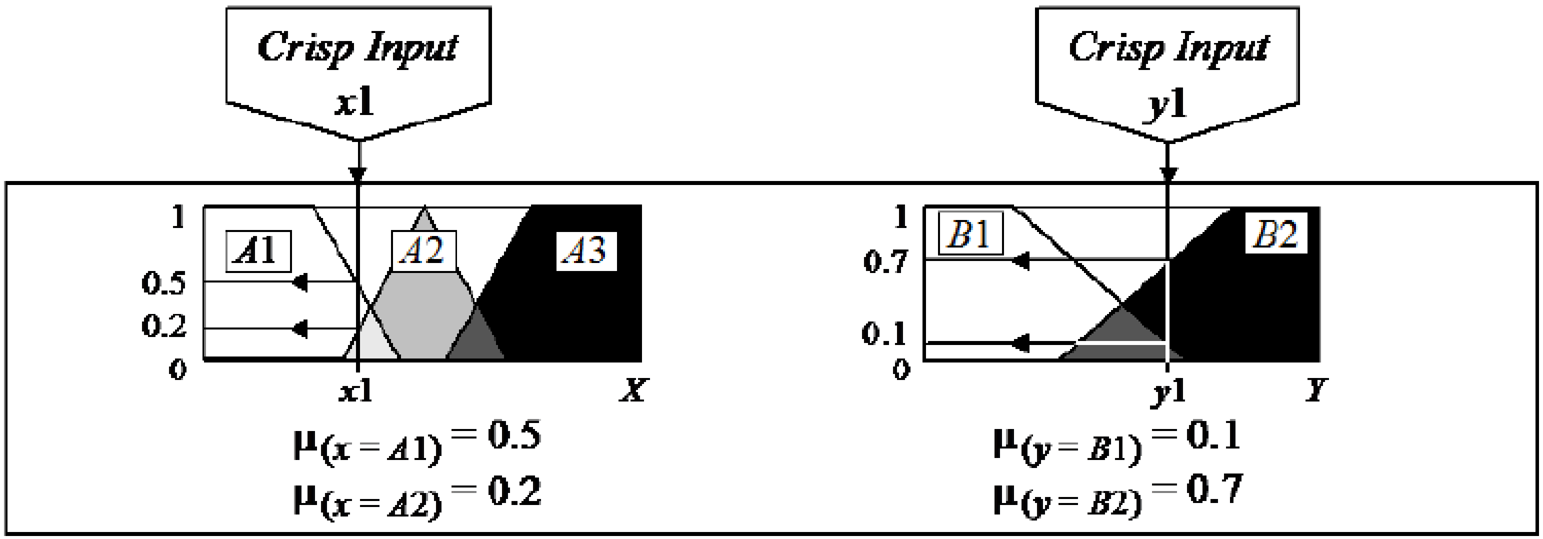
<u>Rule: 1</u> IF project_funding is adequate OR project_staffing is small THEN risk is low

<u>Rule: 2</u> IF project_funding is marginal AND project_staffing is large THEN risk is normal

<u>Rule: 3</u> IF project_funding is inadequate THEN risk is high

Step1: Fuzzification

- The first step is to take the crisp inputs, x1 and y1 (*project funding* and *project staffing*), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



Step2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.
- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.

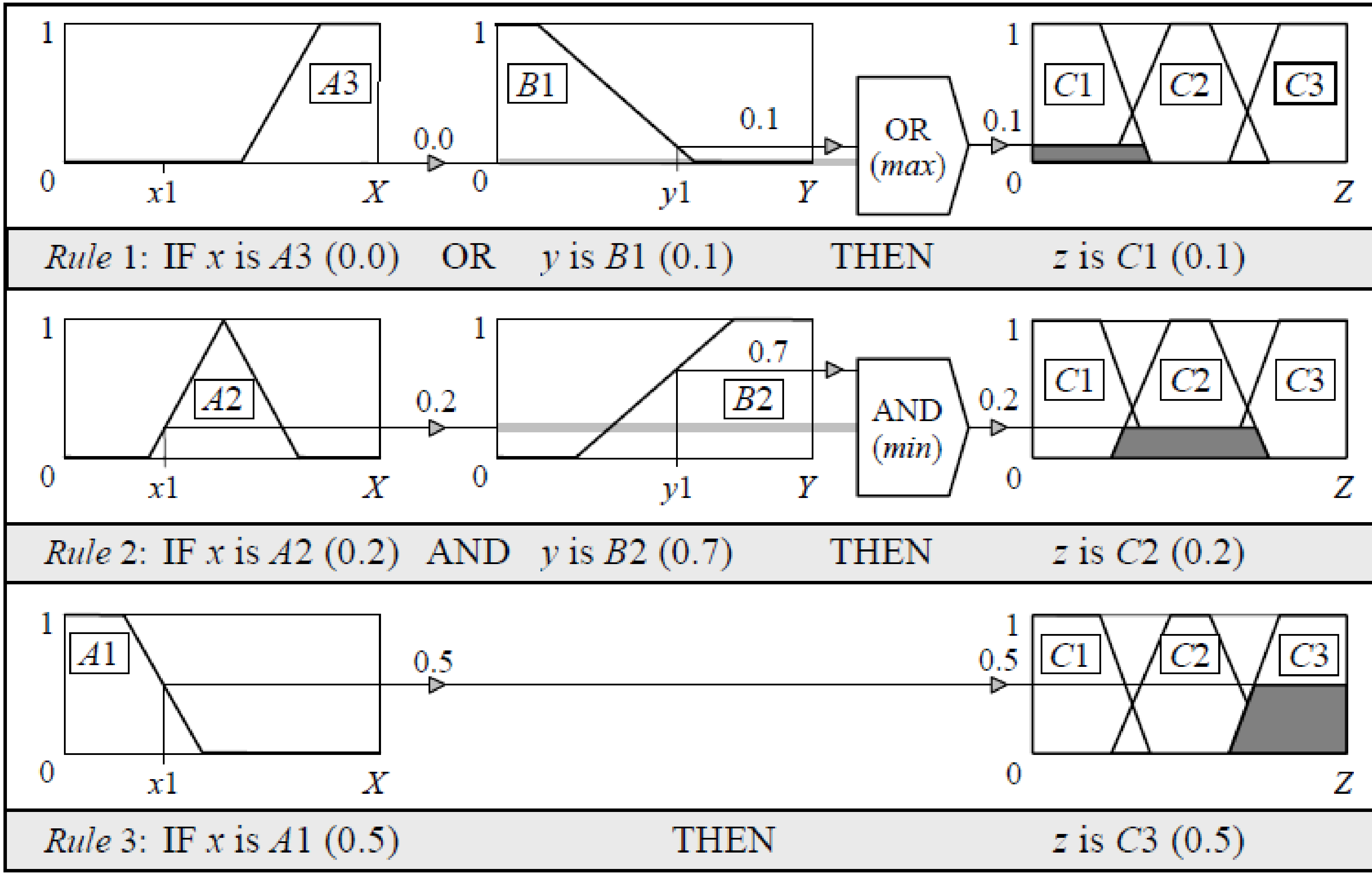
RECALL: To evaluate the disjunction of the rule antecedents, we use the **OR** fuzzy operation. Typically, fuzzy expert systems make use of the classical fuzzy operation union:

$$\mu_{A \cup B}(x) = \max\left[\mu_A(x), \mu_B(x)\right]$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the **AND** fuzzy operation intersection:

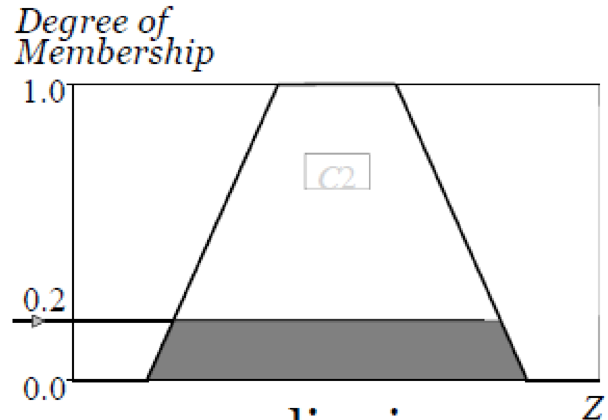
$$\mu_{A \cap B}(x) = \min\left[\mu_A(x), \mu_B(x)\right]$$

Step2: Rule Evaluation

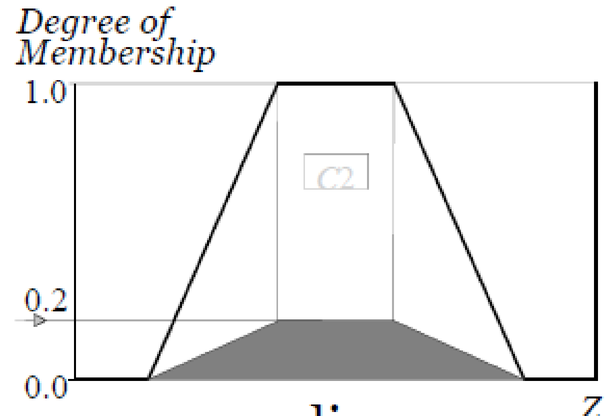


Step2: Rule Evaluation

- Now the result of the antecedent evaluation can be applied to the membership function of the consequent.
- The most common method is to cut the consequent membership function at the level of the antecedent truth. This method is called **clipping** (alpha-cut).
  - Since the top of the membership function is sliced, the clipped fuzzy set loses some information.
  - However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.
- While clipping is a frequently used method, **scaling** offers a better approach for preserving the original shape of the fuzzy set.
  - The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent.
  - This method, which generally loses less information, can be very useful in fuzzy expert systems.



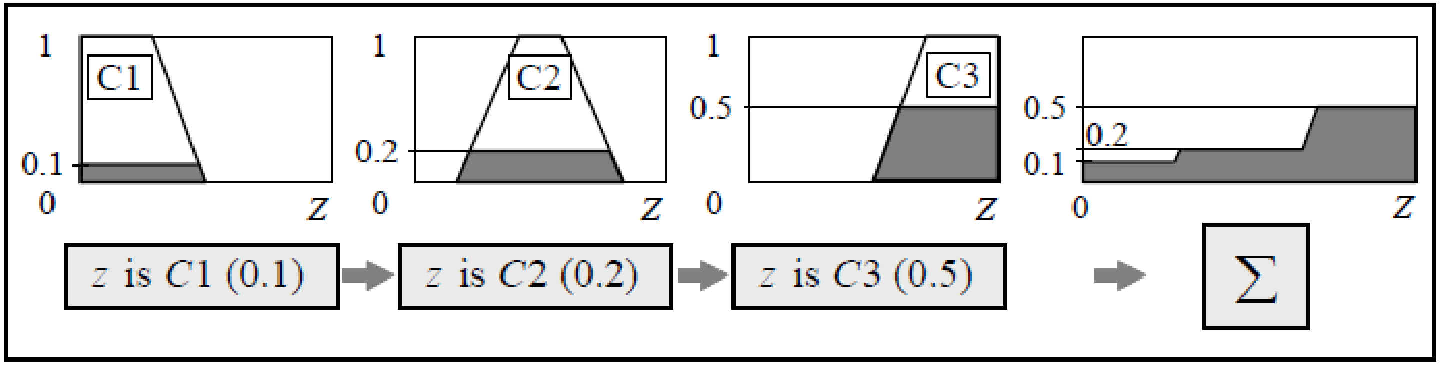
clipping



scaling

Step3: Aggregation of the Rule Outputs

- Aggregation is the process of unification of the outputs of all rules.
- We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.
- The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.



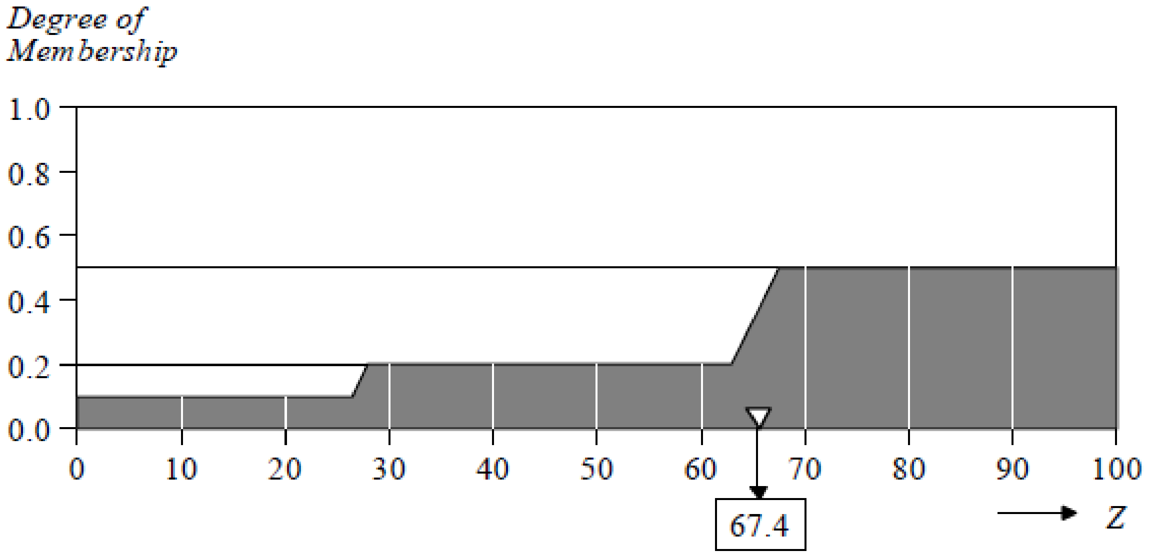
Step 4: Defuzzification

- The last step in the fuzzy inference process is defuzzification.
- Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.
- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.
- There are several defuzzification methods, but probably the most popular one is the **centroid technique**. It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this **centre of gravity** (**COG**) can be expressed as:

$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

Step 4: Defuzzification

- Centroid defuzzification method finds a point representing the centre of gravity of the aggregated fuzzy set *A*, on the interval [*a*, *b*].
- A reasonable estimate can be obtained by calculating it over a sample of points.



$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$

Fuzzy Inference System: Sugeno inference (optional)

- This method was introduced by Michio Sugeno in 1985.
- It uses a singleton, as the membership function of the consequent.
- Sugeno fuzzy inference is similar to Mamdani method. Sugeno changed only a rule consequent. Instead of a fuzzy set, he used a mathematical function of the input variable.
- The format of the Sugeno-style fuzzy rule is

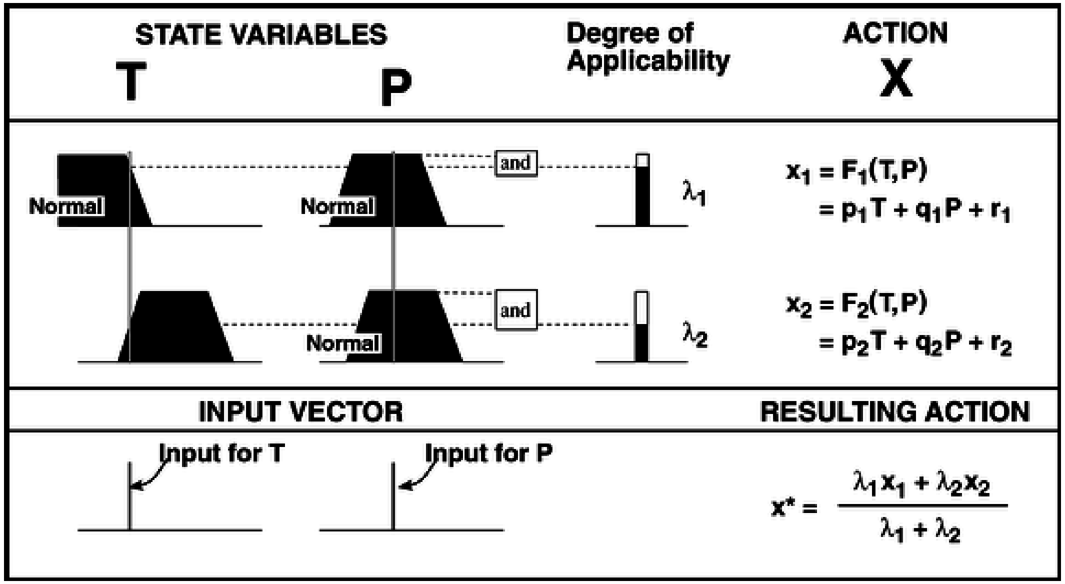
IF	x is A
	AND y is B
THEN	z is f(x,y)

- The most used is zero-order Sugeno fuzzy model, where k is a constant.

IF	x is A
	AND y is B
THEN	z is k

- In Sugeno inference, the output of each fuzzy rule is constant. All consequent membership functions are represented by singleton spikes

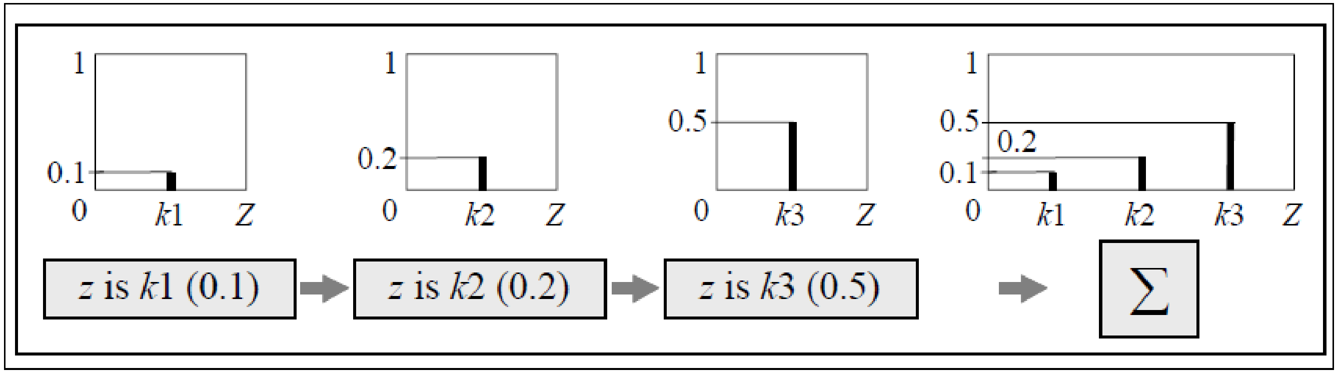
Fuzzy Inference System: Sugeno inference (optional)



A two inputs, two rules Sugeno FIS, p_n, q_n, and r_n are user-defined constants

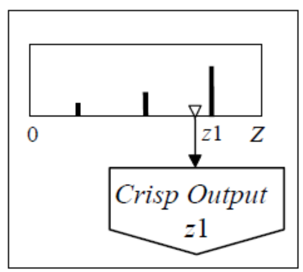
- The Sugeno FIS is quite similar to the Mamdani FIS. But the output of Sugeno FIS is a crisp number computed by multiplying each input by a constant and then adding up the results.
- "Rule strength" in this example is referred to as "degree of applicability" and the output is referred to as the "action". Also notice that there is no output distribution, only a "resulting action" which is the mathematical combination of the rule strengths (degree of applicability) and the outputs (actions).

A Sugeno FIS example



$$WA = \frac{\mu(k1) \times k1 + \mu(k2) \times k2 + \mu(k3) \times k3}{\mu(k1) + \mu(k2) + \mu(k3)} = \frac{0.1 \times 20 + 0.2 \times 50 + 0.5 \times 80}{0.1 + 0.2 + 0.5} = 65$$

Sugeno-style defuzzification



Mamdani FIS

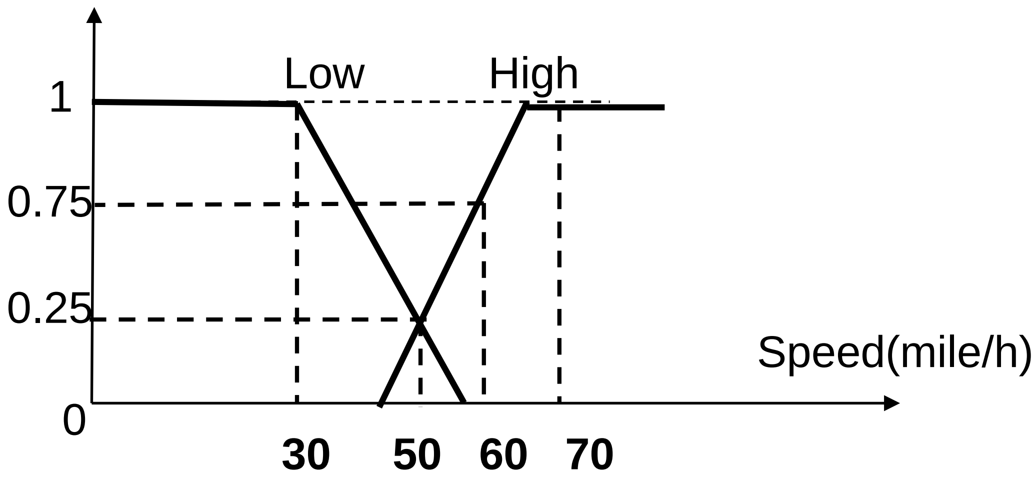
- Output membership function is present
- Crisp result is obtained through defuzzification of rules' consequent
- Non-continuous output surface
- MISO (Multiple Input Single Output) and MIMO (Multiple Input Multiple Output) systems
- Expressive power and Interpretable rule consequents
- Less flexibility in system design

Sugeno FIS

- No output membership function is present
- No defuzzification: crisp result is obtained using weighted average of the rules' consequent
- Continuous output surface
- Only MISO systems
- Loss of interpretability
- More flexibility in system design

Exercise - solution

- Driving speed



- Two fuzzy sets: low and high
- 30 gives a degree of 1 to low
- 50 gives a degree of 0.25 to low and 0.25 to high
- 60 gives a degree of 0.75 to high
- 70 gives a degree of 1 to high