Pre-Calculus in Brief with Python, Colab, GitHub, and LaTeX PCiB - Version 0.3

November 10, 2023

Available on GitHub at:
https://GitHub.com/nicholaskarlson/PCiB

Contents

Preface

This text, *Pre-Calculus in Brief - PCiB* aspires to be more than just another math book. This book strives to foster collaborative math writing. Note that this book has very few references. The reader is encouraged to use resources available on the Web to fact-check. This book's view on "causation" and facts is heavily influenced by Mosteller and Tukey [MT77].

Redefining the Role of the Reader

Pre-Calculus in Brief (PCiB) is an endeavor to reshape how math is written, understood, and studied. It's not just a passive read but an open-source approach to math, aiming to encourage students to become proactive learners.

This project strives to break the traditional mold of math education and invites readers and professional mathematicians to participate actively.

A Dynamic Relationship with Math

Pre-Calculus in Brief is not just a book but a movement and methodology, heralding a new era in how we approach, consume, and interact with math. By positioning the reader as an integral part of the math-book process, PCiB fosters a dynamic relationship with math, making mathematics more accessible, proactive, and relevant. In this shifting paradigm, we are all potential mathematicians, creators of interesting and relevant ways to learn and study math.

Please fork the LaTeX source code for PCiB (available on GitHub) and create your own book that chooses the facts and exercises most relevant to you! Also, starring the PCiB project on GitHub would be greatly appreciated! Thanks for reading PCiB!

Chapter 1

Introduction to PCiB

Welcome to PCiB on GitHub

Pre-Calculus in Brief, abbreviated PCiB, isn't merely a passive read. It's an endeavor to reshape how math is written, studied, and taught. By presenting an open-source approach to math, the goal is to encourage everyone to become proactive readers and writers of math.

Fostering a Proactive Engagement with Math

Pre-Calculus in Brief is a call for a renewed engagement with mathematics. PCiB is an endeavor to reshape how math is written, understood, and studied. It's not just a passive read but an open-source approach to math, aiming to encourage students to become proactive learners.

This project strives to break the traditional mold of math education and encourages readers and professional mathematicians to participate actively.

Please fork the LaTeX source code for PCiB (available on GitHub) and create your own book on Pre-Calculus that chooses the content most relevant to you! Also, starring the PCiB project on GitHub would be greatly appreciated! Thanks for reading PCiB!

Chapter 2

Open-Source Ethos

The Spirit of Shared Knowledge and Collaboration

Math, like software, is better when it's open. PCiB draws inspiration from the open-source software movement; this section elucidates how a collaborative, transparent, and shared approach can enhance our understanding of math. Here, we look at the philosophy behind open-source and how it beautifully combines with the study of mathematics.

Open-Source Math: Preserving Tradition Through Collaborative Exploration

Mathematics, like software, thrives when it embraces openness and transparency. PCiB takes a leaf from the proven benefits of the open-source software model; this section highlights how a collaborative and transparent method can improve and deepen our grasp of math and its texts. Here, we explore the principles of open source and how these principles align with the development of mathematics and its texts.

Understanding the Open-Source Ethos

The open-source paradigm revolves around shared ownership, collaboration, and the free exchange of knowledge. In the software realm, this approach has led to groundbreaking innovations built and enhanced by a global community of skilled contributors. United by a mutual objective, these individuals pool their diverse talents and insights to improve and share software solutions for broader public benefit.

Advantages of the Open-Source Framework in Math

Collective Insight

Mirroring the collaborative essence of open-source software, many individuals can offer their perspectives and knowledge, making math texts more robust and varied.

Enhancement and Accuracy

Open platforms foster an environment of constructive criticism, ensuring prompt identification and correction of inaccuracies. This meticulous peer review can help provide a credible and current mathematical text.

Universal Access

Much as open-source software promotes free access and modification, open-source math prioritizes universal accessibility. This ensures mathematics knowledge isn't restricted to a select few but is available to all curious minds.

Potential Challenges

Despite its advantages, melding open-source with math has potential pitfalls. The volume of contributions can complicate accuracy verification processes.

However, the very community championing this open-source approach to math can serve as its vigilant protector. They can ensure that contributions undergo rigorous evaluation and referencing, akin to the meticulous checks within the open-source software community.

Conclusion: Reinvigorating Our Experience with Math

Adopting an open-source perspective to the approach of math signifies a refreshed approach. It beckons a worldwide community to collaborate and forge a comprehensive and exciting math text. In this refreshed approach, every individual can play a part, both as a contributor and a learner. Math texts, through this lens, evolve and flourish, reflecting the collective input of active participants.

Chapter 3

Introduction to GitHub

The Hub for Modern Collaboration

Harnessing GitHub: A New Frontier in Collaborative Math Writing

At the heart of our collaborative math endeavor lies GitHub, a platform traditionally associated with code but now repurposed for our endeavor. This section provides a primer on GitHub, laying the foundation for those unfamiliar and offering insights into its transformative potential for collective math writing, learning, and teaching.

A Brief Introduction to GitHub

Originally conceptualized as a platform for developers, GitHub is a repository hosting service that facilitates version control using Git. At its core, it allows multiple users to work on a project simultaneously, tracking changes and ensuring that the latest version of a project is always accessible. Over the years, GitHub has grown beyond its initial software-centric confines, becoming a hub for all kinds of collaborative projects, from writing to data science and now to math.

Repurposing GitHub for Math Texts

Version Control

Math writing, like software, is dynamic and constantly evolving. As new sources or perspectives emerge, math texts may need revisions. GitHub's version control ensures that every change made to a document is tracked, enabling mathematicians to see how math texts evolve over time.

Collaborative Writing

Multiple contributors can work on a single math text simultaneously. This multi-user capability ensures diverse viewpoints can be seamlessly integrated, making the math text richer and more comprehensive.

Review and Feedback

Just as developers review and comment on code, mathematicians can provide feedback on written content. This feature encourages rigorous peer review, ensuring accuracy and credibility.

Open Access

Math texts on GitHub can be made public, granting anyone access to read, contribute, or fork the text into their own versions. This workflow democratizes math texts, making the creation process a collective endeavor rather than the domain of a select few.

Transparency

All changes and contributions are logged, providing a clear trail of the evolution of a mathematical text. This transparency bolsters the credibility of the text hosted on the platform.

Community Building

Beyond just writing, GitHub fosters a community of mathematicians, enthusiasts, and readers who can discuss, debate, and engage in meaningful dialogues about math and available math texts on GitHub.

Conclusion: Envisioning a Collaborative Mathematical Landscape

Embracing GitHub as a tool for collaborative math signifies more than just a shift in approach; it heralds a new era of inclusivity, transparency, and dynamism in writing, learning, and math teaching.

Chapter 4

Encouragement to Fork

Invitation to Dive Deep and Make It Your Own

PCiB isn't a static entity. It thrives on evolution, adaptation, and diversification, much like math itself. We encourage readers to "fork" (a term soon to be discussed) and create their own versions of this book. Read this section to understand the essence of "forking" and how it can be the starting point of your unique math journey.

The Concept of Forking: A Brief Overview

In the realm of software development, particularly in platforms like GitHub, "forking" refers to the act of creating a copy of a project, allowing one to make changes independently of the original. In this context, forking PCiB enables readers to take the base content and adapt, modify, and expand upon it, tailoring the narrative to resonate with their perspectives, insights, and understanding.

How to Begin Your Forking Journey

Start Small: You don't need to rewrite entire chapters. Begin by adding annotations, insights, or even footnotes to existing content. As you grow more confident, you can expand and modify larger sections.

Engage with the Community: Share your forked version with other readers. This encourages discourse, debate, and constructive feedback, allowing your text to be refined and enhanced.

Celebrate input: Encourage others around you to fork and create their own versions. The more in-depth the input, the deeper our collective understanding of math potentially becomes.

Conclusion: The Power of Collective Math

The invitation to fork PCiB isn't just about creating different versions of a book. It's a call to embrace collective writing, learning, and teaching. By embracing the essence of forking, math is not just something we read but something we actively shape, share, and pass on.

Chapter 5

More About GitHub

Discovering the Power of Collaborative Tools

Diving deeper into the world of GitHub, this chapter provides a comprehensive overview. Beyond its technicalities, we explore how GitHub emerged as a revolutionary platform for collaboration and how it can be leveraged for those interested in writing, teaching, and learning about math.

The Genesis of GitHub

GitHub began as a platform designed for software developers to manage and track changes to their codebase. Launched in 2008, it swiftly gained traction due to its user-friendly interface and efficient version control system powered by Git. Over the years, it evolved from a mere repository hosting service to a dynamic hub of collaboration, housing millions of projects and engaging tens of millions of users worldwide.

GitHub: More than Just Code

While GitHub's origins are rooted in code collaboration, its adaptable nature has made it a favored platform for various non-code projects. Writers, designers, educators, and researchers have discovered the potential of GitHub as a tool for:

Document Collaboration

With its built-in version control, contributors can track changes, revert to previous versions, and seamlessly merge updates.

Project Management

With features like "issues" and "milestones," teams can organize tasks, set goals, and monitor progress.

Open Access & Transparency

Public repositories allow for open contributions, ensuring transparency and fostering a sense of collective ownership.

Collaborative Writing

Multiple contributors can simultaneously work on a single document, with every change being tracked and attributed, facilitating teamwork on extensive projects like books or research papers.

Engaging the Public

With the platform's inherent transparency, researchers can make their work-in-progress accessible to the public, inviting insights, corrections, and contributions.

Case Study: PCiB's Use of GitHub

PCiB's journey on GitHub is a testament to the platform's potential in mathematical endeavors. By hosting the book on GitHub, the following is possible:

Feedback Loop

Readers can raise "issues," pointing out inaccuracies, suggesting enhancements, or even recommending new sections or topics.

Forking

As previously discussed, readers can "fork" the repository, creating their unique versions of the book while staying connected to the original.

Regular Updates

With math being dynamic, the book can be regularly updated, with new versions being released as and when significant changes are incorporated.

Challenges and Considerations

While GitHub offers many advantages, it's essential to understand its limitations:

Learning Curve

For those unfamiliar with Git or version control, there can be an initial learning curve.

Data Overwhelm

With vast amounts of data and contributions, ensuring quality and accuracy can be challenging.

Diverse Audience Management

Catering to both tech-savvy and non-tech audiences might require creating additional resources or tutorials to ensure inclusivity.

Conclusion: GitHub – A Paradigm Shift in Collaboration

The rise of GitHub marks a significant shift in how we perceive and participate in collaborative projects. Its adaptability, transparency, and user-centric design make it a powerful tool, not just for coders but for anyone passionate about collective endeavors. In the realm of mathematics, GitHub promises a future where texts are continually refined, expanded, and enriched by a global community.

Chapter 6

Forking Process

The Heart of Collaboration on GitHub

The beauty of open-source lies in its democratization of content creation. In this section, we demystify the process of "forking" on GitHub, guiding you step-by-step on how to take PCiB and create a version uniquely yours.

Understanding Forking

Before diving into the specifics, it's crucial to understand what "forking" means in the context of GitHub. In the simplest terms, to "fork" a project means to create a personal copy of someone else's project. Forking allows you to freely experiment with changes without affecting the original project. Forking is akin to taking a book you admire and making a copy to write your notes, edits, or additional chapters without altering the original book.

Why Fork?

Experimentation

It provides a safe space where you can test out ideas, make changes, or introduce new content.

Personalization

For projects like PCiB, it allows readers to customize the content, tailor it to their perspectives, or even localize it for specific audiences.

Collaboration

If you believe your changes have broad appeal, you can propose that they be incorporated back into the original project, enriching it with your unique contributions.

Step-by-Step Forking Guide

Set Up Your GitHub Account

If you don't have an account on GitHub, you'll need to create one. Visit GitHub's official site and sign up.

Navigate to the PCiB Repository

Once logged in, search for the PCiB project or navigate to its URL directly.

Click the 'Fork' Button

The fork button is located at the top right corner of the repository page; this button will create a copy of PCiB in your account.

Clone Your Forked Repository

Forking allows you to have a local copy on your computer, making editing and experimentation easier. Use the command: `git clone [URL of your forked repo]`.

Make Your Changes

Using your preferred tools, introduce the edits, additions, or modifications you desire.

Commit and Push Changes

Once satisfied, save these changes (known as a "commit") and then "push" them to your forked repository on GitHub.

Optional – Create a Pull Request

If you believe your changes should be incorporated into the original PCiB repository, you can create a "pull request." A pull request notifies the original authors of your suggestions.

Things to Keep in Mind

Stay Updated

The original PCiB project may undergo updates. It's a good practice to regularly "pull" from the original repo to keep your fork up-to-date.

Engage with the Community

Open-source thrives on community interactions. Engage in discussions, seek feedback, and please remain open to constructive criticism.

Conclusion: Embracing the Forking Culture

Forking is more than just a technical process; it symbolizes the ethos of open-source — a world where knowledge is not hoarded but shared, refined, and built upon collectively. By forking PCiB or any other project, you're not just creating a personal copy; you're becoming a part of a global movement that values collaboration, innovation, and the shared pursuit of knowledge. So, embark on this journey, make your unique mark, and contribute to the ever-evolving corpus of collective wisdom.

Chapter 7

Editing and Customizing

Tailoring Repositories to Suit Your Needs

Now, let's build upon the forking process; this segment delves into the next steps. How can you edit and customize your version of PCiB? What tools and techniques are available at your disposal? Embark on this informative journey as we guide you through the intricacies of editing on GitHub.

Understanding the GitHub Workspace

Before diving into the specifics of editing, it's essential to familiarize yourself with the GitHub workspace. Think of it as a digital toolshed where each tool serves a unique function:

- **Repository (Repo)**: This is the project's main folder where all your project's files are stored and where you track all changes.
- **Branches**: These are parallel versions of a repository, allowing you to work on features or edits without altering the main project.
- **Commits**: This is a saved change in the repository, akin to saving a file after making edits.
- **Pull Requests**: This is how you notify the main project of desired changes, proposing that your edits be merged with the original.

Editing Files Directly on GitHub

For minor changes, you might opt to edit directly on GitHub:

1. Navigate to the File: Within your forked PCiB repository, find the file you want to edit.

2. Click the Pencil Icon: This button allows you to edit the file.

3. Make Your Edits: Modify the content as needed.

4. Save and Commit: Below the editing pane, you'll see a "commit changes" section. Add a brief note summarizing your changes and click 'Commit.'

Editing Files Locally

For extensive customization:

1. Clone Your Repository: Use a tool like Git to clone (download) your forked repo to your local computer.

2. Edit Using Your Preferred Tools: This could range from text editors to specialized software, depending on the file type.

3. Commit and Push: After making your changes, save them (commit) and then upload (push) them to your GitHub repository.

Utilizing Branches for Extensive Customization

Branches are especially useful for significant overhauls or when working on different versions:

1. Create a New Branch: From your main project page, use the branch dropdown to type in a new branch name and create it.

2. Switch to Your Branch: Ensure you're working in this new parallel environment.

3. Make and Commit Changes: As you would in the main project.

4. Merging: Once satisfied with your edits in the branch, you can merge these changes back into the main project or keep them separate as a different version.

Exploring Additional Tools and Extensions

GitHub's ecosystem is rich with tools and extensions to enhance your editing experience:

- **GitHub Desktop**: An application that simplifies the process of managing your repositories without using command-line tools.
- **Markdown Editors**: Since many GitHub files (like READMEs) are written in Markdown, tools like StackEdit or Dillinger can be invaluable.
- **Extensions for Browsers**: Tools like Octotree can help in navigating repositories more effortlessly.

Conclusion: The Art of Tailored Content

Editing and customizing on GitHub might seem daunting initially, but with practice, it transforms into a manageable workflow. Many people find that the ability to take a project like PCiB and mold it into something uniquely theirs is empowering. It's a testament to the open-source community's ethos, where shared knowledge becomes the canvas and our collective edits, the brushstrokes, crafting an ever-evolving masterpiece. As you embark on your customization journey, remember that every edit, no matter how small, contributes to the project potentially in significant ways.

Chapter 8

Engaging with the Community

Joining the Global Conversation

The Significance of the GitHub Community

The digital age has bestowed upon us the gift of connectivity. On platforms like GitHub, this connectivity transcends borders, disciplines, and ideologies, culminating in a melting pot of diverse ideas and knowledge. For mathematicians and math enthusiasts, GitHub offers a space not only to store and manage content but also to engage with an audience that is passionate, informed, and eager to contribute.

1. Discussions and Debates

One of the most enriching aspects of the GitHub community is the plethora of discussions that unfold:

- **Issues**: A core feature of GitHub, "issues" allow users to raise questions, report problems, or propose enhancements.
- **GitHub Discussions**: A newer feature, Discussions, acts like a community forum. It's an excellent place for extended conversations, brainstorming, and sharing ideas or resources.

2. Collaborative Content Creation

Beyond solitary endeavors, GitHub shines in its collaborative capabilities:

- **Pull Requests**: If you have made an alteration to a math text or added a new perspective, pull requests are the way to propose these changes to the original repository owner. Pull requests foster a collaborative spirit, where content isn't static but continually evolving with community input.

- **Fork and Merge**: As you've learned, forking allows you to create your version of a repository. Engaging with the Community means you can merge changes from others into your fork, blending a mixture of diverse insights.

3. Building and Nurturing Networks

Connections made on GitHub often spill over into lasting professional relationships:

- **Following and Followers**: Like on social media platforms, you can follow contributors whose work resonates with you. Following contributors creates a curated feed of updates and also allows you to be part of a more extensive network.
- **GitHub Stars**: If a particular project or repository impresses you, give it a star! Starring not only bookmarks the project for you but also shows appreciation to the creator.

4. Learning and Growing Through Feedback

The Community's feedback is an invaluable asset:

- **Code Reviews**: Although traditionally for software, text writers can use this feature to receive feedback on their methodologies or approaches, refining their work.
- **Community Insights**: The "insights" tab on a repository provides analytics. For text writers, this can give a sense of which topics garner more attention and interest.

5. Participating in Community Events

GitHub often hosts and sponsors events:

- **Hackathons**: While traditionally for coders, these events can be repurposed for text writer content creation, where participants collaboratively tackle projects or themes.
- **Webinars and Workshops**: These events can range from mastering GitHub's technical side to thematic discussions on math topics.

A Project of Collective Wisdom

Math, in many ways, is a collective endeavor. GitHub can provide a dynamic Community. By engaging with this Community you can become an active participant in the creation of mathematical texts.

Chapter 9

Pre-Calculus Basic Topics

Introduction

In this chapter, we start talking about actual pre-calculus topics. Here we will introduce geometry, algebra, and trigonometry, with details provided in later chapters.

Chapter 10

Essential Geometry for Calculus

Geometry, with its focus on shapes, sizes, and the properties of space, is fundamental to understanding and applying calculus concepts. In this chapter, we explore key geometric principles and their applications in calculus.

10.1 Coordinate Geometry

Coordinate geometry, or analytic geometry, combines algebra and geometry. It is crucial for calculus. In this section, we will delve into the fundamental aspects of coordinate geometry, illustrating each concept with diagrams and graphs.

10.1.1 The Cartesian Plane

The Cartesian plane is a two-dimensional plane formed by the intersection of a vertical line (y-axis) and a horizontal line (x-axis). Points on this plane are identified by coordinates (x, y).

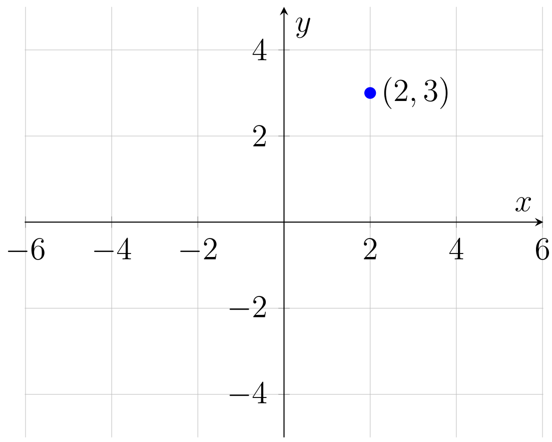


Figure 10.1: A point $(2, 3)$ on the Cartesian plane

10.1.2 Graphing Linear Equations

Linear equations in two variables can be graphed on the Cartesian plane. The graph of a linear equation forms a straight line.

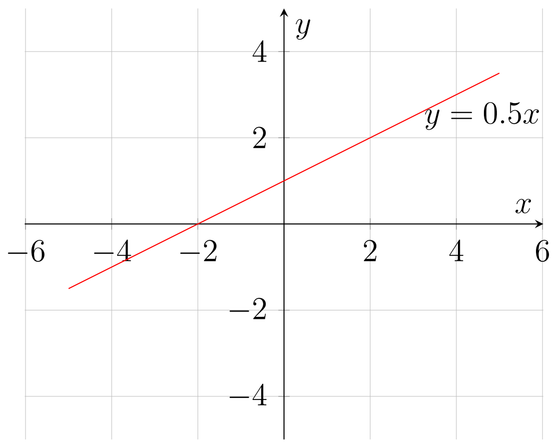


Figure 10.2: Graph of the linear equation $y = 0.5x + 1$

10.1.3 Slope of a Line

The slope of a line is a measure of its steepness and direction. It is defined as the ratio of the change in y to the change in x between two points on the line.

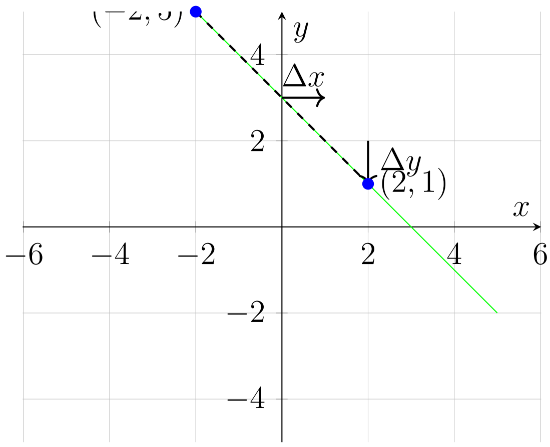


Figure 10.3: Illustration of the slope of a line

Exercise Calculate the slope of the line connecting points $(-2, 5)$ and $(2, 1)$.

Example Problems

To further solidify your understanding of coordinate geometry, here are some example problems with solutions.

Problem 1: Find the coordinates of the midpoint of the line segment joining the points $A(1, 2)$ and $B(4, 6)$.

Solution: The midpoint M of a line segment with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the formula:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Substituting the given values:

$$M = \left(\frac{1 + 4}{2}, \frac{2 + 6}{2}\right) = \left(\frac{5}{2}, 4\right)$$

Therefore, the coordinates of the midpoint are $\left(\frac{5}{2}, 4\right)$.

Problem 2: Determine the slope of the line passing through the points $(3, -2)$ and $(7, 6)$.

Solution: The slope m of the line passing through points (x_1, y_1) and (x_2, y_2) is calculated as:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting the given points:

$$m = \frac{6 - (-2)}{7 - 3} = \frac{8}{4} = 2$$

Hence, the slope of the line is 2.

Problem 3: Graph the linear equation $2x - 3y = 6$ on the Cartesian plane.

Solution: To graph the equation $2x - 3y = 6$, first solve for y:

$$3y = 2x - 6$$

$$y = \frac{2}{3}x - 2$$

Now plot two or more points that satisfy the equation and draw the line through them.

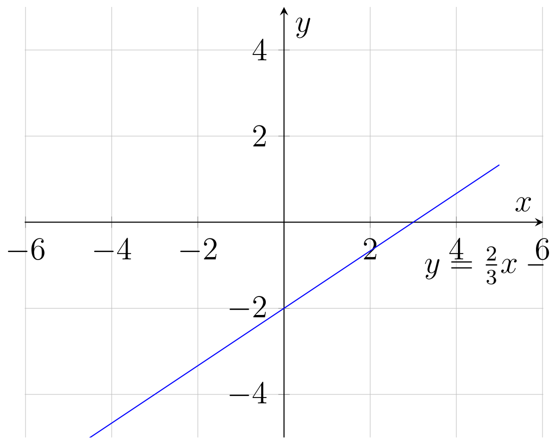


Figure 10.4: Graph of the equation $2x - 3y = 6$

10.1.4 The Cartesian Plane - Details

The Cartesian plane is a fundamental concept in coordinate geometry, named after the French mathematician René Descartes. It is a two-dimensional plane with two perpendicular axes: the horizontal axis (x-axis) and the vertical axis (y-axis). The intersection of these axes forms the origin, labeled as O.

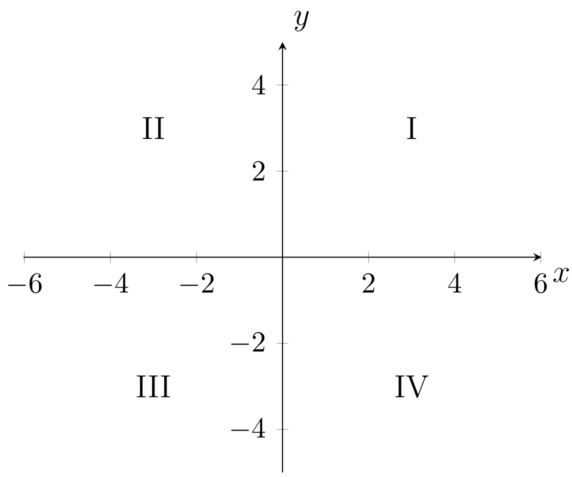


Figure 10.5: The Cartesian Plane with Quadrants

Points in the Cartesian plane are denoted by coordinates (x, y), where x is the horizontal distance from the origin, and y is the vertical distance. The plane is divided into four quadrants based on the sign of the coordinates:

- Quadrant I: $x > 0, y > 0$
- Quadrant II: $x < 0, y > 0$
- Quadrant III: $x < 0, y < 0$
- Quadrant IV: $x > 0, y < 0$

Distance Between Two Points

The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the Cartesian plane can be calculated using the distance formula derived from the Pythagorean theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Exercise Calculate the distance between the points $A(1, 2)$ and $B(4, 6)$.

Solution: Using the distance formula:

$$d = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Exercise Identify the quadrant in which the point $(-3, 4)$ is located.

Solution: The point $(-3, 4)$ is in Quadrant II because the x-coordinate is negative and the y-coordinate is positive.

Problem 4: Locate and label the point $C(-2, 3)$ on the Cartesian plane and state which quadrant it lies in.

Solution: The point $C(-2, 3)$ lies in Quadrant II as the x-coordinate is negative and the y-coordinate is positive.

Problem 5: Find the coordinates of the midpoint of the line segment connecting points $D(4, -1)$ and $E(-2, 3)$.

Solution: The midpoint M has coordinates given by $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Therefore, $M = \left(\frac{4-2}{2}, \frac{-1+3}{2}\right) = (1, 1)$.

Problem 6: Determine the slope of the line passing through points $F(1, 2)$ and $G(3, -2)$.

Solution: The slope m is given by $m = \frac{y_2-y_1}{x_2-x_1}$. Therefore, $m = \frac{-2-2}{3-1} = \frac{-4}{2} = -2$.

Problem 7: Given a line with the equation $y = 3x + 1$, identify a point that lies on this line and plot it on the Cartesian plane.

Solution: Choose any value for x and solve for y. For example, if $x = 2$, then $y = 3(2) + 1 = 7$. So, the point $(2, 7)$ lies on the line.

10.1.5 Slope and Equation of a Line

The concept of the slope and the equation of a line are fundamental in both geometry and algebra. The slope of a line measures its steepness and direction. It is calculated as the ratio of the vertical change to the horizontal change between two points on the line.

Definition of Slope

The slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope-Intercept Form

The slope-intercept form of a line's equation is $y = mx + b$, where m is the slope and b is the y-intercept (the y-coordinate of the point where the line crosses the y-axis).

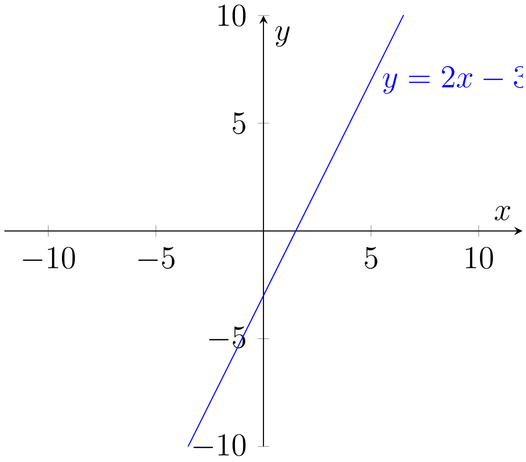


Figure 10.6: Line with slope $m = 2$ and y-intercept $b = -3$

Point-Slope Form

The point-slope form of a line's equation is $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line.

Standard Form

The standard form of a line's equation is $Ax + By = C$, where A, B, and C are constants, and A and B are not both zero.

Exercise Write the equation of the line passing through the point $(3, -2)$ with a slope of 4 in point-slope and slope-intercept forms.

Solution: Using the point-slope form: $y + 2 = 4(x - 3)$. In slope-intercept form, it becomes $y = 4x - 14$.

Problem 8: Find the slope of the line passing through the points $A(1, 3)$ and $B(4, 11)$.

Solution: The slope m is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Therefore, $m = \frac{11-3}{4-1} = \frac{8}{3}$.

Problem 9: Write the equation of a line in slope-intercept form that has a slope of -5 and passes through the point $(2, 3)$.

Solution: Using the point-slope form, the equation is $y - 3 = -5(x - 2)$. Expanding and rearranging gives $y = -5x + 13$ in slope-intercept form.

Problem 10: Convert the equation $2x - 3y = 6$ into slope-intercept form and identify the slope and y-intercept.

Solution: Rearranging the equation gives $3y = 2x - 6$ and then $y = \frac{2}{3}x - 2$. The slope is $\frac{2}{3}$ and the y-intercept is -2.

Problem 11: Determine whether the lines given by the equations $4y - 12x = 5$ and $2y - 6x = -3$ are parallel, perpendicular, or neither.

Solution: Convert each equation into slope-intercept form: $y = 3x + \frac{5}{4}$ and $y = 3x - \frac{3}{2}$. Both lines have the same slope 3, so they are parallel.

10.1.6 Graphing Linear Equations and Inequalities

Graphing linear equations and inequalities is a fundamental aspect of coordinate geometry and calculus. Linear equations form straight lines when graphed, and their general form is $y = mx + b$ where m is the slope and b is the y-intercept.

Graphing Linear Equations

To graph a linear equation, we can plot points that satisfy the equation and then connect these points with a straight line.

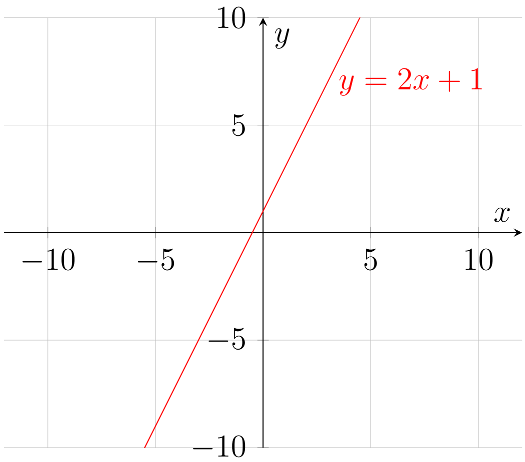


Figure 10.7: Graph of the linear equation $y = 2x + 1$

Graphing Linear Inequalities

Graphing linear inequalities involves shading a region of the Cartesian plane. The boundary line is drawn as either solid (for $\leq$ or $\geq$) or dashed (for $<$ or $>$).

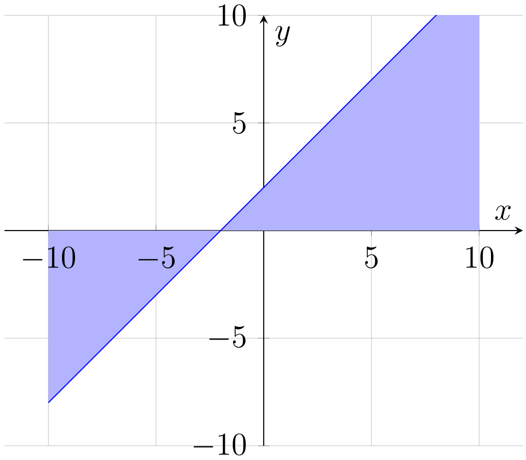


Figure 10.8: Graph of the linear inequality $y < x + 2$

Exercise Graph the linear equation $y = -\frac{1}{2}x + 3$ on the Cartesian plane.

Exercise Graph the linear inequality $y \geq 3x - 2$ and identify the shaded region.

Problem 12: Graph the linear equation $y = -x + 4$ and label two points that lie on the line.

Solution: To graph $y = -x + 4$, plot two points that satisfy the equation, such as (0,4) and (4,0), and draw a straight line through them.

Problem 13: Graph the inequality $y > 2x - 3$. Indicate the region that satisfies the inequality.

Solution: First, graph the line $y = 2x - 3$ with a dashed line since the inequality is strict ($>$). Then, shade the region above the line, as it represents $y > 2x - 3$.

Problem 14: Determine whether the point (3, -2) lies on the graph of the equation $2y = 4x - 6$.

Solution: Substitute $x = 3$ and $y = -2$ into the equation. If the equation holds true, the point lies on the graph.

$$2(-2) = 4(3) - 6$$
$$-4 = 12 - 6$$
$$-4 = 6$$

This is false, so the point (3, -2) does not lie on the graph.

Problem 15: Sketch the inequality $y \leq -\frac{1}{2}x + 1$ and identify the shaded region.

Solution: Graph the line $y = -\frac{1}{2}x + 1$ with a solid line (since the inequality includes $\leq$). Shade the area below the line to represent $y \leq -\frac{1}{2}x + 1$.

10.1.7 Slope of a Curve

The concept of the slope of a curve at a point is fundamental in calculus. It involves finding the slope of the tangent line to the curve at that point. This concept is a precursor to derivatives, which are essentially the slope of a curve at a given point.

Tangent to a Curve

A tangent to a curve at a given point is a straight line that touches the curve at that point without crossing it. The slope of this tangent line represents the rate of change of the curve at that point.

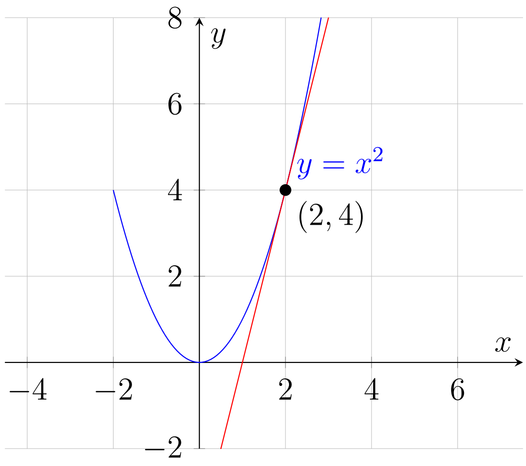


Figure 10.9: Slope of the curve $y = x^2$ at the point $x = 2$

In this example, the curve $y = x^2$ has a tangent line at $x = 2$. The slope of this tangent line can be calculated using the derivative of $y = x^2$.

Exercise Find the slope of the tangent to the curve $y = x^2$ at $x = 2$.

Solution: The derivative of $y = x^2$ is $y' = 2x$. At $x = 2$, the slope is $y' = 2(2) = 4$. Thus, the slope of the tangent at $x = 2$ is 4.

Problem 16: Consider the curve given by $y = 3x^2 - 2x + 1$. Find the slope of the tangent to the curve at $x = 1$.

Solution: First, find the derivative of $y = 3x^2 - 2x + 1$ which is $y' = 6x - 2$. At $x = 1$, the slope is $y' = 6(1) - 2 = 4$. Thus, the slope of the tangent at $x = 1$ is 4.

Problem 17: Determine the slope of the tangent line to the curve $y = \sqrt{x}$ at the point $x = 4$.

Solution: The derivative of $y = \sqrt{x}$ is $y' = \frac{1}{2\sqrt{x}}$. At $x = 4$, the slope is $y' = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Therefore, the slope of the tangent at $x = 4$ is $\frac{1}{4}$.

Problem 18: Find the equation of the tangent line to the curve $y = x^3$ at the point where $x = -2$.

Solution: The derivative of $y = x^3$ is $y' = 3x^2$. The slope at $x = -2$ is $y' = 3(-2)^2 = 12$. The tangent line has the equation $y - y_1 = m(x - x_1)$, where $m = 12$ and $x_1 = -2, y_1 = (-2)^3 = -8$. Thus, the equation is $y + 8 = 12(x + 2)$.

Problem 19: Calculate the slope of the tangent to the curve $y = \frac{1}{x}$ at $x = 3$.

Solution: The derivative of $y = \frac{1}{x}$ is $y' = -\frac{1}{x^2}$. At $x = 3$, the slope is $y' = -\frac{1}{3^2} = -\frac{1}{9}$. Hence, the slope of the tangent at $x = 3$ is $-\frac{1}{9}$.

10.2 Basic Geometric Figures

Geometry, the study of shapes and figures, begins with understanding the basics: points, lines, and planes. These elements form the foundation of more complex geometric concepts.

10.2.1 Points

A point represents a location in space. It has no size, area, length, or any other dimensional attribute. In diagrams, points are usually marked with a dot.

Figure 10.10: A point labeled A

10.2.2 Lines

A line is a straight one-dimensional figure having no thickness and extending infinitely in both directions. A line segment is a part of a line that is bounded by two distinct end points.

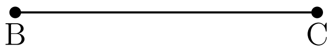


Figure 10.11: A line segment with end points B and C

10.2.3 Planes

A plane is a flat, two-dimensional surface that extends infinitely far. Planes are usually represented by a parallelogram in diagrams.

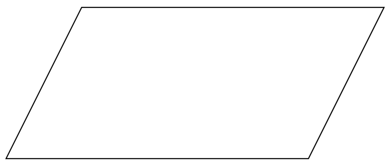


Figure 10.12: Representation of a plane

10.2.4 Properties of Basic Figures

Points

Points are often used to denote a specific location or a position in space.

Lines

Lines can be characterized by their slope, which is a measure of how steep the line is. Parallel lines have the same slope, while perpendicular lines have slopes that are negative reciprocals of each other.

Planes

Planes can be defined by three non-collinear points. They are used in geometry to discuss figures like triangles, rectangles, and circles.

Exercise Identify whether the following pairs of lines are parallel, perpendicular, or neither:

a) Line 1: $y = \frac{3}{2}x + 4$ and Line 2: $y = -\frac{2}{3}x + 1$

b) Line 3: $y = 5x - 2$ and Line 4: $y = 5x + 3$

Solution:

a) The slopes are $\frac{3}{2}$ and $-\frac{2}{3}$ which are negative reciprocals. So, they are perpendicular.

b) Both lines have a slope of 5, so they are parallel.

Problem 20: Consider three points $A(1, 2)$, $B(3, 4)$, and $C(-1, -2)$. Determine if these points are collinear.

Solution: To check for collinearity, calculate the slope of line segments AB and BC. If the slopes are equal, the points are collinear. The slope of AB is $\frac{4-2}{3-1} = 1$, and the slope of BC is $\frac{-2-4}{-1-3} = 1$. Since the slopes are equal, points A, B, and C are collinear.

Problem 21: Find the midpoint of the line segment connecting points $D(4, -6)$ and $E(-2, 8)$.

Solution: The midpoint M has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Thus, $M = \left(\frac{4-2}{2}, \frac{-6+8}{2}\right) = (1, 1)$.

Problem 22: Given a line with the equation $y = -3x + 7$, determine the y-intercept and one other point on the line.

Solution: The y-intercept is the point where $x = 0$, which is $(0, 7)$. For another point, choose any x value, say $x = 2$, and calculate y: $y = -3(2)+7 = 1$. So, another point is $(2, 1)$.

Problem 23: A plane is defined by points $F(1, 2, 3)$, $G(4, 5, 6)$, and $H(-2, -1, 0)$. Write the equation of the plane.

Solution: To find the equation of the plane, use the formula $Ax + By + Cz + D = 0$, where A, B, C are the components of the normal vector to the plane, and D is a constant. This requires more advanced techniques involving vectors and cross products and is beyond the scope of basic geometry.

10.2.5 Points and Lines - Detail

Understanding points and lines is crucial in geometry. Here, we explore their definitions and how they are represented on the Cartesian plane.

Points

A point is a fundamental concept in geometry. It represents a precise location or place in a two-dimensional space. A point has no size, shape, or real dimension.

Figure 10.13: Representation of a point P

Lines

A line is an infinitely long, one-dimensional figure with no thickness. It extends in two opposite directions without end. Lines are often described by a linear equation in various forms such as slope-intercept form, point-slope form, or standard form.

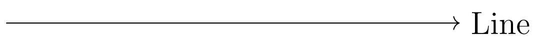


Figure 10.14: Representation of a line

Line Segments

A line segment is a part of a line that is bounded by two distinct endpoints. It has a fixed length.

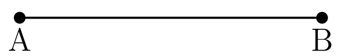


Figure 10.15: A line segment AB

Rays

A ray starts at a point and extends infinitely in one direction. It is represented by a starting point and another point through which it passes.

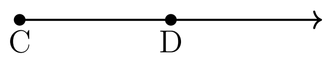


Figure 10.16: A ray CD

Exercise On the Cartesian plane, plot the points A(2, 3) and B(-1, -2). Then, draw the line segment AB.

Exercise Describe how a ray differentiates from a line segment and draw an example of a ray on a Cartesian plane.

Problem 24: Given two points $P(1, 2)$ and $Q(4, 6)$, draw the line segment PQ and calculate its length.

Solution: The line segment PQ can be drawn by plotting the points $P(1, 2)$ and $Q(4, 6)$ on a Cartesian plane and connecting them with a straight line. The length of the line segment is found using the distance formula:

$$PQ = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Problem 25: On a Cartesian plane, plot the ray that starts at $R(0, 0)$ and passes through $S(3, 3)$. Indicate the direction of the ray.

Solution: Plot the point $R(0, 0)$ and draw a line passing through $S(3, 3)$ extending infinitely in the direction away from R. The direction of the ray is towards the upper right quadrant.

Problem 26: If the line segment AB has endpoint $A(2, 3)$ and midpoint $M(5, 7)$, find the coordinates of endpoint B.

Solution: Using the midpoint formula, where M is the midpoint of AB: $M = \left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$, we have $5 = \frac{2+x_B}{2}$ and $7 = \frac{3+y_B}{2}$. Solving for x_B and y_B, we get $x_B = 8$ and $y_B = 11$. So, $B(8, 11)$.

Problem 27: Create a diagram showing a point T and two rays TA and TB such that $\angle ATB$ is a right angle.

Solution: This problem requires drawing a point T and two rays TA and TB that form a right angle at point T. The rays can be drawn on a Cartesian plane with TA and TB perpendicular to each other.

10.2.6 Vectors in Geometry

Introduce the concept of vectors, their representation, and significance in geometry and calculus.

10.2.7 Angles and Their Types

Define angles, discuss different types of angles (acute, obtuse, right, etc.), and their measurement in degrees and radians. Include an expanded section on radians.

10.2.8 Planes and Space

Introduction to the concept of a plane in geometry and its significance in spatial understanding.

10.3 Triangles and Congruence

Triangles form the basis of much of geometric reasoning. This section delves into different types of triangles and the concept of congruence.

10.3.1 Types of Triangles

Discuss different types of triangles based on sides (equilateral, isosceles, scalene) and angles (acute, right, obtuse).

10.3.2 Congruence and Similarity

Define congruence and similarity in the context of triangles, including criteria like SSS, SAS, ASA, and RHS.

10.3.3 Trigonometric Ratios in Right Triangles

Introduce the concept of trigonometric ratios in right triangles and their importance in both geometry and calculus.

10.4 Right Triangles and the Pythagorean Theorem

Right triangles form the basis of trigonometry, which is integral to calculus.

10.4.1 Properties of Right Triangles

Discussion on right triangles, including the definitions and properties specific to them.

10.4.2 The Pythagorean Theorem

The Pythagorean theorem states that in a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides.

Graphical Interpretation of the Pythagorean Theorem

Provide a visual interpretation of the Pythagorean theorem, possibly including a TikZ diagram to illustrate the concept.

10.4.3 Trigonometric Identities

Introduce basic trigonometric identities and their geometric interpretations, linking to their use in calculus.

10.5 Understanding Area

The concept of area is a foundational aspect of geometry and is pivotal in integral calculus.

10.5.1 Basic Principles of Area

Discuss the fundamental principles and units of area measurement.

10.5.2 Area Under a Curve

Introduce the preliminary idea of the area under a curve, connecting it to integral calculus concepts.

10.6 Circle Geometry

The circle is a fundamental shape in both geometry and calculus. This section covers its properties and equations.

10.6.1 Properties of Circles

Discuss the basic properties of circles, including tangents, chords, arc, and sector.

10.6.2 Equation of a Circle

Derive and explain the standard form and general form of the equation of a circle.

10.6.3 Parametric Equations of a Circle

Introduce the concept of parametric equations for a circle and their relevance in calculus.

10.6.4 Area of a Circle and Pi (π)

Explore the relationship between the area of a circle and the mathematical constant π. Discuss the formula $A = \pi r^2$ and its derivation.

10.7 Area Formulas for Common 2-D Shapes

This section will include formulas and explanations for finding the area of common two-dimensional shapes.

10.7.1 Decomposing Complex Shapes

Discuss the method of decomposing complex shapes into simpler ones to find their area. This technique is foundational for understanding the integration of irregular shapes in calculus.

10.7.2 Triangles

Include formulas for the area of different types of triangles (e.g., equilateral, right-angled).

10.7.3 Rectangles and Squares

Discuss the formulas for the area of rectangles and squares.

10.7.4 Other Common Shapes

Include area formulas for other common shapes like parallelograms, trapezoids, and ellipses.

10.8 Volume Formulas for Common 3-D Objects

Understanding the volume of three-dimensional objects is essential for many calculus applications.

10.8.1 Cross-Sectional Area

Introduce the concept of cross-sectional area and its role in determining the volume of 3-D objects, paving the way for understanding volume integration in calculus.

10.8.2 Prisms and Cylinders

Discuss the formulas for finding the volume of prisms and cylinders.

10.8.3 Pyramids and Cones

Include the formulas and methods to calculate the volume of pyramids and cones.

10.8.4 Spheres

Provide the formula for the volume of a sphere and discuss its derivation.

10.9 Applications in Calculus

This final section highlights how geometric concepts find applications in calculus.

10.9.1 Limits and Continuity

Discuss how geometric understanding aids in comprehending limits and continuity in calculus.

10.9.2 Integrals and Area

Explore how integration in calculus is used for finding areas under curves, with a basis in geometric understanding of area.

10.9.3 Real-World Examples in Calculus

Provide real-world examples and case studies demonstrating the use of geometric concepts in various calculus applications.

10.9.4 Geometric Interpretation of Derivatives and Integrals

Explore the geometric interpretation of derivatives (slopes and rates of change) and integrals (area under curves) in calculus.

Enhancements for the Chapter

Visual Elements Emphasize the inclusion of diagrams, graphs, and visual aids throughout the chapter to enhance understanding of geometric concepts.

Bridging Exercises Include exercises at the end of each section designed to bridge the gap between geometric concepts and their applications in calculus, reinforcing learning and application skills.

Conclusion

This chapter serves as the foundation for understanding the vital role of Geometry in calculus. As you progress through your studies, these concepts will become second nature, providing the tools necessary for analyzing and solving a broad range of problems in higher mathematics.

Chapter 11

Topics in Algebra

Basic Ideas Regarding Numbers

Introduction

Algebra is important in pre-calculus studies.

Conclusion

This chapter surveys important topics in algebra relevant to the study of calculus. Please fork the LaTeX source code for PCiB (available on GitHub) and create your own book that chooses the facts and exercises most relevant to you! Also, starring the PCiB project on GitHub would be greatly appreciated! Thanks for reading PCiB!

Chapter 12

Topics in Trigonometry

Understanding the Basic Trig Functions

Introduction

Trigonometry is important in pre-calculus studies.

12.1 The Unit Circle

The unit circle is a fundamental concept in trigonometry. It is a circle with a radius of one unit, centered at the origin (0,0) of the coordinate plane. The unit circle allows us to define all trigonometric functions for any angle, including those greater than 90 degrees.

12.1.1 Definition of the Unit Circle

The unit circle is defined as the set of all points (x, y) that satisfy the equation $x^2 + y^2 = 1$. This circle intersects the x-axis at the points $(1, 0)$ and $(-1, 0)$, and the y-axis at the points $(0, 1)$ and $(0, -1)$.

12.1.2 Angles and the Unit Circle

In the context of the unit circle, an angle θ is defined starting from the positive x-axis and rotating counter-clockwise for positive angles and clockwise for negative angles. This rotation creates an arc on the circle, and the length of this arc is the measure of the angle in radians.

12.1.3 Radians and Degrees

A radian is the measure of an angle corresponding to an arc length equal to the radius of the circle. Since the circumference of the unit circle is 2π, there are 2π radians in a full rotation. To convert radians to degrees, we use the fact that 360 degrees is equivalent to 2π radians. Therefore, to convert an angle from radians to degrees, we multiply by $\frac{180}{\pi}$.

$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

Conversely, to convert from degrees to radians, we multiply by $\frac{\pi}{180}$.

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$

12.1.4 Graphing the Unit Circle

Let's graph the unit circle with the point $(0, 1)$ marked:

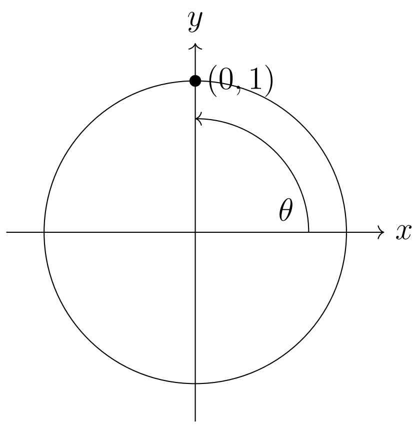


Figure 12.1: The unit circle with the point (0,1) marked.

As you can see from Figure 12.1, the unit circle provides a way to visualize angles and trigonometric functions. Each point on the circle corresponds to the cosine and sine of the angle θ, where the x-coordinate represents the cosine, and the y-coordinate represents the sine.

Conclusion

This chapter surveys important topics in trigonometry relevant to the study of calculus. Please fork the LaTeX source code for PCiB (available on GitHub) and create your own book that chooses the facts and exercises most relevant to you! Also, starring the PCiB project on GitHub would be greatly appreciated! Thanks for reading PCiB!

Chapter 13

Trigonometry for Calculus

Trigonometry is a branch of mathematics that deals with the relationships between the sides and angles of triangles. It is essential for understanding concepts in calculus, as it provides tools for modeling and solving problems involving periodic phenomena.

13.1 Angles and Their Measure

Every study of Trigonometry begins with an understanding of angles. Angles can be measured in two primary ways: degrees and radians. Degrees are traditionally used in many fields and cultures, and there are 360 degrees in a full rotation. Radians, however, provide a more natural approach in mathematics, especially calculus, because they arise from the arc length of a circle.

13.1.1 Degree Measure

One degree is defined as $\frac{1}{360}$ of a full rotation. The degree measure is subdivided into minutes and seconds, where one degree contains 60 minutes and one minute contains 60 seconds.

13.1.2 Radian Measure

A radian is a measure based on the radius of a circle. It is the angle created at the center of a circle by an arc whose length is equal to the radius of the circle. Since the circumference of a circle is 2π times the radius, a full rotation contains 2π radians.

13.1.3 Converting Between Degrees and Radians

To convert from degrees to radians, we use the relationship that 180 degrees is equal to π radians. Therefore, to convert x degrees to radians, we multiply x by $\frac{\pi}{180}$.

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$

Conversely, to convert from radians to degrees, we multiply the radian measure by $\frac{180}{\pi}$.

$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

13.1.4 Visualizing Angles with TikZ

To better understand how angles are measured, we can visualize them using a unit circle. Below, we depict an angle in both degrees and radians.

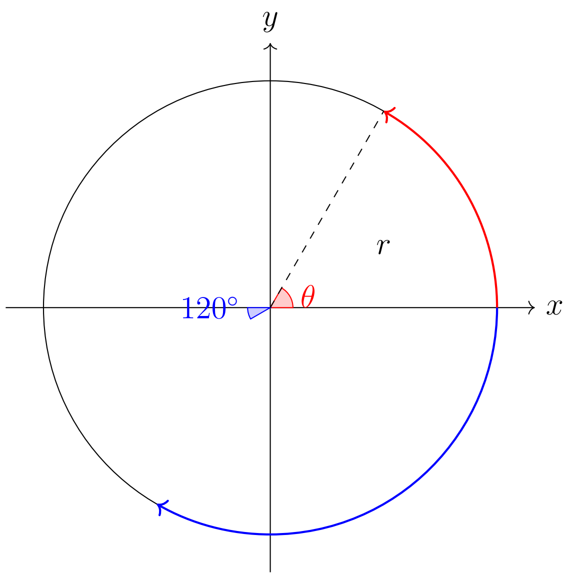


Figure 13.1: Visualization of an angle in degrees and radians on the unit circle.

Figure 13.1 shows an angle θ measured in radians (in red) and the same angle measured in degrees (in blue). This visual aid helps to understand the relationship between the two units of measurement.

13.2 Exercises on Angles and Their Measures

The following problems are designed to reinforce the concepts of angle measurement in degrees and radians as well as the conversion between them. Attempt to solve these problems before checking the provided solutions.

13.2.1 Problems

1. Convert 45° into radians.

2. Convert $\frac{\pi}{3}$ radians into degrees.

3. Find the radian measure of the angle subtended by an arc of length 10 cm in a circle of radius 20 cm.

4. If the angle θ in radians satisfies the equation $3\theta = \pi$, what is θ in degrees?

5. An angle is $\frac{3}{4}$ of a full circle. What is this angle in radians?

6. A clock's minute hand covers 30 degrees in 5 minutes. How many radians does it cover in 20 minutes?

13.2.2 Solutions

1. To convert 45° into radians, multiply by $\frac{\pi}{180}$:

$$45^\circ \times \frac{\pi}{180} = \frac{\pi}{4} \text{ radians}$$

2. To convert $\frac{\pi}{3}$ radians into degrees, multiply by $\frac{180}{\pi}$:

$$\frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$$

3. For an arc length $s = 10$ cm and radius $r = 20$ cm, the radian measure θ is given by $\theta = \frac{s}{r}$:

$$\theta = \frac{10 \text{ cm}}{20 \text{ cm}} = \frac{1}{2} \text{ radian}$$

4. If $3\theta = \pi$ radians, then $\theta = \frac{\pi}{3}$ radians. To convert this to degrees:

$$\frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$$

5. An angle that is $\frac{3}{4}$ of a full circle in radians is:

$$\frac{3}{4} \times 2\pi = \frac{3\pi}{2} \text{ radians}$$

6. In 20 minutes, the minute hand covers 4 times the distance it would in 5 minutes. Thus:

$$30^\circ \times 4 \times \frac{\pi}{180} = 2\pi \text{ radians}$$

13.2.3 Degree Measure

The measurement of an angle in degrees is a reflection of the rotation from the angle's initial side to its terminal side. One full rotation around a circle is equal to 360 degrees, meaning that a degree represents $\frac{1}{360}$th of a full rotation. This form of measurement originates from the ancient Babylonians, who had a base-60 number system and divided the circle into 360 parts.

To visualize an angle measured in degrees, consider the following figure created using the TikZ package. It shows an angle of 45 degrees, which is an eighth of a full 360-degree rotation.

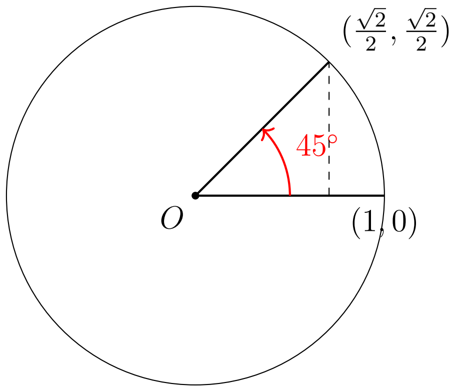


Figure 13.2: Illustration of a 45° angle in standard position.

The figure above illustrates an angle whose terminal side has rotated 45° from the positive x-axis, which is often referred to as the angle's initial side. The radius of the circle has a length of 1, hence the name "unit circle." The coordinates $(1, 0)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ correspond to the points where the radius intersects the x-axis and the circle, respectively. The dashed line represents the y-component of the terminal side's endpoint, helping to visualize the 45° angle as a right triangle within the circle.

Angles in degrees are widely used in various fields, including navigation, astronomy, and in everyday situations like measuring temperatures and geographic coordinates. Understanding how to measure angles in degrees and relate them to rotations is fundamental in trigonometry.

Practice Problems

Problem 1: Convert the following degree measurements to radians. (Note: π radians $= 180°$)

a. $30°$

b. $90°$

c. $270°$

Problem 2: Identify the quadrant in which the terminal side of each angle lies.

a. $60°$

b. $150°$

c. $210°$

d. $330°$

Problem 3: For each angle given, sketch the angle in standard position on the unit circle and label the point of intersection with the circle.

a. $45°$

b. $135°$

c. $225°$

d. $315°$

Problem 4: Determine the reference angle for each of the following:

a. $120°$

b. $250°$

c. $300°$

Problem 5: If the minute hand of a clock is pointing at 12 and it rotates to point at 3, what is the measure of the angle of rotation in degrees?

Solutions:

Problem 1:

a. $30° \times \frac{\pi}{180°} = \frac{\pi}{6}$

b. $90° \times \frac{\pi}{180°} = \frac{\pi}{2}$

c. $270° \times \frac{\pi}{180°} = \frac{3\pi}{2}$

Remarks:

- The problems above are designed to provide practice with converting between degrees and radians, identifying angle positions, and understanding the unit circle.

- The concept of a reference angle is important for understanding the standard trigonometric functions, as it is the acute angle formed by the terminal side of the given angle and the horizontal axis.

- Problem 5 illustrates a real-life application of angle measurement.

13.2.4 Radian Measure

The radian measure of an angle is the length of the arc on the unit circle subtended by the angle. One radian is the angle subtended by an arc length equal to the radius of the circle. Since the circumference of a unit circle is 2π times the radius, a full rotation around the circle is 2π radians.

Why Use Radians?

In calculus, radian measure is preferred because it provides a direct relationship between the angle measure and the arc length. This relationship simplifies the computation of derivatives and integrals involving trigonometric functions. For instance, when using radians, the derivative of $\sin x$ is simply $\cos x$, and the derivative of $\cos x$ is $-\sin x$. Such simplicity is not evident when using degrees.

Visualizing Radian Measure

To visualize an angle in radian measure, we plot it on the unit circle:

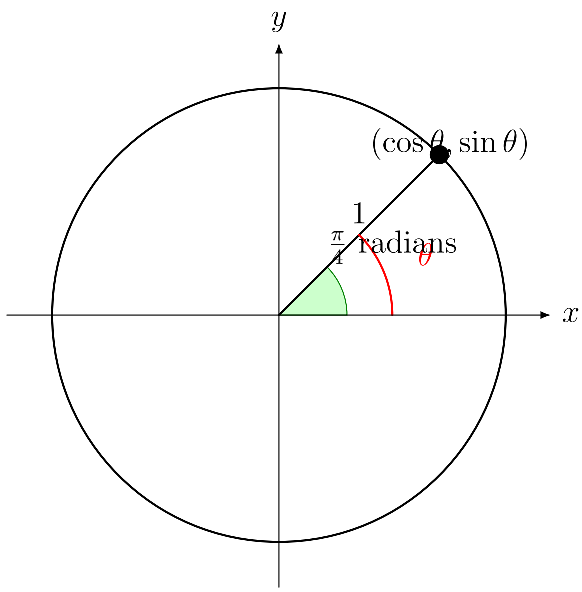


Figure 13.3: Visualization of an angle in radian measure on the unit circle.

Exercises

1. Convert the following angles from degrees to radians:

   a. $180°$

   b. $360°$

   c. $90°$

2. Show that the length of an arc subtended by a central angle of θ radians in a circle of radius r is given by $r\theta$.

3. Find the radian measure of the central angle of a circle with radius 10 units that subtends an arc of length 15 units.

Solutions

1. To convert degrees to radians, use the conversion factor $\frac{\pi \text{ radians}}{180°}$:

   a. $180° \times \frac{\pi}{180°} = \pi$ radians

   b. $360° \times \frac{\pi}{180°} = 2\pi$ radians

   c. $90° \times \frac{\pi}{180°} = \frac{\pi}{2}$ radians

2. The length of an arc l is directly proportional to the angle θ in radians, such that $l = r\theta$.

3. The radian measure of the angle is $\theta = \frac{l}{r} = \frac{15 \text{ units}}{10 \text{ units}} = 1.5$ radians.

Practice Problems

Here are some practice problems to help you solidify your understanding of the radian measure:

Problem 1: Convert the following angles from radians to degrees:

a. $\frac{\pi}{6}$ radians

b. $\frac{2\pi}{3}$ radians

c. $-\frac{\pi}{4}$ radians

Problem 2: An arc on a circle of radius 8 units has a measure of 2 radians. Find the length of the arc.

Problem 3: If a sector of a circle with radius 5 units has an area of 10 square units, what is the angle in radians of the sector?

Problem 4: A central angle of $\frac{5\pi}{6}$ radians is subtended at the center of a circle with radius 12 units. Find the area of the corresponding sector.

Problem 5: The wheel of a bicycle has a diameter of 70 cm. If the wheel rolls without slipping and turns through an angle of $\frac{\pi}{2}$ radians, how far has the bicycle traveled?

Solutions to Practice Problems

Solution to Problem 1: To convert radians to degrees, multiply by $\frac{180^\circ}{\pi}$.

a. $\frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$

b. $\frac{2\pi}{3} \times \frac{180^\circ}{\pi} = 120^\circ$

c. $-\frac{\pi}{4} \times \frac{180^\circ}{\pi} = -45^\circ$

Solution to Problem 2: The length L of the arc can be found using the formula $L = r\theta$.

$$L = 8 \text{ units} \times 2 \text{ radians} = 16 \text{ units}$$

Solution to Problem 3: Use the area formula for a sector $A = \frac{1}{2}r^2\theta$ and solve for θ.

$$\theta = \frac{2A}{r^2} = \frac{2 \times 10}{5^2} = \frac{20}{25} = \frac{4}{5} \text{ radians}$$

Solution to Problem 4: The area A of the sector is given by $A = \frac{1}{2}r^2\theta$.

$$A = \frac{1}{2} \times 12^2 \times \frac{5\pi}{6} = 72 \times \frac{5\pi}{6} = 60\pi \text{ square units}$$

Solution to Problem 5: The distance D traveled is equal to the length of the arc formed by the wheel, which is $D = r\theta$, where r is the radius.

$$D = \frac{70}{2} \text{ cm} \times \frac{\pi}{2} = 35\pi \text{ cm}$$

13.3 Trigonometric Functions

The six trigonometric functions are fundamental in both Trigonometry and calculus. These functions are sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot). They are defined as ratios of sides in a right triangle or as certain coordinates of points on the unit circle.

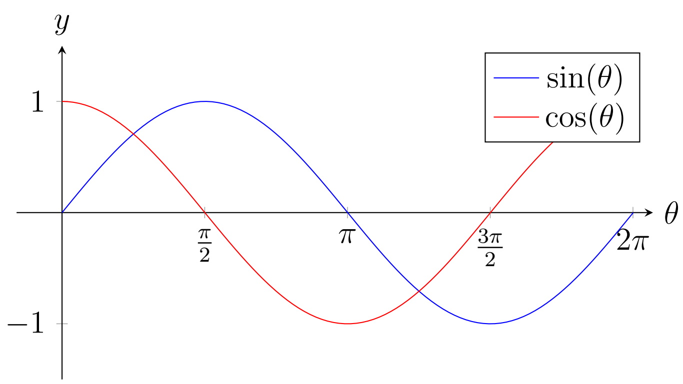
13.3.1 Sine and Cosine

The sine function (sin) represents the ratio of the length of the opposite side to the length of the hypotenuse in a right triangle. The cosine function (cos) represents the ratio of the length of the adjacent side to the length of the hypotenuse.

For a point (x, y) on the unit circle corresponding to an angle θ, the sine and cosine values are y and x, respectively.

Graph of Sine and Cosine Functions

We can graph the sine and cosine functions using their values from the unit circle.



The sine function starts at 0, reaches its maximum at $\frac{\pi}{2}$, returns to 0 at π, reaches its minimum at $\frac{3\pi}{2}$, and completes the cycle at 2π.

The cosine function starts at 1, drops to 0 at $\frac{\pi}{2}$, reaches its minimum at π, returns to 0 at $\frac{3\pi}{2}$, and completes the cycle at 2π.

13.3.2 Tangent Function

The tangent function (tan) is defined as the ratio of the sine to the cosine of an angle in a right triangle, or $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. It represents the slope of the line that intersects the unit circle at an angle θ.

13.3.3 Reciprocal Functions

The reciprocal functions, cosecant (csc), secant (sec), and cotangent (cot), are the reciprocals of the sine, cosine, and tangent functions, respectively.

- Cosecant ($\csc(\theta)$) is the reciprocal of sine: $\csc(\theta) = \frac{1}{\sin(\theta)}$
- Secant ($\sec(\theta)$) is the reciprocal of cosine: $\sec(\theta) = \frac{1}{\cos(\theta)}$
- Cotangent ($\cot(\theta)$) is the reciprocal of tangent: $\cot(\theta) = \frac{1}{\tan(\theta)}$

13.3.4 Practice Problems

To better understand trigonometric functions and their properties, try solving the following problems:

1. Evaluate $\sin\left(\frac{\pi}{6}\right)$, $\cos\left(\frac{\pi}{6}\right)$, and $\tan\left(\frac{\pi}{6}\right)$.
2. Given that a point P on the unit circle corresponding to an angle θ has coordinates $P(\frac{\sqrt{3}}{2}, \frac{1}{2})$, find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.
3. If $\cos(\theta) = -\frac{1}{2}$ and θ is in the third quadrant, what is $\sin(\theta)$?
4. Sketch the graph of the function $f(\theta) = 2\sin(\theta)$ for $0 \leq \theta \leq 2\pi$.
5. Using the graph of $f(\theta) = \cos(\theta)$, find the values of θ where the function is negative.
6. Solve the equation $\tan(\theta) = \sqrt{3}$ for $0 \leq \theta < 2\pi$.
7. Prove that $\sec^2(\theta) - \tan^2(\theta) = 1$ for all values of θ where the secant and tangent functions are defined.

8. Find the exact value of $\cot\left(\frac{3\pi}{4}\right)$.

9. Determine the amplitude and period of the function $g(\theta) = 3\cos(2\theta)$.

Answers to Practice Problems:

1. For $\frac{\pi}{6}$:
    - $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
    - $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
    - $\tan\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}}$

2. With $P(\frac{\sqrt{3}}{2}, \frac{1}{2})$:
    - $\sin(\theta) = \frac{1}{2}$
    - $\cos(\theta) = \frac{\sqrt{3}}{2}$
    - $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\sqrt{3}}$

3. For $\cos(\theta) = -\frac{1}{2}$ in the third quadrant:
    - $\sin(\theta)$ must also be negative in the third quadrant, and $\sin^2(\theta) + \cos^2(\theta) = 1$, so $\sin(\theta) = -\frac{\sqrt{3}}{2}$.

13.3.5 Sine and Cosine

The sine and cosine functions are fundamental to trigonometry. They are defined for a given angle θ within a right-angled triangle as follows:

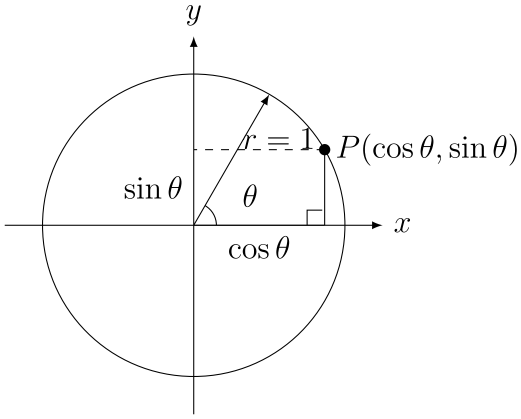
- The **sine** of angle θ, written as $\sin(\theta)$, is the ratio of the length of the opposite side to the length of the hypotenuse.

- The **cosine** of angle θ, written as $\cos(\theta)$, is the ratio of the length of the adjacent side to the length of the hypotenuse.

However, these definitions are not limited to angles between 0 and $\frac{\pi}{2}$ radians (or 0 and 90 degrees). By extending the concept of the unit circle, where the radius is 1 and the center is at the origin of a coordinate plane, we can define sine and cosine for any real angle.

Sine and Cosine on the Unit Circle

Consider the unit circle with a point $P(x, y)$ that corresponds to an angle θ measured from the positive x-axis. The coordinates x and y of point P on the unit circle are equal to $\cos(\theta)$ and $\sin(\theta)$, respectively.



The unit circle approach allows for a seamless transition of sine and cosine from acute angles to any angle in the coordinate plane. The definitions on the unit circle are as follows:

- For any angle θ, $\sin(\theta)$ is the y-coordinate of the corresponding point on the unit circle.
- For any angle θ, $\cos(\theta)$ is the x-coordinate of the corresponding point on the unit circle.

Properties of Sine and Cosine Functions

The sine and cosine functions have several important properties that are useful in various mathematical analyses:

- They are periodic functions with a period of 2π radians (or 360 degrees).
- The sine function is odd: $\sin(-\theta) = -\sin(\theta)$.
- The cosine function is even: $\cos(-\theta) = \cos(\theta)$.
- The maximum value of both functions is 1, and the minimum value is -1.
- They satisfy the Pythagorean identity: $\sin^2(\theta) + \cos^2(\theta) = 1$.

Understanding these functions and their properties is crucial for solving problems involving periodic behavior and waves, and they are the building blocks for defining the other trigonometric functions.

Example Problems

Below are some problems that help solidify the understanding of sine and cosine functions:

1. Given that $P(x, y)$ is a point on the unit circle corresponding to an angle θ, and $x = \frac{1}{2}$, find y and hence $\sin(\theta)$ and $\cos(\theta)$.

2. Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta) = 1$ using the definition of sine and cosine on the unit circle.

3. If $\sin(\theta) = \frac{3}{5}$ and θ is in the first quadrant, find $\cos(\theta)$.

4. For an angle θ with $\cos(\theta) = -\frac{\sqrt{2}}{2}$ and $\frac{\pi}{2} < \theta < \pi$, determine $\sin(\theta)$.

5. Using the unit circle, find the exact values of $\sin(\frac{\pi}{6})$, $\sin(\frac{\pi}{4})$, $\sin(\frac{\pi}{3})$, $\cos(\frac{\pi}{6})$, $\cos(\frac{\pi}{4})$, and $\cos(\frac{\pi}{3})$.

6. Sketch the unit circle and mark the points where $\sin(\theta) = \cos(\theta)$. What are the angles θ at these points?

Answers to Example Problems

Here are the answers to the example problems provided:

1. Since $x = \cos(\theta) = \frac{1}{2}$, we have $y = \sin(\theta) = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

2. To prove the identity, we use the fact that for any point on the unit circle, the distance from the origin to the point is 1. Hence, $x^2+y^2 = 1$. Since $x = \cos(\theta)$ and $y = \sin(\theta)$, it follows that $\cos^2(\theta) + \sin^2(\theta) = 1$.

3. Since θ is in the first quadrant, both sine and cosine are positive. Using the Pythagorean identity, $\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

4. In the second quadrant, sine is positive and cosine is negative. Using the Pythagorean identity, $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2}$.

5. For the specified angles, the exact values can be determined from the known special angles on the unit circle: $\sin(\frac{\pi}{6}) = \frac{1}{2}$, $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $\cos(\frac{\pi}{3}) = \frac{1}{2}$.

6. On the unit circle, $\sin(\theta) = \cos(\theta)$ at $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$. These correspond to the points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, respectively.

Note: The answers provided are meant to guide students in understanding the concepts. Instructors should verify all computations and explanations before use.

13.3.6 Tangent and Cotangent

The tangent of an angle in a right-angled triangle is the ratio of the length of the opposite side to the length of the adjacent side. In the unit circle, tangent is the y-coordinate divided by the x-coordinate when the line made by the angle intersects the circle. Symbolically, for an angle θ, this is expressed as:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Conversely, the cotangent is the reciprocal of the tangent function. It represents the ratio of the length of the adjacent side to the length of the opposite side, or using the unit circle, the x-coordinate divided by the y-coordinate:

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

These functions are undefined when their denominators are zero (i.e., when $\cos(\theta) = 0$ for tangent, and when $\sin(\theta) = 0$ for cotangent). This occurs at odd multiples of $\frac{\pi}{2}$ for tangent and at integer multiples of π for cotangent.

The graphs of these functions illustrate their periodic nature and the locations of their asymptotes, which correspond to the angles where the functions are undefined.

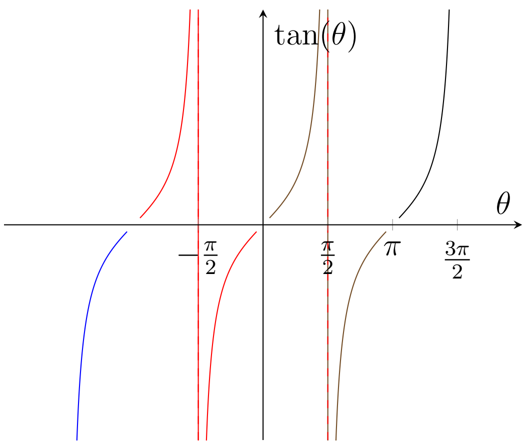


Figure 13.4: Tangent function graph showing asymptotes.

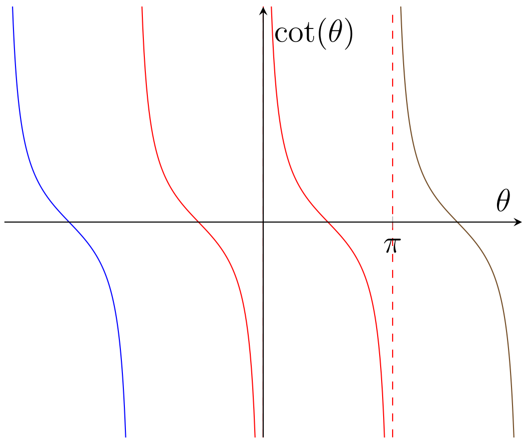


Figure 13.5: Cotangent function graph showing asymptotes.

In calculus, the tangent function is particularly important when it comes to determining the slope of a curve at a point, which is essential for understanding the concept of derivatives. Cotangent can similarly be used, especially when dealing with the slope of reciprocal functions.

13.3.7 Exercises on Tangent and Cotangent Functions

In this section, we provide some example problems involving the tangent and cotangent functions. These exercises are designed to reinforce the concepts discussed and help students gain a deeper understanding of these trigonometric functions.

Exercise Calculate the tangent of $\frac{\pi}{4}$ and verify that it equals 1.

Solution: Using the unit circle or the definition of the tangent function, we have:

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Exercise Find all solutions in the interval $[0, 2\pi)$ for the equation $\tan(\theta) = \sqrt{3}$.

Solution: We know that $\tan(\theta) = \sqrt{3}$ when $\theta = \frac{\pi}{3}$, but since tangent has a period of π, the other solution in the given interval is $\theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$.

Exercise Evaluate $\cot(\pi)$ and explain why it takes its particular value.

Solution: Cotangent is the reciprocal of the tangent function. Since $\tan(\pi) = 0$, and since the cotangent is undefined when its sine component is 0 (which it is at π), $\cot(\pi)$ is undefined. This reflects the asymptote of the cotangent graph at π.

Exercise Determine the exact value of $\cot\left(-\frac{\pi}{6}\right)$ using the cotangent definition.

Solution: Using the cotangent definition and known values of sine and cosine for $\frac{\pi}{6}$, we get:

$$\cot\left(-\frac{\pi}{6}\right) = \frac{\cos\left(-\frac{\pi}{6}\right)}{\sin\left(-\frac{\pi}{6}\right)} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

Exercise Sketch the graph of $\tan(\theta)$ and $\cot(\theta)$ from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{3\pi}{2}$ and label the asymptotes.

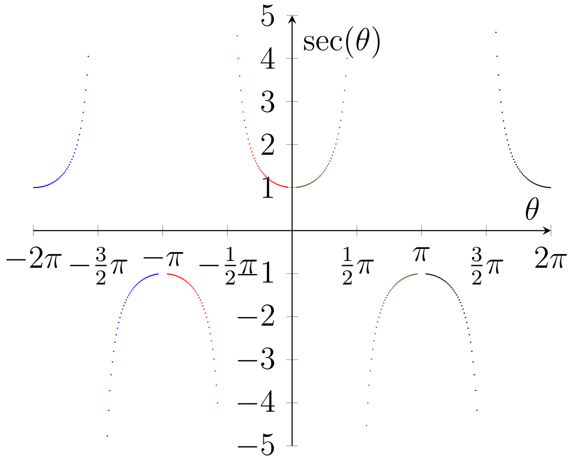
Solution: Students should refer to the graphs provided in the section above and reproduce them by hand, marking the asymptotes at $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$ for the tangent function and at $\theta = 0$, $\theta = \pi$ for the cotangent function.

These exercises require students to apply their understanding of the unit circle, the definitions of tangent and cotangent, and the properties of these functions such as periodicity and asymptotes. Students should attempt these exercises without a calculator to improve their understanding of the trigonometric functions.

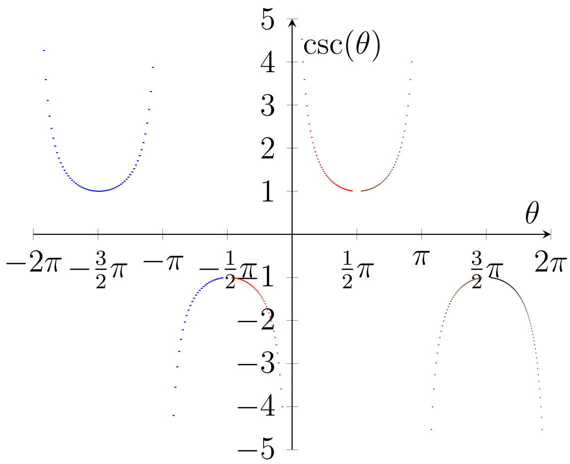
13.3.8 Secant and Cosecant

The secant (sec) and cosecant (csc) functions are the reciprocals of the cosine and sine functions, respectively. These functions are less commonly encountered in basic trigonometry but play an important role in calculus, especially in integration that involves trigonometric substitution. In sum, the reciprocal functions of cosine and sine are less commonly used but equally important, especially in certain types of integration.

Graph of the Secant Function



Graph of the Cosecant Function



These graphs demonstrate the behavior of the secant and cosecant functions and their asymptotic nature at points where their respective sine and cosine functions are zero. Understanding these graphs is essential for solving trigonometric equations and in calculus, particularly when dealing with integrals that require trigonometric identities or substitutions involving these functions.

13.3.9 Practice Problems

Here are some problems to test your understanding of the secant and cosecant functions:

1. Find the secant and cosecant of the following angles:

   (a) 0 radians

   (b) $\frac{\pi}{4}$ radians

   (c) $\frac{\pi}{2}$ radians

   (d) $\frac{3\pi}{4}$ radians

   Remember that secant and cosecant are undefined for certain angles where cosine and sine are zero, respectively.

2. Given that $\sin(\theta) = \frac{3}{5}$ in the second quadrant, find $\csc(\theta)$.

3. If $\cos(\theta) = -\frac{1}{2}$ and θ is in the third quadrant, determine $\sec(\theta)$.

4. For an angle θ where $0 < \theta < \frac{\pi}{2}$, if $\sec(\theta) = 3$, find the exact value of $\csc(\theta)$.

5. A point P on the terminal side of angle θ in standard position has coordinates $P(-2, -3)$. Calculate $\sec(\theta)$ and $\csc(\theta)$.

6. Prove the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$.

7. Show that for any angle θ, $\csc(\theta) \cdot \sin(\theta) = 1$.

Solutions to these problems will reinforce your understanding of secant and cosecant and prepare you for their use in calculus.

13.4 Graphs of Trigonometric Functions

Understanding the graphs of trigonometric functions is essential for mastering calculus, as they represent periodic behavior.

13.5 Graphs of Trigonometric Functions

The graphs of trigonometric functions are a visual representation of their behavior and are essential for understanding periodic phenomena in calculus. In this section, we'll explore the sine and cosine functions, which form the basis for all other trigonometric functions.

13.5.1 Graph of the Sine Function

The sine function, denoted by $y = \sin(x)$, represents the y-coordinate of a point on the unit circle as it rotates through an angle of x radians. It is periodic with a period of 2π and has a range from -1 to 1.

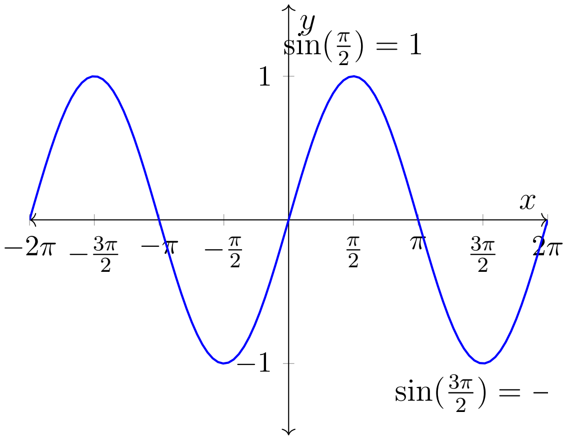


Figure 13.6: Graph of the sine function over two periods.

13.5.2 Graph of the Cosine Function

The cosine function, denoted by $y = \cos(x)$, represents the x-coordinate of a point on the unit circle as it rotates through an angle of x radians. Like the sine function, it has a period of 2π and ranges from -1 to 1.

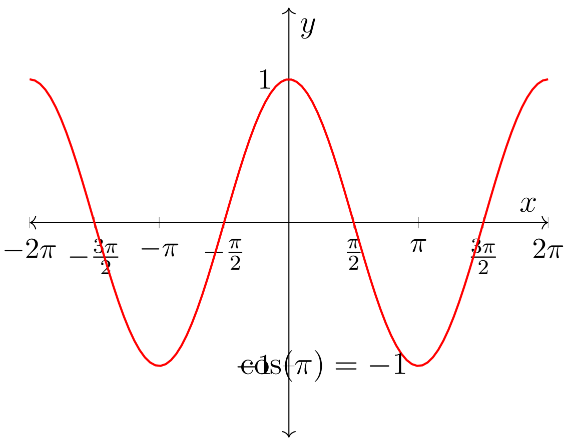


Figure 13.7: Graph of the cosine function over two periods.

Both of these functions exhibit properties that are useful in calculus, including symmetry, periodicity, and amplitude. Sine and cosine functions are the building blocks of more complex trigonometric functions, which can be constructed by shifting, stretching, or compressing these basic graphs.

13.5.3 Exercises

Practice the concepts we've discussed by working through the following exercises.

Graph Identification

Exercise 1. Identify the function graphed below. What are the amplitude, period, and phase shift of the function?

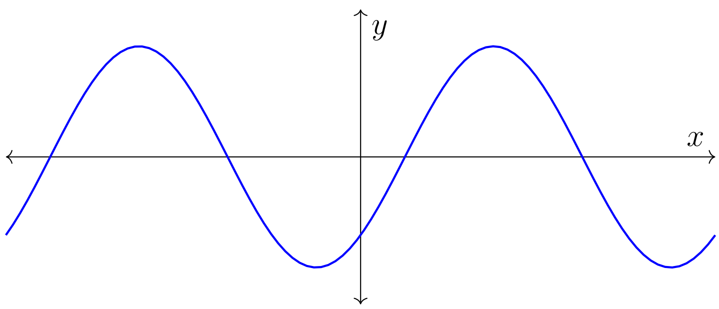


Figure 13.8: Identify the function and its characteristics.

Answer: The graph shows $y = 1.5\sin(x - \frac{\pi}{4})$. The amplitude is 1.5, the period is 2π, and the phase shift is $\frac{\pi}{4}$ to the right.

Function Properties

Exercise 2. What is the amplitude and period of the function $y = 3\cos(2x)$?

Answer: The amplitude of the function is 3, and the period is $\frac{\pi}{1}$, which is π since the coefficient of x in the cosine function affects the period as $\frac{2\pi}{\text{coefficient}}$.

Calculating Values

Exercise 3. Calculate the exact value of $\sin(\frac{3\pi}{4})$ and $\cos(\frac{3\pi}{4})$.

Answer: By using the unit circle or Pythagorean identities:

$$\sin(\frac{3\pi}{4}) = \sin(\pi - \frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$\cos(\frac{3\pi}{4}) = \cos(\pi - \frac{\pi}{4}) = -\cos(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$$

Real-World Application

Exercise 4. A Ferris wheel with a diameter of 50 meters makes one complete revolution every 2 minutes. Write a function that represents the height of a passenger car above the ground over time, assuming it starts at the lowest point at $t = 0$.

Answer: Let the height function be $h(t)$. The amplitude is the radius of the Ferris wheel, which is 25 meters. The period is the time for one revolution, which is 2 minutes, or 120 seconds. Using the sine function, we get:

$$h(t) = 25\sin\left(\frac{2\pi}{120}t\right) + 25$$

This function accounts for the fact that the Ferris wheel starts at the lowest point, so we add 25 to shift the graph up.

Challenge Problems

Exercise 5. Given the function $y = 2\sin(x) + \cos(2x)$, find the first derivative using trigonometric identities.

Answer: First, express $\cos(2x)$ using a double-angle identity:

$$y = 2\sin(x) + 1 - 2\sin^2(x)$$

Then, take the derivative with respect to x:

$$y' = 2\cos(x) - 4\sin(x)\cos(x)$$

These exercises encourage students to apply their understanding of the graphs of sine and cosine functions to solve problems and understand real-world applications. Adjust the complexity and nature of the exercises to match the level of your audience.

13.5.4 Periodicity

Trigonometric functions are inherently periodic; they repeat their values in regular intervals along the domain. This periodic nature allows us to predict the behavior of these functions beyond the basic interval of 0 to 2π for sine and cosine, and $-\pi/2$ to $\pi/2$ for tangent and cotangent. In sum, The concept of period, amplitude, and phase shift is introduced here to understand how trigonometric functions behave over intervals.

Defining Periodicity

The *period* of a function is the smallest positive interval after which the function's values repeat. For $\sin(x)$ and $\cos(x)$, the period is 2π because every 2π units, the cycle of the sine and cosine curve repeats. For $\tan(x)$ and $\cot(x)$, the period is π since their values repeat after every π units along the x-axis.

Amplitude

The amplitude of a trigonometric function is a measure of its vertical stretch or compression, represented by a coefficient in front of the function. For example, in the function $y = A\sin(x)$, the amplitude is $|A|$. This determines the maximum and minimum values of the function or, in other words, how "tall" or "short" the waves of the sine or cosine graph appear.

Phase Shift

Phase shift refers to the horizontal displacement of the basic function. If a function takes the form $y = \sin(x - C)$ or $y = \cos(x - C)$, the phase shift is C, which translates the graph to the right or left along the x-axis.

Graphing Trigonometric Functions

To graph trigonometric functions that involve amplitude, period, and phase shifts, the following general form can be considered:

$$y = A\sin(B(x - C)) + D \quad \text{or} \quad y = A\cos(B(x - C)) + D$$

Where:

- A represents the amplitude.
- B affects the period of the function, with the actual period being $\frac{2\pi}{|B|}$.

- C represents the phase shift.
- D represents the vertical shift.

Graphical Illustration

Let's illustrate a sine function with a period alteration and phase shift.

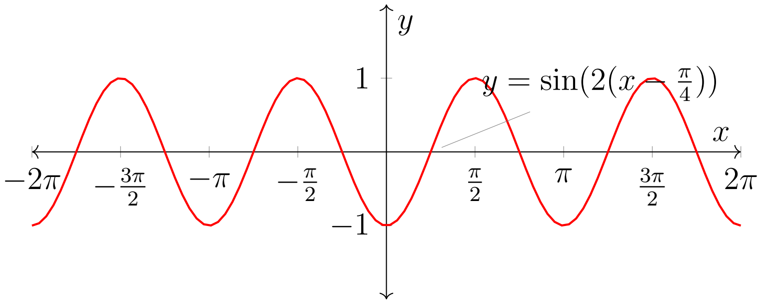


Figure 13.9: Graph of $y = \sin(2(x - \frac{\pi}{4}))$ showing a period of π and a phase shift of $\frac{\pi}{4}$.

Understanding these properties enables us to graph complex trigonometric functions and predict their behavior across any range of x values. This comprehension is vital in calculus where the manipulation of these functions' properties can be necessary for solving integrals and derivatives involving trigonometric functions.

Practice Problems

To further enhance understanding of periodicity in trigonometric functions, here are several practice problems:

1. Given the function $f(x) = 3\sin(2x)$, find the amplitude, period, and whether there is any phase shift.
2. Sketch the graph of $g(x) = \cos(x - \frac{\pi}{3})$ and identify the phase shift and period.
3. For the function $h(x) = 2\sin(\frac{x}{2} + \pi) + 1$, determine the amplitude, period, phase shift, and vertical shift. Then, sketch one period of the function's graph.
4. What is the period and phase shift of the function $j(x) = \tan(3x - \frac{\pi}{4})$?
5. If $k(x) = \sin(Bx)$ has a period of 3π, find the value of B.

6. Sketch the graph of $m(x) = -2\cos(4x + \frac{\pi}{2}) - 3$ and indicate the amplitude, period, phase shift, and vertical shift.

Solutions:

1. The amplitude of $f(x)$ is $|3| = 3$, the period is $\frac{2\pi}{|2|} = \pi$, and there is no phase shift.

2. The graph of $g(x)$ would show a cosine curve shifted to the right by $\frac{\pi}{3}$ units, with a period of 2π.

3. For $h(x)$, the amplitude is $|2| = 2$, the period is $\frac{2\pi}{|\frac{1}{2}|} = 4\pi$, the phase shift is $-\pi$ (shifted π units to the left), and the vertical shift is 1 unit upwards.

4. The function $j(x)$ has a period of $\frac{\pi}{|3|} = \frac{\pi}{3}$ and a phase shift of $\frac{\pi}{4}$ to the right.

5. Since the period of $k(x)$ is 3π, $B = \frac{2\pi}{3\pi} = \frac{2}{3}$.

6. The graph of $m(x)$ would be an inverted cosine curve (due to the "-2" amplitude) stretched by a factor of $\frac{1}{4}$ (since the period is $\frac{2\pi}{|4|} = \frac{\pi}{2}$), shifted to the left by $-\frac{\pi}{8}$, and shifted downward by 3 units.

13.5.5 Transformations

Trigonometric functions, much like any other functions, can undergo transformations such as scaling, reflection, and translation. These transformations allow us to alter the amplitude, period, and phase of the sine and cosine functions to fit various scenarios. In this section, we explore each of these transformations and how they apply to trigonometric functions. In sum, we explore how the basic sine and cosine graphs can be scaled, reflected, and translated to fit various scenarios.

In sum, we have begun to explore how the basic sine and cosine graphs can be scaled, reflected, and translated to fit various scenarios.

Scaling

Scaling a function can either compress or stretch its graph vertically or horizontally. The amplitude and period of trigonometric functions are affected by vertical and horizontal scaling respectively.

Vertical Scaling: If $f(x) = \sin(x)$, then $g(x) = A\sin(x)$ represents a vertical scaling by a factor of A. If $A > 1$, the function is stretched; if $0 < A < 1$, it is compressed.

Horizontal Scaling: Horizontal scaling affects the period of the trigonometric functions. For $f(x) = \sin(x)$, the function $h(x) = \sin(Bx)$ has a period of $\frac{2\pi}{B}$, scaling the period by a factor of $\frac{1}{B}$.

Reflection

Reflection flips the graph of the function over a specific axis.

Reflection over the X-axis: If $f(x) = \sin(x)$, then $g(x) = -\sin(x)$ is a reflection of f over the x-axis.

Reflection over the Y-axis: For $f(x) = \sin(x)$, the function $h(x) = \sin(-x) = -\sin(x)$ (since sine is an odd function) reflects f over the y-axis.

Translation

Translation shifts the graph of the function vertically or horizontally without altering its shape.

Vertical Translation: The function $f(x) = \sin(x) + D$ represents a vertical shift by D units. If $D > 0$, the graph shifts up; if $D < 0$, it shifts down.

Horizontal Translation (Phase Shift): For $f(x) = \sin(x)$, the function $g(x) = \sin(x - C)$ shifts the graph to the right by C units if $C > 0$ and to the left if $C < 0$.

To visualize these transformations, consider the following graphs created with the TikZ package.

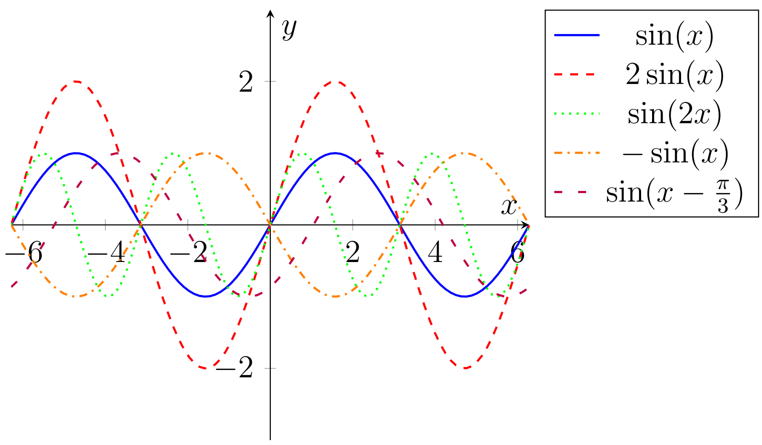


Figure 13.10: Graphs showing transformations of the sine function.

As seen in Figure 13.10, each transformation alters the original sine function in a unique way, illustrating the concepts of scaling, reflection, and translation. Through understanding these transformations, one can analyze and predict the behavior of more complex trigonometric functions.

13.5.6 Practice Problems

In this section, we provide several problems that allow students to apply their knowledge of transformations of trigonometric functions. Students should sketch the graph of each transformed function and identify key characteristics such as amplitude, period, phase shift, and vertical shift.

Problem 28: Graph the function $f(x) = 3\sin(2x)$ and describe the amplitude and period of the function.

Problem 29: Sketch the function $g(x) = \cos(x) - 2$. What is the vertical shift and how does it affect the graph of the standard cosine function?

Problem 30: Consider the function $h(x) = -\frac{1}{2}\sin(\pi x) + 1$. Graph the function and determine the amplitude, period, and vertical shift.

Problem 31: Graph the function $p(x) = \sin(x + \frac{\pi}{4})$ and describe the phase shift. How does this affect the starting point of the sine wave?

Problem 32: The function $q(x) = \sin(-x)$ represents a reflection. Sketch the graph and explain how the graph is transformed from the basic sine function.

Problem 33: Graph $r(x) = 2\cos(x - \frac{\pi}{3}) + 1$ and indicate all transformations from the parent cosine function.

For each problem, students should identify the type of transformation applied and use their knowledge of the unit circle and trigonometric properties to construct accurate graphs. These problems reinforce concepts of amplitude, period, phase shift, and vertical shift, which are pivotal in understanding trigonometric functions in calculus.

Answers to Practice Problems

Note to the instructor: Solutions are provided for reference to guide students through the problem-solving process.
Solution: For $f(x) = 3\sin(2x)$, the amplitude is 3 and the period is $\frac{\pi}{1}$, as the function is stretched vertically by a factor of 3 and horizontally compressed by a factor of 2.

Solution: The function $g(x) = \cos(x) - 2$ is shifted downward by 2 units. The amplitude and period remain the same as the standard cosine function.

13.6 Trigonometric Identities

Trigonometric identities are fundamental to manipulating and simplifying trigonometric expressions. They also play a crucial role in solving trigonometric equations, both in trigonometry and calculus. In this section, we will outline some of the most important identities and their applications.

In sum, trigonometric identities are vital tools for simplifying expressions and solving equations in calculus.

13.6.1 Pythagorean Identities

These identities are derived from the Pythagorean theorem and relate the square of the sine and cosine of an angle to 1.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
1 + \tan^2(\theta) &= \sec^2(\theta) \\
1 + \cot^2(\theta) &= \csc^2(\theta)
\end{aligned}$$

13.6.2 Angle Sum and Difference Identities

These identities express the sine, cosine, and tangent of the sum or difference of two angles in terms of the sines and cosines of those angles.

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\tan(\alpha \pm \beta) &= \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}
\end{aligned}$$

13.6.3 Double Angle Identities

These identities give the sine, cosine, and tangent of double angles, which are useful in various calculus applications.

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}
\end{aligned}$$

13.6.4 Half Angle Identities

These identities are useful when integrating trigonometric functions and in solving trigonometric equations.

$$\begin{aligned}
\sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos(\theta)}{2} \\
\cos^2\left(\frac{\theta}{2}\right) &= \frac{1+\cos(\theta)}{2} \\
\tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}
\end{aligned}$$

13.6.5 Product-to-Sum and Sum-to-Product Identities

These identities are useful in calculus for integrating products of sines and cosines and for simplifying expressions.

Product-to-Sum Identities:

$$\begin{aligned}
\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \\
\cos(\alpha)\cos(\beta) &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\
\sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]
\end{aligned}$$

Sum-to-Product Identities:

$$\begin{aligned}
\sin(\alpha) \pm \sin(\beta) &= 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right) \\
\cos(\alpha) + \cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
\cos(\alpha) - \cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}$$

13.6.6 Reduction Formulas

Reduction formulas are a type of recurrence relation that express trigonometric functions of a larger angle in terms of functions of a smaller angle. These are particularly useful in integration and series expansion of trigonometric functions.

Each subsection could be further expanded with examples, proofs, and applications of these identities in calculus. Moreover, practice problems could be provided to help students gain proficiency in using these identities to simplify and solve trigonometric equations.

13.6.7 Practice Problems

To apply the trigonometric identities outlined above, try solving the following problems. These will test your understanding and ability to manipulate the identities to simplify expressions and solve equations.

Problem 34: Use the Pythagorean identity to find $\sin(\theta)$ if $\cos(\theta) = \frac{3}{5}$ and θ is in the first quadrant.

Solution: Since $\cos(\theta) = \frac{3}{5}$ and θ is in the first quadrant, where sine is positive,

$$\begin{aligned}
\sin(\theta) &= \sqrt{1 - \cos^2(\theta)} \\
&= \sqrt{1 - \left(\frac{3}{5}\right)^2} \\
&= \sqrt{1 - \frac{9}{25}} \\
&= \sqrt{\frac{16}{25}} \\
&= \frac{4}{5}.
\end{aligned}$$

Problem 35: Prove the double angle identity for sine using the angle sum identity for sine:

$$\sin(2\theta) = ?$$

Solution: Using the angle sum identity for sine, $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, let $\alpha = \beta = \theta$:

$$\begin{aligned} \sin(2\theta) &= \sin(\theta+\theta) \\ &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\ &= 2\sin(\theta)\cos(\theta). \end{aligned}$$

Problem 36: Simplify the expression using sum-to-product identities: $\sin(x) - \sin(y)$.

Solution: Using the sum-to-product identity $\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$,

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

Problem 37: Find the exact value of $\tan\left(\frac{\pi}{8}\right)$ using the half-angle identity.

Solution: Using the half-angle identity for tangent, $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{\sin(\theta)}$, let $\theta = \frac{\pi}{4}$:

$$\begin{aligned} \tan\left(\frac{\pi}{8}\right) &= \frac{1-\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} \\ &= \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= \sqrt{2} - 1. \end{aligned}$$

13.6.8 Pythagorean Identities

The Pythagorean identities are a direct consequence of the Pythagorean theorem applied to a right triangle on the unit circle. For any angle θ, the following fundamental relationships hold true:

- The primary Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- Derived from the primary identity, the other two Pythagorean identities are:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

These identities are immensely useful in simplifying trigonometric expressions, solving trigonometric equations, and converting between trigonometric functions. Let's visualize the primary Pythagorean identity on the unit circle:

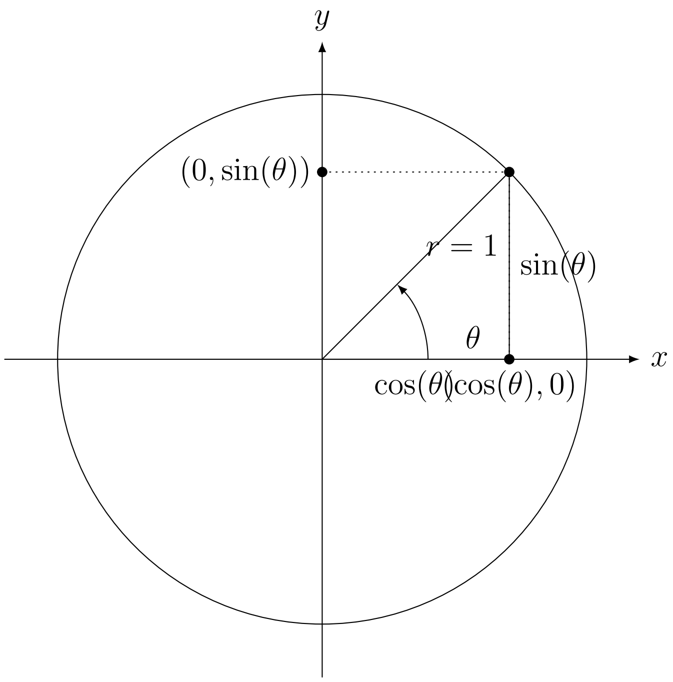


Figure 13.11: Visualization of the primary Pythagorean identity on the unit circle.

As the figure demonstrates, for any point on the unit circle at an angle θ from the positive x-axis, the x-coordinate is $\cos(\theta)$ and the y-coordinate is $\sin(\theta)$. Since the radius of the unit circle is 1, by the Pythagorean theorem, we have that $\cos^2(\theta) + \sin^2(\theta) = 1^2$.

Example: To see these identities in action, consider the angle $\frac{\pi}{4}$. For this angle, we know that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. Therefore,

$$\begin{aligned}
\sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) &= \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 \\
&= \frac{2}{4} + \frac{2}{4} \\
&= \frac{4}{4} \\
&= 1,
\end{aligned}$$

which confirms the primary Pythagorean identity.

13.6.9 Practice Problems

To further solidify your understanding of Pythagorean identities, try solving the following problems:

1. Verify the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$ for $\theta = \frac{\pi}{3}$.
2. Prove that $\csc^2(\theta) - \cot^2(\theta) = 1$ using the primary Pythagorean identity.
3. Simplify the expression $\sin^2(x) \cdot \sec^2(x)$ using Pythagorean identities.
4. Given that $\sin(\alpha) = \frac{3}{5}$ and α is in the first quadrant, find the values of $\cos(\alpha)$, $\tan(\alpha)$, and $\cot(\alpha)$.
5. If $\cos(\beta) = \frac{1}{2}$ and β is an acute angle, determine the exact values of $\sin(\beta)$, $\tan(\beta)$, and $\sec(\beta)$.
6. Show that $\sin(\theta) \cdot \tan(\theta) = \frac{\sin^2(\theta)}{\cos(\theta)}$ and simplify the expression.

Solutions to Practice Problems:

Note to instructor: The solutions provided below are for your reference and should not be shared with the students until after they have attempted to solve the problems on their own.

1. For $\theta = \frac{\pi}{3}$, $\tan(\theta) = \sqrt{3}$ and $\sec(\theta) = 2$. Then,
$$\tan^2\left(\frac{\pi}{3}\right) + 1 = (\sqrt{3})^2 + 1 = 3 + 1 = 4 = \sec^2\left(\frac{\pi}{3}\right).$$
2. Starting with $\csc^2(\theta) = 1 + \cot^2(\theta)$,
$$\csc^2(\theta) - \cot^2(\theta) = (1 + \cot^2(\theta)) - \cot^2(\theta) = 1.$$

3. Simplifying $\sin^2(x) \cdot \sec^2(x)$:

$$\sin^2(x) \cdot \sec^2(x) = \sin^2(x) \cdot \frac{1}{\cos^2(x)} = \tan^2(x).$$

4. Given $\sin(\alpha) = \frac{3}{5}$:

$$\cos(\alpha) = \sqrt{1 - \sin^2(\alpha)} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5},$$

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{3/5}{4/5} = \frac{3}{4},$$

$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{4}{3}.$$

5. If $\cos(\beta) = \frac{1}{2}$:

$$\sin(\beta) = \sqrt{1 - \cos^2(\beta)} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2},$$

$$\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3},$$

$$\sec(\beta) = \frac{1}{\cos(\beta)} = 2.$$

6. For $\sin(\theta) \cdot \tan(\theta)$:

$$\sin(\theta) \cdot \tan(\theta) = \sin(\theta) \cdot \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sin^2(\theta)}{\cos(\theta)}.$$

13.6.10 Sum and Difference Formulas

The sum and difference formulas are a cornerstone in trigonometry, allowing us to find the sine, cosine, and tangent of the sum or difference of two angles using the known values of these functions for the two angles. They can be stated as follows:

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\tan(\alpha \pm \beta) &= \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}
\end{aligned}$$

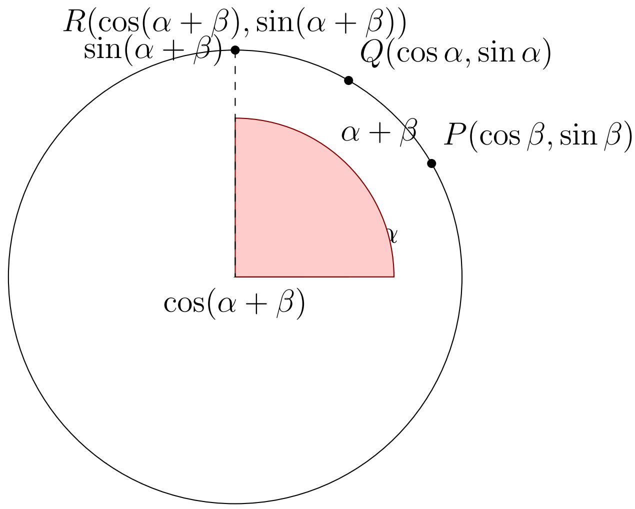


Figure 13.12: Geometric representation of sum of angles in a unit circle.

These identities are particularly useful in calculus for integrating products of sine and cosine functions and solving trigonometric equations.

To gain a better understanding and reinforce these concepts, it is advised to work through some example problems:

1. Given that $\sin(\pi/6) = 1/2$ and $\cos(\pi/3) = 1/2$, calculate $\sin(\pi/6 + \pi/3)$.

2. If $\cos(45°) = \sin(45°) = \frac{\sqrt{2}}{2}$, find the value of $\cos(45° - 45°)$.

3. Determine $\tan(75°)$ using the tangent sum formula, given that $75° = 45° + 30°$.

The above figure aids in visualizing how the sum of angles is represented within the unit circle, which can further help in understanding how the sum and difference formulas are derived geometrically.

Practice Problems

To ensure a solid grasp of the sum and difference formulas, practice the following problems:

1. Evaluate $\sin(75°)$ using the sum formula for sine with $\alpha = 45°$ and $\beta = 30°$.

2. If $\cos(60°) = \frac{1}{2}$ and $\sin(45°) = \frac{\sqrt{2}}{2}$, find the value of $\cos(60° - 45°)$.

3. Using the difference formula for tangent, calculate $\tan(15°)$ knowing that $\tan(45°) = 1$ and $\tan(30°) = \frac{\sqrt{3}}{3}$.

4. Verify the identity $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2(\alpha)-\sin^2(\beta)$ for $\alpha = 60°$ and $\beta = 45°$.

5. A point P moves in such a way that the angle α it makes with the positive x-axis varies. If α increases at a constant rate of 30° per second, express the x-coordinate of P, which is $\cos(\alpha)$, as a function of time t in seconds.

These problems combine the use of the sum and difference formulas with some algebraic manipulation and the fundamental trigonometric values for special angles. They serve as a good practice for students to apply these formulas in different contexts, which is a skill that will be beneficial in more advanced studies of mathematics, such as calculus.

13.6.11 Double-Angle and Half-Angle Formulas

The double-angle formulas are derived from the sum formulas and provide the trigonometric functions of an angle that is double another. Similarly, the half-angle formulas give the functions of half of a given angle. These are particularly useful in integration and in solving trigonometric equations in calculus.

In conclusion, these formulas are useful for integrating powers of sine and cosine functions.

Double-Angle Formulas

For any angle θ, the double-angle formulas are as follows:

- Sine: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

- Cosine: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

- Tangent: $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

The cosine double-angle formula has three variants, each useful depending on the context, especially in integration where one expression might be more convenient than the others.

Half-Angle Formulas

The half-angle formulas are derived using the double-angle formulas and the Pythagorean identity. They allow us to express trigonometric functions of $\frac{\theta}{2}$ in terms of θ:

- Sine: $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- Cosine: $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- Tangent: $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$

The signs for the half-angle formulas depend on the quadrant in which the resulting half-angle lies. These formulas are essential in calculus for simplifying the integration of trigonometric functions and solving trigonometric equations.

Visualization of Double-Angle and Half-Angle

To better understand the double-angle and half-angle concepts, it is often helpful to visualize them on the unit circle and through their graphs.

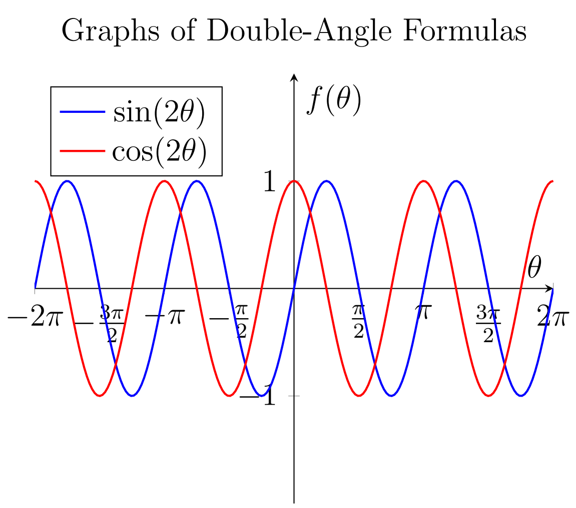


Figure 13.13: Graphs of $\sin(2\theta)$ and $\cos(2\theta)$.

In the graph, we see the effect of doubling the angle on the sine and cosine functions, where the frequency of the waveforms is doubled, reflecting the periodic nature of these functions.

Problems for Reinforcement

After reviewing the double-angle and half-angle formulas and observing their graphs, tackle the following problems to test your understanding:

1. Simplify the expression $\sin(4\theta)$ using the double-angle formula for sine.
2. Given that $\cos(\theta) = \frac{3}{5}$ and θ is in the first quadrant, find the exact value of $\cos(2\theta)$.
3. Use the half-angle formula to determine $\sin\left(\frac{\pi}{8}\right)$, given the cosine and sine of $\frac{\pi}{4}$.
4. Derive the tangent half-angle formula starting from the sine and cosine half-angle formulas.
5. Solve the equation $2\sin^2(\theta) - \sin(\theta) - 1 = 0$ for θ in the interval $[0, 2\pi)$.

Problems for Reinforcement

Problem 1: Prove that $\cos(4\theta) = 1 - 8\sin^2(\theta) + 8\sin^4(\theta)$ using double-angle formulas.

Solution:

$$\begin{aligned}
\cos(4\theta) &= \cos(2 \cdot 2\theta) \\
&= 2\cos^2(2\theta) - 1 \quad \text{(Double-angle formula)} \\
&= 2(2\cos^2(\theta) - 1)^2 - 1 \\
&= 2(4\cos^4(\theta) - 4\cos^2(\theta) + 1) - 1 \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1 \\
&= 8(1 - \sin^2(\theta))^2 - 8(1 - \sin^2(\theta)) + 1 \\
&= 1 - 8\sin^2(\theta) + 8\sin^4(\theta).
\end{aligned}$$

Problem 2: If $\tan(\theta) = 3$, find the value of $\tan(2\theta)$.

Solution:

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \quad \text{(Tangent double-angle formula)} \\
&= \frac{2 \cdot 3}{1 - 3^2} \\
&= \frac{6}{1 - 9} \\
&= -\frac{2}{3}.
\end{aligned}$$

Problem 3: Determine $\cos\left(\frac{\theta}{2}\right)$ if $\sin(\theta) = \frac{1}{2}$ and θ is in the first quadrant.

Solution:

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}} \quad \text{(Cosine half-angle formula)}$$

Since $\sin^2(\theta) + \cos^2(\theta) = 1,$

$$\begin{aligned}
\cos(\theta) &= \sqrt{1-\sin^2(\theta)} \\
&= \sqrt{1-\left(\frac{1}{2}\right)^2} \\
&= \sqrt{1-\frac{1}{4}} \\
&= \sqrt{\frac{3}{4}} \\
&= \frac{\sqrt{3}}{2}.
\end{aligned}$$

$$\begin{aligned}
\text{Then, } \cos\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{3}}{4}} \\
&= \frac{\sqrt{2+\sqrt{3}}}{2}.
\end{aligned}$$

Problem 4: Using the double-angle formulas, express $\sin^2(\theta)$ in terms of $\cos(2\theta)$.

Solution:

$$\begin{aligned}
\cos(2\theta) &= 1 - 2\sin^2(\theta) \quad \text{(Cosine double-angle formula)} \\
\sin^2(\theta) &= \frac{1-\cos(2\theta)}{2}.
\end{aligned}$$

Problem 5: Solve for θ if $\sin(2\theta) = \sqrt{2}$, assuming θ is in the second quadrant.

Solution:

$$\begin{aligned}
\sin(2\theta) &= \sqrt{2} \\
2\sin(\theta)\cos(\theta) &= \sqrt{2} \quad \text{(Sine double-angle formula)} \\
&\text{Since } \theta \text{ is in the second quadrant, } \sin(\theta) > 0 \text{ and } \cos(\theta) < 0. \\
\text{Let } \sin(\theta) &= \frac{\sqrt{2}}{2}, \text{ then } \cos(\theta) = -\frac{\sqrt{2}}{2}, \\
\text{and } \theta &= \frac{3\pi}{4} \text{ or } \theta = \frac{5\pi}{4} \text{ to satisfy the quadrant condition.}
\end{aligned}$$

13.7 Solving Trigonometric Equations

Solving trigonometric equations is a fundamental skill in calculus. These equations can often be solved by using algebraic techniques combined with trigonometric identities. In this section, we will explore various methods for solving trigonometric equations and apply these methods to more complex problems.

In sum, this section looks at how to solve basic trigonometric equations and how these principles apply to more complex equations in calculus.

13.7.1 Algebraic Methods

The first step in solving a trigonometric equation is to isolate the trigonometric function. This can be done through algebraic manipulation, such as adding or subtracting terms on both sides, factoring, or using the quadratic formula if applicable.

Example

Solve for θ in the equation $2\sin(\theta) + \sqrt{3} = 0$.

Solution:

$$\begin{aligned}
2\sin(\theta) &= -\sqrt{3} \\
\sin(\theta) &= -\frac{\sqrt{3}}{2}.
\end{aligned}$$

Now, we find the value of θ within the desired interval, typically $[0, 2\pi)$.

13.7.2 Using Identities

Trigonometric identities, such as the Pythagorean identities, sum and difference formulas, and double-angle formulas, can be used to transform complex equations into simpler forms that are easier to solve.

Example

Solve for x in the equation $\cos^2(x) - \sin(x) = 0$.

Solution: Using the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$, we substitute $\sin^2(x)$ with $1 - \cos^2(x)$:

$$\begin{aligned}\cos^2(x) - \sin(x) &= 0\\ 1 - \sin^2(x) - \sin(x) &= 0\\ \sin^2(x) + \sin(x) - 1 &= 0.\end{aligned}$$

Solving this quadratic equation in $\sin(x)$, we find the values of x within the interval $[0, 2\pi)$.

13.7.3 Inverse Trigonometric Functions

When the trigonometric function is isolated, we can use inverse trigonometric functions to find the angle corresponding to a given trigonometric ratio.

Example

Solve for α in the equation $\tan(\alpha) = 1$.

Solution:

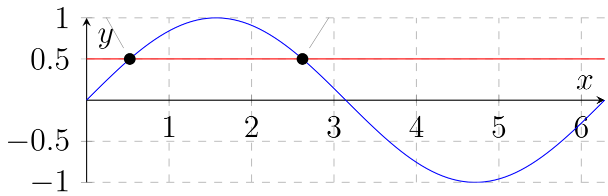
$$\begin{aligned}\alpha &= \arctan(1)\\ \alpha &= \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}.\end{aligned}$$

Since tangent has a period of π, we include $k\pi$ in the solution.

13.7.4 Graphical Solutions

Graphical methods can also be used to find solutions to trigonometric equations. By plotting the functions on both sides of the equation, the solutions are the x-coordinates of the intersection points.

Graphical Solution of $y = \sin(x)$ and $y = \frac{1}{2}$



These graphical solutions correspond to the equation $y = \sin(x)$ intersecting $y = \frac{1}{2}$, indicating the solutions for x where $\sin(x) = \frac{1}{2}$.

13.7.5 Challenges in Calculus

In calculus, trigonometric equations often arise within the context of differential equations, integrals, or series. The principles of solving simple trigonometric equations are applied, but often with additional techniques from calculus.

Example

Solve for θ in the equation $\frac{d}{d\theta}\sin(\theta) + \cos(\theta) = 0$, assuming θ is in radians.

Solution: First, we find the derivative of $\sin(\theta)$:

$$\begin{aligned}\frac{d}{d\theta}\sin(\theta) &= \cos(\theta)\\ \cos(\theta) + \cos(\theta) &= 0\\ 2\cos(\theta) &= 0.\end{aligned}$$

Solving for θ, we get $\theta = \frac{\pi}{2} + k\pi$, where k is an integer.

These are just introductory examples. More complex equations can be explored, including those involving multiple angles, products of trigonometric functions, and inverse trigonometric functions. Practice problems with varying difficulty should be provided to help students gain proficiency in solving trigonometric equations.

13.7.6 Practice Problems

To solidify your understanding of solving trigonometric equations, try to solve the following practice problems. Remember to use algebraic manipulation, trigonometric identities, inverse functions, and graphical insights where appropriate.

1. Solve for x in the equation $2\cos(x) - 1 = 0$ for x in the interval $[0, 2\pi)$.
2. Find all angles θ that satisfy the equation $\tan(\theta) + \sqrt{3} = 0$ for θ in the interval $[-\pi, \pi)$.
3. Determine the solution set for $\sin(2x) = \cos(x)$ within the interval $[0, 2\pi)$.
4. Solve $\sin^2(x) - \sin(x) - 6 = 0$ for all x in the interval $[0, 2\pi)$.
5. Using the double-angle formula, solve the equation $\sin(x) - \cos(2x) = 0$ for x in the interval $[0, 2\pi)$.
6. Graphically determine the solutions to $\cos(x) = \sin(x)$ using a plot. Hint: Consider the lines $y = \cos(x)$ and $y = \sin(x)$ and find their points of intersection.
7. For the differential equation $\frac{d}{dx}\sin(x) + \sin(x) = 0$, find the general solution.
8. If $\cos(3x) = \frac{1}{2}$, find all solutions for x within the interval $[0, 2\pi)$.
9. Prove that the equation $\tan(x) = \cot(x)$ has no solutions for x in the interval $(0, \pi)$.
10. Solve the equation $3\sin(x) + 4\cos(x) = 2$ by expressing it in the form $R\sin(x + \alpha)$, where $R > 0$ and $0 \leq \alpha < 2\pi$.

Solution to Problem 1:

$$\begin{aligned} 2\cos(x) - 1 &= 0 \\ \cos(x) &= \frac{1}{2}. \end{aligned}$$

The solutions for x in the interval $[0, 2\pi)$ are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Note: It is beneficial to check your solutions with a graphing calculator or software that can plot trigonometric functions and their intersections.

13.7.7 Linear Trigonometric Equations

Techniques for solving first-order trigonometric equations are outlined here.

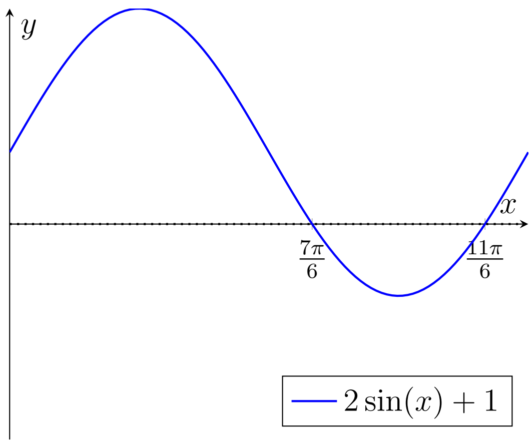
Linear trigonometric equations are of the form $A \cdot \text{trig\_function}(x) + B = 0$, where A and B are constants, and the trig_function can be sin, cos, or tan. To solve such equations, we isolate the trigonometric function and then find the angle x that satisfies the equation.

For example, consider the equation $2\sin(x) + 1 = 0$. To solve for x, we first isolate $\sin(x)$:

$$2\sin(x) + 1 = 0 \Rightarrow \sin(x) = -\frac{1}{2}.$$

We then look for all angles x that have a sine of $-\frac{1}{2}$. In the interval $[0, 2\pi)$, these angles would be $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

To visualize the solution, we can plot the equation $y = 2\sin(x) + 1$ and identify the points where the graph intersects the x-axis.



This graph clearly shows the points of intersection corresponding to the solutions of our linear trigonometric equation.

13.7.8 Practice Problems

Problem 1: Solve the trigonometric equation $3\cos(x) = 1.5$.

Problem 2: Find all solutions in the interval $[0, 2\pi)$ for the equation $\tan(x) - \sqrt{3} = 0$.

Problem 3: Determine the general solution for $\sin(x) + \frac{1}{2} = 0$.

13.7.9 Solutions to Practice Problems

Solution to Problem 1: To find the value of x we divide both sides by 3:

$$\cos(x) = \frac{1.5}{3} = 0.5.$$

The solution to this equation is the angles where the cosine is 0.5. In the interval $[0, 2\pi)$, these are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Solution to Problem 2: We can start by adding $\sqrt{3}$ to both sides:

$$\tan(x) = \sqrt{3}.$$

The solution to this equation is the angles where the tangent is $\sqrt{3}$, which corresponds to $x = \frac{\pi}{3}$ and $x = \frac{4\pi}{3}$ within the interval $[0, 2\pi)$.

Solution to Problem 3: Subtracting $\frac{1}{2}$ from both sides gives us:

$$\sin(x) = -\frac{1}{2}.$$

The general solution includes all angles where the sine is $-\frac{1}{2}$. In the interval $[0, 2\pi)$, the solutions are $x = \frac{7\pi}{6} + 2k\pi$ and $x = \frac{11\pi}{6} + 2k\pi$ where k is any integer.

13.7.10 Inverse Trigonometric Functions

The inverse trigonometric functions are introduced as a means of solving trigonometric equations and their relevance in integration.

The inverse trigonometric functions, often denoted as arcsin, arccos, and arctan, provide the angle that corresponds to a given trigonometric value. These functions are the inverses of the sine, cosine, and tangent functions, respectively, within their domains. They are essential in solving trigonometric equations and are frequently encountered in calculus, particularly within integration and differentiation problems where trigonometric substitution is involved.

Arcsine Function

The function $y = \arcsin(x)$ returns the angle whose sine is x. The domain of $\arcsin(x)$ is $[-1, 1]$, and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Arccosine Function

Similarly, $y = \arccos(x)$ gives the angle whose cosine is x. The domain of $\arccos(x)$ is $[-1, 1]$, and its range is $[0, \pi]$.

Arctangent Function

Lastly, $y = \arctan(x)$ finds the angle whose tangent is x. The domain of $\arctan(x)$ is all real numbers, and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Graphs of Inverse Trigonometric Functions

Here we provide a visual representation of the inverse trigonometric functions using the ‘tikz‘ and ‘pgfplots‘ packages.

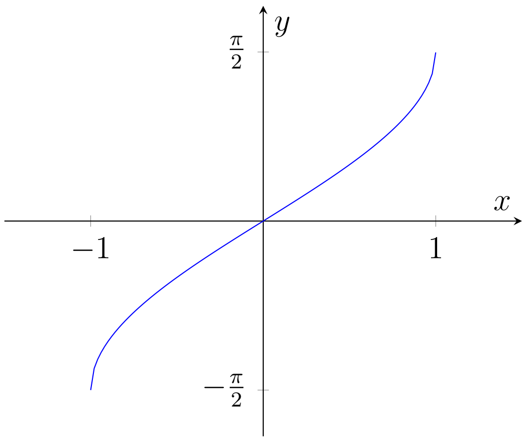


Figure 13.14: Graph of the arcsine function

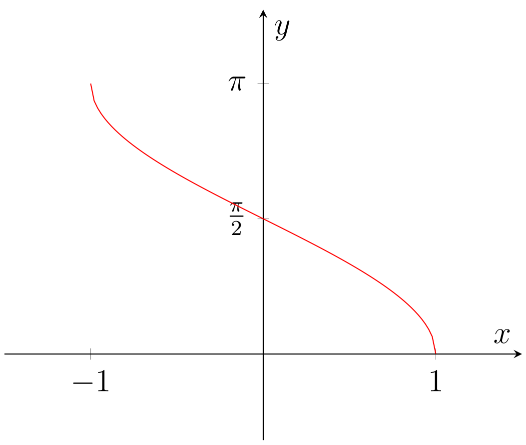


Figure 13.15: Graph of the arccosine function

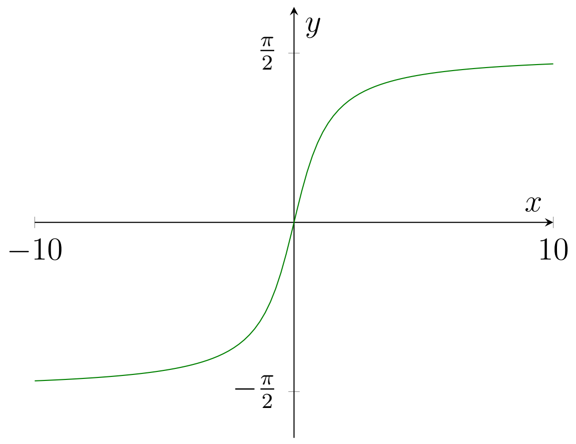


Figure 13.16: Graph of the arctangent function

Practice Problems

Solve the following problems involving inverse trigonometric functions.

Problem 1: Find the value of x if $x = \arcsin(\frac{1}{2})$.

Solution 1: We know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, therefore $x = \frac{\pi}{6}$.

Problem 2: Solve for θ in the equation $\cos(\theta) = \frac{\sqrt{2}}{2}$.

Solution 2: We recognize that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, thus $\theta = \frac{\pi}{4}$ or $\theta = \frac{7\pi}{4}$ since cosine is positive in the first and fourth quadrants.

Problem 3: Evaluate $\arctan(1)$.

Solution 3: Since $\tan\left(\frac{\pi}{4}\right) = 1$, we have $\arctan(1) = \frac{\pi}{4}$.

Problem 4: Determine the value of α given that $\sin(\alpha) = -\frac{1}{2}$ and α is in the fourth quadrant.

Solution 4: In the fourth quadrant, the reference angle for α is $\frac{\pi}{6}$. Since sine is negative in the fourth quadrant, $\alpha = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Problem 5: If $x = \arccos(-\frac{1}{2})$, find the exact value of x.

Solution 5: The value $\frac{1}{2}$ has an arccosine of $\frac{\pi}{3}$, and since the cosine is negative, we are in the second quadrant: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Problem 6: Express $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$ in terms of π.

Solution 6: Since $\frac{5\pi}{6}$ is outside the range of the arcsine function, we must find an equivalent angle within $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)$, we have $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}$.

Problem 7: Solve for y in the equation $\arctan(y) - \arctan(1) = 0$.

Solution 7: Since $\arctan(1) = \frac{\pi}{4}$, we have $\arctan(y) = \frac{\pi}{4}$, which means $y = \tan\left(\frac{\pi}{4}\right) = 1$.

These problems encourage students to apply their knowledge of inverse trigonometric functions to find angles corresponding to given trigonometric values and to solve equations involving these functions. The solutions are presented immediately after each problem for self-checking.

13.8 Applications of Trigonometry

Trigonometry is not just a theoretical mathematical discipline; it has numerous practical applications in science, engineering, music, astronomy, and various other fields. In this section, we will explore how trigonometry models real-world phenomena and its specific applications in calculus.

13.8.1 Sound Waves

Sound waves can be modeled as sine or cosine functions, representing the periodic nature of the vibrations that produce sound. The frequency and amplitude of these waves determine the pitch and volume of the sound, respectively.

13.8.2 Electrical Engineering

In electrical engineering, trigonometry is used to model alternating currents (AC). The voltage in AC circuits varies with time, and trigonometric functions can represent the cyclical variations in voltage and current.

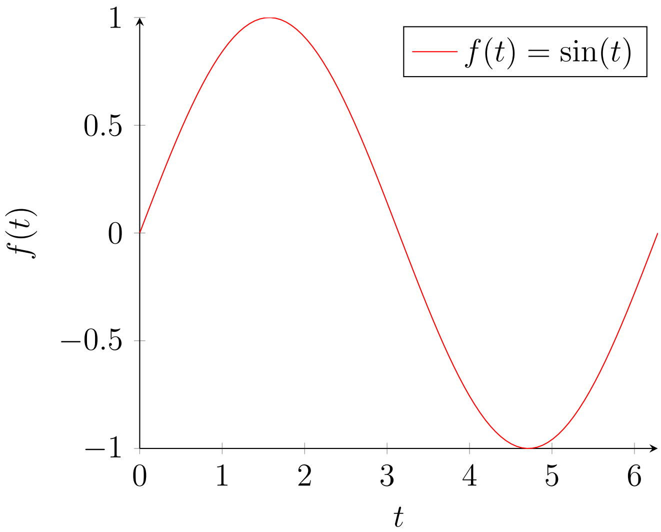


Figure 13.17: Graph of a sound wave modeled by a sine function.

13.8.3 Astronomy

Astronomers use trigonometry to calculate distances to stars and other celestial objects. The parallax method, which relies on trigonometric ratios, can measure the distance based on the apparent shift of an object when viewed from different positions.

13.8.4 Calculus Applications

In calculus, trigonometry is essential in solving integrals and derivatives of trigonometric functions. It is also used to solve differential equations that model wave motion, harmonic oscillators, and other dynamic systems.

Trigonometry is deeply woven into the fabric of calculus. The periodic properties of trigonometric functions make them ideal for describing cyclic phenomena. For instance, the derivative of the sine function is the cosine function, and this relationship is fundamental in solving problems involving motion.

These applications illustrate the utility of trigonometry as a tool for modeling and understanding the world around us. Its capacity to describe circular motion and oscillatory behavior makes it invaluable in many branches of science and technology.

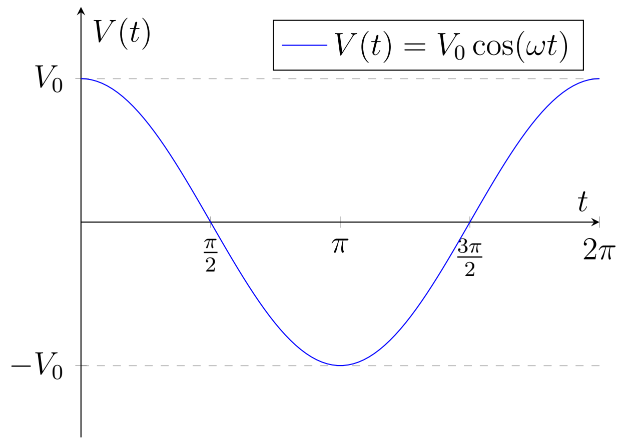


Figure 13.18: Voltage in an AC circuit as a function of time modeled by a cosine function.

13.8.5 Practice Problems

To solidify your understanding of the applications of trigonometry in various fields, try to solve the following problems:

1. **Sound Waves:** A sound wave is modeled by the equation $f(t) = 3\sin(2\pi \cdot 440t)$, where $f(t)$ is the displacement of the wave at time t and the constant 440 represents the frequency in hertz (Hz). What is the amplitude and period of this sound wave?

2. **Electrical Engineering:** An alternating current (AC) in an electrical circuit is represented by the function $V(t) = V_0 \sin(\omega t + \phi)$, where $V_0 = 120$ V is the maximum voltage, $\omega = 377$ rad/s is the angular frequency, and $\phi = \frac{\pi}{6}$ rad is the phase shift. Determine the instantaneous voltage at $t = 0.01$ seconds.

3. **Astronomy:** If a star exhibits a parallax angle of 0.05 arcseconds as observed from the Earth, and the baseline distance is 1 astronomical unit (AU), calculate the distance to the star in parsecs. Recall that 1 parsec corresponds to an angle of 1 arcsecond.

4. **Calculus Application:** Find the integral $\int \sin^2(x)\,dx$. Hint: Use the trigonometric identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ to simplify the integral.

5. **Modeling Motion:** A Ferris wheel with a radius of 10 meters makes a complete rotation every 40 seconds. If a passenger gets on at the lowest point at $t = 0$, derive the function that models the height of the passenger above the ground over time.

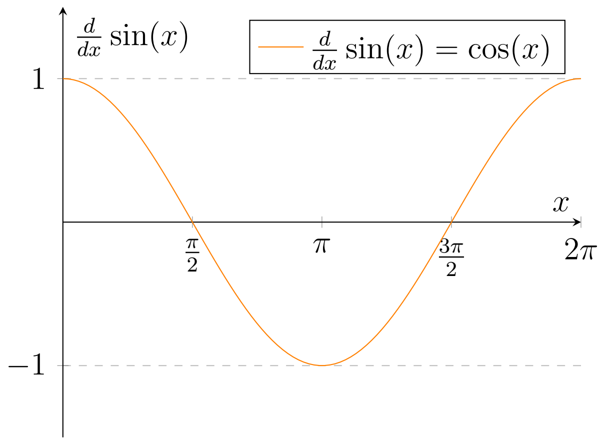


Figure 13.19: The derivative of the sine function with respect to x is the cosine function.

Each problem incorporates trigonometric functions and requires an understanding of their properties to solve. The astronomy problem involves angular measurements and distances, the sound wave problem explores the wave's properties, the electrical engineering problem relates to AC circuits, and the calculus problem requires knowledge of integrals and trigonometric identities. The modeling motion problem provides an application of periodic functions in mechanical motion.

These problems are designed to be challenging and to encourage students to apply trigonometric principles to a variety of contexts.

13.8.6 Harmonic Motion

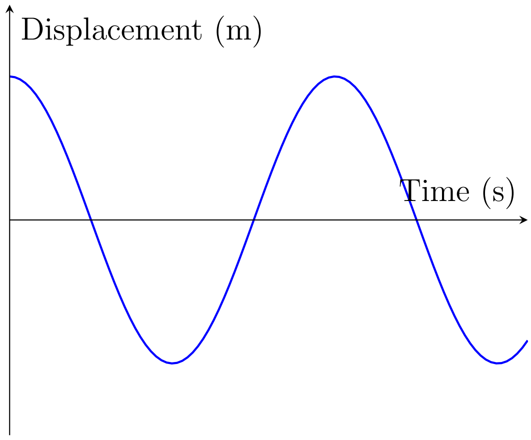
Harmonic motion is a type of periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement. Simple harmonic motion (SHM) can be described by the following equation, which represents a sinusoidal function:

$$x(t) = A\cos(\omega t + \phi)$$

where:

- $x(t)$ is the displacement from the equilibrium position at time t,
- A is the amplitude of the motion,
- ω is the angular frequency, and
- ϕ is the phase angle.

This model can be visualized using a graph of the cosine function, which depicts the oscillation of an object in SHM.



The graph above shows the displacement of an object in simple harmonic motion over time. The maximum displacement on either side of the equilibrium position is known as the amplitude A, and the time it takes to complete one full cycle is called the period T. The relationship between the period and the angular frequency is given by $T = \frac{2\pi}{\omega}$.

In the context of calculus, we can derive the velocity and acceleration of an object in SHM by taking the first and second derivatives of the displacement function with respect to time:

$$v(t) = \frac{dx}{dt} = -A\omega \sin(\omega t + \phi)$$

$$a(t) = \frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x(t)$$

The acceleration $a(t)$ is always directed towards the equilibrium position, and its magnitude is proportional to the displacement, which is characteristic of simple harmonic motion.

13.8.7 Practice Problems on Harmonic Motion

Solve the following problems related to simple harmonic motion:

1. An object is undergoing simple harmonic motion with an amplitude of 5 cm and a period of 2 seconds. Find the angular frequency ω and write the equation for the displacement $x(t)$ as a function of time.

2. Given the displacement function $x(t) = 3\sin(2t + \frac{\pi}{4})$, determine the following:

   (a) The amplitude of motion.

   (b) The angular frequency and the period of the motion.

   (c) The phase shift.

   (d) The initial displacement at $t = 0$.

3. If the displacement of a particle is described by the equation $x(t) = 4\cos(5t)$, calculate the velocity and acceleration of the particle at $t = \pi$.

4. A particle moves with simple harmonic motion in such a way that its acceleration at any time t is given by $a(t) = -16x(t)$. If the maximum displacement is 2 meters, find the displacement equation $x(t)$ assuming the initial phase is zero.

5. A mass-spring system has a mass of 0.5 kg and a spring constant $k = 100\,\mathrm{N/m}$. Determine:

   (a) The angular frequency of the system.

   (b) The period of the oscillations.

   (c) The equation of motion if the initial displacement is 0.1 m and initial velocity is zero.

Solutions

Here you can provide the solutions for the above problems, or you may choose to leave them for the students to solve.

1.

2.

3.

4.

5.

13.8.8 Trigonometric Substitution

This section will discuss how trigonometric substitution is used in integration, a technique that simplifies integrals involving square roots. Trigonometric substitution is a technique for evaluating integrals that enables us to simplify integrals involving square roots by exploiting trigonometric identities. This method is especially useful when dealing with expressions under the square root that resemble the Pythagorean theorem, such as $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$. By substituting x with a trigonometric expression involving θ, the integral can often be rewritten in a more manageable form.

The basic idea is to substitute variables in the integral with trigonometric functions that transform the integrand into a trigonometric identity, which can be easily integrated. The three main forms of substitution are:

- For integrands containing $\sqrt{a^2 - x^2}$, use $x = a\sin(\theta)$.
- For integrands containing $\sqrt{a^2 + x^2}$, use $x = a\tan(\theta)$.
- For integrands containing $\sqrt{x^2 - a^2}$, use $x = a\sec(\theta)$.

After substituting, the integral is typically in terms of θ, which can be integrated using standard trigonometric integrals. Finally, it's necessary to convert back to the original variable using the inverse trigonometric functions.

Example 1: Evaluate the integral $\int \frac{dx}{\sqrt{9-x^2}}$.

Solution: Use the substitution $x = 3\sin(\theta)$, hence $dx = 3\cos(\theta)d\theta$ and $\sqrt{9 - x^2} = 3\cos(\theta)$. The integral becomes

$$\int \frac{3\cos(\theta)d\theta}{3\cos(\theta)} = \int d\theta = \theta + C.$$

To return to the variable x, we use $\sin^{-1}(\frac{x}{3}) = \theta$, so the solution is

$$\sin^{-1}\left(\frac{x}{3}\right) + C.$$

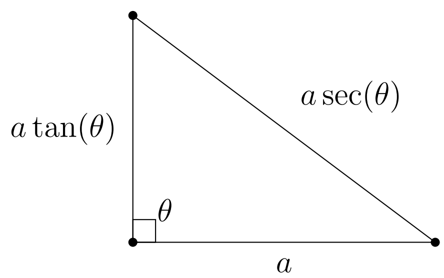
Example 2: Determine the integral $\int \frac{x^2}{\sqrt{x^2+4}}dx$.

Solution: Use the substitution $x = 2\tan(\theta)$, which implies $dx = 2\sec^2(\theta)d\theta$ and $\sqrt{x^2 + 4} = 2\sec(\theta)$. The integral can then be written as

$$\int \frac{4\tan^2(\theta)2\sec^2(\theta)d\theta}{2\sec(\theta)} = 4\int \tan^2(\theta)\sec(\theta)d\theta.$$

This can be integrated using trigonometric identities and the substitution method for integrals.

Note: For each trigonometric substitution, it's important to draw a right triangle to relate the substitution back to the original variable for finding the bounds of integration or for writing the final answer in terms of the original variable.



In these diagrams, the hypotenuse represents x after substitution, and the other two sides represent the expressions involved in the integral after substitution.

13.8.9 Practice Problems on Trigonometric Substitution

The following practice problems will help reinforce the concept of trigonometric substitution in integrals. Solve the given integrals using appropriate trigonometric substitutions.

Problem 1: Evaluate the integral $\int \frac{dx}{x^2\sqrt{4x^2-1}}$.

Problem 2: Determine the integral $\int \frac{\sqrt{x^2+2x+1}}{x}dx$. (Hint: Complete the square for the expression under the square root first.)

Problem 3: Compute the integral $\int \sqrt{25-x^2}\,dx$.

Problem 4: Calculate the integral $\int \frac{dx}{(x^2+4)^{3/2}}$.

Problem 5: Find the integral $\int x^3\sqrt{x^2+9}\,dx$.

Remember to use the trigonometric substitution rules outlined in Section 13.8.8:

- For $\sqrt{a^2-x^2}$, substitute $x = a\sin(\theta)$.
- For $\sqrt{a^2+x^2}$, substitute $x = a\tan(\theta)$.
- For $\sqrt{x^2-a^2}$, substitute $x = a\sec(\theta)$.

For each problem, ensure that you:

1. Identify the correct form of trigonometric substitution to use.

2. Draw a right triangle to relate trigonometric substitution back to the original variable if necessary.

3. Solve the integral in terms of θ.

4. Convert back to the original variable x using the appropriate inverse trigonometric function.

Note: You may need to use additional trigonometric identities or integration techniques to find the final solution.

Conclusion

This chapter serves as the foundation for understanding the vital role of Trigonometry in calculus. As you progress through your studies, these concepts will become second nature, providing the tools necessary for analyzing and solving a broad range of problems in higher mathematics.

Appendix I

Basic GitHub Guide

A Quick Start to Your GitHub Journey

Welcome to the fascinating world of GitHub, a platform that has revolutionized the way we collaborate on projects, share code, and build software together. Whether you are a programmer, a writer, or a mathematician, GitHub provides a set of powerful tools to help you collaborate with others, manage your projects, and contribute to the vast world of open-source software. In this guide, we will walk you through the foundational steps to get started with GitHub, helping you to navigate, contribute, and make the most out of this incredible platform.

Creating Your GitHub Account

The first step to joining the GitHub community is to create an account. Here's how you can do it:

1. Visit the GitHub website.
2. Click on the "Sign up" button.
3. Fill in the required information, including your username, email address, and password.
4. Verify your account and complete the sign-up process.

Once you have created your account, take a moment to explore your new GitHub dashboard. Here, you will find a variety of tools and features that will help you manage your projects, collaborate with others, and discover new and interesting repositories.

Creating Your First Repository

A repository (or “repo”) is a digital directory where you can store your project files. Here’s how you can create your first repository:

1. From your GitHub dashboard, click on the “New” button to create a new repository.
2. Give your repository a name and provide a brief description.
3. Initialize this repository with a README file. (This is an optional step, but it’s a good practice to include a README file in every repository to explain what your project is about.)
4. Click “Create repository.”

Congratulations! You have just created your first GitHub repository. You can now start adding files, collaborating with others, and managing your project right from GitHub.

Making Changes and Commits

GitHub uses Git, a version control system, to keep track of changes made to your project. Here’s a quick guide on how to make changes and commits:

1. Navigate to your repository on GitHub.
2. Find the file you want to edit, and click on it.
3. Click the pencil icon to start editing.
4. Make your changes and then scroll down to the “Commit changes” section.
5. Provide a commit message that explains the changes you made.
6. Choose whether you want to commit directly to the main branch or create a new branch for your changes.
7. Click “Commit changes.”

Your changes are now saved, and a new commit is created. Every commit has a unique ID, making it easy to track changes, revert to previous versions, and collaborate with others.

Collaborating with Others

One of the biggest strengths of GitHub is its collaborative nature. Here are some ways you can collaborate with others:

- **Forking:** You can fork a repository, create your own copy, make changes, and then propose those changes back to the original project.
- **Issues:** Use issues to report bugs, request new features, or start a discussion with the community.
- **Pull Requests:** Propose changes to a project by creating a pull request. This allows others to review your changes, discuss them, and eventually merge them into the project.

Conclusion: Embarking on Your GitHub Adventure

Now that you have a basic understanding of GitHub and how it works, you are ready to embark on your GitHub adventure. Explore repositories, contribute to open-source projects, collaborate with others, and build amazing things together. Remember, the GitHub community is vast and supportive, and there is a wealth of knowledge and resources available to help you along the way. Happy coding!

Appendix II

Basic LaTeX Guide

A Quick Start to Your LaTeX Journey

Welcome to the immersive world of LaTeX, a typesetting system widely used for creating scientific and professional documents due to its powerful handling of formulas and bibliographies. This guide is designed to offer you the foundational steps to grasp the basics of LaTeX, enabling you to craft documents of high typographic quality akin to this book.

Setting Up Your LaTeX Environment

Before you can start creating documents with LaTeX, you need to set up a working LaTeX environment on your computer. Here's how you can do it:

1. Download and install a TeX distribution, which includes LaTeX. For Windows, MiKTeX is a popular choice, while Mac users might prefer MacTeX, and TeX Live is widely used on Linux.

2. Install a LaTeX editor. Some popular options include TeXShop (for Mac), TeXworks (cross-platform), and Overleaf (an online LaTeX editor).

3. Ensure that your TeX distribution and LaTeX editor are properly configured and integrated.

Creating Your First LaTeX Document

Once your LaTeX environment is set up, you are ready to create your first LaTeX document. Follow these steps:

1. Open your LATEX editor and create a new document.
2. Insert the following code to set up a basic LATEX document:

```
\documentclass{article}
\begin{document}
Hello, \LaTeX\ world!
\end{document}
```

3. Save your document with a .tex file extension.
4. Compile your document using your LATEX editor. This process converts your .tex file into a PDF document.
5. View the output PDF and admire your first LATEX creation.

Understanding LATEX Commands and Environments

LATEX documents are created using a series of commands and environments. Commands typically start with a backslash \ and are used to format text, insert special characters, or execute functions. Environments are used to define specific sections of your document that require special formatting.

- **Commands:** For example, \{italics} will render the word "italics" in italic font.
- **Environments:** To create a bulleted list, you would use the *itemize* environment:

```
\begin{itemize}
    \item First item
    \item Second item
\end{itemize}
```

Adding Structure to Your Document

LATEX makes it easy to structure your documents with sections, subsections, and chapters. Here's how you can add structure:

```
\section{Introduction}
This is the introduction of your document.
\subsection{Background}
This subsection provides background information.
\subsubsection{Details}
This is a subsubsection for more detailed information.
```

Including Mathematical Formulas

LaTeX excels at typesetting mathematical formulas. Use the *equation* environment or the $ sign for inline formulas. For example:

```
The quadratic formula is \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).
```

Adding Images and Tables

You can also include images and tables in your LaTeX documents:

- **Images:** Use the *graphicx* package and the *includegraphics* command.
- **Tables:** Use the *tabular* environment to create tables.

Compiling Your Document

LaTeX documents need to be compiled to produce a PDF. This can be done through your LaTeX editor. If your document includes bibliographies or cross-references, you may need to compile multiple times.

Conclusion: Embracing the Power of LaTeX

Congratulations! You have taken your first steps into the world of LaTeX. With practice, you will discover that LaTeX is a powerful tool for creating professional-quality documents, from simple articles to complex books. Embrace the learning curve, explore the vast array of packages available, and join the community of LaTeX users who are ready to help you on your journey. Happy typesetting!

Bibliography

[MT77] F. Mosteller and J. W. Tukey. *Data Analysis and Regression: A Second Course in Statistics*. Addison-Wesley Pub Co, Reading, MA, 1977.